Acknowledgements

The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful.

Henri Poincaré (Mathematician)

Studying the abstract branch of mathematics known as (noncommutative) algebraic geometry has been a fascinating journey for the last four years. Many people might have prejudices about the usefulness of such exact research. This is rather unfortunate, because abstract results often lie at the heart of applications at a later point in time. 3D-computer animations, the Google Pagerank algorithm and Global Positioning Systems are just a few examples of inventions which would not exist without abstract algebraic results invented decades, or even centuries before the application. Therefor I am grateful to everyone who supports and agrees with the importance of abstract research. Without them, this thesis would not be possible. In particular I am very grateful for the opportunities the research foundation Flanders (FWO) and the University of Hasselt have given me through their financial and logistic support.

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Overview

This thesis serves as an investigation of noncommutative surfaces. In particular we consider the construction of noncommutative surfaces as well as birational transformations between them. Many of these noncommutative surfaces will be noncommutative del Pezzo surfaces. As such, the work in this thesis focuses on noncommutative algebraic geometry, a field of mathematics which tries to generalize algebraic geometry to noncommutative algebra. Although there is no naïve way of generalizing geometric objects like varieties to a noncommutative setting, it is still possible to carry across some of the techniques and intuition. For this, one investigates (algebraic) constructions which are closely related to geometry and which are amenable to noncommutative generalizations. This gives rise to essentially two ways of doing noncommutative algebraic geometry.

One approach is to investigate rings which mimic the behavior of homogeneous coordinate rings. The study of Artin-Schelter regular algebras as introduced in [AS87] is central in this approach. These algebras have been classified for GKdim $= 2, 3$ [AS87, ATVdB90, ATVdB91, Ste96] and they are closely related to so called twisted homogeneous coordinate rings [AVdB90]. For example, all connected graded domains of Gelfand-Kirillov dimension 2 which are finitely generated in degree 1, are in sufficiently high degrees isomorphic to twisted homogeneous coordinate rings ([AS95, Theorem 0.2]) and hence from a categorical point of view, these are no different from commutative curves ([AVdB90, Theorem 1.3]). Hence the first "nontrivial" class of AS-regular algebras, are those of Gelfand-Kirillov dimension 3. In [Art97] Artin posed a conjecture classifying noetherian, connected graded domains of Gelfand–Kirillov dimension 3 up to birational equivalence. This conjecture is still far from being proved.

In Chapter 1 we make a small step towards understanding Artin's conjecture, by investigating birational equivalences between AS-regular algebras of Gelfand–Kirillov dimension 3. In particular we show that quadratic and cubic Sklyanin algebras[1]

[1]Sklyanin algebras were introduced and investigated in [Skl82, Skl83, OF89]. Genererically AS-regular algebras of Gelfand–Kirillov dimension 3 are Sklyanin algebras

(which serve as noncommutative $\mathbb{P}^2$ and noncommutative $\mathbb{P}^1 \times \mathbb{P}^1$) have isomorphic function fields (Theorem 1.4.1). For this we construct noncommutative versions of the classical birational transformations $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ and the Cremona transform $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Moreover we show that the latter factors as a composition of the first two (Theorem 1.6.4) and that all such noncommutative transformations are invertible (Proposition 1.7.4 and Theorem 1.7.6). Finally in §1.11 and §1.12 we relate our constructions to the noncommutative blowups as introduced in [RSS14, VdB01].

In Chapter 2 up to Chapter 5 we focus on another approach of doing noncommutative algebraic geometry, namely a categorical approach. This approach is inspired by the Gabriel-Rosenberg reconstruction theorem ([Bra13, Gab62, Ros98]) which states that a scheme is uniquely defined (up to isomorphism) by it's category of quasicoherent sheaves (up to equivalence of categories). I.e. there is a 1-1-correspondence

$$\{\text{schemes } X\}/\cong \xleftrightarrow{1-1} \{\text{categories } \operatorname{Qcoh}(X)\}/\cong$$

As the category of quasicoherent sheaves on a scheme is always Grothendieck [EE05], one could think of each Grothendieck category as (a representation of) a noncommutative scheme. This formalism was introduced by Van den Bergh in [VdB01] and is covered briefly in §1.2.

As one wants to understand noncommutative varieties instead of arbitrary noncommutative schemes, one often imposes conditions on these Grothendieck categories (or their derived categories). For example, [dTdV16] provides a numerical classification of possible *noncommutative smooth projective surfaces* with an exceptional sequence of length 4. Their numerical classification gives rise to two types. One type corresponding to $\mathbb{P}^1 \times \mathbb{P}^1$ and a another type describing a family K_n indexed by the natural numbers. For $n = 0$ and 1 there are commutative models: $\mathbb{P}^2 \cup \{\bullet\}$ and $\mathbb{F}_1 = \operatorname{Bl}_x(\mathbb{P}^2)$. For $n \geq 2$ however there exist no commutative models. Using the degree δ as in op.cit. the case $n = 2$ should correspond to a noncommutative del Pezzo surface.

In Chapter 3 we exhibit the theory of noncommutative $\mathbb{P}^1$-bundles as defined in [VdB12] to construct a *symmetric sheaf* $\mathbb{Z}$*-algebra* $\mathbb{S}(\mathcal{E})$ and an exceptional collection

$$\left(\Pi_1^*(\mathcal{O}_{\mathbb{P}^1}), \Pi_1^*(\mathcal{O}_{\mathbb{P}^1}(1)), \Pi_0^*(\mathcal{O}_{\mathbb{P}^1}), \Pi_0^*(\mathcal{O}_{\mathbb{P}^1}(1)\right)$$

on the (derived) category of graded $\mathbb{S}(\mathcal{E})$-modules corresponding to the numerical type K_2 in [dTdV16]. Moreover we prove that $\mathbb{S}(\mathcal{E})_{i,j}$ is locally free (as a bimodule) of a finite, explicitely known ranks (Corollary 3.4.5) and that $\operatorname{Gr}(\mathbb{S}(\mathcal{E}))$ is a locally noetherian category (Theorem 3.3.1). We prove these results by reducing to an affine setting (Lemma 3.3.13 and Lemma 3.3.22) where the results on $\mathbb{S}(\mathcal{E})$ follow by similar results on so-called generalized preprojective algebras. These algebras are of independent interest and are the topic of the second chapter of this thesis. I.e. in Chapter 2 we generalize the notion of a Frobenius algebra over a field to a relative Frobenius pair of commutative rings S/R. To such a relative Frobenius pair we associate a generalized preprojective algebra $\Pi_R(S)$. This construction produces an R-family of algebras which gives the classical preprojective algebra associated to the quiver with one central vertex and n outgoing arrows at the geometric generic fibre $S = R^{\oplus n}$. Our main focus lies on the situation where S has rank 4 over R. In this situation we can prove the following:

- for each $d \in \mathbb{N}$: $(\Pi_R(S))_d$ is a projective R-module of finite, explicitely known rank (Theorem 2.3.9)
- $\left(\mathrm{Z}(\Pi_R(S))\right)_d$ is a split submodule of $\Pi_R(S)_d$ for each $d \in \mathbb{N}$; as such it is projective of finite, explicitely known rank (Theorem 2.4.1, Theorem 2.4.2 and Theorem 2.4.3)
- $\Pi_R(S)$ is finite over its center and it is noetherian if R is noetherian. (Theorem 2.5.1)
- If R and S have finite global dimension, then so does $\Pi_R(S)$ and we have an explicit upper bound for $\mathrm{gl.\,dim}(\Pi_R(S))$ (Theorem 2.6.1)

In Chapter 4 we return to the numerical classification from [dTdV16] as described above and construct noncommutative models for K_n for all $n \geq 2$. For this construction we start from a quadratic AS-regular algebra which is finite over its center (by [ATVdB90, Theorem 7.1] these algebras are well understood). Such an algebra induces a maximal order $\mathcal{S}$ on $\mathbb{P}^2$ (Lemma 4.5.1). If $x \in \mathbb{P}^2$ is any point outside of the ramification divisor of $\mathcal{S}$, we can pullback $\mathcal{S}$ to $\mathrm{Bl}_x \mathbb{P}^2$ to find a maximal order on $\mathbb{F}_1$. We then generalize Orlovs blowup formula [Orl92, Theorem 4.3] to our setting of maximal orders (Theorem 4.5.11) and use this to show that $\mathrm{coh}(p^*\mathcal{S})$ admits a full and strong exceptional sequence

$$(E_1, E_2, E_3, E_4) = (p^*\mathcal{S}_0, p^*\mathcal{S}_1, p^*\mathcal{S}_2, p^*\mathcal{F})$$

whose images form a basis of $\mathrm{K}_0(Z_m) \cong \mathbb{Z}^4$ for which the Gram matrix is given by a matrix K'_m (Theorem 4.5.7). Under the action of the signed braid group $\Sigma\mathrm{B}_4$ we can mutate K'_m into K_m (Proposition 4.2.4). We have hence obtained noncommutative models for all K_m appearing in the numerical classification of [dTdV16].

Finally in Chapter 5 we compare the noncommutative model for K_2 constructed in Chapter 3 with the noncommutative model for K'_2 constructed in Chapter 4 using the theory of Clifford algebras. In particular we show (Corollary 5.5.4 together with

Lemma 3.3.6) that the noncommutative surfaces constructed in Chapters 3 and 4 are isomorphic. I.e. we show that for every noncommutative $\mathbb{P}^1$-bundle $\operatorname{qgr}(\mathbb{S}(\mathcal{E}))$ on $\mathbb{P}^1$ there is a quaternionic order $\operatorname{coh}(p^*\mathcal{S})$ on $\mathbb{F}_1$ such that

$$\operatorname{qgr}(\mathbb{S}(\mathcal{E})) \cong \operatorname{coh}(p^*\mathcal{S})$$

and vice versa. As such we give a noncommutative version of the classical isomorphism $\underline{\operatorname{Proj}}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \cong \operatorname{Bl}_x \mathbb{P}^2$. Moreover (see Table 5.4) we relate this comparison to the classification of pencils of binary quartics and nets of conics as in [Wal77, Wal98].

Nederlandstalige samenvatting

Het hoofdonderwerp van deze thesis zijn zogenaamde niet-commutatieve oppervlakken. In het bijzonder bestuderen we de constructie van en birationale transformaties tussen niet-commutatieve oppervlakken, die tevens vaak niet-commutatieve del Pezzo oppervlakken zijn. Zodoende valt deze thesis in het domein van de niet-commutatieve algebraïsche meetkunde, een domein waarin resultaten uit algbraïsche meetkunde vertaald worden naar niet-commutatieve algebra.

Hoewel er geen naïeve manier is om meetkundige objecten (zoals variëteiten) te veralgemenen naar een niet-commutatieve setting, is het toch mogelijk om veel van de technieken en intuïtie van commutatieve algebraïsche meetkunde uit te buiten in dit onderzoeksdomein. De reden daarvoor is dat er een aantal algebraïsche constructies zijn die erg gerelateerd zijn aan de meetkundige constructies en die wél een duidelijke veralgemening toelaten naar niet-commutatieve settings. Essentieel ontstaan er zo twee manieren om aan niet-commutatieve algebraïsche meetkunde te doen.

De eerste aanpak bestudeert ringen waarvan de eigenschappen gelijkaardig zijn aan die van homogene coördinaatringen. De studie van Artin-Schelter reguliere algebra's, zoals geïntroduceerd in [AS87] staat centraal in deze aanpak. Deze algebra's zijn geclassificeerd voor Gelfand–Kirillov dimensie 2 en 3 [AS87, ATVdB90, ATVdB91, Ste96] en ze staan sterk in verband met zogenaamde getwiste homogene coördinaatringen [AVdB90]. Zo zijn, op een eindig-dimensionale vectorruimte na, alle geconnecteerde, gegradeerde domeinen van Gelfand–Kirillov dimensie 2 isomorf met een getwiste homogene coördinaatring ([AS95, Stelling 0.2]). Zodoende zijn deze vanuit een categorisch standpunt niet te onderscheiden van commutatieve krommen ([AVdB90, Stelling 1.3]). De eerste "niet-triviale" klasse van AS-reguliere algebra's die we dus zouden kunnen bestuderen, zijn deze van Gelfand-Kirillov dimensie 3. In [Art97] voorspelde Artin een classificatie van noetherse, geconnecteerde, gegradeerde domeinen van Gelfand–Kirillov dimensie 3 tot op birationale equivalentie. Tot op heden is dit vermoeden nog niet aangetoond.

In Hoofdstuk 1 wordt een kleine stap gemaakt met het oog op het begrijpen van

Artin's vermoeden: we bestuderen er birationale equivalenties tussen AS-reguliere algebra's van Gelfand–Kirillov dimensie 3. In het bijzonder tonen we aan dat kwadratische en cubische Sklyanin algebras[2] (die we kunnen beschouwen als niet-commutatieve $\mathbb{P}^2$'s en $\mathbb{P}^1 \times \mathbb{P}^1$'s) isomorfe functie-lichamen hebben (Stelling 1.4.1). Om dit resultaat te bekomen, construeren we niet-commutatieve versies van de klassieke birationale transformaties $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ en de Cremona transformatie $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Bovendien tonen we aan dat deze laatste te schrijven is als een samenstelling van de eerste twee (Stelling 1.6.4) en dat iedere niet-commutatieve birationale transformatie die op deze manier bekomen wordt, inverteerbaar is (Propositie 1.7.4 en Stelling 1.7.6). Ten slotte relateren we in §1.11 en §1.12 onze constructies aan de niet-commutieve *blowups* zoals geïntroduceerd in [RSS14, VdB01].

In Hoofstukken 2 tot en met 5 ligt onze focus op een andere manier om aan niet-commutatieve meetkunde te doen, namelijk een categorische manier. Deze aanpak is geïnspireerd door de reconstructie stelling van Gabriel en Rosenberg ([Bra13, Gab62, Ros98]) die zegt dat een schema (op isomorfisme na) uniek bepaald is door de geassocieerde categorie van quasicoherente schoven (waarbij we die laatste slechts op equivalentie van categorieën na moeten kennen). Met andere woorden, er is een 1-1-correspondentie

$$\{\text{schema's } X\}/\cong \xleftrightarrow{1-1} \{\text{categorieën } \operatorname{Qcoh}(X)\}/\cong$$

Gezien de categorie van quasicoherente schoven op een schema altijd een Grothendieck categorie is [EE05], kan men over elke Grothendieck categorie denken als zijnde een niet-commutatief schema. Dit formalisme werd ingevoerd door Van den Bergh in [VdB01] en is kort samengevat in §1.2.

Gezien onze focus ligt op niet-commutatieve variëteiten in de plaats van de meer algemene niet-commutatieve schema's, legt men vaak extra voorwaarden op deze Grothendieck categorieën (of hun afgeleide categorieën). Zo geeft [dTdV16] bijvoorbeeld een numerieke classificatie van mogelijke *niet-commutatieve gladde projectieve oppervlakken* met een exceptionele collectie van lengte 4. De numerieke classificatie die door hun bekomen wordt, geeft aanleiding tot twee types van niet-commutatieve gladde projectieve oppervlakken. Het ene type correspondeert met $\mathbb{P}^1 \times \mathbb{P}^1$, het andere type is een familie K_n geïndexeerd door de strikt-positieve gehele getallen. Voor $n = 0$ en $n = 1$ bestaan er commutatieve modellen: $\mathbb{P}^2 \cup \{\bullet\}$ en $\mathbb{F}_1 = \mathrm{Bl}_x(\mathbb{P}^2)$. Voor $n \geq 2$ bestaan er echter geen commutatieve modellen meer. Gebruik makende van een *graad* δ zoals in op.cit., weten we dat het geval $n = 2$ een niet-commutatief del Pezzo oppervlak zou moeten geven.

[2]Sklyanin algebra's zijn geïntroduceerd en bestudeer in onder andere [Skl82, Skl83, OF89]. Generiek gezien is elke AS-reguliere algebra van Gelfand–Kirillov dimensie 3 een Sklyanin algebra.

In Hoofdstuk 3 gebruiken we de theorie van niet-commutatieve $\mathbb{P}^1$-bundels zoals in [VdB12] om een *symmetrische schoof* $\mathbb{Z}$*-algebra* $\mathbb{S}(\mathcal{E})$ te construeren, tesamen met een exceptionele collectie

$$\left(\Pi_1^*(\mathcal{O}_{\mathbb{P}^1}), \Pi_1^*(\mathcal{O}_{\mathbb{P}^1}(1)), \Pi_0^*(\mathcal{O}_{\mathbb{P}^1}), \Pi_0^*(\mathcal{O}_{\mathbb{P}^1}(1)\right)$$

op de (afgeleide) categorie van gegradeerde $\mathbb{S}(\mathcal{E})$-modules. Deze data geven een (niet-commutatief) model voor het numerieke type K_2 van de classificatie uit [dTdV16]. Daarnaast tonen we aan dat elke $\mathbb{S}(\mathcal{E})_{i,j}$ lokaal vrij is (als bimoduul) van eindige, expliciet gekende rang (Gevolg 3.4.5) en dat $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$ een lokaal noetherse categorie is (Stelling 3.3.1). Het bewijs van al deze resultaten steunt op een reductie naar een affiene setting (Stelling 3.3.13 en Lemma 3.3.22) waar de resultaten over $\mathbb{S}(\mathcal{E})$ een gevolg zijn van gelijkaardige resultaten over zogenaamde *veralgemeende preprojectieve algebra's*. Deze algebra's zijn nog voor tal van andere redenen interessant en vormen het onderwerp voor het tweede hoofdstuk van deze thesis.

Zodoende veralgemenen we in Hoofdstuk 2 de notie van Frobenius algebra over een lichaam tot relatieve Frobenius paren van commutatieve ringen S/R. Aan zo een relatief Frobenius paar associëren we een veralgemeende preprojectieve algebra $\Pi_R(S)$. Deze constructie geeft een R-familie van algebra's waarbij de klassieke preprojectieve algebra geassocieerd aan de quiver met een centrale knoop en n vertrekkende pijlen verschijnt in de meetkundige vezel $S = R^{\oplus n}$. We focussen ons vooral op de situaties waar S rang 4 heeft over R. In die situatie bekomen we de volgende resultaten:

- Voor elke $d \in \mathbb{N}$ is $(\Pi_R(S))_d$ een projectieve R-module van eindige, expliciet gekende rang (Stelling 2.3.9)

- $\left(\mathrm{Z}(\Pi_R(S))\right)_d$ is een gesplitste deelmodule van $\Pi_R(S)_d$ voor elke $d \in \mathbb{N}$; zodoende is $\left(\mathrm{Z}(\Pi_R(S))\right)_d$ projectief van eindige, expliciet gekende rang (Lemma 2.4.1, Lemma 2.4.2 en Stelling 2.4.3)

- $\Pi_R(S)$ is eindig over zijn centrum en is noethers wanneer R noethers is. (Stelling 2.5.1)

- Als R en S eindige globale dimensie hebben, dan ook $\Pi_R(S)$. We hebben in dat geval een expliciete bovengrens voor $\mathrm{gl.dim}(\Pi_R(S))$ (Stelling 2.6.1)

In Hoofdstuk 4 keren we terug naar de numerieke classificatie uit [dTdV16] en construeren niet-commutatieve modellen voor K_n voor elke $n \geq 2$. In deze constructie starten we met een kwadratische AS-reguliere algebra die eindig is over zijn centrum (dankzij [ATVdB90, Stelling 7.1] zijn deze algebra's bijzonder goed begrepen). Zo een algebra induceert een maximaal order $\mathcal{S}$ op $\mathbb{P}^2$ (Lemma 4.5.1). Als $x \in \mathbb{P}^2$ een punt is dat niet op de ramificatie divisor van $\mathcal{S}$ ligt, dan kunnen we $\mathcal{S}$ terugtrekken tot $\mathrm{Bl}_x \mathbb{P}^2$, waar het een maximaal order geeft op $\mathbb{F}_1$. We veralgemenen dan de *blowup formula* van Orlov [Orl92, Stelling 4.3] naar onze setting van maximale orders (Stelling 4.5.11) en gebruiken dit om aan te tonen dat $\mathrm{coh}(p^*\mathcal{S})$ een volle en sterke, exceptionele collectie heeft:

$$(E_1, E_2, E_3, E_4) = (p^*\mathcal{S}_0, p^*\mathcal{S}_1, p^*\mathcal{S}_2, p^*\mathcal{F})$$

De beelden van deze objecten vormen een basis voor $\mathrm{K}_0(Z_m) \cong \mathbb{Z}^4$ waarbij de Gram matrix gegeven is door een matrix K'_m (Stelling 4.5.7). De actie van de *signed braid group* $\Sigma\mathrm{B}_4$ laat ons toe de matrix K'_m te muteren tot K_m (Propositie 4.2.4). Zodoende bekomen we niet-commutatieve modellen voor alle K_m die verschijnen in de numerieke classificatie van [dTdV16].

Tot slot vergelijken we in Hoofdstuk 5 de niet-commutatieve modellen voor K_2 zoals geconstrueerd in Hoofdstukken 3 en 4. In het bijzonder tonen we aan (Gevolg 5.5.4 tesamen met Lemma 3.3.6) dat de niet-commutatieve oppervlakken geconstrueerd in Hoofstukken 3 en 4 isomorf zijn. We tonen immers aan dat er voor elke niet-commutatieve $\mathbb{P}^1$-bundel $\mathrm{qgr}(\mathbb{S}(\mathcal{E}))$ op $\mathbb{P}^1$ een quaternion order $\mathrm{coh}(p^*\mathcal{S})$ op $\mathbb{F}_1$ is zodat

$$\mathrm{qgr}(\mathbb{S}(\mathcal{E})) \cong \mathrm{coh}(p^*\mathcal{S})$$

en vice versa. Zodoende geven we een niet-commutatieve versie van het klassieke isomorfisme $\underline{\mathrm{Proj}}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \cong \mathrm{Bl}_x \mathbb{P}^2$. Tot slot relateren we in Tabel 5.4 onze vergelijking met de classificatie van "pencils of binary quartics" met die van "nets of conics" zoals in [Wal77, Wal98].

Chapter 0

Preliminaries

To be ignorant of what occurred before you were born is to remain always a child.

Marcus Tullius Cicero ((Roman) politician)

In noncommutative algebraic geometry one tries to generalize schemes or rather varieties to a more general (noncommutative) setting. A priori it is not clear how one should do this, because a variety is something like a curve or a surface and these geometric objects do not have obvious generalizations. Instead we look at closely related notions. As a first approach, recall that by the Gabriel-Rosenberg reconstruction theorem ([Bra13, Gab62, Ros98]), a scheme is uniquely determined (up to isomorphism) by its category of quasicoherent sheaves (up to equivalence of categories). I.e. there is a 1-1-correspondence

$$\{\text{schemes } X\}/\cong \stackrel{1-1}{\longleftrightarrow} \{\text{categories } \operatorname{Qcoh}(X)\}/\cong$$

As the category of quasicoherent sheaves on a scheme is always Grothendieck [EE05], as a first approach we could simply state that a noncommutative scheme is a Grothendieck category. So we can think of any Grothendieck category as being the category of quasicoherent sheaves on some object we cannot always describe directly. This formalism was introduced by Van den Bergh in [VdB01] and it was shown that in this context it is possible to blow up points on (certain) noncommutative surfaces. As most of the noncommutative surfaces we encounter have a more down-to-earth construction, we refer the interested reader to op. cit. or [Smi00] for the details of this formalism. Some, but not all, of these details will also be covered in §1.2.

A large class of noncommutative schemes one encounters are noncommutative projective varieties induced by well-behaved graded rings. This is inspired by the correspondence between projective varieties and their (commutative) homogeneous coordinate rings. Of particular interest is the following theorem by Serre:

Theorem (Serre's Theorem on projective schemes)**.** *Let A be a $\mathbb{N}$-graded, commutative algebra, (finitely) generated by A_1,then there is an (explicit) equivalence of categories:*

$$\mathrm{Qcoh}(\mathrm{Proj}(A)) \cong \mathrm{QGr}(A) := \mathrm{Gr}(A)/\,\mathrm{Tors}(A)$$

Where torsion A-modules are colimits of right-bounded A-modules.

As the latter category also makes sense for noncommutative k-algebras, it makes sense to consider noncommutative versions of homogeneous coordinate rings and the corresponding category $\mathrm{QGr}(A)$ will be a noncommutative scheme as in the above formalism. Moreover for a large class of algebras one can recover A from $\mathrm{QGr}(A)$ (up to a finite dimensional vector space), see [AZ94, Theorem 4.5].

One often thinks of $\mathrm{QGr}(A)$ as being a reprentation of a (nonexisting) scheme $\mathrm{Proj}(A)$. The category $\mathrm{QGr}(A)$ is only reasonably behaved when A satisfies suitable homological conditions (see for example [AZ94, Pol05]). Under these conditions we think of $\mathrm{QGr}(A)$ as representing a noncommutative variety instead of just a noncommutative scheme. Inspired by the commutative setting, we think of $\mathrm{QGr}(A)$ as representing a (smooth) noncommutative n-fold whenever A has global dimension $n+1$.

Of particular interest will be so called Artin-Schelter regular algebras, more concretely (as we are interested in noncommutative surfaces) the ones of global dimension three. We introduce these algebras, as well as their generalizations to the level of $\mathbb{Z}$-algebras in §0.1, §0.2 and §0.3.

In §0.4 and §0.5 we introduce $\mathbb{Z}$-domains and $\mathbb{Z}$-fields of fractions as generalizations of the obvious graded counterparts. Moreover we prove that quadrics (which are introduced at the end of §0.3) admit $\mathbb{Z}$-domains and $\mathbb{Z}$-fields of fractions.

Finally in §0.6 we prove (Theorem 0.6.6) that for sufficiently well behaved $\mathbb{Z}^2$-algebras R the category $\mathrm{QGr}(R)$ is equivalent to $\mathrm{QGr}(R_\Delta)$ where R_Δ is a *diagonal-like* sub-$\mathbb{Z}$-algebra. This technical result is a generalization of a well known fact for multi-homogeneous algebras (i.e. $\mathbb{Z}^n$-graded algebras). Our main application of Theorem 0.6.6 lies in §1.11.

0.1 Artin-Schelter regular algebras

Artin-Schelter regular algebras were introduced in [AS87] as follows:

Definition 0.1.1. A connected graded algebra $A = k \oplus A_1 \oplus A_2 \oplus \dots$ is said to be Artin-Schelter-regular of dimension d if the following conditions are satisfied:

- A has (graded right) global dimension d
- A has Gelfand–Kirilov dimension d
- A satisfies the Gorenstein-condition with respect to d; i.e. there is an integer l such that
$$\operatorname{Ext}^i_{\operatorname{Gr} A}(k_A, A) \cong \begin{cases} {}_A k(l) & i = d, \\ 0 & i \neq d. \end{cases}$$

Three dimensional AS-regular algebras were subsequently classified in [AS87, ATVdB90, Ste96]. Throughout we will only consider three-dimensional AS-regular algebras generated in degree one. For such algebras A there are two possibilities:

1. (the "quadratic case") A is generated by three elements which satisfy three quadratic relations. In this case A has Hilbert series $1/(1-t)^3$, i.e. the same Hilbert series as a polynomial ring in three variables.

2. (the "cubic case") A is generated by two elements satisfying two cubic relations. In this case A has Hilbert series $1/(1-t)^2(1-t^2)$.

For use below we define (r, s) to be respectively the number of generators of A and the degrees of the relations. Thus $(r, s) = (3, 2)$ or $(2, 3)$ depending on whether A is quadratic or cubic.

As mentioned above, to such a noncommutative algebra A we can associate a (nonexisting) object $\operatorname{Proj}(A)$ which is defined by its category of *quasi-coherent sheaves*:
$$\operatorname{Qcoh}(\operatorname{Proj}(A)) := \operatorname{QGr}(A) = \operatorname{Gr}(A)/\operatorname{Tors}(A)$$

When A is a quadratic three-dimensional AS-regular algebra, $\operatorname{Proj} A$ may be thought of as a noncommutative $\mathbb{P}^2$. Similarly if A is cubic, $\operatorname{Proj} A$ may be viewed as a noncommutative $\mathbb{P}^1 \times \mathbb{P}^1$. The rationale for this is explained in [VdB11].

The classification of three-dimensional AS-regular algebras A is in terms of suitable geometric data $(Y, \mathcal{L}, \sigma)$ where Y is a k-scheme, $\mathcal{L}$ is a line bundle on Y and σ is an automorphism of Y. More precisely: in the quadratic case Y is either $\mathbb{P}^2$ (the "linear case") or Y is embedded as a divisor of degree 3 in $\mathbb{P}^2$ (the "elliptic case") and $\mathcal{L}$ is the restriction of $\mathcal{O}_{\mathbb{P}^2}(1)$. In the cubic case Y is either $\mathbb{P}^1 \times \mathbb{P}^1$ (the "linear case") or Y is embedded as a divisor of bidegree (2,2) in $\mathbb{P}^1 \times \mathbb{P}^1$ (the "elliptic case") and $\mathcal{L}$

is the restriction of $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)$. The geometric data must also satisfy an additional numerical condition ([ATVdB90, Theorem 2]) which we will not discuss here.

Starting from the geometric data $(Y,\mathcal{L},\sigma)$ we construct a so-called "twisted homogeneous coordinate ring" $B = B(Y,\mathcal{L},\sigma)$. It is an $\mathbb{N}$-graded ring such that

$$B_n = \Gamma(Y,\mathcal{L}\otimes\sigma^*\mathcal{L}\otimes\ldots\otimes\sigma^{(n-1)*}\mathcal{L}) \tag{0.1}$$

with product $a\cdot b = a\otimes\sigma^{n*}b$ for $a\in B_n$. The corresponding AS-regular algebra $A = A(Y,\sigma,\mathcal{L})$ is obtained from B by dropping all relations in degree $> s$. By virtue of the construction there is a graded surjective k-algebra homomorphism

$$A \to B \tag{0.2}$$

which is an isomorphism in the linear case and it has a kernel generated by a normal element g in degree $s+1$ in the elliptic case.

By [AVdB90, Theorem 1.3] there is an equivalence of categories

$$\mathrm{Qcoh}(Y)\cong\mathrm{QGr}(B):\mathcal{F}\mapsto\bigoplus_{i\geq 0}\Gamma\left(Y,\mathcal{F}\otimes\mathcal{L}\otimes\sigma^*\mathcal{L}\otimes\ldots\sigma^{(i-1)*}\mathcal{L}\right) \tag{0.3}$$

As such the embedding of categories (induced by (0.2))

$$\mathrm{QGr}(B)\hookrightarrow\mathrm{QGr}(A)$$

can be interpreted as the noncommutative scheme $\mathrm{QGr}(A)$ "containing" the commutative scheme Y. In the linear case both categories coincide and hence the noncommutative scheme is actually a commutative one. In the elliptic case Y is said to be a "divisor" in $\mathrm{QGr}(A)$ [VdB01, Section 3.6].

It is well-known (see for example [ATVdB91]) that there is a 1-1-correspondence between the points of Y and so called *point modules*:

Definition 0.1.2. Let A be a connected graded algebra and $M\in\mathrm{Gr}(A)$. We say M is a point module if

- $M_0 = k$
- M is generated by M_0
- $\dim_k(M_i) = 1$ for all $i\geq 0$

For an AS-regular algebra $A = A(Y,\mathcal{L},\sigma)$ the correspondence between point modules and points on Y is based on the fact that each point module over A is actually a B-module. As such the point modules correspond to the sheaves $\mathcal{O}_p$ via (0.3). Moreover whenever M is a point module, it is customary to refer to $\pi(M)\in\mathrm{QGr}(A)$ as a point module as well.

Throughout this thesis we also encounter natural generalizations of point modules. E.g. we will encounter shifted point modules (which are generated by their bottom degree which need no longer be 0, see for example Lemma 1.3.7) or fat point modules (where the Hilbert series is constant but greater than 1, see §4.3).

The following results are well-known for AS-regular algebras:

Theorem 0.1.3. *Let A be a 3-dimensional AS-regular algebra, then*

i) A is a noetherian domain and in particular it has a graded field of fractions. ([ATVdB91, Theorem 3.9] and [ATVdB90, Theorem 8.1])

ii) $A = A(Y, \mathcal{L}, \sigma)$ is finite over its center if and only if σ has finite order. In this case A is a maximal order. ([ATVdB90, Theorem 7.1])

Remark 0.1.4. *Inspired by the commutative setting, one refers to the degree zero part* $\mathrm{Frac}_0(A)$ *of the graded quotient ring* $\mathrm{Frac}(A)$ *as the* function field of A or $\mathrm{Proj}(A)$.

Of particular interest are three dimensional AS-regular algebras for which Y is a smooth elliptic curve, σ is a translation such that $\sigma^{s+1} \neq \mathrm{id}$ and $\mathcal{L}$ is a line bundle of degree r. In this case we call the corresponding AS-regular algebra $A = A(Y, \mathcal{L}, \sigma)$ a *Sklyanin algebra* and the normal element g is actually central.

Remark 0.1.5. *Since any two line bundles of the same degree on a smooth elliptic curve are related by a translation, which necessarily commutes with σ, it is easy to see that the resulting Sklyanin algebra depends up to isomorphism only on (E, σ). So we sometimes drop $\mathcal{L}$ from the notation. Furthermore* $\mathrm{QGr}(A)$ *does not change if we compose σ with a translation by a point of order $s+1$ (See for example [ATVdB91, §8]). In other words* $\mathrm{QGr}(A)$ *depends only on σ^{s+1}. This observation agrees with Theorem 1.4.1 below.*

0.2 $\mathbb{Z}$-algebras

In this section we recall some definitions and facts on $\mathbb{Z}$-algebras. We refer the reader to [Sie11] or sections 3 and 4 of [VdB11] for a more thorough introduction. Recall that a $\mathbb{Z}$-algebra is defined as an algebra R (without unit) with a decomposition

$$R = \bigoplus_{m,n \in \mathbb{Z}} R_{m,n} \tag{0.4}$$

such that addition is degree-wise and multiplication satisfies

$$R_{m,n} R_{n,j} \subset R_{m,j} \tag{0.5}$$

and

$$R_{m,n} R_{i,j} = 0 \text{ if } n \neq i \tag{0.6}$$

Moreover there are local units $e_n \in R_{n,n}$ such that $\forall x \in R_{m,n} : e_m x = x = x e_n$.

