# The polynomiality of the Poisson center and semi-center of a Lie algebra and Dixmier's fourth problem 

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#### Abstract

. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over an algebraically closed field $k$ of characteristic zero. We provide necessary and also some sufficient conditions in order for its Poisson center and semi-center to be polynomial algebras over $k$. This occurs for instance if $\mathfrak{g}$ is quadratic of index 2 with $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$ and also if $\mathfrak{g}$ is nilpotent of index at most 2. The converse holds for filiform Lie algebras of type $L_{n}, Q_{n}, R_{n}$ and $W_{n}$.


We show how Dixmier's fourth problem for an algebraic Lie algebra $\mathfrak{g}$ can be reduced to that of its canonical truncation $\mathfrak{g}_{\Lambda}$. Moreover, Dixmier's statement holds for all Lie algebras of dimension at most eight. The nonsolvable, indecomposable ones among them possess a polynomial Poisson center and semi-center.

## 1. Introduction

Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field $k$ of characteristic zero, with basis $x_{1}, \ldots, x_{n}$. Let $U(\mathfrak{g})$ be its enveloping algebra with center $Z(U(\mathfrak{g}))$ and semi-center $S z(U(\mathfrak{g})$ ), i.e. the subalgebra of $U(\mathfrak{g})$ generated by the semi-invariants of $U(\mathfrak{g})$. Denote by $D(\mathfrak{g})$ the quotient division ring of $U(\mathfrak{g})$ with center $Z(D(\mathfrak{g}))$. In this paper we address the following problems:

1) When are $Z(U(\mathfrak{g}))$ and $S z(U(\mathfrak{g}))$ polynomial algebras over $k$ ?
2) Is $Z(D(\mathfrak{g}))$ always rational over $k$ ? (Dixmier's fourth problem [D6, p.354]).

In order to simplify things we consider the symmetric algebra $S(\mathfrak{g})$ which we identify with the polynomial algebra $k\left[x_{1}, \ldots, x_{n}\right]$.

We equip $S(\mathfrak{g})$ with its natural Poisson structure. Its Poisson center $Y(\mathfrak{g})$ coincides with the algebra $S(\mathfrak{g})^{\mathfrak{g}}$ of invariants. By a celebrated result of Michel Duflo [Du1, Du2, Du3] there exists an algebra isomorphism between $Z(U(\mathfrak{g}))$ and $Y(\mathfrak{g})$. Rentschler and Vergne [RV] later extended this to an algebra isomorphism between $S z(U(\mathfrak{g}))$ and the semi-center $S z(S(\mathfrak{g}))$, which is usually denoted by $S y(\mathfrak{g})$. Furthermore, $Z(D(\mathfrak{g}))$ is isomorphic with $R(\mathfrak{g})^{\mathfrak{g}}$, the subfield of invariants of $R(\mathfrak{g})$, where $R(\mathfrak{g})$ is the quotient field of $S(\mathfrak{g})$. Therefore it suffices to deal with both problems in $S(\mathfrak{g})$ and $R(\mathfrak{g})$, where things are easier and where it is possible to use MAPLE for the less trivial calculations. Our first objective is to collect necessary (see 3.1) and sufficient (see 3.2) conditions in order to have polynomiality.
The index $i(\mathfrak{g})$ of $\mathfrak{g}$ (see 2.1) will play a major role. However, an alternative index $j(\mathfrak{g})$ (see 2.2) will perform better in the nonalgebraic case. For instance we have the following.

## Theorem 1.

$$
j(\mathfrak{g})=\operatorname{trdeg}_{k} R(\mathfrak{g})^{\mathfrak{g}}=\operatorname{trdeg}_{k} Z(D(\mathfrak{g})) \leq i(\mathfrak{g})
$$

Moreover, equality occurs if $\mathfrak{g}$ is ad-algebraic or if $\mathfrak{g}$ has no proper semi-invariants in $S(\mathfrak{g})$.

For brevity we will call $\mathfrak{g}$ coregular if $Y(\mathfrak{g})$ is a polynomial algebra over $k$.

Definition 17. Let $p_{\mathfrak{g}} \in S(\mathfrak{g})$ be the fundamental semi-invariant of $\mathfrak{g}$ (see 2.6). We say that $\mathfrak{g}$ satisfies the Joseph-Shafrir conditions, JS for short, if $\mathfrak{g}$ is unimodular for which $p_{\mathfrak{g}}$ is an invariant and $\operatorname{trdeg}_{k} Y(\mathfrak{g})=i(\mathfrak{g})$. (For example JS is satisfied if $\mathfrak{g}$ has no proper semi-invariants (Remark 2)).

The following criterion for coregularity will be employed quite often.

Corollary 19. (Short version)
Let $\mathfrak{g}$ be a Lie algebra satisfying JS with center $Z(\mathfrak{g})$. If $\mathfrak{g}$ is coregular then

$$
3 i(\mathfrak{g})+2 \operatorname{deg} p_{\mathfrak{g}} \leq \operatorname{dim} \mathfrak{g}+2 \operatorname{dim} Z(\mathfrak{g})
$$

This inequality imposes a strong upperbound on $i(\mathfrak{g})$. Therefore coregularity becomes a rare phenomenon for nonabelian Lie algebras having a large index. This is especially true if $i(\mathfrak{g})=\operatorname{dim} \mathfrak{g}-2$ as it is in the following result (see also Proposition 40 and Corollary 41).

Theorem 20. Let $\mathfrak{g}$ be unimodular, having an abelian ideal $\mathfrak{h}$ of codimension one. Then the following are equivalent:
(1) $\mathfrak{g}$ is coregular and $\operatorname{trdeg}_{k} Y(\mathfrak{g})=i(\mathfrak{g})$
(2) $\mathfrak{g}$ is coregular and ad-algebraic
(3) $\operatorname{dim}[\mathfrak{g}, \mathfrak{g}] \leq 2$

The conditions in the above results cannot be weakened as shown by the examples 24-26.
As an application of this theorem we consider the nilradical $N$ of the parabolic subalgebra $P$ of type $(1,1, n-2)$ inside $s l(n)$ and show that

$$
N \text { is not coregular } \Leftrightarrow n \geq 5
$$

(see Proposition 28). This extends an example by A. Hersant [J2, 8.5].

At the end of section 3 we consider a Lie algebra $\mathfrak{g}$ for which $Y(\mathfrak{g})$ is saturated with quotient field $R(\mathfrak{g})^{\mathfrak{g}}$. Then there exist irreducible, proper semi-invariants $v_{1}, \ldots, v_{t} \in S(\mathfrak{g})$ such that $S y(\mathfrak{g})=Y(\mathfrak{g})\left[v_{1}, \ldots, v_{t}\right]$ is a polynomial ring over $Y(\mathfrak{g})$. In particular, if $Y(\mathfrak{g})$ is polynomial over $k$, then so is $S y(\mathfrak{g})$ (Theorem 36).

The above conditions are satisfied if $j(\mathfrak{g})=\operatorname{dim} Z(\mathfrak{g})$ because then $Y(\mathfrak{g})=S(Z(\mathfrak{g}))$ and $R(\mathfrak{g})^{\mathfrak{g}}=R(Z(\mathfrak{g}))$. Hence both $Y(\mathfrak{g})$ and $S y(\mathfrak{g})$ are polynomial over $k$, while $R(\mathfrak{g})^{\mathfrak{g}}$ is rational over $k$. Moreover, $v_{1}, \ldots, v_{t}$ are then precisely the irreducible factors of a special semi-invariant $p_{\mathfrak{g}}^{\prime} \in S(\mathfrak{g})$, which takes over the role of the fundamental semi-invariant $p_{\mathfrak{g}}$ (Theorem 37). This is illustrated in Example 39 and applied to some Lie algebras such as $L_{8,25}$ (see Example 58) of section 5 .

In section 4 we study the coregularity for Lie algebras of index at most two. The following is one of the main results:

Theorem 45. Any nilpotent Lie algebra with index at most two is coregular.

Its proof is constructive and we can give a useful characterization of the generator(s) of $Y(\mathfrak{g})$. (see the claim within the proof). This is used in the following application to the major types of filiform Lie algebras:

## Theorem 51.

(1) If $\mathfrak{g}$ is of type $Q_{n}$ or $W_{n}$ then $\mathfrak{g}$ is coregular since $i(\mathfrak{g}) \leq 2$.
(2) If $\mathfrak{g}$ is of type $L_{n}$ or $R_{n}$ then

$$
\mathfrak{g} \text { is coregular } \quad \Leftrightarrow \quad i(\mathfrak{g}) \leq 2
$$

Theorem 45 cannot be extended to the solvable case as there exists a solvable Lie algebra of index two which is not coregular (Example 23.).

Theorem 52. Let $\mathfrak{g}$ be a quadratic Lie algebra. Then $\mathfrak{g}$ is coregular if one of the following conditions is satisfied:
(i) $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$ and $i(\mathfrak{g})=2$
(ii) $\mathfrak{g}$ is nilpotent and $i(\mathfrak{g})=3$

In section 5 we verify, case by case, that any nonsolvable, indecomposable Lie algebra of dimension at most eight satisfies the two problems we raised in the beginning (Theorem 53).
However, in dimension nine, we exhibit a counterexample (Example 59).

Section 6 is devoted to Dixmier's fourth problem.
We list some important classes where this question is known to have a positive answer and we prove that it is also the case for all Lie algebras of dimension at most 8 (Proposition 63). The following is the main result of this section:

Theorem 66. Let $\mathfrak{g}$ be an algebraic Lie algebra for which the field $Z\left(D\left(\mathfrak{g}_{\Lambda}\right)\right)$ is freely generated by semi-invariants $u_{1}, \ldots, u_{s}$ of $U(\mathfrak{g})$. Then $Z(D(\mathfrak{g}))$ is rational over $k$.

As an application we obtain a result by Panyushev [Pa1], namely $Z(D(\mathfrak{g}))$ is rational over $k$ if $\mathfrak{g}$ is any biparabolic subalgebra of a simple Lie algebra of type $A$ or $C$ (Corollary 67).

Some of the results of [AOV2] are used in section 5. Therefore we briefly discuss the well known Gelfand-Kirillov conjecture. This is a much stronger statement than Dixmier's fourth problem (see also Example 60 and Proposition 62). In the Appendix we correct the proof of an example from [GK] showing the existence of nonalgebraic Lie algebras satisfying the Gelfand-Kirillov conjecture (Example 71).

## 2. Preliminaries

## $2.1 i(\mathfrak{g})$, the index of $\mathfrak{g}$

Let $k$ be an algebraically closed field of characteristic zero and let $\mathfrak{g}$ be a Lie algebra over $k$ with basis $x_{1}, \ldots, x_{n}$. For each $\xi \in \mathfrak{g}^{*}$ we consider its stabilizer

$$
\mathfrak{g}(\xi)=\{x \in \mathfrak{g} \mid \xi([x, y])=0 \text { for all } y \in \mathfrak{g}\}
$$

The minimal value of $\operatorname{dim} \mathfrak{g}(\xi)$ is called the index of $\mathfrak{g}$ and is denoted by $i(\mathfrak{g})$ [D6, 1.11.6; TY, 19.7.3]. Put $c(\mathfrak{g})=(\operatorname{dim} \mathfrak{g}+i(\mathfrak{g})) / 2$. This integer will play an important role throughout this paper. An element $\xi \in \mathfrak{g}^{*}$ is called regular if $\operatorname{dim} \mathfrak{g}(\xi)=i(\mathfrak{g})$. The set $\mathfrak{g}_{\text {reg }}^{*}$ of all regular elements of $\mathfrak{g}^{*}$ is an open dense subset of $\mathfrak{g}^{*}$.
We put $\mathfrak{g}_{\text {sing }}^{*}=\mathfrak{g}^{*} \backslash \mathfrak{g}_{\text {reg }}^{*}$. Clearly, codim $\mathfrak{g}_{\text {sing }}^{*} \geq 1$. Following [JS] we call $\mathfrak{g}$ singular if equality holds and nonsingular otherwise. For instance, any semi-simple Lie algebra $\mathfrak{g}$ is nonsingular since codim $\mathfrak{g}_{\text {sing }}^{*}=3$. We recall from [D6, 1.14.13] that

$$
i(\mathfrak{g})=\operatorname{dim} \mathfrak{g}-\operatorname{rank}_{R(\mathfrak{g})}\left(\left[x_{i}, x_{j}\right]\right)
$$

In particular, $\operatorname{dim} \mathfrak{g}-i(\mathfrak{g})$ is an even number.

## $2.2 j(\mathfrak{g})$, the alternative index of $\mathfrak{g}$

Let $H$ be the algebraic hull of ad $\mathfrak{g}$ in Der $\mathfrak{g}$ ([C, p.173; TY, 24.5.4]), i.e. the smallest algebraic Lie subalgebra $H$ of Der $\mathfrak{g}$ containing ad $\mathfrak{g}$. Let $\xi \in \mathfrak{g}^{*}$ and put

$$
\mathfrak{g}[\xi]=\{x \in \mathfrak{g} \mid \xi(E x))=0 \text { for all } E \in H\}
$$

This is an ideal of $\mathfrak{g}(\xi)$ which contains the center $Z(\mathfrak{g})$ of $\mathfrak{g}$. Clearly, $\mathfrak{g}[\xi]=\mathfrak{g}(\xi)$ if $\mathfrak{g}$ is ad-algebraic (i.e. ad $\mathfrak{g}=H$ ). Let $E_{1}, \ldots, E_{m}$ be a basis of $H$. Then it is easily seen that

$$
\operatorname{dim} \mathfrak{g}[\xi]=\operatorname{dim} \mathfrak{g}-\operatorname{rank}\left(\xi\left(E_{i} x_{j}\right)\right)
$$

We denote by $j(\mathfrak{g})$ the minimal value of $\operatorname{dim} \mathfrak{g}[\xi], \xi \in \mathfrak{g}^{*}$. Then

$$
j(\mathfrak{g})=\operatorname{dim} \mathfrak{g}-\operatorname{rank}_{R(\mathfrak{g})}\left(E_{i} x_{j}\right)
$$

Clearly, $\operatorname{dim} Z(\mathfrak{g}) \leq j(\mathfrak{g}) \leq i(\mathfrak{g})$ and $j(\mathfrak{g})=i(\mathfrak{g})$ if $\mathfrak{g}$ is ad-algebraic. For the first part of the following we refer to [O1, O2; RV, p.401].

## Theorem 1.

$$
j(\mathfrak{g})=\operatorname{trdeg}_{k} R(\mathfrak{g})^{\mathfrak{g}}=\operatorname{trdeg}_{k} Z(D(\mathfrak{g})) \leq i(\mathfrak{g})
$$

Moreover, equality occurs if one of the following conditions is satisfied:
(1) $\mathfrak{g}$ is ad-algebraic
(2) $\mathfrak{g}$ has no proper semi-invariants in $S(\mathfrak{g})$ (or equivalently in $U(\mathfrak{g})$ ) [OV, Proposition 4.1].

### 2.3 Commutative polarizations of $\mathfrak{g}$

Suppose $\mathfrak{g}$ admits a commutative Lie subalgebra $\mathfrak{h}$ such that $\operatorname{dim} \mathfrak{h}=c(\mathfrak{g})$, i.e. $\mathfrak{h}$ is a commutative polarization (notation: CP) with respect to any $\xi \in \mathfrak{g}_{\text {reg }}^{*}$ [D6, 1.12]. These CP's occur frequently in the nilpotent case [O7, O8]. If in addition $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ then we call $\mathfrak{h}$ a CP-ideal (notation: CPI). If a solvable Lie algebra $\mathfrak{g}$ admits a CP then it also admits a CPI [EO, Theorem 4.1].

### 2.4 The Poisson algebra $S(\mathfrak{g})$ and its center

The symmetric algebra $S(\mathfrak{g})$, which we identify with $k\left[x_{1}, \ldots, x_{n}\right]$, has a natural Poisson algebra structure, the Poisson bracket of $f, g \in S(\mathfrak{g})$ given by:

$$
\{f, g\}=\sum_{i=1}^{n} \sum_{j=1}^{n}\left[x_{i}, x_{j}\right] \frac{\partial f}{\partial x_{i}} \frac{\partial g}{\partial x_{j}}
$$

In particular, $S(\mathfrak{g}),\{$,$\} is a Lie algebra for which \mathfrak{g}$ is a Lie subalgebra since for any two elements $x, y \in \mathfrak{g}$ we have that $\{x, y\}=[x, y]$. Also, for all $f, g, h \in S(\mathfrak{g})$ :

$$
\begin{equation*}
\{f, g h\}=\{f, g\} h+g\{f, h\} \tag{*}
\end{equation*}
$$

It now easily follows that the center of $S(\mathfrak{g}),\{$,$\} is equal to$

$$
\{f \in S(\mathfrak{g}) \mid\{x, f\}=0 \quad \forall x \in \mathfrak{g}\}
$$

and since $\{x, f\}=\operatorname{ad} x(f)$ this clearly coincides with $Y(\mathfrak{g})=S(\mathfrak{g})^{\mathfrak{g}}$, the subalgebra of invariant polynomials of $S(\mathfrak{g})$.
The Poisson bracket has a unique extension to the quotient field $R(\mathfrak{g})$ of $S(\mathfrak{g})$ such that $(*)$ holds in $R(\mathfrak{g})$. It follows that $R(\mathfrak{g}),\{$,$\} is a Lie algebra with center R(\mathfrak{g})^{\mathfrak{g}}$,
the subfield of rational invariants of $R(\mathfrak{g}) . R(\mathfrak{g})$ is called the rational Poisson algebra [V, p. 311].

Let $A$ be a Poisson commutative subalgebra of $S(\mathfrak{g})$ (i.e. $\{f, g\}=0$ for all $f, g \in A$ ). Then it is well-known that $\operatorname{trdeg}_{k}(A) \leq c(\mathfrak{g})$. $A$ is called complete if equality holds and strongly complete if it is also a maximal Poisson commutative subalgebra. According to Sadetov there always exists a complete Poisson commutative subalgebra of $S(\mathfrak{g})$ [Sa]. For example, suppose $\mathfrak{g}$ admits a commutative polarization (CP) $\mathfrak{h}$. Then $S(\mathfrak{h})$ is a polynomial, strongly complete subalgebra of $S(\mathfrak{g})$ and its quotient field $R(\mathfrak{h})$ is a maximal Poisson commutative subfield of $R(\mathfrak{g})$ [O4, Theorem 14].

### 2.5 The semi-center $S y(\mathfrak{g})$ of $S(\mathfrak{g})$

Let $\lambda \in \mathfrak{g}^{*}$. We denote by $S(\mathfrak{g})_{\lambda}$ the set of all $f \in S(\mathfrak{g})$ such that ad $x(f)=\lambda(x) f$ for all $x \in \mathfrak{g}$. Any element $f \in S(\mathfrak{g})_{\lambda}$ is said to be a semi-invariant w.r.t. the weight $\lambda$. We call $f$ a proper semi-invariant if $\lambda \neq 0$. Clearly, $S(\mathfrak{g})_{\lambda} S(\mathfrak{g})_{\mu} \subset S(\mathfrak{g})_{\lambda+\mu}$ for all $\lambda, \mu \in \mathfrak{g}^{*}$. Let $f, g \in S(\mathfrak{g})$. If $f g$ is a nonzero semi-invariant of $S(\mathfrak{g})$, then so are $f$ and $g$.
The sum of all $S(\mathfrak{g})_{\lambda}, \lambda \in \mathfrak{g}^{*}$, is direct and it is a nontrivial factorial subalgebra $S y(\mathfrak{g})$ of $S(\mathfrak{g})$ [D3, Mo, LO]. Moreover, it is Poisson commutative [OV, p. 308].
Any nonzero semi-invariant can be written uniquely as a product of irreducible semiinvariants.
Suppose $h \in R(\mathfrak{g}), h \neq 0$. Then $h \in R(\mathfrak{g})^{\mathfrak{g}}$ if and only if $h$ can be written as a quotient of two semi-invariants of the same weight.

Remark 2. Assume that $\mathfrak{g}$ has no proper semi-invariants (as it is if the radical of $\mathfrak{g}$ is nilpotent). Then $R(\mathfrak{g})^{\mathfrak{g}}$ is the quotient field of $S(\mathfrak{g})^{\mathfrak{g}}=Y(\mathfrak{g})$. In particular,

$$
\operatorname{trdeg}_{k} Y(\mathfrak{g})=\operatorname{trdeg}_{k} R(\mathfrak{g})^{\mathfrak{g}}=i(\mathfrak{g})
$$

by Theorem 1. Also, $\mathfrak{g}$ is unimodular (i.e. $\operatorname{tr}(\operatorname{ad} x)=0$ for all $x \in \mathfrak{g}$ ) by [DDV, Thm. 1.11] and its proof.

The weights of the semi-invariants of $S(\mathfrak{g})$ form an additive semi-group $\Lambda(\mathfrak{g})$, which is not necessarily finitely generated [DDV, p. 322]. However, the subgroup $\Lambda_{R}(\mathfrak{g})$ of $\mathfrak{g}^{*}$ generated by $\Lambda(\mathfrak{g})$ is a finitely generated free abelian group [NO, Theorem 1.3], [FJ2, p. 1519].
Next, we denote by $\mathfrak{g}_{\Lambda}$ the intersection of $\operatorname{ker} \lambda, \lambda \in \Lambda(\mathfrak{g}) . \mathfrak{g}_{\Lambda}$ is a characteristic ideal
of $\mathfrak{g}$ which contains $[\mathfrak{g}, \mathfrak{g}]$. It is called the canonical truncation of $\mathfrak{g}$.

Lemma 3. Let $u \in S(\mathfrak{g})$ be a nonzero semi-invariant with weight $\lambda \in \Lambda(\mathfrak{g})$. Denote by $C(u)\left(\right.$ resp. $\left.C_{R}(u)\right)$ the centralizer of $u$ in $S(\mathfrak{g})$ (resp. $R(\mathfrak{g})$ ). Then we have

$$
C(u)=S(\operatorname{ker} \lambda) \quad \text { and } \quad C_{R}(u)=R(\operatorname{ker} \lambda)
$$

Proof. We may assume that $\lambda \neq 0$. Choose a basis $x_{1}, x_{2}, \ldots, x_{n}$ of $\mathfrak{g}$ such that $x_{2}, \ldots, x_{n}$ is a basis of ker $\lambda$ and such that $\lambda\left(x_{1}\right)=1$. Since $u \in S(\mathfrak{g})_{\lambda}$ we have $\{f, u\}=d_{\lambda}(f) u$ for all $f \in S(\mathfrak{g})$ [OV, p.308]. Then the first equality follows from

$$
f \in C(u) \quad \Leftrightarrow \quad d_{\lambda}(f)=0 \quad \Leftrightarrow \quad \frac{\partial f}{\partial x_{1}}=0 \quad \Leftrightarrow \quad f \in k\left[x_{2}, \ldots, x_{n}\right]=S(\operatorname{ker} \lambda)
$$

Next, we take a nonzero $h \in C_{R}(u)$. We may write $h=f / g$ for some nonzero, relatively prime $f, g \in S(\mathfrak{g})$. From $h g=f$ we deduce $h\{g, u\}=\{f, u\}$ since $\{h, u\}=0$. Hence,

$$
h d_{\lambda}(g) u=d_{\lambda}(f) u
$$

Simplification gives

$$
d_{\lambda}(g) f=d_{\lambda}(f) g
$$

Now suppose $d_{\lambda}(f) \neq 0$. Then $f$, being coprime with $g$, divides $d_{\lambda}(f)$, contradicting the fact that $\operatorname{deg} d_{\lambda}(f)<\operatorname{deg} f$.
Therefore $d_{\lambda}(f)=0$ and thus $f \in S(\operatorname{ker} \lambda)$. Similarly, $g \in S(\operatorname{ker} \lambda)$ and so $h=f / g \in$ $R(\operatorname{ker} \lambda)$. Consequently, $C_{R}(u) \subset R(\operatorname{ker} \lambda)$. The other inclusion is obvious.

Using this lemma one can now apply the same approach as in [DNO, pp. 331$334]$ and [MO, pp. 213-214] in order to obtain the following. In fact (1), (2), (3) do not require for $k$ to be algebraically closed. See also [BGR, F, FJ2, RV].

