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HOPF DENSE GALOIS EXTENSIONS WITH APPLICATIONS

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ABSTRACT. Let H be a finite dimensional Hopf algebra, and let A be a left H -module algebra. Motivated by the study of the isolated singularities of A^H and the endomorphism ring $\text{End}_{A^H}(A)$, we introduce the concept of Hopf dense Galois extensions in this paper. Hopf dense Galois extensions yield certain equivalences between the quotient categories over A and A^H . A special class of Hopf dense Galois extensions consists of the so-called densely group graded algebras, which are weaker versions of strongly graded algebras. A weaker version of Dade's Theorem holds for densely group graded algebras. As applications, we recover the classical equivalence of the noncommutative projective scheme over a noetherian \mathbb{N} -graded algebra A and its d -th Veronese subalgebra $A^{(d)}$ respectively. Hopf dense Galois extensions are also applied to the study of noncommutative graded isolated singularities.

INTRODUCTION

The study of Hopf algebra actions on Artin-Schelter regular algebras is always an important subject in the field of noncommutative algebraic geometry. Many interesting results are obtained in recent years (for instance, [Ue, MU, CKWZ1, CKWZ2, KKZ1, KKZ2, KKZ3, EW, WW] etc.). Let H be a Hopf algebra, and let A be a left H -module algebra. Assume A is also an Artin-Schelter regular algebra. In general, the invariant subalgebra A^H of A is not regular. Indeed, it was proved in [KKZ2] that the invariant subalgebra of a finite group action on a quantum polynomial algebra is regular if and only if the group is generated by quasi-reflections. If the Hopf algebra action satisfies some good conditions, say the homological determinant of H is trivial (for the terminology, see [JZ, KKZ1]), then the invariant subalgebra A^H is Artin-Schelter Gorenstein (cf. [KKZ1, KKZ3]). So, in this case, one asks about the properties of the singularities of A^H . Ueyama called a noetherian connected graded algebra A a *graded isolated singularity* if the quotient category $\text{tail } A$ has finite global dimension (cf. [Ue]). He proved in [Ue] that a finite group action on an Artin-Schelter regular algebra of global dimension 2 with trivial homological determinant always yields a graded isolated singularity.

Let G be a finite group which acts homogeneously on an Artin-Schelter regular algebra A . If the invariant subalgebra A^G is a graded isolated singularity, then there is an equivalence of abelian categories $\text{tail } A * G \cong \text{tail } A^G$ (cf. [Ue, MU]), where $A * G$ is the skew group algebra of A and G . More generally, Mori and Ueyama call a finite group action on A *ample* if there is an equivalence $\text{tail } A * G \cong \text{tail } A^G$ (cf. [MU]). We found that the whole theory of graded isolated singularities and ample group actions on Artin-Schelter regular algebras fits into a more general theory, *Hopf dense Galois extensions*, which we will introduce in this paper. Our aim of this paper is to develop a general theory on Hopf dense Galois extensions, and then in this framework to understand the finite dimensional Hopf-actions on Artin-Schelter regular algebras.

Key words and phrases. Hopf dense Galois extension, densely graded algebra, quotient category.

Let H be a finite dimensional Hopf algebra, and A a left H -module algebra. We may view A as a right H^* -comodule algebra with a coaction map $\rho : A \rightarrow A \otimes H^*$. Recall that the algebra extension A/A^H is called a *right H^* -Galois extension* if the map $\beta : A \otimes_{A^H} A \rightarrow A \otimes H^*$, $a \otimes b \mapsto \sum_{(b)} ab_{(0)} \otimes b_{(1)}$ is an isomorphism, where $\rho(b) = \sum_{(b)} b_{(0)} \otimes b_{(1)}$. When we study H -actions on infinite dimensional algebras, say Artin-Schelter regular algebras, the canonical map β is not often an isomorphism. So, we have to modify the definition of the classical Hopf Galois extension. We require that the map β is almost surjective, that is, the cokernel of β is finite dimensional. In this case, we call A/A^H a *right H^* -dense Galois extension* (for more details, see Section 1). It turns out that this is an efficient way to study the noncommutative projective scheme over the invariant subalgebra A^H , especially when A^H is a graded isolated singularity.

The paper is organized as follows. In Section 1, we introduce the definition of Hopf dense Galois extensions. In Section 2, we prove some general equivalences of abelian categories for Hopf dense Galois extensions. Let H be a finite dimensional semisimple Hopf algebra, and let A be a left H -module algebra. In the theory of the classical Hopf Galois extensions, the module ${}_{A\#H}A$ is a projective generator of the category of left $A\#H$ -modules, and $\text{End}({}_{A\#H}A) \cong A\#H$ as algebras (cf. [CFM, Theorem 1.9]). However, for the Hopf dense Galois extensions, these properties are not true in general. We need more restrictions on the algebra A . In Section 3, we show that if A satisfies some additional homological properties, then we have an isomorphism $\text{End}({}_{A\#H}A) \cong A\#H$. The main results of Sections 2 and 3 are the following theorem (cf. Theorems 2.3 and 3.8). The notions in the theorem will be introduced in the corresponding sections.

Theorem. *Let H be a finite dimensional semisimple Hopf algebra, and let A be a left H -module algebra. Assume that A is also noetherian (on both sides). Let $B = A\#H$, $R = A^H$ and t a nonzero integral of H . The following statements (i) to (iv) are equivalent.*

- (i) A/R is a right H^* -dense Galois extension.
- (ii) For any finitely generated right B -module M , $T(M)$ is finite dimensional, where $T(M)$ is the largest t -torsion submodule of M (cf. Section 2).
- (iii) $B/(BtB)$ is finite dimensional.
- (iv) $-\otimes_B \mathcal{A} : \text{QMod } B \rightarrow \text{QMod } R$ is an equivalence of abelian categories.

If we assume further that $\text{depth } A_A \geq 2$, then the above statements implies that

(v) *the natural map*

$$A\#H \longrightarrow \text{End}({}_{A\#H}A), a\#h \mapsto [b \mapsto a(h \cdot b)]$$

is an isomorphism of algebras (cf. Section 3).

The statement (v) above follows from a more general result about equivalences of quotient categories (cf. Theorem 3.6). We remark that a part of the theorem above was obtained in [MU, Theorem 2.13] in the case that H is a finite group algebra and A is \mathbb{N} -graded.

A special class of Hopf dense Galois extensions are the so called *densely group graded algebras* which we will introduce in Section 4. It is a weaker version of strongly graded algebras (cf. [NV]). We have the following version of Dade's Theorem (cf. Theorem 4.7).

Theorem. *Let G be a finite group, and let $A = \bigoplus_{g \in G} A_g$ be a G -graded algebra. Assume that A is a noetherian algebra. Then the following are equivalent.*

- (i) *A is a densely G -graded algebra.*
- (ii) *For any finitely generated G -graded A -module $M = \bigoplus_{g \in G} M_g$, if M_e is finite dimensional, then M itself is finite dimensional.*
- (iii) *The functor $(-)_e : \text{QGrMod}_G A \rightarrow \text{QMod } A_e$ is an equivalence of abelian categories.*

For any \mathbb{Z} -graded algebra $A = \bigoplus_{i \in \mathbb{Z}} A_i$, we may view A as a \mathbb{Z}_d -graded algebra. Indeed, if we set $B_{\bar{i}} = \bigoplus_{r \in \mathbb{Z}} A_{rd+i}$ for all $\bar{i} \in \mathbb{Z}_d$ ($0 \leq i \leq d-1$), then $B = \bigoplus_{\bar{i} \in \mathbb{Z}_d} B_{\bar{i}}$ is a \mathbb{Z}_d -graded algebra. Note that $B_{\bar{0}} = A^{(d)} = \bigoplus_{r \in \mathbb{Z}} A_{rd}$ is the d th Veronese subalgebra of A . As an applications of densely graded algebras, in Section 5 we recover the classical result about the equivalence of the categories of the noncommutative projective schemes $\text{tail } A^{(d)}$ and $\text{tail } A$. Indeed, we have the following more general result (cf. Theorem 5.2), which is a consequence of a more general result (cf. Theorem 4.9).

Theorem. *Let $A = A_0 \oplus A_1 \oplus \cdots$ be a locally finite noetherian \mathbb{N} -graded algebra and let d be a fixed positive integer. The following are equivalent.*

- (i) *There is an integer $p > 0$ such that, for all $n \geq p$ and $0 \leq s \leq d-1$,*

$$A_{nd} = \sum_{\substack{i+j=n \\ i \geq 0, j \geq 1}} A_{id+s} A_{jd-s}.$$

- (ii) *For any finitely generated right graded A -module M , if $M^{(d)} = \bigoplus_{n \in \mathbb{Z}} M_{nd}$ is finite dimensional, then M itself is finite dimensional.*
- (iii) *$(-)^{(d)} : \text{Tail } A \rightarrow \text{Tail } A^{(d)}$ is an equivalence of abelian categories.*
- (iv) *$(-)^{(d)} : \text{tail } A \rightarrow \text{tail } A^{(d)}$ is an equivalence of abelian categories.*

Note that the condition (i) in the above theorem is relatively easy to verify. For example, if the graded algebra A is generated by elements in A_0 and A_1 , then (i) is satisfied. Hence we recover the classical result (cf. Theorem 5.1). We remark that Mori also provided some equivalent conditions for the equivalence of $\text{tail } A$ and $\text{tail } A^{(d)}$ in [Mo, Theorem 3.5]. Note that it is assumed in [Mo] that A is a connected graded algebra satisfying further homological conditions. In our case, we only assume that A is a locally finite noetherian algebra, and we prove the result in the framework of Hopf dense Galois extension, which seems to be a new way to understand noncommutative projective schemes.

In Section 6, we provide another application of Hopf dense Galois extensions to graded isolated singularities. We prove that the invariant subalgebra A^H is a graded isolated singularity if A/A^H is a H^* -dense Galois extension and A has finite global dimension (cf. Corollary 6.2). Some further results about the endomorphism ring of A_{A^H} are obtained.

Throughout the paper, \mathbb{k} is a field of characteristic zero. All the algebras and modules are over \mathbb{k} , and the symbol \otimes we means $\otimes_{\mathbb{k}}$. For the basic properties of classical Galois extensions, we refer to the references [CFM] and [Mon]. For the basic properties of quotient categories, we refer to the books [St] and [PP].

1. HOPF DENSE GALOIS EXTENSIONS

Let H be a Hopf algebra with the coproduct Δ and antipode S . Let A be a right H -comodule algebra, that is, A is an algebra together with a right H -comodule action $\rho : A \rightarrow A \otimes H$, satisfying the compatibility condition

$$\rho(ab) = \sum_{(b)(a)} a_{(0)}b_{(0)} \otimes a_{(1)}b_{(1)},$$

where we use Sweedler's sigma notation. The coinvariants of the H -coaction are defined to be $A^{coH} = \{a \in A \mid \rho(a) = a \otimes 1\}$, which is a subalgebra of A . If H is finite dimensional, then H^* is a Hopf algebra. The right H -coaction on A induces a left H^* -action by setting $f \cdot a = \sum_{(a)} f(a_{(1)})a_{(0)}$. Then A is a left H^* -module algebra. The invariant subalgebra of H^* -action on A is defined to be $A^{H^*} = \{a \in A \mid f \cdot a = f(1)a, \forall f \in H^*\}$. One has $A^{coH} = A^{H^*}$ in case H is finite dimensional.