Notation 0.2.1. *If A is a graded algebra, it gives rise to a $\mathbb{Z}$-algebra $\check{A}$ via*

$$\check{A}_{m,n} = A_{n-m}$$

In particular the notion of a $\mathbb{Z}$-algebra is a generalization of a ($\mathbb{Z}$)-graded algebra. Based on this, most graded notions have a natural $\mathbb{Z}$-algebra counterpart. For example we say that a $\mathbb{Z}$-algebra R is positively graded if $R_{m,n} = 0$ for $m > n$.

A graded R-module is an R-module M together with a decomposition

$$M = \bigoplus_{n\in\mathbb{Z}} M_n$$

such that the R-action on M satisfies

$$M_m R_{m,n} \subset M_n \tag{0.7}$$

and

$$M_m R_{i,n} = 0 \text{ if } i \neq m$$

The category of graded R-modules is denoted $\mathrm{Gr}(R)$ and similar to the graded case we use the notation $\mathrm{QGr}(R) := \mathrm{Gr}(R)/\operatorname{Tors}(R)$.

Remark 0.2.2. *If A is a graded algebra, there is an obvious isomorphism of categories* $\mathrm{Gr}(A) = \mathrm{Gr}(\check{A})$ *by identifying* $A(n) \in \mathrm{Gr}(A)$ *with* $e_{-n}\check{A} \in \mathrm{Gr}(\check{A})$*.*

Similarly $\mathrm{QGr}(A) = \mathrm{QGr}(\check{A})$*.*

Definition 0.2.3. Let R be a $\mathbb{Z}$-algebra. Then for each $n \in \mathbb{Z}$ we define $R(n)$ by setting $(R(n))_{i,j} = R_{i+n,j+n}$ with obvious multiplication. We say R is n-periodic if there is a $\mathbb{Z}$-algebra-isomorphism $R \cong R(n)$.

Lemma 0.2.4. *Let R be a $\mathbb{Z}$-algebra. Then there exists a graded algebra A such that $R = \check{A}$ if and only if R is 1-periodic.*

Proof. The "only if" part is obvious from the definition of $\check{A}$. The "if" part is proven in [VdB11, Lemma 3.4]. □

In §3.3.3 we extend the equivalences in Remark 0.2.2 to exact embeddings for n-periodic $\mathbb{Z}$-algebras. In particular we prove the following:

Lemma (See Lemma 3.3.17 and Proposition 3.3.20)**.** *Let R be a n-periodic $\mathbb{Z}$-algebra. Then there exists a graded algebra $\overline{R}$ and an exact embedding*

$$\mathrm{Gr}(R) \hookrightarrow \mathrm{Gr}(\overline{R})$$

Moreover the essential image is a direct summand of $\mathrm{Gr}(\overline{R})$*.*

Although we do not need it in Chapter 3, we mention that there is a second way of associating a graded algebra $\tilde{R}$ to a periodic $\mathbb{Z}$-algebra. This construction is closely related to the construction of $\overline{R}$, with the sole difference that the grading is different in these algebras. Therefor we state the following folklore result without proof:

Lemma. *Let R be a $\mathbb{Z}$-algebra and let n be a positive integer. Define the $\mathbb{Z}$-algebra $\widetilde{R^n}$ as follows:*

$$\left(\widetilde{R^n}\right)_{i,j} := \begin{bmatrix} R_{ni,nj} & R_{ni,nj+1} & \cdots & R_{ni,nj+n-1} \\ R_{ni+1,nj} & R_{ni+1,nj+1} & \cdots & R_{ni+1,nj+n-1} \\ \vdots & \vdots & \ddots & \vdots \\ R_{ni+n-1,nj} & R_{ni+n-1,nj+1} & \cdots & R_{ni+n-1,nj+n-1} \end{bmatrix}$$

Then the following results hold

- *There is an equivalence of categories* $\mathrm{Gr}(R) \cong \mathrm{Gr}(\widetilde{R^n})$.
- *If R is n-periodic, the $\mathbb{Z}$-algebra $\widetilde{R^n}$ is $1-$periodic and hence there exists a graded algebra $\tilde{R}$ such that* $\mathrm{Gr}(R) \cong \mathrm{Gr}(\tilde{R})$

Remark 0.2.5. *From a categorical point of view a $\mathbb{Z}$-algebra R is nothing but a k-linear category $\mathcal{R}$ whose objects are given by the integers. The homogeneous elements of the algebra then correspond to morphisms between two such integers via* $\mathrm{Hom}_{\mathcal{R}}(-j,-i) = R_{i,j}$ *and multiplication in R corresponds to composition of morphisms in $\mathcal{R}$.*

Remark 0.2.6. *More generally one can consider G-algebras where G is some (index) set (see [Sie11] for more details on such algebras). Similar to the above, such algebras admit a category* $\mathrm{Gr}(R)$*. One can also make sense of* $\mathrm{Tors}(R)$ *and* $\mathrm{QGr}(R)$ *whenever there is a partial order on I. We will, for example, encounter $\mathbb{Z}^2$-algebras in §0.6 and §1.8.*

0.3 AS-regular $\mathbb{Z}$-algebras

Definition 0.3.1. Let R be a $\mathbb{Z}$-algebra, then R is said to be connected, if it is positively graded, $\dim_k(R_{m,n}) < \infty$ for each m, n and $R_{m,m} \cong k$ for all m. We say R is generated in degree 1 if $R_{m,m+1}R_{m+1,n} = R_{m,n}$ holds for all $m < n$. If R is a connected $\mathbb{Z}$-algebra, generated in degree 1, we denote $S_{n,R} = e_nR/(e_nR)_{\geq n+1}$. I.e. $S_{n,R}$ is the unique R-module concentrated in degree n where it is equal to the base field k.

We can now give the definition of an AS-regular $\mathbb{Z}$-algebra as in [VdB11]

Definition 0.3.2. A $\mathbb{Z}$-algebra R over k is said to be AS-regular if the following conditions are satisfied:

i) R is connected and generated in degree 1

ii) $\dim_k(R_{m,n})$ is bounded by a polynomial in $n-m$

iii) The projective dimension of $S_{n,R}$ is finite and bounded by a number independent of n

iv) $\forall n \in \mathbb{N} : \sum_{i,j} \dim_k\left(\operatorname{Ext}^i_{\operatorname{Gr}(R)}(S_{j,R}, e_n R)\right) = 1$ (the "Gorenstein condition")

Remark 0.3.3. *It is immediate that if a graded algebra A is AS-regular, then $\check{A}$ is AS-regular in the above sense.*

$\mathbb{Z}$-algebra analogues of three dimensional quadratic and cubic AS-regular algebras were classified in [VdB11] (following [BP94] in the quadratic case). Similar to the graded case, the classification of three-dimensional quadratic and cubic AS-regular $\mathbb{Z}$-algebras in terms of geometric data $(Y, (\mathcal{L}_i)_{i\in\mathbb{Z}})$ where Y is a k-scheme and $(\mathcal{L}_i)_{i\in\mathbb{Z}}$ is an *elliptic helix of line bundles on* Y. We will give a (slightly restrictive) definition for elliptic helices in Definition 1.3.2.

Starting from the geometric data one first constructs a $\mathbb{Z}$-algebra analogue $B = B(Y, (\mathcal{L}_i)_i)$ of the twisted homogeneous coordinate ring as introduced in (0.1):

$$B_{i,j} := \begin{cases} \Gamma(Y, \mathcal{L}_i \otimes \mathcal{L}_{i+1} \otimes \dots \mathcal{L}_{j-1}) & \text{if } i \le j \\ 0 & \text{if } i > j \end{cases} \tag{0.8}$$

Similarly to in (0.3) there is an equivalence of categories.

$$\operatorname{Qcoh}(Y) \cong \operatorname{QGr}(B) : \mathcal{F} \mapsto \bigoplus_{i\ge 0} \Gamma(Y, \mathcal{F} \otimes \mathcal{L}_0 \otimes \mathcal{L}_1 \otimes \dots \mathcal{L}_{i-1}) \tag{0.9}$$

(see [VdB11, Corollary 5.5.9]). $r = \dim_k(B_{i,i+1})$ does not depend on i and equals 3 (in the quadratic case) or 2 (in the cubic case). $A(Y, (\mathcal{L}_i)_i)$ is then obtained from B by only preserving the relations in degree $(i, i+s)$ for $s = 5 - r$. In particular there is a surjective $\mathbb{Z}$-algebra morphism

$$A(Y, (\mathcal{L}_i)_i) \to B(Y, (\mathcal{L}_i)_i)$$

giving rise to an embedding of categories as in the graded case:

$$\operatorname{QGr}(B(Y, (\mathcal{L}_i)_i)) \hookrightarrow \operatorname{QGr}(A(Y, (\mathcal{L}_i)_i))$$

Moreover it is shown that $\dim_k(A_{i,i+n})$ does not depend on i; it hence makes sense to write $h(n) := \dim_k(A_{i,i+n})$ and one checks that $h(n)$ coincides with the Hilbert series in the graded case.

In the special case that A is 3-dimensional quadratic AS-regular algebra with geometric data $(Y, \sigma, \mathcal{L})$, the elliptic helix corresponding to $\check{A}$ is $(\mathcal{L}^{\sigma^i})_i$. This follows immediately from the construction of A from $(Y, \sigma, \mathcal{L})$ as given in [ATVdB90] (see §0.1 for an outline).

The following theorem gives a partial converse to Remark 0.3.3
(see Definition 0.2.3 for the definition of periodicity of a $\mathbb{Z}$-algebra.)

Theorem 0.3.4. *([VdB11, Theorem 4.2.2. and Proposition 6.6])*

- *Let R be a quadratic AS-regular $\mathbb{Z}$-algebra, then there exists a quadratic AS-regular algebra A such that $R = \check{A}$.*
- *Let R be a cubic AS-regular $\mathbb{Z}$-algebra, then R is 2-periodic. In particular if we denote $R^{(2)}$ for its 2-Veronese (i.e. $R^{(2)}_{i,j} = R_{2i,2j}$), then there exists a $\mathbb{Z}$-graded algebra A such that $R^{(2)} = \check{A}$. Moreover there is a 4-dimensional AS-regular algebra D with Hilbert series $\frac{1}{(1-t)^4}$ together with a regular normal element $C \in D_2$ such that $A \cong D/(C)$.*

Sketch of the ideas.

- It is shown that if $R = A(Y, (\mathcal{L}_i)_i)$ is a quadratic AS-regular $\mathbb{Z}$-algebra, then there is a $\psi \in \operatorname{Aut}(Y)$ such that $\mathcal{L}_i \cong \psi^{i*}\mathcal{L}_0$ holds for all i.
- It is shown that if $R = A(Y, (\mathcal{L}_i)_i)$ is a cubic AS-regular $\mathbb{Z}$-algebra, then there is an $\alpha \in \operatorname{Aut}(Y)$ such that $\mathcal{L}_{i+2} \cong \alpha^*\mathcal{L}_i$ holds for all i. The existence of D and $C \in D_2$ is obtained by suitably generalizing the results in [BVdB98, §3].

□

In particular there exist cubic AS-regular $\mathbb{Z}$-algebras are not 1-periodic. To distinguish cubic AS-regular algebras from the more general cubic AS-regular $\mathbb{Z}$-algebras, one often refers to the latter as *quadrics*.

0.4 $\mathbb{Z}$-domains and $\mathbb{Z}$-fields of fractions

In this section we give the natural generalizations of "domain" and "field of fractions" for $\mathbb{Z}$-algebras. Among other generalizations, these notions can also be found in [CN16, §2].

Definition 0.4.1. Let R be a $\mathbb{Z}$-algebra. Then we say that R is a $\mathbb{Z}$-domain if the following condition is satisfied:

$$\forall i, j, k \in \mathbb{Z}, \forall x \in R_{i,j}, \forall y \in R_{j,k} : xy = 0 \Rightarrow x = 0 \quad \text{or} \quad y = 0$$

i.e. a product of two nonzero elements can only be zero if it was forced to be zero by (0.6).

It is well known that three dimensional AS-regular algebras are domains (Theorem 0.1.3). We extend this result to the level of quadrics and show:

Theorem 0.4.2. *Let $A = A(Y, (\mathcal{L}_i)_i)$ be a quadric. Then A is a $\mathbb{Z}$-domain.*

Proof. The proof of this theorem is postponed to §0.5. □

Let R be a $\mathbb{Z}$-algebra and $\mathcal{R}$ the associated category as in Remark 0.2.5. Let W be a collection of homogeneous elements and let $\mathcal{W}$ be the corresponding collection of morphisms. We then say that R is localizable at W if $(\mathcal{R}, \mathcal{W})$ admits a calculus of fractions (see for example [GZ67]). In this case we define $R[W^{-1}]$ as the $\mathbb{Z}$-algebra associated to $\mathcal{R}[\mathcal{W}^{-1}]$, again using Remark 0.2.5.

Using the theory of (right) fractions for a category one easily checks that the following definition makes sense. Essentially, this definition coincides with the usual definition of a graded ring of fractions; however, we need to take extra care of the degrees of all elements in order to avoid products which are trivially zero.

Definition 0.4.3. Let R be a $\mathbb{Z}$-domain, then R admits a $\mathbb{Z}$-field of (right) fractions if the following condition is satisfied:

$$\forall r \in R_{l,i}, s \in R_{l,j} \smallsetminus \{0\} : \exists n \in \mathbb{Z} : \exists r' \in R_{j,n}, s' \in R_{i,n} \smallsetminus \{0\} : rs' = sr'$$

The elements of $\operatorname{Frac}(R)_{i,j}$ are equivalence classes of couples (r, s) where $r \in R_{i,l}$, $s \in R_{j,l} \smallsetminus \{0\}$ for some $l \in \mathbb{Z}$. The equivalence relation is given by

$$\forall l_1, l_2 \in \mathbb{Z}, \forall r_1 \in R_{i,l_1}, r_2 \in R_{i,l_2}, s_1 \in R_{j,l_1} \smallsetminus \{0\}, s_2 \in R_{j,l_2} \smallsetminus \{0\} :$$
$$(r_1, s_1) \sim (r_2, s_2)$$
$$\Updownarrow$$
$$\exists l_3 \in \mathbb{Z}, \exists x \in R_{l_1,l_3} \smallsetminus \{0\}, \exists y \in R_{l_2,l_3} \smallsetminus \{0\} : r_1 x = r_2 y \text{ and } s_1 x = s_2 y$$

The following is obvious from the definition:

Proposition 0.4.4. *Let A be a graded domain. Then*

A admits a (graded) field of (right) fractions

$$\Updownarrow$$

$\check{A}$ admits a $\mathbb{Z}$-field of (right) fractions

Moreover in this case $\widecheck{Frac(A)} = Frac(\check{A})$

Recall that graded domains admit fields of (right-)fractions when they are graded (right-)noetherian or when they have subexponential growth. A similar result can be found in [CN16]:

Proposition 0.4.5. *Let R be a $\mathbb{Z}$-domain such that each e_iR is a uniform module (i.e. for all nonzero $M, N \subset e_iR : M \cap N \neq 0$). Then R admits a $\mathbb{Z}$-field of (right) fractions.*

Proof. We need to show that for all $r \in R_{l,i}, s \in R_{l,j} \setminus \{0\}$ there exists an $n \in \mathbb{Z}$ and elements $r' \in R_{j,n}, s' \in R_{i,n} \setminus \{0\}$ such that $rs' = sr'$. If $r = 0$ then it suffices to take $r' = 0$. If $r \neq 0$ the existence of r' and s' follows from the fact that rR and sR are nonzero submodules of the uniform module e_lR. $\square$

Similarly the following will be shown in §0.5:

Theorem 0.4.6. *Let A be a quadric. Then A admits a $\mathbb{Z}$-field of fractions.*

Remark 0.4.7. *As three dimensional, quadratic AS-regular $\mathbb{Z}$-algebras are 1-periodic (Theorem 0.3.4), the existence of their $\mathbb{Z}$-field of fractions is automatic from Proposition 0.4.4.*

Remark 0.4.8. *Inspired by Remark 0.1.4 it is customary to refer to $Q_{0,0}$ as the function field of A (with $Q = \mathrm{Frac}(A)$), the associated noncommutative scheme* $\mathrm{Proj}(A)$ *or its representation* $\mathrm{QGr}(A)$. *We will do so throughout this paper.*

0.5 Quadrics admit $\mathbb{Z}$-fields of fractions

In this section (which is completely based on the appendix of our paper [Pre16]) we prove the following

Theorem (Theorem 0.4.2 and 0.4.6)**.** *Let A be a quadric, then A is a $\mathbb{Z}$-domain and A admits a $\mathbb{Z}$-field of fractions.*

The proof of this theorem is based on several preliminary results.

Notation 0.5.1. *Throughout this section A will always be a quadric. Moreover for any A-module M we let* $\mathrm{p.dim}(M)$ *and* $\mathrm{GKdim}(M)$ *denote the projective and Gelfand–Kirillov dimension respectively.*

0.5.1 Preliminary results

Some lemmas

Lemma 0.5.2. *Let M be a finitely generated left- or right-A-module and assume* $\mathrm{p.dim}(M) \leq 1$, *then* $\mathrm{GKdim}(M) \geq 2$.

Proof. Upon replacing the projective modules $A(-i)$ by e_iA or Ae_i one can copy the proof of [ATVdB91, Proposition 2.41] $\square$

Lemma 0.5.3. *Let M be a finitely generated right-A-module and let S_i denote the simple module $e_iA/e_iA_{>i}$, then*

$$pd(M) \leq 2 \;\Rightarrow\; \forall i \in \mathbb{Z} : \mathrm{Hom}_A(S_i, M) = 0$$

Proof. Upon replacing the projective modules $A(-i)$ by e_iA one can copy the proof of [ATVdB91, Proposition 2.46 (i)] □

For use below, we introduce the notion of *multiplicity of a graded module* below. Apart from the fact that we consider graded modules over a $\mathbb{Z}$-algebra in stead of a graded algebra, this notion was already defined in [ATVdB91, §2]

Definition 0.5.4. Let A be a $\mathbb{Z}$-algebra and M a graded A-module. The *multiplicity* $e(M)$ of M is defined as the leading coefficient of the series expension of $h_M(t)$ in terms of $(1-t)^{-1}$.

Lemma 0.5.5. *Let $i \in \mathbb{Z}$ be any integer and M be some graded submodule of e_iA then* $\mathrm{GKdim}(M) = 3 \;\Leftrightarrow\; \mathrm{GKdim}(e_iA/M) < 3$.

Proof. $\mathrm{GKdim}(e_iA/M) < 3 \Rightarrow \mathrm{GKdim}(M) = 3$ is trivial. Let us prove the other direction.

Assume by way of contradiction that $\mathrm{GKdim}(M) = \mathrm{GKdim}(e_iA/M) = 3$. As both M and e_iA/M are nonzero we have $e(M) > 0$ and $e(e_iA/M) > 0$. However as they have equal GKdim, we have $e(e_iA) = e(M) + e(e_iA/M)$. A direct computation shows that $e(e_jA) = \frac{1}{2}$ holds for all j. Similar to the proof of [ATVdB91, Proposition 2.21 (iii)] we then know that $e(M)$ and $e(e_iA/M)$ must be a nonnegative multiple of $\frac{1}{2}$. Contradiction! □

We now introduce a homogeneous ideal N of A in a similar fashion as was done in [ATVdB91]:

1. e_iA is a noetherian object in $\mathrm{Gr}(A)$. In particular any ascending chain of submodules of e_iA must stabilize. This allows us to set N_i to be the largest submodule of e_iA of GKdim ≤ 2.

2. Define N as $\bigoplus_{i\in\mathbb{Z}} N_i$. Then N is a homogeneous two-sided ideal of A. To see why N also has the structure of a left ideal, note that if $a \in A_{i,j}$ then aN_j is a submodule of e_iA of GKdim ≤ 2. This implies that $aN_j \subset N_i$ for otherwise $aN_j + N_i$ would be a strictly larger submodule than N_i but it still has GKdim ≤ 2.

Remark 0.5.6. *Recall that A, being a quadric, is 2-periodic (Theorem 0.3.4). I.e. there is an isomorphism $A \cong A(2)$. There is an induced isomorphism*

$$N \cong N(2)$$

To see this, fix any $i \in \mathbb{Z}$ and let $f_i : e_iA \to e_{i+2}A(2)$ be the induced isomorphism. Then $f_i(e_iN)$ has GKdim = 2*. In particular, being an $A(2)$-submodule of $e_{i+2}A(2)$ we must have $f_i(e_iN) \subset e_{i+2}N(2)$. By considering f_i^{-1} we see that this must in fact be an equality.*

Lemma 0.5.7. *Let N be as above, then $\overline{A} := A/N$ is a $\mathbb{Z}$-domain.*

Proof. Let $b \in A_{i,j} \setminus N_{i,j}$, we then need to show that the induced morphism

$$e_j\overline{A} \xrightarrow{\bar{b}\cdot} e_i\overline{A}$$

is injective. For this, consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker(b\cdot) \arrow[r] \arrow[d, two heads] & e_jA \arrow[r] \arrow[d, two heads] & bA \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \ker(\bar{b}\cdot) \arrow[r] & e_j\overline{A} \arrow[r] & \overline{bA} \arrow[r] & 0
\end{tikzcd}$$

Now suppose by way of contradiction $\ker(\bar{b}\cdot) \neq 0$. By construction $e_j\overline{A} = e_jA/N_j$ does not contain submodules of GKdim ≤ 2, hence GKdim $\left(\ker(\bar{b}\cdot)\right) = 3$. This implies that GKdim $(\ker(b\cdot)) = 3$ as well. By Lemma 0.5.5 we must have GKdim $(bA) < 3$, hence also GKdim $\left(\overline{bA}\right) < 3$. As $\overline{bA} \subset e_i\overline{A}$ and $e_i\overline{A}$ does not contain submodules of GKdim ≤ 2 we must have $\overline{bA} = 0$, contradicting the fact that $\bar{b} \neq 0$. □

Dual modules

Next we introduce the notion of dualization of (right-)A-modules. Throughout this section we will use A^{op} to denote the opposite algebra of A. A^{op} is a $\mathbb{Z}$-algebra by setting $(A^{op})_{i,j} = (A_{-j,-i})^{op}$. With this $\mathbb{Z}$-algebra structure graded right-A^{op}-modules can be identified with graded left-A-modules; for example e_iA^{op} naturally corresponds to Ae_{-i}. It hence makes sense to let $\mathrm{Gr}(A^{op})$ denote the categories of graded left A-modules.

Let M be a graded right-A-module, then $\bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Gr}(A)}(M, e_iA)$ naturally has the structure of a graded left-A-module via:

$$A_{ij} \otimes \mathrm{Hom}_{\mathrm{Gr}(A)}(M, e_jA) \to \mathrm{Hom}_{\mathrm{Gr}(A)}(M, e_iA) : x \otimes f \mapsto x \cdot f$$

where

$$(x \cdot f)(m) := x \cdot f(m)$$

We denote this graded left-A-module by M^* or $\mathrm{Hom}_{\mathrm{Gr}(A)}(M, A)$ and it is called the dual of M. One easily checks that this induces a left-exact functor

$$\mathrm{Hom}_{\mathrm{Gr}(A)}(-, A) = (-)^* : \mathrm{Gr}(A) \to \mathrm{Gr}(A^{op}).$$

Note that as $\mathrm{Hom}_{\mathrm{Gr}(A)}(e_iA, e_jA) \cong A_{j,i}$ we naturally have

$$\mathrm{Hom}_{\mathrm{Gr}(A)}(e_iA, A) = (e_iA)^* = Ae_i.$$

This allows us to define the right derived functors

$$\mathbf{R}\mathrm{Hom}_{\mathrm{Gr}(A)}(-, A) : D^b_f(\mathrm{Gr}(A)) \to D^b_f(\mathrm{Gr}(A^{op})).$$

If $C^\bullet$ is some object in $D^b_f(\mathrm{Gr}(A))$ which is represented by a bounded exact complex of finitely generated projectives, say

$$0 \longrightarrow \bigoplus_{i\in\mathbb{Z}} e_iA^{\oplus l_{i,n}} \xrightarrow{\cdot M_n} \bigoplus_{i\in\mathbb{Z}} e_iA^{\oplus l_{i,n-1}} \to \ldots \to \bigoplus_{i\in\mathbb{Z}} e_iA^{\oplus l_{i,m}} \longrightarrow 0$$

where M_n is some matrix whose entries are homogeneous elements in A, then $\mathbf{R}\mathrm{Hom}_{\mathrm{Gr}(A)}(C^\bullet, A)$ is represented by the complex

$$0 \longleftarrow \bigoplus_{i\in\mathbb{Z}} Ae_i^{\oplus l_{i,n}} \xleftarrow{M_n\cdot} \bigoplus_{i\in\mathbb{Z}} Ae_i^{\oplus l_{i,n-1}} \leftarrow \ldots \leftarrow \bigoplus_{i\in\mathbb{Z}} Ae_i^{\oplus l_{i,m}} \longleftarrow 0$$

(where each term in position j in the original complex gives rise to a term in position $-j$ in the new complex) Similar to the graded case we use the shorthand notation $(C^\bullet)^D := \mathbf{R}\mathrm{Hom}_{\mathrm{Gr}(A)}(C^\bullet, A)$ If M is some graded right-A-module, then we denote $\mathrm{Ext}^i_{\mathrm{Gr}(A)}(M, A) := R^i \mathrm{Hom}_{\mathrm{Gr}(A)}(M, A) = h^i\left(M^D\right)$.

Remark 0.5.8. *If we introduce* $\mathbf{R}\mathrm{Hom}_{\mathrm{Gr}(A^{op})}(-, A) : D^b_f(\mathrm{Gr}(A^{op})) \to D^b_f(\mathrm{Gr}(A))$ *in an analogous way, then* $((-)^D)^D \cong \mathrm{id}$ *holds, giving rise to a biduality spectral sequence as in the graded case.*

For a bounded complex $C^\bullet$ of (finitely generated, graded right-)A-modules (or A^{op}-modules) we define the Hilbert series of $C^\bullet$ as

$$h_{C^\bullet}(t) = \sum_{i\in\mathbb{Z}} h_i(C^\bullet)t^i \text{ with } h_i(C^\bullet) = \sum_{j\in\mathbb{Z}} (-1)^j \dim_k\left((C^j)_i\right)$$

and similar to Definition 0.5.4 we denote $e(C^\bullet)$ to be the leading coefficient of the series expension of $h_{C^\bullet}(t)$ in terms of $(1-t)^{-1}$ and $\mathrm{GKdim}(C^\bullet)$ as the highest power of $(1-t)^{-1}$ in this expansion, i.e. the order of pole of $h_{C^\bullet}(t)$

Remark 0.5.9. *As was noted in [ATVdB91, §2], for graded modules this definition for the Gelfand–Kirillov–dimension coincides with the usual one as in [KL91].*

We then have the following:

Lemma 0.5.10. *Let* $C^\bullet \in D^b_f(\mathrm{Gr}(A))$*, then we have the following equality of rational functions:*

$$h_{(C^\bullet)^D}(t) = -t^4 \cdot h_{C^\bullet}(t^{-1})$$

Proof. By linearity of the definition of $h_{C^\bullet}$, it suffices to prove the equality in case $C^\bullet$ is given by some projective e_iA concentrated in position j. In this case $(C^\bullet)^D$ is given by Ae_i (hence $e_{-i}A^{op}$) concentrated in position $(-j)$ such that

$$\begin{aligned} h_{(C^\bullet)^D}(t) = (-1)^{-j}\cdot\frac{t^{-i}}{(1-t^2)(1-t)^2} &= (-1)^j\cdot\frac{(t^{-1})^i}{-t^4(1-(t^{-1})^2)(1-t^{-1})^2}\\ &= -t^{-4}\cdot\frac{(t^{-1})^i}{(1-(t^{-1})^2)(1-t^{-1})^2}\\ &= -t^{-4}\cdot h_{C^\bullet}(t^{-1}) \end{aligned}$$ □

Corollary 0.5.11. *Let $C^\bullet$ be a bounded complex of (fin. gen.) right A-modules. Let $m = \mathrm{GKdim}(C^\bullet)$, then*

i) $\mathrm{GKdim}((C^\bullet)^D) = m$

ii) $e((C^\bullet)^D) = (-1)^{m+1}e(C^\bullet)$

Proof. Suppose

$$h_{C^\bullet}(t) = \frac{\sum_{i=0}^\infty \alpha_i(1-t)^i}{(1-t)^m}$$

with $\alpha_0 = e(C^\bullet) \neq 0$. Then we need to show that

$$h_{(C^\bullet)^D}(t) = \frac{\sum_{i=0}^\infty \widetilde{\alpha_i}(1-t)^i}{(1-t)^m}$$

with $\widetilde{\alpha_0} = (-1)^{m+1}\alpha_0$.

First note that for each $n \in \mathbb{Z}$ we can write:

$$t^n = 1 + \sum_{j=1}^\infty \beta_{n,j}(1-t)^j$$

with $\beta_{n,j} \in k^*$. Then by Lemma 0.5.10 we have

$$\begin{aligned} h_{(C^\bullet)^D}(t) &= \frac{\sum_{i=0}^\infty \alpha_i(-t^{-4})(1-t^{-1})^i}{(1-t^{-1})^m}\\ &= \frac{\sum_{i=0}^\infty -\alpha_i(-1)^i t^{-i-4}(1-t)^i}{(-1)^m t^{-m}(1-t)^m}\\ &= \frac{\sum_{i=0}^\infty \alpha_i(-1)^{m+i+1} t^{m-i-4}(1-t)^i}{(1-t)^m}\\ &= \frac{\sum_{i=0}^\infty \widetilde{\alpha_i}(1-t)^i}{(1-t)^m} \end{aligned}$$

where

$$\widetilde{\alpha_i} = (-1)^{m+i+1}\left(\alpha_i + \sum_{j=1}^i (-1)^j\beta_{m-i+j-4,j}\alpha_{i-j}\right)$$ □

Lemma 0.5.12. *Let M be a finitely generated right-A-module, then* $\mathrm{Ext}^3_{\mathrm{Gr}(A)}(M, A)$ *is a finite dimensional k-vectorspace.*

Proof. This is an immediate generalization of [ATVdB91, Proposition 2.46(ii)]. □

We now prove some more results on the homogeneous ideal N as above:

Lemma 0.5.13. *For each $i \in \mathbb{Z}$ we have* $\mathrm{GKdim}(Ne_i) \leq 2$.

Proof. By construction we know that for each i we have $\mathrm{GKdim}(e_iN) \leq 2$. In particular there is for each i a degree 2 polynomial P_i such that $\dim_k(N_{i,i+l}) \leq P_i(l)$ holds for all l sufficiently large. Now fix some $i \in \mathbb{Z}$, we must show that there is a degree 2 polynomial Q such that $\dim_k(N_{i-l,i}) \leq Q(l)$. For this, recall that the 2-periodicity of A descends to N (see Remark 0.5.6). In particular we have

$$\dim_k(N_{i-l,i}) = \begin{cases} \dim_k(N_{i,i+l}) & \text{if } l \text{ is even} \\ \dim_k(N_{i+1,i+l+1}) & \text{if } l \text{ is odd} \end{cases}$$

Without loss of generality we can now assume that $P_i(l) \geq P_{i+1}(l)$ holds for all l sufficiently large. We can finish the proof by setting $Q = P_i$.

□

Lemma 0.5.14. *Fix some $i \in \mathbb{Z}$ and let $I \subset Ae_i$ be the left-annihilator of e_iN, then* $\mathrm{GKdim}(I) = 3$.

Proof. As e_iN is a submodule of the noetherian right A-module e_iA, it is finitely generated. I.e. there are elements $x_{i,i} \in A_{i,i}, x_{i,i+1} \in A_{i,i+1}, \ldots, x_{i,n} \in A_{i,n}$ such that

$$e_iN = \sum_{j=i}^{n} x_{i,j}A = \sum_{j=i}^{n} x_{i,j}e_jA$$

Let I_j be the left annihilator of $x_{i,j}$, i.e.

$$I_j = \{a \in Ae_i \mid ax_{i,j} = 0$$

Then there is an exact sequence of left A-modules

$$0 \longrightarrow I_j \longrightarrow Ae_i \xrightarrow{\cdot x_{i,j}} Ae_j$$

such that $Ae_i/I_j \cong Ax_{i,j}$.

Moreover $I = \bigcap_j I_j$ such that

$$\mathrm{GKdim}(A/I) \leq max_j \,\mathrm{GKdim}(Ax_{i,j}) \leq max_j \,\mathrm{GKdim}(Ae_j) = 2$$

The result now follows from Lemma 0.5.5.

□

Lemma 0.5.15. *For each $i \in \mathbb{Z}$ we have:*

(i) e_iN is a second syzygy

(ii) $pd(e_iN) \leq 1$

Proof. (ii) obviously follows from (i), so we only need to prove e_iN is a second syzygy. By Lemmas 0.5.14 and 0.5.13 we know there exists an element $b \in A_{ji}$ such that $bN = 0$ while $b \notin Ne_i$. Hence $e_iN \subset \ker(b\cdot)$ while $e_i\overline{A} \xrightarrow{b\cdot} e_j\overline{A}$ is injective by Lemma 0.5.7. The proof now follows as this implies that we have a left exact sequence

$$0 \longrightarrow e_iN \longrightarrow e_iA \xrightarrow{b\cdot} e_jA$$

□

0.5.2 Proof of Theorem 0.4.2

Let N be as above. By Lemma 0.5.7 it suffices to prove that $N = 0$. Suppose by way of contradiction that this is not the case. Without loss of generality we can assume $e_0N \neq 0$. Then by Lemma 0.5.15 $pd(e_0N) \leq 1$ which by Lemma 0.5.2 implies $\mathrm{GKdim}(e_0N) \geq 2$. As by construction $\mathrm{GKdim}(e_0N) \leq 2$, we have $\mathrm{GKdim}(e_0N) = 2$. Let $(e_0N)^D = \mathbf{R}\mathrm{Hom}(e_0N, A)$ denote the dual complex as above, by the projective dimension of e_0N, this complex only has homology at position 0 and 1. By Lemma 0.5.15 e_0N is a second syzygy and hence we have

$$\mathrm{H}^1\left((e_0N)^D\right) = \mathrm{Ext}^1(e_0N, A) \cong \mathrm{Ext}^3(M, A)$$

for some module M. Lemma 0.5.12 then implies that $h^1((e_0N)^D)$ is finite dimensional. In particular the Gelfand–Kirillov dimension and multiplicity of $(e_0N)^D$ are solely determined by $(e_0N)^*$. Corollary 0.5.11 gives

$$e((e_0N)^*) = e((e_0N)^D) = -e(e_0N).$$

A contradiction!