## Theorem 4.

1. $C(S y(\mathfrak{g}))=S\left(\mathfrak{g}_{\Lambda}\right)$ and $C_{R}(S y(\mathfrak{g}))=R\left(\mathfrak{g}_{\Lambda}\right)$
2. $\mathfrak{g}_{\Lambda}$ has no proper semi-invariants and so $R\left(\mathfrak{g}_{\Lambda}\right)^{\mathfrak{g}_{\Lambda}}$ is the quotient field of $Y\left(\mathfrak{g}_{\Lambda}\right)$. Also $\operatorname{trdeg}_{k} Y\left(\mathfrak{g}_{\Lambda}\right)=i\left(\mathfrak{g}_{\Lambda}\right)$
3. $S(\mathfrak{g})^{\mathfrak{g}_{\Lambda}}=Y\left(\mathfrak{g}_{\Lambda}\right)$ and $R(\mathfrak{g})^{\mathfrak{g}_{\Lambda}}=R\left(\mathfrak{g}_{\Lambda}\right)^{\mathfrak{g}_{\Lambda}}$
4. $c\left(\mathfrak{g}_{\Lambda}\right)=c(\mathfrak{g})$ (use [OV, Lemma 3.7] and [O7, Proposition 3.2])
5. $S y(\mathfrak{g}) \subset Y\left(\mathfrak{g}_{\Lambda}\right)=S y\left(\mathfrak{g}_{\Lambda}\right)$ and equality occurs if $\mathfrak{g}$ is almost algebraic or if $\mathfrak{g}$ is Frobenius (i.e. $i(\mathfrak{g})=0$ )
6. Suppose $\mathfrak{h}$ is a CP-ideal of $\mathfrak{g}$. Then $\mathfrak{h} \subset \mathfrak{g}_{\Lambda}\left[\right.$ EO, p. 141] and $Y\left(\mathfrak{g}_{\Lambda}\right) \subset S(\mathfrak{h})$

### 2.6 The fundamental semi-invariant $p_{\mathfrak{g}}$

Definition 5. Put $t=\operatorname{dim} \mathfrak{g}-i(\mathfrak{g})$, which is the rank of the structure matrix $B=\left(\left[x_{i}, x_{j}\right]\right) \in M_{n}(R(\mathfrak{g}))$, where $x_{1}, \ldots, x_{n}$ is an arbitrary basis of $\mathfrak{g}$. Assume first that $\mathfrak{g}$ is nonabelian. Then the greatest common divisor $q_{\mathfrak{g}}$ of the $t \times t$ minors in $B$ is a nonzero semi-invariant of $S(\mathfrak{g})$ [DNO, pp. 336-337]. If $\mathfrak{g}$ is abelian we put $q_{\mathfrak{g}}=1$. Next, let $p_{\mathfrak{g}}$ be the greatest common divisor of the Pfaffians of the principal $t \times t$ minors in $B$. In particular, $\operatorname{deg} p_{\mathfrak{g}} \leq(\operatorname{dim} \mathfrak{g}-i(\mathfrak{g})) / 2$. By [OV, Lemma 2.1] $p_{\mathfrak{g}}^{2}=q_{\mathfrak{g}}$ up to a nonzero scalar multiplier. We call $p_{\mathfrak{g}}$ the fundamental semi-invariant of $S(\mathfrak{g})$ (instead of $q_{\mathfrak{g}}$ as we did in [OV, p. 309]).

Remark 6. [OV, p. 307]
$\mathfrak{g}$ is singular if and only if $p_{\mathfrak{g}} \notin k$
Example 7. Let $\mathfrak{g}$ be a nonabelian Lie algebra with center $Z(\mathfrak{g})$. $\mathfrak{g}$ is called square integrable (SQ.I.) if $i(\mathfrak{g})=\operatorname{dim} Z(\mathfrak{g})$. For instance any Heisenberg Lie algebra is square integrable.
Choose a basis $x_{1}, \ldots, x_{t}, x_{t+1}, \ldots, x_{n}$ such that $x_{t+1}, \ldots, x_{n}$ is a basis of $Z(\mathfrak{g})$.
Then, $t=\operatorname{dim} \mathfrak{g}-\operatorname{dim} Z(\mathfrak{g})=\operatorname{dim} \mathfrak{g}-i(\mathfrak{g})$, which is the rank of the matrix $\left(\left[x_{i}, x_{j}\right]\right)_{1 \leq i, j \leq t}$. By the above, its Pfaffian coincides with $p_{\mathfrak{g}}$ (up to a nonzero scalar). Hence, $\operatorname{deg} p_{\mathfrak{g}}=(\operatorname{dim} \mathfrak{g}-i(\mathfrak{g})) / 2 \geq 1$ and so $\mathfrak{g}$ is singular. In particular, any Frobenius Lie algebra $\mathfrak{g}$ is singular.

Lemma 8. [J5, Lemma 2.3] Let $\mathfrak{g}$ be an algebraic Lie algebra. Then $p_{\mathfrak{g}_{\Lambda}}$ divides $p_{\mathfrak{g}}$.

### 2.7 Frobenius Lie algebras

A Lie algebra $\mathfrak{g}$ is called Frobenius if there is a linear functional $\xi \in \mathfrak{g}^{*}$ such that the alternating bilinear $B_{\xi}(x, y)=\xi([x, y]), x, y \in \mathfrak{g}$, is nondegenerate, i.e. $i(\mathfrak{g})=0$. The name was suggested to us by George Seligman because of its obvious resemblance with the notion of an associative Frobenius algebra. They came about in connection with Jacobson's problem on the characterization of Lie algebras having a primitive universal enveloping algebra. It turns out that:
$U(\mathfrak{g})$ is primitive if and only if $Z(D(\mathfrak{g}))=k$, i.e. $j(\mathfrak{g})=0[\mathrm{O} 1, \mathrm{O} 2]$.

In particular, $U(\mathfrak{g})$ is primitive if $\mathfrak{g}$ is Frobenius and the converse holds if $\mathfrak{g}$ is adalgebraic by Theorem 1 .
Frobenius Lie algebras form a large class and they appear naturally in different areas. For example many parabolic and biparabolic (seaweed) subalgebras of semisimple Lie algebras are Frobenius [CGM, CMW, DY, E1, E2, E3, CV, JS, PY2, O3], including most Borel subalgebras of simple Lie algebras [EO, p. 146]. A Frobenius biparabolic Lie algebra $\mathfrak{g}$ satisfies interesting properties. For instance $\mathfrak{g}_{\Lambda}=[\mathfrak{g}, \mathfrak{g}]$ [J5, Proposition 7.6], which is not true for all Frobenius Lie algebras as the following demonstrates (this answers a question by Joseph [J5, Remark 7.6].

Example 9. Let $L$ be the Lie algebra over $k$ with basis $x_{1}, x_{2}, x_{3}, x_{4}$ and nonvanishing brackets $\left[x_{1}, x_{3}\right]=x_{3},\left[x_{1}, x_{4}\right]=x_{4},\left[x_{2}, x_{3}\right]=x_{4}$.
Consider its structure matrix $B=\left(\left[x_{i}, x_{j}\right]\right)$. Clearly $\operatorname{det} B=x_{4}^{4} \neq 0$. Hence $i(L)=0$ by 2.1 and $p_{L}=x_{4}^{2}$.
$x_{4}$ is the only irreducible semi-invariant of $S(L)$ (see below). Its weight $\lambda \in L^{*}$ is determined by $\lambda\left(x_{1}\right)=1, \lambda\left(x_{2}\right)=\lambda\left(x_{3}\right)=\lambda\left(x_{4}\right)=0$.
Consequently, $L_{\Lambda}=\operatorname{ker} \lambda=\left\langle x_{2}, x_{3}, x_{4}\right\rangle$, while $[L, L]=\left\langle x_{3}, x_{4}\right\rangle$ (which happens to be a CPI of $L$ ). Moreover $S y(L)=k\left[x_{4}\right]=Y\left(L_{\Lambda}\right)$.

We now collect some useful facts on semi-invariants from [O3, DNO]. Let $\mathfrak{g}$ be a Frobenius Lie algebra with basis $x_{1}, \ldots, x_{n}$. Then $n$ is even and $\mathfrak{g}$ has a trivial center. The Pfaffian $\operatorname{Pf}\left(\left[x_{i}, x_{j}\right]\right) \in S(\mathfrak{g})$ is homogeneous of degree $\frac{1}{2} \operatorname{dim} \mathfrak{g}$ and $\left(\operatorname{Pf}\left(\left[x_{i}, x_{j}\right]\right)\right)^{2}=\operatorname{det}\left(\left[x_{i}, x_{j}\right]\right) \neq 0$ by 2.1. Hence $p_{\mathfrak{g}}=\operatorname{Pf}\left(\left[x_{i}, x_{j}\right]\right)$. We put $\Delta(\mathfrak{g})=\operatorname{det}\left(\left[x_{i}, x_{j}\right]\right)$ (which is well determined up to nonzero scalar multipliers). $p_{\mathfrak{g}}$ is a semi-invariant with weight $\tau$, where $\tau(x)=\operatorname{tr}(\operatorname{ad} x), x \in \mathfrak{g}$.
Moreover, any semi-invariant of $S(\mathfrak{g})$ is homogeneous. It is also a semi-invariant under the action of Der $\mathfrak{g}$.

Theorem 10. Let $\mathfrak{g}$ be Frobenius. Decompose $p_{\mathfrak{g}}$ into a product of irreducible factors:

$$
p_{\mathfrak{g}}=v_{1}^{m_{1}} \ldots, v_{r}^{m_{r}}, \quad m_{i} \geq 1
$$

Then:
(1) $v_{1}, \ldots, v_{n}$ are the only (up to nonzero scalars) irreducible semi-invariants of $S(\mathfrak{g})$, say with weights $\lambda_{1}, \ldots, \lambda_{r} \in \Lambda(\mathfrak{g})$.
(2) $S y(\mathfrak{g})=k\left[v_{1}, \ldots, v_{r}\right]=Y\left(\mathfrak{g}_{\Lambda}\right)$, a polynomial algebra over $k$.
(3) $r=\operatorname{dim} \mathfrak{g}-\operatorname{dim} \mathfrak{g}_{\Lambda}=i\left(\mathfrak{g}_{\Lambda}\right)$
(4) $\lambda_{1}, \ldots, \lambda_{r}$ are linearly independent over $k$. They generate the semi-group $\Lambda(\mathfrak{g})$ and $\mathfrak{g}_{\Lambda}=\cap \operatorname{ker} \lambda_{i}, i=1, \ldots, r$
(5) $\mathfrak{h}_{i}=\operatorname{ker} \lambda_{i}$ is an ideal of $\mathfrak{g}$ of index one and $Y\left(\mathfrak{h}_{i}\right)=k\left[v_{i}\right]$ and $R\left(\mathfrak{h}_{i}\right)^{\mathfrak{h}_{i}}=k\left(v_{i}\right)$
(6) $m_{1} \lambda_{1}+\ldots+m_{r} \lambda_{r}=\tau(*)$ and $m_{1} \operatorname{deg} v_{1}+\ldots+m_{r} \operatorname{deg} v_{r}=\operatorname{deg} p_{\mathfrak{g}}=\frac{1}{2} \operatorname{dim} \mathfrak{g}(* *)$
(7) (Joseph [J5, 2.2]) Suppose in addition that $\mathfrak{g}$ is algebraic.

Then $p_{\mathfrak{g}_{\Lambda}}=v_{1}^{m_{1}-1} \ldots v_{r}^{m_{r}-1}$. In particular,
$\mathfrak{g}_{\Lambda}$ is nonsingular $\Leftrightarrow \quad m_{i}=1$ for all $i=1, \ldots, r$.
Remark 11. Each semi-invariant $v_{i}$ is determined by its weight $\lambda_{i}$ (up to a nonzero scalar multiplier) [Indeed, suppose that also $v \in S(\mathfrak{g})_{\lambda_{i}}, v \neq 0$. Then $v v_{i}^{-1} \in R(\mathfrak{g})^{\mathfrak{g}}=k$, i.e. $v=a v_{i}$ for some nonzero $\left.a \in k\right]$. Therefore $\lambda_{i}$ will provide information on $v_{i}$. For example its multiplicity $m_{i}$ (by $(*)$ since $\lambda_{1}, \ldots, \lambda_{r}$ are linearly independent over $k$ ), $\operatorname{deg} v_{i}$ and $(* *)$ can be obtained directly from ( $*$ ). To demonstrate this we take $\xi \in \mathfrak{g}_{r e g}^{*}$, i.e. $\xi\left(p_{\mathfrak{g}}\right) \neq 0$ (we extend $\xi$ to an algebra endomorphism of $S(\mathfrak{g}))$ and hence also $\xi\left(v_{i}\right) \neq 0$. By [O3, p. 21] there exists a unique element $x_{\xi} \in \mathfrak{g}$ such that $\xi \circ$ ad $x_{\xi}=\xi$ (Nowadays $x_{\xi}$ is called a principal element of $\mathfrak{g})$. From ad $x_{\xi}\left(v_{i}\right)=\lambda_{i}\left(x_{\xi}\right) v_{i}$ we get $\xi\left(\operatorname{ad} x_{\xi}\left(v_{i}\right)\right)=\lambda_{i}\left(x_{\xi}\right) \xi\left(v_{i}\right)$, which we can rewrite as $\left(\operatorname{deg} v_{i}\right) \xi\left(v_{i}\right)=\lambda_{i}\left(x_{\xi}\right) \xi\left(v_{i}\right)$ since $v_{i}$ is homogeneous. Simplification yields $\operatorname{deg} v_{i}=\lambda_{i}\left(x_{\xi}\right)$. On the other hand, $\tau\left(x_{\xi}\right)=\operatorname{tr}\left(\operatorname{ad} x_{\xi}\right)=\frac{1}{2} \operatorname{dim} \mathfrak{g}$ [O3, Theorem 3.3]. Substitution in $(*)$ gives us $(* *)$.

### 2.8 The Frobenius semi-radical $F(\mathfrak{g})$

Put $F(\mathfrak{g})=\sum_{\xi \in \mathfrak{g}_{\text {reg }}^{*}} \mathfrak{g}(\xi)$. This is a characteristic ideal of $\mathfrak{g}$ containing $Z(\mathfrak{g})$ and for which $F(F(\mathfrak{g}))=F(\mathfrak{g})$. It can also be characterized as follows: $R(\mathfrak{g})^{\mathfrak{g}} \subset R(F(\mathfrak{g}))$ and if $\mathfrak{g}$ is algebraic then $F(\mathfrak{g})$ is the smallest Lie subalgebra of $\mathfrak{g}$ with this property. Similar results hold in $D(\mathfrak{g})$ [O5 Proposition 2.4, Theorem 2.5] Also, $F(\mathfrak{g}) \subset \mathfrak{g}_{\Lambda}$. As a special case we have the following:

Remark 12. $Y(\mathfrak{g}) \subset S(F(\mathfrak{g}))($ respectively $Z(U(\mathfrak{g})) \subset U(F(\mathfrak{g})))$ and $F(\mathfrak{g})$ is the smallest Lie subalgebra of $\mathfrak{g}$ with this property in case $\mathfrak{g}$ is an algebraic Lie algebra without proper semi-invariants.

In case $\mathfrak{g}$ is square integrable we notice that $F(\mathfrak{g})=Z(\mathfrak{g})($ since $\mathfrak{g}(\xi)=Z(\mathfrak{g})$ for all
regular $\xi \in \mathfrak{g}^{*}$ ) which forces $R(\mathfrak{g})^{\mathfrak{g}}=R(Z(\mathfrak{g}))$. See also Remark 38. In particular, $Y(\mathfrak{g})=S(Z(\mathfrak{g}))$, which is a polynomial algebra.

If $\mathfrak{g}$ admits a CP $\mathfrak{h}$ then $F(\mathfrak{g})$ is commutative (since $F(\mathfrak{g}) \subset \mathfrak{h})$. Clearly,

$$
F(\mathfrak{g})=0 \text { if and only if } \mathfrak{g} \text { is Frobenius }
$$

For this reason $F(\mathfrak{g})$ is called the Frobenius semi-radical of $\mathfrak{g}$. At the other end of the spectrum we have the Lie algebras for which $F(\mathfrak{g})=\mathfrak{g}$, which we call quasi-quadratic. These are unimodular and they do not possess any proper semi-invariants. They form a large class, which include all quadratic Lie algebras (and hence all abelian and semi-simple Lie algebras) [O5].

## 3. General results

### 3.1 Necessary conditions for polynomiality

Theorem 13. [OV, Theorem 1.1] Let $\mathfrak{g}$ be a Lie algebra for which the semi-center $S y(\mathfrak{g})$ is freely generated by homogeneous elements $f_{1}, \ldots, f_{r}$.
Then

$$
\sum_{i=1}^{r} \operatorname{deg} f_{i} \leq c(\mathfrak{g})
$$

Definition 14. A Lie algebra $\mathfrak{g}$ is called coregular if $Y(\mathfrak{g})$ is a polynomial algebra over $k$.

Proposition 15. [OV, Proposition 1.6]. Assume that $\mathfrak{g}$ is nonabelian, without proper semi-invariants. If $\mathfrak{g}$ is coregular then $\operatorname{codim} \mathfrak{g}_{\text {sing }}^{*} \leq 3$.

Theorem 16. [O8, Theorem 26]. Let $\mathfrak{g}$ be a nonabelian, algebraic, unimodular Lie algebra such that $\operatorname{trdeg}_{k} Y(\mathfrak{g})=i(\mathfrak{g})$. Suppose that $\mathfrak{g}$ admits a CP. If $\mathfrak{g}$ is coregular then codim $\mathfrak{g}_{\text {sing }}^{*} \leq 2$.

Definition 17. We say that $\mathfrak{g}$ satisfies the Joseph-Shafrir conditions, JS for short, if $\mathfrak{g}$ is unimodular for which $p_{\mathfrak{g}}$ is an invariant and $\operatorname{trdeg}_{k} Y(\mathfrak{g})=i(\mathfrak{g})$.
Note that JS is satisfied if $\mathfrak{g}$ has no proper semi-invariants by Remark 2.

The following sum rule is an extension of [OV, Proposition 1.4].

Theorem 18. [JS, Theorem 2.2]
Assume that $\mathfrak{g}$ satisfies JS and that $Y(\mathfrak{g})$ is freely generated by homogeneous elements $f_{1}, \ldots, f_{r}$. Then

$$
\sum_{i=1}^{r} \operatorname{deg} f_{i}=c(\mathfrak{g})-\operatorname{deg} p_{\mathfrak{g}}
$$

Corollary 19. Assume that $\mathfrak{g}$ satisfies JS and that $Y(\mathfrak{g})$ is freely generated by homogeneous elements $f_{1}, \ldots, f_{r}$. Then $3 i(\mathfrak{g})+2 \operatorname{deg} p_{\mathfrak{g}} \leq \operatorname{dim} \mathfrak{g}+2 \operatorname{dim} Z(\mathfrak{g})$ Moreover, equality occurs if and only if $\operatorname{deg} f_{i} \leq 2, i: 1, \ldots, r$.

Proof. Clearly $r=\operatorname{trdeg}_{k} Y(\mathfrak{g})=i(\mathfrak{g})$. We apply a similar argument as in [OV, Corollary 1.3]. The observation that $\operatorname{deg} f_{i} \geq 2$ unless $f_{i} \in Z(\mathfrak{g})$ combined with the preceding theorem yields:

$$
\operatorname{dim} Z(\mathfrak{g})+2(i(\mathfrak{g})-\operatorname{dim} Z(\mathfrak{g})) \leq \sum_{i=1}^{r} \operatorname{deg} f_{i}=\frac{1}{2}(\operatorname{dim} \mathfrak{g}+i(\mathfrak{g}))-\operatorname{deg} p_{\mathfrak{g}}
$$

(and here equality occurs precisely when $\operatorname{deg} f_{i} \leq 2$ for all $i=1, \ldots, r$ )
$\Leftrightarrow-2 \operatorname{dim} Z(\mathfrak{g})+4 i(\mathfrak{g}) \leq \operatorname{dim} \mathfrak{g}+i(\mathfrak{g})-2 \operatorname{deg} p_{\mathfrak{g}}$
$\Leftrightarrow 3 i(\mathfrak{g})+2 \operatorname{deg} p_{\mathfrak{g}} \leq \operatorname{dim} \mathfrak{g}+2 \operatorname{dim} Z(\mathfrak{g})$

Theorem 20. Let $\mathfrak{g}$ be unimodular, having an abelian ideal $\mathfrak{h}$ of codimension one. Then the following are equivalent:
(1) $\mathfrak{g}$ is coregular and $\operatorname{trdeg}_{k} Y(\mathfrak{g})=i(\mathfrak{g})$
(2) $\mathfrak{g}$ is coregular and ad-algebraic
(3) $\operatorname{dim}[\mathfrak{g}, \mathfrak{g}] \leq 2$

Proof. Clearly $\mathfrak{g}$ is solvable and we may assume that $\mathfrak{g}$ is not abelian. Then the center $Z(\mathfrak{g})$ is contained in $\mathfrak{h}$ (otherwise $\mathfrak{g}=\mathfrak{h}+Z(\mathfrak{g})$ which is abelian). Choose $x_{0} \in \mathfrak{g} \backslash \mathfrak{h}$ and let $x_{1}, \ldots, x_{m}, x_{m+1}, \ldots, x_{n}$ be a basis of $\mathfrak{h}$ such that $x_{m+1}, \ldots, x_{n}$ is a basis of $Z(\mathfrak{g})$. Then with respect to the basis $x_{0}, x_{1}, \ldots, x_{m}, \ldots, x_{n}$ of $\mathfrak{g}$ we have that $\operatorname{rank}\left(\left[x_{i}, x_{j}\right]\right)=2$. Therefore, $i(\mathfrak{g})=\operatorname{dim} \mathfrak{g}-2$ and $c(\mathfrak{g})=\operatorname{dim} \mathfrak{g}-1$. So $\mathfrak{h}$ is a CPI of $\mathfrak{g}$.
We may assume that $m \geq 2$ (otherwise the result is trivial). Put $U=\left\langle x_{1}, \ldots, x_{m}\right\rangle$. Then

$$
\mathfrak{g}=k x_{0} \oplus \mathfrak{h}=k x_{0} \oplus U \oplus Z(\mathfrak{g})
$$

Because $\mathfrak{h}$ is abelian we see that

$$
[\mathfrak{g}, \mathfrak{g}]=\left[x_{0}, \mathfrak{h}\right]=\left[x_{0}, U\right] \quad \text { and } \quad C\left(x_{0}\right)=k x_{0} \oplus Z(\mathfrak{g})
$$

where $C\left(x_{0}\right)$ is the centralizer of $x_{0}$ in $\mathfrak{g}$.
So $\mathfrak{g}=U \oplus C\left(x_{0}\right)$. Therefore $\operatorname{dim} C\left(x_{0}\right)=\operatorname{dim} \mathfrak{g}-\operatorname{dim} U$. Next, we observe that

$$
\left.\operatorname{ad} x_{0}\right|_{U}: U \rightarrow[\mathfrak{g}, \mathfrak{g}], \quad x \mapsto\left[x_{0}, x\right]
$$

is a linear bijection. This implies that $\operatorname{dim} U=\operatorname{dim}[\mathfrak{g}, \mathfrak{g}]$ and also that $\left[x_{0}, x_{1}\right], \ldots,\left[x_{0}, x_{m}\right]$ are linearly independent and a fortiori relatively prime. Since they are the Pfaffians of the principal $2 \times 2$ minors of $\left(\left[x_{i}, x_{j}\right]\right)$, their greatest common divisor is $p_{\mathfrak{g}}=1$. We now proceed as follows:
$(1) \Rightarrow(3)$ :
As the JS-conditions are satisfied, we may apply the preceding corollary:

$$
3 i(\mathfrak{g})+2 \operatorname{deg} p_{\mathfrak{g}} \leq \operatorname{dim} \mathfrak{g}+2 \operatorname{dim} Z(\mathfrak{g})<\operatorname{dim} \mathfrak{g}+2 \operatorname{dim} C\left(x_{0}\right)
$$

Hence, $3(\operatorname{dim} \mathfrak{g}-2)<\operatorname{dim} \mathfrak{g}+2(\operatorname{dim} \mathfrak{g}-\operatorname{dim}[\mathfrak{g}, \mathfrak{g}])$
Consequently, $\operatorname{dim}[\mathfrak{g}, \mathfrak{g}]<3$.
$(3) \Rightarrow(2)$ :
So, suppose $\operatorname{dim}[\mathfrak{g}, \mathfrak{g}] \leq 2$. Clearly we may assume that $\mathfrak{g}$ is indecomposable. This implies that $Z(\mathfrak{g}) \subset[\mathfrak{g}, \mathfrak{g}]$ (otherwise there is a $z \in Z(\mathfrak{g})$ such that $z \notin[\mathfrak{g}, \mathfrak{g}]$. But then we could split off the abelian Lie algebra $k z$ ). Hence, $\operatorname{dim} Z(\mathfrak{g}) \leq 2$ and $\operatorname{dim} U=\operatorname{dim}[\mathfrak{g}, \mathfrak{g}] \leq 2$. Thus $\operatorname{dim} \mathfrak{g} \leq 5$. We now distinguish two cases
(i) ad $x_{0}$ is not nilpotent

Decompose $\mathfrak{h}$ into the generalized weight spaces w.r.t. ad $x_{0}$ :

$$
\mathfrak{h}=\mathfrak{h}^{o} \oplus \mathfrak{h}^{\lambda_{1}} \oplus \ldots \oplus \mathfrak{h}^{\lambda_{q}}, \quad \lambda_{i} \in k \backslash\{0\}
$$

Hence, $\mathfrak{h}^{\lambda_{1}} \oplus \ldots \oplus \mathfrak{h}^{\lambda_{q}} \subset\left[x_{0}, \mathfrak{h}\right]=[\mathfrak{g}, \mathfrak{g}]$. Put $m_{i}=\operatorname{dim} \mathfrak{h}^{\lambda_{i}}$.
Then $m_{1}+\ldots+m_{q} \leq \operatorname{dim}[\mathfrak{g}, \mathfrak{g}] \leq 2$. On the other hand, since $\mathfrak{g}$ is unimodular,

$$
m_{1} \lambda_{1}+\ldots+m_{q} \lambda_{q}=\operatorname{tr}\left(\operatorname{ad} x_{o}\right)=0
$$

This forces $q=2, m_{1}=m_{2}=1$ and $\lambda_{2}=-\lambda_{1}$.
So, $\mathfrak{h}_{\lambda_{1}} \oplus \mathfrak{h}_{-\lambda_{1}}=\mathfrak{h}^{\lambda_{1}} \oplus \mathfrak{h}^{-\lambda_{1}}=[\mathfrak{g}, \mathfrak{g}]$. In particular, $Z(\mathfrak{g})=0$
and $[\mathfrak{g}, \mathfrak{g}]=U=\mathfrak{h}$. Choose nonzero $y_{1} \in \mathfrak{h}_{\lambda_{1}}, y_{2} \in \mathfrak{h}_{-\lambda_{1}}$ and put $y_{0}=\left(1 / \lambda_{1}\right) x_{0}$. Then, $y_{0}, y_{1}, y_{2}$ is a basis for $\mathfrak{g}$ with nonzero brackets: $\left[y_{0}, y_{1}\right]=y_{1},\left[y_{0}, y_{2}\right]=$ $-y_{2}$. Clearly $\mathfrak{g}$ is algebraic and also coregular since $Y(\mathfrak{g})=k\left[y_{1} y_{2}\right]$.
(ii) ad $x_{0}$ is nilpotent.