Recall from [KT] that the algebra extension A/A^{coH} is said to be *right H -Galois* if the map

$$\beta : A \otimes_{A^{coH}} A \rightarrow A \otimes H, a \otimes b \mapsto (a \otimes 1)\rho(b)$$

is surjective. In this case, A^{coH} shares many common properties with the smash product $A \# H^*$. Indeed, A^{coH} is Morita equivalent to $A \# H^*$ (cf. [CFM, Theorem 1.2]). In many interesting examples in noncommutative algebraic geometry, the algebra extension A/A^{coH} may not be H -Galois, but still the coinvariant subalgebra A^{coH} shares some common properties with $A \# H^*$. It seems necessary to introduce a more general definition than the one of an H -Galois extension.

We first introduce some terminologies. Let B be an algebra, and M and N left B -modules. Let $f : M \rightarrow N$ be a left B -module morphism. We say that f is *almost surjective* if the image of f is cofinite, that is, $N/\text{im}(f)$ is finite dimensional. We say that f is *almost injective* if $\ker f$ is finite dimensional. If f is both almost surjective and almost injective, we say that f is an *almost isomorphism*.

Definition 1.1. *Let H be a Hopf algebra, and let A be a right H -comodule algebra. We say that the algebra extension A/A^{coH} is a right H -dense Galois extension if the map*

$$\beta : A \otimes_{A^{coH}} A \rightarrow A \otimes H, a \otimes b \mapsto (a \otimes 1)\rho(b)$$

is almost surjective.

Note that if A is finite dimensional, then any coaction of a finite dimensional Hopf algebra on A yields an H -dense Galois extension. Hence the definition is trivial for finite dimensional comodule algebras. When the comodule algebra A is infinite dimensional, there will be many interesting examples in noncommutative algebraic geometry. We give the following example. More examples will be given in following sections.

Example 1.2. Let $\sigma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and let $G = \langle \sigma \rangle$ be the cyclic group of order 2. Let $A = \mathbb{k}[x, y]$ be the polynomial algebra. We view A as an \mathbb{N} -graded algebra. Then G acts homogeneously on A . Let $\mathbb{k}G$ be the group algebra, and let $H = \mathbb{k}G^*$. Then A is a right H -comodule algebra. Let e be the unit of G . Then $\{e, \sigma\}$ is a basis of $\mathbb{k}G$. Let $\{e^*, \sigma^*\}$ be the dual basis of H . Set $1_H = e^* + \sigma^*$ and $h = e^* - \sigma^*$. Then 1_H is the unit of H and $h^2 = 1_H$. The induced H -coaction $\rho : A \rightarrow A \otimes H$

is defined by

$$\rho(x^i y^j) = \begin{cases} x^i y^j \otimes 1_H, & i + j \text{ is even;} \\ x^i y^j \otimes h, & i + j \text{ is odd.} \end{cases}$$

Then one sees that $A^{coH} = \bigoplus_{n \geq 0} A_{2n}$. The map $\beta : A \otimes_{A^{coH}} A \rightarrow A \otimes H$ is as follows

$$\beta(x^i y^j \otimes x^r y^s) = \begin{cases} x^{i+r} y^{j+s} \otimes 1_H, & r + s \text{ is even;} \\ x^{i+r} y^{j+s} \otimes h, & r + s \text{ is odd.} \end{cases}$$

Now it is easy to see that $\dim((A \otimes H)/\text{im } \beta) = 1$. Hence A/A^{coH} is a right H -dense Galois extension.

Similar to the case of H -Galois extensions, we have the following property of H -dense Galois extensions.

Proposition 1.3. *Let H be a finite dimensional Hopf algebra, and let A be a left H -module algebra. Then A/A^H is a right H^* -dense Galois extension if and only if the map $[\ , \] : A \otimes_{A^H} A \rightarrow A \# H, a \otimes b \mapsto (a \# t)(b \# 1)$ is almost surjective, where t is a nonzero integral of H .*

Proof. Since H is finite dimensional, the map $\theta : H^* \rightarrow H$ defined by $\theta(f) = t \leftarrow f$ is a right H^* -module isomorphism, where ' \leftarrow ' is the right H^* -module action on H . As showed in the proof of [CFM, Theorem 1.2], $[\ , \] = (\text{id} \otimes \theta) \circ \beta$ as maps. Hence $[\ , \]$ is almost surjective if and only if β is almost surjective.

2. GENERAL EQUIVALENCES OF QUOTIENT CATEGORIES

Let B be a noetherian algebra. We write $\text{Mod } B$ for the category of right B -modules, and $\text{mod } B$ for the full subcategory of $\text{Mod } B$ consisting of finitely generated right B -modules. Let M_B be a right B -module. We call M a *torsion module* if for any $m \in M$, the submodule mB is finite dimensional. Note that every B -module admits a largest torsion submodule. We write $\text{Tor } B$ for the full subcategory of $\text{Mod } B$ consisting of all the torsion modules. Let $\text{tor } B$ be the intersection of $\text{mod } B$ and $\text{Tor } B$.

Since B is noetherian, we see that $\text{Tor } B$ (resp. $\text{tor } B$) is a Serre subcategory of $\text{Mod } B$ (resp. $\text{mod } B$). Define

$$\text{QMod } B := \frac{\text{Mod } B}{\text{Tor } B}, \quad \text{qmod } B := \frac{\text{mod } B}{\text{tor } B}.$$

Then both $\text{QMod } B$ and $\text{qmod } B$ are abelian categories.

Throughout the rest of this section, let H be a finite dimensional semisimple Hopf algebra, and let A be a left H -module algebra. Set $R = A^H$ to be the invariant subalgebra of A , and set $B := A \# H$. Assume that A is noetherian. Then both B and R are noetherian, moreover, A viewed as a left (or a right) R -module is finitely generated (cf. [Mon]).

We introduce another torsion class in the category of B -modules. Let $t \in H$ be the integral in H such that $\varepsilon(t) = 1$. Hence $t^2 = t$. We view t as an element of B by identifying t with $1 \# t$. The ideal of B generated by t is written as BtB . We call a right B -module M a *t -torsion module* if for every element $m \in M$ we have mBt is finite dimensional. Let \mathcal{T} be the full subcategory of $\text{Mod } B$ consisting of all the t -torsion modules. By the definition, we see that $\text{Tor } B$ is a full subcategory

of \mathcal{T} . Let N be a right B -module. Write $T(N)$ for the sum of all the t -torsion submodules of N . Then $T(N)$ is the largest t -torsion submodule of N . Hence, we indeed obtain a functor

$$T : \text{Mod } B \rightarrow \mathcal{T}.$$

Note that $R \cong tBt$ as algebras. We have $Mt \in \text{Mod } R$ for any $M \in \text{Mod } B$.

Lemma 2.1. (i) *If $M \in \mathcal{T}$ is a finitely generated B -module, then Mt is finite dimensional.*

(ii) *$M \in \mathcal{T}$ if and only if $Mt \in \text{Tor } R$.*

(iii) *\mathcal{T} is a localizing Serre subcategory of $\text{Mod } B$.*

Proof. The first statement is clear.

(ii) Assume $M \in \mathcal{T}$. For $m \in M$, we have $(mt)R \subset mBt$. Hence the R -submodule of Mt generated by mt is finite dimensional. Therefore, $Mt \in \text{Tor } R$. On the contrary, assume $Mt \in \text{Tor } R$. By assumption, A is a noetherian algebra. Hence A , viewed as a right R -module, is finitely generated. So, Bt is a finitely generated R -module. Take an arbitrary element $m \in M$. We see that mBt is finitely generated R -module. On the other hand, $mBt \subset Mt$, implying that mBt is a finitely generated module in $\text{Tor } R$. Hence mBt is finite dimensional. Therefore $M \in \mathcal{T}$.

(iii) One sees that \mathcal{T} is closed under taking arbitrary direct sums. Let M be a B -module, and let K be a submodule of M . Consider an exact sequence $0 \rightarrow K \hookrightarrow M \xrightarrow{g} N \rightarrow 0$. If $M \in \mathcal{T}$, then one sees both K and N are in \mathcal{T} . Assume K and N are in \mathcal{T} . Take an element $m \in M$. The exact sequence above induces an exact sequence $0 \rightarrow mB \cap K \rightarrow mB \rightarrow g(m)B \rightarrow 0$ of right B -modules, which induces an exact sequence $0 \rightarrow (mB \cap K)t \rightarrow mBt \rightarrow g(m)Bt \rightarrow 0$ since the functor $(-)t : \text{Mod } B \rightarrow \text{Mod } R$ is exact. Since K and N are in \mathcal{T} , both $(mB \cap K)t$ and $g(m)Bt$ are finite dimensional. Hence mBt is finite dimensional. Therefore M is in \mathcal{T} . \square

Consider the bimodules ${}_B A_R$. The functor $- \otimes_B A : \text{Mod } B \rightarrow \text{Mod } R$ has a right adjoint functor $\text{Hom}_R({}_B A_R, -)$. Recall that $R \cong tBt$ as algebras, $Bt \cong A$ as B - R -bimodules. For any right R -module N , we have the following isomorphisms of right R -modules

$$\begin{aligned} \text{Hom}_R({}_B A_R, N) \otimes_B A_R &\cong \text{Hom}_R(Bt, N) \otimes_B Bt \\ &\cong \text{Hom}_B(tB, \text{Hom}_R(Bt, N)) \\ &\cong \text{Hom}_R(tB \otimes_B Bt, N) \\ &\cong N. \end{aligned}$$

The isomorphisms are natural on N . Hence we have a natural isomorphism $(- \otimes_B A) \circ \text{Hom}_R({}_B A_R, -) \cong \text{id}_{\text{Mod } R}$.

Consider the quotient category $\text{QMod } R = \frac{\text{Mod } R}{\text{Tor } R}$. Let $\pi : \text{Mod } R \rightarrow \text{QMod } R$ be the natural projection functor. Since $\text{Tor } R$ is a localizing subcategory, π has a right adjoint functor $\omega : \text{QMod } R \rightarrow \text{Mod } R$. Thus we have the following pairs of adjoint functors:

$$\text{Mod } B \begin{array}{c} \xrightarrow{- \otimes_B A_R} \\ \xleftarrow{\text{Hom}_R({}_B A_R, -)} \end{array} \text{Mod } R \begin{array}{c} \xrightarrow{\pi} \\ \xleftarrow{\omega} \end{array} \text{QMod } R.$$

Let $F = \pi \circ (- \otimes_B A_R)$ and $G = \text{Hom}_R({}_B A_R, -) \circ \omega$. Then (F, G) is a pair of adjoint functors. Since $(- \otimes_B A_R) \circ \text{Hom}_R({}_B A_R, -)$ is isomorphic to $\text{id}_{\text{Mod } R}$ and $\pi \circ \omega$ is isomorphic to $\text{id}_{\text{QMod } R}$, it

follows that FG is isomorphic to $\text{id}_{\text{QMod } R}$. Since H is semisimple, ${}_B A$ is a projective module. Hence $-\otimes_B A_R$ is an exact functor. Therefore F is an exact functor. Then a classical result of torsion theory (cf. [PP, Theorem 7.11, Chapter 4]) shows that $\ker F$ is a localizing Serre subcategory of $\text{Mod } B$ and F induces an isomorphism of abelian categories

$$(1) \quad \overline{F} : \frac{\text{Mod } B}{\ker F} \cong \text{QMod } R.$$

Let us check the objects in $\ker F$. For $M \in \text{Mod } B$, we have $F(M) = \pi(M \otimes_B A) = \pi(Mt)$. Hence $F(M) = 0$ if and only if $Mt \in \text{Tor } R$. By Lemma 2.1(ii), $M \in \ker F$ if and only if $M \in \mathcal{T}$.

In summarizing, we have the following result, which may be viewed as a generalization of [VZ, Theorem 2.4], and also can be compared to [Ga, Theorem 4.6].