0.5.3 Proof of Theorem 0.4.6

By Theorem 0.4.2 and Proposition 0.4.5 it suffices to prove that all e_iA are uniform modules. To show this, fix any i and nonzero $M \subset e_iA$. Then $\mathrm{GKdim}(M) = 3$. To see this let x be any nonzero element in $M_j \subset A_{i,j}$, then by Theorem 0.4.2 we have

$$3 = \mathrm{GKdim}(e_jA) = \mathrm{GKdim}(xe_jA) \leq \mathrm{GKdim}(M) \leq \mathrm{GKdim}(e_iA) = 3$$

Now let N be a nonzero submodule of e_iA. Then as above, $\mathrm{GKdim}(N) = 3$. Suppose by way of contradiction that $M \cap N = 0$, then the following composition is a monomorphism:

$$N \hookrightarrow e_iA \longrightarrow e_iA/M$$

such that $\mathrm{GKdim}(e_iA/M) = 3$. This gives a contradiction with Lemma 0.5.5. Hence for any nonzero $M, N \subset e_A$ we must have $M \cap N \neq 0$, s that e_iA is a uniform module.

0.6 Diagonal-like subalgebras

Multi-homogeneous algebras (i.e. $\mathbb{Z}^n$-graded algebras with with $n > 1$) appear frequently in the literature [Cha00, CHTV97, KSK⁺09, STV98, Tru00]. These multi-homogeneous algebras S inherit many properties from diagonal subalgebras S_Δ. Moreover to each multi-homogeneous algebra S one associates a projective scheme (Multi)Proj(S) and given suitable conditions on S this projective scheme coincides with (Multi)Proj(S_Δ) (see for example [Tru00, Lemma 1.3]). For a bi-homogeneous algebra such a diagonal subalgebra is simply a $\mathbb{Z}^2$-graded algebra and as such (Multi)Proj(S) is isomorphic to the Proj of a graded algebra. We generalize this result to the level of $\mathbb{Z}^2$-algebras and prove (Theorem 0.6.6) that for sufficiently well behaved $\mathbb{Z}^2$-algebras R the category QGr(R) is equivalent to QGr(R_Δ) where R_Δ is a *diagonal-like* sub-$\mathbb{Z}$-algebra.

Throughout this section R is a $\mathbb{Z}^2$-algebra over some field k. $\mathbb{Z}^2$-algebras are defined completely analogous to $\mathbb{Z}$-algebras as in §0.2, i.e. R is a k-algebra together with a decomposition

$$R = \bigoplus_{(i,j),(m,n)\in\mathbb{Z}^2} R_{(i,j),(m,n)}$$

such that addition is degree-wise and multiplication satisfies

$$R_{(a,b),(i,j)}R_{(i,j),(m,n)} \subset R_{(a,b),(m,n)}$$

and

$$R_{(a,b),(c,d)}R_{(i,j),(m,n)} = 0 \text{ whenever } (c,d) \neq (i,j)$$

Moreover there are local units $e_{(i,j)} \in R_{(i,j),(i,j)}$ such that for each $x \in R_{(a,b),(m,n)}$:

$$e_{(a,b)}x = x = xe_{(m,n)}$$

The category of graded R-modules is defined completely similar to (0.7). Throughout this section we also make the assumption that R is noetherian in the sense that Gr(R) is locally noetherian, moreover we assume that each $R_{(i,j),(m,n)}$ is a finite dimensional vectorspace and $R_{(i,j),(i,j)} = k$.

Notation 0.6.1. *Let R be a $\mathbb{Z}^2$-algebra. Then we denote R_+ for the $\mathbb{Z}^2$-subalgebra:*

$$(R_+)_{(i,j),(m,n)} = \begin{cases} R_{(i,j),(m,n)} & \text{if } i \leq m \text{ and } j \leq n \\ 0 & \text{else} \end{cases}$$

Definition 0.6.2. Let R be a $\mathbb{Z}^2$-algebra and let R_+ be as above. We say that R_+ is *generated in degree* $(0,1)$ *and* $(1,0)$ if each homogeneous element in R_+ can be written as a product of elements of degree $(0,1)$ or $(1,0)$.

More precisely:

$$\forall i,j,m,n \in \mathbb{Z}, i < m, j \leq n : R_{(i,j),(i+1,j)} \otimes R_{(i+1,j),(m,n)} \to R_{(i,j),(m,n)} \text{ is surjective}$$
and
$$\forall i,j,m,n \in \mathbb{Z}, i \leq m, j < n : R_{(i,j),(i,j+1)} \otimes R_{(i,j+1),(m,n)} \to R_{(i,j),(m,n)} \text{ is surjective}$$

Remark 0.6.3. *We do not require R itself to be generated in degree $(0,1)$ and $(1,0)$. This is not needed for the results in this section. Moreover if $R = \tilde{A}$ as in §1.8, then R is not generated in degree $(0,1)$ and $(1,0)$ but R_+ is.*

Definition 0.6.4. Let R be a $\mathbb{Z}^2$-algebra such that R_+ is generated in degree $(0,1)$ and $(1,0)$ and let M be a graded R-module. We say M is *right-upper-bounded* if there exist $i_0, j_0 \in \mathbb{Z} : \forall i \geq i_0, j \geq j_0 : M_{(i,j)} = 0$. M is said to be *torsion* if it is a direct limit of right-upper-bounded modules. We denote $\operatorname{Tors}(R)$ for the full subcategory of torsion modules in $\operatorname{Gr}(R)$.

The assumption that R is noetherian implies that $\operatorname{Tors}(R)$ is a Serre subcategory of $\operatorname{Gr}(R)$ and as such we can define a quotient category

$$\operatorname{QGr}(R) := \operatorname{Gr}(R)/\operatorname{Tors}(R)$$

The main result of this section is that we can understand $\operatorname{QGr}(R)$ in terms of *diagonal-like* sub-$\mathbb{Z}$-algebras of R.

Definition 0.6.5. Let $\Delta : \mathbb{Z} \to \mathbb{Z}^2$ and let R be a $\mathbb{Z}^2$-algebra, then we denote R_Δ the induced $\mathbb{Z}$-subalgebra:

$$(R_\Delta)_{i,j} := R_{\Delta(i),\Delta(j)}$$

We say Δ (or R_Δ) is *diagonal-like* if $\pi_1 \circ \Delta$ and $\pi_2 \circ \Delta$ are (eventually) strictly increasing, where $\pi_i : \mathbb{Z}^2 \to \mathbb{Z}$ are the projections on the components.

Theorem 0.6.6. *Let R be a $\mathbb{Z}^2$-algebra and let R_Δ be a diagonal-like sub-$\mathbb{Z}$-algebra. Then there is an equivalence of categories*

$$\operatorname{QGr}(R) \cong \operatorname{QGr}(R_\Delta) \tag{0.10}$$

Proof. There is a restriction functor $F : \operatorname{Gr}(R) \to \operatorname{Gr}(R_\Delta)$ defined by $F(M)_i := M_{\Delta(i)}$. F obviously maps torsion modules to torsion modules and hence induces a functor $F : \operatorname{QGr}(R) \to \operatorname{QGr}(R_\Delta)$. We define a right exact functor $G : \operatorname{Gr}(R_\Delta) \to \operatorname{Gr}(R)$ via

$$G(e_i R_\Delta) = e_{\Delta(i)} R$$

at the level of objects. As $\operatorname{Hom}_{R_\Delta}(e_i R_\Delta, e_j R_\Delta)$ and $\operatorname{Hom}_R(e_{\Delta(i)} R, e_{\Delta(j)} R)$ are both canonically isomorphic to $R_{\Delta(j),\Delta(i)}$,

$$G : \operatorname{Hom}_{R_\Delta}(e_i R_\Delta, e_j R_\Delta) \longrightarrow \operatorname{Hom}_R(e_{\Delta(i)} R, e_{\Delta(j)} R)$$

is chosen to be $\operatorname{id}_{R_{\Delta(j),\Delta(i)}}$.

We now claim that G sends torsion modules to torsion modules. As G is right exact and commutes with direct sums, it is compatible with direct limits. As such, it suffices to check that G sends finitely generated, right bounded R_Δ-modules to right-upper-bounded R-modules. To show this, let M be a finitely generated, right bounded R_Δ-module. There is a presentation

$$\bigoplus_m e_{i_m} R_\Delta \xrightarrow{f} \bigoplus_{n=0}^{n_0} e_{j_n} R_\Delta \longrightarrow M \longrightarrow 0$$

and there is a $u_0 \in \mathbb{Z}$ such that f is surjective in all degrees u with $u \geq u_0$. Moreover we can assume $u_0 \geq j_n$ for all n. Now write $\Delta(u_0) = (a_0, b_0)$. The fact that R_+ is assumed to be generated in degree $(0,1)$ and $(1,0)$ implies that the induced map

$$G(f) : \bigoplus_m e_{\Delta(i_m)} R \longrightarrow \bigoplus_{n=0}^{n_0} e_{\Delta(j_n)} R$$

is surjective in all degrees (a,b) with $a \geq a_0, b \geq b_0$. This implies $G(M)_{(a,b)} = 0$ for all such a, b, hence $G(M)$ is right-upper-bounded. In particular the functor G induces a functor $G : \mathrm{QGr}(R_\Delta) \to \mathrm{QGr}(R)$.

It is immediate that $G \circ F = \mathrm{id}$. By Lemma 0.6.7 below it now suffices to check that $F(G(e_{\Delta(i)}R)) = e_{\Delta(i)}R$. This is however obvious. $\square$

Lemma 0.6.7. *The collection $\{e_{\Delta(i)}R \mid i \in \mathbb{Z}\}$ (or rather the set of corresponding objects in $\mathrm{QGr}(R)$) forms a set of generators for $\mathrm{QGr}(R)$.*

Proof. As the collection $\{e_{(m,n)}R \mid m, n \in \mathbb{Z}\}$ forms a set of generators for $\mathrm{Gr}(R)$ and hence also for $\mathrm{QGr}(R)$ it suffices to show that every $e_{(m,n)}R$ is a quotient of a direct sum of objects $e_{\Delta(i)}R$ in $\mathrm{QGr}(R)$. For this, fix some $m, n \in \mathbb{Z}$. We claim that there are surjective maps (in $\mathrm{QGr}(R)$)

$$e_{(m+1,n)}R^{\oplus N} \longrightarrow e_{(m,n)}R \tag{0.11}$$

and

$$e_{(m,n+1)}R^{\oplus N'} \longrightarrow e_{(m,n)}R \tag{0.12}$$

As Δ was assumed to be diagonal-like, there exist integers $a, b, i \in \mathbb{Z}$ such that $a, b \geq 0$ and $\Delta(i) = (m+a, n+b)$. In particular the surjective maps (0.11) and (0.12) give rise to a surjective map $e_{\Delta(i)}R^{\oplus N''} \to e_{(m,n)}R$. Hence the lemma follows from the claims. As both claims are similar we only prove (0.11). To show this, let

$$N = \dim_k(\mathrm{Hom}(e_{m+1,n}R, e_{m,n}R)) = \dim_k(R_{(m,n),(m+1,n)}).$$

As we assumed R to be positively generated in degree (1,0) and (0,1) there is a map

$$e_{(m+1,n)}R^{\oplus N} \longrightarrow e_{(m,n)}R$$

in $\mathrm{Gr}(R)$ whose cokernel lives in degrees (x,y) with either $x \leq m$ or $y < n$. As such this cokernel is torsion and the induced map in $\mathrm{QGr}(R)$ is surjective. $\square$

Chapter 1

Noncommutative versions of some classical birational transformations.

No rational argument will have a rational effect on a man who does not want to adopt a rational attitude.

Karl R. Popper (Philosopher)

This chapter is based on the results obtained by the author and Michel Van den Bergh in [PVdB16], [Pre16] and [Pre17].

1.1 Introduction

Convention 1.1.1. *Throughout this chapter, k will always denote an algebraically closed field of characteristic zero. Moreover we only consider Sklyanin algebras $A = A(Y, \mathcal{L}, \sigma)$ for which the order of σ is sufficiently large. Concretely, we always assume $\sigma^{s+1} \neq \mathrm{id}$ where, as above, s is the degree of the relations in A.*

As mentioned in the previous chapter, the study of noncommutative surfaces naturally leads to the study of graded (noncommutative) k-algebras A of global dimension and Gelfand–Kirilov dimension equal to three. A classification of (well-behaved classes of) such algebras is hence one of the ultimate goals of this area of research. In [Art97] Artin posed a conjecture classifying noetherian, connected graded domains of Gelfand–Kirillov dimension 3 into a short list of cases of which Sklyanin algebras are the generic case. This classification was upto birational equivalence, where two rings R and S are said to be birationally equivalent if their function fields (see Remark 0.1.4) $\mathrm{Frac}(R)_0$ and $\mathrm{Frac}(S)_0$ are isomorphic. A proof of this conjecture is far beyond the scope of this thesis. Instead in this chapter we focus on understanding birational equivalences between cubic and quadratic Sklyanin-algebras.

One of the main results of this chapter is the following result announced in [SVdB01]. A similar result by Rogalski-Sierra-Stafford was announced in [Sie14].

Theorem (Theorem 1.4.1)**.** *If A, A' are a cubic and a quadratic Sklyanin algebra respectively with geometric data (Y, σ) and (Y, ψ) such that $\sigma^4 = \psi^3$. Then* $\operatorname{Proj} A$ *and* $\operatorname{Proj} A'$ *have the same function field.*

Remark 1.1.2. *In Theorem 1.4.1 we actually prove a stronger result in the sense that we allow A to be a so-called cubic Sklyanin $\mathbb{Z}$-algebra (see Definition 1.3.1 and Remark 0.4.8). Throughout this introduction we will only state the main results for (graded, three dimensional) Sklyanin algebras although most of them also hold for cubic Sklyanin $\mathbb{Z}$-algebras.*

The proof of Theorem 1.4.1 is geometric. In the commutative case the passage from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$ goes by blowing up a point p and then contracting the strict transforms of the two rulings through this point. One may short circuit this construction by considering a suitable linear system on $\mathbb{P}^1 \times \mathbb{P}^1$ with base point in p.

Recall that if $\delta = (H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{L}) - 0)/k^*$ is some linear system on $\mathbb{P}^1 \times \mathbb{P}^1$, then it defines a rational map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ where n is the dimension of the linear system (i.e. $\mathcal{L}$ has $n+1$ linearly independent global sections). This map is defined everywhere except for the base points of δ. In particular the classical birational transformation $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ should be given by a 2-dimensional linear system with exactly one base point p. It is well known that choosing $\mathcal{L}$ to be $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)(-p)$ gives such a linear system. Hence the birational transformation is induced by the isomorphism

$$\mathbb{P}^2 \cong \operatorname{Proj}\big(\Gamma(\mathbb{P}^1 \times \mathbb{P}^1, \operatorname{Sym}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)(-p))\big)$$

It is this construction that we generalize first. Hence we need to know the following:

0. We start from a noncommutative surface represented by $\operatorname{QGr}(A)$ where $A = A(Y, \mathcal{L}, \sigma)$ is a cubic Sklyanin algebra.
1. We need to make sense of sheaf-like objects on a noncommutative surface and we should be able to repeatedly take the tensor product of these objects.
2. We need to make sense of the ideal "sheaf", of an analogue of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)$ and of symmetric algebras.
3. We need to make sense of global sections.

These questions are solved in the following way

1. We introduce bimodules in §1.2. In order to have a nice abelian category, these should be defined very abstractly (opposite category of left exact functors). But

the bimodules we encounter are much more down to earth. For example, the bimodules we encounter on the curve Y are given by classical sheaves.

2. Let p be a point on Y, then there is a bimodule o_p corresponding to p. Similarly there is a bimodule o_X corresponding to the identity functor on $X := \mathrm{QGr}(A)$ and there is a surjection $o_X \twoheadrightarrow o_p$. We let m_p be the kernel. The correct analogue of $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)$ is given by the double shift functor $o(2)$. (see Remark 1.3.4 for the rationale of this argument). In order to make a symmetric algebra out of these data (m_p and the shift functor $o(2)$ do not commute!) we need to let σ (or rather $\tau = \sigma^4$) act on d in order for the multiplication on the symmetric algebra to make sense.

3. We define $\Gamma(X,-) = \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X \otimes -)$ where $\mathcal{O}_X = \pi(A)$. A priori this functor need not be left exact anymore, but for our applications it will be.

So what we will actually do is the following: let A be a cubic Sklyanin algebra, let $\check{A}$ be the associated $\mathbb{Z}$-algebra and $\check{A}^{(2)}$ its 2-Veronese, i.e.

$$\check{A}^{(2)}_{i,j} = \check{A}_{2i,2j} = A_{2j-2i}$$

We will construct a sub-$\mathbb{Z}$-algebra D of $\check{A}^{(2)}$. This construction depends on the choice of a point $p \in Y$. Our goal is to prove that D is a 3-dimensional quadratic Artin-Schelter-regular $\mathbb{Z}$-algebra in the sense of [VdB11]. Again invoking [VdB11] this $\mathbb{Z}$-algebra must correspond to a 3-dimensional quadratic Artin-Schelter-regular *graded algebra* A'. It will turn out that A' is in fact a Sklyanin algebra and that the geometric data of A and A' are related as in Theorem 1.4.1. Note that the use of $\mathbb{Z}$-algebras is essential here as there is no direct embedding $A' \hookrightarrow A^{(2)}$ of graded rings. The above construction as well as the proof of Theorem 1.4.1 are covered in §1.3 and §1.4.

In §1.5 we use a similar technique to construct noncommutative versions of the classical transformation $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ as inclusions $\check{A}' \hookrightarrow \check{A}$ where A is a quadratic Sklyanin algebra and A' is a cubic Sklyanin algebra.

In §1.6 we use these techniques once more to construct noncommutative versions of the "Cremona transform" $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Commutatively this birational transformation is obtained by blowing up the three vertices of a triangle in $\mathbb{P}^2$ and then contracting the sides. Algebraically there is a corresponding inclusion $k[x,y,z] \hookrightarrow k[x,y,z]^{(2)}$. A noncommutative Cremona transform is obtained as an inclusion $\check{A}' \hookrightarrow \check{A}^{(2)}$ where both A and A' are quadratic Sklyanin algebras. As in the commutative case (see figure 1.1) a noncommutative Cremona transform naturally factors through $\mathbb{P}^1 \times \mathbb{P}^1$, giving the main result of §1.6:

Theorem (Theorem 1.6.4). *Let $\gamma : \check{A}' \hookrightarrow \check{A}^{(2)}$ be a noncommutative Cremona transform between quadratic Sklyanin algebras A and A'. Then there exists a cubic Sklyanin algebra A'' such that γ factors as $\gamma_2 \circ \gamma_1$ where $\gamma_1 : \check{A}' \hookrightarrow \check{A}''^{(2)}$ and $\gamma_2 : \check{A}'' \hookrightarrow \check{A}$ are noncommutative versions of $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ and $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ as in §1.3 and §1.5 respectively.*

Inspired by Theorem 1.6.4 we introduce quadratic transforms in §1.7 as any inclusion $\check{A}' \hookrightarrow \check{A}^{(2^v)}$ between quadratic or cubic Sklyanin algebras which factors as a composition of noncommutative versions of $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ or $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ as in §1.3 or §1.5. Our main result on quadratic transforms is that they are invertible in the following sense:

Theorem (Proposition 1.7.4 and Theorem 1.7.6). *Let $\gamma : \check{A}' \hookrightarrow \check{A}^{(2^v)}$ be a quadratic transform between cubic or quadratic Sklyanin algebras. Then there exists a quadratic transform $\delta : \check{A} \hookrightarrow \check{A}'^{(2^w)}$ such that $\delta \circ \gamma$ and $\gamma \circ \delta$ induce the identity on the function-fields of A' and A respectively.*

We first show (Lemma 1.7.7) that it suffices to prove the case where $\gamma : \check{A}' \hookrightarrow \check{A}^{(2)}$ is a noncommutative version of $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ as in §1.3 or $\gamma : \check{A}' \hookrightarrow \check{A}$ is a noncommutative version of $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ as in §1.5. Both cases are proven individually in §1.8. The proof is quite technical and uses a $\mathbb{Z}^2$-algebra $\tilde{A}$ "containing" both $\check{A}$ and $\check{A}'$ as respectively a column and a row. Moreover we show that the constructions in §1.3 and §1.5 are "mutually inverse" whenever certain geometric conditions (see (1.70) or (1.79)) are satisfied.

In §1.9 we prove (Theorem 1.9.6) that the $\mathbb{Z}^2$-algebras $\tilde{A}$ as appearing in §1.8 are noetherian in the sense that $\mathrm{Gr}(\tilde{A})$ is a locally noetherian category. As in the graded case, this result is based on the fact (Theorem 1.10.5) that a certain generalization of a twisted homogeneous coordinate ring is noetherian.

In §1.11 we combine Theorem 1.9.6 with the technical result from Theorem 0.6.6 to conclude that $\mathrm{QGr}(\tilde{A}) \cong \mathrm{QGr}(\tilde{A}_\Delta)$ for each diagonal-like $\mathbb{Z}$-subalgebra $\tilde{A}_\Delta$ of $\tilde{A}$. Moreover when we focus on a specific Δ the methods in §1.8 provide us with inclusions

$$\tilde{A}_\Delta \hookrightarrow \check{A}^{(4)} \quad \text{and} \quad \tilde{A}_\Delta \hookrightarrow \check{A}'^{(3)} \tag{1.1}$$

We show that these inclusions are simultaneously compatible with the 1-periodicity of $\check{A}^{(4)}$ and $\check{A}'^{(3)}$. This implies that there is a graded algebra T for which $\check{T} \cong \tilde{A}_\Delta$ and the inclusions in (1.1) give rise to inclusions $T \hookrightarrow A^{(4)}$ and $T \hookrightarrow A'^{(3)}$. Finally, we check that these inclusions give T the construction of a noncommutative blowup $A^{(4)}(p)$ and $A'^{(3)}(p' + q')$ as in [RSS14], resulting in the main result of §1.11:

Theorem (Theorem 1.11.3)**.** *Let $\gamma : \check{A}' \hookrightarrow \check{A}^{(2)}$ be a quadratic transform constructed starting from a point $p \in Y$ as in §1.3. Let $\delta : \check{A} \hookrightarrow \check{A}'$ be the inverse birational transform constructed starting from a points $p', q' \in Y$ as in §1.8 and let $\tilde{A}$, $\tilde{A}'$ be the associated $\mathbb{Z}^2$-algebras. Then A and A' contain a common blowup*

$$A^{(4)}(p) \cong T \cong A'^{(3)}(p' + q')$$

and there are equivalences of categories

$$\mathrm{QGr}(\tilde{A}) \cong \mathrm{QGr}(T) \cong \mathrm{QGr}(\tilde{A}')$$

Remark 1.1.3. *With the exception of the equivalence of categories the above result was announced independently by Rogalski, Sierra and Stafford in [RSS16, Theorem 1.7].*

In §1.12 we show how the results from §1.8-1.11 can be obtained for the noncommutative Cremona transformations introduced in §1.6.

Finally §1.13 introduces the theory of I-bases for (quadratic) Sklyanin algebras. We use this theory in the technical proofs of Lemma 1.8.4 and Proposition 1.12.2.

1.2 Bimodules over noncommutative schemes

It will be convenient to use the formalism of noncommutative geometry used in [VdB01] which we summarize here. For more details we refer to op. cit.. See also [Smi00]. We will change the terminology and notations slightly to be more compatible with current conventions. Recall from the previous chapter that in this formalism a noncommutative scheme is represented by a Grothendieck category (i.e. an abelian category with a generator and exact filtered colimits) and that each commutative scheme X induces a noncommutative scheme $\mathrm{Qcoh}(X)$. To emphasize that we think of noncommutative schemes as geometric objects, we will use this notation more generaly. I.e. we will denote noncommutative schemes by roman capitals X, Y, $\ldots$. whereas we use the notation $\mathrm{Qcoh}(X)$ to explicitely denote the category represented by a noncommutative scheme X.

A morphism $\alpha : X \to Y$ between noncommutative schemes will be a right exact functor $\alpha^* : \mathrm{Qcoh}(Y) \to \mathrm{Qcoh}(X)$ possessing a right adjoint (denoted by α_*). In this way the noncommutative schemes form a category (more accurately: a two-category).

If X is a noncommutative scheme then we think of objects in $\mathrm{Qcoh}(X)$ as sheaves of right modules on X. To define the analogue of a sheaf of algebras on X however we need a category of bimodules on X (see [VdB96] for the case where X is commutative).

The most obvious way to proceed is to define the category $\mathrm{BiMod}(X-Y)$ of $X-Y$-bimodules as the right exact functors $\mathrm{Qcoh}(X) \to \mathrm{Qcoh}(Y)$ commuting with direct limits. The action of a bimodule $\mathcal{N}$ on an object $\mathcal{M} \in \mathrm{Qcoh}(X)$ is written as $\mathcal{M} \otimes_X \mathcal{N}$.

If we define the "tensor product" of bimodules as composition then we can define algebra objects on X as algebra objects in the category of $X-X$-bimodules and in this we may extend much of the ordinary commutative formalism. For example the identity functor $\mathrm{Qcoh}(X) \to \mathrm{Qcoh}(X)$ is a natural analogue of the structure sheaf, and as such it will be denoted by o_X. If $\mathcal{A}$ is an algebra object on X then it is routine to define an abelian category $\mathrm{Mod}(\mathcal{A})$ of right-$\mathcal{A}$-modules. We have $\mathrm{Mod}(o_X) = \mathrm{Qcoh}(X)$. Unraveling all the definitions it turns out that $-\otimes_X-$ (the "tensor product" (composition) in the monoidal category $\mathrm{BiMod}(X-X)$) and $-\otimes_{o_X}-$ (the tensor product over the algebra o_X) have the same meaning. We will use both notations, depending on the context.

Unfortunately $\mathrm{BiMod}(X-Y)$ appears not to be an abelian category and this represents a technical inconvenience which is solved in [VdB01] by embedding the category $\mathrm{BiMod}(X-Y)$ into a larger category $\mathrm{BIMOD}(X-Y)$ consisting of "weak bimodules". The category $\mathrm{BIMOD}(X-Y)$ is opposite to the category of left exact functors $\mathrm{Qcoh}(Y) \to \mathrm{Qcoh}(X)$. Since left exact functors are determined by their values on injectives, they trivially form an abelian category. $\mathrm{BiMod}(X-Y)$ is the full subcategory of $\mathrm{BIMOD}(X-Y)$ consisting of functors having a left adjoint. Or equivalently: functors commuting with direct products.

This being said, these technical complication will be invisible in this chapter as all bimodules we encounter will be in $\mathrm{BiMod}(X-Y)$.

If $\mathcal{M} \in \mathrm{Qcoh}(X)$ then we define the *global sections* of $\mathcal{M}$ as

$$\Gamma(X,\mathcal{M}) = \mathrm{Hom}_X(\mathcal{O}_X,\mathcal{M}) \tag{1.2}$$

Similarly we define the global sections of an $X-X$-bimodule $\mathcal{N}$ as in [VdB01, Section 3.5]:

$$\Gamma(X,\mathcal{N}) := \mathrm{Hom}(\mathcal{O}_X,\mathcal{O}_X \otimes_{o_X} \mathcal{N}) \tag{1.3}$$

Use of the functor $\Gamma(X,-)$ on bimodules requires some care since it is a priori not left exact. However in our applications it will be left exact, by [VdB01, Lemma 8.2.1.].

Note that $\mathcal{N}$ is an algebra object in the category of bimodules then $\Gamma(X,\mathcal{N})$ is in fact an algebra for purely formal reasons. The same holds true for graded algebras and $\mathbb{Z}$-algebras.

If A be a graded algebra then the associated noncommutative scheme $X = \mathrm{Proj}\, A$ is represented by the category $\mathrm{Qcoh}(X) := \mathrm{QGr}(A) = \mathrm{Gr}(A)/\mathrm{Tors}(A)$, as discussed above. We denote the quotient functor $\mathrm{Gr}(A) \to \mathrm{QGr}(A)$ by π. The object πA is denoted by $\mathcal{O}_X$. The "shift by n" functor $\mathrm{Qcoh}(X)$ is written as $\mathcal{M} \mapsto \mathcal{M}(n)$ and the corresponding bimodule is written as $o_X(n)$. In particular $o_X = o_X(0)$ and

$$\mathcal{O}_X(n) = \mathcal{O} \otimes_{o_X} o_X(n) = \pi(A(n)) \tag{1.4}$$

In the light of Remark 0.2.2 this formalism can be extended to $\mathbb{Z}$-algebras: if R is any reasonably behaved $\mathbb{Z}$-algebra (i.e. a connected noetherian $\mathbb{Z}$-algebra), it makes sense to consider the noncommutative scheme X defined by $\mathrm{Qcoh}(X) = \mathrm{QGr}(R)$. The "shift by n" functor $\mathrm{Qcoh}(X) \to \mathrm{Qcoh}(X)$ only exists if R is n-periodic as in Definition 0.2.3. Nevertheless the objects $\mathcal{O}_X(i)$ exist for all $i \in \mathbb{Z}$ and are defined by

$$\mathcal{O}_X(i) = \pi(e_{-i}R) \tag{1.5}$$

It is an easy exercise to check that if R in n-periodic there are isomorphisms

$$\mathcal{O}_X(i) \otimes o_X(n) \cong \mathcal{O}_X(i+n)$$

for all i.

1.3 Construction of a noncommutative map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$

1.3.1 Some notations and technical results

Throughout this section and the subsequent sections we define noncommutative birational transformations between quadratic and cubic Sklyanin algebras (see §0.1 for their definitions). Moreover we also consider the following generalization of cubic Sklyanin algebras:

Definition 1.3.1. Let $A = A(Y, (\mathcal{L}_i)_{i\in\mathbb{Z}})$ be a cubic AS-regular algebra. By Theorem 0.3.4 there exists an $\alpha \in \mathrm{Aut}(Y)$ such that $\alpha^*\mathcal{L}_i \cong \mathcal{L}_{i+2}$ holds for all i. A is called a *cubic Sklyanin $\mathbb{Z}$-algebra* if the following conditions are satisfied:

- Y is a smooth elliptic curve
- α is an automorphism given by translation. Moreover $\alpha^2 \neq \mathrm{id}$

In particular if $A = A(Y, \mathcal{L}, \sigma)$ is cubic Sklyanin algebra with $\sigma^4 \neq \mathrm{id}$, $\check{A}$ is a cubic Sklyanin $\mathbb{Z}$-algebra.

As mentioned in §0.3 AS-regular $\mathbb{Z}$-algebras are classified in terms of a k-scheme Y and an elliptic helix $(\mathcal{L}_i)_{i\in\mathbb{Z}}$. We will only consider the special case of quadratic/cubic Sklyanin ($\mathbb{Z}$-)algebras. In particular Y will always be an elliptic curve for us. In this restricted setting elliptic helices are defined as follows:

Definition 1.3.2. Let Y be an elliptic curve and $(\mathcal{L}_i)_{i\in\mathbb{Z}}$ a collection of line bundles on Y then $(\mathcal{L}_i)_{i\in\mathbb{Z}}$ is called a *quadratic elliptic helix* if

- $\deg(\mathcal{L}_i) = 3$
- $\mathcal{L}_0 \ncong \mathcal{L}_1$
- $\mathcal{L}_i \otimes \mathcal{L}_{i+1}^{\otimes -2} \otimes \mathcal{L}_{i+2} \cong \mathcal{O}_Y$

And it is called a *cubic elliptic helix* if

- $\deg(\mathcal{L}_i) = 2$
- $\mathcal{L}_0 \ncong \mathcal{L}_2$
- $\mathcal{L}_i \otimes \mathcal{L}_{i+1}^{\otimes -1} \otimes \mathcal{L}_{i+2}^{\otimes -1} \otimes \mathcal{L}_{i+3} \cong \mathcal{O}_Y$

For the remainder of this section A is a cubic Sklyanin $\mathbb{Z}$-algebra with associated elliptic curve Y, elliptic helix $(\mathcal{L}_i)_{i\in\mathbb{Z}}$, $\alpha \in \mathrm{Aut}(Y)$ and twisted homogeneous coordinate $\mathbb{Z}$-algebra $B(Y,(\mathcal{L}_i)_{i\in\mathbb{Z}})$ (see (0.8)). We let X be the noncommutative scheme represented by $\mathrm{QGr}(A)$ and define the objects $\mathcal{O}_X(i)$ as in (1.5). We can use these objects to recover the algebra A as follows:

$$\begin{aligned}
\mathrm{Hom}(\mathcal{O}_X(-j),\mathcal{O}_X(-i)) &= \mathrm{Hom}_{\mathrm{QGr}(A)}(\pi(e_jA),\pi(e_iA)) \\
&= \lim_{n\to\infty} \mathrm{Hom}_{\mathrm{Gr}(A)}(e_jA_{\geq n}, e_iA) \\
&= \mathrm{Hom}_{\mathrm{Gr}(A)}(e_jA, e_iA) \qquad (1.6)\\
&= A_{i,j}
\end{aligned}$$

where the second equality follows from the fact that e_jA and e_iA are finitely generated A-modules (see [AZ94, Proposition 7.2] for the graded case). The third equality follows from the Gorenstein condition as this implies

$$\mathrm{Hom}_{\mathrm{Gr}(A)}(e_jA/e_jA_{\geq} n, e_iA) = \mathrm{Ext}^1_{\mathrm{Gr}(A)}(e_jA/e_jA_{\geq} n, e_iA) = 0$$

We will also need the following cohomology result:

Lemma 1.3.3. *Let A be an AS-regular $\mathbb{Z}$-algebra of dimension 3 with Hilbert function h and let $X = \mathrm{QGr}(A)$ and $\mathcal{O}_X(i) = \pi(e_{-i}A)$. Let $s = 2$ for quadratic AS-regular $\mathbb{Z}$-algebras and $s = 3$ for cubic AS-regular $\mathbb{Z}$-algebras (i.e. quadrics). Then we have*

$$\mathrm{Ext}^n(\mathcal{O}_X(i),\mathcal{O}_X(j)) = 0 \text{ for } n \neq 0,2$$

$$\dim_k\left(\mathrm{Ext}^2(\mathcal{O}_X(i),\mathcal{O}_X(j))\right) = \begin{cases} 0 & \text{if } j \geq i-s \\ h(i-s-1-j) & \text{if } j \leq i-s-1 \end{cases}$$

Proof. This is a $\mathbb{Z}$-algebra version of [AZ94, Theorem 8.1]. $\square$

Similarly we can define objects $\mathcal{O}_Y(i) := \pi(e_{-i}B) \in \mathrm{QGr}(B)$. Taking in mind the equivalence of categories $\mathrm{QGr}(B) \cong \mathrm{Qcoh}(Y)$ (see (0.9)) these objects can be identified with

$$\begin{cases} \mathcal{L}_{-i}\mathcal{L}_{-i+1}\dots\mathcal{L}_{-1} & \text{if } i > 0 \\ \mathcal{O}_Y & \text{if } i = 0 \\ (\mathcal{L}_0\mathcal{L}_1\dots\mathcal{L}_{-i-1})^{-1} & \text{if } i < 0 \end{cases} \tag{1.7}$$

Remark 1.3.4. *In particular in case A is a* linear *quadric (we do not consider these algebras in our construction, but the interested reader can find the definition in [VdB11, Proposition 5.1.2.]) we have $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$, $\mathcal{L}_0 \cong \mathcal{O}_Y(1,0)$ and $\mathcal{L}_1 \cong \mathcal{O}_Y(0,1)$. Using this, one identifies the object $\mathcal{O}_Y(2) \in \mathrm{QGr}(B)$ with $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1) \in \mathrm{Qcoh}(Y)$. This explains why $\mathcal{O}_Y(2)$ or rather the bimodule $o_Y(2)$ serves as a noncommutative analogue of $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)$ in our construction.*

The $\mathbb{Z}$-algebra morphism $A \to B$ induces an inclusion functor

$$\mathrm{Qcoh}(Y) \hookrightarrow \mathrm{Qcoh}(X).$$

The left adjoint (corresponding to $-\otimes_A B$) is denoted by

$$-\otimes_{o_X} o_Y : \mathrm{Qcoh}(X) \longrightarrow \mathrm{Qcoh}(Y)$$

and obviously $\mathcal{O}_X(i) \otimes_{o_X} o_Y \cong \mathcal{O}_Y(i)$.