In this case $\mathfrak{g}$ is nilpotent and thus algebraic. Consulting [D2, O7], the following are the only indecomposable nilpotent Lie algebras of dimension at most 5 having a commutative ideal of codimension one and such that $\operatorname{dim}[\mathfrak{g}, \mathfrak{g}] \leq 2: \mathfrak{g}_{3}, \mathfrak{g}_{4}, \mathfrak{g}_{5,2}$, which are all coregular.
$(2) \Rightarrow(1)$ : Denote by $Q(Y(\mathfrak{g}))$ the quotient field of $Y(\mathfrak{g})$.
It suffices to show that $Q(Y(\mathfrak{g}))=R(\mathfrak{g})^{\mathfrak{g}}(\bullet)$, because then we obtain at once that

$$
\operatorname{trdeg}_{k} Y(\mathfrak{g})=\operatorname{trdeg}_{k} R(\mathfrak{g})^{\mathfrak{g}}=i(\mathfrak{g})
$$

by Theorem 1 since $\mathfrak{g}$ is ad-algebraic.
Let ad $x_{o}=S+N$ be the Jordan decomposition of ad $x_{0}$, with $S$ and $N$ its semisimple and nilpotent components. As $\mathfrak{g}$ is ad-algebraic we can find $s, y \in \mathfrak{g}$ such that $S=\operatorname{ad} s$ and $N=\operatorname{ad} y$.
We distinguish 2 cases:
a) $\operatorname{ad} s(\mathfrak{h}) \neq 0$, i.e. $s \notin \mathfrak{h}$. Then we replace $x_{0}$ by a suitable nonzero scalar multiple of $s$, which is diagonalizable with integer eigenvalues having zero sum. By the same argument as in the proof of [O8, Example 28] we obtain (•).
b) $\operatorname{ad} s(\mathfrak{h})=0$. Then ad $y(\mathfrak{h}) \neq 0\left(\right.$ since ad $\left.x_{0}(\mathfrak{h}) \neq 0\right)$, i.e. $y \notin \mathfrak{h}$. Then we replace $x_{0}$ by $y$. It follows that $\mathfrak{g}$ is nilpotent for which $(\bullet)$ is well known (since $\mathfrak{g}$ has no proper semi-invariants).

Remark 21. A more direct approach for the implication $(2) \Rightarrow(3)$ goes as follows. By assumption $\mathfrak{g}$ is nonabelian, unimodular, coregular and ad-algebraic. As above it then also satisfies $\operatorname{trdeg}_{k} Y(\mathfrak{g})=i(\mathfrak{g})$. Moreover $\mathfrak{h}$ is a CP of $\mathfrak{g}$. Next, we observe that

$$
\mathfrak{g}_{\text {sing }}^{*}=\left\{\xi \in \mathfrak{g}^{*} \mid \xi\left(\left[x_{0}, x_{i}\right]\right)=0, i=1, \ldots, m\right\}
$$

Then, $\operatorname{dim}[\mathfrak{g}, \mathfrak{g}]=m=\operatorname{codim} \mathfrak{g}_{\text {sing }}^{*} \leq 2$ by Theorem 16 .
Remark 22. In the list of all indecomposable nilpotent Lie algebras of dimension at most seven [O7,O8] there are only 6 Lie algebras with an abelian ideal of codimension one and with $\operatorname{dim}[\mathfrak{g}, \mathfrak{g}]>2$, namely $8,25,156,157,158,159$. None of these is coregular as predicted by Theorem 20.
Examples 24-26 show that none of the conditions such as unimodular, algebraic and
$\operatorname{trdeg}_{k} Y(\mathfrak{g})=i(\mathfrak{g})$ can be removed from Theorem 18 and Corollary 19. In all, except for Example 26, $\mathfrak{h}=\left\langle x_{2}, x_{3}, x_{4}\right\rangle$ is an abelian ideal of codimension one.

Example 23. Let $\mathfrak{g}$ be the solvable Lie algebra with basis $x_{1}, x_{2}, x_{3}, x_{4}$ and with nonzero brackets

$$
\left[x_{1}, x_{2}\right]=x_{2},\left[x_{1}, x_{3}\right]=x_{3},\left[x_{1}, x_{4}\right]=-2 x_{4}
$$

Clearly, $\mathfrak{g}$ is unimodular, algebraic and $\operatorname{dim}[\mathfrak{g}, \mathfrak{g}]=3>2$. Hence $\mathfrak{g}$ is not coregular by Theorem 20 (see also [JS, 8.4] and [O8, Example 28]). This can also be seen directly.
Indeed, $Y(\mathfrak{g})=k\left[f_{1}, f_{2}, f_{3}\right]$ where $f_{1}=x_{2}^{2} x_{4}, f_{2}=x_{3}^{2} x_{4}, f_{3}=x_{2} x_{3} x_{4}$ with $f_{1} f_{2}=f_{3}^{2}$. In particular, $Y(\mathfrak{g})$ is not factorial. Note that $S y(\mathfrak{g})=k\left[x_{2}, x_{3}, x_{4}\right]$, which is polynomial, and $R(\mathfrak{g})^{\mathfrak{g}}=k\left(f_{1}, f_{3}\right)$. So $\operatorname{trdeg}_{k}\left(Y(\mathfrak{g})=2=i(\mathfrak{g})\right.$ while $p_{\mathfrak{g}}=1$. Thus JS is satisfied. Finally, codim $\mathfrak{g}_{\text {sing }}^{*}=3$.

## Example 24.

Let $\mathfrak{g}$ be the Lie algebra with basis $x_{1}, x_{2}, x_{3}, x_{4}$ and nonzero brackets

$$
\left[x_{1}, x_{2}\right]=x_{2},\left[x_{1}, x_{3}\right]=x_{3},\left[x_{1}, x_{4}\right]=-x_{4}
$$

Clearly, $\mathfrak{g}$ is algebraic, but not unimodular. Also, $p_{\mathfrak{g}}=1$ and $Z(\mathfrak{g})=0$. Put $f_{1}=x_{2} x_{4}$ and $f_{2}=x_{3} x_{4}$. Then $Y(\mathfrak{g})=k\left[f_{1}, f_{2}\right]$, so $\mathfrak{g}$ is coregular and $\operatorname{trdeg}_{k} Y(\mathfrak{g})=$ $2=i(\mathfrak{g})$. Moreover,

$$
S y(\mathfrak{g})=k\left[x_{2}, x_{3}, x_{4}\right] \text { and } R(\mathfrak{g})^{\mathfrak{g}}=k\left(f_{1}, f_{2}\right)
$$

However,

$$
\begin{gathered}
\operatorname{deg} f_{1}+\operatorname{deg} f_{2}=4>3=c(\mathfrak{g})-\operatorname{deg} p_{\mathfrak{g}} \\
3 i(\mathfrak{g})+2 \operatorname{deg} p_{\mathfrak{g}}=6>4=\operatorname{dim} \mathfrak{g}+2 \operatorname{dim} Z(\mathfrak{g})
\end{gathered}
$$

and $\operatorname{dim}[\mathfrak{g}, \mathfrak{g}]=3>2$.

## Example 25.

Let $\mathfrak{g}$ be the Lie algebra with basis $x_{1}, x_{2}, x_{3}, x_{4}$ and nonzero brackets

$$
\left[x_{1}, x_{2}\right]=x_{2}+x_{3},\left[x_{1}, x_{3}\right]=x_{3},\left[x_{1}, x_{4}\right]=-2 x_{4}
$$

$\mathfrak{g}$ is unimodular, but not algebraic (not even almost algebraic).
Again, $p_{\mathfrak{g}}=1$ and $Z(\mathfrak{g})=0 . \mathfrak{g}$ is coregular since $Y(\mathfrak{g})=k\left[x_{3}^{2} x_{4}\right]$.

However, $\operatorname{trdeg}_{k} Y(\mathfrak{g})=1<2=i(\mathfrak{g})$. Clearly,

$$
S y(\mathfrak{g})=k\left[x_{3}, x_{4}\right] \text { and } R(\mathfrak{g})^{\mathfrak{g}}=k\left(x_{3}^{2} x_{4}\right)
$$

In particular, $j(\mathfrak{g})=1<i(\mathfrak{g})$. Also,

$$
3 i(\mathfrak{g})+2 \operatorname{deg} p_{\mathfrak{g}}=6>4=\operatorname{dim} \mathfrak{g}+2 \operatorname{dim} Z(\mathfrak{g})
$$

and $\operatorname{dim}[\mathfrak{g}, \mathfrak{g}]=3>2$. Finally, we notice that $\mathfrak{g}_{\Lambda}=\left\langle x_{2}, x_{3}, x_{4}\right\rangle=F(\mathfrak{g})$ and $S y(\mathfrak{g}) \neq k\left[x_{2}, x_{3}, x_{4}\right]=Y\left(\mathfrak{g}_{\Lambda}\right)$.

## Example 26.

Consider the 9 -dimensional solvable Lie algebra $\mathfrak{g}$ with basis $x_{0}, x_{1}, \ldots, x_{8}$ and nonzero brackets

$$
\begin{aligned}
& {\left[x_{0}, x_{1}\right]=5 x_{1},\left[x_{0}, x_{2}\right]=10 x_{2},\left[x_{0}, x_{3}\right]=-13 x_{3},\left[x_{0}, x_{4}\right]=-8 x_{4},} \\
& {\left[x_{0}, x_{5}\right]=-3 x_{5},\left[x_{0}, x_{6}\right]=2 x_{6},\left[x_{0}, x_{7}\right]=7 x_{7},\left[x_{1}, x_{3}\right]=x_{4},} \\
& {\left[x_{1}, x_{4}\right]=x_{5},\left[x_{1}, x_{5}\right]=x_{6},\left[x_{1}, x_{6}\right]=x_{7},\left[x_{2}, x_{3}\right]=x_{5},\left[x_{2}, x_{4}\right]=x_{6},\left[x_{2}, x_{5}\right]=x_{7} .}
\end{aligned}
$$

Then, $\mathfrak{g}$ is algebraic and unimodular with codim $\mathfrak{g}_{\text {sing }}^{*}=3$. In particular, $p_{\mathfrak{g}}=1$.
$\mathfrak{g}$ is coregular since $Y(\mathfrak{g})=k\left[x_{8}\right]$, but $\operatorname{trdeg}_{k} Y(\mathfrak{g})=1<3=i(\mathfrak{g})$.
Note that $c(\mathfrak{g})=6$ and $\operatorname{deg} x_{8}=1<6=c(\mathfrak{g})-\operatorname{deg} p_{\mathfrak{g}}$.
So the sum rule fails in these circumstances. Furthermore, $\mathfrak{g}_{\Lambda}=\left\langle x_{1}, \ldots, x_{8}\right\rangle$. By (5) of Theorem 4 and [DDV, p.323]

$$
S y(\mathfrak{g})=Y\left(\mathfrak{g}_{\Lambda}\right)=k\left[x_{7}, x_{8}, f, g, h\right]
$$

where

$$
\begin{aligned}
f & =3 x_{4} x_{7}^{2}-3 x_{5} x_{6} x_{7}+x_{6}^{3} \\
g & =4 x_{3} x_{7}^{2}-2 x_{5}^{2} x_{7}^{2}-4 x_{4} x_{6} x_{7}^{2}+4 x_{5} x_{6}^{2} x_{7}-x_{6}^{4} \\
h & =\left(f^{4}+g^{3}\right) / x_{7}^{3}
\end{aligned}
$$

Hence $S y(\mathfrak{g})$ is not polynomial. Note that $\mathfrak{g}_{\Lambda}$ is isomorphic to a central extension of the nilpotent Lie algebra with number 152 of [O8, p.109]. Finally, $R(\mathfrak{g})^{\mathfrak{g}}=$ $k\left(x_{8}, f^{4} g^{-3}, f g x_{7}^{-2}\right)$ and $F(\mathfrak{g})=\left\langle x_{3}, x_{4}, x_{5}, x_{6}, x_{7}, x_{8}\right\rangle$ which is a CPI of $\mathfrak{g}$.

Remark 27. The preceding example is a central extension of example (58) of [DDV, p.322], which turned out to be a counterexample to Bolsinov's completeness criterion for Mishchenko-Fomenko subalgebras [O8, Counterexample 20]. Inspired
by this, Bolsinov obtained an interesting and useful adaptation of his original criterion by considering an alternative definition for Mishchenko-Fomenko subalgebras [Bo]. See also [JS, Theorem 7.2].

Let $\mathfrak{g}$ be a semi-simple Lie algebra, $B$ a Borel subalgebra of $\mathfrak{g}$. Then it is well known that the nilradical of $B$ is coregular [J2, 4.7], see also Corollary 32. An example by A. Hersant shows that a similar result does not hold in general if we replace $B$ by an arbitrary parabolic subalgebra of $\mathfrak{g}$ [J2, 8.5].
We will now give a short proof of an extension of this example.

## Proposition 28.

Let $N$ be the nilradical of the parabolic subalgebra $P$ of type $(1,1, n-2)$ inside $\operatorname{sl}(n)$, with $n \geq 3$. Then

$$
N \text { is coregular } \Leftrightarrow n \leq 4 \quad \Leftrightarrow \quad i(N) \leq 3
$$

## Proof.

Let $\left(E_{i j}\right), i, j=1, \ldots, n$, be the standard basis for $g l(n)$. Then

$$
\left\{E_{12}, E_{13}, \ldots, E_{1 n} ; E_{23}, \ldots, E_{2 n}\right\}
$$

is a basis for $N$ (so $\operatorname{dim} N=2 n-3$ ), with nonzero brackets

$$
\left[E_{12}, E_{23}\right]=E_{13},\left[E_{12}, E_{24}\right]=E_{14}, \ldots,\left[E_{12}, E_{2 n}\right]=E_{1 n}
$$

Clearly, $[N, N]=\left\langle E_{13}, E_{14}, \ldots, E_{1 n}\right\rangle$ and $\operatorname{dim}[N, N]=n-2$.
$N$ admits an abelian ideal of codimension one, namely $H=\left\langle E_{13}, \ldots, E_{1 n} ; E_{23}, \ldots, E_{2 n}\right\rangle$.
$N$, being nilpotent, is ad-algebraic and $i(N)=2 n-5$.
By Theorem 20:

$$
N \text { is coregular } \Leftrightarrow \quad \operatorname{dim}[N, N] \leq 2 \quad \Leftrightarrow \quad n \leq 4 \quad \Leftrightarrow \quad i(N) \leq 3
$$

### 3.2 Sufficient conditions for polynomiality

We exhibit some methods which will be used in sections 4 and 5 . The first one is very efficient for proving coregularity, provided one has candidates for the generating invariants. It is an extension of [PPY, Theorem 1.2]. See also [Pa2, Theorem 1.2].

Theorem 29. [JS, 5.7], [Sh]
Assume that $f_{1}, \ldots, f_{r} \in Y(\mathfrak{g}), r=i(\mathfrak{g})$, are algebraically independent homogeneous invariants such that

$$
\left.\sum_{i=1}^{r} \operatorname{deg} f_{i} \leq c(\mathfrak{g})\right)-\operatorname{deg} p_{\mathfrak{g}}
$$

Then, equality holds and $Y(\mathfrak{g})=k\left[f_{1}, \ldots, f_{r}\right]$. In particular, $\operatorname{trdeg}_{k} Y(\mathfrak{g})=i(\mathfrak{g})$.

Theorem 30. [O7, Theorem 3.5] (The Frobenius method).
Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $k$. Assume that there exists a torus $T \subset \operatorname{Derg}$ (i.e. an abelian subalgebra consisting of semi-simple derivations of $\mathfrak{g}$ ) such that the semi-direct product $L=T \oplus \mathfrak{g}$ is Frobenius. Let $f_{1}, \ldots, f_{r}$ be the irreducible factors of $p_{L}$ (equivalently of $\Delta(L)$ ). Then the following hold:
(1) $S y(\mathfrak{g})=S y(L)=Y\left(L_{\Lambda}\right)=k\left[f_{1}, \ldots, f_{r}\right]$, a polynomial algebra.
(2) $\operatorname{dim} T=i(\mathfrak{g})$ and $r=i\left(L_{\Lambda}\right)=\operatorname{dim} L-\operatorname{dim} L_{\Lambda}$.
(3) $\Lambda(\mathfrak{g})=\left\{\left.\lambda\right|_{\mathfrak{g}} \mid \lambda \in \Lambda(L)\right\}$ and $\mathfrak{g}_{\Lambda}=\mathfrak{g} \cap L_{\Lambda}$
(4) If $\mathfrak{g}$ has no proper semi-invariants (i.e. $\mathfrak{g}=\mathfrak{g}_{\Lambda}$ ) then $\mathfrak{g}=L_{\Lambda}$ and

$$
Y(\mathfrak{g})=k\left[f_{1}, \ldots, f_{r}\right]
$$

Remark 31. Although this method does not always work, it has some significant advantages. First of all it is relatively simple: it comes down to showing that the determinant $\Delta(L)$ of the structure matrix of $L$ is not zero. In addition, there is no need to have prior knowledge of candidates for the generating (semi-) invariants. In fact, we get them as a bonus since they are precisely the irreducible factors of the determinant above (or equivalently of $p_{L}$ ). This method works rather well if $\mathfrak{g}$ is nilpotent. It will also be useful in sections 4 and 5. In [O7,O8] the Poisson center has been determined explicitly for the 159 cases of the indecomposable nilpotent Lie algebras of dimension at most seven (here a family is counted as one Lie algebra). It turns out that 132 of them are coregular. Among the latter, 67 Lie algebras were treated successfully with this method [O7, 5].

Corollary 32. See also [J2, 4.7]. Let $\mathfrak{g}$ be a simple Lie algebra with triangular decomposition $\mathfrak{g}=N^{-} \oplus H \oplus N$. Then the nilradical $N$ of the Borel subalgebra $B=H \oplus N$ is coregular.

Proof. There exists a torus $T \subset \operatorname{ad}_{N} H \subset \operatorname{Der} N$ such that the semi-direct product $L=T \oplus N$ is Frobenius. Hence $Y(N)$ is polynomial by (4) of Theorem 30. Indeed, in case $\mathfrak{g}$ is not of type $A_{n}, n \geq 2 ; D_{2 t+1}, t \geq 2$; or $E_{6}$ then it suffices to take $T=\operatorname{ad}_{N} H$ because then $T \oplus N=B$, which is Frobenius (for more details see [O7, Corollary 3.6]). The existence of $T$ if $\mathfrak{g}$ is of type $A_{n}$ is easy to verify. The remaining cases were done by Rupert Yu (unpublished).

## Question 33. (Rupert Yu)

Suppose $\mathfrak{g}$ is a Lie algebra for which there exists a derivation $d \in \operatorname{Derg}$ such that the semidirect product $L=k d \oplus \mathfrak{g}$ is Frobenius. Does this imply that $S y(\mathfrak{g})=S y(L)$ ?

We know this is true if $d$ is diagonalizable by Theorem 30. However the following is a counterexample for the general case.

Example 34. Let $\mathfrak{g}$ be the 5 -dimensional Lie algebra with basis $x_{1}, \ldots, x_{5}$ and nonzero brackets: $\left[x_{1}, x_{3}\right]=x_{3}-x_{4},\left[x_{1}, x_{4}\right]=x_{4},\left[x_{1}, x_{5}\right]=x_{5},\left[x_{2}, x_{3}\right]=x_{5}$. $\mathfrak{g}$ is solvable of index one, but it is not almost algebraic. One verifies that

$$
Y(\mathfrak{g})=k, S y(\mathfrak{g})=k\left[x_{4}, x_{5}\right] \text { and } R(\mathfrak{g})^{\mathfrak{g}}=k\left(x_{4} / x_{5}\right)
$$

Note that $\operatorname{trdeg}_{k} Y(\mathfrak{g})=0<1=i(\mathfrak{g})$. Also, $j(\mathfrak{g})=i(\mathfrak{g})$.
Next we take the derivation $d \in$ Der $\mathfrak{g}$ given by

$$
d\left(x_{1}\right)=-x_{2}, d\left(x_{2}\right)=0, d\left(x_{3}\right)=x_{4}, d\left(x_{4}\right)=x_{5}, d\left(x_{5}\right)=0
$$

Clearly $d$ is nilpotent. Consider $L=k d \oplus \mathfrak{g}$. Then, $\Delta(L)=x_{5}^{6} \neq 0$.
Hence $L$ is Frobenius and $S y(L)=k\left[x_{5}\right]$ (by Theorem 30), which does not coincide with $S y(\mathfrak{g})$.

Definition 35. $Y(\mathfrak{g})$ is said to be saturated if for some nonzero $u, v \in S(\mathfrak{g})$, $u v \in Y(\mathfrak{g})$ implies that so are $u$ and $v$. In particular, $Y(\mathfrak{g})$ is factorial. Note that the condition $u, v \in S(\mathfrak{g})$ may be replaced by $u, v \in S y(\mathfrak{g})$ because $u v \in Y(\mathfrak{g})$ implies that $u$ and $v$ are semi-invariants and thus belong to $S y(\mathfrak{g})$.
We now recall when $S y(\mathfrak{g})$ is a polynomial ring over $Y(\mathfrak{g})$. Clearly, in order for this to happen $Y(\mathfrak{g})$ must be saturated.