Theorem 2.2. *Let H be a finite dimensional semisimple Hopf algebra, and let A be a noetherian left H -module algebra. Let $B = A\#H$ and $R = A^H$. Then we have an equivalence of abelian categories*

$$\text{Mod } B/\mathcal{T} \cong \text{QMod } R.$$

Since $\text{Tor } B$ is a full subcategory of \mathcal{T} , it is indeed a Serre subcategory of \mathcal{T} . Hence we have a quotient category $\mathcal{T}/\text{Tor } B$. Moreover, $\mathcal{T}/\text{Tor } B$ is a full subcategory of $\text{QMod } B$. Since $-\otimes_B A$ is an exact functor, it induces an exact functor

$$(2) \quad -\otimes_B A : \text{QMod } B \longrightarrow \text{QMod } R.$$

The exact functor F defined above induces an exact sequence of abelian categories

$$(3) \quad 0 \longrightarrow \mathcal{T}/\text{Tor } B \longrightarrow \text{QMod } B \xrightarrow{-\otimes_B A} \text{QMod } R \longrightarrow 0.$$

Theorem 2.3. *With the same notions as in Theorem 2.2. The following are equivalent.*

- (i) A/R is a right H^* -dense Galois extension;
- (ii) For any finitely generated right B -module M , $T(M)$ is finite dimensional;
- (iii) $B/(BtB)$ is finite dimensional;
- (iv) $-\otimes_B A : \text{QMod } B \longrightarrow \text{QMod } R$ is an equivalence of abelian categories.

Proof. The equivalence of (i) and (iii) is Proposition 1.3. That (ii) implies (iii) is obvious since $B/(BtB)$ is a t -torsion B -module.

(iii) \implies (ii). Let M_B be a finitely generated module. Since A is (and hence B is) noetherian, $T(M)$ is finitely generated. By Lemma 2.1(i), $T(M)t$ is finite dimensional R -module. Hence $T(M)t \otimes_R A$ is finite dimensional since A is finitely generated as a left R -module. Since the right multiplication map $T(M)t \otimes_R A \longrightarrow T(M)tA$ is surjective, it follows that $T(M)tA$ is finite dimensional. Note that $T(M)tA = T(M)tB$ since $(A\#t)(A\#1) = BtB$. Hence $T(M)/(T(M)tA)$ is indeed a finitely generated B/BtB -module. By the hypothesis (iii), $B/(BtB)$ is finite dimensional, hence $T(M)/(T(M)tA)$ is finite dimensional. Together with the property that $T(M)tA$ is finite dimensional, we obtain that $T(M)$ is finite dimensional.

(ii) \implies (iv). Take an object $M \in \mathcal{T}$. For an arbitrary element $m \in M$, mB is an object in \mathcal{T} . By the condition (ii), $mB = T(mB)$ is finite dimensional. Hence $M \in \text{Tor } B$. Therefore $\text{Tor } B = \mathcal{T}$. From the exact sequence (3), we obtain that $-\otimes_B A : \text{QMod } B \longrightarrow \text{QMod } R$ is an equivalence.

(iv) \implies (ii). The condition (iv) implies $\text{Tor } B = \mathcal{T}$. If M is a finitely generated B -module, so is $T(M)$ since B is noetherian. Then $T(M) \in \mathcal{T} = \text{Tor } B$ is finite dimensional. \square

3. ENDOMORPHISM RINGS

Let H be a finitely dimensional semisimple Hopf algebra, and let A be a left H -module algebra. Recall from [CFM, Theorem 1.2] that if A is a right H^* -Galois extension, then there is an isomorphism $\text{End}_{A^H}(A) \cong A \# H$. In the Hopf dense Galois case, this property is not true in general. However, we will see that a similar property holds if we put more restrictions on A .

The discussions in this section fit in a more general setting. Let R be a noetherian algebra. Let $\text{Mod } R$ be the category of right R -modules, and $\text{Tor } R$ be the subcategory of torsion modules. Consider the torsion functor $\tau : \text{Mod } R \rightarrow \text{Tor } R$ sending each module $M \in \text{Mod } R$ to its largest torsion submodule. Note that τ is a left exact functor. We write $R^i\tau$ for the i th right derived functor of τ .

Let M_R be a finitely generated R -module. We define the *depth* of M to be

$$\text{depth}(M) = \min\{i \mid R^i\tau(M) \neq 0\}.$$

Lemma 3.1. *Let R be a noetherian algebra, and M_R a finitely generated R -module. Then the following are equivalent.*

- (i) $\text{depth}(M) \geq d$.
- (ii) $\text{Ext}_R^i(S, M) = 0$ for every $i < d$ and every finite dimensional simple module S_R .
- (iii) $\text{Ext}_R^i(K, M) = 0$ for every $i < d$ and every finite dimensional module K_R .

Proof. (i) \iff (ii). Take a minimal injective resolution of M as: $0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow \dots \rightarrow I^i \rightarrow \dots$. Assume $\text{depth}(M) \geq d$. We claim that I^i is torsion free for all $i < d$. If $d = 0$, nothing needs to be proved. We assume $d > 0$. Since $R^0\tau(M) = \tau(M)$, we see M is torsion free. Hence the injective enveloping I^0 of M is also torsion free. Let i be the smallest number such that I^i is not torsion free. Then $\text{Soc}(I^i) \cap \tau(M) \neq 0$. Since the injective resolution is assumed to be minimal, we see $R^i\tau(M) \neq 0$, which contradicts to the assumption that $\text{depth}(M) \geq d$. The claim follows. By the minimality of the injective resolution again, for any finite dimensional simple module S_R , we have $\text{Ext}_R^i(S, M) = \text{Hom}_R(S, I^i) = 0$ for all $i < d$.

On the contrary, assume that $\text{Ext}_R^i(S, M) = 0$ for every $i < d$ and every finite dimensional simple module S_R . Since the injective resolution is minimal, we have $\text{Hom}_R(S, I^i) = \text{Ext}_R^i(S, M) = 0$ for $i < d$. Hence I^i is torsion free for every $i < d$, which in turn implies $\text{depth}(M) \geq d$.

(ii) \iff (iii). It suffices to show that (ii) implies (iii). Since any finite dimensional module K_R contains a simple submodule, the result follows from an easy induction on the dimension of K . \square

Given noetherian algebras R and B and a B - R -bimodule ${}_B M_R$, we are interested in when M defines an equivalence between the quotient categories $\text{QMod } R$ and $\text{QMod } B$. We need some terminology.

Let ${}_B P$ be a B -module. We say that ${}_B P$ is *almost flat* if given finitely generated modules K_B and N_B and an injective morphism $f : K_B \rightarrow N_B$, the map $f \otimes_B P : K \otimes_B P \rightarrow N \otimes_B P$ is almost

injective. We say that ${}_B P$ is *faithful* if it satisfies the following condition: for any finitely generated module N_B , if $N \otimes_B M$ is finite dimensional then N_B itself is finite dimensional. We say ${}_B P$ is a *weaker generator* of $\text{Mod } B$, if there is a finite index set I and an almost surjective B -module morphism $\oplus_{i \in I} P_i \rightarrow B/\tau(B)$, where P_i is a cofinite submodule of ${}_B P$ for every $i \in I$.

We have the following examples of almost flat modules.

Example 3.2. (i) Let B a noetherian algebra. Let ${}_B Q$ be a finitely generated projective B -module. If ${}_B P$ is a cofinite submodule of ${}_B Q$, then ${}_B P$ is an almost flat module. Indeed, let X_B and Y_B be finitely generated modules, and let $f : X_B \rightarrow Y_B$ be an injective morphism. From the exact sequence $0 \rightarrow P \xrightarrow{\iota} Q \xrightarrow{p} Q/P \rightarrow 0$, we obtain the following commutative diagram:

$$\begin{array}{ccccccc} Y \otimes_B P & \xrightarrow{Y \otimes_B \iota} & Y \otimes_B Q & \xrightarrow{Y \otimes_B p} & Y \otimes_B (Q/P) & \longrightarrow & 0 \\ f \otimes_B P \uparrow & & f \otimes_B Q \uparrow & & f \otimes_B (Q/P) \uparrow & & \\ X \otimes_B P & \xrightarrow{X \otimes_B \iota} & X \otimes_B Q & \xrightarrow{X \otimes_B p} & X \otimes_B (Q/P) & \longrightarrow & 0. \end{array}$$

Hence we have $(f \otimes_B Q) \circ (X \otimes_B \iota) = (Y \otimes_B \iota) \circ (f \otimes_B P)$. Since Q is projective, $f \otimes_B Q$ is injective. Hence $\ker[(f \otimes_B Q) \circ (X \otimes_B \iota)] = \ker(X \otimes_B \iota)$. Since Q/P is finite dimensional and X_B is finitely generated, we have that $\text{Tor}_1^B(X, Q/P)$ is finite dimensional. Then $\ker(X \otimes_B \iota)$ is finite dimensional. Therefore $\ker[(Y \otimes_B \iota) \circ (f \otimes_B P)]$ is finite dimensional, which implies that $\ker(f \otimes_B P)$ is finite dimensional.

(ii) Let B be a noetherian algebra. If J is a cofinite left ideal of B , then J , as a left B -module, is faithful and almost flat. By (i), J is almost flat. Now assume X_B is a finitely generated B -module and $X \otimes_B J$ is finite dimensional. The exact sequence $X \otimes_B J \rightarrow X \rightarrow X \otimes_B (B/J) \rightarrow 0$ implies that X is finite dimensional. Hence J is faithful.

Lemma 3.3. *Let B be a noetherian algebra, and let ${}_B P$ be a finitely generated B -module. Then ${}_B P$ is almost flat if and only if for any almost injective B -module morphism $f : K_B \rightarrow N_B$ with K_B and N_B finitely generated, the kernel of the morphism $f \otimes_B P$ is finite dimensional.*

Proof. Assume that ${}_B P$ is almost flat. Let $f : K_B \rightarrow N_B$ be an almost injective morphism. We have an injective morphism $\bar{f} : K/\ker f \rightarrow N$ induced by f . Then $\ker(\bar{f} \otimes_B P)$ is finite dimensional. Let $p : K \rightarrow K/\ker f$ be the natural projection map. Then the morphism $f \otimes_B P$ is the composition $K \otimes_B P \xrightarrow{p \otimes_B P} K/\ker f \otimes_B P \xrightarrow{\bar{f} \otimes_B P} N \otimes_B P$. Hence we have an exact sequence of B -modules $(\ker f) \otimes_B P \rightarrow \ker(f \otimes_B P) \rightarrow \ker(\bar{f} \otimes_B P) \rightarrow 0$. Since f is almost injective, $\ker f$ is finite dimensional. Since ${}_B P$ is finitely generated, $(\ker f) \otimes_B P$ is finite dimensional. Hence $\ker(f \otimes_B P)$ is finite dimensional. \square

Lemma 3.4. *Let R and B be noetherian algebras, and let ${}_B M_R$ be a B - R -bimodule which is finitely generated on both sides. Assume $g : X_B \rightarrow Y_B$ is an injective B -module morphism where X_B and Y_B are not necessary finitely generated. If M is almost flat as a left B -module, then the kernel of the morphism $g \otimes_B M$ is a torsion R -module.*

Proof. Note that $Y = \varinjlim Y'$ where the limit runs over all the finitely generated submodules Y' respectively. For any finitely generated submodule $Y' \subseteq Y$, set $X' = g^{-1}(Y' \cap g(X))$. Then we see $X = \varinjlim X'$. By the definition, the kernel of the map $g|_{X'} \otimes_B M : X' \otimes_B M \rightarrow Y' \otimes_B M$ is

finite dimensional, hence it is a torsion module. Let $K' = \ker(g|_{X'} \otimes_B M)$. So, we have an exact sequence $0 \rightarrow K' \rightarrow X' \otimes_B M \rightarrow Y' \otimes_B M$. Taking direct limit on these exact sequence and using the property that direct limits are exact and commute with the tensor functor, we obtain that the kernel of $g \otimes_B M$ is a torsion R -module. \square

Lemma 3.5. *Let R , B and ${}_B M_R$ be as in Lemma 3.4. Assume that ${}_B M$ is a faithful and almost flat B -module. For a right B -module N , $N \otimes_B M$ is a torsion R -module if and only if N is a torsion B -module.*

Proof. Assume that $N \otimes_B M$ is a torsion R -module. Take any finitely generated submodule K_B of N_B , and let $\iota : K \rightarrow N$ be the inclusion map. Since ${}_B M$ is almost flat, $\ker(\iota \otimes_B M)$ is a torsion R -module by Lemma 3.4. Since K_B and M_R are finitely generated, $K \otimes_B M$ is a finitely generated R -module. Hence $\ker(\iota \otimes_B M)$ a finitely generated torsion R -module, and therefore it is finite dimensional. Since $N \otimes_B M$ is a torsion R -module, the image of $\iota \otimes_B M$ is a torsion R -module. Hence $K \otimes_B M$ is a torsion R -module. As $K \otimes_B M$ is also finitely generated as an R -module, it follows that $K \otimes_B M$ is finite dimensional. By the assumption, ${}_B M$ is faithful, we obtain that K_B is finite dimensional. Therefore, N_B is a torsion module.