We will routinely regard a sheaf of $\mathcal{O}_Y$-modules $\mathcal{N}$ as an object in $\mathrm{BiMod}(Y-Y)$ by identifying it with the functor $-\otimes_{\mathcal{O}_Y}\mathcal{N}$. It is easy to see that the resulting functor

$$\mathrm{Qcoh}(Y) \longrightarrow \mathrm{BiMod}(Y-Y) \subset \mathrm{BIMOD}(Y-Y) \tag{1.8}$$

is fully faithful and exact.

Similarly we regard an $Y-Y$-bimodule $\mathcal{M}$ as an $X-X$-bimodule by defining the corresponding functor to be

$$\mathrm{Qcoh}(X) \xrightarrow{-\otimes_{o_X} o_Y} \mathrm{Qcoh}(Y) \xrightarrow{-\otimes_{o_Y}\mathcal{M}} \mathrm{Qcoh}(Y) \hookrightarrow \mathrm{Qcoh}(X) \tag{1.9}$$

In this way o_Y becomes an $X-X$-bimodule and one checks that it is in fact an algebra quotient of o_X. Note that o_Y now denotes both an algebra on X and an algebra on Y (the identity functor) but for both interpretations we have $\mathrm{Mod}(o_Y) \cong \mathrm{Qcoh}(Y)$.

For use in the sequel we also write

$$o_X(-Y) = \ker(o_X \longrightarrow o_Y)$$

Similar to the graded case, for each i there is an element $g_i \in A_{i,i+4} \setminus \{0\}$ and a short exact sequence of A-modules

$$0 \longrightarrow g_iA \longrightarrow e_iA \longrightarrow e_iB \longrightarrow 0$$

Moreover by Theorem 0.4.2 the map $e_{i+4}A \xrightarrow{\cdot g_i} g_iA$ is an isomorphism. From this we obtain the following identification for all $j \in \mathbb{Z}$:

$$\mathcal{O}_X(j) \otimes o_X(-Y) \cong \mathcal{O}_X(j-4) \tag{1.10}$$

Finally Theorem 0.3.4 and Definition 1.3.2 imply

$$o_Y(2) \cong \alpha_*(- \otimes_{\mathcal{O}_Y} \mathcal{L}_0 \otimes_{\mathcal{O}_Y} \mathcal{L}_1) \tag{1.11}$$

(we may verify this by applying both sides on the objects $\mathcal{O}_Y(n)$ as these generate $\mathrm{Qcoh}(Y) \cong \mathrm{QGr}(B)$)

1.3.2 A subalgebra D of $A^{(2)}$

As mentioned in the introduction of this chapter we start the construction of a non-commutative $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ by first constructing a sub-$\mathbb{Z}$-algebra D of $A^{(2)}$ where A is a cubic Sklyanin-$\mathbb{Z}$-algebra.

We starting by fixing a point $p \in Y$. Using (1.8) we can associate a Y-bimodule o_p to the sheaf $\mathcal{O}_p$. By (1.9) this can also be seen as an X-bimodule. It is not hard to see that o_p is a quotient of o_Y and o_X and we define

$$m_{p,Y} = \ker(o_Y \longrightarrow o_p) \tag{1.12}$$

$$m_p = \ker(o_X \longrightarrow o_p) \tag{1.13}$$

A priori these elements only exist in $\mathrm{BIMOD}(Y-Y)$ and $\mathrm{BIMOD}(X-X)$. However we can easily see that $m_{p,Y} \in \mathrm{BiMod}(Y-Y)$ as $m_{p,Y}$ corresponds to an ordinary ideal sheaf in $\mathcal{O}_Y$ (see (1.8) above). The fact that $m_p \in \mathrm{BiMod}(X-X)$ follows by applying [VdB01, Corollary 5.5.6].

Finally consider the following bimodules over X, respectively Y:

$$(\mathcal{D}_Y)_{m,n} = \begin{cases} o_Y(-2m) \otimes_{o_Y} m_{\tau^{-m}p,Y} \dots m_{\tau^{-n+1}p,Y} \otimes_{o_Y} o_Y(2n) & \text{if } n \geq m \\ 0 & \text{if } n < m \end{cases} \tag{1.14}$$

$$\mathcal{D}_{m,n} = \begin{cases} o_X(-2m) \otimes_{o_X} m_{\tau^{-m}p} \dots m_{\tau^{-n+1}p} \otimes_{o_X} o_X(2n) & \text{if } n \geq m \\ 0 & \text{if } n < m \end{cases} \tag{1.15}$$

where $\tau = \alpha^2$ and $m_{\tau^{-k}p} \dots m_{\tau^{-l}p}$ is the image of

$$m_{\tau^{-l}p} \otimes_X \cdots \otimes_X m_{\tau^{-l}p} \longrightarrow o_X \otimes_X \cdots \otimes_X o_X = o_X$$

A priori this image lies only in $\mathrm{BIMOD}(X-X)$ but with the same method as the proof of [VdB01, Proposition 6.1.1] one verifies that it lies in fact in $\mathrm{BiMod}(X-X)$.

The collections of bimodules $\mathcal{D} \overset{\text{def}}{=} \bigoplus_{m,n} \mathcal{D}_{m,n}$, $\mathcal{D}_Y \overset{\text{def}}{=} \bigoplus_{m,n} (\mathcal{D}_Y)_{m,n}$ represent $\mathbb{Z}$-algebra objects respectively in $\operatorname{BiMod}(X - X)$ and $\operatorname{BiMod}(Y - Y)$. For example the product

$$\mathcal{D}_{m,n} \otimes_{o_X} \mathcal{D}_{n,l}$$

is given by

$$\begin{aligned} o_X(-2m) \otimes m_{\tau^{-m}p} \dots m_{\tau^{-n+1}p} \otimes o_X(2n) \otimes o_X(-2n) \otimes m_{\tau^{-n}p} \dots m_{\tau^{-l+1}p} \otimes o_X(2l) \to \\ o_X(-2m) \otimes m_{\tau^{-m}p} \dots m_{\tau^{-n+1}p} \otimes m_{\tau^{-n}d} \dots m_{\tau^{-l+1}p} \otimes o_X(2l) \to \\ o_X(-2m) \otimes m_{\tau^{-m}p} \dots m_{\tau^{-n+1}p} m_{\tau^{-n}p} \dots m_{\tau^{-l+1}p} \otimes o_X(2l) \end{aligned}$$

Denote the global sections of $\mathcal{D}_Y$ and $\mathcal{D}$ by D_Y and D respectively. Hence, by definition (see (1.3)) D and D_Y are given by the following $\mathbb{Z}$-algebras:

$$(D_Y)_{m,n} = \begin{cases} \operatorname{Hom}\left(\mathcal{O}_Y(-2n), \mathcal{O}_Y(-2m) \otimes_{o_Y} m_{\tau^{-m}p,Y} \dots m_{\tau^{-n+1}p,Y}\right) & \text{if } n \geq m \\ 0 & \text{if } n < m \end{cases} \tag{1.16}$$

$$D_{m,n} = \begin{cases} \operatorname{Hom}\left(\mathcal{O}_X(-2n), \mathcal{O}_X(-2m) \otimes_{o_X} m_{\tau^{-m}p} \dots m_{\tau^{-n+1}p}\right) & \text{if } n \geq m \\ 0 & \text{if } n < m \end{cases}$$

Using [VdB01, Lemma 8.2.1] (with $\mathcal{E} = \mathcal{O}_X$) and (1.6) the inclusion

$$\mathcal{D}_{m,n} \hookrightarrow o_X(2(n-m))$$

gives rise to an inclusion of $\mathbb{Z}$-algebras

$$D \hookrightarrow \bigoplus_{m,n\in\mathbb{Z}} \operatorname{Hom}_X(\mathcal{O}_X(-2n), \mathcal{O}_X(-2m)) = \bigoplus_{m,n\in\mathbb{Z}} A_{2m,2n} = A^{(2)} \tag{1.17}$$

Our aim is to show that D is a quadratic AS-regular $\mathbb{Z}$-algebra. We will prove this by the following steps:

1. In in §1.3.3 we show that D_Y is a $\mathbb{Z}$-analogue of a twisted homogeneous coordinate ring (see (0.9) in §0.3). I.e. $D_Y \cong B(Y, (\mathcal{G}_i)_{i\in\mathbb{Z}})$ for some elliptic helix $(\mathcal{G}_i)_i$.

2. Using a short exact sequence (see (1.27)) and some cohomology computations (Lemma 1.3.10 and Lemma 1.3.10) we prove that D is generated in degree 1 and compute its Hilbert series.

3. We show (Lemma 1.3.12) that the canonical map $D \to D_Y$ is surjective, such that by a Hilbert series argument we conclude that there must be an isomorphism $D \cong A(Y, (\mathcal{G}_i)_{i\in\mathbb{Z}})$

1.3.3 Analysis of D_Y

We now prove the existence of a quadratic elliptic helix $(\mathcal{G}_i)_i$ (see Definition 1.3.2) such that D_Y is isomorphic to the twisted homogeneous coordinate ring $B(Y,(\mathcal{G}_i)_{i\in\mathbb{Z}})$. I.e. we want to show that

$$(D_Y)_{m,n} \cong \Gamma\left(Y, \mathcal{L}_m \otimes \ldots \otimes \mathcal{L}_{n-1}\right) \tag{1.18}$$

holds for all m,n and that these isomorphisms are compatible with the algebra structure of D_Y and $B(Y,(\mathcal{G}_i)_{i\in\mathbb{Z}})$.

Recall that the functor $-\otimes_{o_Y} m_{p,Y}$ is given by $-\otimes_{\mathcal{O}_Y}\mathcal{O}_Y(-p)$ where $\mathcal{O}_Y(-p)$ is the ideal sheaf of p on Y. Similarly by (1.7) we can identify the objects $\mathcal{O}_Y(i)\in \mathrm{QGr}(B)$ with line bundles in $\mathrm{Qcoh}(Y)$. Using this, we find

$$\begin{aligned}
(D_Y)_{m,n} &= \mathrm{Hom}\left(\mathcal{O}_Y, \mathcal{O}_Y(-2m)\otimes m_{\tau^{-m}p,Y}\ldots m_{\tau^{-n+1}p,Y}\otimes o_Y(2n)\right)\\
&= \mathrm{Hom}\left(\mathcal{O}_Y(-2n), \mathcal{O}_Y(-2m)\otimes m_{\tau^{-m}p,Y}\ldots m_{\tau^{-n+1}p,Y}\right)\\
&= \mathrm{Hom}\Big(\left(\mathcal{L}_0\otimes\ldots\otimes\mathcal{L}_{2n-1}\right)^{-1},\\
&\qquad\qquad \left(\mathcal{L}_0\otimes\ldots\otimes\mathcal{L}_{2m-1}\right)^{-1}\otimes\mathcal{O}_Y(-\tau^{-m}p-\ldots-\tau^{-n+1}p)\Big)\\
&= \mathrm{Hom}\left(\mathcal{O}_Y, \mathcal{O}_Y(-\tau^{-m}p-\ldots-\tau^{-n+1}p)\otimes\mathcal{L}_{2m}\otimes\ldots\otimes\ \mathcal{L}_{2n-1}\right)\\
&= \Gamma\left(Y,\mathcal{O}_Y, \mathcal{O}_Y(-\tau^{-m}p-\ldots-\tau^{-n+1}p)\otimes\mathcal{L}_{2m}\otimes\ldots\otimes\mathcal{L}_{2n-1}\right)
\end{aligned} \tag{1.19}$$

(where without loss of generality we assumed $0\le m\le n$ in the above computations)

In particular, in order to prove (1.18) it suffices to define

$$\mathcal{G}_i = \mathcal{O}_Y(-\tau^{-i}p)\otimes\mathcal{L}_{2i}\otimes\mathcal{L}_{2i+1} \tag{1.20}$$

and to prove the following:

Lemma 1.3.5. *Let $(\mathcal{G}_i)_{i\in\mathbb{Z}}$ be as in (1.20) then:*

i) $\deg(\mathcal{G}_i) = 3$

ii) $\mathcal{G}_0 \ncong \mathcal{G}_1$

iii) $\mathcal{G}_i\otimes\mathcal{G}_{i+1}^{\otimes -2}\otimes\mathcal{G}_{i+2}\cong\mathcal{O}_Y$

Simultaneously we also prove

Lemma 1.3.6. *Let $(\mathcal{G}_i)_{i\in\mathbb{Z}}$ be as in (1.20) and let $\psi\in\mathrm{Aut}(Y)$ be an arbitrary translation satisfying $\psi^3=\alpha^2$, then*

$$\psi^*(\mathcal{G}_i)\cong\mathcal{G}_{i+1}$$

Proof of Lemma 1.3.5 and Lemma 1.3.6. $\deg(\mathcal{G}_i) = 3$ follows immediately from

$$\deg(\mathcal{L}_{2i}) = \deg(\mathcal{L}_{2i+1}) = 2 \text{ and } \deg(\mathcal{O}_Y(-\tau^{-i}p)) = -1.$$

Now let $\beta \in \operatorname{Aut}(Y)$ be a translation for such that $\beta^3 = \alpha$ and $\beta^2 = \psi$ (and hence $\beta^6 = \tau$).

By [VdB11, Theorem 4.2.3] there is an invertible sheaf $\mathcal{N}$ of degree zero on Y such that for each invertible sheaf $\mathcal{M}$ on Y we have the following identity in $\operatorname{Pic}(Y)$:

$$[\beta^* \mathcal{M}] = [\mathcal{M}] + \deg(\mathcal{M}) \cdot [\mathcal{N}] \tag{1.21}$$

In particular $\psi^* \mathcal{G}_i = \beta^{2*}\mathcal{G}_i \cong \mathcal{G}_{i+1}$ follows from the following

$$\begin{aligned}
[\beta^{2*}\mathcal{G}_i] - [\mathcal{G}_{i+1}] =& [\beta^{2*}\mathcal{L}_{2i}] + [\beta^{2*}\mathcal{L}_{2i+1}] + [\beta^{2*}\mathcal{O}_Y(-\tau^{-i}p)] \\
& - [\mathcal{L}_{2i+2}] - [\mathcal{L}_{2i+3}] - [\mathcal{O}_Y(-\tau^{-i-1}p)] \\
=& [\beta^{2*}\mathcal{L}_{2i}] + [\beta^{2*}\mathcal{L}_{2i+1}] + [\beta^{2*}\mathcal{O}_Y(-\tau^{-i}p)] \\
& - [\mathcal{L}_{2i+2}] - [\mathcal{L}_{2i+3}] - [\beta^{6*}\mathcal{O}_Y(-\tau^{-i}p)] \\
=& [\mathcal{L}_{2i}] + [\mathcal{L}_{2i+1}] + [\mathcal{O}_Y(-\tau^{-i}p)] - [\mathcal{L}_{2i+2}] - [\mathcal{L}_{2i+3}] - [\mathcal{O}_Y(-\tau^{-i}p)] \\
& + \Big(2\deg(\mathcal{L}_{2i}) + 2\deg(\mathcal{L}_{2i+1}) + 2\deg(\mathcal{O}_Y(-\tau^{-i}p)) \\
& - 6\deg(\mathcal{O}_Y(-\tau^{-i}p))\Big)[\mathcal{N}] \\
=& [\mathcal{L}_{2i}] + [\mathcal{L}_{2i+1}] - [\mathcal{L}_{2i+2}] - [\mathcal{L}_{2i+3}] + 12[\mathcal{N}] \\
=& [\mathcal{L}_{2i}] - ([\mathcal{L}_{2i+3}] - 3\deg(\mathcal{L}_{2i+3})[\mathcal{N}]) - [\mathcal{L}_{2i+2}] \\
& + ([\mathcal{L}_{2i+1}] + 3\deg(\mathcal{L}_{2i+1}[\mathcal{N}]) \\
=& [\mathcal{L}_{2i}] - [\alpha^{(-1)*}\mathcal{L}_{2i+3}] - [\mathcal{L}_{2i+2}] + [\alpha^*\mathcal{L}_{2i+1}] \\
=& [\mathcal{L}_{2i}] - [\mathcal{L}_{2i+1}] - [\mathcal{L}_{2i+2}] + [\mathcal{L}_{2i+3}] \\
=& [\mathcal{L}_{2i} \otimes \mathcal{L}_{2i+1}^{\otimes -1} \otimes \mathcal{L}_{2i+2}^{\otimes -1} \otimes \mathcal{L}_{2i+3}] \\
=& [\mathcal{O}_Y] \\
=& 0
\end{aligned}$$

In the second equality we used $\mathcal{O}_Y(-\tau^{-i-1}p) = \tau^*\mathcal{O}_Y(-\tau^{-i}p) = \beta^{6*}\mathcal{O}_Y(-\tau^{-i}p)$, in the seventh equality we used $\beta^{3*}\mathcal{L}_j = \alpha^*\mathcal{L}_j \cong \mathcal{L}_{j+2}$ and in the ninth equality used the fact that the $(\mathcal{L}_i)_{i\in\mathbb{Z}}$ is a cubic elliptic helix.

The proof of $\mathcal{G}_i \otimes \mathcal{G}_{i+1}^{\otimes -2} \otimes \mathcal{G}_{i+2} \cong \mathcal{O}_Y$ is slightly longer, but completely similar to the above.

Now assume by way of contradiction that $\mathcal{G}_0 \cong \mathcal{G}_1$. By the above this implies $\mathcal{G}_0 \cong \psi^*\mathcal{G}_0$ and hence $3[\mathcal{N}] = 0$ which in turn implies $\psi^3 = \mathrm{id}$. This is impossible as $\psi^3 = \alpha^2 \neq \mathrm{id}$. □

Upto now, we have proven that there are isomorphisms of vector spaces

$$(D_Y)_{m,n} \cong B(Y, (\mathcal{G}_i)_{i\in\mathbb{Z}})_{m,n}$$

as in (1.18). A routine but somewhat tedious verification shows that these isomorphism send the products in D_Y (induced by the algebra structure of $\mathcal{D}_Y$) to the obvious product in $B(Y, (\mathcal{G}_i)_{i\in\mathbb{Z}})$ (corresponding to the tensor product (of global sections) of line bundles). Hence (1.18) induces an isomorphism of $\mathbb{Z}$-algebras:

$$D_Y \cong B(Y, (\mathcal{G}_i)_{i\in\mathbb{Z}})$$

1.3.4 Showing that D is AS-regular

For use below recall some commutation formulas. Recall from (1.11) that $o_Y(2) = \alpha_*(- \otimes_{\mathcal{O}_Y} \mathcal{L}_0 \otimes_{\mathcal{O}_Y} \mathcal{L}_1)$. This implies

$$o_p \otimes_{o_Y} o_Y(2) = o_Y(2) \otimes_{o_Y} o_{\alpha p}$$

(we may see this by applying both sides to an object in $\mathrm{Qcoh}(Y)$). Using the definitions of m_p, $m_{p,Y}$ (see (1.12), (1.13)) we deduce from this

$$\begin{aligned} m_{p,Y} \otimes_{o_Y} o_Y(2) &= o_Y(2) \otimes_{o_Y} m_{\alpha p,Y} \\ m_p \otimes_{o_X} o_X(2) &= o_X(2) \otimes_{o_X} m_{\alpha p} \end{aligned} \tag{1.22}$$

Similar formulas hold for longer products of m's such as the ones appearing in the definition of $(\mathcal{D}_Y)_{m,n}$ and $\mathcal{D}_{m,n}$.

For the sequel we need a resolution of $\mathcal{O}_X(j) \otimes m_p$. To construct this, we first need the following $\mathbb{Z}$-algebra analogue of [ATVdB91, Proposition 6.7.i]:

Lemma 1.3.7. *Let A be as above and let p be a point in Y. Let $i \in \mathbb{Z}$ and let P_i be the shifted point module generated in degree i (i.e. a graded right A-module generated in degree i with Hilbert function $1, 1, 1, 1, \ldots$) corresponding to the point p. Then the minimal projective resolution for P_i is given by*

$$0 \longrightarrow e_{i+3}A \longrightarrow e_{i+1}A \oplus e_{i+2}A \longrightarrow e_iA \longrightarrow P_i \longrightarrow 0 \tag{1.23}$$

Proof. The proof in [ATVdB91, Proposition 6.7.i] can be adapted to a $\mathbb{Z}$-algebra version as follows:

First we note that the correspondence between point modules and points $p \in Y$ as established in [ATVdB91] (see also Definition 0.1.2 in §0.1) is generalized to $\mathbb{Z}$-algebras by [VdB01, §5] to obtain a 1-1-correspondence between point modules (truncated in degree i) and points on Y, so the correspondence between $p \in Y$ and P_i is well defined.

Next we generalize line modules to modules of the form e_jA/aA where $a \in A_{j,j+1}$. As A is a $\mathbb{Z}$-domain (Theorem 0.4.2) we know that line modules have the desired

Hilbert series and have projective dimension 1. The characterization of line modules as in [ATVdB91, Corollary 2.43] generalizes to this new definition of line modules if we replace the modules $A(-i)$ in the graded case by e_iA.

Finally we find (1.23) by generalizing the proof of [ATVdB91, Proposition 6.7.i]. This proof uses the fact that point modules in the graded case have projective dimension 2. A careful observation however shows that projective dimension ≤ 2 suffices for the proof. This in turn can be shown by a variation of [ATVdB91, Proposition 2.46.i]. $\square$

Lemma 1.3.8. *Let A and P_i be as above. Then there is a complex of the following form:*

$$0 \to e_{i+5}A \xrightarrow{(\zeta,0)} e_{i+4}A^{\oplus 2} \oplus e_{i+3}A \longrightarrow e_{i+2}A^{\oplus 3} \longrightarrow e_iA \longrightarrow P_i \to 0 \tag{1.24}$$

where ζ is part of the minimal resolution of S_{i+1} as given in [VdB01, Definition 4.1.1.]

$$0 \to e_{i+5}A \xrightarrow{\zeta} e_{i+4}A^{\oplus 2} \xrightarrow{\varepsilon} e_{i+2}A^{\oplus 2} \xrightarrow{\delta_0} e_{i+1}A \xrightarrow{\gamma} S_{i+1} \to 0 \tag{1.25}$$

Moreover the complex (1.24) *is exact everywhere except at e_iA where it has one-dimensional cohomology, concentrated in degree $i+1$.*

Proof. By Lemma 1.3.7 we know P_i has the following (minimal) resolution:

$$0 \to e_{i+3}A \longrightarrow e_{i+1}A \oplus e_{i+2}A \longrightarrow e_iA \longrightarrow P_i \to 0 \tag{1.26}$$

Combining this with the minimal resolution of S_{i+1} (recall the modules S_j were defined in §0.3) as given in (1.25) we obtain the following

$$\begin{tikzcd}
 & & 0 & & & \\
 & & S_{i+1} \arrow[u] & & & \\
0 \arrow[r] & e_{i+3}A \arrow[r, "\alpha"] \arrow[dr, "\exists\eta"'] & e_{i+1}A \oplus e_{i+2}A \arrow[u, "\gamma"'] \arrow[r, "\beta"] & e_iA \arrow[r, "\varphi"] & P_i \arrow[r] & 0 \\
 & & e_{i+2}A^{\oplus 2} \oplus e_{i+2}A \arrow[u, "\delta_0 \oplus \mathrm{id}"'] & & & \\
 & & e_{i+4}A^{\oplus 2} \arrow[u, "{(\varepsilon, 0)}"'] & & & \\
 & & e_{i+5}A \arrow[u, "\zeta"'] & & & \\
 & & 0 \arrow[u] & & &
\end{tikzcd}$$

Put $\delta = \delta_0 \oplus \mathrm{id}$. The existence of the map η such that $\delta \circ \eta = \alpha$ follows from the projectivity of $e_{i+3}A$ and the fact that $\gamma \circ \alpha$ is zero by degree reasons. By diagram chasing one easily finds that $\ker(\beta \circ \delta) = \mathrm{im}(\varepsilon) \oplus \mathrm{im}(\eta)$ and hence we end up with the following complex:

$$0 \longrightarrow e_{i+5}A \xrightarrow{\zeta \oplus 0} e_{i+4}A^{\oplus 2} \oplus e_{i+3}A \xrightarrow{\eta \oplus \varepsilon} e_{i+2}A^{\oplus 3} \xrightarrow{\beta \circ \delta} e_i A \xrightarrow{\varphi} P_i \longrightarrow 0$$

Using diagram chasing again one easily checks that this complex is exact everywhere except at $e_i A$. We then conclude with a Hilbert series argument. □

Note that

$$\mathcal{O}_X(j) \otimes m_p = \pi\left(\ker(e_{-j}A \longrightarrow P_{-j})\right)$$

where P_{-j} is as in Lemma 1.3.8.

In particular (1.24) induces a resolution for $\mathcal{O}_X(j) \otimes m_p$ of the form

$$0 \longrightarrow \mathcal{J}(j-2) \oplus \mathcal{O}_X(j-3) \longrightarrow \mathcal{O}_X(j-2)^{\oplus 3} \longrightarrow \mathcal{O}_X(j) \otimes m_p \longrightarrow 0 \tag{1.27}$$

where

$$\mathcal{J} := \mathrm{coker}\left(\mathcal{O}_X(-3) \xrightarrow{\zeta} \mathcal{O}_X(-2)^{\oplus 2}\right) \tag{1.28}$$

We will now prove some vanishing results. An object in $\mathrm{Qcoh}(X)$ will be said to have finite length if it is a finite extension of objects of the form $\mathcal{O}_p$, $p \in Y$. Likewise an object in $\mathrm{BiMod}(X - X)$ will be said to have finite length if it is a finite extension of o_p for $p \in Y$. The objects of finite length are fully understood, see [VdB01, Chapter 5]. Note that by [VdB01, Proposition 5.5.2] o_p is a simple object in $\mathrm{BiMod}(X - X)$ so the Jordan-Holder theorem applies to finite length bimodules.

Lemma 1.3.9. *A finite length object in* $\mathrm{Qcoh}(X)$ *has no higher cohomology.*

Proof. For an object of the form o_p this follows from [VdB01, Proposition 5.1.2] with $\mathcal{F} = \mathcal{O}_X$. The general case follows from the long exact sequence for Ext. □

Lemma 1.3.10. *With the notations as above we have for $i - j \leq 3$:*

$$\mathrm{Ext}^2(\mathcal{O}_X(i), \mathcal{O}_X(j) \otimes m_{\tau^{-m}p} \dots m_{\tau^{-n+1}p}) = 0$$

and for $i - j \leq 2m - 2n + 2$:

$$\mathrm{Ext}^1(\mathcal{O}_X(i), \mathcal{O}_X(j) \otimes m_{\tau^{-m}p} \dots m_{\tau^{-n+1}p}) = 0$$

Proof. For Ext^2 this follows from the fact that

$$\mathcal{O}_X(j) \otimes m_{\tau^{-m}d} \dots m_{\tau^{-n+1}d} \subset \mathcal{O}_X(j)$$

with finite length cokernel. As such, Lemma 1.3.9 implies

$$\operatorname{Ext}^2(\mathcal{O}_X(i), \mathcal{O}_X(j) \otimes m_{\tau^{-m}p} \dots m_{\tau^{-n+1}p}) = \operatorname{Ext}^2(\mathcal{O}_X(i), \mathcal{O}_X(j))$$

and the result follows from the standard vanishing properties on $\operatorname{QGr}(A)$ as in Lemma 1.3.3.

For Ext^1 we prove this result by induction on $n - m$. By Lemma 1.3.3 the result holds for all m, n with $n \le m$. Moreover for technical reasons we claim at this point of the proof that the result also holds for $n = m + 1$, $i = j$. By this claim (and the inequality $i - j \le 2m - 2n + 2$) we are only left to prove the result (by induction on $n - m$) for all i, j, m, n with $n - m > 0$ and $i - j \le -1$. The induction step proceeds as follows: From [VdB01, Theorem 5.5.10] and the fact that $m_{\tau^{-m}p} \dots m_{\tau^{-n+1}p} \subset o_X$ with finite length cokernel we may deduce that the kernel of the obvious surjective map

$$m_{\tau^{-m}p} \otimes_X m_{\tau^{-m-1}p} m_{\tau^{-m-2}p} \dots m_{\tau^{-n+1}p} \longrightarrow m_{\tau^{-m}p} \dots m_{\tau^{-n+1}p}$$

has finite length. Using [VdB01, Lemma 8.2.1] we see that this remains the case if we tensor on the left with $\mathcal{O}_X(j)$. Thus we obtain a short exact sequence in $\operatorname{Qcoh}(X)$

$$0 \to \text{f.l.} \to \mathcal{O}_X(j) \otimes m_{\tau^{-m}p} \otimes_X m_{\tau^{-m-1}p} \dots m_{\tau^{-n+1}p} \to \mathcal{O}_X(j) \otimes m_{\tau^{-m}p} \dots m_{\tau^{-n+1}p} \to 0$$

from which, Lemma 1.3.9 leads us to

$$\begin{aligned} \operatorname{Ext}^1(\mathcal{O}_X(i), &\mathcal{O}_X(j) \otimes m_{\tau^{-m}p} \dots m_{\tau^{-n+1}p}) \\ &= \operatorname{Ext}^1(\mathcal{O}_X(i), \mathcal{O}_X(j) \otimes m_{\tau^{-m}p} \otimes m_{\tau^{-m-1}p} \dots m_{\tau^{-n+1}p}) \end{aligned}$$

From (1.27) we obtain an exact sequence

$$\begin{array}{c} \operatorname{Tor}_1^{o_X}(\mathcal{O}_X(j) \otimes m_{\tau^{-m}p}, m_{\tau^{-m-1}p} \dots m_{\tau^{-n+1}p})) \\ \downarrow \\ \mathcal{J}(j-2) \otimes m_{\tau^{-m-1}p} \dots m_{\tau^{-n+1}p} \oplus \mathcal{O}_X(j-3) \otimes m_{\tau^{-m-1}p} \dots m_{\tau^{-n+1}p}) \\ \downarrow \\ (\mathcal{O}_X(j-2) \otimes m_{\tau^{-m-1}p} \dots m_{\tau^{-n+1}p})^{\oplus 3} \\ \downarrow \\ \mathcal{O}_X(j) \otimes m_{\tau^{-m}p} \otimes_X \otimes m_{\tau^{-m-1}p} \dots m_{\tau^{-n+1}p} \\ \downarrow \\ 0 \end{array} \tag{1.29}$$

One deduces again, for example using [VdB01, Theorem 5.5.10], that $\operatorname{Tor}_1^{o_X}(\mathcal{O}_X(j) \otimes m_{\tau^{-m}p}, m_{\tau^{-m-1}p} \dots m_{\tau^{-n+1}p}))$ has finite length.