Theorem 36. [DNOW, Theorem 6]
Assume that
(i) $R(\mathfrak{g})^{\mathfrak{g}}=Q(Y(\mathfrak{g}))$, the quotient field of $Y(\mathfrak{g})$.
(ii) $Y(\mathfrak{g})$ is saturated.

Then the following hold:
(1) $S(\mathfrak{g})$ has at most a finite number of irreducible proper semi-invariants $v_{1}, \ldots, v_{t}$. Let $\lambda_{1}, \ldots, \lambda_{t} \in \Lambda(\mathfrak{g})$ be their weights.
(2) $S(\mathfrak{g})_{\lambda_{i}}=Y(\mathfrak{g}) v_{i}, i=1, \ldots, t$.
(3) Each $v_{i}$ is a semi-invariant for all derivations $d \in \operatorname{Der} \mathfrak{g}$.
(4) $S y(\mathfrak{g})=Y(\mathfrak{g})\left[v_{1}, \ldots, v_{t}\right]$, a polynomial ring over $Y(\mathfrak{g})$. In particular, if $Y(\mathfrak{g})$ is polynomial over $k$, then the same holds for $S y(\mathfrak{g})$. [We don't know if the converse holds. It is a special case of the Zariski cancellation problem. For the general question Susumu Oda claims to have a proof [Oda], but some experts are skeptical]

We now look at a special case of Theorem 36.

Theorem 37. [DNOW, Proposition 16 and Theorem 18] Let $x_{1}, \ldots, x_{s}, x_{s+1}, \ldots, x_{n}$ be a basis such that $x_{1}, \ldots, x_{s}$ is a basis of $Z(\mathfrak{g})$. Let $E_{1}, \ldots, E_{m}$ be a basis of the algebraic hull $H$ of ad $\mathfrak{g}$. Then the following conditions are equivalent:
(1) $j(\mathfrak{g})=\operatorname{dim} Z(\mathfrak{g})$ (i.e. $\mathfrak{g}[\xi]=Z(\mathfrak{g})$ ) for some $\xi \in \mathfrak{g}^{*}$ )
(2) $R(\mathfrak{g})^{\mathfrak{g}}=R(Z(\mathfrak{g}))$
(3) $Z(D(\mathfrak{g}))=D(Z(\mathfrak{g}))=k\left(x_{1}, \ldots, x_{s}\right)$, a rational extension of $k$.
(4) The localization $U(\mathfrak{g})_{S}$, where $S=U(Z(\mathfrak{g})) \backslash\{0\}$, is primitive.

Moreover, these conditions imply that:
(a) $Y(\mathfrak{g})=S(Z(\mathfrak{g}))=k\left[x_{1}, \ldots, x_{s}\right]$, which is saturated.
(b) $S(\mathfrak{g})$ admits at most a finite number of irreducible, proper semi-invariants $v_{1}, \ldots, v_{t}$.
(c) $S y(\mathfrak{g})=Y(\mathfrak{g})\left[v_{1}, \ldots, v_{t}\right]=k\left[x_{1}, \ldots, x_{s}, v_{1}, \ldots, v_{t}\right]$, a polynomial algebra over $k$.
(d) $v_{1}, \ldots, v_{t}$ are precisely the irreducible factors, not in $Y(\mathfrak{g})$, of $p_{\mathfrak{g}}^{\prime} \in S(\mathfrak{g})$, the latter being the greatest common divisor of the $r \times r$ minors of the $m \times n$ $\operatorname{matrix}\left(E_{i} x_{j}\right)$, where

$$
r=\operatorname{rank}\left(E_{i} x_{j}\right)=\operatorname{dim} \mathfrak{g}-\operatorname{dim} Z(\mathfrak{g})
$$

Remark 38. Suppose $\mathfrak{g}$ is square integrable, i.e. $i(\mathfrak{g})=\operatorname{dim} Z(\mathfrak{g})$ which forces $j(\mathfrak{g})=\operatorname{dim} Z(\mathfrak{g})$. So, the above conditions are satisfied and we may replace the matrix $\left(E_{i} x_{j}\right)$ by the structure matrix $\left(\left[x_{i}, x_{j}\right]\right)$ of $\mathfrak{g}$. Consequently $v_{i}, \ldots, v_{t}$ are then precisely the irreducible factors, not in $Y(\mathfrak{g})$, of $p_{\mathfrak{g}}$.

Example 39. Let $\mathfrak{g}$ be the 4 -dimensional Lie algebra with basis $x_{1}, x_{2}, x_{3}, x_{4}$ and nonzero brackets $\left[x_{1}, x_{2}\right]=x_{2}+x_{3},\left[x_{1}, x_{3}\right]=x_{4}$.
Clearly, $i(\mathfrak{g})=2, Z(\mathfrak{g})=\left\langle x_{4}\right\rangle$ and $p_{\mathfrak{g}}=1$.
$\operatorname{dim} Z(\mathfrak{g})=1<i(\mathfrak{g})$, so $\mathfrak{g}$ is not square integrable. Obviously, in this situation $p_{\mathfrak{g}}$ is useless in order to compute the remaining semi-invariants. Next, we introduce $E_{1}, E_{2} \in \operatorname{Der} \mathfrak{g}$ as follows:
$E_{1}\left(x_{1}\right)=0, E_{1}\left(x_{2}\right)=x_{2}+x_{2}+x_{4}, E_{1}\left(x_{3}\right)=E_{1}\left(x_{4}\right)=0$
$E_{2}\left(x_{1}\right)=0, E_{2}\left(x_{2}\right)=-x_{4}, E_{2}\left(x_{3}\right)=x_{4}, E_{2}\left(x_{4}\right)=0$.
In fact, $E_{1}$ and $E_{2}$ are the semi-simple and nilpotent components of ad $x_{1}$.
Hence they belong to the algebraic hull $H$ of ad $\mathfrak{g}$. One verifies that

$$
E_{1}, E_{2}, E_{3}=\operatorname{ad} x_{2}, E_{4}=a d x_{3}
$$

form a basis of $H$. We now observe the matrix $\left(E_{i} x_{j}\right)$ :

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $E_{1}$ | 0 | $x_{2}+x_{3}+x_{4}$ | 0 | 0 |
| $E_{2}$ | 0 | $-x_{4}$ | $x_{4}$ | 0 |
| $E_{3}$ | $-x_{2}-x_{3}$ | 0 | 0 | 0 |
| $E_{4}$ | $-x_{4}$ | 0 | 0 | 0 |

which is of rank 3. By Theorem 1

$$
\operatorname{trdeg}_{k} R(\mathfrak{g})^{\mathfrak{g}}=j(\mathfrak{g})=\operatorname{dim} \mathfrak{g}-\operatorname{rank}\left(E_{i} x_{j}\right)=1
$$

Since $j(\mathfrak{g})=1=\operatorname{dim} Z(\mathfrak{g})$ we can apply Theorem 37 . Hence,

$$
Y(\mathfrak{g})=k\left[x_{4}\right] \text { and } R(\mathfrak{g})^{\mathfrak{g}}=k\left(x_{4}\right)
$$

Furthermore, the matrix $\left(E_{i} x_{j}\right)$ has only 2 nonzero $3 \times 3$ minors, namely

$$
-x_{4}\left(x_{2}+x_{3}\right)\left(x_{2}+x_{3}+x_{4}\right) \text { and }-x_{4}^{2}\left(x_{2}+x_{3}+x_{4}\right)
$$

Their greatest common divisor is $p_{\mathfrak{g}}^{\prime}=x_{4}\left(x_{2}+x_{3}+x_{4}\right)$. By (c) of Theorem 37 we may conclude that

$$
S y(\mathfrak{g})=Y(\mathfrak{g})\left[x_{2}+x_{3}+x_{4}\right]=k\left[x_{4}, x_{2}+x_{3}+x_{4}\right]
$$

## 4. Coregularity for Lie algebras with index at most two

Motivation: Due to Corollary 19 there are not many nonabelian, coregular Lie algebras with a large index. This is especially true if the index is maximal (Proposition 40 and Corollary 41). On the other hand, we will encounter quite a few Lie algebras for which the coregularity implies that their index is at most two (Proposition 42, Proposition 50, Theorem 51, subsection 5.1).

## Proposition 40

Assume that $\mathfrak{g}$ is an indecomposable Lie algebra which satisfies JS and for which $i(\mathfrak{g})=\operatorname{dim} \mathfrak{g}-2$. If $\mathfrak{g}$ is coregular then either $\mathfrak{g}=\operatorname{sl}(2, k)$ or $\mathfrak{g}$ is solvable with $\operatorname{dim} \mathfrak{g} \leq 6$.

Proof. First we notice that $Z(\mathfrak{g}) \subset[\mathfrak{g}, \mathfrak{g}]$ as $\mathfrak{g}$ is indecomposable. Next we claim that $\operatorname{dim} Z(\mathfrak{g}) \leq \frac{1}{2} \operatorname{dim} \mathfrak{g}$.
Indeed, take $\xi \in \mathfrak{g}_{\text {reg }}^{*}$. Then the stabilizer $\mathfrak{g}(\xi)$ is abelian [D6, 1.11.7] of dimension $i(\mathfrak{g})=\operatorname{dim} \mathfrak{g}-2$ and $Z(\mathfrak{g}) \subset \mathfrak{g}(\xi)$. There exists a basis
$x_{1}, x_{2}, x_{3}, \ldots, x_{p}, x_{p+1}, \ldots, x_{n}$ of $\mathfrak{g}$ such that $x_{3}, \ldots, x_{n}$ is a basis of $\mathfrak{g}(\xi)$ and $x_{p+1}, \ldots, x_{n}$ is a basis of $Z(\mathfrak{g})$. It suffices to show that $\operatorname{dim}[\mathfrak{g}, \mathfrak{g}] \leq p$, because then

$$
2 \operatorname{dim} Z(\mathfrak{g}) \leq \operatorname{dim}[\mathfrak{g}, \mathfrak{g}]+\operatorname{dim} Z(\mathfrak{g}) \leq p+\operatorname{dim} Z(\mathfrak{g})=\operatorname{dim} \mathfrak{g}
$$

Clearly, $\left[x_{1}, x_{2}\right] \neq 0$ and the structure matrix $M=\left(\left[x_{i}, x_{j}\right]\right)_{1 \leq i, j \leq n}$ of $\mathfrak{g}$ has rank $r=n-i(\mathfrak{g})=2$.
We may assume that $\left[x_{1}, x_{3}\right] \neq 0$ and that $p>3$ (if $p=3$ then $\operatorname{dim}[\mathfrak{g}, \mathfrak{g}]=$ $\left.\operatorname{dim}\left\langle\left[x_{1}, x_{2}\right],\left[x_{1}, x_{3}\right],\left[x_{2}, x_{3}\right]\right\rangle \leq 3=p\right)$. This implies that the following submatrix $A$ of $M$ has rank one (otherwise rank $M=4$ )

$$
A=\binom{\left[x_{1}, x_{3}\right] \ldots\left[x_{1}, x_{p}\right]}{\left[x_{2}, x_{3}\right] \ldots\left[x_{2}, x_{p}\right]}
$$

Using the fact that its nonzero entries have degree one, it is not difficult to see that we have to consider the following two cases:
(1) Each column of $A$ is a scalar multiple of the first one. Then,

$$
\operatorname{dim}[\mathfrak{g}, \mathfrak{g}]=\operatorname{dim}\left\langle\left[x_{1}, x_{2}\right],\left[x_{1}, x_{3}\right],\left[x_{2}, x_{3}\right]\right\rangle \leq 3<p
$$

(2) The second row of $A$ is a scalar multiple of the first one. Then,

$$
\operatorname{dim}[\mathfrak{g}, \mathfrak{g}]=\operatorname{dim}\left\langle\left[x_{1}, x_{2}\right],\left[x_{1}, x_{3}\right], \ldots,\left[x_{1}, x_{p}\right]\right\rangle \leq p
$$

This establishes the claim.
Application of Corollary 19 gives us:

$$
3(\operatorname{dim} \mathfrak{g}-2)=3 i(\mathfrak{g}) \leq \operatorname{dim} \mathfrak{g}+2 \operatorname{dim} Z(\mathfrak{g}) \leq 2 \operatorname{dim} \mathfrak{g}
$$

Consequently, $\operatorname{dim} \mathfrak{g} \leq 6$. Hence $\mathfrak{g}$ is solvable or $\mathfrak{g}=\operatorname{sl}(2, k)$ (otherwise $i(\mathfrak{g})<$ $\operatorname{dim} \mathfrak{g}-2$ by [AOV2, pp. 554-559] or by subsection 5.1).

Corollary 41. Suppose $\mathfrak{g}$ is an indecomposable nilpotent Lie algebra with $i(\mathfrak{g})=$ $\operatorname{dim} \mathfrak{g}-2$. If $\mathfrak{g}$ is coregular then by the above and $[O 7,5] \mathfrak{g}$ is isomorphic to one of the following:

$$
\mathfrak{g}_{3}, \quad \mathfrak{g}_{4}, \quad \mathfrak{g}_{5,2}, \quad \mathfrak{g}_{5,4}, \quad \mathfrak{g}_{6,3}
$$

Proposition 42. Assume that $\mathfrak{g}$ satisfies JS with $\operatorname{dim} Z(\mathfrak{g}) \leq 1$. If $\mathfrak{g}$ is coregular then $i(\mathfrak{g})$ is 1 or 2 in each of the following cases:
(1) $7 \neq \operatorname{dim} \mathfrak{g} \leq 8$
(2) $\operatorname{dim} \mathfrak{g}=7$ and $\mathfrak{g}$ is singular
(3) $\operatorname{dim} \mathfrak{g}=9$ or 10 and $\operatorname{deg} p_{\mathfrak{g}} \geq 2$

Proof. Again the main tool will be Corollary 19. Being unimodular, $\mathfrak{g}$ is not Frobenius [O3, Theorem 3.3]. Hence, $i(\mathfrak{g}) \geq 1$.
(1) First we suppose $\operatorname{dim} \mathfrak{g}$ (and hence also $i(\mathfrak{g})$ ) is even. Then

$$
3 i(\mathfrak{g}) \leq 3 i(\mathfrak{g})+2 \operatorname{deg} p_{\mathfrak{g}} \leq \operatorname{dim} \mathfrak{g}+2 \operatorname{dim} Z(\mathfrak{g}) \leq 10
$$

implies that $i(\mathfrak{g})=2$.
On the other hand, if $\operatorname{dim} \mathfrak{g}$ (and hence also $i(\mathfrak{g})$ ) is odd, then

$$
3 i(\mathfrak{g}) \leq 3 i(\mathfrak{g})+2 \operatorname{deg} p_{\mathfrak{g}} \leq \operatorname{dim} \mathfrak{g}+2 \operatorname{dim} Z(\mathfrak{g}) \leq 7
$$

forces $i(\mathfrak{g})=1$.
(2) By assumption $\operatorname{dim} \mathfrak{g}=7$ and $\operatorname{deg} p_{\mathfrak{g}} \geq 1$ since $\mathfrak{g}$ is singular. Hence,

$$
3 i(\mathfrak{g})<3 i(\mathfrak{g})+2 \operatorname{deg} p_{\mathfrak{g}} \leq \operatorname{dim} \mathfrak{g}+2 \operatorname{dim} Z(\mathfrak{g}) \leq 9
$$

and thus $i(\mathfrak{g})=1$.
(3) follows at once from $\operatorname{deg} p_{\mathfrak{g}} \geq 2$ and

$$
3 i(\mathfrak{g})+2 \operatorname{deg} p_{\mathfrak{g}} \leq \operatorname{dim} \mathfrak{g}+2 \operatorname{dim} Z(\mathfrak{g}) \leq 12
$$

Proposition 43. Let $\mathfrak{g}$ be a Lie algebra with $j(\mathfrak{g}) \leq 1$ (as it is when $i(\mathfrak{g}) \leq 1$.
(a) If $j(\mathfrak{g})=0$ then $Z(\mathfrak{g})=0$. By Theorem 37

$$
Y(\mathfrak{g})=k, R(\mathfrak{g})^{\mathfrak{g}}=k \quad \text { and } \quad S y(\mathfrak{g}) \text { is polynomial. }
$$

(b) Let $j(\mathfrak{g})=1$ (in particular $\operatorname{trdeg}_{k} R(\mathfrak{g})^{\mathfrak{g}}=1$ ) and suppose $Y(\mathfrak{g}) \neq k$. Choose a homogeneous element $v \in Y(\mathfrak{g}) \backslash k$ of smallest degree. Then by [O3, Lemma 3.8], $Y(\mathfrak{g})=k[v]$ and $R(\mathfrak{g})^{\mathfrak{g}}=k(v)$. Now, assume in addition that $Y(\mathfrak{g})$ is saturated. Then $v$ is irreducible and by Theorem 36 there are irreducible semi-invariants $v_{1}, \ldots, v_{t}$ in $S(\mathfrak{g})$ such that

$$
S y(\mathfrak{g})=Y(\mathfrak{g})\left[v_{1}, \ldots, v_{t}\right]=k\left[v, v_{1}, \ldots, v_{t}\right]
$$

which is a polynomial algebra.

Now we need to recall a special case of a result by Dixmier [D1, p. 333]:

Theorem 44. Let $\mathfrak{g}$ be a nilpotent Lie algebra and let

$$
0=\mathfrak{g}_{0} \subset \mathfrak{g}_{1} \ldots \subset \mathfrak{g}_{n}=\mathfrak{g}
$$

be a sequence of ideals of $\mathfrak{g}$ such that for each $j: 1, \ldots, n$, $\operatorname{dim} \mathfrak{g}_{j}=j$ and $\left[\mathfrak{g}, \mathfrak{g}_{j}\right] \subset$ $\mathfrak{g}_{j-1}$. Choose $x_{j} \in \mathfrak{g}_{j} \backslash \mathfrak{g}_{j-1}$. Suppose $j_{1}<j_{2}<\ldots<j_{r}$ are the indices $j \geq 1$ such that

$$
S\left(\mathfrak{g}_{j-1}\right) \cap Y(\mathfrak{g}) \neq S\left(\mathfrak{g}_{j}\right) \cap Y(\mathfrak{g})
$$

(1) Then for each such $j$ there is a nonzero element $b_{j} \in S\left(\mathfrak{g}_{j-1}\right) \cap Y(\mathfrak{g})$ and $c_{j} \in S\left(\mathfrak{g}_{j-1}\right)$ such that $a_{j}=b_{j} x_{j}+c_{j} \in S\left(\mathfrak{g}_{j}\right) \cap Y(\mathfrak{g})$.

In (2), (3), (4) $a_{j}, b_{j}, c_{j}$ are chosen to satisfy (1).
(2) $Y(\mathfrak{g}) \subset k\left[a_{j_{1}}, \ldots, a_{j_{r}}, b_{j_{1}}^{-1}, \ldots, b_{j_{r}}^{-1}\right]$
(3) $R(\mathfrak{g})^{\mathfrak{g}}$ is the quotient field of $Y(\mathfrak{g})$. It is the field generated by $a_{j_{1}}, \ldots, a_{j_{r}}$, which are algebraically independent over $k$. In particular, $r=i(\mathfrak{g})$.
(4) $Y(\mathfrak{g}) \subset k\left[a_{j_{1}}, \ldots, a_{j_{r}}, a^{-1}\right]$ for some nonzero $a \in k\left[a_{j_{1}}, \ldots, a_{j_{r}}\right]$

## Theorem 45.

Any nilpotent Lie algebra $\mathfrak{g}$ with $i(\mathfrak{g}) \leq 2$ is coregular.

Proof. We observe that

$$
1 \leq \operatorname{dim} Z(\mathfrak{g}) \leq i(\mathfrak{g}) \leq 2
$$

If $\operatorname{dim} Z(\mathfrak{g})=i(\mathfrak{g})$ then $\mathfrak{g}$ is square integrable and therefore $Y(\mathfrak{g})=S(Z(\mathfrak{g}))$ which is polynomial by Remark 38 (or section 2.8). So it suffices to deal with the case where $\operatorname{dim} Z(\mathfrak{g})=1$ and $i(\mathfrak{g})=2$. (hence $\operatorname{dim} \mathfrak{g}$ is even)
Choose a sequence of ideals

$$
0=\mathfrak{g}_{0} \subset \mathfrak{g}_{1} \subset \ldots \subset \mathfrak{g}_{n}=\mathfrak{g}
$$

with the same properties as in Theorem 44. In particular, $x_{1}, \ldots, x_{n}$ is a basis of $\mathfrak{g}$ with $x_{i} \in \mathfrak{g}_{i} \backslash \mathfrak{g}_{i-1}, i=1, \ldots, n$. Note that $\mathfrak{g}_{1}=\left\langle x_{1}\right\rangle=Z(\mathfrak{g})$. So,

$$
x_{1} \in S\left(\mathfrak{g}_{1}\right) \cap Y(\mathfrak{g})=k\left[x_{1}\right], \text { but } x_{1} \notin S\left(\mathfrak{g}_{0}\right) \cap Y(\mathfrak{g})=k
$$

Since $i(\mathfrak{g})=2$ there is only one more $j>1$ such that

$$
S\left(\mathfrak{g}_{j-1}\right) \cap Y(\mathfrak{g}) \subsetneq S\left(\mathfrak{g}_{j}\right) \cap Y(\mathfrak{g})
$$

and there is a nonzero.

$$
b \in S\left(\mathfrak{g}_{j-1}\right) \cap Y(\mathfrak{g})=S\left(\mathfrak{g}_{1}\right) \cap Y(\mathfrak{g})=k\left[x_{1}\right] \quad \text { and } \quad c \in S\left(\mathfrak{g}_{j-1}\right)
$$

such that

$$
a=b x_{j}+c \in S\left(\mathfrak{g}_{j}\right) \cap Y(\mathfrak{g})
$$

by Theorem 44. We may assume that $a$ is of smallest degree with these properties. Clearly the homogeneous components of $a$ also belong to $S\left(\mathfrak{g}_{j}\right) \cap Y(\mathfrak{g})$. Among them we let $f$ be a homogeneous component not contained in $S\left(\mathfrak{g}_{j-1}\right) \cap Y(\mathfrak{g})=k\left[x_{1}\right]$. Note that $\operatorname{deg} f=\operatorname{deg} a$. Then up to a nonzero scalar multiplier:
$f=x_{1}^{m} x_{j}+u$ for some $m$ and homogeneous $u \in S\left(\mathfrak{g}_{j-1}\right)$ of degree $m+1$.
Clearly $u \neq 0$ (otherwise $x_{1}^{m} x_{j} \in Y(\mathfrak{g})$ and hence $x_{j} \in Z(\mathfrak{g})=\mathfrak{g}_{1}$, contradiction)
Also, $m \geq 1$ (if $m=0$ then $u \in \mathfrak{g}_{j-1}$ and $f=x_{j}+u \in Z(\mathfrak{g})=\mathfrak{g}_{1}=\left\langle x_{1}\right\rangle$, contradiction).
Moreover, $u$ is not divisible by $x_{1}$ (otherwise $f / x_{1}$, which belongs to $S\left(\mathfrak{g}_{j}\right) \cap Y(\mathfrak{g})$ but not to $S\left(\mathfrak{g}_{j-1}\right) \cap Y(\mathfrak{g})=k\left[x_{1}\right]$, would be of a smaller degree than $f$, contradiction).

Claim. Suppose we have an element $f=x_{1}^{m} x_{j}+u \in Y(\mathfrak{g})$ with $m \geq 1, j>1$ and nonzero homogeneous $u \in S\left(\mathfrak{g}_{j-1}\right)$ of degree $m+1$ and not divisible by $x_{1}$. Then $Y(\mathfrak{g})=k\left[x_{1}, f\right]$, which is polynomial.

First we see that $f \in S\left(\mathfrak{g}_{j}\right) \cap Y(\mathfrak{g}), x_{1}^{m} \in S\left(\mathfrak{g}_{j-1}\right) \cap Y(\mathfrak{g})=k\left[x_{1}\right]$ and $u \in S\left(\mathfrak{g}_{j-1}\right)$. By Theorem 44, $x_{1}$ and $f$ are algebraically independent over $k$ and

$$
Y(\mathfrak{g}) \subset k\left[x_{1}, f, x_{1}^{-m}\right] \subset k\left[x_{1}, f, x_{1}^{-1}\right]
$$

So, it suffices to show that $Y(\mathfrak{g})=k\left[x_{1}, f\right](*)$. For this we need the following lemmas:

Lemma A: Let $P \in k[X]$ be a polynomial. If $x_{1}$ divides $P(f)$ then $P=0$ and so $P(f)=0$.