Conversely, assume that N_B is a torsion B -module. Note that $N = \varinjlim K$ where K runs over all the finite dimensional submodules of N_B . Since direct limits commute with left adjoint functor, we have $N \otimes_B M = (\varinjlim K) \otimes_B M \cong \varinjlim (K \otimes_B M)$. Since ${}_B M$ is finitely generated, $K \otimes_B M$ is finite dimensional. Hence we obtain that $\varinjlim (K \otimes_B M)$ is a torsion R -module. \square

Let ${}_B M_R$ be a B - R -bimodule which is finitely generated on both sides. Then we have functors $-\otimes_B M : \text{Mod } B \rightarrow \text{Mod } R$, and $-\otimes_B M : \text{mod } B \rightarrow \text{mod } R$. By Lemmas 3.4 and 3.5, the tensor functor $-\otimes_B M$ sends torsion B -modules to torsion R -modules and almost isomorphisms to almost isomorphisms. It induces functors $-\otimes_B \mathcal{M} : \text{QMod } B \rightarrow \text{QMod } R$ and $-\otimes_B \mathcal{M} : \text{qmod } B \rightarrow \text{qmod } R$, such that the following diagrams commute

$$\begin{array}{ccc} \text{Mod } B & \xrightarrow{-\otimes_B M} & \text{Mod } R \\ \pi \downarrow & & \downarrow \pi \\ \text{QMod } B & \xrightarrow{-\otimes_B \mathcal{M}} & \text{QMod } R \end{array} \qquad \begin{array}{ccc} \text{mod } B & \xrightarrow{-\otimes_B M} & \text{mod } R \\ \pi \downarrow & & \downarrow \pi \\ \text{qmod } B & \xrightarrow{-\otimes_B \mathcal{M}} & \text{qmod } R. \end{array}$$

Let M' be another B - R -bimodule, and let $f : M \rightarrow M'$ be an almost isomorphism of B - R -bimodules. It is not hard to see that f induces a natural isomorphism $-\otimes_B \mathcal{M} \rightarrow -\otimes_B \mathcal{M}'$.

We have the following Morita type theorem for quotient categories.

Theorem 3.6. *Let R and B be noetherian algebras, and let ${}_B M_R$ be a bimodule which is finitely generated on both sides. Assume $\text{depth } M_R \geq 2$ and $\text{depth } B_B \geq 2$. Then the following are equivalent.*

- (i) *The functor $-\otimes_B \mathcal{M} : \text{qmod } B \rightarrow \text{qmod } R$ is an equivalence of abelian categories.*
- (ii) *M_R is a weaker generator, the natural map $B \rightarrow \text{End}(M_R), b \mapsto [m \mapsto bm]$ is an isomorphism of algebras, and the left B -module ${}_B M$ is faithful and almost flat.*
- (iii) *The functor $-\otimes_B \mathcal{M} : \text{QMod } B \rightarrow \text{QMod } R$ is an equivalence of abelian categories.*

Proof. (i) \implies (ii). Set $F = - \otimes_B \mathcal{M}$. By the commutative diagrams above the theorem, we have $F(\pi(B_B)) = \pi(M_R)$. Let I be a right ideal of B such that B/I is finite dimensional. Since $\text{depth } B_B \geq 2$, we see $\text{Hom}_B(B, B) \cong \text{Hom}_B(I, B)$. Hence the projection functor $\pi : \text{mod } B \rightarrow \text{qmod } B$ induces an isomorphism of algebras $\text{End}(B_B) \xrightarrow{\pi} \text{End}_{\text{qmod } B}(\pi(B_B))$. Similar, since $\text{depth } M_R \geq 2$, we have an isomorphism of algebra $\text{End}(M_R) \xrightarrow{\pi} \text{End}_{\text{qmod } R}(\pi(M_R))$. By the commutative diagram above the theorem and the assumption that F is an equivalence of abelian categories, the natural map $B \rightarrow \text{End}(M_R)$ is an isomorphism.

Assume $\pi(R) = F(\pi(N))$ for some $N \in \text{mod } B$. Since N_B is finitely generated, there is a finite set Λ and an epimorphism $g : \bigoplus_{i \in \Lambda} B_i \rightarrow N$, where $B_i \cong B$ for all $i \in \Lambda$. By the commutative diagrams above the theorem again, we obtain an epimorphism $\varrho : \pi(\bigoplus_{i \in \Lambda} B_i \otimes_B M) \rightarrow \pi(R)$ in $\text{qmod } R$. Note that there is a cofinite submodule P of $\bigoplus_{i \in \Lambda} B_i \otimes_B M$ and a right R -module morphism $h : P \rightarrow R/\tau(R)$ such that $\pi(h) = \varrho$. Since ϱ is epimorphic, h is almost surjective. Write $M_i = B_i \otimes_B M \cong M$. We view M_i as a submodule of $\bigoplus_{i \in \Lambda} M_i$ through the natural injective map. Let $P_i = M_i \cap P$ for all $i \in \Lambda$. Since P is cofinite, M_i/P_i is finite dimensional. Since Λ is a finite set, $\bigoplus_{i \in \Lambda} P_i$ is a cofinite submodule of $\bigoplus_{i \in \Lambda} M_i$, which implies that $\bigoplus_{i \in \Lambda} P_i$ is also a cofinite submodule of P . Denote by $\varsigma : \bigoplus_{i \in \Lambda} P_i \rightarrow P$ the inclusion map. Let h_i be the restriction $h|_{P_i} : P_i \rightarrow R/\tau(R)$ for all $i \in \Lambda$. The maps $\{h_i\}_{i \in \Lambda}$ define a morphism $h' : \bigoplus_{i \in \Lambda} P_i \rightarrow R/\tau(R)$ such that $h\varsigma = h'$. Since both h and ς are almost surjective, h' is almost surjective. Hence M_R is a weaker generator.

For $K_B \in \text{mod } B$, assume $K \otimes_B M$ is finite dimensional. Then we see $F(\pi(K)) \cong \pi(K \otimes_B M) = 0$. Since F is an equivalence, $\pi(K) = 0$. So, K_B is a torsion B -module, and hence K is finite dimensional. Therefore, ${}_B M$ is faithful. Let $g : X \rightarrow Y$ be an injective morphism in $\text{mod } B$. From the commutative diagram above the theorem and the assumption that F is an equivalence, we see that $\pi(g \otimes_B M) : \pi(X \otimes_B M) \rightarrow \pi(Y \otimes_B M)$ is an injective morphism in $\text{qmod } R$. Then $\ker(g \otimes_B M)$ is a torsion R -module, and hence $\ker(g \otimes_B M)$ is finite dimensional. Therefore ${}_B M$ is almost flat.

(ii) \implies (iii). Since M_R is a weaker generator in $\text{Mod } R$, one sees that $\pi(M)$ is a generator in $\text{QMod } R$. By Popescu-Gabriel's Theorem (cf. [St, Theorem X.4.1]), the functor $\text{Hom}_{\text{QMod } R}(\pi(M), -) : \text{QMod } R \rightarrow \text{Mod } B$ is fully faithful, and has a left adjoint functor. Since π has a right adjoint functor $\omega : \text{QMod } R \rightarrow \text{Mod } R$, one sees that there is a natural isomorphism $\text{Hom}_{\text{QMod } R}(\pi(M), -) \cong \text{Hom}_R(M, -) \circ \omega$. Hence the left adjunction of $\text{Hom}_{\text{QMod } R}(\pi(M), -)$ is $\pi \circ (- \otimes_B M) : \text{Mod } B \rightarrow \text{QMod } R$. Write Ψ for $\pi \circ (- \otimes_B M)$. Since ${}_B M$ is almost flat, Ψ is an exact functor, and hence it induces an equivalence of abelian categories $\text{Mod } B/\ker \Psi \rightarrow \text{QMod } R$, where $\ker \Psi$ is the full subcategory of $\text{Mod } B$ consisting of objects K such that $\Psi(K) = 0$. Since ${}_B M$ is faithful, we see $\ker \Psi = \text{Tor } B$ by Lemma 3.5. Hence $\text{Mod } B/\ker \Psi = \text{QMod } B$. Therefore the functor $- \otimes_B \mathcal{M}$ induced by Ψ is an equivalence.

(iii) \implies (i). This is clear since $\text{qmod } B$ (resp. $\text{qmod } R$) is a full subcategory of $\text{QMod } B$ (resp. $\text{QMod } R$) consisting of noetherian objects. \square

Let us go back to the Hopf dense Galois extensions. Let H be a finite dimensional semisimple Hopf algebra, and let A be a left H -module algebra. We assume further that A is a noetherian algebra. As before, set $B = A\#H$ and $R = A^H$.

Lemma 3.7. *Let A and H be as above. If $\text{depth } A_A \geq 2$, then $\text{depth } B_B \geq 2$ and $\text{depth } A_R \geq 2$.*

Proof. Let K_B be a finite dimensional B -module. We view K and B as right A -modules, then $\text{Ext}_A^i(K, B)$ is a right H -module for all $i \geq 0$. By [HVZ, Corollary 2.9], we have $\text{Ext}_B^i(K, B) = \text{Ext}_A^i(K, B)^H$ for all $i \geq 0$, where $\text{Ext}_A^i(K, B)^H$ is the subspace of invariants of the H -action. Since B is a free A -module and $\text{depth } A_A \geq 2$, we have $\text{Ext}_A^i(K, B) = 0$ for all $i < 2$ by Lemma 3.1. Therefore $\text{Ext}_B^i(K, B) = 0$ for all $i < 2$. Hence by Lemma 3.1 again $\text{depth } B_B \geq 2$.