It is clear that $\mathcal{O}_X(j-3) \otimes m_{\tau^{-m-1}p} \dots m_{\tau^{-n+1}p}$ has no finite length subobjects. We claim this is the same for $\mathcal{J}(j-2) \otimes_X m_{\tau^{-m-1}p} \dots m_{\tau^{-n+1}p}$. Indeed tensoring the short exact sequence (obtained from (1.25)):

$$0 \to \mathcal{J}(j-2) \longrightarrow \mathcal{O}_X(j-2)^{\oplus 2} \longrightarrow \mathcal{O}_X(j-1) \to 0 \tag{1.30}$$

on the right with $m_{\tau^{-m-1}p} \dots m_{\tau^{-n+1}p}$ and using Tor-vanishing [VdB01, Theorem 8.2.1] we obtain an inclusion

$$\mathcal{J}(j-2) \otimes m_{\tau^{-m-1}p} \dots m_{\tau^{-n+1}p} \hookrightarrow \mathcal{O}_X(j-2)^{\oplus 2} \otimes m_{\tau^{-m-1}p} \dots m_{\tau^{-n+1}p} \tag{1.31}$$

In particular $\mathcal{J}(j-2) \otimes m_{\tau^{-m-1}p} \dots m_{\tau^{-n+1}p}$ is torsion free. We conclude that (1.29) becomes in fact a short exact sequence

$$\begin{array}{c} 0 \\ \downarrow \\ \mathcal{J}(j-2) \otimes m_{\tau^{-m-1}p} \dots m_{\tau^{-n+1}p} \oplus \mathcal{O}_X(j-3) \otimes m_{\tau^{-m-1}p} \dots m_{\tau^{-n+1}p}) \\ \downarrow \\ (\mathcal{O}_X(j-2) \otimes m_{\tau^{-m-1}p} \dots m_{\tau^{-n+1}p})^{\oplus 3} \\ \downarrow \\ \mathcal{O}_X(j) \otimes m_{\tau^{-m}p} \otimes_X \otimes m_{\tau^{-m-1}p} \dots m_{\tau^{-n+1}p} \\ \downarrow \\ 0 \end{array} \tag{1.32}$$

We find that $\operatorname{Ext}^1(\mathcal{O}_X(i), \mathcal{O}_X(j) \otimes m_{\tau^{-m}p} \otimes m_{\tau^{-m-1}p} \dots m_{\tau^{-n+1}p})$ is sandwiched between a direct sum of copies of

$$\operatorname{Ext}^1(\mathcal{O}_X(i), \mathcal{O}_X(j-2) \otimes m_{\tau^{-m-1}p} \dots m_{\tau^{-n+1}p})$$

(= 0 by the induction hypothesis),

$$\operatorname{Ext}^2(\mathcal{O}_X(i), \mathcal{O}_X(j-3) \otimes m_{\tau^{-m-1}p} \dots m_{\tau^{-n+1}p})$$

(= 0 by the first part of the proof) and a direct sum of copies of

$$\operatorname{Ext}^2(\mathcal{O}_X(i), \mathcal{J}_X(j-2) \otimes m_{\tau^{-m-1}p} \dots m_{\tau^{-n+1}p}). \tag{1.33}$$

Hence it suffices to show that (1.33) is trivial as well.

For this, first note that similar to the construction of (1.31) we can start from (1.28) to obtain a short exact sequence

$$\begin{array}{c} 0 \\ \downarrow \\ \mathcal{O}_X(j-5) \otimes m_{\tau^{-m-1}p} \dots m_{\tau^{-n+1}p}) \\ \downarrow \\ \mathcal{O}_X(j-4) \otimes m_{\tau^{-m-1}p} \dots m_{\tau^{-n+1}p})^{\oplus 2} \\ \downarrow \\ \mathcal{J}(j-2) \otimes m_{\tau^{-m-1}p} \dots m_{\tau^{-n+1}p} \\ \downarrow \\ 0 \end{array}$$

In particular, (1.33) is trivial, as this Ext^2 is sandwiched between a direct sum of copies of $\mathrm{Ext}^2(\mathcal{O}_X(i), \mathcal{O}_X(j-4) \otimes m_{\tau^{-m-1}p} \otimes m_{\tau^{-m-1}p} \dots m_{\tau^{-n+1}p})$ (= 0 by the Ext^2-vanishing and the assumption $i - j \le -1$) and $\mathrm{Ext}^3(\mathcal{O}_X(i), \mathcal{O}_X(j-5) \otimes m_{\tau^{-m-1}p} \otimes m_{\tau^{-m-1}p} \dots m_{\tau^{-n+1}p})$ (= 0 as $\mathrm{gl.dim}(X) = 2$). This concludes the induction step.

We only need to prove the claim

$$\mathrm{Ext}^1(\mathcal{O}_X(i), \mathcal{O}_X(i) \otimes m_{\tau^{-m}p}) = 0$$

The exact sequence in (1.26) induces a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(i-3) \longrightarrow \mathcal{O}_X(i-2) \oplus \mathcal{O}_X(i-1) \longrightarrow \mathcal{O}_X(i) \otimes_X m_{\tau^{-m}p} \longrightarrow 0$$

By considering the associated long exact sequence of cohomology groups, we see that $\mathrm{Ext}^1(\mathcal{O}_X(i), \mathcal{O}_X(i) \otimes m_{\tau^{-m}p})$ is sandwiched between
$\mathrm{Ext}^1(\mathcal{O}_X(i), \mathcal{O}_X(i-1))$, $\mathrm{Ext}^1(\mathcal{O}_X(i), \mathcal{O}_X(i-2))$ and $\mathrm{Ext}^2(\mathcal{O}_X(i), \mathcal{O}_X(i-3))$. This proves the claim as the latter three Ext-groups vanish by Lemma 1.3.3. □

We may now draw some conclusions.

Lemma 1.3.11. *D is generated in degree one.*

Proof. We need to show for $n > m$ that

$$\Gamma(X, \mathcal{D}_{m,m+1}) \otimes_k \Gamma(X, \mathcal{D}_{m+1,n}) \longrightarrow \Gamma(X, \mathcal{D}_{m,n})$$

is surjective.

The kernel of $\mathcal{D}_{m,m+1} \otimes_X \mathcal{D}_{m+1,n} \to \mathcal{D}_{m,n}$ has finite length whence by Lemma 1.3.9: $\Gamma(X, \mathcal{D}_{m,m+1} \otimes_X \mathcal{D}_{m+1,n}) \to \Gamma(X, \mathcal{D}_{m,n})$ is surjective. Hence it is sufficient to show that

$$\Gamma(X, \mathcal{D}_{m,m+1}) \otimes_k \Gamma(X, \mathcal{D}_{m+1,n}) \longrightarrow \Gamma(X, \mathcal{D}_{m,m+1} \otimes_X \mathcal{D}_{m+1,n})$$

is surjective. To prove this, we tensor (1.27) (with $j = 2$ and p replaced by $\alpha^{m+1}\tau^{-m}p$ (recall (1.22))) on the right with $\mathcal{D}_{m+1,n}$.

Using the short hand notation $(j)\mathcal{M} = \mathcal{O}_X(j) \otimes_{o_X} \mathcal{M}$ for each bimodule $\mathcal{M}$, this gives

$$\begin{array}{c}\mathrm{Tor}_1^{o_X}((0)\mathcal{D}_{m,m+1}, \mathcal{D}_{m+1,n})\\ \downarrow \\ \mathcal{J}\otimes\mathcal{D}_{m+1,n}\oplus(-1)\mathcal{D}_{m+1,n} \longrightarrow (0)\mathcal{D}_{m+1,n}^{\oplus 3} \longrightarrow (0)\mathcal{D}_{m,m+1}\otimes_X\mathcal{D}_{m+1,n}\to 0\end{array}$$

This resolution is actually of the form

$$\begin{array}{c}\mathrm{Tor}_1^{o_X}((0)\mathcal{D}_{m,m+1}, \mathcal{D}_{m+1,n})\\ \downarrow \\ \mathcal{J}\otimes\mathcal{D}_{m+1,n}\oplus(-1)\mathcal{D}_{m+1,n} \to D_{m,m+1}\otimes_k(0)\mathcal{D}_{m+1,n} \to (0)\mathcal{D}_{m,m+1}\otimes_X\mathcal{D}_{m+1,n}\to 0\end{array} \tag{1.34}$$

Since $\mathrm{Tor}_1^{o_X}((0)\mathcal{D}_{m,m+1}, \mathcal{D}_{m+1,n})$ has finite length ([VdB01, Theorem 5.5.10]) and $\mathcal{J}\otimes\mathcal{D}_{m+1,n}\oplus(-1)\mathcal{D}_{m+1,n}\subset\mathcal{O}_X(2n-2m-2)^{\oplus 3}\oplus\mathcal{O}_X(2n-2m-3)$ has no finite length submodules the vertical arrow is zero.

Hence we must show that

$$H^1(X,(-1)\mathcal{D}_{m+1,n}) = \mathrm{Ext}^1(\mathcal{O}_X(-2n),\mathcal{O}_X(-2m-3)\otimes m_{\tau^{-m-1}p}\dots m_{\tau^{-n+1}p}) = 0$$

and

$$H^1(X,\mathcal{J}\otimes_X\mathcal{D}_{m+1,n}) = \mathrm{Ext}^1(\mathcal{O}_X(-2n),\mathcal{J}\otimes o_X(-2m-2)\otimes m_{\tau^{-m-1}p}\dots m_{\tau^{-n+1}p}) = 0$$

Both equalities follow from Lemma 1.3.10 and its proof. □

Our next result is that D has the "correct" Hilbert function. That is

$$\dim_k(D_{m,m+a}) = \begin{cases}\frac{(a+1)(a+2)}{2} & \text{if } a\geq 0\\ 0 & \text{if } a<0\end{cases} \tag{1.35}$$

The case $a<0$ is trivial so we consider $a\geq 0$. [VdB01, Corollary 5.2.4] tells us that the colength of $\mathcal{D}_{m,m+a}$ inside $o_X(2a)$ is

$$\frac{a(a+1)}{2}$$

Using the fact that $H^1(X,\mathcal{D}_{m,m+a}) = 0$ (by Lemma 1.3.10) we obtain (for $a\geq 0$)

$$\dim_k(D_{m,m+a}) = \frac{(2a+2)^2}{4} - \frac{a(a+1)}{2} = \frac{(a+1)(a+2)}{2} \tag{1.36}$$

Hence (1.35) holds.

Lemma 1.3.12. *The canonical map $D \to D_Y$ is surjective.*

Proof. As D and D_Y are both generated in degree 1 (for D_Y this is proved in the same way as for $B(Y, \sigma, \mathcal{L})$, see [ATVdB90, Lemma 7.1]) or [VdB11, Lemma 5.5.4], it suffices to check that $D_{m,m+1} \to (D_Y)_{m,m+1}$ is surjective. To show this, consider the following commuting diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \mathcal{O}_X(-Y) \arrow[r] \arrow[d] & \mathcal{O}_X \otimes_X m_p \arrow[r] \arrow[d] & \mathcal{O}_X \otimes_X m_{p,Y} \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_X(-Y) \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{O}_Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathcal{O}_p \arrow[r] \arrow[d] & \mathcal{O}_p \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The bottom two rows and the first column are obviously exact. The third column is equal to

$$0 \longrightarrow \mathcal{O}_Y(-p) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_p \longrightarrow 0$$

and hence is exact. The exactness of the middle column follows as usual from [VdB01, Lemma 8.2.1]. Hence we can apply the Snake lemma to the above diagram and find the following exact sequence:

$$0 \longrightarrow \mathcal{O}_X(-Y) \longrightarrow \mathcal{O}_X \otimes_X m_p \longrightarrow \mathcal{O}_X \otimes_X m_{p,Y} \longrightarrow 0$$

Since the above obviously remains true when we replace p by $\alpha^{-j}p$ and since $o_X(2)$ is an invertible bimodule we get an exact sequence

$$0 \to \mathcal{O}_X(-Y) \otimes_X o_X(2) \longrightarrow \mathcal{O}_X \otimes_X \mathcal{D}_{m,m+1} \longrightarrow \mathcal{O}_X \otimes_X (\mathcal{D}_Y)_{m,m+1} \to 0$$

The surjectivity of $D_{m,m+1} \to (D_Y)_{m,m+1}$ then follows from

$$H^1(X, \mathcal{O}_X(-Y) \otimes_X o_X(2)) = H^1(X, \mathcal{O}_X(-2)) = 0$$

using (1.10)) as well as the standard vanishing results for X (see Lemma 1.3.3). □

By Lemma 1.3.12 the map $D \to D_Y$ is surjective, one checks using (1.35) that $D_{m,n} \to D_{Y,m,n}$ is an isomorphism for $n \le m+2$. Thus D and D_Y have the same quadratic relations. Let D' be the quadratic AS-regular $\mathbb{Z}$-algebra associated to $(Y, (\mathcal{G}_i)_{i\in\mathbb{Z}})$ (see §0.3) with $\mathcal{G}_i$ as in (1.20). Then since D' is quadratic, and has the same quadratic relations as D_Y we obtain a surjective map $D' \to D$. Moreover, as D'

has no relations of degree ≥ 3, this map induces a surjective morphism $D' \to D$. Since D' and D have the same Hilbert series by (1.35) this must in fact be an isomorphism. Hence $D \cong A(Y, \mathcal{G}_i)_{i\in\mathbb{Z}})$.

Finally set $\mathcal{G} = \mathcal{G}_0$ and let $\psi : Y \to Y$ be as in Lemma 1.3.6. Then Lemma 1.3.6 together with the proof of [VdB11, Theorem 4.2.2.] we find

$$D \cong \check{A}'$$

where

$$A' = A(Y, \mathcal{G}, \psi)$$

Using this, the inclusion $D \subset A^{(2)}$ provided by (1.17) gives rise to an inclusion

$$\gamma : \check{A}' \hookrightarrow A^{(2)} \tag{1.37}$$

Finishing the construction of our noncommutative version of $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$.

1.4 Noncommutative function fields

The goal of this section is to prove the following theorem:

Theorem 1.4.1. *Let $A = A(Y, (\mathcal{L}_i)_{i\in\mathbb{Z}})$ be a cubic Sklyanin $\mathbb{Z}$-algebra with $\alpha \in \mathrm{Aut}(Y)$ as in Definition 1.3.1 and let $A' = A(Y, \mathcal{G}, \psi)$ is a quadratic Sklyanin algebra such that $\alpha^2 = \psi^3$. Then* Proj A *and* Proj A' *have isomorphic function fields. (see Remark 0.4.8 and Theorem 0.4.6 for the definition and existence of these function fields)*

Throughout this section we fix A and A' as in the above theorem. By Remark 0.1.4 we know that, upto isomorphism, A' does not depend on the choice of $\mathcal{G}$. In particular, we can choose $\mathcal{G} = \mathcal{G}_0 = \mathcal{O}_Y(-p) \otimes \mathcal{L}_0 \otimes \mathcal{L}_1$ as in §1.3 and the techniques from the that section provide us with an inclusion

$$\gamma : \check{A}' \hookrightarrow A^{(2)}$$

which induces a morphism

$$\mathrm{Frac}(\gamma) : \mathrm{Frac}(A')_0 \cong \mathrm{Frac}(\check{A}')_{0,0} \longrightarrow \mathrm{Frac}(A)_{0,0} : a/s \mapsto \gamma(a)/\gamma(s)$$

As $\mathrm{Frac}(\gamma) \neq 0$ and $\mathrm{Frac}(A')_0$ and $\mathrm{Frac}(A)_{0,0}$ are division rings, $\mathrm{Frac}(\gamma)$ is automatically injective. We will prove that it is surjective as well. So given any $a, s \in A_{0,2i} \backslash \{0\}$ we need to find an $n \in \mathbb{N}$ and $h \in A_{2i,2(i+n)}$ such that ah and sh lie in the image of γ. We claim we can find such an h only depending on i and not on a or s. For this consider the following map:

$$\Gamma\big(X, o_X(2i)\big) \otimes \Gamma\big(X, o_X(2n) \otimes_X \mathcal{I}_{i,n}\big) \longrightarrow \Gamma\big(X, o_X(2(i+n) \otimes_X \mathcal{I}_{i,n})\big)$$

where $\mathcal{I}_{i,n}$ is the ideal in o_X such that $o_X(2(i+n)) \otimes_X \mathcal{I}_{i,n} = \mathcal{D}_{0,i+n}$. More concretely we can use (1.15) and (1.22) to see that $\mathcal{I}_{i,n}$ is explicitely given by

$$\mathcal{I}_{i,n} = m_{\alpha^{-i-n}p} m_{\alpha^{-i-n-2}p} \dots m_{\alpha^{-3i-3n+2}p} \subset o_X$$

In particular $\Gamma\big(X, o_X(2(i+n) \otimes_X \mathcal{I}_{i,n})\big)$ lies in the image of γ and surjectivity of $\mathrm{Frac}(\gamma)$ follows if we can choose an n such that

$$\dim_k \left(\Gamma\big(X, o_X(2n) \otimes_X \mathcal{I}_{i,n}\big)\right) \neq \{0\} \tag{1.38}$$

By [VdB01, Corollary 5.2.4] the codimension of $\Gamma\big(X, o_X(2N) \otimes_X \mathcal{I}_{i,n}\big)$ inside $A_{2i,2i+2n}$ is at most

$$\frac{(i+n)(i+n+1)}{2} = \frac{n^2}{2} + \text{lower degree terms}$$

On the other hand $\dim_k(A_{2i,2i+2n}) = (n+1)^2$ so for n sufficiently large (1.38) will be fulfilled.

1.5 Construction of a noncommutative map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$

In this section we construct a partial converse to (1.37). For this we fix a quadratic Sklyanin algebra $A = A(Y, \mathcal{G}, \psi)$. In particular Y is an elliptic curve, $\mathcal{G}$ is a degree 3 line bundle and $\psi \in \mathrm{Aut}(Y)$ is a translation such that $\tau := \psi^3 \neq \mathrm{id}$. Moreover, we also fix points $q, r \in Y$ such that for all $i \in \mathbb{Z}$: $\tau^i(q) \neq r$. Starting from these geometric data, we construct a cubic Sklyanin $\mathbb{Z}$-algebra $A' = A(Y, (\mathcal{L}_i)_{i\in\mathbb{Z}})$ together with an inclusion:

$$\gamma : A' \hookrightarrow A^{(2)} \tag{1.39}$$

which induces an isomorphism of the function fields.

Moreover, we find an explicit condition (see Remark 1.5.2) such that A' is a $\mathbb{Z}$-algebra associated to a (graded) cubic Sklyanin algebra.

As before, we start by considering $\mathbb{Z}$-algebras D and D_Y. These are defined slightly different from their counterparts in (1.16). First we define for each $i \in \mathbb{Z}$ a point $d_i \in Y$ via

$$d_i = \begin{cases} \tau^{-j} q & \text{if } i = 2j \\ \tau^{-j} r & \text{if } i = 2j+1 \end{cases} \tag{1.40}$$

where $\tau = \psi^3$. We define $m_{d_i,Y}$ and m_{d_i} as in (1.13) and introduce $\mathbb{Z}$-algebra D_Y and

D via

$$(D_Y)_{m,n} = \begin{cases} \mathrm{Hom}(\mathcal{O}_Y(-n), \mathcal{O}_Y(-m) \otimes_{o_Y} m_{d_m,Y} \dots m_{d_{n-1},Y}) & \text{if } n \geq m \\ 0 & \text{if } n < m \end{cases}$$

$$D_{m,n} = \begin{cases} \mathrm{Hom}(\mathcal{O}_X(-n), \mathcal{O}_X(-m) \otimes_{o_X} m_{d_m} \dots m_{d_{n-1}}) & \text{if } n \geq m \\ 0 & \text{if } n < m \end{cases} \tag{1.41}$$

Again using [VdB01, Lemma 8.2.1], there is an obvious inclusion $D \hookrightarrow \check{A}$.

Remark 1.5.1. *As in §1.3, these $\mathbb{Z}$-algebras are the global sections of the corresponding $\mathbb{Z}$-algebra objects in* $\mathrm{BiMod}(Y - Y)$ *and* $\mathrm{BiMod}(X - X)$ *respectively:*

$$(D_Y)_{m,n} = \begin{cases} o_Y(-m) \otimes_{o_Y} m_{d_m,Y} \dots m_{d_{n-1},Y} \otimes o_Y(n) & \text{if } n \geq m \\ 0 & \text{if } n < m \end{cases}$$

$$D_{m,n} = \begin{cases} o_X(-m) \otimes_{o_X} m_{d_m} \dots m_{d_{n-1}} \otimes o_X(n) & \text{if } n \geq m \\ 0 & \text{if } n < m \end{cases}$$

Next we mimic the results in §1.3.3. I.e. we need to show the existence of a cubic elliptic helix $(\mathcal{L}_i)_{i\in\mathbb{Z}}$ such that

$$D_Y \cong B(Y, (\mathcal{L}_i)_{i\in\mathbb{Z}}) \tag{1.42}$$

Computations completely analogous to the ones in Lemma 1.3.5 and Lemma 1.3.6 show that if we define $\mathcal{L}_i$ via

$$\mathcal{L}_i = \mathcal{O}_Y(-d_i) \otimes \psi^{i*}\mathcal{G} \tag{1.43}$$

the following hold.

- $\deg \mathcal{L}_i = 2$
- $\mathcal{L}_0 \ncong \mathcal{L}_2$.
- $\mathcal{L}_i \otimes \mathcal{L}_{i+1}^{-1} \otimes \mathcal{L}_{i+2}^{-1} \otimes \mathcal{L}_{i+3} \cong \mathcal{O}_Y$.
- $\alpha^*(\mathcal{L}_i) \cong \mathcal{L}_{i+2}$ where α is an arbitrary translation satisfying $\alpha^2 = \psi^3$.
- The isomorphism in (1.42) holds

In particular D_Y is isomorphic to the twisted homogeneous coordinate ring associated to the cubic elliptic helix $(\mathcal{L}_i)_{i\in\mathbb{Z}}$.

Remark 1.5.2. *If moreover $q = \alpha^{-1}p$ then $\mathcal{L}_{i+1} \cong \sigma^*\mathcal{L}_i$ for an arbitrary translation $\sigma : Y \to Y$ such that $\sigma^4 = \psi^3$.*

In order to adapt the results in §1.3.4 we need a resolution for $\mathcal{O}_X(-i) \otimes m_{d_n}$. We construct such a resolution based on the following:

Lemma 1.5.3. *Let $A = A(Y, \mathcal{L}, \psi)$ be a quadratic AS-regular algebra of dimension 3, let $p \in Y$ be a point and let P be the corresponding point module. Then the minimal resolution of P has the following form*

$$0 \longrightarrow A(-2) \longrightarrow A(-1)^{\oplus 2} \longrightarrow A \longrightarrow P \longrightarrow 0$$

Proof. By [ATVdB91, Proposition 6.7] it suffices to prove $\varepsilon(P) = 1$ where as in [ATVdB91, (2.23)] $\varepsilon(P)$ is defined by

$$\varepsilon(P) := \frac{e(P)}{e(A)}$$

where $e(P)$ and $e(A)$ are the leading coefficients of the series expension of $h_P(t)$ and $h_A(t)$ respectively in powers of $\frac{1}{1-t}$. The result follows as $e(P) = e(A) = 1$ by the following identities:

$$h_P(t) = \sum_{n=0}^{\infty} t^n = \frac{1}{1-t} \text{ and } h_A(t) = \sum_{n=0}^{\infty} \frac{n^2+3n+2}{2} t^n = \frac{1}{(1-t)^3} \qquad \square$$

From Lemma 1.5.3 we obtain the following resolution for $\mathcal{O}_X(-i) \otimes m_{d_n}$ in $\operatorname{QGr}(A)$:

$$0 \to \mathcal{O}(-i-3) \longrightarrow \mathcal{O}(-i-2)^{\oplus 2} \longrightarrow \mathcal{O}(-i) \otimes m_{d_n} \to 0 \tag{1.44}$$

Using this resolution, the computations in Lemma 1.3.10 can be adapted to obtain the following:

Lemma 1.5.4. *With the notations as above we have for $i - j \le 2$:*

$$\operatorname{Ext}^2(\mathcal{O}_X(i), \mathcal{O}_X(j) \otimes m_{m_d} \dots m_{d_{n-1}}) = 0$$

and for $i - j \le m - n + 1$:

$$\operatorname{Ext}^1(\mathcal{O}_X(i), \mathcal{O}_X(j) \otimes m_{d_m} \dots m_{d_{n-1}}) = 0$$

To prove that D is generated in degree 1 we can mimic the proof of Lemma 1.3.11: Lemma 1.3.9 implies that it suffices to show

$$\Gamma(X, \mathcal{D}_{m,m+1}) \otimes_k \Gamma(X, \mathcal{D}_{m+1,n}) \longrightarrow \Gamma(X, \mathcal{D}_{m,m+1} \otimes_X \mathcal{D}_{m+1,n})$$

is surjective. For this we tensor (1.44) on the right with $\mathcal{D}_{m+1,n}$ (for an appropriate choice of j and p). An argument of torsion objects versus torsion free objects shows that we find a short exact sequence

$$0 \to (-1)\mathcal{D}_{m+1,n} \to D_{m,m+1} \otimes_k (0)\mathcal{D}_{m+1,n} \to (0)\mathcal{D}_{m,m+1} \otimes \mathcal{D}_{m+1,n} \to 0$$

Hence we must show that

$$H^1(X, (-1)\mathcal{D}_{m+1,n}) = \operatorname{Ext}^1(\mathcal{O}_X(-n), \mathcal{O}_X(-m-1) \otimes m_{d_{m+1}} \dots m_{d_{n-1}}) = 0$$

Which follows from Lemma 1.5.4.

For checking that D has the correct Hilbert series we see that (using computation similar to [VdB01, Corollary 5.2.4]) the colength of $m_{d_m} \dots m_{d_{m+i-1}}$ inside o_X is

$$\begin{cases} 2 \cdot \dfrac{a(a+1)}{2} = a(a+1) & \text{if } i = 2a \\ \dfrac{(a+2)(a+1)}{2} + \dfrac{a(a+1)}{2} = (a+1)^2 & \text{if } i = 2a+1 \end{cases}$$

Using Lemma 1.5.4 this implies

$$\dim D_{m,m+i} = \begin{cases} \dfrac{(2a+1)(2a+2)}{2} - a(a+1) = (a+1)^2 & \text{if } i = 2a \\ \dfrac{(2a+2)(2a+3)}{2} - (a+1)^2 = (a+1)(a+2) & \text{if } i = 2a+1 \end{cases} \tag{1.45}$$

i.e. D has the Hilbert series of a cubic AS-regular $\mathbb{Z}$-algebra.

The proof of Lemma 1.3.12 can be copied almost literally (the only difference lies in the fact that for the current choice of $X = \operatorname{Proj}(A)$ we have $\mathcal{O}_X(-Y) \cong \mathcal{O}_X(-3)$ instead of $\mathcal{O}_X(-4)$) to conclude that the obvious morphism $D \to D_Y$ is surjective. As D has the Hilbert series of a quadric, it hence must be a quadric, more specifically: $D \cong A(Y, (\mathcal{L}_i)_{i \in \mathbb{Z}})$. As Y is an elliptic curve and as we found a translation α (satifying $\alpha^2 = \psi^3 \neq \mathrm{id}$) such that $\mathcal{L}_{i+2} \cong \alpha^* \mathcal{L}_i$ we conclude that D is a cubic Sklyanin $\mathbb{Z}$-algebra.

Replacing D by A' we have hence constructed the desired inclusion as in (1.39).

Lastly we check that $\gamma : A' \to \check{A}$ induces an isomorphism of the function fields. As in §1.4 it is immediate that

$$\operatorname{Frac}(\gamma) : \operatorname{Frac}(A')_{0,0} \longrightarrow \operatorname{Frac}(A)_0$$

is injective. For surjectivity we need to prove that for any fixed $i \in \mathbb{N}$ there is some $n \in \mathbb{N}$ such that

$$\operatorname{Hom}(\mathcal{O}_X(-i-n), \mathcal{O}_X(-n) \otimes m_{d_0} \dots m_{d_{i+n-1}}) \neq 0 \tag{1.46}$$

holds. Now note that the colength of $m_{d_0} \dots m_{d_{i+n-1}}$ inside o_X is given by a^2 or $a(a+1)$ if $i+n = 2a$ or $2a+1$ respectively. Hence this colength equals $\frac{n^2}{4} + \omega n + \theta$, where ω and θ are constants depending on the parity of n. In particular, for n large, the codimension of $\operatorname{Hom}(\mathcal{O}_X(-i-n), \mathcal{O}_X(-i) \otimes m_{d_0} \dots m_{d_{i+n-1}})$ inside $A_{i,i+n}$ grows at most like $\frac{n^2}{4}$. Now, as

$$\dim_k A_{i,i+n} = \frac{n^2+3n+2}{2} > \frac{n^2}{4},$$

it follows that (1.46) is satisfied for n sufficiently large.

1.6 Noncommutative Cremona Transformations

In this section we construct a noncommutative version of the *Cremona transform* $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. This (commutative) birational transformation is obtained by blowing up the three vertices of a triangle in $\mathbb{P}^2$ and then contracting the sides. Similarly the (commutative) birational transformation $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is obtained by blowing up two points in $\mathbb{P}^2$ and contracting the line through these two points. So one sees that the Cremona transform factors through $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$. In a picture, this looks like Figure 1.1

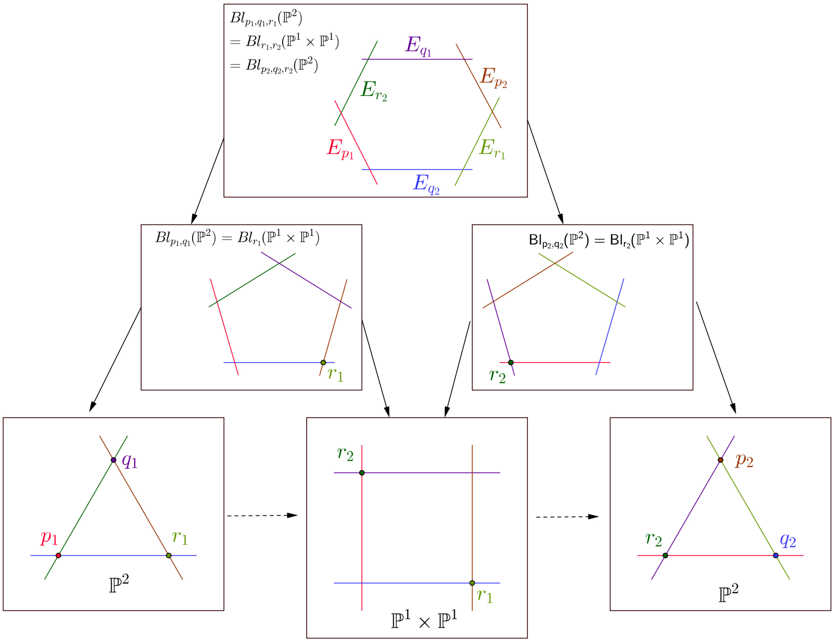

Figure 1.1: The commutative Cremona transform

Our goal is to construct a *noncommutative Cremona transform* which factors (Theorem 1.6.4) through the noncommutative versions of $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ as in (1.37) and (1.39). We will construct such a noncommutative Cremona transform as an inclusion of quadratic Sklyanin algebras. For this recall that the (commutative) Cremona transform corresponds algebraically to an inclusion $k[x,y,z] \hookrightarrow k[u,v,w]^{(2)}$. For example, it is classical (e.g. [Har97, Example V.4.2.3.]) that if we blow up the points

$$p = (1:0:0),\ \ q = (0:1:0) \text{ and } r = (0:0:1),$$

then the associated Cremona transform is given by $x \mapsto vw, y \mapsto uw, z \mapsto uv$.

Hence, we construct a noncommutative Cremona transform as an inclusion

$$\gamma : \check{A}' \hookrightarrow \check{A}^{(2)}$$

where both A and A' are quadratic Sklyanin algebras.

Throughout this section we fix a quadratic Sklyanin algebra $A = A(Y, \mathcal{G}, \psi)$ and a collection of non-collinear points $p, q, r \in Y$. Moreover, for technical reasons we assume that p, q and r lie in different τ-orbits where $\tau = \psi^3$. We denote

$$d = p + q + r \tag{1.47}$$

for the corresponding divisor on Y. As above we consider $\mathcal{O}_d$ as a $Y-Y$-bimodule but to avoid confusion we write it as o_d. Following (1.9) we also consider o_d as an X-bimodule. It is not hard to see that o_d is a quotient of o_Y and o_X and we define bimodules m_d and $m_{d,Y}$ as in (1.12) and (1.13):

$$m_{d,Y} = \ker(o_Y \longrightarrow o_d) \tag{1.48}$$

$$m_d = \ker(o_X \longrightarrow o_d) \tag{1.49}$$

As in the previous sections, the inclusion $\check{A}' \hookrightarrow \check{A}^{(2)}$ is constructed by proving that there is a sub-$\mathbb{Z}$-algebra D of $A^{(2)}$ such that D is a quadratic AS-regular $\mathbb{Z}$-algebra. Hence, there exists a quadratic AS-regular algebra A' such that $\check{A}' \cong D$. An investigation of the geometric triple classifying A' shows that A' is in fact a Sklyanin algebra.

Since this construction is completely similar to the ones in §1.3 and §1.5, we will only mention the steps in the construction which are essentially different.

First we define $\mathbb{Z}$-algebras D_Y and D via

$$(D_Y)_{m,n} = \begin{cases} \operatorname{Hom}\left(\mathcal{O}_Y(-2n), \mathcal{O}_Y(-2m) \otimes_{o_Y} m_{\tau^{-m}d,Y} \dots m_{\tau^{-n+1}d,Y}\right) & \text{if } n \geq m \\ 0 & \text{if } n < m \end{cases} \tag{1.50}$$

$$D_{m,n} = \begin{cases} \operatorname{Hom}\left(\mathcal{O}_X(-2n), \mathcal{O}_X(-2m) \otimes_{o_X} m_{\tau^{-m}d} \dots m_{\tau^{-n+1}d}\right) & \text{if } n \geq m \\ 0 & \text{if } n < m \end{cases}$$

with $\tau = \psi^3$. We have an obvious inclusion $D \subset \check{A}^{(2)}$ and one checks that if we define

$$\mathcal{G}'_i := \mathcal{O}_Y(-\tau^{-i}d) \otimes \psi^{2i*}\mathcal{G} \otimes \psi^{(2i+1)*}\mathcal{G} \tag{1.51}$$

Then $(\mathcal{G}'_i)_{i \in \mathbb{Z}}$ forms a quadratic elliptic helix on Y as defined in Definition 1.3.2 and there is an isomorphism

$$D_Y \cong B(Y, (\mathcal{G}'_i))$$

Moreover, if $\psi' \in \mathrm{Aut}(Y)$ is a translation such that $\psi^3 = \psi'^3$ (for example $\psi' = \psi$), then

$$\psi'^* \mathcal{G}'_i \cong \mathcal{G}'_{i+1} \tag{1.52}$$

Next we want to prove vanishing results as in Lemma 1.3.10 and Lemma 1.5.4. For the noncommutative Cremona transform this is based on the following lemma.