Proof. Let $I$ be the ideal of $S(L)$ generated by $x_{1}$. We identify the quotient algebra by $k\left[x_{2}, \ldots, x_{n}\right]$. Denote by $f_{1}$ and $u_{1}$ the canonical images of $f$ and $u$. By assumption we obtain $P\left(u_{1}\right)=P\left(f_{1}\right)=0$.
But $u_{1} \in k\left[x_{2}, \ldots, x_{n}\right]$ is nonzero (as $x_{1}$ does not divide $u$ ) of degree $m+1$. Consequently, $P=0$ and $P(f)=0$.

Lemma B. Let $q \in S(\mathfrak{g})$ be such that $x_{1} q \in k\left[x_{1}, f\right]$. Then also $q \in k\left[x_{1}, f\right]$.

Proof. By assumption there are $h_{i} \in k[f]$ such that, for some $r$

$$
x_{1} q=x_{1}^{r} h_{r}+\ldots+x_{1}^{2} h_{2}+x_{1} h_{1}+h_{0}
$$

Clearly, $x_{1}$ divides $h_{0}$, which implies that $h_{0}=0$ by Lemma A. Consequently,

$$
q=x_{1}^{r-1} h_{r}+\ldots+x_{1} h_{2}+h_{1} \in k\left[x_{1}, f\right]
$$

We can now show $(*)$ as follows:
Take $q \in Y(\mathfrak{g}) \subset k\left[x_{1}, f, x_{1}^{-1}\right]$, i.e. $x_{1}^{t} q \in k\left[x_{1}, f\right]$ for some $t$.
By applying Lemma B times we may conclude that $q \in k\left[x_{1}, f\right]$. Therefore $Y(\mathfrak{g}) \subset k\left[x_{1}, f\right]$. The other inclusion is obvious.
Finally, we provide a formula for $\operatorname{deg} f$. First, $c(\mathfrak{g})=\frac{1}{2}(\operatorname{dim} \mathfrak{g}+i(\mathfrak{g}))=\frac{n}{2}+1$. Then, by Theorem 18 :

$$
\operatorname{deg} x_{1}+\operatorname{deg} f=c(\mathfrak{g})-\operatorname{deg} p_{\mathfrak{g}}=\frac{n}{2}+1-\operatorname{deg} p_{\mathfrak{g}}
$$

Consequently, $\operatorname{deg} f=\frac{n}{2}-\operatorname{deg} p_{\mathfrak{g}}$ (in particular, $\operatorname{deg} f=\frac{n}{2}$ if $\mathfrak{g}$ is nonsingular)
Remark 46. Yakimova has informed us that Theorem 45 can also be derived from a result of Michel Brion on linear actions of unipotent groups [ Br ].

Remark 47. Theorem 45 fails if
(1) the condition on $i(\mathfrak{g})$ is replaced by $i(\mathfrak{g})=3$.

Indeed, the standard filiform Lie algebra $\mathfrak{g}_{5,5}$ has index 3, but is not coregular [O7, p. 1304]. This was already known by Dixmier [D2, Proposition 2].
(2) nilpotent is replaced by solvable.

Indeed, Example 23 is solvable of index 2, but it is not coregular.
Question 48. Let $\mathfrak{g}$ be nilpotent with $i(\mathfrak{g}) \leq \operatorname{dim} Z(\mathfrak{g})+1$. Does this imply that $\mathfrak{g}$ is coregular?
This is true for all indecomposable nilpotent Lie algebras of dimension at most seven [O7, O8].

We will now examine the coregularity of the major types of filiform Lie algebras, presented in [GK1, p.41].

## Definition 49.

Consider the descending central series of $\mathfrak{g}$

$$
C^{\prime}(\mathfrak{g})=\mathfrak{g}, C^{2}(\mathfrak{g})=[\mathfrak{g}, \mathfrak{g}], \ldots, C^{i}(\mathfrak{g})=\left[\mathfrak{g}, C^{i-1}(\mathfrak{g})\right], \ldots
$$

An $n$-dimensional Lie algebra $\mathfrak{g}$ is called filiform if $\operatorname{dim} C^{i}(\mathfrak{g})=n-i, i=2, \ldots, n$. In particular, $C^{n}(\mathfrak{g})=0$ (and thus $\mathfrak{g}$ is nilpotent) and $Z(\mathfrak{g})=C^{n-1}(\mathfrak{g})$ is 1dimensional.

Combining Proposition 42 with Theorem 45 yields:

Proposition 50. Let $\mathfrak{g}$ be an 8 -dimensional filiform Lie algebra. Then

$$
\mathfrak{g} \text { is coregular } \Leftrightarrow \quad i(\mathfrak{g})=2
$$

Theorem 51. Let $\mathfrak{g}$ be an $n$-dimensional filiform Lie algebra. Then
(1) If $\mathfrak{g}$ is of type $Q_{n}$ or $W_{n}$ then $i(\mathfrak{g}) \leq 2$, so $\mathfrak{g}$ is coregular.
(2) If $\mathfrak{g}$ is of type $L_{n}$ or $R_{n}$ then

$$
\mathfrak{g} \text { is coregular } \quad \Leftrightarrow \quad i(\mathfrak{g}) \leq 2
$$

## Proof.

(1) a) Suppose $\mathfrak{g}$ is of type $Q_{n}$.

Basis of $\mathfrak{g}: x_{1}, \ldots, x_{n}, n=2 q$.
Nonzero brackets: $\left[x_{1}, x_{i}\right]=x_{i+1}, i=2, \ldots, n-2$ and $\left[x_{j}, x_{n-j+1}\right]=(-1)^{j+1} x_{n}$, $j=2, \ldots, q$.
We observe that

$$
i(\mathfrak{g})=2, \quad Z(\mathfrak{g})=\left\langle x_{n}\right\rangle, \quad p_{\mathfrak{g}}=x_{n}^{q-2}
$$

So, $\mathfrak{g}$ is coregular by Theorem 45 . Next put

$$
f=2 x_{1} x_{n}+(-1)^{q+1} x_{q+1}^{2}+2 \sum_{i=3}^{q}(-1)^{i} x_{i} x_{n-i+2} \in Y(\mathfrak{g})
$$

which satisfies the conditions of the claim in the proof of Theorem 45. Therefore $Y(\mathfrak{g})=k\left[x_{n}, f\right]$.

As an alternative solution we can use the Frobenius method:
Consider the torus $T=\left\langle t_{1}, t_{2}\right\rangle \subset$ Der $\mathfrak{g}$, where

$$
t_{1}=\operatorname{diag}(0,1,1, \ldots, 1,2), \quad t_{2}=\operatorname{diag}(1,1,2,3, \ldots, n-2, n-1)
$$

see $[\mathrm{R}, \mathrm{p} .4]$. Then the semi-direct product $L=T \oplus \mathfrak{g}$ is Frobenius. Indeed,

$$
\Delta(L)=x_{n}^{n-2} f^{2} \neq 0 . \text { Hence } p_{L}=x_{n}^{q-1} f
$$

By (4) of Theorem 30:

$$
Y(\mathfrak{g})=S y(L)=k\left[x_{n}, f\right]
$$

since $\mathfrak{g}$ is nilpotent (and hence has no proper semi-invariants) and so $\mathfrak{g}=L_{\Lambda}$.
b) Suppose $\mathfrak{g}$ is of type $W_{n}$

Basis: $\quad x_{1}, \ldots, x_{n}$
Nonzero brackets: $\left[x_{i}, x_{j}\right]=(j-i) x_{i+j}, i<j$ and $i+j \leq n$
b1) $n=2 q+1$. Then $Z(\mathfrak{g})=\left\langle x_{n}\right\rangle$ and $\operatorname{dim} Z(\mathfrak{g})=1=i(\mathfrak{g})$, i.e. $\mathfrak{g}$ is square integrable. Consequently, $Y(\mathfrak{g})=k\left[x_{n}\right]$ by Remark 38.
b2) $n=2 q$. Then $i(\mathfrak{g})=2$ and thus $\mathfrak{g}$ is coregular by Theorem 45 .
For example, if $n=8$ then $Y(\mathfrak{g})=k\left[x_{8}, f\right]$ where

$$
f=64 x_{4} x_{8}^{3}-16 x_{6}^{2} x_{8}^{2}-32 x_{5} x_{7} x_{8}^{2}+24 x_{6} x_{7}^{2} x_{8}-5 x_{7}^{4}
$$

(2) i) Suppose $\mathfrak{g}$ is of type $L_{n}$, the standard filiform Lie algebra,

Basis: $x_{1}, \ldots, x_{n}, n \geq 3$
Nonzero brackets: $\left[x_{1}, x_{i}\right]=x_{i+1}, i=2, \ldots, n-1$.
Clearly, $i(\mathfrak{g})=n-2$ and $\mathfrak{h}=\left\langle x_{2}, \ldots, x_{n}\right\rangle$ is an abelian ideal of codimension one of $\mathfrak{g}$. Also, $[\mathfrak{g}, \mathfrak{g}]=\left\langle x_{3}, \ldots, x_{n}\right\rangle$ and $\operatorname{dim}[\mathfrak{g}, \mathfrak{g}]=n-2$.
By Theorem 20:

$$
\mathfrak{g} \text { is coregular } \quad \Leftrightarrow \quad \operatorname{dim}[\mathfrak{g}, \mathfrak{g}] \leq 2 \quad \Leftrightarrow \quad n \leq 4 \quad \Leftrightarrow \quad i(\mathfrak{g}) \leq 2
$$

See also [OV, Example 1.7] and [O8, Example 27].
[The Lie algebras $1,2,8,25,159$ of [O7, O8] are of this type].
ii) Suppose $\mathfrak{g}$ is of type $R_{n}$.

Basis: $x_{1}, \ldots, x_{n}, n \geq 5$.
Nonzero brackets: $\left[x_{1}, x_{i}\right]=x_{i+1}, i=2, \ldots, n-1 ;\left[x_{2}, x_{j}\right]=x_{j+2}, j=$ $3, \ldots, n-2$.
Since $i(\mathfrak{g})=n-4$, it suffices to show that

$$
\mathfrak{g} \text { is coregular } \quad \Leftrightarrow \quad n \leq 6
$$

First suppose $\mathfrak{g}$ is coregular. Note that $c(\mathfrak{g})=\frac{1}{2}(n+n-4)=n-2$. Then $\mathfrak{h}=\left\langle x_{3}, x_{4}, \ldots, x_{n}\right\rangle$ is an $(n-2)$-dimensional abelian ideal of $\mathfrak{g}$ and so is a CP of $\mathfrak{g}$. Next, it is not difficult to see that

$$
\mathfrak{g}_{\text {sing }}^{*}=\left\{f \in \mathfrak{g}^{*} \mid f\left(x_{5}\right)=\ldots=f\left(x_{n}\right)=0\right\}
$$

Hence, codim $\mathfrak{g}_{\text {sing }}^{*}=n-4$. By Theorem 16 codim $\mathfrak{g}_{\text {sing }}^{*} \leq 2$, i.e. $n \leq 6$.
Conversely, if $n \leq 6$ then $i(\mathfrak{g}) \leq 2$ and so $\mathfrak{g}$ is coregular by Theorem 45. [The Lie algebras $6,27,151$ of [O7, O8] are of this type]

Remark. There are coregular filiform Lie algebras of index larger than 2. For instance the Lie algebra 106 of [O8, p.104].

Theorem 52. Let $\mathfrak{g}$ be a quadratic Lie algebra. Then $\mathfrak{g}$ is coregular if one of the following conditions is satisfied:
(i) $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$ and $i(\mathfrak{g})=2$
(ii) $\mathfrak{g}$ is nilpotent and $i(\mathfrak{g})=3$

## Proof.

(i) Since $i(\mathfrak{g})=2$ we have that $n=\operatorname{dim} \mathfrak{g}$ is even and $c(\mathfrak{g})=\frac{n+2}{2}=\frac{n}{2}+1$. We may assume that $n \geq 4$. $\mathfrak{g}$ being quadratic, admits a nondegenerate, symmetric, invariant bilinear form $b$ (such a Lie algebra is sometimes called regular quadratic) [FS]. It is easy to verify that w.r.t. $b$ we obtain that

$$
Z(\mathfrak{g})=[\mathfrak{g}, \mathfrak{g}]^{\perp} \neq 0 \text { since }[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}
$$

$\mathfrak{g}$ is a fortiori quasi quadratic (2.8), i.e.

$$
\mathfrak{g}=F(\mathfrak{g})=\sum_{\xi \in \mathfrak{g}_{\text {reg }}^{*}} \mathfrak{g}(\xi)
$$

In particular, $\mathfrak{g}(\xi) \neq \mathfrak{g}(\eta)$ for some $\xi, \eta \in \mathfrak{g}_{\text {reg }}^{*}$, which we extend to algebra endomorphisms of $S(\mathfrak{g})$. Both $\mathfrak{g}(\xi)$ and $\mathfrak{g}(\eta)$ contain $Z(\mathfrak{g})$ and are of dimen$\operatorname{sion} i(\mathfrak{g})=2$. It follows that $\operatorname{dim} Z(\mathfrak{g})=1$. Hence, we can find a basis $x_{1}, x_{2}, \ldots, x_{n-1}, x_{n}$ of $\mathfrak{g}$ such that

$$
\mathfrak{g}(\xi)=\left\langle x_{n-1}, x_{n}\right\rangle, \quad \mathfrak{g}(\eta)=\left\langle x_{1}, x_{n}\right\rangle \quad \text { and } \quad Z(\mathfrak{g})=\left\langle x_{n}\right\rangle
$$

By 2.1 the rank $r$ of the structure matrix $M=\left(\left[x_{i}, x_{j}\right]\right)$ is given by

$$
r=\operatorname{dim} \mathfrak{g}-i(\mathfrak{g})=n-2
$$

Clearly, $\left\langle x_{1}, \ldots, x_{n-2}\right\rangle \oplus \mathfrak{g}(\xi)=\mathfrak{g}$.
Next, we consider the $r \times r$ submatrix $A=\left(\left[x_{i}, x_{j}\right]\right)_{1 \leq i, j \leq n-2}$ with Pfaffian $p \in S(\mathfrak{g})$. Then $\xi(p) \neq 0$, indeed

$$
\xi(p)^{2}=\xi\left(p^{2}\right)=\xi(\operatorname{det} A)=\operatorname{det}\left(\xi\left(\left[x_{i}, x_{j}\right]\right) \neq 0\right.
$$

since $\xi$ is regular. Similarly, we observe that

$$
\left\langle x_{2}, \ldots, x_{n-1}\right\rangle \oplus \mathfrak{g}(\eta)=\mathfrak{g}
$$

We put $B=\left(\left[x_{i}, x_{j}\right]\right)_{2 \leq i, j \leq n-1}$ with Pfaffian $q \in S(\mathfrak{g})$. As before we get $\eta(q) \neq 0$. On the other hand, $\eta(p)=0$ because

$$
\eta(p)^{2}=\eta\left(p^{2}\right)=\eta(\operatorname{det} A)=\operatorname{det}\left(\eta\left(\left[x_{i}, x_{j}\right]\right)_{1 \leq i, j \leq n-2}\right)=0
$$

since the first row of the matrix is

$$
\left(\eta\left(\left[x_{1}, x_{1}\right]\right), \eta\left(\left[x_{1}, x_{2}\right]\right), \ldots, \eta\left(\left[x_{1}, x_{n-2}\right]\right)\right)
$$

which is zero because $x_{1} \in \mathfrak{g}(\eta)$.
Consequently, $p$ and $q$ are principal $r \times r$ Pfaffians of the structure matrix $M$ of $\mathfrak{g}$ of the same degree, namely $\frac{n-2}{2}=\frac{n}{2}-1$. By the above $p$ is not a scalar multiple of $q$. The fundamental semi-invariant $p_{\mathfrak{g}}$, being the GCD of all principal $r \times r$ Pfaffians (Definition 5), divides both $p$ and $q$. Therefore,

$$
\operatorname{deg} p_{\mathfrak{g}} \leq\left(\frac{n}{2}-1\right)-1=\frac{n}{2}-2
$$

Next, let $y_{1}, \ldots, y_{n}$ be the dual basis of $x_{1}, \ldots, x_{n}$ w.r.t. $b$, i.e. $b\left(x_{i}, y_{j}\right)=\delta_{i j}$ for all $i, j: 1, \ldots, n$. Then,

$$
f=x_{1} y_{1}+\ldots+x_{n} y_{n} \in Y(\mathfrak{g})
$$

is the well known Casimir element of $S(\mathfrak{g})$. Finally, $x_{n}$ and $f$ are algebraically independent, homogeneous elements of $Y(\mathfrak{g})$ such that

$$
\operatorname{deg} x_{n}+\operatorname{deg} f=3=\left(\frac{n}{2}+1\right)-\left(\frac{n}{2}-2\right) \leq c(\mathfrak{g})-\operatorname{deg} p_{\mathfrak{g}}
$$

By Theorem 29 equality holds (in particular $\operatorname{deg} p_{\mathfrak{g}}=\frac{n}{2}-2$ ) and $Y(\mathfrak{g})=k\left[x_{n}, f\right]$.
(ii) Since $i(\mathfrak{g})=3$ we have that $n=\operatorname{dim} \mathfrak{g}$ is odd and $c(\mathfrak{g})=\frac{1}{2}(n+3)$. We may assume that $n \geq 5$ (otherwise $\mathfrak{g}$ is abelian). $\mathfrak{g}$ being quadratic, admits a nondegenerate, symmetric, invariant bilinear form $b$.
$\mathfrak{g}$ is a fortiori quasi quadratic, i.e.

$$
\mathfrak{g}=F(\mathfrak{g})=\sum_{\xi \in \mathfrak{g}_{r e g}^{*}} \mathfrak{g}(\xi)
$$

In particular, $\mathfrak{g}(\xi) \neq \mathfrak{g}(\eta)$ for some $\xi, \eta \in \mathfrak{g}_{\text {reg }}^{*}$ which we extend to algebra endomorphisms of $S(\mathfrak{g})$. Both $\mathfrak{g}(\xi)$ and $\mathfrak{g}(\eta)$ contain $Z(\mathfrak{g})$ and are of dimension $i(\mathfrak{g})=3$.
On the other hand, $\operatorname{dim} Z(\mathfrak{g}) \geq 2$ because $\mathfrak{g}$ is nilpotent and quasi quadratic [O5, Corollary 3.6]. Therefore $\operatorname{dim} Z(\mathfrak{g})=2$. Hence there exists a basis $x_{1}, x_{2}, \ldots, x_{n-2}, x_{n-1}, x_{n}$ of $\mathfrak{g}$ such that $\mathfrak{g}(\xi)=\left\langle x_{n-2}, x_{n-1}, x_{n}\right\rangle, \mathfrak{g}(\eta)=$ $\left\langle x_{1}, x_{n-1}, x_{n}\right\rangle$ and $Z(\mathfrak{g})=\left\langle x_{n-1}, x_{n}\right\rangle$. The rank $r$ of the structure matrix $M=\left(\left[x_{i}, x_{j}\right]\right)$ is given by

$$
r=\operatorname{dim} \mathfrak{g}-i(\mathfrak{g})=n-3
$$

Similar to the proof of (i) we can find principal $r \times r$ Pfaffians $p$ and $q$ of degree $\frac{1}{2}(n-3)$ such that one is not a scalar multiple of the other.
Since the fundamental semi-invariant $p_{\mathfrak{g}}$ divides both $p$ and $q$, we get

$$
\operatorname{deg} p_{\mathfrak{g}} \leq \frac{1}{2}(n-3)-1=\frac{1}{2}(n-5)
$$

Next, let $f \in S(\mathfrak{g})$ be the Casimir element w.r.t. $b$. Then $x_{n-1}, x_{n}, f$ are algebraically independent, homogeneous elements of $Y(\mathfrak{g})$ such that

$$
\operatorname{deg} x_{n-1}+\operatorname{deg} x_{n}+\operatorname{deg} f=4=\frac{1}{2}(n+3)-\frac{1}{2}(n-5) \leq c(\mathfrak{g})-\operatorname{deg} p_{\mathfrak{g}}
$$

By Theorem 29 equality holds (in particular $\operatorname{deg} p_{\mathfrak{g}}=\frac{1}{2}(n-5)$ ) and $Y(\mathfrak{g})=$ $k\left[x_{n-1}, x_{n}, f\right]$.

## 5. Polynomiality for nonsolvable Lie algebras of dimension at most eight

 Because of Example 23 we now restrict ourselves to the nonsolvable case.Theorem 53. Let $L$ be a nonsolvable, indecomposable Lie algebra with $\operatorname{dim} L \leq 8$. Then

1) $Y(L)$ and $S y(L)$ are polynomial algebras over $k$ (and hence so are $Z(U(L))$ and $S z(U(L)))$
2) $R(L)^{L}$ (and hence also $Z(D(L))$ ) is rational over $k$.

Proof. This will proceed case by case using the classification provided to us by B. Komrakov. See also [Tu]. In [AOV2] the algebraic ones among them were shown to satisfy the following well known
Gelfand-Kirillov conjecture [GK]. Let $L$ be an algebraic Lie algebra over $k$. Then $D(L)$ is isomorphic to a Weyl skew field $D_{n}(F)$ over a rational extension $F$ of $k$. In particular, $Z(D(L))$, which is isomorphic to $F$, is also rational over $k$.

Over the years positive, but also some negative, answers have been obtained [BGR, J1, Mc, N, AOV1, AOV2, O6, Pr]. See also Appendix.
As we will use some results of [AOV2], we will employ the same notation.
Before we list in 5.1 for each case the results of the verification of the theorem, we would like to present in detail some typical examples in order to exhibit the various procedures used in the proof.

Example 54. Let $L$ be the semi-direct product $L_{6,3}=s l(2, k) \oplus H$ of $s l(2, k)$ with the 3 -dimensional Heisenberg Lie algebra $H$ with basis $h, x, y, e_{0}, e_{1}, e_{2}$ and nonzero brackets:
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=e_{0},\left[h, e_{1}\right]=-e_{1},\left[x, e_{1}\right]=e_{0}$, $\left[y, e_{0}\right]=e_{1},\left[e_{0}, e_{1}\right]=e_{2}$
Clearly, $i(L)=2$ and $c(L)=\frac{1}{2}(6+2)=4$. One verifies that $p_{L}=1$ and also that $L$ is quasi quadratic (i.e. $F(L) \stackrel{2}{=} L$ ), so there are no CP's. Next, $Y(L)$ contains the following homogeneous, algebraically independent elements:

$$
e_{2} \quad \text { and } \quad f=e_{2}\left(h^{2}+4 x y\right)+2\left(e_{0} e_{1} h+e_{1}^{2} x-e_{0}^{2} y\right)
$$

Because $\operatorname{deg} e_{2}+\operatorname{deg} f=4=c(L)-\operatorname{deg} p_{L}$ we may conclude that $Y(L)=k\left[e_{2}, f\right]$ by Theorem 29. Since $[L, L]=L, L$ is algebraic, without proper semi-invariants. Therefore

$$
S y(L)=Y(L) \quad \text { and } \quad R(L)^{L}=k\left(e_{2}, f\right)
$$

Finally, $M=k\left[e_{1}, e_{2}, e_{0}^{2}-2 e_{2} x, f\right]$ is a polynomial, complete, Poisson commutative subalgebra of $S(L)$.