Let K_R be a finite dimensional R -module. We have the following isomorphisms $\text{Hom}_B(K \otimes_R A_B, B) \cong \text{Hom}_R(K, \text{Hom}_B({}_R A_B, B)) \cong \text{Hom}_R(K, {}_B A_R)$. These isomorphisms induce the following spectral sequence

$$E_2^{pq} = \text{Ext}_B^p(\text{Tor}_q^R(K, {}_R A_B), B) \implies \text{Ext}_R^{p+q}(K, {}_B A_R).$$

Since K_R is finite dimensional and ${}_R A$ is a finitely generated left R -module, $\text{Tor}_q^R(K, {}_R A_B)$ is finite dimensional for all $q \geq 0$. Since $\text{depth } B_B \geq 2$, we have $\text{Ext}_B^p(\text{Tor}_q^R(K, {}_R A_B), B) = 0$ for $p < 2$ by Lemma 3.1. Hence $\text{Ext}_R^i(K, {}_B A_R) = 0$ for $i < 2$. By Lemma 3.1, $\text{depth}(A_R) \geq 2$. \square

As a special case of Theorem 3.6, we obtain the following result.

Theorem 3.8. *Let H be a finite dimensional semisimple Hopf algebra, and let A be a left H -module algebra. Assume that A/A^H is a right H^* -dense Galois extension. If A is noetherian and $\text{depth } A_A \geq 2$, then the natural map*

$$A\#H \longrightarrow \text{End}(A_{A^H}), a\#h \mapsto [b \mapsto a(h \cdot b)]$$

is an isomorphism of algebras.

Proof. This is a direct consequence of Lemma 3.7 and Theorems 2.3 and 3.6. \square

4. DENSELY GRADED ALGEBRAS

Let G be a group, and let $A = \bigoplus_{g \in G} A_g$ be a G -graded algebra. Recall that A is called a *strongly graded algebra* if $A_g A_h = A_{gh}$ for all $g, h \in G$. We introduce a weaker version of strongly graded algebra, which is a special class of Hopf dense Galois extensions.

Definition 4.1. *Let G be a group, and let $A = \bigoplus_{g \in G} A_g$ be a G -graded algebra. We call A a densely graded algebra if (i) there are only finitely many elements $g \in G$ such that $A_g A_{g^{-1}} \neq A_e$, and (ii) $A_e/A_g A_{g^{-1}}$ is finite dimensional for all $g \in G$.*

Example 4.2. (i) One may check that if G is an infinite group, then the conditions in the definition imply that A is indeed strongly graded (see Lemma 4.3 below). Hence we are interested in the case that G is a finite group in this section.

(ii) Let $A = \mathbb{k}[x, y]/(x^2 - y^2)$, let $G = \mathbb{Z}_2$. We view the polynomial algebra $\mathbb{k}[x, y]$ as a G -graded algebra by setting the degree of x to be 0 and the degree of y to be 1. It follows that the element $x^2 - y^2$ is a homogeneous element of degree 0, which makes A to be a G -graded algebra. We have $A_0 = \mathbb{k}[x]$ and $A_1 = y\mathbb{k}[x]$. So, $A_0^2 = A_0$, $A_1 A_0 = A_0 A_1 = A_1$ and $A_0/A_1 A_1$ has dimension 2. Hence A is a densely G -graded algebra.

(iii) Let A be an algebra and let M be an A - A -bimodule. Assume $\varphi : M \otimes_A M \longrightarrow A$ is an A - A -bimodule morphism such that $A/\varphi(M \otimes_A M)$ is finite dimensional and $m\varphi(n \otimes l) = \varphi(m \otimes n)l$

for any $m, n, l \in M$. Set $B = A \oplus M$, and define a multiplication on B as follows:

$$(a, m)(b, n) = (ab + \varphi(m \otimes n), an + mb), \text{ for any } a, b \in A \text{ and } n, m \in M.$$

Set $B_0 = A$ and $B_1 = M$. Then B is a densely \mathbb{Z}_2 -graded algebra. We call B a *densely semitrivial extension* of A by M (compared to [NV, Example 1.3.3]).

Lemma 4.3. *Let $A = \bigoplus_{g \in G} A_g$ be a densely G -graded algebra. If G is a infinite group, then A is strongly graded.*

Proof. Let $S = \{g \in G \mid A_g A_{g^{-1}} = A_e\}$. Then S is a semigroup. Indeed, for any $g, h \in S$, we have $A_e = A_g A_h A_{h^{-1}} A_{g^{-1}} \subseteq A_{gh} A_{(gh)^{-1}} \subseteq A_e$. Hence $gh \in S$. Now let $K = \{g \in S \mid g^{-1} \in S\}$. Then K is a subgroup of G . Since $G \setminus S$ is finite, $G \setminus K$ is finite. We claim that $G \setminus K = \emptyset$. Otherwise, take an element $g \in G$ but $g \notin K$. Then we have $gk \notin K$ for any $k \in K$, which contradicts with the fact that $G \setminus K$ is finite. The claim follows. Hence $S = K = G$, that is, A is strongly graded. \square

We have the following equivalent definition of densely graded algebra.

Proposition 4.4. *Let G be a finite group, and let $A = \bigoplus_{g \in G} A_g$ be a G -graded algebra. Assume that A_g is finitely generated as a left A_e -module for every $g \in G$. Then A is a densely graded algebra if and only if $A_{gh}/A_g A_h$ is finite dimensional, for any $g, h \in G$.*

Proof. The ‘‘if’’ part is trivial. We next prove the ‘‘only if’’ part. For $g \in G$, we have an exact sequence of right A_e -modules $0 \rightarrow A_g A_{g^{-1}} \rightarrow A_e \rightarrow A_e/A_g A_{g^{-1}} \rightarrow 0$. For any $h \in G$, applying $-\otimes_{A_e} A_{gh}$ on the sequence, we obtain the following exact sequence

$$A_g A_{g^{-1}} \otimes_{A_e} A_{gh} \rightarrow A_{gh} \rightarrow (A_e/A_g A_{g^{-1}}) \otimes_{A_e} A_{gh} \rightarrow 0,$$

from which we obtain an isomorphism $A_{gh}/(A_g A_{g^{-1}} A_{gh}) \cong (A_e/A_g A_{g^{-1}}) \otimes_{A_e} A_{gh}$. By Definition 4.1, $A_e/A_g A_{g^{-1}}$ is finite dimensional. Since A_{gh} is a finitely generated A_e -module, $(A_e/A_g A_{g^{-1}}) \otimes_{A_e} A_{gh}$ is finite dimensional. Hence $A_{gh}/(A_g A_{g^{-1}} A_{gh})$ is finite dimensional. Note that $A_g A_{g^{-1}} A_{gh} \subseteq A_g A_h$. We finally obtain that $A_{gh}/A_g A_h$ is finite dimensional for any $g, h \in G$. \square

Let $A = \bigoplus_{g \in G} A_g$ be a G -graded algebra. Define a comodule action $\rho : A \rightarrow A \otimes \mathbb{k}G$ by $\rho(a) = a \otimes g$ for any $a \in A_g$ and $g \in G$. Then A is a right $\mathbb{k}G$ -comodule algebra. On the contrary, any right $\mathbb{k}G$ -comodule algebra A can be viewed as a G -graded algebra by setting components $A_g = \{a \in A \mid \rho(a) = a \otimes g\}$ for every $g \in G$. The following result is a direct consequence of Proposition 4.4 and the definition of Hopf dense Galois extensions.

Corollary 4.5. *With the hypotheses on A and G as in Proposition 4.4, A is a densely G -graded algebra if and only if A/A_e is a right $\mathbb{k}G$ -dense Galois extension.*

Let $A = \bigoplus_{g \in G} A_g$ be a G -graded algebra. We denote by $\text{GrMod}_G A$ the category of right G -graded A -module. One of the most important results for strongly graded algebras is Dade’s Theorem (cf. [Da, Theorem 2.8], see also [NV]), which establishes an equivalence between the abelian categories $\text{GrMod}_G A$ and $\text{Mod } A_e$. In our case, these two categories are no longer equivalent in general. We next prove a weaker version of Dade’s Theorem.

We may define two torsion classes in the abelian category $\text{GrMod}_G A$. Let M be a right G -graded A -module. We call M a *G -graded torsion module* if the following condition is satisfied: for any cyclic

graded submodule $N = mA$ generated by a homogeneous element $m \in M$, we have that N_e is finite dimensional. Let \mathcal{T}_G be the full subcategory of $\text{GrMod}_G A$ consisting of all the graded torsion modules. For any G -graded right A -module N , we write $T_G(N)$ for the sum of all the G -graded torsion submodule of N . Then $T_G(N)$ is the largest G -graded torsion submodule of N . In this way, we indeed obtain a functor

$$(4) \quad T_G : \text{GrMod}_G A \longrightarrow \mathcal{T}_G.$$

Another torsion class in $\text{GrMod}_G A$ is the graded version of the one defined in previous section. Denote by $\text{GrTor}_G A$ the full subcategory of $\text{GrMod}_G A$ consisting of G -graded A -modules M such that mA is finite dimensional for every homogeneous element $m \in M$. One sees that $\text{GrTor}_G A$ is a full subcategory of \mathcal{T}_G . In general, \mathcal{T}_G is not equal to $\text{GrTor}_G A$. However, we have the following result.

Lemma 4.6. *Let G be a finite group, and let A be a G -graded algebra which is also noetherian. If A is densely G -graded, then $\mathcal{T}_G = \text{GrTor}_G A$.*

Proof. For $M \in \mathcal{T}_G$, let $m \in M$ be a homogenous element. Set $N = mA$, which is a G -graded submodule of M . Then N_e is finite dimensional. Since A is noetherian and G is a finite group, A_e is a noetherian subalgebra of A and A_g is a finitely generated A_e -module on both sides for every $g \in G$. Then $N_e \otimes_{A_e} A_g$ is finite dimensional for every $g \in G$. The right multiplication of A on N gives a surjective map $N_e \otimes_{A_e} A_g \longrightarrow N_e A_g$. Hence $N_e A_g$ is finite dimensional. For every $g \in G$, there is an exact sequence

$$0 \longrightarrow A_{g^{-1}} A_g \longrightarrow A_e \longrightarrow A_e / A_{g^{-1}} A_g \longrightarrow 0.$$

Applying the functor $N_g \otimes_{A_e} -$ to the exact sequence, we obtain the following exact sequence

$$N_g \otimes_{A_e} A_{g^{-1}} A_g \longrightarrow N_g \longrightarrow N_g \otimes_{A_e} (A / A_{g^{-1}} A_g) \longrightarrow 0.$$

Hence $N_g \otimes_{A_e} (A_e / A_{g^{-1}} A_g) \cong N_g / N_g A_{g^{-1}} A_g$. Since A is a densely G -graded algebra, $A_e / A_{g^{-1}} A_g$ is finite dimensional. Since A_g is finitely generated as a right A_e -module for every $g \in G$ and $N = mA$, we have that N_g is a finitely generated A_e -module. Hence $N_g \otimes_{A_e} (A / A_{g^{-1}} A_g)$ is finite dimensional, which in turn implies that $N_g / N_g A_{g^{-1}} A_g$ is finite dimensional. Note that $N_g A_{g^{-1}} A_g \subseteq N_e A_g$. Hence $N_g / N_e A_g$ is finite dimensional. As $N_e A_g$ is finite dimensional, it follows that N_g is finite dimensional for every $g \in G$. Hence $N = mA$ is finite dimensional as G is a finite group. Therefore, $M \in \text{GrTor}_G A$. \square

If A is a noetherian algebra, then both \mathcal{T}_G and $\text{GrTor}_G A$ are localizing Serre subcategories of $\text{GrMod}_G A$. Hence we have quotient categories

$$\text{GrMod}_G A / \mathcal{T}_G, \quad \text{and} \quad \text{QGrMod}_G A := \frac{\text{GrMod}_G A}{\text{GrTor}_G A}.$$

The functor $(-)_e : \text{GrMod}_G A \longrightarrow \text{Mod } A_e$ sending $M = \bigoplus_{g \in G} M_g$ to M_e is an exact functor. One sees that $(-)_e$ sends objects in \mathcal{T}_G to the objects in $\text{Tor } A_e$. Note that $\text{GrTor}_G A$ is a full subcategory of \mathcal{T}_G . We see that $(-)_e$ induces functors (we use the same symbol)

$$(-)_e : \text{GrMod}_G A / \mathcal{T}_G \longrightarrow \text{QMod } A_e, \quad \text{and} \quad (-)_e : \text{QGrMod}_G A \longrightarrow \text{QMod } A_e,$$

which make the following diagram commute

$$(5) \quad \begin{array}{ccc} \text{QGrMod}_G A & \xrightarrow{(-)_e} & \text{QMod } A_e, \\ \pi \downarrow & \nearrow & \\ \text{GrMod}_G A/\mathcal{T}_G & & \end{array}$$

where π is the natural projection functor.