Lemma 1.6.1. *Let A be a quadratic AS-regular algebra of dimension 3. Let q_1, q_2, q_3 be distinct non-collinear points in Y and let Q_1, Q_2, Q_3 be the corresponding point modules. Pick an m in $(Q_1 \oplus Q_2 \oplus Q_3)_0$ whose three components are non-zero and let $M = mA$. Then the minimal resolution of M has the following form*

$$0 \longrightarrow A(-3)^{\oplus 2} \longrightarrow A(-2)^{\oplus 3} \longrightarrow A \longrightarrow M \longrightarrow 0$$

Proof. Let g be the normalizing element of degree three in A and let $B = A/gA$. Using the explicit category equivalence $\mathrm{Qcoh}(B) \cong \mathrm{QGr}(Y)$ [AVdB90], one easily proves that the map $B_{\geq 1} \to M_{\geq 1}$ is surjective. Whence the corresponding map $u : A_{\geq 1} \to M_{\geq 1}$ is also surjective.

Consider the exact sequence

$$0 \longrightarrow \ker u \longrightarrow A_{\geq 1} \longrightarrow M_{\geq 1} \longrightarrow 0$$

Tensoring this exact sequence with k yields an exact sequence

$$\mathrm{Tor}_1^A(M_{\geq 1}, k) \longrightarrow \ker u \otimes_A k \longrightarrow A_{\geq 1} \otimes_A k \xrightarrow{\overline{u}} M_{\geq 1} \otimes k \longrightarrow 0$$

Now both $A_{\geq 1}$ and $M_{\geq 1}$ are generated in degree one and furthermore

$$\dim_k (A_1) = \dim_k (M_1) = 3.$$

Hence it follows that $\bar{u}$ is an isomorphism. Therefore $\ker u \otimes_A k$ is a quotient of $\mathrm{Tor}_1^A(M_{\geq 1}, k)$. From the fact that $M_{\geq 1}$ is a sum of shifted point modules we compute that $\mathrm{Tor}_1^A(M_{\geq 1}, k) = k(-2)^3$. Thus $\ker u$ is a quotient of $A(-2)^3$. Now using the fact that M has no torsion and hence has projective dimension two we may now complete the full resolution of M using a Hilbert series argument. $\square$

Using Lemma 1.6.1, the computations in Lemma 1.3.10 can be adapted to obtain:

Lemma 1.6.2. *With the notations as above we have for $i - j \leq 2$:*

$$\mathrm{Ext}^2(\mathcal{O}_X(i), \mathcal{O}_X(j) \otimes m_{\tau^{-m}d} \ldots m_{\tau^{-n+1}d}) = 0$$

and for $i - j \leq 2m - 2n + 1$:

$$\mathrm{Ext}^1(\mathcal{O}_X(i), \mathcal{O}_X(j) \otimes m_{\tau^{-m}d} \ldots m_{\tau^{-n+1}d}) = 0$$

which immediately gives rise to

Lemma 1.6.3. *D is generated in degree one and has the Hilbert function of a quadratic Sklyanin algebra, i.e.*

$$h_D(a) = \dim_k(D_{m,m+a}) = \frac{(a+1)(a+2)}{2}$$

The proof of Lemma 1.3.12 still holds in the current setting, allowing us to conclude that the canonical map $D \to D_Y$ is surjective. As D has the "correct" Hilbert series, as $D_Y \cong B(Y,(\mathcal{G}'_i)_{i\in\mathbb{Z}})$ and as $D_{m,m+a} \cong (D_Y)_{m,m+a}$ for $a = 0,1,2$ we obtain

$$A(Y,(\mathcal{G}'_i)_{i\in\mathbb{Z}}) \cong D \subset \check{A}^{(2)} \tag{1.53}$$

such that by (1.52) we find an inclusion

$$\gamma : \check{A}' \hookrightarrow \check{A}^{(2)} \tag{1.54}$$

where $A' = A(Y,\mathcal{G}'_0,\psi')$.

The main result from this section is that noncommutative Cremona transforms have a factorisation as in the commutative setting:

Theorem 1.6.4. *Let $A = A(Y,\mathcal{G},\psi)$ and A' be quadratic Sklyanin algebras and let $\gamma : \check{A}' \hookrightarrow \check{A}^{(2)}$ be a noncommutative Cremona transformation as in (1.54). Denote $d = p+q+r$ as in loc. cit. and assume p,q and r are non-collinear points lying in different τ-orbits where $\tau = \psi^3$. Then there exists a cubic Sklyanin $\mathbb{Z}$-algebra A'' and inclusions $\gamma_1 : \check{A}' \hookrightarrow A''^{(2)}$, $\gamma_2 : A'' \hookrightarrow \check{A}$ such that γ_1 is as in (1.37), γ_2 is as in (1.39) and $\gamma = \gamma_2 \circ \gamma_1$.*

Before we can prove this we need the following technical result:

Lemma 1.6.5. *Let p and q be points such that $p \neq \tau^i q$ for $i \in \{-1,0,1\}$. Then $m_{p+q} = m_p m_q$. (where m_{p+q} is defined completely analogous to (1.47) and (1.49))*

Proof. Consider the following diagram:

$$\begin{tikzcd}
m_p \otimes o_q \arrow[r, "a_1"] & o_q & \\
m_p \arrow[u, "g_1"] \arrow[r, "a_2"] & o_X \arrow[u, "g_2"] \arrow[r, "b_2"] & o_p \\
m_p \otimes m_q \arrow[u, "f_1"] \arrow[r, "a_3"'] & m_q \arrow[u, "f_2"'] \arrow[r, "b_3"'] & o_p \otimes m_q \arrow[u, "f_3"']
\end{tikzcd}$$

The middle row and column are exact sequences by (1.13). The last row is obtained by applying $- \otimes m_q$ to the middle row, it is therefor automatically right exact. Similarly the first column is right exact.

We can identify $m_p m_q$ both with the image of f_1 and a_3 (and hence the kernel of g_1 and b_3). To see this recall that $m_p m_q$ is defined as the image of

$$f : m_p \otimes m_q \longrightarrow o_X \otimes o_X \longrightarrow o_X$$

The identification then follows as $f = f_2 \circ a_3 = a_2 \circ f_1$ and as a_2 and f_2 are monomorphisms.

Next we claim that a_1 and f_3 are in fact isomorphisms. The proof follows from this claim as it implies $m_p m_q$ serves both as a kernel for $b_2 \circ f_2$ and $g_2 \circ a_2$. So it is the pullback of

$$\begin{tikzcd} m_p \arrow[r, "a_2"] & o_X \\ & m_q \arrow[u, "f_2"'] \end{tikzcd}$$

On the other hand $o_{p+q} = o_p \oplus o_q$ such that m_{p+q}, being the kernel of $o_X \xrightarrow{b_2 \oplus g_2} o_{p+q}$, also is a pullback of the above diagram. In particular $m_{p+q} \cong m_p m_q$.

It hence remains to prove the above claim. As the argument is the same for both morphisms we only explain this for a_1. Note that this map is obtained by tensoring $m_p \hookrightarrow o_X$ with o_q. As such, the cokernel of a_1 is given by $o_p \otimes o_q$ whereas the kernel is a subobject of $\mathcal{T}or_1(o_p, o_q)$ (see [VdB01, §3] for the definition of $\mathcal{T}or_i$). In particular it suffices to show $o_p \otimes o_q = 0 = \mathcal{T}or_1(o_p, o_q)$. By [VdB01, Lemma 5.5.1] we need to show $\mathrm{Hom}_X(\mathcal{O}_p, \mathcal{O}_q) = 0 = \mathrm{Ext}^1_X(\mathcal{O}_p, \mathcal{O}_q)$. For Hom_X this follows from the fact that $\mathcal{O}_p$ and $\mathcal{O}_q$ are simple. For Ext^1, we can use [VdB01, Proposition 5.1.2 and (5.3)] to reduce the computations to showing $\mathrm{Ext}^1_Y(\mathcal{O}_p, \mathcal{O}_q) = 0 = \mathrm{Hom}_Y(\mathcal{O}_p, \mathcal{O}_{\tau q})$. The latter follows as p, q and τq are different points by assumption. □

We can now finish the main result of this section:

Proof of Theorem 1.6.4. Recall from (1.50) and (1.53) that $\gamma : \check{A}' \hookrightarrow \check{A}^{(2)}$ was constructed as follows:

$$\begin{array}{c} (\check{A}')_{m,n} = \mathrm{Hom}_X(\mathcal{O}_X(-2n), \mathcal{O}_X(-2m) \otimes_{o_X} m_{\tau^{-m}d} \dots m_{\tau^{-n+1}d}) \\ \big\downarrow \\ \check{A}_{2m,2n} = \mathrm{Hom}_X(\mathcal{O}_X(-2n), \mathcal{O}_X(-2m)) \end{array} \tag{1.55}$$

We construct $\gamma_2 : A'' \hookrightarrow \check{A}$ with respect to the points q, r as in (1.39). I.e.

$$(A'')_{m,n} = \mathrm{Hom}_X(\mathcal{O}_X(-n), \mathcal{O}_X(-m) \otimes_{o_X} m_{d_m} \dots m_{d_{n-1}})$$

where

$$d_i = \begin{cases} \tau^{-j} q & \text{if } i = 2j \\ \tau^{-j} r & \text{if } i = 2j+1 \end{cases}$$

By Lemma 1.6.5 for each i we can write $m_{d_{2i}} m_{d_{2i+1}} = m_{\tau^{-i}(p+q)}$. In particular the inclusions $m_{\tau^{-i}(p+q+r)} \hookrightarrow m_{\tau^{-i}(q+r)}$ give rise to an inclusion $\gamma_1 : \check{A}' \hookrightarrow A''^{(2)}$ such that $\gamma = \gamma_2 \circ \gamma_1$. It hence remains to show that γ_1 is in fact an inclusion as in (1.37). For this we need to prove the existence of a point p' such that

$$\gamma_1((\check{A}')_{m,n}) = \mathrm{Hom}_{X'}(\mathcal{O}_{X'}(-2n), \mathcal{O}_{X'}(-2m) \otimes_{o_{X'}} m'_{\tau^{-m}p'} \ldots m'_{\tau^{-n+1}p'}) \tag{1.56}$$

where $\mathrm{Qcoh}(X') = \mathrm{QGr}(A')$ and $m'_{p'} = \ker(o_{X'} \to o_{p'})$. As A' is generated in degree 1, it suffices to check (1.56) for $n = m+1$ in which case the left hand side of (1.56) equals

$$\Gamma(Y, \psi^{2m*}\mathcal{L} \otimes \psi^{2m+1*}\mathcal{L} \otimes \mathcal{O}_Y(-\tau^{-m}p - \tau^{-m}q - \tau^{-m}r))$$

and the right hand side equals

$$\Gamma(Y, \psi^{2m*}\mathcal{L} \otimes \mathcal{O}_Y(\tau^{-m}q) \otimes \psi^{2m+1*}\mathcal{L} \otimes \mathcal{O}_Y(-\tau^{-m}r) \otimes \mathcal{O}_Y(\tau^{-m}p'))$$

. Hence the theorem is proven by choosing $p' = p$. □

Remark 1.6.6. *It follows immediately from Theorem 1.6.4 that a noncommutative Cremona transform $\gamma : \check{A}' \hookrightarrow \check{A}^{(2)}$ induces an isomorphism of the function fields:* $\mathrm{Frac}(A')_0 \cong \mathrm{Frac}(A)_0$.

1.7 Quadratic transforms and inner morphisms

Throughout this section as well as the subsequent sections we will identify a graded algebra A with the associated $\mathbb{Z}$-algebra $\check{A}$ whenever there is no confusion possible.

Inspired by Theorem 1.6.4 we make the following definition:

Definition 1.7.1. Let A, A' be three dimensional (quadratic or cubic) Sklyanin algebras. An inclusion $A' \hookrightarrow A^{(v)}$ is called a quadratic transform if it can be written as a composition of inclusions as in (1.37) and (1.39).

Remark 1.7.2. *By construction a quadratic transform always induces an isomorphism of function fields.*

It immediately follows from the definition that if $A' \hookrightarrow A^{(v)}$ is a quadratic transform, then $v = 2^n$ for some nonnegative integer n. By construction our noncommutative versions of $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ and $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ as in (1.37) and (1.39) are quadratic transforms. Theorem 1.6.4 implies a noncommutative Cremona transform as in (1.54) is a quadratic transform as well.

The main goal of this section (as well as §1.8) is to prove that in a suitable sense quadratic transforms are invertible:

Definition 1.7.3. Let A be a $\mathbb{Z}$-algebra such that $Q := \mathrm{Frac}(A)$ exists. We say that an injective morphism of $\mathbb{Z}$-algebras $\phi : A \hookrightarrow A^{(v)}$ is *inner* if there exist $z_m \in Q_{vm,m} - \{0\}$ such that for $a \in A_{m,n}$ we have $\phi(a) = z_m a z_n^{-1}$. Moreover, we require

$$\begin{aligned} z_m &\in A_{vm,m} && \text{if } m < 0 \\ z_0 &= 1 && \\ z_m^{-1} &\in A_{m,vm} && \text{if } m > 0 \end{aligned} \tag{1.57}$$

The following is clear

Proposition 1.7.4. *If $A \to A^{(v)}$ is inner then the induced map $Q_{0,0} \to (Q^{(v)})_{0,0} = Q_{0,0}$ is the identity.*

Definition 1.7.5. Let A and A' be $\mathbb{Z}$-algebras such that $\mathrm{Frac}(A)$ and $\mathrm{Frac}(A')$ exist. An inclusion $\gamma : A' \hookrightarrow A^{(w)}$ is said to be *invertible* if there exists an inclusion $\delta : A \hookrightarrow A'^{(v)}$ such that $\delta \circ \gamma : A' \hookrightarrow A'^{(vw)}$ and $\gamma \circ \delta : A \hookrightarrow A^{(vw)}$ are inner.

We can now state the main result of this paper:

Theorem 1.7.6. *Assume that $\gamma : A' \hookrightarrow A^{(2^m)}$ is a quadratic transform between (three dimensional) Sklyanin $\mathbb{Z}$-algebras. Write $\gamma = \gamma_1 \circ \ldots \circ \gamma_t$ with the γ_j as in* (1.37) *and* (1.39)*. Then γ is invertible and the "inverse" δ can be chosen as a quadratic transform $\delta : A \to A'^{(2^n)}$ with $|n - m| \le 1$.*

We will call δ as in Theorem 1.7.6 an inverse quadratic transform to γ.

The following reduces the amount of work for proving Theorem 1.7.6 dramatically.

Lemma 1.7.7. *Let $\gamma_1 : A' \hookrightarrow A^{(w_1)}$ and $\gamma_2 : A'' \hookrightarrow A'^{(w_2)}$ be invertible as in Definition 1.7.5. Then $\gamma_1 \circ \gamma_2$ is invertible as well.*

Proof. By assumption there exist inclusions $\delta_1 : A \hookrightarrow A'^{(v_1)}$ and $\delta_2 : A' \hookrightarrow A''^{(v_2)}$ such that $\delta_1 \circ \gamma_1$, $\delta_2 \circ \gamma_2$, $\gamma_1 \circ \delta_1$ and $\gamma_2 \circ \delta_2$ are inner. We now claim $(\delta_2 \circ \delta_1) \circ (\gamma_1 \circ \gamma_2)$ and $(\gamma_1 \circ \gamma_2) \circ (\delta_2 \circ \delta_1)$ are inner as well. As both proofs are analogous, we only prove the latter. Let $z_{1,m} \in \mathrm{Frac}\, A_{w_1 v_1 m,m} - \{0\}$ and $z_{2,m} \in \mathrm{Frac}\, A'_{w_2 v_2 m,m} - \{0\}$ be as in Definition 1.7.3. Then for each $a \in A_{m,n}$:

$$\begin{aligned} ((\gamma_1 \circ \gamma_2) \circ (\delta_2 \circ \delta_1))(a) &= \gamma_1 \left((\gamma_2 \circ \delta_2)(\delta_1(x))\right) \\ &= \gamma_1 \left(z_{2,v_1 m} \delta_1(x) z_{2,v_1 n}^{-1}\right) \\ &= \gamma_1(z_{2,v_1 m})(\gamma_1 \circ \delta_1)(x) \gamma_1(z_{2,v_1 n})^{-1} \\ &= \gamma_1(z_{2,v_1 m})(z_{1,m} x z_{1,n}^{-1}) \gamma_1(z_{2,v_1 n})^{-1} \\ &= \left(\gamma_1(z_{2,v_1 m}) z_{1,m}\right) x \left(\gamma_1(z_{2,v_1 n}) z_{1,n}\right)^{-1} \end{aligned}$$

Moreover, obviously

$$\begin{aligned} \gamma_1(z_{2,v_2m})z_{1,m} &\in \mathrm{Frac}(A)_{w_1v_1w_2v_2m,m} \setminus \{0\} \\ \gamma_1(z_{2,v_2m})z_{1,m} &\in A \quad \text{if } m < 0 \\ \gamma_1(z_{2,0}))z_{1,0} &= 1 \\ (\gamma_1(z_{2,v_2m})z_{1,m})^{-1} &\in A \quad \text{if } m > 0 \end{aligned} \qquad \square$$

The proof of Theorem 1.7.6 then follows from the following 2 theorems which are proven in the next section:

Theorem 1.7.8. *Let $\gamma : A' \hookrightarrow \check{A}$ be a quadratic transform as in* (1.39)*. Then γ is invertible and the inverse can be chosen as an inclusion $\delta : \check{A} \hookrightarrow A'^{(2)}$ as in* (1.37)*.*

Theorem 1.7.9. *Let $\gamma : \check{A}' \hookrightarrow A^{(2)}$ be a quadratic transform as in* (1.37)*. Then γ is invertible and the inverse can be chosen as an inclusion $\delta : A \hookrightarrow \check{A}'$ which is as in* (1.39)*.*

Remark 1.7.10. *Our approach to proving Theorems 1.7.8 and 1.7.9 is as follows: we first construct an inclusion of $\mathbb{Z}$-algebras $\delta : A \hookrightarrow A'^{(v)}$ such that the composition $\gamma \circ \delta : A \to A^{(wv)}$ is inner. Afterwards we will show that δ is in fact a quadratic transform and that $\delta \circ \gamma$ is inner as well.*

1.8 Inverting quadratic transforms between quadratic Sklyanin algebras and cubic Sklyanin $\mathbb{Z}$-algebras

In this section we finish the proof of Theorem 1.7.6 by proving Theorem 1.7.8 and Theorem 1.7.9. The proofs of these theorems are intertwined:

In §1.8.2 we prove that if $\gamma : A' \hookrightarrow \check{A}$ is as in (1.39), then there is a quadratic transform $\delta : \check{A} \hookrightarrow A'^{(2)}$ such that $\gamma \circ \delta$ is inner. In §1.8.1 we prove that if $\gamma' : \check{A}' \hookrightarrow A^{(2)}$ is as in (1.37), then there is a quadratic transform $\delta' : A \hookrightarrow \check{A}'$ such that $\gamma' \circ \delta'$ is inner. Finally in §1.8.3 we prove that these constructions are each others inverses. I.e. if we were to construct δ out of γ as in §1.8.2, set $\gamma' = \delta$ and compute δ' as in §1.8.1, then $\delta' = \gamma$. This allows us to conclude that not only $\gamma \circ \delta$, but also $\delta \circ \gamma$ is inner (as it is equal to $\gamma' \circ \delta'$). The analogous results are true if we were to start from γ'.

1.8.1 $\mathbb{Z}^2$-algebras associated to a noncommutative map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$

Throughout this subsection $\gamma : A' \hookrightarrow \check{A}$ will be a quadratic transform between a quadratic Sklyanin algebra $A = A(Y, \mathcal{G}, \psi)$ and a cubic Sklyanin $\mathbb{Z}$-algebra $A' = A(Y, (\mathcal{L}_i)_{i\in\mathbb{Z}})$. Recall from §1.5 that the construction of γ is based on the choice of two points $q, r \in Y$, which lie in different τ-orbits for $\tau = \psi^3$. We will use the notation from this section, in particular d_i is defined as in (1.40).

Moreover, we use the notation $\mathcal{G}_i = \psi^{*i}\mathcal{G}$ such that $\check{A} = A(Y, (\mathcal{G}_i)_{i\in\mathbb{Z}})$. As in §1.7 we will however identify A with $\check{A}$ whenever there is no confusion possible. We will also use the notations X and $\mathcal{O}_X(i)$ as in §1.5.

We "glue" the $\mathbb{Z}$-algebras A and A' into a single $\mathbb{Z}^2$-algebra:

$$\tilde{A}_{(i,j),(m,n)} := \begin{cases} \mathrm{Hom}_X(\mathcal{O}_X(-m-n), \mathcal{O}_X(-i-j)m_{d_j}\dots m_{d_{n-1}} & \text{if } n > j \\ \mathrm{Hom}_X(\mathcal{O}_X(-m-n), \mathcal{O}_X(-i-j)) & \text{if } n \leq j \end{cases}$$

With X and $\mathcal{O}_X(i)$ as in Lemma 1.3.3.
Note that

$$\tilde{A}_{(i,0),(m,0)} = A_{i,m}$$

and

$$\tilde{A}_{(0,j),(0,n)} = \begin{cases} \mathrm{Hom}_X(\mathcal{O}_X(-n), \mathcal{O}_X(-j)m_{d_j}\cdots m_{d_{n-1}}) & \text{if } n \geq j \\ 0 & \text{if } n < j \end{cases}$$

such that

$$\tilde{A}_{(0,j),(0,n)} = A'_{j,n}$$

In other words $\tilde{A}$ "contains" both A and A'.

Remark 1.8.1. *By construction $\tilde{A}_{(i,j),(m,n)} \subset A_{i+j,m+n}$, therefor $\tilde{A}$ contains no non-trivial zero divisors as A is a domain. i.e. using the terminology of Definition 0.4.1, $\tilde{A}$ is a $\mathbb{Z}^2$-domain.*

Remark 1.8.2. *Geometrically one should think of $\tilde{A}$, as well as the $\mathbb{Z}^2$-algebras appearing in §1.8.2 and §1.12, as noncommutative blowups of the Sklyanin algebras underlying the construction. This intuition is justified by Theorem 1.11.3 and Corollary 1.12.3.*

We now give some results on the dimensions of certain $\tilde{A}_{(i,j),(m,n)}$. Let $h(n)$ be the Hilbert function of A and $h'(n)$ be the Hilbert function of A'. I.e.

$$\forall i \in \mathbb{Z} : \dim_k(A_{i,i+n}) = h(n) \text{ and } \dim_k(A'_{i,i+n}) = h'(n) \tag{1.58}$$

The following easy properties of $\tilde{A}$ are immediate from the definition:

Proposition 1.8.3. *Let $\tilde{A}$ be as above then*

i) $\dim_k \tilde{A}_{(i,j),(i,j+b)} = h'(b)$ *holds for all $b, i, j \in \mathbb{Z}$ with $b \geq 0$*

ii) $\dim_k \tilde{A}_{(i,j),(i+a,j+b)} = h(a+b)$ *holds for all $a, b, i, j \in \mathbb{Z}$ with $a \geq 0, b \leq 0$*

More interesting is the following:

Lemma 1.8.4. *Let $a \le 0$, then:*

$$\dim_k \tilde{A}_{(i,j),(i+a,j+b)} = \dim_k \tilde{A}_{(i,j),(i,j+b+2\cdot a)} = h'(b+2\cdot a)$$

Proof. The case $a=-1$ can be done analogously to (1.45), using Lemma 1.5.4. For $a \le -2$ we can no longer use the vanishing results from Lemma 1.5.4 and the proof is based on the existence of an "I-basis" (see [TV96] for the definition and construction of an I-basis). As we will only use the case $a=-1$, we refer the interested reader to §1.13 for the details of the proof for $a \le -2$. □

As a result of Lemma 1.8.4 both $\tilde{A}_{(i,j),(i-1,j+2)}$ and $\tilde{A}_{(i,j),(i+1,j-1)}$ are one dimensional. Let $\delta_{i,j}$ and $\gamma_{i,j}$ be nonzero elements in these spaces. We can then visualise $\tilde{A}$ on a 2-dimensional square grid.

$$\begin{tikzcd}
(3,0) \arrow[r] \arrow[drr, dotted, "\delta_{3,0}"] & (3,1) \arrow[r] \arrow[drr, dotted, "\delta_{3,1}"] & (3,2) \arrow[r] & (3,3) \\
(2,0) \arrow[r] \arrow[u] \arrow[drr, dotted, "\delta_{2,0}"] & (2,1) \arrow[r] \arrow[u] \arrow[ul, dotted, "\gamma_{2,1}"] \arrow[drr, dotted, "\delta_{2,1}"] & (2,2) \arrow[r] \arrow[u] \arrow[ul, dotted, "\gamma_{2,2}"] & (2,3) \arrow[u] \arrow[ul, dotted, "\gamma_{2,3}"] \\
(1,0) \arrow[r] \arrow[u] \arrow[drr, dotted, "\delta_{1,0}"] & (1,1) \arrow[r] \arrow[u] \arrow[ul, dotted, "\gamma_{1,1}"] \arrow[drr, dotted, "\delta_{1,1}"] & (1,2) \arrow[r] \arrow[u] \arrow[ul, dotted, "\gamma_{1,2}"] & (1,3) \arrow[u] \arrow[ul, dotted, "\gamma_{1,3}"] \\
(0,0) \arrow[r] \arrow[u] & (0,1) \arrow[r] \arrow[u] \arrow[ul, dotted, "\gamma_{0,1}"] & (0,2) \arrow[r] \arrow[u] \arrow[ul, dotted, "\gamma_{0,2}"] & (0,3) \arrow[u] \arrow[ul, dotted, "\gamma_{0,3}"]
\end{tikzcd}$$

The vertical arrows represent three dimensional vector spaces, whereas the horizontal arrows represent two dimensional vector spaces and the dotted arrows represent one dimensional vector spaces (labeled by $\gamma_{i,j}$ and $\delta_{i,j}$).

Now consider the following diagram

$$\begin{tikzcd}
(i,j) \arrow[rr] \arrow[dr, dotted, "\delta_{i,j}"'] \arrow[drrr] & & (i,j+b) \arrow[dr, dotted, "\delta_{i,j+b}"] & \\
& (i-1,j+2) \arrow[rr] & & (i-1,j+b+2)
\end{tikzcd}$$

From Lemma 1.8.4 we conclude that the vector spaces on the solid arrows all have the same dimension. Hence since $\tilde{A}$ is a $\mathbb{Z}^2$-domain we have an isomorphism of vector spaces:

$$\delta_{i,j}^{-1} \cdot \delta_{i,j+b} : \tilde{A}_{(i,j),(i,j+b)} \longrightarrow \tilde{A}_{(i-1,j+2),(i-1,j+b+2)} \tag{1.59}$$

Whenever $2i + j = 2m + n$ and $i \geq m$ we write

$$\delta_{(i,j),(m,n)} = \delta_{i,j}\delta_{i-1,j+2}\dots\delta_{m+1,n-2} \in \tilde{A}_{(i,j),(m,n)} \tag{1.60}$$

when $i < m$ we define

$$\delta_{(i,j),(m,n)} := \delta^{-1}_{(m,n),(i,j)} \in \mathrm{Frac}(A)_{i+2j,m+2n} \tag{1.61}$$

In particular we always have

$$\delta_{(i,j),(k,l)}\delta_{(k,l),(m,n)} = \delta_{(i,j),(m,n)}$$

From (1.59) we obtain an isomorphism

$$\delta_{(0,2i+j),(i,j)} \cdot \delta_{(i,j+b),(0,2i+j+b)} : \tilde{A}_{(i,j),(i,j+b)} \longrightarrow \tilde{A}_{(0,2i+j),(0,2i+j+b)}$$

Now note that there is always an inclusion

$$\cdot\delta_{(m,n),(i,n+2m-2i)} : \tilde{A}_{(i,j),(m,n)} \longrightarrow \tilde{A}_{(i,j),(i,n+2m-2i)} \tag{1.62}$$

If $m \geq i$ this follows from the fact that $\tilde{A}$ is a $\mathbb{Z}^2$-domain. Lemma 1.8.4 tells us the map is also well defined if $m < i$, in which case case (1.62) even is an isomorphism.

Summarizing we obtain an inclusion

$$\delta_{(0,2i+j),(i,j)} \cdot \delta_{(m,n),(0,2m+n)} : \tilde{A}_{(i,j),(m,n)} \longrightarrow \tilde{A}_{(0,2i+j),(0,2m+n)}$$

And hence an inclusion

$$\delta_{(0,2i),(i,0)} \cdot \delta_{(m,0),(0,2m)} : A_{i,m} = \tilde{A}_{(i,0),(m,0)} \to \tilde{A}_{(0,2i),(0,2m)} = A'_{2i,2m}$$

One easily checks that these inclusions are compatible with multiplication on A and A' such that we get an inclusion of algebras

$$\delta : A \hookrightarrow A'^{(2)}$$

Our goal is to show that δ is in fact a quadratic transform as in Definition 1.7.1. Moreover, we want $\gamma \circ \delta$ to be inner (as stated in the introduction of this section, the proof to show that $\delta \circ \gamma$ is inner is postponed to §1.8.3). We first prove the latter:

Let $\gamma_{(i,j),(m,n)}$ be defined analogously to $\delta_{(i,j),(m,n)}$. I.e. for $i + j = m + n$ we set

$$\gamma_{(i,j),(m,n)} = \begin{cases} \gamma_{i,j}\gamma_{i+1,j-1}\dots\gamma_{m-1,n+1} \in \tilde{A}_{(i,j),(m,n)} & \text{if } i \leq m \\ \gamma^{-1}_{(m,n),(i,j)} \in \mathrm{Frac}(A)_{i+j,m+n} & \text{if } i > m \end{cases} \tag{1.63}$$

It is easy to see that the quadratic transform $\gamma : A' \to A$ we started with is given by

$$\gamma_{(j,0),(0,j)} \cdot \gamma_{(0,n),(n,0)} : A'_{j,n} = \tilde{A}_{(0,j),(0,n)} \to \tilde{A}_{(2j,0),(2n,0)} = A_{2j,2n}$$

So the composition $\gamma \circ \delta : A \to A^{(2)}$ is given by

$$\gamma_{(2i,0),(0,2i)}\delta_{(0,2i),(i,0)} \cdot \delta_{(m,0),(0,2m)}\gamma_{(0,2m),(2m,0)} : A_{i,m} \to A_{2i,2m}$$

This composition is inner with

$$z_i = \gamma_{(2i,0),(0,2i)}\delta_{(0,2i),(i,0)} \in \mathrm{Frac}(A)_{2i,i}$$

One easily checks that the elements z_i indeed satisfy the conditions in (1.57).

We now prove that δ is a quadratic transform. Concretely, we need to show the existence of a point $p' \in Y$ such that

$$\begin{aligned}\delta(A_{i,i+1}) &= \Gamma(Y, \mathcal{L}_{2i}\mathcal{L}_{2i+1}(-\tau^{-i}p')) \\ &= \Gamma(Y, \mathcal{G}_{2i}\mathcal{G}_{2i+1}(-d_{2i} - d_{2i+1} - \tau^{-i}p'))\end{aligned} \tag{1.64}$$

We first define $\tilde{B}$ like $\tilde{A}$ but starting from B instead of from A. We find (using (1.5) and (1.7))

$$\tilde{B}_{(i,j),(m,n)} := \begin{cases} \Gamma(Y, \mathcal{G}_{i+j}\mathcal{G}_{i+j+1}\ldots\mathcal{G}_{m+n}(-d_j - \ldots - d_{n-1})) & \text{if } n > j \\ \Gamma(Y, \mathcal{G}_{i+j}\mathcal{G}_{i+j+1}\ldots\mathcal{G}_{m+n}) & \text{if } n \le j \end{cases} \tag{1.65}$$

Similar to Lemma 1.3.12 one can show

Lemma 1.8.5. *The canonical map*

$$\tilde{A}_{(i,j),(m,n)} \longrightarrow \tilde{B}_{(i,j),(m,n)}$$

is an epimorphism in the first quadrant (i.e. $m \ge i$, $n \ge j$).

Recall that by the definition of $\delta : A \to A'^{(2)}$ we have for each $x \in A_{i,i+1}$

$$\delta_{i,0}\delta_{i-1,2}\cdots\delta_{1,2i-2} \cdot \delta(x) = x \cdot \delta_{i+1,0}\delta_{i,2}\cdots\delta_{1,2i} \tag{1.66}$$

when $i \ge 0$ and

$$\delta(x) \cdot \delta_{0,2i+2}\delta_{-1,2i+4}\ldots\delta_{i+2,-2} = \delta_{0,2i}\delta_{-1,2i+2}\ldots\delta_{i+1,-2} \cdot x \tag{1.67}$$

when $i < 0$. Hence in order to prove the existence of p' in (1.64) we have to understand (the product of) the image(s) $\overline{\delta}_{i,j}$ of $\delta_{i,j}$ in

$$\tilde{B}_{(i,j),(i-1,j+2)} = \Gamma(Y, \mathcal{G}_{i+j}(-d_j - d_{j+1}))$$

As $\mathcal{G}_{i+j}(-d_j - d_{j+1})$ has degree 1 on Y we can choose a point $p'_{i,j}$ defined by

$$d_j + d_{j+1} + p'_{i,j} \sim [\mathcal{G}_{i+j}] \tag{1.68}$$

such that

$$\tilde{B}_{(i,j),(i-1,j+2)} = \Gamma(Y, \mathcal{G}_{i+j}(-d_j - d_{j+1})) = \Gamma(Y, \mathcal{G}_{i+j}(-d_j - d_{j+1} - p'_{i,j})) \tag{1.69}$$

Lemma 1.8.6. *Define p' by the following identitiy*

$$q + r + \tau p' \sim [\mathcal{G}_0] \tag{1.70}$$

Then $p'_{i,j} = \tau^{-i+1} p'$.