Example 55. Let $L$ be the 7-dimensional algebraic Lie algebra $L_{7,9}$ with basis $h, x, y, e_{0}, e_{1}, e_{2}, e_{3}$ and with nonzero brackets:
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=e_{0},\left[h, e_{1}\right]=-e_{1},\left[x, e_{1}\right]=e_{0}$, $\left[y, e_{0}\right]=e_{1},\left[e_{0}, e_{1}\right]=e_{2},\left[e_{3}, e_{0}\right]=e_{0},\left[e_{3}, e_{1}\right]=e_{1},\left[e_{3}, e_{2}\right]=2 e_{2}$.
Note that $L$ is the semi-direct product of $L_{6,3}$ with ad $e_{3}$, which is semi-simple. One verifies that $i(L)=1, c(L)=4$ and $p_{L}=1$. Also

$$
F(L)=\left\langle h, x, y, e_{0}, e_{1}, e_{2}\right\rangle=[L, L]=L_{6,3}
$$

Since this is not commutative there are no CP's. The above implies that $L_{\Lambda}=[L, L]$. By (5) of Theorem 4 we observe that

$$
S y(L)=Y\left(L_{\Lambda}\right)=Y\left(L_{6,3}\right)=k\left[e_{2}, f\right]
$$

where $f=e_{2}\left(h^{2}+4 x y\right)+2\left(e_{0} e_{1} h+e_{1}^{2} x-e_{0}^{2} y\right)$.
Furthermore, $e_{2}$ and $f$ are irreducible semi-invariants with the same weight $\lambda \in L^{*}$ for which $\lambda\left(e_{3}\right)=2$ and $\lambda([L, L])=0$.
Consequently, $Y(L)=k$ and $R(L)^{L}=k\left(e_{2}^{-1} f\right)$.
Finally, we know from the previous example that $M=k\left[e_{1}, e_{2}, e_{0}^{2}-2 e_{2} x, f\right]$ is a polynomial, complete, Poisson commutative subalgebra of $S\left(L_{6,3}\right)$. Hence $\operatorname{trdeg}_{k} M=$ $c\left(L_{6,3}\right)=c\left(L_{\Lambda}\right)=c(L)$, the latter by (4) of Theorem 4. Therefore, $M$ is also complete in $S(L)$.

Example 56. Let $\mathfrak{g}$ be the 8 -dimensional algebraic Lie algebra $L_{8,17}$ with basis $h, x, y, e_{0}, e_{1}, e_{2}, e_{3}, e_{4}$ and nonzero brackets:
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=e_{0},\left[h, e_{1}\right]=-e_{1},\left[h, e_{2}\right]=e_{2}$, $\left[h, e_{3}\right]=-e_{3},\left[x, e_{1}\right]=e_{0},\left[x, e_{3}\right]=e_{2},\left[y, e_{0}\right]=e_{1},\left[y, e_{2}\right]=e_{3},\left[e_{2}, e_{4}\right]=e_{0}$, $\left[e_{3}, e_{4}\right]=e_{1}$.
One verifies that $i(\mathfrak{g})=2, c(\mathfrak{g})=5, p_{\mathfrak{g}}=1$. Also $\mathfrak{g}$ is quasi-quadratic (i.e. $F(\mathfrak{g})=\mathfrak{g}$ ) and so it has no proper semi-invariants (2.8). Moreover, there are no CP's as $F(\mathfrak{g})$ is not commutative (2.8).
Next, consider the semi-direct product $L=T \oplus \mathfrak{g}$, where $T=\left\langle t_{1}, t_{2}\right\rangle \subset$ Der $\mathfrak{g}$ with

$$
t_{1}=\operatorname{diag}(0,0,0,1,1,0,0,1), \quad t_{2}=\operatorname{diag}(0,0,0,1,1,1,1,0)
$$

Then

$$
\Delta(L)=4\left(e_{1} e_{2}-e_{0} e_{3}\right)^{2}\left(e_{0} e_{1} h+e_{1}^{2} x-e_{0}^{2} y+e_{1} e_{2} e_{4}-e_{0} e_{3} e_{4}\right)^{2} \neq 0
$$

Hence, $L$ is a 10 -dimensional Frobenius Lie algebra. By (4) of Theorem 30 we may conclude that

$$
Y(\mathfrak{g})=k\left[f_{1}, f_{2}\right]=S y(\mathfrak{g}) \quad \text { and } \quad R(\mathfrak{g})^{\mathfrak{g}}=k\left(f_{1}, f_{2}\right)
$$

where $f_{1}=e_{1} e_{2}-e_{0} e_{3}$ and $f_{2}=e_{0} e_{1} h+e_{1}^{2} x-e_{0}^{2} y+\left(e_{1} e_{2}-e_{0} e_{3}\right) e_{4}$.
Finally, $M=k\left[e_{0}, e_{1}, e_{2}, e_{3}, f_{2}\right]$ is a polynomial, complete Poisson commutative subalgebra of $S(\mathfrak{g})$.

Remark 57. The same Frobenius method can be used to show the theorem for $L_{8,2}$ [O7, p.1302]. The theorem also holds for $L_{6,4}, L_{8,19}, L_{8,20}(\alpha \neq-1), L_{8,28}$ (since they are Frobenius) as well as for their canonical truncations $L_{5}, L_{7,1}, L_{7,2}$ (apply Theorem 10).

Example 58. Let $L$ be the 8-dimensional non algebraic Lie algebra $L_{8,25}$ with basis $h, x, y, e_{0}, e_{1}, e_{2}, e_{3}, e_{4}$ and nonzero brackets:
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=e_{0},\left[h, e_{1}\right]=-e_{1},\left[x, e_{1}\right]=e_{0}$, $\left[y, e_{0}\right]=e_{1},\left[e_{4}, e_{0}\right]=e_{0},\left[e_{4}, e_{1}\right]=e_{1},\left[e_{4}, e_{2}\right]=-e_{3}$.
One verifies that $i(L)=2, c(L)=5, p_{L}=1, Z(L)=\left\langle e_{3}\right\rangle$ and

$$
F(L)=\left\langle h, x, y, e_{0}, e_{1}, e_{2}, e_{3}\right\rangle=L_{\Lambda}
$$

Next, we put $E_{1}=\operatorname{ad} h, E_{2}=\operatorname{ad} x, E_{3}=\operatorname{ad} y, E_{4}=\operatorname{ad} e_{0}, E_{5}=\operatorname{ad} e_{1}, E_{6}=\operatorname{ad} e_{2}$, $E_{7}\left(e_{0}\right)=e_{0}, E_{7}\left(e_{1}\right)=e_{1}$ and zero on others, $E_{8}\left(e_{2}\right)=-e_{3}$ and zero on others. Then

$$
\text { ad } e_{4}=E_{7}+E_{8}
$$

is the decomposition of ad $e_{4}$ into its semi-simple and nilpotent components. It follows that $E_{1}, E_{2}, \ldots, E_{8}$ is a basis for the algebraic hull $H$ of ad $L$.
Next, we rename the basis

$$
h, x, y, e_{0}, e_{1}, e_{2}, e_{3}, e_{4} \quad \text { by } \quad x_{1}, x_{2}, x_{3}, \ldots, x_{8}
$$

Then we see that $\operatorname{rank}\left(E_{i} x_{j}\right)=7$ and so

$$
j(L)=\operatorname{dim} L-\operatorname{rank}\left(E_{i} x_{j}\right)=1=\operatorname{dim} Z(L)
$$

So, we can apply Theorem 37. First one verifies that $p_{L}^{\prime}=e_{3}\left(e_{0} e_{1} h+e_{1}^{2} x-e_{0}^{2} y\right)$.
Hence, $f=e_{0} e_{1} h+e_{1}^{2} x-e_{0}^{2} y$ is the only proper irreducible semi-invariant and we may conclude that

$$
Y(L)=k\left[e_{3}\right], \quad R(L)^{L}=k\left(e_{3}\right), \quad S y(L)=Y(L)[f]=k\left[e_{3}, f\right]
$$

Finally, $M=k\left[e_{0}, e_{1}, e_{2}, e_{3}, f\right]$ is a polynomial, complete, Poisson commutative subalgebra of $S(L)$.

The result on $S y(L)$ can also be seen as follows. Clearly, $L=L_{\Lambda} \oplus k e_{4}$ while $L_{\Lambda}=L_{5} \times\left\langle e_{2}, e_{3}\right\rangle$ (direct product).
Now $Y\left(L_{5}\right)=k[f][\mathrm{O} 7, \mathrm{p} .1301]$.
By (5) of Theorem 4:

$$
S y(L) \subset Y\left(L_{\Lambda}\right)=k\left[e_{2}, e_{3}, f\right]
$$

On the other hand, $e_{3}$ is an invariant and $f$ is a semi-invariant for $L$, indeed

$$
\{x, f\}=\lambda(x) f \text { for all } x \in L
$$

where $\lambda \in L^{*}, \lambda\left(e_{4}\right)=2$ and zero on others. Consequently,

$$
k\left[e_{3}, f\right] \subset S y(L) \subset k\left[e_{2}, e_{2}, f\right]
$$

Now take any semi-invariant $g \in S y(L)$. Then already $g \in k\left[e_{2}, e_{3}, f\right]$. As $g$ is a semi-invariant for ad $L$ it is also one under the action of $H$ [C, p.208]. In particular,

$$
E_{8}(g)=a g \text { for a suitable } a \in k
$$

But $E_{8}$ is nilpotent and so $a=0$.
Next, we consider

$$
-e_{3} \frac{\partial g}{\partial e_{2}}=-x_{7} \frac{\partial g}{\partial x_{6}}=\sum_{j=1}^{8} E_{8}\left(x_{j}\right) \frac{\partial g}{\partial x_{j}}=E_{8}(g)=0
$$

which implies that $\frac{\partial g}{\partial e_{2}}=0$ and so $g \in k\left[e_{3}, f\right]$. Therefore $S y(L)=k\left[e_{3}, f\right]$.

### 5.1. List of indecomposable nonsolvable Lie algebras of dimension $\leq 8$

The main purpose is to show that for each member $L$ of the list $Y(L), S y(L)$ and $R(L)^{L}$ satisfy the requirements of Theorem 53 by giving their explicit description. In particular, $L$ is coregular. It will turn out that $i(L) \leq 2$, which is hardly surprising in view of Proposition 42. In addition we will provide the Frobenius semi-radical $F=F(L)$ and if it exists a CP-ideal (CPI).
$M$ will be a polynomial, complete, Poisson commutative subalgebra of $S(L)$. Other abbreviations are: $i=i(L), c=c(L), p=p_{L}, Y=Y(L), S y=S y(L), R^{I}=R(L)^{L}$.

Furthermore, $h, x, y$ will be the standard basis of $s l(2, k)$. $W_{n}$ will be its $(n+1)$ dimensional irreducible module with standard basis $e_{0}, e_{1}, \ldots, e_{n}$. In particular,

$$
h \cdot e_{i}=(n-2 i) e_{i}, x \cdot e_{i}=(n-i+1) e_{i-1}, y \cdot e_{i}=(i+1) e_{i+1}
$$

for all $i$ and $e_{-1}=e_{n+1}=0$.

## I. $L$ is algebraic

For this the possible parameters need to be rational numbers, but we will briefly indicate what happens if they are not.
I.0. $\operatorname{dim} L=3$
0. $\operatorname{sl}(2, k)$ (simple and hence quadratic)

Basis: $h, x, y$
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h$.
$i=1, c=2, p=1, F=s l(2, k)$, no CP's,
$Y=k[f]=S y$, where $f=h^{2}+4 x y$
$R^{I}=k(f), M=k[h, f]$.
I.1. $\operatorname{dim} L=5$

1. $L_{5}=s l(2, k) \oplus W_{1}$ (quasi quadratic)

Basis: $h, x, y, e_{0}, e_{1}$
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=e_{0},\left[h, e_{1}\right]=-e_{1},\left[x, e_{1}\right]=e_{0}$, $\left[y, e_{0}\right]=e_{1}$.
$i=1, c=3, p=1, F=L_{5}$, no CP's,
$Y=k[f]=S y$, where $f=e_{0} e_{1} h+e_{1}^{2} x-e_{0}^{2} y$
$R^{I}=k(f), M=k\left[e_{0}, e_{1}, f\right]$.

## I.2. $\operatorname{dim} L=6$, with basis $h, x, y, e_{0}, e_{1}, e_{2}$

2. $L_{6,1}=s l(2, k) \oplus W_{2}$ (quadratic)
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=2 e_{0},\left[h, e_{2}\right]=-2 e_{2}$,
$\left[x, e_{1}\right]=2 e_{0},\left[x, e_{2}\right]=e_{1},\left[y, e_{0}\right]=e_{1},\left[y, e_{1}\right]=2 e_{2}$
$i=2, c=4, p=1 F=L_{6,1}$, no CP's
$Y=k\left[f_{1}, f_{2}\right]=S y, f_{1}=e_{1}^{2}-4 e_{0} e_{2}, f_{2}=e_{1} h+2 e_{2} x-2 e_{0} y$
$R^{I}=k\left(f_{1}, f_{2}\right), M=k\left[e_{0}, e_{1}, e_{2}, f_{2}\right]$.
3. $L_{6,3}=s l(2, k) \oplus H$ (quasi quadratic) (see Example 54)

$$
\begin{aligned}
& {[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=e_{0},\left[h, e_{1}\right]=-e_{1},\left[x, e_{1}\right]=e_{0},} \\
& {\left[y, e_{0}\right]=e_{1},\left[e_{0}, e_{1}\right]=e_{2}} \\
& i=2, c=4, p=1 F=L_{6,3}, \text { no CP's } \\
& Y=k\left[e_{2}, f\right]=S y, f=e_{2}\left(h^{2}+4 x y\right)+2\left(e_{0} e_{1} h+e_{1}^{2} x-e_{0}^{2} y\right), R^{I}=k\left(e_{2}, f\right), \\
& M=k\left[e_{1}, e_{2}, e_{0}^{2}-2 e_{2} x, f\right] .
\end{aligned}
$$

4. $L_{6,4}=L_{5} \oplus k e_{2}$ (Frobenius)
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=e_{0},\left[h, e_{1}\right]=-e_{1}$,
$\left[x, e_{1}\right]=e_{0},\left[y, e_{0}\right]=e_{1},\left[e_{2}, e_{0}\right]=e_{0},\left[e_{2}, e_{1}\right]=e_{1}$.
$i=0, c=3, p=e_{0} e_{1} h+e_{1}^{2} x-e_{0}^{2} y, F=0$, no CP's,
$Y=k, S y=k[p], R^{I}=k, M=k\left[e_{0}, e_{1}, p\right]$.

## I.3. $\operatorname{dim} L=7$, with basis $h, x, y, e_{0}, e_{1}, e_{2}, e_{3}$

5. $L_{7,1}=s l(2, k) \oplus W_{3}$
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=3 e_{0}$,
$\left[h, e_{1}\right]=e_{1},\left[h, e_{2}\right]=-e_{2},\left[h, e_{3}\right]=-3 e_{3},\left[x, e_{1}\right]=3 e_{0}$
$\left[x, e_{2}\right]=2 e_{1},\left[x, e_{3}\right]=e_{2},\left[y, e_{0}\right]=e_{1},\left[y, e_{1}\right]=2 e_{2}$,
$\left[y, e_{2}\right]=3 e_{3}$.
$i=1, c=4, p=1, F=W_{3}=C P I$
$Y=k[f]=k\left[W_{3}\right]^{S L(2)}=S y, R^{I}=k(f)$,
$f=4 e_{0} e_{2}^{3}-e_{1}^{2} e_{2}^{2}-18 e_{0} e_{1} e_{2} e_{3}+27 e_{0}^{2} e_{3}^{2}+4 e_{1}^{3} e_{3}$
$M=k\left[e_{0}, e_{1}, e_{2}, e_{3}\right]$
6. $L_{7,2}=\operatorname{sl}(2, k) \oplus W_{1} \oplus W_{1}$
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=e_{0}$,
$\left[h, e_{1}\right]=-e_{1},\left[h, e_{2}\right]=e_{2},\left[h, e_{3}\right]=-e_{3},\left[x, e_{1}\right]=e_{0}$
$\left[x, e_{3}\right]=e_{2},\left[y, e_{0}\right]=e_{1},\left[y, e_{2}\right]=e_{3}$.
$i=1, c=4, p=e_{0} e_{3}-e_{1} e_{2}, F=W_{1} \oplus W_{1}=C P I$,
$Y=k[p]=S y, R^{I}=k(p), M=k\left[e_{0}, e_{1}, e_{2}, e_{3}\right]$.
7. $L_{7,7}=L_{6,1} \oplus k e_{3}$
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=2 e_{0}$,
$\left[h, e_{2}\right]=-2 e_{2},\left[x, e_{1}\right]=2 e_{0},\left[x, e_{2}\right]=e_{1},\left[y, e_{0}\right]=e_{1}$,
$\left[y, e_{1}\right]=2 e_{2},\left[e_{3}, e_{0}\right]=e_{0},\left[e_{3}, e_{1}\right]=e_{1},\left[e_{3}, e_{2}\right]=e_{2}$.
$i=1, c=4, p=1, F=L_{6,1}=\left(L_{7,7}\right)_{\Lambda}$, no CP's.
$Y=k, S y=k\left[f_{1}, f_{2}\right], f_{1}=e_{1}^{2}-4 e_{0} e_{2}, f_{2}=e_{1} h+2 e_{2} x-2 e_{0} y$,
$R^{I}=k\left(f_{2}^{2} / f_{1}\right), M=k\left[e_{0}, e_{1}, e_{2}, f_{2}\right]$.
8. $L_{7,8}(\alpha \neq 0)=\left(L_{5} \times k e_{2}\right) \oplus k e_{3}$
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=e_{0}$,
$\left[h, e_{1}\right]=-e_{1},\left[x, e_{1}\right]=e_{0},\left[y, e_{0}\right]=e_{1},\left[e_{3}, e_{0}\right]=e_{0}$
$\left[e_{3}, e_{1}\right]=e_{1},\left[e_{3}, e_{2}\right]=\alpha e_{2}$
$i=1, c=4, p=1, F=L_{5} \times k e_{2}=\left(L_{7,8}\right)_{\Lambda}$, no CP's
$Y=k, S y=k\left[e_{2}, f\right], f=e_{0} e_{1} h+e_{1}^{2} x-e_{0}^{2} y$,
$R^{I}=k\left(e_{2}^{r} f^{s}\right), r, s \in \mathbb{Z}$ coprime such that $r \alpha+2 s=0$,
$M=k\left[e_{0}, e_{1}, e_{2}, f\right]$.
[If $\alpha \notin \mathbb{Q}$, then $R^{I}=k$ ]
9. $L_{7,9}=L_{6,3} \oplus k e_{3}$ (see Example 55)
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=e_{0},\left[h, e_{1}\right]=-e_{1}$,
$\left[x, e_{1}\right]=e_{0},\left[y, e_{0}\right]=e_{1},\left[e_{0}, e_{1}\right]=e_{2},\left[e_{3}, e_{0}\right]=e_{0}$,
$\left[e_{3}, e_{1}\right]=e_{1},\left[e_{3}, e_{2}\right]=2 e_{2}$.
$i=1, c=4, p=1, F=L_{6,3}=\left(L_{7,9}\right)_{\Lambda}$, no CP's
$Y=k, S y=k\left[e_{2}, f\right], f=e_{2}\left(h^{2}+4 x y\right)+2\left(e_{0} e_{1} h+e_{1}^{2} x-e_{0}^{2} y\right)$
$R^{I}=k\left(f / e_{2}\right), M=k\left[e_{1}, e_{2}, e_{0}^{2}-2 e_{2} x, f\right]$