Theorem 4.7 (Dade's Theorem). *Let G be a finite group, and let $A = \bigoplus_{g \in G} A_g$ be a G -graded algebra. Assume that A is a noetherian algebra. Then the following are equivalent.*

- (i) A is a densely G -graded algebra.
- (ii) For any finitely generated G -graded A -module $M = \bigoplus_{g \in G} M_g$, if M_e is finite dimensional, then M itself is finite dimensional.
- (iii) The functor $(-)_e : \text{QGrMod}_G A \rightarrow \text{QMod } A_e$ is an equivalence of abelian categories.

Proof. Let $\mathbb{k}G$ be the group algebra and let $H := \mathbb{k}G^*$ be its dual Hopf algebra. Then A is an H -module algebra, and $A^H = A_e$. Set $B = A \# H$. It is well known that any G -graded A -module M can be viewed as a right B -module, and vice versa. Hence we may (and we do) identify the abelian category $\text{Mod } B$ with $\text{GrMod}_G A$. Under this view, we may identify the torsion subcategory \mathcal{T} (cf. Section 2) with \mathcal{T}_G , and $\text{Tor } B$ with $\text{GrTor}_G A$. Then the torsion functor $T : \text{Mod } B \rightarrow \text{Tor } B$ coincides with $T_G : \text{GrMod}_G A \rightarrow \mathcal{T}_G$. Hence we have $\text{GrMod}_G A/\mathcal{T}_G = \text{Mod } B/\mathcal{T}$ and $\text{QGrMod}_G A = \text{QMod } B$.

(i) \implies (iii). By Corollary 4.5, A/A_e is a right H -dense Galois extension. By Lemma 4.6, we have $\mathcal{T} = \mathcal{T}_G = \text{GrTor}_G A = \text{Tor } B$. The functor $- \otimes_B \mathcal{A} : \text{QMod } B \rightarrow \text{QMod } A^H$ in Theorem 2.3 coincides with the functor $(-)_e : \text{QGrMod}_G A \rightarrow \text{QMod } A_e$. Now Theorem 2.3 implies that the statement (iii) holds.

(iii) \implies (ii). By the commutative diagram (5) and Theorem 2.3, we see that the projection functor $\pi : \text{QGrMod}_G A \rightarrow \text{GrMod}_G A/\mathcal{T}_G$ is an equivalence. Hence $\mathcal{T}_G = \text{GrTor}_G A$. Let $M = \bigoplus_{g \in G} M_g$ be a finitely generated G -graded A -module. If M_e is finite dimensional, then $M \in \mathcal{T}_G = \text{GrTor}_G A$. Since M is finitely generated, it follows that M is finite dimensional.

(ii) \implies (i). Let M be a finitely generated G -graded A -module. Then $T_G(M)$ is also a finitely generated G -graded A -module. Hence $T_G(M)_e$ is finite dimensional since $T_G(M)$ is a G -graded torsion module. By (ii) $T_G(M)$ is finite dimensional. Hence (i) follows from Corollary 4.5 and Theorem 2.3 since we identify the torsion functor T_G with the torsion functor T of Theorem 2.3. \square

Dade's theorem will imply some further equivalences of quotient categories over group graded algebras.

Let G and Γ be groups. Let $A = \bigoplus_{g \in G} A_g$ be a G -graded algebra, and let $B = \bigoplus_{\gamma \in \Gamma} B_\gamma$ be a Γ -graded algebra. Let $M = \bigoplus_{g \in G} M_g$ be a left graded A -module. For $g \in G$, we define the left graded A -module $M(g)$ by setting $M(g)_h = M_{hg}$ for every $h \in G$. Similarly, if $N = \bigoplus_{\gamma \in \Gamma} N_\gamma$ is a right graded B -module, we define the right graded B -module $N(\gamma)$ by setting $N(\gamma)_\theta = N_{\gamma\theta}$ for every $\gamma \in \Gamma$.

Del Rio established a Morita type theorem over the category of right graded A -modules and that of right graded B -modules (cf. [DR]). Recall that a *bigraded A - B -bimodule* is an A - B -bimodule

$$M = \bigoplus_{g \in G, \gamma \in \Gamma} {}_g M_\gamma,$$

such that $amb \in {}_{g'} M_{\gamma'}$ for any $a \in A_{g'}$, $m \in {}_g M_\gamma$ and $b \in B_{\gamma'}$.

Let $L = \bigoplus_{g \in G} L_g$ be a right graded A -module. Define a right graded B -module $L \bar{\otimes}_A M$ as follows (we use the notions as introduced in [Sm])

$$L \bar{\otimes}_A M = \bigoplus_{\gamma \in \Gamma} (L \bar{\otimes}_A M)_\gamma,$$

where the component $(L \bar{\otimes}_A M)_\gamma$ is the image of $\bigoplus_{g \in G} L_{g^{-1}} \otimes {}_g M_\gamma$ through the natural projection $L \otimes M \rightarrow L \bar{\otimes}_A M$.

Let $N = \bigoplus_{\gamma \in \Gamma} N_\gamma$ be a right graded B -module. Define a right graded A -module $\underline{\underline{\text{Hom}}}_B(M, N)$ as follows

$$\underline{\underline{\text{Hom}}}_B(M, N) = \bigoplus_{g \in G} \underline{\underline{\text{Hom}}}_B(M, N)_g,$$

where the component $\underline{\underline{\text{Hom}}}_B(M, N)_g = \text{Hom}_{\text{GrMod}_\Gamma B}(g^{-1} M_*, N)$, in which $g^{-1} M_* = \bigoplus_{\gamma \in \Gamma} g^{-1} M_\gamma$. We have a canonical isomorphism (cf. [DR])

$$(6) \quad \text{Hom}_{\text{GrMod}_\Gamma B}(L \bar{\otimes}_A M, N) \cong \text{Hom}_{\text{GrMod}_G A}(L, \underline{\underline{\text{Hom}}}_B(M, N)).$$

Hence we obtain a pair of adjoint functors

$$\text{GrMod}_G A \begin{array}{c} \xleftarrow{-\bar{\otimes}_A M} \\ \xrightarrow{\underline{\underline{\text{Hom}}}_B(M, -)} \end{array} \text{GrMod}_\Gamma B.$$

Now let G be a group and let Γ be a normal subgroup of G such that the quotient group $\bar{G} = G/\Gamma$ is finite. Let $A = \bigoplus_{g \in G} A_g$ be a noetherian G -graded algebra. For $\gamma \in \Gamma$, set $B_\gamma = A_\gamma$, and let $B = \bigoplus_{\gamma \in \Gamma} B_\gamma$. Then B is a Γ -graded algebra. Since G/Γ is finite and A is noetherian, B is noetherian.

Define a bigraded A - B -bimodule

$$P = \bigoplus_{g \in G, \gamma \in \Gamma} {}_g P_\gamma$$

by setting ${}_g P_\gamma = A_{g\gamma}$. Note that ${}_e P_* := \bigoplus_{\gamma \in \Gamma} {}_e P_\gamma = B$ as a right graded B -module, and ${}_* P_\gamma = A(\gamma)$, which is a shift of the regular graded module ${}_A A$ by degree γ . By the isomorphism (6), we have a pair of adjoint functors $(-\bar{\otimes}_A P, \underline{\underline{\text{Hom}}}_B(P, -))$.

Note that as a left graded A -module $P = \bigoplus_{\gamma \in \Gamma} {}_* P_\gamma$. For any right graded A -module L , we have

$$(7) \quad (L \bar{\otimes}_A P)_\gamma = \bigoplus_{\gamma \in \Gamma} (L \otimes_A ({}_* P_\gamma))_e = \bigoplus_{\gamma \in \Gamma} (L \otimes_A A(\gamma))_e = \bigoplus_{\gamma \in \Gamma} L_\gamma.$$

Hence, for any right graded B -module N , we have the following natural isomorphisms of right graded B -modules

$$\begin{aligned}
 \underline{\underline{\text{Hom}}}_B(P, N) \bar{\otimes}_A P &\cong \bigoplus_{\gamma \in \Gamma} \underline{\underline{\text{Hom}}}_B(P, N)_\gamma \\
 &\cong \bigoplus_{\gamma \in \Gamma} \text{Hom}_{\text{GrMod}_\Gamma B}(\gamma^{-1} P_*, N) \\
 &\cong \bigoplus_{\gamma \in \Gamma} \text{Hom}_{\text{GrMod}_\Gamma B}(\bigoplus_{\theta \in \Gamma} A_{\gamma^{-1}\theta}, N) \\
 &\cong \bigoplus_{\gamma \in \Gamma} \text{Hom}_{\text{GrMod}_\Gamma B}(B(\gamma^{-1}), N) \\
 &\cong N.
 \end{aligned}$$

Therefore, we see

$$(8) \quad (-\bar{\otimes}_A P) \circ \underline{\underline{\text{Hom}}}_B(P, -) \cong \text{id}_{\text{GrMod}_\Gamma B}.$$

From the equations in (7), we see that the functor $-\bar{\otimes}_A P$ can be written in a simpler way. We write $A^{(\Gamma)}$ for $B = \bigoplus_{\gamma \in \Gamma} A_\gamma$. Let L be a right graded A -module, define $L^{(\Gamma)} = \bigoplus_{\gamma \in \Gamma} L_\gamma$. Then $L^{(\Gamma)}$ is a right graded $A^{(\Gamma)}$ -module. By the equations (7), we have

$$(9) \quad (-)^{(\Gamma)} = -\bar{\otimes}_A P : \text{GrMod}_G A \longrightarrow \text{GrMod}_\Gamma A^{(\Gamma)}.$$

Clearly, $(-)^{(\Gamma)}$ is an exact functor, and sends torsion modules to torsion modules. Hence it induces a functor (use the same symbol)

$$(-)^{(\Gamma)} : \text{QGrMod}_G A \longrightarrow \text{QGrMod}_\Gamma A^{(\Gamma)}.$$

Lemma 4.8. *With the notions as above. The following are equivalent.*

- (i) *For any finitely generated right graded A -module L , if $L^{(\Gamma)}$ is finite dimensional, then so is L .*
- (ii) *The functor $(-)^{(\Gamma)} : \text{QGrMod}_G A \longrightarrow \text{QGrMod}_\Gamma A^{(\Gamma)}$ is an equivalence of abelian categories.*

Proof. Consider the following functors

$$\text{GrMod}_G A \begin{array}{c} \xleftarrow{(-)^{\Gamma}} \\ \xrightarrow{\underline{\underline{\text{Hom}}}_B(P, -)} \end{array} \text{GrMod}_\Gamma B \begin{array}{c} \xleftarrow{\pi} \\ \xrightarrow{\omega} \end{array} \text{QGrMod}_\Gamma B,$$

where π is the projection functor and ω is the right adjoint functor of π . We see the functor $\Phi = \pi \circ (-)^{(\Gamma)}$ is left adjoint to $\Psi = \underline{\underline{\text{Hom}}}_B(P, -) \circ \omega$, and moreover $\Phi \circ \Psi \cong \text{id}_{\text{QGrMod}_\Gamma B}$ by equations (8) and (9). By [PP, Theorem 7.11, Chapter 4], we have $\text{QGrMod}_\Gamma B \cong \frac{\text{GrMod}_G A}{\ker \Phi}$.