Proof. As ψ is a translation such that $\mathcal{G}_{i+1} \cong \psi^* \mathcal{G}_i$, there is an invertible sheaf $\mathcal{N}$ of degree zero (see for example (1.21) or [VdB11, Theorem 4.2.3]) such that

$$[\mathcal{G}_n] = [\mathcal{G}_0] + 3n[\mathcal{N}]$$

and

$$\tau^{-i} q \sim q + 3i[\mathcal{N}]$$

As $p'_{i,j}$ is uniquely defined by (1.68) this proves the lemma. □

If p' is as in the above lemma, then (1.69) gives rise to

$$\tilde{B}_{(i,j),(i-1,j+2)} = \Gamma(Y, \mathcal{G}_{i+j}(-d_j - d_{j+1})) = \Gamma(Y, \mathcal{G}_{i+j}(-d_j - d_{j+1} - \tau^{-i+1} p'))$$

In particular $\overline{\delta}_{i,j}$ is a non-zero section of $\mathcal{G}_{i+j}(-d_j - d_{j+1} - \tau^{-i+1}p')$. As the latter has degree zero on Y, $\overline{\delta}_{i,j}$ is everywhere non-zero on Y. Going back to (1.66) (and hence assuming $i \geq 0$, the case $i < 0$ being completely similar) we see that

$$\overline{\delta}_{i,0} \overline{\delta}_{i-1,2} \dots \overline{\delta}_{1,2i-2}$$

is an everywhere non-zero section of

$$\mathcal{G}_i \dots \mathcal{G}_{2i-1}(-d_0 - d_1 - \tau^{-i+1} p' - d_2 - d_3 - \tau^{-i+2} p' - \dots - d_{2i-2} - d_{2i-1} - p')$$

Likewise

$$\overline{x}\ \overline{\delta}_{i+1,0} \overline{\delta}_{i,2} \dots \overline{\delta}_{1,2i}$$

is a section of

$$\mathcal{G}_i \dots \mathcal{G}_{2i+1}(-d_0 - d_1 - \tau^{-i} p' - d_2 - d_3 - \tau^{-i+1} p' - \dots - d_{2i} - d_{2i+1} - p')$$

so that $\overline{\delta(x)}$ is a section of $\mathcal{G}_{2i}\mathcal{G}_{2i+1}(-d_{2i} - d_{2i+1} - \tau^{-i} p')$. This is precisely what we had to show according to (1.64).

1.8.2 $\mathbb{Z}^2$ algebras associated to a noncommutative map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$

Throughout this subsection $\gamma : A' \hookrightarrow A^{(2)}$ will be a quadratic transform between a cubic Sklyanin $\mathbb{Z}$-algebra $A = A(Y, (\mathcal{L}_i)_{i\in\mathbb{Z}})$ and a quadratic Sklyanin algebra $A' = A(Y, \mathcal{G}, \psi)$ (which we identify with the associated $\mathbb{Z}$-algebra $\check{A}' = A(Y, (\mathcal{G}_i)_{i\in\mathbb{Z}})$ where $\mathcal{G}_i = \psi^{i*}\mathcal{G}$). Recall from §1.3 that the construction of γ is based on the choice of a points $p \in Y$. We will use the notation from this section, in particular $\tau = \alpha^2$, where α is as in Definition 1.3.1.

We define the $\mathbb{Z}^2$-algebra $\tilde{A}$ as follows:

$$\tilde{A}_{(i,j),(m,n)} := \begin{cases} \operatorname{Hom}_X(\mathcal{O}_X(-m-2n), \mathcal{O}_X(-i-2j)m_{\tau^{-j}p}\dots m_{\tau^{-n+1}p} & \text{if } n > j \\ \operatorname{Hom}_X(\mathcal{O}_X(-m-2n), \mathcal{O}_X(-i-2j)) & \text{if } n \le j \end{cases}$$

As in the previous section the following easy properties of $\tilde{A}$ are immediate from the definition.:

Proposition 1.8.7. *Let $\tilde{A}$ be as above then*

i) $\tilde{A}_{(i,0),(m,0)} = A_{i,m}$

ii) $\tilde{A}_{(0,j),(0,n)} = A'_{j,n}$

iii) *$\tilde{A}$ contains no nontrivial zero divisors*

iv) $\dim_k \tilde{A}_{(i,j),(i,j+b)} = h'(b)$ *holds for all* $b, i, j \in \mathbb{Z}$ *with* $b \ge 0$ *and* h' *the Hilbert series of* A'

v) $\dim_k \tilde{A}_{(i,j),(i+a,j+b)} = h(a+2b)$ *holds for all* $a, b, i, j \in \mathbb{Z}$ *with* $a \ge 0, b \le 0$ *and* h *the Hilbert series of* A

Where for *iii)* we used the fact that A is a $\mathbb{Z}$-domain as in Theorem 0.4.2.

We also have the following partial analogue of Lemma 1.8.4:

Lemma 1.8.8.

$$\dim_k \tilde{A}_{(i,j),(i-1,j+b)} = \dim_k \tilde{A}_{(i,j),(i,j+b-1)} = h'(b-1)$$

Proof. The computation is completely similar to (1.36), using the Ext^1-vanishing from Lemma 1.3.10. □

Remark 1.8.9. *Although one cannot use an I-basis in the classical sense we expect* $\dim_k \tilde{A}_{(i,j),(i+a,j+b)} = h'(b + 2\cdot a)$ *to hold for all* $a \le 0$.

As a corollary of Lemma 1.8.8 and Proposition 1.8.7iii) we know $\tilde{A}_{(i,j),(i-1,j+2)}$ and $\tilde{A}_{(i,j),(i+1,j-1)}$ are one dimensional. Let $\delta_{i,j}$ and $\gamma_{i,j}$ be nonzero elements in these spaces. We can then visualise $\tilde{A}$ on a 2-dimensional square grid.

$$\begin{tikzcd}
(3,0) \arrow[r] \arrow[dr, dotted, "\delta_{3,0}"] & (3,1) \arrow[r] \arrow[dr, dotted, "\delta_{3,1}"] & (3,2) \arrow[r] \arrow[dr, dotted, "\delta_{3,2}"] & (3,3) \\
(2,0) \arrow[r] \arrow[u] \arrow[dr, dotted, "\delta_{2,0}"] & (2,1) \arrow[r] \arrow[u] \arrow[dr, dotted, "\delta_{2,1}"] & (2,2) \arrow[r] \arrow[u] \arrow[dr, dotted, "\delta_{2,2}"] & (2,3) \arrow[u] \\
(1,0) \arrow[r] \arrow[u] \arrow[dr, dotted, "\delta_{1,0}"] & (1,1) \arrow[r] \arrow[u] \arrow[uul, dotted, "\gamma_{1,1}"] \arrow[dr, dotted, "\delta_{1,1}"] & (1,2) \arrow[r] \arrow[u] \arrow[uul, dotted, "\gamma_{1,2}"] \arrow[dr, dotted, "\delta_{1,2}"] & (1,3) \arrow[u] \arrow[uul, dotted, "\gamma_{1,3}"] \\
(0,0) \arrow[r] \arrow[u] & (0,1) \arrow[r] \arrow[u] \arrow[uul, dotted, "\gamma_{0,1}"] & (0,2) \arrow[r] \arrow[u] \arrow[uul, dotted, "\gamma_{0,2}"] & (0,3) \arrow[u] \arrow[uul, dotted, "\gamma_{0,3}"]
\end{tikzcd}$$

All horizontal arrows represent three dimensional vector spaces whereas the vertical arrows represent two dimensional vector spaces and dotted arrows represent one dimensional vector spaces (labeled by $\gamma_{i,j}$ and $\delta_{i,j}$).

Completely identical to §1.8.2 there is an inclusion

$$\delta_{(0,i),(i,0)} \cdot \delta_{(m,0),(0,m)} : A_{i,m} = \tilde{A}_{(i,0),(m,0)} \to \tilde{A}_{(0,i),(0,m)} = A'_{i,m}$$

(where the elements $\delta_{(i,j),(m,n)}$ are defined as in (1.60) and (1.61). The only difference lying in the fact that $\delta_{(i,j),(m,n)}$ is now only defined when $i + j = m + n$ instead of $i + 2j = m + 2n$.)

The induced inclusion of algebras

$$\delta : A \hookrightarrow A'$$

is such that the composition $\gamma \circ \delta$ is inner with

$$z_i = \gamma_{(i,0),(0,2i)}\delta_{(0,i),(i,0)}$$

(where $\gamma_{(i,j),(m,n)}$ is defined as in (1.63). The only difference lying in the fact that $\gamma_{(i,j),(m,n)}$ is now only defined when $i + 2j = m + 2n$ instead of $i + j = m + n$.)

Our next aim is to show that δ is a quadratic transform. Concretely, we need to show the existence of two points $q', r' \in Y$, lying in different τ-orbits such that if we define d'_i as

$$d'_i = \begin{cases} \tau^{-l}q' & \text{if } i = 2l \\ \tau^{-l}r' & \text{if } i = 2l + 1 \end{cases} \tag{1.71}$$

then we have

$$\delta(A_{i,i+1}) = \Gamma(Y, \mathcal{G}_i(-d'_i)) = \Gamma(Y, \mathcal{L}_{2i}\mathcal{L}_{2i+1}(-\tau^{-i}p - d'_i)) \tag{1.72}$$

We again start by defining a $\mathbb{Z}^2$-algebra $\tilde{B}$. This time it takes the following form:

$$\tilde{B}_{(i,j),(m,n)} := \begin{cases} \Gamma(Y, \mathcal{L}_{i+2j}\mathcal{L}_{i+2j+1}\ldots\mathcal{L}_{m+2n-1}(-\tau^{-j}p - \ldots - \tau^{-n+1}p)) & \text{if } n > j \\ \Gamma(Y, \mathcal{L}_{i+2j}\mathcal{L}_{i+2j+1}\ldots\mathcal{L}_{m+2n-1}) & \text{if } n \leq j \end{cases} \tag{1.73}$$

Again analogously to Lemma 1.3.12 one can show

Lemma 1.8.10. *The canonical map*

$$\tilde{A}_{(i,j),(m,n)} \to \tilde{B}_{(i,j),(m,n)}$$

is an epimorphism in the first quadrant (i.e. $m \geq i$, $n \geq j$).

Recall that for each $x \in A_{i,i+1}$, $\delta(x)$ is related to x and elements $\delta_{(i,j),(m,n)}$ via

$$\delta_{i,0}\delta_{i-1,1}\cdots\delta_{1,i-1} \cdot \delta(x) = x \cdot \delta_{i+1,0}\delta_{i,1}\cdots\delta_{1,i} \tag{1.74}$$

when $i \geq 0$ and

$$\delta(x) \cdot \delta_{0,i+1}\delta_{-1,i+2}\ldots\delta_{i+2,-1} = \delta_{0,i}\delta_{-1,i+1}\ldots\delta_{i+1,-1} \cdot x \tag{1.75}$$

when $i < 0$. Hence in order to understand $\delta(x)$ in $A'_{2i,2i+2} = B'_{2i,2i+2}$ we need to understand (the product of) the image(s) $\overline{\delta}_{i,j}$ of $\delta_{i,j}$ in $\tilde{B}$. First remark that if we choose $q', r' \in Y$ such that

$$\begin{aligned} p + \tau r' &\sim [\mathcal{L}_0] \\ p + q' &\sim [\mathcal{L}_1] \end{aligned} \tag{1.76}$$

then similar to Lemma 1.8.6 we then have for all i, j:

$$\tau^{-j}p + d'_{i-1} \sim [\mathcal{L}_{i+2j}] \tag{1.77}$$

with d'_i as in (1.71). Fix q', r' as above, then $\overline{\delta}_{i,j}$ is a nonzero element of

$$\Gamma(Y, \mathcal{L}_{i+2j}(-\tau^{-j}p - d'_{i-1})) \tag{1.78}$$

As $\mathcal{L}_{i+2j}(-\tau^{-j}p - d'_{i-1})$ has degree zero on Y, $\overline{\delta}_{i,j}$ is everywhere non-zero on Y.

In particular, going back to (1.74) (and hence assuming $i \geq 0$, the case $i < 0$ being completely similar) we see that

$$\overline{\delta}_{i,0}\overline{\delta}_{i-1,1}\ldots\overline{\delta}_{1,i-1}$$

is an everywhere non-zero section of

$$\mathcal{L}_i\ldots\mathcal{L}_{2i-1}(-p - d'_{i-1} - \tau^{-1}p - d'_{i-2} - \ldots - \tau^{-i+1}p - d'_0)$$

Likewise

$$\overline{x}\ \overline{\delta}_{i+1,0}\overline{\delta}_{i,1}\dots\overline{\delta}_{1,i}$$

is a section of

$$\mathcal{L}_i\dots\mathcal{L}_{2i+1}(-p-d'_i-\tau^{-1}p-d'_{i-1}-\dots-\tau^{-i}p-d'_0)$$

so that $\overline{\delta(x)}$ is a section of $\mathcal{L}_{2i}\mathcal{L}_{2i+1}(-\tau^{-i}p-d'_i)$. This is precisely what we had to show according to (1.72).

1.8.3 Invertibility of quadratic transforms

We now show that if γ and δ are as in §1.8.2 or §1.8.1, then $\delta\circ\gamma$ is inner. This boils down to computations on the geometric data associated to γ and δ. First assume $A=A(Y,\mathcal{G},\psi)$ is quadratic and $\gamma: A'\hookrightarrow A$ is constructed with respect to $q,r\in Y$. Then according to Lemma 1.8.6 the quadratic transform $\delta: A\hookrightarrow A'^{(2)}$ is constructed with respect to a point $p'\in Y$ satisfying

$$q+r+\tau p'\sim[\mathcal{G}]=:[\mathcal{G}_0]$$

Using the techniques in §1.8.2 we find a quadratic transform $\tilde{\gamma}: A'\to A$ such that $\delta\circ\tilde{\gamma}$ is inner. It hence suffices to prove that γ and $\tilde{\gamma}$ are in fact the same. In both cases the morphism is determined by the choice of two points (q and r, respectively q'' and r'') on Y and the description of A' as a cubic Sklyanin $\mathbb{Z}$-algebra is given by an elliptic helix $((\mathcal{G}_i)_{i\in\mathbb{Z}}$, respectively $(\mathcal{G}''_i)_{i\in\mathbb{Z}})$. Hence it suffices to show that these geometric data coincide.

By (1.76) we know $\tilde{\gamma}$ is constructed with respect to points $q'',r''\in Y$ satisfying

$$\begin{aligned} p'+\tau r''&\sim[\mathcal{L}_0]\\ p'+q''&\sim[\mathcal{L}_1] \end{aligned}\tag{1.79}$$

with $\mathcal{L}_i$ as in (1.43). Moreover, §1.8.2 constructs A out of an elliptic helix $(\mathcal{G}''_i)_{i\in\mathbb{Z}}$ given by

$$\mathcal{G}''_i=\mathcal{L}_{2i}\otimes\mathcal{L}_{2i+1}\otimes\mathcal{O}_Y(-\tau^{-i}p)$$

We need to check that $q''=q, r''=r$ and $\mathcal{G}''_i\cong\mathcal{G}_i:=\psi^{i*}\mathcal{G}$. To show this, recall from [VdB11, Theorem 4.2.3] that there exists a line bundle $\mathcal{N}$ of degree zero on Y such that for each line bundle $\mathcal{M}$ we have $[\psi^*\mathcal{M}]=[\mathcal{M}]+\deg(\mathcal{M})\cdot[\mathcal{N}]$. Using this, we find:

$$\begin{gathered}
p' + \tau r'' \sim [\mathcal{L}_0] = [\mathcal{G} \otimes \mathcal{O}_Y(-p)] \\
\Downarrow \\
p' + (r'' - 3[\mathcal{N}]) + p \sim [\mathcal{G}] \\
\Downarrow \\
p' + r'' + (p - 3[\mathcal{N}]) \sim [\mathcal{G}] \\
\Downarrow \\
p + r'' + \tau p' \sim q + r + \tau p' \\
\Downarrow \\
r'' = r
\end{gathered}$$

Similarly $q'' = q$. Next we show that the elliptic helix $(\mathcal{G}''_i)_{i\in\mathbb{Z}}$ coincides with $(\mathcal{G}_i)_{i\in\mathbb{Z}}$:

$$\begin{aligned}
[\mathcal{G}''_i] &= [\mathcal{L}_{2i}] + [\mathcal{L}_{2i+1}] + [\mathcal{O}_Y(-\tau^{-i}p')] \\
&= [\mathcal{G}_{2i}] + [\mathcal{G}_{2i+1}] + [\mathcal{O}_Y(-d_{2i} - d_{2i+1})] + [\mathcal{O}_Y(-\tau^{-i}p')] \\
&= [\mathcal{G}_{2i}] + [\mathcal{G}_{2i+1}] + [\mathcal{O}_Y(-d_i - d_{i+1})] - 3i \cdot [\mathcal{N}] + [\mathcal{O}_Y(-\tau^{-i}p')] \\
&= ([\mathcal{G}_{2i}] - 3i \cdot [\mathcal{N}]) + \big([\mathcal{G}_{2i+1}] + [\mathcal{O}_Y(-d_i - d_{i+1} - \tau^{-i}p')]\big) \\
&= [\psi^i_* \mathcal{G}_{2i}] + 0 = [\mathcal{G}_i]
\end{aligned}$$

This finishes the proof of Theorem 1.7.8.

Next we do similar computations in case $A = A(Y, (\mathcal{L}_i)_{i\in\mathbb{Z}})$ is a Sklyanin $\mathbb{Z}$-algebra and $\gamma : A' \hookrightarrow A^{(2)}$ in constructed with respect to a point $p \in Y$ as in §1.8.2. Similar to the above it suffices to prove $p'' = p$ and $\mathcal{L}''_i = \mathcal{L}_i$ where

$$\begin{aligned}
p + \tau r' &\sim [\mathcal{L}_0] \\
p + q' &\sim [\mathcal{L}_1] \\
q' + r' + \tau p'' &\sim [\mathcal{G}_0] \\
\mathcal{L}''_i &= \mathcal{G}_i \otimes \mathcal{O}_Y(-d'_i)
\end{aligned}$$

where $\mathcal{G}_i$ is as in (1.20) and d'_i as in (1.71). First we prove $p'' = p$, for this we take $\mathcal{N}$ a degree zero linebundle on Y such that $[\alpha^*\mathcal{M}] = [\mathcal{M}] + \deg(\mathcal{M}) \cdot [\mathcal{N}]$ with α as in

Definition 1.3.1

$$
\begin{gathered}
q' + r' + \tau p'' \sim [\mathcal{G}_0] = [\mathcal{L}_0 \otimes \mathcal{L}_1 \otimes \mathcal{O}_Y(-p)] \\
\Downarrow \\
q' + r' + (p'' - 2[\mathcal{N}]) + p \sim [\mathcal{L}_0] + [\mathcal{L}_1] \\
\Downarrow \\
q' + (r' - 2[\mathcal{N}]) + p'' + p \sim (p + \tau r') + (p + q') \\
\Downarrow \\
q' + \tau r' + p'' + p \sim p + \tau r' + p + q' \\
\Downarrow \\
p'' = p
\end{gathered}
$$

We now prove that the elliptic helix $(\mathcal{L}''_i)_{i\in\mathbb{Z}}$ coincides with $(\mathcal{L}_i)_{i\in\mathbb{Z}}$. To show this, we fix an $i \in \mathbb{Z}$ and assume without loss of generality that i is even.

$$
\begin{aligned}
[\mathcal{L}''_i] &= [\mathcal{G}_i] + [\mathcal{O}_Y(-d'_i)] \\
&= [\mathcal{L}_{2i}] + [\mathcal{L}_{2i+1}] + [\mathcal{O}_Y(-\tau^{-i}p)] + [\mathcal{O}_Y(-d'_i)] \\
&= [\alpha^{\frac{i}{2}*}\mathcal{L}_i] + [\alpha_*^{\frac{i}{2}}\mathcal{L}_{3i+1}] + [\mathcal{O}_Y(-\tau^{-i}p)] + [\mathcal{O}_Y(-d'_i)] \\
&= [\mathcal{L}_i] + i[\mathcal{N}] + [\mathcal{L}_{3i+1}] - i[\mathcal{N}] + [\mathcal{O}_Y(-\tau^{-i}p)] + [\mathcal{O}_Y(-d'_i)] \\
&= [\mathcal{L}_i] + ([\mathcal{L}_{3i+1}] + [\mathcal{O}_Y(-d'_i)] + [\mathcal{O}_Y(-\tau^{-i}p)]) \\
&= [\mathcal{L}_i]
\end{aligned}
$$

(Where in the last line we used (1.77) to conclude that $[\mathcal{L}_{3i+1}] + [\mathcal{O}_Y(-d'_i)] + [\mathcal{O}_Y(-\tau^{-i}p)] = 0$.) This proves Theorem 1.7.9.

As mentioned above Theorem 1.7.6 now follows by combining Theorem 1.7.8, Theorem 1.7.9 and Lemma 1.7.7.

1.9 $\mathbb{Z}^2$-algebras as in §1.8 are noetherian

In this section we show that the $\mathbb{Z}^2$-algebras $\tilde{A}$ as introduced above are noetherian (in the sense that $\mathrm{Gr}(\tilde{A})$ is locally noetherian). We will state all theorems for $\tilde{A}$ as in §1.8.1 or §1.8.2, but we often only prove the statement for one of the two cases. This choice is justified by the fact that the proofs are essentially the same if we choose $\tilde{A}$ as in §1.8.1 or as in §1.8.2, see also Conjecture 1.9.4.

Throughout this section we identify graded algebras with their associated $\mathbb{Z}$-algebras. $\gamma : A' \to A^{(v)}$ is a noncommutative $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ or $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. I.e. either $v = 2$ and γ is as in (1.37) or $v = 1$ and γ is as in (1.39).

In order to prove that $\tilde{A}$ is noetherian we work through $\tilde{B}$. By construction there is a surjective $\mathbb{Z}^2$-algebra morphism $\tilde{A}_+ \to \tilde{B}_+$ (recall Lemma 1.8.5, Lemma 1.8.10 and Notation 0.6.1). We first check that in sufficiently high degrees this map is given by killing a certain collection of "normalizing" elements $\{g_{(i,j)}\}$. I.e. we prove the following

Lemma 1.9.1. *With the notations as above:*

i) For all $i,j \in \mathbb{Z}$ the map $\tilde{A}_{(i,j),(i+v,j+(3-v))} \to \tilde{B}_{(i,j),(i+v,j+(3-v))}$ has a one dimensional kernel. Let $g_{(i,j)}$ be a nonzero element in this kernel.

ii) For all $a \geq v, b \geq 3-v$ the kernel of $\tilde{A}_{(i,j),(i+a,j+b)} \to \tilde{B}_{(i,j),(i+a,j+b)}$ is given by $g_{(i,j)}\tilde{A}_{(i+v,j+(3-v)),(i+a,j+b)} = \tilde{A}_{(i,j),(i+a-v,j+b-(3-v))}g_{(i+a-v,j+b-(3-v))}$.

Proof. i) follows from Lemma 1.9.3.
In order to prove *ii)* we only prove the first equality and restrict to the case $v = 2$. All other equalities are analogous. Now for each $i,j \in \mathbb{Z}$ take

$$g_{(i,j)} \in \ker\left(\tilde{A}_{(i,j),(i+2,j+1)} \to \tilde{B}_{(i,j),(i+2,j+1)}\right) \setminus \{0\}$$

The following diagram

$$\begin{tikzcd}
& 0 \arrow[d] & \\
& g_{(i,j)}k \otimes \tilde{A}_{(i+2,j+1),(i+a,j+b)} \arrow[r] \arrow[d] & g_{(i,j)}\tilde{A}_{(i+2,j+1),(i+a,j+b)} \arrow[d] \\
& \tilde{A}_{(i,j),(i+2,j+1)} \otimes \tilde{A}_{(i+2,j+1),(i+a,j+b)} \arrow[r] \arrow[d] & \tilde{A}_{(i,j),(i+a,j+b)} \arrow[d] \\
& \tilde{B}_{(i,j),(i+2,j+1)} \otimes \tilde{A}_{(i+2,j+1),(i+a,j+b)} \arrow[r] \arrow[d] & \tilde{B}_{(i,j),(i+a,j+b)} \arrow[d] \\
& 0 & 0
\end{tikzcd}$$

(where the bottom horizontal arrow is given by composing the canonical map $\tilde{A} \to \tilde{B}$ with the multiplication in $\tilde{B}$)

shows

$$g_{(i,j)}\tilde{A}_{(i+2,j+1),(i+a,j+b)} \subset \ker\left(\tilde{A}_{(i,j),(i+2,j+1)} \to \tilde{B}_{(i,j),(i+2,j+1)}\right)$$

Hence it suffices to show that the alternating sum of the dimensions of the spaces in the right column equals zero. This again follows form Lemma 1.9.3. ☐

Corollary 1.9.2. *The collection* $\{g_{(i,j)}\}_{i,j\in\mathbb{Z}}$ *is normalizing in the sense that for every* $a \in \tilde{A}_{(i+v,j+(3-v)),(m,n)}$ *there exists a unique* $a' \in \tilde{A}_{(i,j),(m-v,n-(3-v))}$ *such that* $g_{(i,j)}a = a'g_{(m-v,n-(3-v))}$. *As such, for every right* $\tilde{A}$*-module* M *one can consider the right* $\tilde{A}$*-module* Mg *defined by* $(Mg)_{(i,j)} := M_{(i-v,j-(3-v))}g_{(i-v,j-(3-v))}$.

Lemma 1.9.3. *Let* $\gamma : A' \to A^{(v)}$, $\tilde{A}$ *and* $\tilde{B}$ *be as above. Then one has:*

- *If* $v = 2$*:*

$$\dim_k\left(\tilde{A}_{(i,j),(i+a,j+b)}\right) = \begin{cases} \frac{a^2+4ab+2b^2+4a+6b+4}{4} & \text{for } a,b \geq 0,\ a \text{ even} \\ \frac{a^2+4ab+2b^2+4a+6b+3}{4} & \text{for } a,b \geq 0,\ a \text{ odd} \end{cases}$$

and

$$\dim_k\left(\tilde{B}_{(i,j),(i+a,j+b)}\right) = \begin{cases} 2a+3b & \text{for } a,b \geq 0,\ (a,b) \neq (0,0) \\ 1 & \text{for } (a,b) = (0,0) \end{cases}$$

- *If* $v = 1$*:*

$$\dim_k\left(\tilde{A}_{(i,j),(i+a,j+b)}\right) = \begin{cases} \frac{2a^2+4ab+b^2+6a+4b+4}{4} & \text{for } a,b \geq 0,\ b \text{ even} \\ \frac{2a^2+4ab+b^2+6a+4b+3}{4} & \text{for } a,b \geq 0,\ b \text{ odd} \end{cases}$$

and

$$\dim_k\left(\tilde{B}_{(i,j),(i+a,j+b)}\right) = \begin{cases} 3a+2b & \text{for } a,b \geq 0,\ (a,b) \neq (0,0) \\ 1 & \text{for } (a,b) = (0,0) \end{cases}$$

Proof. For $\tilde{B}$ the equality is trivial as it corresponds to calculating sections of line bundles on an elliptic curve. For $\tilde{A}$ the computation is done in a similar way as in (1.36) using Lemma 1.3.10 for $v = 2$ or as in (1.45) using Lemma 1.5.4 for $v = 1$. □

Inspired by Lemma 1.9.3, Theorem 1.7.8 and Theorem 1.7.9 we claim:

Conjecture 1.9.4. *Let* $\gamma : \check{A}' \to A^{(2)}$ *and* $\delta : A \to \check{A}'$ *be as in* (1.37) *and* (1.39). *Assume moreover that* γ *and* δ *are mutually inverse as in Theorem 1.7.8 and Theorem 1.7.9. Let* $\tilde{A}$ *be as in §1.8.1 and* $\tilde{A}'$ *be as in §1.8.2. Then there exist isomorphisms*

$$\tilde{A}_{(i,j),(m,n)} \cong \tilde{A}'_{(j,i),(n,m)}$$

which are compatible with the algebra structures.

In §1.10 we will slightly generalize the arguments in [AVdB90, §3] to show

Theorem 1.9.5. $\tilde{B}_+$ *is noetherian.*

Using this result as well as Lemma 1.9.1 we can prove

Theorem 1.9.6. *$\tilde{A}$ is noetherian*

Proof. Obviously the objects $e_{(i,j)}\tilde{A}$ generate $\mathrm{Gr}(\tilde{A})$, hence it suffices to show that these are noetherian. Let

$$M_0 \subset M_1 \subset \ldots \subset e_{(i,j)}\tilde{A} \tag{1.80}$$

be an ascending chain of right $\tilde{A}$ modules. We need to show that this sequence stabilizes. This is done in two steps. First we show the existence of an $N_0 \in \mathbb{N}$ such that $(M_n)_{(i+a,j+b)} = (M_{N_0})_{(i+a,j+b)}$ holds for all $n \geq N_0$, $ab \leq 0$. This is based on noetherianity of A and A'. Next we show the existence of an $N_1 \in \mathbb{N}$ such that $(M_n)_{(i+a,j+b)} = (M_{N_1})_{(i+a,j+b)}$ holds for $a, b \geq 0$. Which is heavily based on Theorem 1.9.5. (This also explains why we only need noetherianity of $\tilde{B}$ in the first quadrant in Theorem 1.9.5).

Step 1: Convergence of $\left((M_n)_{(i+a,j+b)}\right)_{n\in\mathbb{N}}$ for $ab \leq 0$

As the cases $a \leq 0$ and $b \leq 0$ are analogous, we only prove the case $a \leq 0$. Similarly we only prove the case $v = 2$. Moreover, without loss of generality we can assume $i = j = 0$. Recall that $\tilde{A}_{(m,n),(m-1,n+2)}$ is a 1-dimensional vector space and that $\delta_{(m,n)}$ is a nonzero element in this vector space. Moreover, as $\tilde{A}$ has no non-trivial zero divisors in the sense that

$$\forall a, b, i, j, m, n \in \mathbb{Z}, x \in \tilde{A}_{(a,b),(i,j)}, y \in \tilde{A}_{(i,j),(m,n)} : xy = 0 \Rightarrow x = 0 \vee y = 0$$

multiplication by these elements defines embeddings of vectorspaces. By the dimension count in Lemma 1.9.3 the embeddings

$$\tilde{A}_{(0,0),(a,b)} \hookrightarrow \tilde{A}_{(0,0),(a-1,b+2)} : x \mapsto x\delta_{(a,b)}$$

are in fact isomorphisms. Moreover, these isomorphisms induce embeddings $(M_n)_{(a,b)} \hookrightarrow (M_n)_{(a-1,b+2)}$. In particular if we define $M_n^c \subset e_0 A'$ via

$$(M_n^c)_m := \{x \in A'_{0,m} \cong \tilde{A}_{(0,0),(0,m)} \mid x\delta^c \in (M_n)_{(-c,m+2c)}\}$$

(where we used the shorthand notation $\delta^c = \delta_{(0,m)}\delta_{(-1,m+2)}\ldots\delta_{(-c+1,m+2c-2)}$.)
we find an ascending chain of A'-submodules of e_0A'. (The A'-structure follows from the fact that $A'_{m,m'} = \tilde{A}_{(0,m),(0,m')}$ is isomorphic to $\tilde{A}_{(-c,m+2c),(-c,m'+2c)}$ via $\delta^{-c}(\cdots)\delta^c$). As A' is noetherian this sequence must stabilize. I.e. for each n there is a natural number c_n such that $M_n^{c_n} = M_n^c$ holds for all $c \geq c_n$. Moreover, (1.80) induces

$$M_0^{c_0} \subset M_1^{c_1} \subset \ldots \subset e_0A'$$

This chain must stabilize as well. In particular there is some N such that $M_n^{c_n} = M_N^{c_N}$ holds for all $n \geq N$. Going back to (1.80) and the definition of M_n^c one sees that $c_N \geq c_n$

must hold for all $n \geq N$. Hence $M_n^c = M_N^c$ for all $n \geq N, c \geq c_N$. Similarly the (finitely many!) chains of inclusions

$$\begin{array}{ccc} M_0^0 \subset M_1^0 \subset & \dots & \subset e_0 A' \\ M_0^1 \subset M_1^1 \subset & \dots & \subset e_0 A' \\ & \vdots & \\ M_0^{c_N-1} \subset M_1^{c_N-1} \subset & \dots & \subset e_0 A' \end{array}$$

must similtanuously stabilize. Hence there exists a $N_0 \in \mathbb{N}$ such that $M_n^c = M_{N_0}^c$ holds for all $n \geq N_0$ and all $c \in \mathbb{N}$. But this is exactly the same as

$$\forall n \geq N_0, a \leq 0, b \in \mathbb{Z} : (M_n)_{(a,b)} = (M_{N_0})_{(a,b)}$$

<u>Step 2: Convergence of $\left((M_n)_{(a,b)}\right)_{n\in\mathbb{N}}$ for $a, b \geq 0$</u>

Let $\tilde{A}_+$ be defined as in Notation 0.6.1 and define $M_{n,(\geq 0,\geq 0)}$ via

$$\left(M_{(\geq a,\geq b)}\right) = \begin{cases} M_{(i,j)} & \text{if } i \geq a, j \geq b \\ 0 & \text{else} \end{cases} \tag{1.81}$$

then $\left((M_{n,(\geq 0,\geq 0)})\right)_{n\in\mathbb{N}}$ defines an ascending chain of $\tilde{A}_+$-submodules of $e_{(0,0)}\tilde{A}_+$. Hence it suffices to show that every $\tilde{A}_+$-submodule of $e_{(0,0)}\tilde{A}_+$ is finitely generated in the sense that it can be written as a quotient of some finite direct sum of projective generators $\bigoplus_{n=1}^{n_0} e_{(i_n,j_n)}\tilde{A}_+$. This is shown in a similar way as in [ATVdB90, Lemma 8.2]:

By way of contradiction, suppose that there is some submodule $L \subset e_{(0,0)}\tilde{A}_+$ which is not finitely generated. Using Zorn's Lemma L can be chosen maximal with respect to the inclusion of submodules. The quotient $\overline{A} = e_{(0,0)}\tilde{A}_+/L$ as well as all its submodules must hence be finitely generated. Now consider the following diagram

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, "\cdot g"'] & e_{(0,0)}\tilde{A}_+ \arrow[r] \arrow[d, "\cdot g"'] & \overline{A} \arrow[r] \arrow[d, "\cdot g"'] & 0 \\ 0 \arrow[r] & L \arrow[r] & e_{(0,0)}\tilde{A}_+ \arrow[r] & \overline{A} \arrow[r] & 0 \end{tikzcd}$$

Applying the Snake Lemma (and the fact that the $g_{(i,j)}$ are normalizing, non-zero divisors) provides an exact sequence

$$0 \longrightarrow K \longrightarrow L/Lg \xrightarrow{\epsilon} e_{(0,0)}\tilde{A}_+/g_{(0,0)}\tilde{A}_+$$

where $K = \ker(\overline{A} \xrightarrow{\cdot g} \overline{A})$. As mentioned above, all submodules of $\overline{A}$ are finitely generated, hence K is finitely generated. We also claim $\operatorname{im}(\epsilon) \subset e_{(0,0)}\tilde{A}_+/g_{(0,0)}\tilde{A}_+$ is finitely generated. From this claim it follows that L/Lg is finitely generated as an extension of finitely generated modules and hence L is finitely generated as well (as the elements $g_{(i,j)}$ live in strictly positive degrees). Thus it suffices to prove the claim. First we prove the existence of a map

$$\bigoplus_{n=1}^{n_1} e_{(i_n,j_n)}\tilde{A}_+ \longrightarrow \operatorname{im}(\epsilon) \tag{1.82}$$

which is surjective in degrees (a,b) with $a \geq v$, $b \geq (3-v)$.