## I.4. $\operatorname{dim} L=8$, with basis $h, x, y, e_{0}, e_{1}, e_{2}, e_{3}, e_{4}$

10. $L_{8,1}=\operatorname{sl}(2, k) \oplus W_{4}$
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=4 e_{0},\left[h, e_{1}\right]=2 e_{1}$,
$\left[h, e_{3}\right]=-2 e_{3},\left[h, e_{4}\right]=-4 e_{4},\left[x, e_{1}\right]=4 e_{0},\left[x, e_{2}\right]=3 e_{1}$,
$\left[x, e_{3}\right]=2 e_{2},\left[x, e_{4}\right]=e_{3},\left[y, e_{0}\right]=e_{1},\left[y, e_{1}\right]=2 e_{2},\left[y, e_{2}\right]=3 e_{3},\left[y, e_{3}\right]=4 e_{4}$.
$i=2, c=5, p=1, F=W_{4}=C P I$,
$Y=k\left[f_{1}, f_{2}\right]=S y, f_{1}=e_{2}^{2}-3 e_{1} e_{3}+12 e_{0} e_{4}$,
$f_{2}=2 e_{2}^{3}-9 e_{1} e_{2} e_{3}+27 e_{0} e_{3}^{2}+27 e_{1}^{2} e_{4}-72 e_{0} e_{2} e_{4}$
$R^{I}=k\left(f_{1}, f_{2}\right), M=k\left[e_{0}, e_{1}, e_{2}, e_{3}, e_{4}\right]$
11. $L_{8,2}=\operatorname{sl}(2, k) \oplus W_{2} \oplus W_{1}$
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=2 e_{0},\left[h, e_{2}\right]=-2 e_{2}$,
$\left[h, e_{3}\right]=e_{3},\left[h, e_{4}\right]=-e_{4},\left[x, e_{1}\right]=2 e_{0},\left[x, e_{2}\right]=e_{1}$,
$\left[x, e_{4}\right]=e_{3},\left[y, e_{0}\right]=e_{1},\left[y, e_{1}\right]=2 e_{2},\left[y, e_{3}\right]=e_{4}$
$i=2, c=5, p=1, F=W_{2} \oplus W_{1}=C P I$,
$Y=k\left[f_{1}, f_{2}\right]=S y, f_{1}=e_{1}^{2}-4 e_{0} e_{2}, f_{2}=e_{0} e_{4}^{2}-e_{1} e_{3} e_{4}+e_{2} e_{3}^{2}$
$R^{I}=k\left(f_{1}, f_{2}\right), M=k\left[e_{0}, e_{1}, e_{2}, e_{3}, e_{4}\right]$
12. $L_{8,13}$ (quasi quadratic)
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=e_{0},\left[h, e_{1}\right]=-e_{1}$,
$\left[h, e_{3}\right]=e_{3},\left[h, e_{4}\right]=-e_{4},\left[x, e_{1}\right]=e_{0},\left[x, e_{4}\right]=e_{3}$,
$\left[y, e_{0}\right]=e_{1},\left[y, e_{3}\right]=e_{4},\left[e_{0}, e_{1}\right]=e_{2} . i=2, c=5, p=1, F=L_{8,13}$, no CP's
$Y=k\left[e_{2}, f\right]=S y, f=2 e_{2}\left(e_{3} e_{4} h+e_{4}^{2} x-e_{3}^{2} y\right)-\left(e_{0} e_{4}-e_{1} e_{3}\right)^{2}$
$R^{I}=k\left(e_{2}, f\right), M=k\left[e_{1}, e_{2}, e_{3}, e_{0}^{2}-2 e_{2} x, f\right]$
13. $L_{8,14}=\left(L_{6,3} \times k e_{3}\right) \oplus k e_{4}$
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=e_{0},\left[h, e_{1}\right]=-e_{1}$,
$\left[x, e_{1}\right]=e_{0},\left[y, e_{0}\right]=e_{1},\left[e_{0}, e_{1}\right]=e_{2},\left[e_{3}, e_{4}\right]=e_{2}$.
$i=2, c=5, p=e_{2}, F=L_{6,3}$, no CP's
$Y=k\left[e_{2}, f\right]=S y, f=e_{2}\left(h^{2}+4 x y\right)+2\left(e_{0} e_{1} h+e_{1}^{2} x-e_{0}^{2} y\right)$,
$R^{I}=k\left(e_{2}, f\right), M=k\left[e_{1}, e_{2}, e_{3}, e_{0}^{2}-2 e_{2} x, f\right]$
14. $L_{8,15}$ (quasi quadratic)
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=e_{0},\left[h, e_{1}\right]=-e_{1}$,
$\left[h, e_{3}\right]=e_{3},\left[h, e_{4}\right]=-e_{4},\left[x, e_{1}\right]=e_{0},\left[x, e_{4}\right]=e_{3}$,
$\left[y, e_{0}\right]=e_{1},\left[y, e_{3}\right]=e_{4},\left[e_{0}, e_{1}\right]=e_{2},\left[e_{3}, e_{4}\right]=e_{2}$.
$i=2, c=5, p=1, F=L_{8,15}$, no CP's
$Y=k\left[e_{2}, f\right]=S y, f=e_{2}^{2}\left(h^{2}+4 x y\right)+2 e_{2}\left(e_{0} e_{1}+e_{3} e_{4}\right) h+2 e_{2}\left(e_{1}^{2}+e_{4}^{2}\right) x-$
$2 e_{2}\left(e_{0}^{2}+e_{3}^{2}\right) y-\left(e_{0} e_{4}-e_{1} e_{3}\right)^{2}$,
$R^{I}=k\left(e_{2}, f\right), M=k\left[e_{1}, e_{2}, e_{4}, e_{0}^{2}+e_{3}^{2}-2 e_{2} x, f\right]$.
15. $L_{8,16}$ (quasi quadratic)
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{1}\right]=3 e_{1},\left[h, e_{2}\right]=e_{2}$,
$\left[h, e_{3}\right]=-e_{3},\left[h, e_{4}\right]=-3 e_{4},\left[x, e_{2}\right]=3 e_{1},\left[x, e_{3}\right]=2 e_{2}$,
$\left[x, e_{4}\right]=e_{3},\left[y, e_{1}\right]=e_{2},\left[y, e_{2}\right]=2 e_{3},\left[y, e_{3}\right]=3 e_{4},\left[e_{1}, e_{4}\right]=e_{0},\left[e_{2}, e_{3}\right]=-3 e_{0}$.
$i=2, c=5, p=1, F=L_{8,16}$, no CP's
$Y=k\left[e_{0}, f\right]=S y, f=3 e_{0}^{2}\left(h^{2}+4 x y\right)+2 e_{0}\left(9 e_{1} e_{4}-e_{2} e_{3}\right) h+4 e_{0}\left(3 e_{2} e_{4}-e_{3}^{2}\right) x+$
$4 e_{0}\left(e_{2}^{2}-3 e_{1} e_{3}\right) y+4 e_{1} e_{3}^{3}-e_{2}^{2} e_{3}^{2}-18 e_{1} e_{2} e_{3} e_{4}+27 e_{1}^{2} e_{4}^{2}+4 e_{2}^{3} e_{4}$,
$R^{I}=k\left(e_{0}, f\right), M=k\left[e_{0}, e_{1}, e_{2}, 3 e_{0} x+e_{2}^{2}-3 e_{1} e_{3}, f\right]$.
16. $L_{8,17}=L_{7,2} \oplus k e_{4}$ (quasi quadratic) (see Example 56)
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=e_{0},\left[h, e_{1}\right]=-e_{1}$,
$\left[h, e_{2}\right]=e_{2},\left[h, e_{3}\right]=-e_{3},\left[x, e_{1}\right]=e_{0},\left[x, e_{3}\right]=e_{2}$,
$\left[y, e_{0}\right]=e_{1},\left[y, e_{2}\right]=e_{3},\left[e_{2}, e_{4}\right]=e_{0},\left[e_{3}, e_{4}\right]=e_{1}$.
$i=2, c=5, p=1, F=L_{8,17}$, no CP's
$Y=k\left[f_{1}, f_{2}\right]=S y, f_{1}=e_{1} e_{2}-e_{0} e_{3}, f_{2}=e_{0} e_{1} h+e_{1}^{2} x-e_{0}^{2} y+\left(e_{1} e_{2}-e_{0} e_{3}\right) e_{4}$ $R^{I}=k\left(f_{1}, f_{2}\right), M=k\left[e_{0}, e_{1}, e_{2}, e_{3}, f_{2}\right]$.
17. $L_{8,18}$ (quasi quadratic)
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=e_{0},\left[h, e_{1}\right]=-e_{1}$,
$\left[h, e_{3}\right]=e_{3},\left[h, e_{4}\right]=-e_{4},\left[x, e_{1}\right]=e_{0},\left[x, e_{4}\right]=e_{3}$,
$\left[y, e_{0}\right]=e_{1},\left[y, e_{3}\right]=e_{4},\left[e_{2}, e_{3}\right]=e_{0},\left[e_{2}, e_{4}\right]=e_{1},\left[e_{3}, e_{4}\right]=e_{2}$.
$i=2, c=5, p=1, F=L_{8,18}$, no CP's
$Y=k\left[f_{1}, f_{2}\right]=S y, f_{1}=2\left(e_{0} e_{4}-e_{1} e_{3}\right)+e_{2}^{2}$,
$f_{2}=e_{0} e_{1} h+e_{1}^{2} x-e_{0}^{2} y+e_{2}\left(e_{0} e_{4}-e_{1} e_{3}\right)+\frac{1}{3} e_{2}^{3}$,
$R^{I}=k\left(f_{1}, f_{2}\right), M=k\left[e_{0}, e_{1}, e_{2}, f_{1}, f_{2}\right]$.
18. $L_{8,19}=L_{7,1} \oplus k e_{4}$ (Frobenius)
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=3 e_{0},\left[h, e_{1}\right]=e_{1}$,
$\left[h, e_{2}\right]=-e_{2},\left[h, e_{3}\right]=-3 e_{3},\left[x, e_{1}\right]=3 e_{0},\left[x, e_{2}\right]=2 e_{1}$,
$\left[x, e_{3}\right]=e_{2},\left[y, e_{0}\right]=e_{1},\left[y, e_{1}\right]=2 e_{2},\left[y, e_{2}\right]=3 e_{3},\left[e_{4}, e_{0}\right]=e_{0},\left[e_{4}, e_{1}\right]=e_{1}$,
$\left[e_{4}, e_{2}\right]=e_{2},\left[e_{4}, e_{3}\right]=e_{3}$.
$i=0, c=4, F=0, C P I=\left\langle e_{0}, e_{1}, e_{2}, e_{3}\right\rangle,\left(L_{8,19}\right)_{\Lambda}=L_{7,1}$,
$p=4 e_{0} e_{2}^{3}-e_{1}^{2} e_{2}^{2}-18 e_{0} e_{1} e_{2} e_{3}+27 e_{0}^{2} e_{3}^{2}+4 e_{1}^{3} e_{3}$,
$Y=k, S y=k[p], R^{I}=k, M=k\left[e_{0}, e_{1}, e_{2}, e_{3}\right]$.
19. $L_{8,20}(\alpha \neq-1)=L_{7,2} \oplus k e_{4}$ (Frobenius)
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=e_{0},\left[h, e_{1}\right]=-e_{1}$,
$\left[h, e_{2}\right]=e_{2},\left[h, e_{3}\right]=-e_{3},\left[x, e_{1}\right]=e_{0},\left[x, e_{3}\right]=e_{2}$,
$\left[y, e_{0}\right]=e_{1},\left[y, e_{2}\right]=e_{3},\left[e_{4}, e_{0}\right]=e_{0},\left[e_{4}, e_{1}\right]=e_{1},\left[e_{4}, e_{2}\right]=\alpha e_{2},\left[e_{4}, e_{3}\right]=\alpha e_{3}$.
$i=0, c=4, p=\left(e_{0} e_{3}-e_{1} e_{2}\right)^{2}, F=0, C P I=\left\langle e_{0}, e_{1}, e_{2}, e_{3}\right\rangle,\left(L_{8,20}\right)_{\Lambda}=L_{7,2}$
$Y=k, S y=k\left[e_{0} e_{3}-e_{1} e_{2}\right], R^{I}=k, M=k\left[e_{0}, e_{1}, e_{2}, e_{3}\right]$.
20. $L_{8,20}(\alpha=-1)=L_{7,2} \oplus k e_{4}$ (quasi quadratic)
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=e_{0},\left[h, e_{1}\right]=-e_{1}$,
$\left[h, e_{2}\right]=e_{2},\left[h, e_{3}\right]=-e_{3},\left[x, e_{1}\right]=e_{0},\left[x, e_{3}\right]=e_{2}$,
$\left[y, e_{0}\right]=e_{1},\left[y, e_{2}\right]=e_{3},\left[e_{4}, e_{0}\right]=e_{0},\left[e_{4}, e_{1}\right]=e_{1},\left[e_{4}, e_{2}\right]=-e_{2},\left[e_{4}, e_{3}\right]=-e_{3}$.
$i=2, c=5, p=1, F=L_{8,20}$, no CP's
$Y=k\left[f_{1}, f_{2}\right]=S y, f_{1}=e_{1} e_{2}-e_{0} e_{3}$,
$f_{2}=\left(e_{1} e_{2}+e_{0} e_{3}\right) h+2 e_{1} e_{3} x-2 e_{0} e_{2} y+\left(e_{1} e_{2}-e_{0} e_{3}\right) e_{4}$
$R^{I}=k\left(f_{1}, f_{2}\right), M=k\left[e_{0}, e_{1}, e_{2}, e_{3}, f_{2}\right]$.
21. $L_{8,21}(\alpha \neq 0, \beta \neq 0)=\left(L_{5} \times\left\langle e_{2}, e_{3}\right\rangle\right) \oplus k e_{4}$
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=e_{0},\left[h, e_{1}\right]=-e_{1}$,
$\left[x, e_{1}\right]=e_{0},\left[y, e_{0}\right]=e_{1},\left[e_{4}, e_{0}\right]=e_{0},\left[e_{4}, e_{1}\right]=e_{1}$,
$\left[e_{4}, e_{2}\right]=\alpha e_{2},\left[e_{4}, e_{3}\right]=\beta e_{3}$.
$i=2, c=5, p=1, F=L_{5} \times\left\langle e_{2}, e_{3}\right\rangle=\left(L_{8,21}\right)_{\Lambda}$, no CP's
$Y=k, S y=k\left[e_{2}, e_{3}, f\right], f=e_{0} e_{1} h+e_{1}^{2} x-e_{0}^{2} y$,
$M=k\left[e_{0}, e_{1}, e_{2}, e_{3}, f\right], R^{I}=k\left(e_{2}^{r_{0}} e_{3}^{s_{o}} f^{t_{0}}, e_{2}^{r_{1}} e_{3}^{s_{1}} f^{t_{1}}\right)$, where $\left(r_{0}, s_{0}, t_{0}\right),\left(r_{1}, s_{1}, t_{1}\right)$ is a basis of the free $\mathbb{Z}$-module $\left\{(r, s, t) \in \mathbb{Z}^{3} \mid \alpha r+\beta s+2 t=0\right\}$.
[If $\alpha \notin \mathbb{Q}$ then $R^{I}=k\left(e_{3}^{s} f^{t}\right)$ where $s, t$ are coprime integers such that $\beta s+2 t=$ 0 . Similarly, if $\beta \notin \mathbb{Q}$. If $\alpha$ and $\beta \notin \mathbb{Q}$ and $\alpha / \beta \notin \mathbb{Q}$ then $\left.R^{I}=k\right]$
22. $L_{8,22}(\alpha \neq 0)=\left(L_{6,1} \times k e_{3}\right) \oplus k e_{4}$
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=2 e_{0},\left[h, e_{2}\right]=-2 e_{2}$,
$\left[x, e_{1}\right]=2 e_{0},\left[x, e_{2}\right]=e_{1},\left[y, e_{0}\right]=e_{1},\left[y, e_{1}\right]=2 e_{2}$,
$\left[e_{4}, e_{0}\right]=e_{0},\left[e_{4}, e_{1}\right]=e_{1},\left[e_{4}, e_{2}\right]=e_{2},\left[e_{4}, e_{3}\right]=\alpha e_{3}$.
$i=2, c=5, p=1, F=L_{6,1} \times k e_{3}=\left(L_{8,22}\right)_{\Lambda}$, no CP's
$Y=k, S y=k\left[e_{3}, f_{1}, f_{2}\right], f_{1}=e_{1}^{2}-4 e_{0} e_{2}$,
$f_{2}=e_{1} h+2 e_{2} x-2 e_{0} y, M=k\left[e_{0}, e_{1}, e_{2}, e_{3}, f_{2}\right]$,
$R^{I}=k\left(f_{2}^{2} / f_{1}, f_{2}^{s} / e_{3}^{t}\right)$ where $s, t$ are coprime integers such that $\alpha=\frac{s}{t}$.
[If $\alpha \notin \mathbb{Q}$ then $R^{I}=k\left(f_{2}^{2} / f_{1}\right)$ ]
23. $L_{8,23}(\alpha \neq 0)=\left(L_{6,3} \times k e_{3}\right) \oplus k e_{4}$
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=e_{0},\left[h, e_{1}\right]=-e_{1}$,
$\left[x, e_{1}\right]=e_{0},\left[y, e_{0}\right]=e_{1},\left[e_{0}, e_{1}\right]=e_{2},\left[e_{4}, e_{0}\right]=e_{0}$,
$\left[e_{4}, e_{1}\right]=e_{1},\left[e_{4}, e_{2}\right]=2 e_{2},\left[e_{4}, e_{3}\right]=\alpha e_{3}$.
$i=2, c=5, p=1, F=L_{6,3} \times k e_{3}=\left(L_{8,23}\right)_{\Lambda}$, no CP's
$Y=k, S y=k\left[e_{2}, e_{3}, f\right], f=e_{2}\left(h^{2}+4 x y\right)+2\left(e_{0} e_{1} h+e_{1}^{2} x-e_{0}^{2} y\right)$,
$M=k\left[e_{1}, e_{2}, e_{3}, e_{0}^{2}-2 e_{2} x, f\right]$,
$R^{I}=k\left(f / e_{2}, e_{2}^{s} e_{3}^{t}\right)$ where $s, t$ are coprime integers such that $2 s+\alpha t=0$.
[If $\alpha \notin \mathbb{Q}$ then $R^{I}=k\left(f / e_{2}\right)$ ]
24. $L_{8,24}=\operatorname{sl}(3, k)$ (simple and hence quadratic)

Basis: (see e.g. [D4])
$h_{\alpha}=E_{11}-E_{22}, h_{\gamma}=E_{22}-E_{33}, x_{\alpha}=E_{12}, x_{\beta}=E_{13}, x_{\gamma}=E_{23}, x_{-\alpha}=E_{21}$, $x_{-\beta}=E_{31}, x_{-\gamma}=E_{32}$, where the $E_{i j}$ are the standard $3 \times 3$ matrices and $\alpha, \gamma, \beta=\alpha+\gamma$ are the positive roots w.r.t. the Cartan subalgebra $H=\left\langle E_{11}-E_{22}, E_{22}-E_{33}\right\rangle$.
$\left[h_{\alpha}, x_{\alpha}\right]=2 x_{\alpha},\left[h_{\alpha}, x_{\beta}\right]=x_{\beta},\left[h_{\alpha}, x_{\gamma}\right]=-x_{\gamma},\left[h_{\alpha}, x_{-\alpha}\right]=-2 x_{-\alpha},\left[h_{\alpha}, x_{-\beta}\right]=$ $-x_{-\beta},\left[h_{\alpha}, x_{-\gamma}\right]=x_{-\gamma},\left[h_{\gamma}, x_{\alpha}\right]=-x_{\alpha},\left[h_{\gamma}, x_{\beta}\right]=x_{\beta},\left[h_{\gamma}, x_{\gamma}\right]=2 x_{\gamma},\left[h_{\gamma}, x_{-\alpha}\right]=$ $x_{-\alpha},\left[h_{\gamma}, x_{-\beta}\right]=-x_{-\beta},\left[h_{\gamma}, x_{-\gamma}\right]=-2 x_{-\gamma},\left[x_{\alpha}, x_{\gamma}\right]=x_{\beta},\left[x_{\alpha}, x_{-\alpha}\right]=h_{\alpha}$, $\left[x_{\alpha}, x_{-\beta}\right]=-x_{-\gamma},\left[x_{\beta}, x_{-\alpha}\right]=-x_{\gamma},\left[x_{\beta}, x_{-\beta}\right]=h_{\alpha}+h_{\gamma},\left[x_{\beta}, x_{-\gamma}\right]=x_{\alpha}$,

$$
\begin{aligned}
& {\left[x_{\gamma}, x_{-\beta}\right]=x_{-\alpha},\left[x_{\gamma}, x_{-\gamma}\right]=h_{\gamma},\left[x_{-\alpha}, x_{-\gamma}\right]=-x_{-\beta}} \\
& i=2, c=5, p=1, F=L_{8,24}, \text { no CP's, } \\
& Y=k\left[f_{1}, f_{2}\right]=S y, \\
& f_{1}=h_{\alpha}^{2}+h_{\alpha} h_{\gamma}+h_{\gamma}^{2}+3\left(x_{\alpha} x_{-\alpha}+x_{\beta} x_{-\beta}+x_{\gamma} x_{-\gamma}\right) \\
& f_{2}=\left(h_{\alpha}+2 h_{\gamma}\right)\left(h_{\alpha}-h_{\gamma}\right)\left(2 h_{\alpha}+h_{\gamma}\right)+9\left(h_{\alpha}+2 h_{\gamma}\right) x_{\alpha} x_{-\alpha}+ \\
& 9\left(h_{\alpha}-h_{\gamma}\right) x_{\beta} x_{-\beta}-9\left(2 h_{\alpha}+h_{\gamma}\right) x_{\gamma} x_{-\gamma}+27\left(x_{\alpha} x_{\gamma} x_{-\beta}+x_{\beta} x_{-\alpha} x_{-\gamma}\right) \\
& R^{I}=k\left(f_{1}, f_{2}\right), M=k\left[f_{1}, f_{2}, f_{3}, f_{4}, f_{5}\right], f_{3}=x_{\beta}, \\
& f_{4}=x_{\alpha}+x_{\gamma}, f_{5}=\left(h_{\alpha}+2 h_{\gamma}\right) x_{\alpha}-\left(2 h_{\alpha}+h_{\gamma}\right) x_{\gamma}+3 x_{\beta}\left(x_{-\alpha}+x_{-\gamma}\right) .
\end{aligned}
$$

The subalgebra $M$ was constructed by means of the argument shift method starting out from the generating invariants $f_{1}, f_{2}$ of $Y\left(L_{8,24}\right)$. By [PY1] $M$ is a polynomial, strongly complete Poisson commutative subalgebra of $S\left(L_{8,24}\right)$. See also [Ta].

## II. $L$ is not algebraic

This includes the families of part I with non rational parameters.
In the remaining cases $L$ will be 8 -dimensional with basis $h, x, y, e_{0}, e_{1}, e_{2}, e_{3}, e_{4}$.
25. $L_{8,25}=\left(L_{5} \times\left\langle e_{2}, e_{3}\right\rangle\right) \oplus k e_{4}$ (see Example 58)
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=e_{0},\left[h, e_{1}\right]=-e_{1}$,
$\left[x, e_{1}\right]=e_{0},\left[y, e_{0}\right]=e_{1},\left[e_{4}, e_{0}\right]=e_{0},\left[e_{4}, e_{1}\right]=e_{1},\left[e_{4}, e_{2}\right]=-e_{3}$
$i=2, c=5, p=1, F=L_{5} \times\left\langle e_{2}, e_{3}\right\rangle=\left(L_{8,25}\right)_{\Lambda}$, no CP's
$Y=k\left[e_{3}\right], S y=k\left[e_{3}, f\right], f=e_{0} e_{1} h+e_{1}^{2} x-e_{0}^{2} y$,
$R^{I}=k\left(e_{3}\right), M=k\left[e_{0}, e_{1}, e_{2}, e_{3}, f\right]$.
26. $L_{8,26}(\alpha \neq 0)=\left(L_{5} \times\left\langle e_{2}, e_{3}\right\rangle\right) \oplus k e_{4}$
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=e_{0},\left[h, e_{1}\right]=-e_{1}$,
$\left[x, e_{1}\right]=e_{0},\left[y, e_{0}\right]=e_{1},\left[e_{4}, e_{0}\right]=\alpha e_{0},\left[e_{4}, e_{1}\right]=\alpha e_{1},\left[e_{4}, e_{2}\right]=e_{2}$,
$\left[e_{4}, e_{3}\right]=e_{3}-e_{2}$.
$i=2, c=5, p=1, F=L_{5} \times\left\langle e_{2}, e_{3}\right\rangle=\left(L_{8,26}\right)_{\Lambda}$, no CP's
$Y=k, S y(L)=k\left[e_{2}, f\right], f=e_{0} e_{1} h+e_{1}^{2} x-e_{0}^{2} y$,
$M=k\left[e_{0}, e_{1}, e_{2}, e_{3}, f\right]$
(i) If $\alpha \in \mathbb{Q}$ then $R^{I}=k\left(f^{t} / e_{2}^{s}\right)$ where $s, t$ are coprime integers such that $\frac{s}{t}=2 \alpha$.
(ii) If $\alpha \notin \mathbb{Q}$ then $R^{I}=k$.
27. $L_{8,26}(\alpha=0)=\left(L_{5} \times\left\langle e_{2}, e_{3}\right\rangle\right) \oplus k e_{4}$ $[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=e_{0},\left[h, e_{1}\right]=-e_{1}$,

$$
\begin{aligned}
& {\left[x, e_{1}\right]=e_{0},\left[y, e_{0}\right]=e_{1},\left[e_{4}, e_{2}\right]=e_{2},\left[e_{4}, e_{3}\right]=e_{3}-e_{2} .} \\
& i=2, c=5, p=1, F=L_{5} \times\left\langle e_{2}, e_{3}\right\rangle=\left(L_{8,26}\right)_{\Lambda}, \text { no CP's } \\
& Y=k[f], f=e_{0} e_{1} h+e_{1}^{2} x-e_{0}^{2} y, S y=k\left[e_{2}, f\right] \\
& R^{I}=k(f), M=k\left[e_{0}, e_{1}, e_{2}, e_{3}, f\right] .
\end{aligned}
$$

28. $L_{8,27}=\left(L_{6,3} \times k e_{3}\right) \oplus k e_{4}$
$[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=e_{0},\left[h, e_{1}\right]=-e_{1}$,
$\left[x, e_{1}\right]=e_{0},\left[y, e_{0}\right]=e_{1},\left[e_{0}, e_{1}\right]=e_{2},\left[e_{4}, e_{0}\right]=e_{0},\left[e_{4}, e_{1}\right]=e_{1}$,
$\left[e_{4}, e_{2}\right]=2 e_{2},\left[e_{4}, e_{3}\right]=2 e_{3}-e_{2}$.
$i=2, c=5, p=1, F=L_{6,3} \times k e_{3}=\left(L_{8,27}\right)_{\Lambda}$, no CP's
$Y=k, S y=k\left[e_{2}, f\right], f=e_{2}\left(h^{2}+4 x y\right)+2\left(e_{0} e_{1} h+e_{1}^{2} x-e_{0}^{2} y\right)$,
$R^{I}=k\left(f / e_{2}\right), M=\left[e_{1}, e_{2}, e_{3}, e_{0}^{2}-2 e_{2} x, f\right]$.
29. $L_{8,28}=L_{7,2} \oplus k e_{4}$ (Frobenius)

$$
\begin{aligned}
& {[h, x]=2 x,[h, y]=-2 y,[x, y]=h,\left[h, e_{0}\right]=e_{0},\left[h, e_{1}\right]=-e_{1}} \\
& {\left[h, e_{2}\right]=e_{2},\left[h, e_{3}\right]=-e_{3},\left[x, e_{1}\right]=e_{0},\left[x, e_{3}\right]=e_{2},\left[y, e_{0}\right]=e_{1},\left[y, e_{2}\right]=e_{3},} \\
& {\left[e_{4}, e_{0}\right]=e_{0},\left[e_{4}, e_{1}\right]=e_{1},\left[e_{4}, e_{2}\right]=e_{2}-e_{0},\left[e_{4}, e_{3}\right]=e_{3}-e_{1} .} \\
& i=0, c=4, p=\left(e_{0} e_{3}-e_{1} e_{2}\right)^{2}, F=0,\left\langle e_{0}, e_{1}, e_{2}, e_{3}\right\rangle=C P I \\
& Y=k, S y=k\left[e_{0} e_{3}-e_{1} e_{2}\right], R^{I}=k, M=k\left[e_{0}, e_{1}, e_{2}, e_{3}\right] .
\end{aligned}
$$

### 5.2. Counterexample in dimension 9

Example 59. Take the semi-direct product $L=s l(2, k) \oplus W_{5}$ in which $W_{5}$ is an abelian ideal. As $[L, L]=L, L$ is algebraic without proper semi-invariants. Since $\operatorname{dim} \operatorname{sl}(2, k)<\operatorname{dim} W_{5}$ we know that the stabilizer $s l(2, k)(f)=0$ for some $f \in W_{5}^{*}$ by [AVE]. This implies that $i(L)=\operatorname{dim} W_{5}-\operatorname{dim} s l(2, k)=3, Y(L)=S\left(W_{5}\right)^{s l(2, k)}$ and also that $W_{5}$ is a CPI of $L$ by [O4, Proposition 17]. One verifies that codim $L_{\text {sing }}^{*}=4$. By Proposition 15 (or by Theorem 16) we may conclude that $L$ is not coregular. On the other hand, $L$ satisfies the Gelfand-Kirillov conjecture by [O6, Proposition 4.3].