(i) \implies (ii). We have to show $\ker \Phi = \text{GrTor}_G A$. Clearly, $\text{GrTor}_G A \subseteq \ker \Phi$ since A is assume to be noetherian. Let L be a right graded A -module such that $\Phi(L) = 0$. Then $L^{(\Gamma)}$ is a Γ -graded torsion $A^{(\Gamma)}$ -module. For any homogeneous element $x \in L$, $(xA)^{(\Gamma)}$ is a finitely generated $A^{(\Gamma)}$ -submodule of $L^{(\Gamma)}$ since by assumption G/Γ is a finite group. Hence $(xA)^{(\Gamma)}$ is finite dimensional. By the condition (i), we obtain that xA is finite dimensional. So, $L \in \text{GrTor}_G A$.

That (ii) implies (i) is clear. □

Note that we assume the quotient group $\bar{G} = G/\Gamma$ is finite. For any $g \in G$, we write \bar{g} for the image of g through the projection map $p : G \rightarrow \bar{G}$. Let $\iota : \bar{G} \rightarrow G$ be a map such that $p \circ \iota = \text{id}$. We view the G -graded algebra A as a \bar{G} -graded algebra in the following way. For $\alpha \in \bar{G}$, set $D_\alpha = \bigoplus_{\gamma \in \Gamma} A_{\iota(\alpha)\gamma}$. Let $D = \bigoplus_{\alpha \in \bar{G}} D_\alpha$. Then we see that D is a \bar{G} -graded algebra, and $D_{\bar{e}} = A^{(\Gamma)}$ as a Γ -graded algebra.

As a consequence of Dade's Theorem 4.7, we have the following result, which was suggested to us by the anonymous referee. We appreciate the referee's comments very much!

Theorem 4.9. *Let G be a group, and let Γ be a normal subgroup of G such that $\bar{G} = G/\Gamma$ is a finite group. Let $A = \bigoplus_{g \in G} A_g$ be a noetherian G -graded algebra, and let D be the \bar{G} -graded algebra defined as above. The following are equivalent.*

- (i) D is a densely \bar{G} -graded algebra.
- (ii) For any finitely generated right graded A -module L , if $L^{(\Gamma)}$ is finite dimensional, then so is L .
- (iii) The functor $(-)^{(\Gamma)} : \text{QGrMod}_G A \rightarrow \text{QGrMod}_\Gamma A^{(\Gamma)}$ is an equivalence of abelian categories.

Proof. The equivalence of (ii) and (iii) has been showed in Lemma 4.8.

(i) \implies (ii). Let L be a finitely generated right graded A -module. We may view L as a right graded D -module in the following way. For $\alpha \in \bar{G}$, set $L'_\alpha = \bigoplus_{\gamma \in \Gamma} L_{\iota(\alpha)\gamma}$ and $L' = \bigoplus_{\alpha \in \bar{G}} L'_\alpha$. Then L' is a right graded D -module. Note that $D_{\bar{e}} = A^{(\Gamma)}$ and $L'_{\bar{e}} = L^{(\Gamma)}$. If $L^{(\Gamma)}$ is finite dimensional, then $L'_{\bar{e}}$ is finite dimensional. Hence L' is finite dimensional by Theorem 4.7. Equivalently, L is finite dimensional.

(ii) \implies (i). By Corollary 4.5 and Definition 1.1, it suffices to show that the map $\beta : D \otimes_{D_{\bar{e}}} D \rightarrow D \otimes \mathbb{k}\bar{G}$, defined by $a \otimes b \mapsto ab \otimes \alpha$ for $a \in D$ and $b \in D_\alpha$, is almost surjective.

The cokernel of the map β is $\text{coker } \beta = \bigoplus_{\alpha \in \bar{G}} \left(\bigoplus_{\alpha' \in \bar{G}} \frac{D_{\alpha\alpha'}}{D_\alpha D_{\alpha'}} \otimes \mathbb{k}\alpha' \right)$. For a fixed $\alpha \in \bar{G}$, we note that $\frac{\bigoplus_{\alpha' \in \bar{G}} D_{\alpha\alpha'}}{\bigoplus_{\alpha' \in \bar{G}} D_\alpha D_{\alpha'}} = \bigoplus_{\alpha' \in \bar{G}} \frac{D_{\alpha\alpha'}}{D_\alpha D_{\alpha'}}$. Temporarily, we set $X = \bigoplus_{\alpha' \in \bar{G}} D_{\alpha\alpha'} = \bigoplus_{\alpha' \in \bar{G}, \gamma \in \Gamma} A_{\iota(\alpha\alpha')\gamma}$ and $Y = \bigoplus_{\alpha' \in \bar{G}} D_\alpha D_{\alpha'} = \sum_{\alpha' \in \bar{G}, \gamma, \gamma' \in \Gamma} A_{\iota(\alpha)\gamma} A_{\iota(\alpha')\gamma'}$. Since $p\iota(\alpha) = \alpha$ and $p\iota(\alpha') = \alpha'$, it follows $\iota(\alpha\alpha')\Gamma = \iota(\alpha)\iota(\alpha')\Gamma$. So, $X = \bigoplus_{\alpha' \in \bar{G}, \gamma \in \Gamma} A_{\iota(\alpha)\iota(\alpha')\gamma}$. We see that both X and Y are right graded A -module.

Indeed, X is just the regular right graded A -module A_A , and Y is the graded A -submodule of X generated by $D_{\alpha \in \bar{G}} = \bigoplus_{\gamma \in \Gamma} A_{\iota(\alpha)\gamma}$. Set $Z = X/Y$. Note that $\dim Z = \dim \bigoplus_{\alpha' \in \bar{G}} \frac{D_{\alpha\alpha'}}{D_\alpha D_{\alpha'}} \otimes \mathbb{k}\alpha'$. Let $Z(\iota(\alpha))$ be the A -module obtained from Z by a shift of degree $\iota(\alpha)$, that is, $Z(\iota(\alpha))_g = Z_{\iota(\alpha)g}$ for every $g \in G$. Applying $(-)^{(\Gamma)}$ to $Z(\iota(\alpha))$, we have $Z(\iota(\alpha))^{(\Gamma)} = \bigoplus_{\gamma \in \Gamma} Z(\iota(\alpha))_\gamma = \bigoplus_{\gamma \in \Gamma} Z_{\iota(\alpha)\gamma}$. Note that $\iota(\alpha)\iota(\alpha')\gamma \in \iota(\alpha)\Gamma$ with $\gamma \in \Gamma$ implies $\iota(\alpha') \in \Gamma$. We have

$$\bigoplus_{\gamma \in \Gamma} Z_{\iota(\alpha)\gamma} = \frac{\bigoplus_{\gamma \in \Gamma} A_{\iota(\alpha)\gamma}}{\sum_{\gamma, \gamma' \in \Gamma} A_{\iota(\alpha)\gamma} A_{\gamma'}} = \frac{\bigoplus_{\gamma \in \Gamma} A_{\iota(\alpha)\gamma}}{\bigoplus_{\gamma \in \Gamma} A_{\iota(\alpha)\gamma}} = 0,$$

that is, $Z(\iota(\alpha))^{(\Gamma)} = 0$. By the condition (ii), we have $Z(\iota(\alpha))$ is finite dimensional. Equivalently, $\bigoplus_{\alpha' \in \bar{G}} \frac{D_{\alpha\alpha'}}{D_\alpha D_{\alpha'}} \otimes \mathbb{k}\alpha'$ is finite dimensional for every $\alpha \in \bar{G}$. Since \bar{G} is finite, we see $\text{coker } \beta$ is finite dimensional. Hence (i) follows. \square

5. APPLICATION I: PROJECTIVE SCHEMES ASSOCIATED TO VERONESE SUBALGEBRAS

Throughout this section, $A = A_0 \oplus A_1 \oplus \cdots$ is an \mathbb{N} -graded algebra. We assume that A is noetherian and that A is locally finite, that is, $\dim_{\mathbb{k}} A_i < \infty$ for all $i \geq 0$.

Given an integer $d \geq 1$, the d th Veronese subalgebra is define to be the \mathbb{N} -graded algebra $A^{(d)}$ whose n th component is A_{nd} . Since A is noetherian, then $A^{(d)}$ is also noetherian. Similarly, if M is a \mathbb{Z} -graded right A -module, we have $M^{(d)} = \bigoplus_{n \in \mathbb{Z}} M_{nd}$, which is a right graded $A^{(d)}$ -module.

Let $\text{GrMod}_{\mathbb{Z}} A$ be the category of the \mathbb{Z} -graded right A -modules. An object $M \in \text{GrMod}_{\mathbb{Z}} A$ is said to be a \mathbb{Z} -graded torsion module if mA is finite dimensional for every $m \in M$. Let $\text{GrTor}_{\mathbb{Z}} A$ be the full subcategory of $\text{GrMod}_{\mathbb{Z}} A$ consisting of all the \mathbb{Z} -graded torsion modules. Let $\text{grmod}_{\mathbb{Z}} A$ be the category of the finitely generated right \mathbb{Z} -graded A -modules, and let $\text{grtor}_{\mathbb{Z}} A = \text{grmod}_{\mathbb{Z}} A \cap \text{GrTor}_{\mathbb{Z}} A$.

Since A is noetherian, $\text{GrTor}_{\mathbb{Z}} A$ (resp. $\text{grtor}_{\mathbb{Z}} A$) is a Serre subcategory of $\text{GrMod}_{\mathbb{Z}} A$ (resp. $\text{grmod}_{\mathbb{Z}} A$). Hence we have a quotient category

$$\text{Tail } A := \frac{\text{GrMod}_{\mathbb{Z}} A}{\text{GrTor}_{\mathbb{Z}} A} \quad (\text{ resp. } \text{tail } A := \frac{\text{grmod}_{\mathbb{Z}} A}{\text{grtor}_{\mathbb{Z}} A}),$$

which is usually called *the noncommutative projective scheme* associated to A (cf. [AZ, Ve]).

There are strong relations between the noncommutative projective scheme associated to A and that associated to $A^{(d)}$. Indeed, there is a natural functor

$$(-)^{(d)} : \text{GrMod}_{\mathbb{Z}} A \longrightarrow \text{GrMod}_{\mathbb{Z}} A^{(d)}, \quad M \mapsto M^{(d)}.$$

Restrictions to the finitely generated modules, we have

$$(-)^{(d)} : \text{grmod}_{\mathbb{Z}} A \longrightarrow \text{grmod}_{\mathbb{Z}} A^{(d)}.$$

Since $(-)^{(d)}$ is exact and sends torsion modules to torsion modules, we obtain functors (use the same symbol)

$$(-)^{(d)} : \text{Tail } A \longrightarrow \text{Tail } A^{(d)}, \quad \text{and} \quad (-)^{(d)} : \text{tail } A \longrightarrow \text{tail } A^{(d)}.$$

The following result is fundamental in noncommutative geometry, which firstly appeared in [Ve] (see also, [AZ, Po, Mo]).

Theorem 5.1. *Assume A is generated by A_0 and A_1 . Then $(-)^{(d)} : \text{Tail } A \longrightarrow \text{Tail } A^{(d)}$ and $(-)^{(d)} : \text{tail } A \longrightarrow \text{tail } A^{(d)}$ are equivalences of abelian categories.*

We will see that this result is a direct consequence of Theorem 4.9. Indeed, Theorem 4.9 also provides a necessary and sufficient condition for the functor $(-)^{(d)} : \text{Tail } A \longrightarrow \text{Tail } A^{(d)}$ to be an equivalence.

Let $G = \mathbb{Z}$ and $\Gamma = d\mathbb{Z}$. Then Γ is a normal subgroup of G and $G/\Gamma = \mathbb{Z}_d$ is finite. The elements of \mathbb{Z}_d are written as $\{\bar{0}, \bar{1}, \dots, \overline{d-1}\}$. We view A as a G -graded algebra by setting $A_i = 0$ for every $i < 0$. As we did in Theorem 4.9, we define a \mathbb{Z}_d -graded algebra $D = \bigoplus_{s=0}^{d-1} D_{\bar{s}}$ by setting $D_{\bar{s}} = \bigoplus_{j \in \mathbb{Z}} A_{jd+s}$ for every $0 \leq s \leq d-1$. Note that $A^{(\Gamma)} = D_{\bar{0}} = \bigoplus_{j \in \mathbb{Z}} A_{jd}$ is just the Veronese subalgebra $A^{(d)}$ of A . Moreover, we also have $\text{GrMod}_{\Gamma} A^{(\Gamma)} = \text{GrMod}_{\mathbb{Z}} A^{(d)}$, $\text{QGrMod}_{\Gamma} A^{(\Gamma)} =$

Tail $A^{(d)}$ and the functor $(-)^{(\Gamma)}$ in Theorem 4.9 coincides with the functor $(-)^{(d)}$. Then we have the following result.

Theorem 5.2. *Let $A = A_0 \oplus A_1 \oplus \cdots$ be a locally finite noetherian \mathbb{N} -graded algebra and let d be a fixed positive integer. The following are equivalent.*

- (i) *There is an integer $p > 0$ such that, for all $n \geq p$ and $0 \leq s \leq d - 1$,*

$$A_{nd} = \sum_{\substack{i+j=n \\ i \geq 0, j \geq 1}} A_{id+s} A_{jd-s}.$$

- (ii) *For any finitely generated right graded A -module M , if $M^{(d)} = \bigoplus_{n \in \mathbb{Z}} M_{nd}$ is finite dimensional, then M itself is finite dimensional.*
 (iii) *$(-)^{(d)} : \text{Tail } A \rightarrow \text{Tail } A^{(d)}$ is an equivalence of abelian categories.*
 (iv) *$(-)^{(d)} : \text{tail } A \rightarrow \text{tail } A^{(d)}$ is an equivalence of abelian categories.*

Proof. With the notions above the theorem, the \mathbb{Z}_d -graded algebra D is densely graded if and only if $D_{\bar{0}}/D_{\bar{s}}D_{\bar{d-s}}$ is finite dimensional for all $\bar{s} \in \mathbb{Z}_d$, which is equivalent to the condition (i). The conditions (ii) and (iii) coincide with corresponding ones in Theorem 4.9. Hence (i) \iff (ii) \iff (iii) follows.

That (iii) implies (iv) is because $\text{tail } A$ (resp. $\text{tail } A^{(d)}$) consists of noetherian objects of $\text{Tail } A$ (resp. $\text{Tail } A^{(d)}$).

(iv) \implies (ii). For any finitely right graded A -module M , write \mathcal{M} for $\pi(M)$. Note that $\mathcal{M}^{(d)} = \pi(\bigoplus_{n \in \mathbb{Z}} M_{nd})$. If $\bigoplus_{n \in \mathbb{Z}} M_{nd}$ is finite dimensional, then $\mathcal{M}^{(d)} = \pi(\bigoplus_{n \in \mathbb{Z}} M_{nd}) = 0$. Since $(-)^{(d)}$ is an equivalence, we have $\mathcal{M} = 0$ in $\text{tail } A$, and hence $M \in \text{grtor}_{\mathbb{Z}} A$. Since M by assumption is finitely generated, M has to be finite dimensional. Therefore the condition (ii) holds. \square

Remark 5.3. (i) If A is generated by A_0 and A_1 , then one sees that the condition (i) of Theorem 5.2 is satisfied. Hence in this case, we recover the classical Theorem 5.1.

(ii) More generally, if there is an integer $p > 0$ such that $A_i A_j = A_{i+j}$ for all $i + j \geq p$, then condition (i) of Theorem 5.2 is also satisfied. Hence in this case, $(-)^{(d)} : \text{Tail } A \rightarrow \text{Tail } A^{(d)}$ is an equivalence of abelian categories.

(iii) Mori also provided some equivalent conditions for the functor $(-)^{(d)}$ to be an equivalence in [Mo, Theorem 3.4]. Note that in that paper, A is assumed to be a coherent connected graded algebra satisfying further homological conditions. In our case, we only assume that A is a locally finite noetherian algebra. Moreover, we obtain the result under the framework of Hopf dense Galois extension, which is a new way to understand noncommutative projective schemes.

6. APPLICATION II: NONCOMMUTATIVE ISOLATED SINGULARITIES

Throughout this section, $A = \bigoplus_{n \in \mathbb{N}} A_n$ is always a locally finite noetherian \mathbb{N} -graded algebra. Following [Ue], A is called a *graded isolated singularity* if the abelian category $\text{tail } A$ has finite global dimension, that is, there is an integer $p \geq 0$ such that $\text{Ext}_{\text{tail } A}^i(\mathcal{M}, \mathcal{N}) = 0$ for all $i > p$ and all $\mathcal{M}, \mathcal{N} \in \text{tail } A$.

Let H be a finite dimensional semisimple Hopf algebra. Assume that A is a left H -module algebra, and that the H -action preserves the grading of A . Then $B(= A\#H)$ is also an \mathbb{N} -graded algebra by setting the degree of elements of H to be zero. Since H is finite dimensional, A can be viewed as a right H^* -comodule algebra. Note that the comodule action $\rho : A \rightarrow A \otimes H^*$ also preserves the grading of A . We may replace the module categories in Theorem 2.3 by the categories of \mathbb{Z} -graded modules. Then Theorem 2.3 reads as follows.

Theorem 6.1. *Let $A = A_0 \oplus A_1 \oplus \dots$ be a locally finite noetherian \mathbb{N} -graded algebra, and let H be a finite dimensional semisimple Hopf algebra. Assume that A is a left H -module and the H -action preserves the grading of A . Set $B = A\#H$ and $R = A^H$, and let t be the integral of H such that $\varepsilon(t) = 1$. Then the following are equivalent.*

- (i) A/R is right H^* -dense Galois extension.
- (ii) For any finitely generated \mathbb{Z} -graded right B -module, if Mt is finite dimensional, then M itself is finite dimensional.
- (iii) B/BtB is finite dimensional.
- (iv) $-\otimes_{\mathcal{B}} \mathcal{A} : \text{Tail } B \rightarrow \text{Tail } R$ is an equivalence of abelian categories.
- (v) $-\otimes_{\mathcal{B}} \mathcal{A} : \text{tail } B \rightarrow \text{tail } R$ is an equivalence of abelian categories.

Proof. The statements (i), (iii) and (iv) are exactly the graded versions of the statements (i), (iii) and (iv) of Theorem 2.3. The graded version of the statement (ii) of Theorem 2.3 can be stated as follows: for any finitely generated right \mathbb{Z} -graded B -module M , $T_{\mathbb{Z}}(M)$ is finite dimensional, where $T_{\mathbb{Z}}(M)$ is the largest \mathbb{Z} -graded B -submodule of M such that $T_{\mathbb{Z}}(M)t$ is finite dimensional. Since B is noetherian, this is equivalent to the statement (ii) in this theorem. Hence (i), (ii), (iii) and (iv) are equivalent.

That (iv) implies (v) is because $\text{tail } A$ (resp. $\text{tail } R$) is full subcategory of $\text{Tail } A$ (resp. $\text{Tail } R$) consisting of noetherian objects, and the restriction of $-\otimes_{\mathcal{B}} \mathcal{A}$ to $\text{tail } A$ is densely onto $\text{tail } R$.

(v) \implies (ii). Let M be a finitely generated right \mathbb{Z} -graded B -module. If Mt is finite dimensional, then $\pi(M) \otimes_{\mathcal{B}} \mathcal{A} \cong \pi(Mt) = 0$ in $\text{tail } R$. Since $-\otimes_{\mathcal{B}} \mathcal{A}$ is an equivalence, $\pi(M)$ is finite dimensional. Hence M is finite dimensional. \square

With assumptions as in Theorem 6.1, we obtain the following.

Corollary 6.2. *If A is of finite global dimension and A/A^H is a right H^* -dense Galois extension, then A^H is a graded isolated singularity.*

Proof. If A is of finite global dimension, then so is B . Then the global dimension of $\text{tail } B$ is finite. By Theorem 6.1, $\text{tail } A^H$ has finite global dimension, and hence A^H is a graded isolated singularity. \square

Remark 6.3. (i) Note that we also have the graded version of Theorem 3.8.

(ii) A general theory of graded isolated singularities was established by Ueyama [Ue] and Mori-Ueyama [MU]. Many interesting examples were presented in those papers in the case that A is a Artin-Schelter regular algebra and H is a cyclic group algebra.

(iii) It is well known that finite subgroups of $SL_2(\mathbb{k})$ acting on the polynomial algebra $\mathbb{k}[x, y]$ yield isolated singularities. As for the noncommutative case, finite dimensional Hopf actions on

quantum planes are well understood by Chan, Kirkman, Walton and Zhang in [CKWZ1]. If G is a finite group which acts homogeneous on a quantum plane A and the homological determinant of G is trivial, then the condition (iii) of Theorem 6.1 is always satisfied (cf. [Ue]). Hence the invariant subalgebra A^G is graded isolated by Corollary 6.2.

Recall from [MM] that a locally finite noetherian \mathbb{N} -graded algebra A is called an *Artin-Schelter Gorenstein algebra* if $\text{injdim } A_A = \text{injdim}_A A = n < \infty$, and

$$\text{Ext}_A^i(A_0, A) \cong \begin{cases} 0, & i \neq n \\ A_0^*(l), & i = n \end{cases}$$

both in $\text{GrMod}_{\mathbb{Z}} A$ and in $\text{GrMod}_{\mathbb{Z}} A^{op}$, where (l) is the degree shift functor.

Corollary 6.4. *Let H be a semisimple Hopf algebra, and let A be an \mathbb{N} -graded left H -module algebra. Assume that the H -action preserves the grading of A , and that A is an Artin-Schelter Gorenstein algebra with injective dimension $\text{injdim } A_A = \text{injdim } A_A = n \geq 2$. If A/A^H is a Hopf H^* -dense Galois extension, then the natural map*

$$A\#H \longrightarrow \text{End}(A_{A^H}), a\#h \mapsto [b \mapsto a(h \cdot b)]$$

is an isomorphism of \mathbb{N} -graded algebras.

Proof. Since A is Artin-Schelter Gorenstein and the injective dimension of A is not less than 2, it follows that $\text{depth } A_A \geq 2$ (we remark that in graded case, the depth of A is considered in $\text{GrMod}_{\mathbb{Z}} A$). By the graded version of Theorem 3.8 (cf. Remark 6.3 (i)), we have $A\#H \cong \text{End}(A_{A^H})$ as graded algebra. \square

Finally, we remark that a similar result was already obtained in [MU, Theorem 3.7] under the much stronger assumptions that A is a noetherian Artin-Schelter regular algebra and H is a finite group algebra.

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