Now suppose $k=\mathbb{C}$. Then $W_{5}$ may be considered as the vector space of binary forms of degree 5 with complex coefficients (the quintics) on which $S L(2, \mathbb{C})$ acts. The algebra of invariants $\mathbb{C}\left[W_{5}\right]^{S L(2, \mathbb{C})}$ which is isomorphic to $S\left(W_{5}\right)^{s l(2, \mathbb{C})}$, has been studied already in the 19th century by Sylvester, among others. At first 3 algebraically independent invariants $I_{4}, I_{8}, I_{12}$ were found of degrees $4,8,12$. In 1854 Hermite discovered an invariant $I_{18}$ of degree 18 and he showed that

$$
\mathbb{C}\left[W_{5}\right]^{S L(2, \mathbb{C})}=\mathbb{C}\left[I_{4}, I_{8}, I_{12}, I_{18}\right]
$$

with the following relation:

$$
16 I_{18}^{2}=I_{4} I_{8}^{4}+8 I_{8}^{3} I_{12}-2 I_{4}^{2} I_{8}^{2} I_{12}-72 I_{4} I_{8} I_{12}^{2}-432 I_{12}^{3}+I_{4}^{3} I_{12}^{2}
$$

In particular, $\mathbb{C}\left[W_{5}\right]^{S L(2, \mathbb{C})}$ is not polynomial.
The explicit forms of $I_{4}, I_{8}, I_{12}, I_{18}$ were given in papers by Cayley. $I_{18}$ has 848 monomials with very large coefficients! See [D5, p.41].

## Example 60.

We conclude this section by considering the semi-direct product $L=s l(2, k) \oplus W_{2} \oplus W_{2}$ with standard basis $h, x, y ; e_{0}, e_{1}, e_{2} ; e_{3}, e_{4}, e_{5}$.
We know that this is a counterexample to the Gelfand-Kirillov conjecture [AOV1]. However, besides this, its behaviour is rather tame.
Indeed, $[L, L]=L$ so $L$ is algebraic and it has no proper semi-invariants. Also, $F(L)=W_{2} \oplus W_{2}$ is a CPI of $L, i(L)=3, c(L)=6$ and $p_{L}=1$.
Furthermore, $L$ is coregular as
$Y(L)=k\left[f_{1}, f_{2}, f_{3}\right]=S y(L)$, where $f_{1}=e_{1}^{2}-4 e_{0} e_{2}, f_{2}=e_{4}^{2}-4 e_{3} e_{5}$, $f_{3}=e_{1} e_{4}-2 e_{2} e_{3}-2 e_{0} e_{5}$ by Theorem 29. Consequently, $R(L)^{L}=k\left(f_{1}, f_{2}, f_{3}\right)$.

## 6. Dixmier's fourth problem

Let $L$ be a finite dimensional Lie algebra over an algebraically closed field $k$ of characteristic zero. Then we know that the field $Z(D(L))$ is isomorphic with $R(L)^{L}$ and hence is an extension of finite type of $k$. [RV, p.401], [D6, 10.5.6].
In his book Enveloping Algebras Dixmier raised the following problem [D6, p.354] and proved it for $L$ solvable (In fact he even showed it for $L$ completely solvable over an arbitrary field $k$ of characteristic zero [D6, Proposition 4.4.8]. It also holds for solvable $L$ over $k=\mathbb{R}[\mathrm{Be}]$ ).

## Problem 61.

Is $Z(D(L))$ rational over $k$ ? (i.e. is it a purely transcendental extension of $k$ ?).
To our knowledge this problem is still open. Notice that the Gelfand-Kirillov conjecture is a much stronger condition.
Obviously, Dixmier's question has a positive answer if $L$ is coregular without proper semi-invariants, since then $Z(D(L))$ is precisely the quotient field of $Z(U(L))$, the latter being polynomial. This is especially the case for $L$ semi-simple and also for the canonical truncation $\mathfrak{g}_{\Lambda}$ of a Frobenius Lie algebra $\mathfrak{g}\left(\right.$ since $Z\left(U\left(\mathfrak{g}_{\Lambda}\right)\right)=S z(U(\mathfrak{g}))$,
which is polynomial [DNO]). Dixmier's statement is also true for a Lie algebra $L$ for which $j(L)=\operatorname{dim} Z(L)$ (Theorem 37).

Proposition 62. Consider the semi-direct product $L=s l(2, k) \oplus W_{n}$, where $W_{n}$ is the $(n+1)$-dimensional irreducible $s l(2, k)$-module. Then $Z(D(L))$ is rational over k .

On the other hand, $L$ does not satisfy the Gelfand-Kirillov conjecture if $n$ is even and $n \geq 6$. [O6, Proposition 4.3].

Proof. We may assume that $n \geq 5$, since we verified it for $n=1,2,3,4$ (see $L_{5}, L_{6,1}, L_{7,1}, L_{8,1}$ of 5.1). Hence, $\operatorname{dim} \operatorname{sl}(2, k)=3<n+1=\operatorname{dim} W_{n}$. We now follow the same argument as in Example 59. By [AVE]

$$
s l(2, k)(f)=0 \text { for some } f \in W_{n}^{*}
$$

which by [O4, Proposition 17] implies that $i(L)=\operatorname{dim} W_{n}-\operatorname{dim} s l(2, k)=n+1-3=$ $n-2$ and

$$
Z(D(L))=R\left(W_{n}\right)^{s l(2, k)}=R\left(W_{n}\right)^{S L(2, k)}
$$

where the latter is rational over $k[\mathrm{BK}]$.

Next, Dixmier's result above combined with Theorem 53 yields: (since the rationality of $R(L)^{L}$ is preserved under taking direct products)

Proposition 63. Assume $L$ is a Lie algebra over $k$ of dimension at most 8. Then $Z(D(L))$ is rational over $k$.

Lemma 64. Let $L$ be an algebraic Lie algebra. Then there exists a torus $T \subset L$ such that $L=L_{\Lambda} \oplus T$ and a basis $t_{1}, \ldots, t_{r}$ of $T$ such that ad $t_{1}, \ldots$, ad $t_{r}$ have rational eigenvalues and such that $\lambda\left(t_{i}\right) \in \mathbb{Q}$ for all $\lambda \in \Lambda(L), i=1, \ldots, r$.

Proof. As $L$ is algebraic, so is ad $L$. By [C, p.324] $L$ admits the following decomposition

$$
L=S \oplus N \oplus A
$$

where $S$ is a semi-simple Lie subalgebra of $L, N$ is the nilradical of $L$ and $A$ is a torus of $L$ (i.e. an abelian Lie subalgebra of $L$ such that $\operatorname{ad}_{L} A$ consists of semi-simple elements). Also, $[A, S]=0, R=N \oplus A$ is the (solvable) radical of $L$ and $\operatorname{ad}_{L} A$ is
algebraic. It follows that also $R$ is algebraic [C, p.309].
By [Mc, Theorem 3.3] there is a basis $a_{1}, \ldots, a_{s}$ of $A$ such that $\operatorname{ad}_{N} a_{1}, \ldots, \operatorname{ad}_{N} a_{s}$ have rational eigenvalues. The same holds for $a d_{L} a_{1}, \ldots, a d_{L} a_{s}$ since $A$ is abelian and $[A, S]=0$. Because $S=[S, S]$ and each $x \in N$ acts locally nilpotent on $U(L)$, we see that $S \oplus N \subset L_{\Lambda}$. Hence, $L_{\Lambda}+A=L$. We may assume that $L_{\Lambda} \neq L$ (otherwise take $T=0$ ). Let $t_{1}$ be the first one among $a_{1}, \ldots, a_{s}$ such that $t_{1} \notin L_{\Lambda}$. Next, consider $L_{\Lambda} \oplus k t_{1} \subset L$ and so on. After a number of steps we obtain

$$
L_{\Lambda} \oplus\left\langle t_{1}, \ldots, t_{r}\right\rangle=L
$$

where $t_{1}, \ldots, t_{r}$ are linearly independent over $k$ and $\operatorname{ad}_{L} t_{1}, \ldots, \operatorname{ad}_{L} t_{r}$ have rational eigenvalues. So it suffices to put $T=\left\langle t_{1}, \ldots, t_{r}\right\rangle$. Let $x_{1}, \ldots, x_{n}$ be a basis of $L$ such that for all $i=1, \ldots, r ; j=1, \ldots, n$ : ad $t_{i}\left(x_{j}\right)=q_{i j} x_{j}$ for some $q_{i j} \in \mathbb{Q}$. Next, let $u \in U(L)$ be a nonzero semi-invariant with weight $\lambda$, which can be written as

$$
u=\sum_{m} \alpha_{m} x_{1}^{m_{1}} \ldots x_{n}^{m_{n}} \text { for some } \alpha_{m} \in k
$$

and $m=\left(m_{1}, \ldots, m_{n}\right)$. Now observe that

$$
\sum_{m} \lambda\left(t_{i}\right) \alpha_{m} x_{1}^{m_{1}} \ldots x_{n}^{m_{n}}=\lambda\left(t_{i}\right) u=\operatorname{ad} t_{i}(u)=\sum_{m}\left(\sum_{j=1}^{n} m_{j} q_{i j}\right) \alpha_{m} x_{1}^{m_{1}} \ldots x_{n}^{m_{n}}
$$

Now, select $m=\left(m_{1}, \ldots, m_{n}\right)$ such that $\alpha_{m} \neq 0$. Then we may conclude that $\lambda\left(t_{i}\right)=\sum_{j=1}^{n} m_{j} q_{i j} \in \mathbb{Q}, i=1, \ldots, r$.

## Lemma 65.

(1) $Z(D(L)) \subset Z\left(D\left(L_{\Lambda}\right)\right)$
(2) Assume $L$ is almost algebraic. Then the field $Z\left(D\left(L_{\Lambda}\right)\right)$ is generated by the semi-invariants of $U(L)$.
(3) If $L$ is algebraic then

$$
\operatorname{trdeg}_{k} Z\left(D\left(L_{\Lambda}\right)\right)-\operatorname{trdeg}_{k}\left(Z(D(L))=\operatorname{dim} L-\operatorname{dim} L_{\Lambda}\right.
$$

Proof. We recall that $S z(U(L)) \subset Z\left(U\left(L_{\Lambda}\right)\right)$ and equality occurs if $L$ is almost algebraic [DNO, Theorem 1.19].
(1) Because any nonzero $z \in Z(D(L))$ can be written as $z=u v^{-1}$, where $u, v$ are nonzero semi-invariants of $U(L)$ with the same weight [RV, Théorème 4.4], [DNO, Corollary 1.10], we deduce at once that

$$
Z(D(L)) \subset Q(S z(U(L))) \subset Q\left(Z\left(U\left(L_{\Lambda}\right)\right)\right)=Z\left(D\left(L_{\Lambda}\right)\right)
$$

the latter since $L_{\Lambda}$ has no proper semi-invariants by 2 of Theorem 4.
(2) By definition $S z(U(L))$ is generated by the semi-invariants of $U(L)$. Hence, the same holds for its quotient field $Q(S z(U(L)))=Q\left(Z\left(U\left(L_{\Lambda}\right)\right)\right)=Z\left(D\left(L_{\Lambda}\right)\right)$.
(3) If $L$ is algebraic then so is $L_{\Lambda}$ [DNO, Proposition 1.14]. By 4 of Theorem 4 we know that $c\left(L_{\Lambda}\right)=c(L)$. Therefore

$$
\operatorname{dim} L_{\Lambda}+i\left(L_{\Lambda}\right)=2 c\left(L_{\Lambda}\right)=2 c(L)=\operatorname{dim} L+i(L)
$$

and hence $i\left(L_{\Lambda}\right)-i(L)=\operatorname{dim} L-\operatorname{dim} L_{\Lambda}$.
On the other hand,

$$
i(L)=\operatorname{trdeg}_{k} Z(D(L)) \quad \text { and } \quad i\left(L_{\Lambda}\right)=\operatorname{trdeg}_{k} Z\left(D\left(L_{\Lambda}\right)\right)
$$

by Theorem 1 .
The following is the main result of this section. It proved to be a useful tool in obtaining the explicit description of $R(L)^{L}$ in the list of 5.1.

Theorem 66. Let $L$ be an algebraic Lie algebra for which the field $Z\left(D\left(L_{\Lambda}\right)\right)$ is freely generated by semi-invariants $u_{1}, \ldots, u_{s}$ of $U(L)$. Then $Z(D(L))$ (and also $\left.R(L)^{L}\right)$ is rational over $k$.

Proof. By Lemma 64 there is a torus $T \subset L$ such that $L=L_{\Lambda} \oplus T$ and a basis $t_{1}, \ldots, t_{r}$ of $T$ such that for all $i=1, \ldots, r ; j=1, \ldots, s$ :

$$
\begin{equation*}
\text { ad } t_{i}\left(u_{j}\right)=a_{i j} u_{j} \text { for some } a_{i j} \in \mathbb{Q} \tag{*}
\end{equation*}
$$

We may assume that $a_{i j} \in \mathbb{Z}$, since we can replace $t_{i}$ by a suitable integer multiple of itself. Put $A=\left(a_{i j}\right) \in \mathbb{Z}^{r \times s}$ and $d_{i}=\operatorname{ad} t_{i}$. For any $m=\left(m_{1}, \ldots, m_{s}\right) \in \mathbb{Z}^{s}$ we set

$$
u^{m}=u_{1}^{m_{1}} \ldots u_{s}^{m_{s}}
$$

Clearly, for all $i=1, \ldots r$

$$
d_{i}\left(u^{m}\right)=\operatorname{ad} t_{i}\left(u^{m}\right)=\left(\sum_{j=1}^{s} a_{i j} m_{j}\right) u^{m}=(A m)_{i} u^{m}
$$

Next, we consider

$$
\begin{aligned}
K & =\left\{m=\left(m_{1}, \ldots, m_{s}\right) \in \mathbb{Z}^{s} \mid d_{i}\left(u^{m}\right)=0, i=1, \ldots, r\right\} \\
& =\left\{m \in \mathbb{Z}^{s} \mid A m=0\right\}
\end{aligned}
$$

$K$ is a free $\mathbb{Z}$-module, being a submodule of the free $\mathbb{Z}$-module $\mathbb{Z}^{s}$.
Let $k_{1}, \ldots, k_{q}$ be a basis of $K$, then

$$
z_{1}=u^{k_{1}}, \ldots, z_{q}=u^{k_{q}} \in Z(D(L))
$$

(since they are annihilated by both ad $L_{\Lambda}$ and ad $T$ )

## Claim:

(1) $z_{1}, \ldots, z_{q}$ are algebraically independent over $k$.
(2) $Z(D(L))=k\left(z_{1}, \ldots, z_{q}\right)$
(1) By assumption $u_{1}, \ldots, u_{s}$ are algebraically independent over $k$, which is equivalent with the fact that
$u^{m}=u_{1}^{m_{1}} \ldots u_{s}^{m_{s}}, m=\left(m_{1}, \ldots, m_{s}\right) \in \mathbb{N}^{s}$ (and even $m \in \mathbb{Z}^{s}$ ) are linearly independent over $k(* *)$. Similarly, we have to show that $z_{1}^{m_{1}} \ldots z_{q}^{m_{q}}$, $m=\left(m_{1}, \ldots, m_{q}\right) \in \mathbb{N}^{q}$ are linearly independent over $k$. Clearly,

$$
z_{1}^{m_{1}} \ldots z_{q}^{m_{q}}=\left(u^{k_{1}}\right)^{m_{1}} \ldots\left(u^{k_{q}}\right)^{m_{q}}=u^{\sum m_{i} k_{i}}
$$

and these are indeed linearly independent over $k$ by $(* *)$, since $k_{1}, \ldots, k_{q}$ are linearly independent over $\mathbb{Z}$.
(2) We already know that $k\left(z_{1}, \ldots, z_{q}\right) \subset Z(D(L))$. On the other hand, take $0 \neq z \in Z(D(L))$. Then $z \in Z\left(D\left(L_{\Lambda}\right)\right)=k\left(u_{1}, \ldots, u_{s}\right)$ by (1) of the previous lemma. So, $z=v w^{-1}$, where $v, w$ are nonzero coprime elements of the polynomial algebra $k\left[u_{1}, \ldots, u_{s}\right]$ in the variables $u_{1}, \ldots, u_{s}$. We consider the degree of $v$ and $w$ with respect to these variables. Obviously $v w=w v$ and thus $z=v w^{-1}=w^{-1} v$. Because $z \in Z(D(L))$ it is annihilated by each derivation $d_{i}=\operatorname{ad} t_{i}, i=1, \ldots, r$. Therefore,

$$
z d_{i}(w)=d_{i}(z) w+z d_{i}(w)=d_{i}(z w)=d_{i}(v)
$$

and so $v d_{i}(w)=w d_{i}(v)$. This implies that $v$ divides $d_{i}(v)$ as $v$ and $w$ are coprime. But $\operatorname{deg}\left(d_{i}(v)\right) \leq \operatorname{deg}(v)$ (by $(*)$ ) and thus $d_{i}(v)=\lambda_{i} v$ for a suitable $\lambda_{i} \in k$. It follows that $d_{i}(w)=\lambda_{i} w$. We can find nonzero $a_{m} \in k$ and $M \subset \mathbb{N}^{s}$ such that $v=\sum_{m \in M} a_{m} u^{m}$. Similarly, $w=\sum_{n \in N} b_{n} u^{n}$. From

$$
\sum_{m \in M} a_{m}(A m)_{i} u^{m}=\sum_{m \in M} a_{m} d_{i}\left(u^{m}\right)=d_{i}(v)=\lambda_{i} v=\sum_{m \in M} a_{m} \lambda_{i} u^{m}
$$

we obtain $(A m)_{i}=\lambda_{i}, m \in M, i=1, \ldots, r$.
Similarly, $(A n)_{i}=\lambda_{i}, n \in N, i=1, \ldots, r$. Hence,
$(A(n-m))_{i}=(A n)_{i}-(A m)_{i}=0$.
Consequently, $A(n-m)=0$ and thus $n-m \in K$ for all $n \in N, m \in M$.
We can find $\alpha_{i} \in \mathbb{Z}$ such that $n-m=\sum_{i=1}^{q} \alpha_{i} k_{i}$. Therefore,

$$
u^{n-m}=u^{\sum \alpha_{i} k_{i}}=\left(u^{k_{1}}\right)^{\alpha_{1}} \ldots\left(u^{k_{q}}\right)^{\alpha_{q}}=z_{1}^{\alpha_{1}} \ldots z_{q}^{\alpha_{q}} \in k\left(z_{1}, \ldots, z_{q}\right)
$$

Finally,

$$
\begin{aligned}
z=v w^{-1} & =\left(\sum_{m} a_{m} u^{m}\right)\left(\sum_{n} b_{n} u^{n}\right)^{-1}=\sum_{m} a_{m}\left(u^{-m} \sum_{n} b_{n} u^{n}\right)^{-1} \\
& =\sum_{m} a_{m}\left(\sum_{n} b_{n} u^{n-m}\right)^{-1} \in k\left(z_{1}, \ldots, z_{q}\right)
\end{aligned}
$$

This establishes the claim, i.e. $Z(D(L))$ is rational over $k$.

The following result by Panyushev [Pa1] can now be derived from Joseph's work on biparabolics.

Corollary 67. Let $L$ be a biparabolic (seaweed) subalgebra of a simple Lie algebra of type $A$ or $C$. Then $R(L)^{L}$ (and hence also $Z(D(L))$ ) is rational over $k$.

Proof. $L$ is algebraic [F, 6.4]. By [J3, J4] $Y\left(L_{\Lambda}\right)$ is freely generated by some semi-invariants of $S(L)$, say $v_{1}, \ldots, v_{s}$. By the Duflo isomorphism the same holds for $Z\left(U\left(L_{\Lambda}\right)\right)$ and also for its quotient field, which is $Z\left(D\left(L_{\Lambda}\right)\right)$. By Theorem 66 $Z(D(L))$ is rational over $k$.

Remark 68. Most (but not all $[\mathrm{Y}]$ ) (bi)parabolic subalgebras of semi-simple Lie
algebras have canonical truncations whose Poisson centers are freely generated by semi-invariants [F, FJ, FJ2, J3, J4]. Hence, we may then draw the same conclusion as above.
Let $L$ be a finite dimensional algebraic Lie algebra over $k$ and $G$ its algebraic adjoint group. $G$ and $L$ act on $L^{*}$ via the coadjoint action. We identify $R(L)$ with the field of rational functions on $L^{*}$.

Definition 69. (See e.g. [TY1]) An affine slice of $L$ is an affine subspace $V$ of $L^{*}$ such that there exists an open subset $U$ of $V$ verifying the following conditions:
(1) The set G.U is dense in $L^{*}$
(2) $T_{f}(G . f) \cap T_{f}(U)=\{0\}$ for all $f \in U$
(3) $G$.f $\cap U=\{f\}$ for all $f \in U$

Such an affine slice exists for the coadjoint action for certain truncated [J6, J7] and non truncated [TY1] biparabolic subalgebras of a semi-simple Lie algebra.

Theorem 70. (Tauvel, Yu [TY1, Theorem 3.3.1])
Let $L$ be algebraic with algebraic adjoint group $G$. Suppose there exists an affine slice for the coadjoint action of $L$. Then $R(L)^{G}$ is rational aver $k$. Hence the same holds for $Z(D(L))$ (since the latter is isomorphic to $R(L)^{G}$ by [RV, 4.5]).

## 7. Appendix

Example 71. Let $L$ be the nonalgebraic Lie algebra over $\mathbb{C}$ with basis $x, y, z, t$ and nonzero brackets.

$$
[x, y]=y, \quad[x, z]=\alpha z, \quad[y, z]=t, \quad[x, t]=(1+\alpha) t
$$

with $\alpha$ irrational.
This example was introduced in [GK, p.522] in order to demonstrate the existence of nonalgebraic Lie algebras satisfying the Gelfand-Kirillov conjecture. However the proof is incorrect, as the given Weyl generators of $D(L)$, namely

$$
p_{1}=y t^{-1}, \quad q_{1}=z, \quad p_{2}=(1+\alpha)^{-1} t, \quad q_{2}=y z t^{-2} x t^{-1}
$$

do not satisfy the necessary requirements, for instance $\left[q_{1}, q_{2}\right]=-((x+1) t+\alpha y z) z t^{-3} \neq 0$. Probably a term is missing in $q_{2}$.
We now present a very short proof. First we observe that $L$ can be considered as the semi-direct product $\mathfrak{g} \oplus W$ of the Lie algebra $\mathfrak{g}=\langle x, y\rangle,[x, y]=y$, with its representation space $W=\langle z, t\rangle$. Since $L$ is Frobenius $\left(\Delta(L)=(1+\alpha)^{2} t \neq 0\right)$ we see that

$$
i(L)=0=\operatorname{dim} W-\operatorname{dim} \mathfrak{g}
$$

Then $L$ satisfies the Gelfand-Kirillov conjecture by Theorem 1.1 combined with Proposition 2.1 of [O6].

We conclude by producing explicitly a set of Weyl generators of $D(L)$ as follows: As $W=\langle z, t\rangle$ we have $R(W)=\mathbb{C}(z, t)$ and $R(W)^{\mathfrak{g}}=Z(D(L))=\mathbb{C}$. Next, we put $q_{1}=z, q_{2}=t$. By the proof of [O6, Theorem 1.1] $p_{1}, p_{2}$ are the solutions of the following system of equations:

$$
x=\left[x, q_{1}\right] p_{1}+\left[x, q_{2}\right] p_{2} \quad y=\left[y, q_{1}\right] p_{1}+\left[y, q_{2}\right] p_{2}
$$

which simplifies to

$$
x=\alpha z p_{1}+(1+\alpha) t p_{2} \quad \text { and } \quad y=t p_{1}
$$

Hence, $p_{1}=t^{-1} y$ and $p_{2}=(1+\alpha)^{-1} t^{-1}\left(x-\alpha z t^{-1} y\right)$.

Then, $p_{1}, p_{2}, q_{1}, q_{2}$ form a set of Weyl generators over $\mathbb{C}$ of $D(L)$ by the proof of $\left[\right.$ O6, Theorem 1.1]. Hence, $D(L) \cong D_{2}(\mathbb{C})$.

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