Note that the elements in $\operatorname{im}(\epsilon)$ living in these degrees actually form a $\tilde{B}_+$-submodule by Lemma 1.9.1. By Theorem 1.9.5 it must be finitely generated as a $\tilde{B}_+$-module and hence also as an $\tilde{A}_+$-module. I.e. (1.82) exists.

Next we prove that $\operatorname{im}(\epsilon)/(\operatorname{im}(\epsilon))_{\geq v,\geq(3-v)}$ is finitely generated as well. As the proof for $v = 1$ and $v = 2$ is analogous, we only prove the latter. I.e. we need to prove the existence of a map

$$\bigoplus_{n=1}^{n_2} e_{(i_n,j_n)}\tilde{A}_+ \longrightarrow \operatorname{im}(\epsilon) \tag{1.83}$$

which is surjective in degrees (a,b) with $a < 2$ or $b = 0$. Note that $(\operatorname{im}(\epsilon)_{(a,0)})_{a\in\mathbb{N}}$ forms an A-submodule of e_0A. Hence there is an epimorphism

$$\bigoplus_{m=1}^{m_2} e_{(i_m)}A \longrightarrow (\operatorname{im}(\epsilon)_{(a,0)})_{a\in\mathbb{N}}$$

giving rise to a map

$$\bigoplus_{m=1}^{m_2} e_{(i_m,0)}\tilde{A}_+ \longrightarrow \operatorname{im}(\epsilon) \tag{1.84}$$

which is surjective in degrees $(a,0)$. Similarly we can construct an $m_2' \in \mathbb{N}$ and a map

$$\bigoplus_{m=1}^{m_2'} e_{(0,j_m)}\tilde{A}_+ \longrightarrow \operatorname{im}(\epsilon) \tag{1.85}$$

which is surjective in degrees $(0,b)$. Finally using the fact that

$$(\tilde{A}_+)_{(1,b),(1,b')} \cong (\tilde{A}_+)_{(0,b+2),(0,b'+2)} \cong A'_{b+2,b'+2}$$

via $\delta^{-1}(\cdots)\delta$, we get a structure for $(\operatorname{im}(\epsilon)_{(1,b)})_{b\in\mathbb{N}}$ as a A' submodule of e_0A'. Hence there is a surjection

$$\bigoplus_{m=1}^{m_2''} e_{(j_m)}A' \longrightarrow (\operatorname{im}(\epsilon)_{(1,b)})_{b\in\mathbb{N}}$$

Note that we can assume $j_m \geq 2$ for all m such that we find a map

$$\bigoplus_{m=1}^{m_2''} e_{(1,j_m-2)}\tilde{A}_+ \longrightarrow \operatorname{im}(\epsilon) \tag{1.86}$$

which is surjective in degrees $(1,b)$. Finally combining (1.84), (1.85) and (1.86) we find the required map (1.83), proving the claim and hence the theorem. □

1.10 Ample $\mathbb{Z}^2$-sequences and noetherianity of $\tilde{B}_+$

The goal of this section is to prove a $\mathbb{Z}^2$-version of [AVdB90, Theorem 1.4]. For this we introduce some ad hoc definitions:

Definition 1.10.1. Let Y be a noetherian variety. A projective $\mathbb{Z}^2$-sequence on Y consists of two collections of line bundles $\{\mathcal{L}_{(i,j)}\}_{i,j\in\mathbb{Z}}, \{\mathcal{G}_{(i,j)}\}_{i,j\in\mathbb{Z}}$ satisfying

$$\mathcal{L}_{(i,j)}\mathcal{G}_{(i+1,j)} = \mathcal{G}_{(i,j)}\mathcal{L}_{(i,j+1)} \tag{1.87}$$

Remark 1.10.2. *We refer to these $\mathbb{Z}^2$-sequences as "projective" because they obviously generalize projective sequences as in [Pol05]*

Of particular importance are ample $\mathbb{Z}^2$-sequences: Given a projective $\mathbb{Z}^2$-sequence $\{\mathcal{L}_{(i,j)}\}_{i,j\in\mathbb{Z}}, \{\mathcal{G}_{(i,j)}\}_{i,j\in\mathbb{Z}}$, we associate a sheaf of $\mathbb{Z}^2$-algebras $\mathcal{B}$ to it via

$$\mathcal{B}_{(i,j),(m,n)} = \begin{cases} \mathcal{L}_{(i,j)}\mathcal{L}_{(i+1,j)}\dots\mathcal{L}_{(m-1,j)}\mathcal{G}_{(m,j)}\mathcal{G}_{(m,j+1)}\dots\mathcal{G}_{(m,n-1)} & \text{if } i\le m, j\le n \\ \left(\mathcal{L}_{(m,j)}\mathcal{L}_{(m+1,j)}\dots\mathcal{L}_{(i-1,j)}\right)^{-1}\mathcal{G}_{(m,j)}\dots\mathcal{G}_{(m,n-1)} & \text{if } i>m, j\le n \\ \mathcal{L}_{(i,j)}\mathcal{L}_{(i+1,j)}\dots\mathcal{L}_{(m-1,j)}\left(\mathcal{G}_{(m,n)}\dots\mathcal{G}_{(m,j-1)}\right)^{-1} & \text{if } i\le m, j>n \\ \left(\mathcal{L}_{(m,j)}\mathcal{L}_{(m+1,j)}\dots\mathcal{L}_{(i-1,j)}\right)^{-1}\left(\mathcal{G}_{(m,n)}\dots\mathcal{G}_{(m,j-1)}\right)^{-1} & \text{if } i>m, j>n \end{cases} \tag{1.88}$$

We also associate a $\mathbb{Z}^2$-algebra $B = B\left(\{\mathcal{L}_{(i,j)}\}_{i,j\in\mathbb{Z}}, \{\mathcal{G}_{(i,j)}\}_{i,j\in\mathbb{Z}}\right)$ to the sequence via

$$B_{(i,j),(m,n)} = \begin{cases} \Gamma(Y, \mathcal{B}_{(i,j),(m,n)}) & \text{if } i\le m, j\le n \\ 0 & \text{else} \end{cases} \tag{1.89}$$

Definition 1.10.3. Let $\{\mathcal{L}_{(i,j)}\}_{i,j\in\mathbb{Z}}, \{\mathcal{G}_{(i,j)}\}_{i,j\in\mathbb{Z}}$ be a projective $\mathbb{Z}^2$-sequence and let $\mathcal{B}$ be the associated sheaf of $\mathbb{Z}^2$-algebras, then we say the sequence is an ample sequence if for each coherent sheaf $\mathcal{F}$ on Y and for each $i, j \in \mathbb{Z}$ there exist $i_0, j_0 \in \mathbb{Z}$ such that

$$H^q(Y, \mathcal{F} \otimes_Y \mathcal{B}_{(i,j),(m,n)}) = 0 \tag{1.90}$$

holds for all $q > 0$ and $m \ge i_0, n \ge j$ or $m \ge i, n \ge j_0$.

The following is immediate

Proposition 1.10.4. *Let $\gamma : A' \to A^{(v)}$ be as above and let $\tilde{B}$ be as in* (1.65) *or* (1.73) *(for $v = 1, 2$ respectively). Then $\tilde{B}_+$ is isomorphic to an algebra of the form* (1.88). *More concretely:*

- *For $v = 1$, using the notation as in §1.8.1: define $\mathcal{L}_{(i,j)} = \mathcal{G}_{i+j} = \psi^{(i+j)*}\mathcal{G}$ and $\mathcal{G}_{(i,j)} = \mathcal{G}_{i+j}(-d_j)$.*
- *Similarly, for $v = 2$, using the notation as in §1.8.2: define $\mathcal{L}_{(i,j)} = \mathcal{L}_{i+2j}$ and $\mathcal{G}_{(i,j)} = \mathcal{L}_{i+2j}\mathcal{L}_{i+2j+1}(-\tau^{-j}p)$.*

Then $\{\mathcal{L}_{(i,j)}\}_{i,j\in\mathbb{Z}}, \{\mathcal{G}_{(i,j)}\}_{i,j\in\mathbb{Z}}$ *is an ample* $\mathbb{Z}^2$*-sequence and*

$$B\left(\{\mathcal{L}_{(i,j)}\}_{i,j\in\mathbb{Z}}, \{\mathcal{G}_{(i,j)}\}_{i,j\in\mathbb{Z}}\right) = \tilde{B}_+$$

Proof. It is obvious that $\tilde{B}_+$ equals the $\mathbb{Z}^2$-algebra associated to the sequence

$$\{\mathcal{L}_{(i,j)}\}_{i,j\in\mathbb{Z}}, \{\mathcal{G}_{(i,j)}\}_{i,j\in\mathbb{Z}}$$

and that (1.87) is satisfied. It only remains to show that the associated sheaf of $\mathbb{Z}^2$-algebras $\mathcal{B}$ satisfies the condition (1.90) in Definition 1.10.3. For this let $\mathcal{F}$ be a coherent sheaf on Y. As Y is a nonsingular curve, it suffices to prove

$$H^1(Y, \mathcal{F}\otimes\mathcal{B}_{(i,j),(m,n)}) = 0$$

for sufficiently high m,n.

[Har97, Exercice II.6.11(c)] gives us the existence of a line bundle $\mathcal{N}$ and a torsion sheaf $\mathcal{T}$ together with a short exact sequence:

$$0 \longrightarrow \mathcal{N}^{\oplus r} \longrightarrow \mathcal{F} \longrightarrow \mathcal{T} \longrightarrow 0$$

where r is the rank of $\mathcal{F}$. As $\mathcal{T}$ is torsion, its support is zero dimensional and hence $\mathcal{T}\otimes\mathcal{B}_{(i,j),(m,n)}$ has no higher cohomology. In particular there is an epimorphism

$$H^1(Y, \mathcal{N}\otimes\mathcal{B}_{(i,j),(m,n)})^{\oplus r} \longrightarrow\!\!\!\!\!\rightarrow H^1(Y, \mathcal{F}\otimes\mathcal{B}_{(i,j),(m,n)})$$

Hence it suffices to show $H^1(Y, \mathcal{N}\otimes\mathcal{B}_{(i,j),(m,n)}) = 0$ for sufficiently high m,n. By Riemann-Roch the latter follows when $\deg(\mathcal{N}\otimes\mathcal{B}_{(i,j),(m,n)}) \geq 2g-2$ where g is the genus of Y. Finally this condition is satisfied for m,n big enough as $\deg(\mathcal{B}_{(i,j),(m,n)})$ is a strictly increasing in function of the variables m and n. $\square$

Now Theorem 1.9.5 follows from Proposition 1.10.4 and the following:

Theorem 1.10.5. *Let* $\{\mathcal{L}_{(i,j)}\}_{i,j\in\mathbb{Z}}, \{\mathcal{G}_{(i,j)}\}_{i,j\in\mathbb{Z}}$ *be an ample* $\mathbb{Z}^2$ *sequence on a noetherian, projective scheme* Y*, then* $B := B\left(\{\mathcal{L}_{(i,j)}\}_{i,j\in\mathbb{Z}}, \{\mathcal{G}_{(i,j)}\}_{i,j\in\mathbb{Z}}\right)$ *is noetherian in the sense that* $\operatorname{Gr}(B)$ *is a locally noetherian category.*

Remark 1.10.6. *It was shown in [Cha00] that, under suitable conditions, twisted bi-homogeneous coordinate rings are noetherian. The* $(\check{-})$*-construction from Notation 0.2.1 which turns graded algebras into* $\mathbb{Z}$*-algebras can be generalized to turn bi-homogeneous algebras into* $\mathbb{Z}^2$*-algebras, tri-homogeneous algebras into* $\mathbb{Z}^3$*-algebras, etc. Moreover, it is not hard to see that in this way we can turn a twisted bi-homogeneous coordinate ring into a* $\mathbb{Z}^2$*-algebra of the form* (1.89)*. As such, Theorem 1.10.5 is a generalization of [Cha00, Theorem 5.2].*

The proof of Theorem 1.10.5 follows from a chain of lemmas:

Lemma 1.10.7. *Let $\mathcal{F}$ be a coherent sheaf on Y, then for all $i,j \in \mathbb{Z} : \mathcal{F} \otimes \mathcal{B}_{(i,j),(m,n)}$ is generated by its global sections for sufficiently large m,n, in the sense that there exist $i_0, j_0 \in \mathbb{Z}$ such that $\mathcal{F} \otimes_Y \mathcal{B}_{(i,j),(m,n)}$ is generated by global sections whenever $m \geq i, n \geq j_0$ or $m \geq i_0, n \geq j$*

Proof. This is a straightforward generalization of [AVdB90, Proposition 3.2.ii and Lemma 3.3]. □

Lemma 1.10.8. *We have two pairs of functors between* $\mathrm{Gr}(B)$, $\mathrm{Gr}(\mathcal{B})$ *and* $\mathrm{Qcoh}(Y)$*:*

$$\begin{tikzcd}
\mathrm{Gr}(B) \arrow[r, bend left, "\widetilde{(-)} := - \otimes_B \mathcal{B}"] & \mathrm{Gr}(\mathcal{B}) \arrow[l, bend left, "\Gamma_* := \bigoplus_{n\in\mathbb{Z}} \Gamma(Y, (-)_n)"] \arrow[r, bend left, "(-)_{(0,0)}"] & \mathrm{Qcoh}(Y) \arrow[l, bend left, "- \otimes_Y e_{(0,0)}\mathcal{B}"]
\end{tikzcd}$$

Moreover, these satisfy the following properties:

i) $(-)_{(0,0)}$ *and* $-\otimes_Y e_{(0,0)}\mathcal{B}$ *are quasi-inverses and define an equivalence of categories*

ii) $\widetilde{(-)}$ *is left adjoint to* Γ_*

iii) Γ_* *is exact modulo torsion modules*

iv) Let $M \in \mathrm{Gr}(B)$ *and define* $\overline{M} := \Gamma_*(\widetilde{M})$*. Then there is a natural map* $M \to \overline{M}$*. Moreover, the kernel and cokernel of this map are torsion.*

v) $\widetilde{(-)}$ *is exact*

Proof. *i*) This is standard and follows from the fact that $\mathcal{B}$ is strongly graded (i.e. $\mathcal{B}_{(a,b),(i,j)}\mathcal{B}_{(i,j),(m,n)} = \mathcal{B}_{(a,b),(m,n)}$ holds for all $a,b,i,j,m,n \in \mathbb{Z}$).

ii) The functor $\widetilde{(-)}$ is defined by considering B as well as each graded B-module as constant sheaves on Y. As such, $\mathcal{B}$ obtains a natural left B-structure. The fact that $\widetilde{(-)}$ is left adjoint to Γ_* follows immediately from this construction.

iii) Analogous to [AVdB90, Lemma 3.7.(ii)]

iv) The existence of the natural map $M \to \overline{M}$ follows from the adjunction in (2). The fact that the kernel and cokernel are torion are a straightforward generalization of [AVdB90, Lemma 3.13.(i) and (iii)]

v) Analogous to [AVdB90, Lemma 3.13.(iv)], also using [AVdB90, Lemma 3.7.(iii)]. □

Remark 1.10.9. *The above lemma implies that up to isomorphism any graded $\mathcal{B}$-module $\mathcal{M}$ can be written as $\mathcal{F} \otimes_Y e_{(0,0)}\mathcal{B}$ for some quasi-coherent sheaf $\mathcal{F}$. In the special case that $\mathcal{F}$ is coherent, $\mathcal{M}$ is called coherent as well.*

Corollary 1.10.10. *Let $\mathcal{M}$ be a coherent $\mathcal{B}$-module. Then there exist $i, j, n \in \mathbb{Z}$, $n \geq 0$ such that $\mathcal{M}$ is a quotient of $\left(e_{(i,j)}\mathcal{B}\right)^{\oplus n}$*

Proof. There is a coherent sheaf $\mathcal{F}$ such that $\mathcal{M} = \mathcal{F} \otimes_Y e_{(0,0)}\mathcal{B}$. By Lemma 1.10.7 there are $i, j \in \mathbb{Z}$ such that $\mathcal{F} \otimes \mathcal{B}_{(0,0),(i,j)}$ is generated by global sections. From this it follows that $\mathcal{M}$ is a quotient of $\Gamma(Y, \mathcal{F} \otimes \mathcal{B}_{(0,0),(i,j)}) \otimes_k e_{(i,j)}\mathcal{B}$ □

Lemma 1.10.11. *For each $i, j \in \mathbb{Z}, \forall a, b \in \mathbb{Z} \cup \{-\infty\}$ we have that $\left(e_{(i,j)}B\right)_{(\geq a, \geq b)}$ is finitely generated. (Recall: the notation $M_{\geq a, \geq b}$ was introduced in* (1.81)*)*

Proof. (inspired by [AVdB90, p.261-262])
Without loss of generality we assume $i = j = 0$. Moreover, replacing a, b by $\max(a, 0)$, $\max(b, 0)$ we can assume $a, b \geq 0$. By Lemma 1.10.7 there is an $m_0 \geq a$ such that $\mathcal{B}_{(0,0),(m_0,b)}$ is generated by sections. I.e. there is a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow B_{(0,0),(m_0,b)} \otimes_k \mathcal{O}_Y \longrightarrow \mathcal{B}_{(0,0),(m_0,b)} \longrightarrow 0$$

As $\{\mathcal{L}_{(i,j)}\}_{i,j \in \mathbb{Z}}, \{\mathcal{G}_{(i,j)}\}_{i,j \in \mathbb{Z}}$ is an ample sequence (see Definition 1.10.3) there is $m_1 > m_0$ such that $H^1(Y, \mathcal{F} \otimes_Y \mathcal{B}_{(m_0,b),(m,n)}) = 0$ for all $m \geq m_1$ and $n \geq b$. In particular, applying Γ_* to the above sequence provides a surjective morphism for all $m \geq m_1$ and $n \geq b$

$$B_{(0,0),(m_0,b)} \otimes_k B_{(m_0,b),(m,n)} \twoheadrightarrow B_{(0,0),(m,n)}$$

I.e. the natural map

$$B_{(0,0),(m_0,b)} \otimes_k e_{(m_0,b)}B \longrightarrow \left(e_{(0,0)}B\right)_{(\geq a, \geq b)}$$

is surjective in degrees (m, n) with $m \geq m_1$ and $n \geq b$. Similarly there is a $n_0 \geq b$ and $n_1 > n_0$ such that

$$B_{(0,0),(a,n_0)} \otimes_k e_{(a,n_0)}B \longrightarrow \left(e_{(0,0)}B\right)_{(\geq a, \geq b)}$$

is surjective in degrees (m, n) with $m \geq a$ and $n \geq n_0$. Combining the above two morphisms we find a surjective morphism

$$\bigoplus_{\substack{a \leq m < m_1 \\ b \leq n < n_1}} B_{(0,0),(m,n)} \otimes_k e_{(m,n)}B \twoheadrightarrow \left(e_{(0,0)}B\right)_{(\geq a, \geq b)}$$

As $\dim_k(B_{(0,0),(m,n)}) < \infty$ for all m, n this finishes the proof.

□

Lemma 1.10.12. *Let* $0 \to M \to M' \to M'' \to 0$ *be an exact sequence in* $\mathrm{Gr}(B)$*.*

i) If M *and* M'' *are finitely generated, then so is* M'*.*

ii) If M' *is finitely generated and there exist* $a, b, c, d \in \mathbb{Z}$ *such that* $M''_{(i,j)} = 0$ *if* $i \geq c$ *and* $j \geq d$ *or if* $i < a$ *or if* $j < b$ *(i.e. there exists a* $N \in \mathrm{Gr}(B)$ *such that* $M'' = N_{(\geq a, \geq b)}/N_{(\geq c, \geq d)}$*), then* M *is finitely generated.*

Proof. *i*) This is standard.

ii) By induction it suffices to show the lemma in case $c = a, d = b+1$ or $c = a+1, d = b$. Without los of generality we may assume the latter, i.e. $M''_{(i,j)} = 0$ if $i \neq a$ or $j < b$. As M' is finitely generated there exists as surjective morphism

$$\bigoplus_{n=1}^{N} e_{(i_n, j_n)} B \twoheadrightarrow M' \tag{1.91}$$

For each n, consider the composition

$$e_{(i_n, j_n)} B \longrightarrow M''$$

As this is a morphism of B-modules, it can only be nonzero if $i_n = a, j_n \geq b$. Moreover, for each such n, Lemma 1.10.11 implies that $\left(e_{(a, j_n)} B\right)_{(\geq a+1, \geq j_n)}$ is finitely generated, say

$$\bigoplus_{x_n=1}^{x_{n,0}} e_{(i_{x_n}, j_{x_n})} B \twoheadrightarrow \left(e_{(a, j_n)} B\right)_{(\geq a+1, \geq j_n)} \tag{1.92}$$

Combining the above we obtain a morphism

$$\left(\bigoplus_{\substack{n=1 \\ i_n \neq a \text{ or } j_n < b}}^{N} e_{(i_n, j_n)} B \right) \oplus \left(\bigoplus_{\substack{n=1 \\ i_n = a, j_n \geq b}}^{N} \left(\bigoplus_{x_n=1}^{x_{n,0}} e_{(i_{x_n}, j_{x_n})} B \right) \right) \longrightarrow M \tag{1.93}$$

and by construction this morphism is surjective in all degrees (i, j) for which $i \neq a$ or $j < b$.

Now consider the $\mathbb{Z}$-algebra B' defined by

$$B'_{n,m} := B_{(a, b+n), (a, b+m)}$$

By construction $B'_{n,m}$ is the twisted homogeneous coordinate ring with respect to the ample sequence $\left(\mathcal{G}_{(a, b+n)}\right)_{n \in \mathbb{Z}}$. It is standard that B' is a noetherian $\mathbb{Z}$-algebra (see for example [AVdB90] or [Pol05]). Moreover, define B'-modules L and L' by

$$L_m := M_{(a, b+m)} \text{ and } L'_m = M'_{(a, b+m)}$$

Recall that M' was finitely generated as in (1.91). For each n we know that $\left(e_{(i_n,j_n)}B\right)_{(\geq a,\geq b)}$ is finitely generated, say

$$\bigoplus_{u_n=1}^{u_{n,0}} e_{(i_{u_n},j_{u_n})}B \twoheadrightarrow \left(e_{(i_n,j_n)}B\right)_{(\geq a,\geq b)}$$

As such, we obtain that L' is finitely generated. For the interested reader we mention that the explicit surjective morphism is given by

$$\bigoplus_{n=1}^{N}\left(\bigoplus_{\substack{u_n=1\\ i_{u_n}=a, j_{u_n}\geq b}}^{u_{n,0}} e_{j_{u_n}-b}B'\right) \twoheadrightarrow L'$$

As B' is noetherian, $L \subset L'$ is finitely generated as well, say

$$\bigoplus_{v=1}^{v_0} e_{j_v}B' \twoheadrightarrow L$$

The induced morphism

$$\bigoplus_{v=1}^{v_0} e_{(a,b+j_v)}B \twoheadrightarrow M \tag{1.94}$$

is surjective in all degrees (i,j) with $i=a, j\geq b$. Combining (1.93) and (1.94) we obtain that M is finitely generated as required. □

The following lemma will be crucial in the proof of Theorem 1.10.5:

Lemma 1.10.13. *Let $M \in \operatorname{Gr}(B)$ be such that $\widetilde{M}$ is coherent, then for each $a,b\in\mathbb{Z}$: $\overline{M}_{(\geq a,\geq b)}$ is a finitely generated graded B-module.*

Proof. (inspired by [AVdB90, Lemma 3.17]) As $\tilde{M}$ is coherent, Corollary 1.10.10 provides us with $i,j,n\in\mathbb{Z}$, $n\geq 0$ and a surjective morphism

$$\left(e_{(i,j)}B\right)^{\oplus n} \twoheadrightarrow \tilde{M}$$

Let $\mathcal{K}$ be the kernel of this morphism. Then we have a long exact sequence

$$0 \longrightarrow \Gamma_*(\mathcal{K}) \xrightarrow{f} e_{(i,j)}B^{\oplus n} \longrightarrow \overline{M} \longrightarrow H^1(\mathcal{K})$$

Truncation turns this into an exact sequence

$$0 \longrightarrow \operatorname{coker}(f)_{(\geq a,\geq b)} \longrightarrow \overline{M}_{(\geq a,\geq b)} \longrightarrow H^1(\mathcal{K})_{(\geq a,\geq b)}$$

$\operatorname{coker}(f)_{(\geq a,\geq b)}$ is finitely generated as a quotient of $e_{(i,j)}B^{\oplus n}_{(\geq a,\geq b)}$, which in turn is finitely generated by Lemma 1.10.11. $\mathcal{K}$ is coherent as a $\mathcal{B}$-submodule of $e_{(i,j)}\mathcal{B}^{\oplus n}$, hence by the definition of an ample sequence, $H^1(\mathcal{K})_{(\geq a,\geq b)}$ is concentrated in finitely many degrees. As Y is projective, $H^1(\mathcal{K})_{(\geq a,\geq b)}$ is finite dimensional and in particular finitely generated. The result now follows from Lemma 1.10.12. □

We can now finish the proof of the main theorem of this section

Proof of Theorem 1.10.5. As $\{e_{(i,j)}B \mid i,j \in \mathbb{Z}\}$ obviously serves as a set of generators for $\mathrm{Gr}(B)$ it suffices to show that these modules are noetherian. Hence let M be a submodule of some $e_{(a,b)}B$. By Lemma 1.10.8(v) this induces an embedding

$$\tilde{M} \hookrightarrow e_{(a,b)}\mathcal{B}$$

Thus $\tilde{M}$ is coherent and Lemma 1.10.13 implies $\overline{M}_{(\geq a,\geq b)}$ is a finitely generated module. As $M = M_{(\geq a,\geq b)}$ the natural map $M \to \overline{M}$ factors through $\overline{M}_{(\geq a,\geq b)}$. Now consider the diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d] & e_{(a,b)}B \arrow[d] \\ \overline{M}_{(\geq a,\geq b)} \arrow[r] & \overline{e_{(a,b)}B} \end{tikzcd}$$

As the upper horizontal arrow and the right vertical arrow are injective, so is the left vertical arrow. We obtain a short exact sequence

$$0 \longrightarrow M \longrightarrow \overline{M}_{(\geq a,\geq b)} \longrightarrow \left(\overline{M}/M\right)_{(\geq a,\geq b)} \longrightarrow 0 \tag{1.95}$$

By Lemma 1.10.8(iv) we have that $\overline{M}/M$ and hence also $\left(\overline{M}/M\right)_{(\geq a,\geq b)}$ is torsion. Being a quotient of a finitely generated module, $\left(\overline{M}/M\right)_{(\geq a,\geq b)}$ is also finitely generated and hence concentrated in finitely many degrees. In particular there exist $c,d \in \mathbb{Z}$ such that $\left(\left(\overline{M}/M\right)_{(\geq a,\geq b)}\right)_{(i,j)} = 0$ for $i > c, j > d$. As such, (1.95) satisfies the assumptions in Lemma 1.10.12, implying M is finitely generated. □

1.11 $\mathbb{Z}^2$-algebras in §1.8 as noncommutative blowups

In this section we show that the $\mathbb{Z}^2$-algebras introduced in §1.8 are equivalent (at the level of QGr) with graded algebras which are simultaneously a noncommutative blowup of a quadratic and a cubic Sklyanin algebra.

We first recall the relevant definitions

Definition. (See [RSS15, Assumption 2.1] of [RSS14, §1])
A connected graded algebra $S = k \oplus S_1 \oplus S_2 \oplus \ldots$ is said to be elliptic if it is a domain and if there exists a central element $g \in S_1$ such that

$$S/gS \cong B(Y, \mathcal{M}, \tau)$$

where $B(Y, \mathcal{M}, \tau)$ is the twisted homogeneous coordinate ring for a smooth elliptic curve Y, a line bundle $\mathcal{M}$ of degree μ and $\tau \in \mathrm{Aut}(Y)$ of infinite order.

One often refers to μ as the degree of the elliptic algebra S.

Remark 1.11.1. *Let $A = A(Y, \mathcal{L}, \sigma)$ be a cubic Sklyanin algebra for which σ has infinite order. Then the 4-Veronese ring $A^{(4)}$ is an elliptic algebra of degree 8. To see this note that there is a central element $g \in A_1^{(4)} = A_4$ such that*

$$A^{(4)}/gA^{(4)} \cong B(Y, \mathcal{L} \otimes \sigma^* \mathcal{L} \otimes \sigma^{2*}\mathcal{L} \otimes \sigma^{3*}\mathcal{L}, \sigma^4)$$

Similarly if $A' = A(Y, \mathcal{G}, \psi)$ is a quadratic Sklyanin algebra for which ψ has infinite order. Then the 3-Veronese ring $A'^{(3)}$ is an elliptic algebra of degree 9 because there is a central element $g' \in A_1'^{(3)} = A_3'$ for which

$$A'^{(3)}/g'A'^{(3)} \cong B(Y, \mathcal{G} \otimes \psi^* \mathcal{G} \otimes \psi^{2*}\mathcal{G}, \psi^3)$$

Let S be an elliptic algebra of degree μ and let d be an effective divisor on Y for which $\deg(d) < \mu$. The noncommutative blowups $S(d)$ of S at d is introduced in [RSS14, Definition 5.1 and Definition 4.1]. As we are only interested in situations where $\deg(d) < \mu - 1$, we can use [RSS14, Theorem 5.3(4)] to obtain the following shorter definition:

Definition 1.11.2. Let S be an elliptic algebra of degree μ and let d be an effective divisor on Y for which $\deg(d) < \mu - 1$. The noncommutative blowup $S(d)$ is the subalgebra of S generated in degree 1 by

$$S(d)_1 := \{x \in S_1 \mid x + gS \in B(Y, \mathcal{M}(-d)) \subset B(Y, \mathcal{M}) = (S/gS)_1\}$$

These noncommutative blowups have many satisfactory properties :

Theorem. *([RSS14, Theorem 5.3.]) With the notation as above:*

- *$S(d)$ has hilbert series $h_{S(d)}(t) = h_S(t) - \dfrac{\deg(d)}{(1-t)^3}$*
- *$S(d)$ is* g-divisible*, i.e. $S(d) \cap gS = gS(d)$*
- *$S(d)/gS(d) \cong B(Y, \mathcal{M}(-d), \tau)$, in particular $S(d)$ is an elliptic algebra of degree $\mu - \deg(d)$*
- *$S(d)$ is strongly noetherian, Auslander-Gorenstein, Cohen-Macaulay and a maximal order.*
- *$S(d)$ satisfies the χ-conditions as in [AZ94]*
- *$S(d)$ has cohomological dimension 2*

We can now state the main result of this section. We fix a cubic Sklyanin algebra $A = A(Y, \mathcal{L}, \sigma)$ and a quadratic Sklyanin algebra $A' = A(Y, \mathcal{G}, \psi)$. Let $\gamma : \check{A}' \to \check{A}^{(2)}$ and $\delta : \check{A} \to \check{A}'$ be as in (1.37) and (1.39). Assume moreover that γ and δ are mutually inverse as in Theorem 1.7.8 and Theorem 1.7.9. In particular γ is constructed starting a point p and δ is constructed starting from points q, r and these points satisfy suitable compatibility relations (see (1.70) and (1.76)). Moreover, we assume $\tau = \sigma^4 = \psi^3$ to be of infinite order. Finally let $\tilde{A}$ be the $\mathbb{Z}^2$-algebra as in §1.8.2 (i.e. constructed starting from δ) and let $\tilde{A}'$ be the $\mathbb{Z}^2$-algebra as in §1.8.1 (i.e. constructed starting from δ).

All these algebras are related via commons blowups as follows:

Theorem 1.11.3. *A and A' contain a common blowup*

$$T \cong A^{(4)}(p) \cong A'^{(3)}(q + r)$$

and there are equivalences of categories

$$\mathrm{QGr}(\tilde{A}) \cong \mathrm{QGr}(T) \cong \mathrm{QGr}(\tilde{A}')$$

We devote this section to the proof of Theorem 1.11.3.

As $\tilde{A}$ is noetherian (Theorem 1.9.6) we can apply Theorem 0.6.6. Hence for each diagonal-like $\Delta : \mathbb{Z} \to \mathbb{Z}^2$ there is an equivalence of categories $\mathrm{QGr}(\tilde{A}) \cong \mathrm{QGr}(\tilde{A}_\Delta)$. Throughout this section we focus on a specific choice of Δ: $\Delta(i) = (2i, i)$. For this choice of Δ we have the following:

$$\left(\tilde{A}_\Delta\right)_{i,j} = \tilde{A}_{(2i,i),(2j,j)} = \begin{cases} \mathrm{Hom}_X(\mathcal{O}_X(-4j), \mathcal{O}_X(-4i) m_{\tau^{-i}p} \dots m_{\tau^{-j+1}p} & \text{if } j \geq i \\ 0 & \text{else} \end{cases}$$

In particular there is an obvious inclusion

$$\tilde{A}_\Delta \hookrightarrow \widetilde{(A^{(4)})} \tag{1.96}$$

Moreover, $\widetilde{(A^{(4)})}$ is 1-periodic as it is induced by the graded algebra $A^{(4)}$. The equality

$$o_X(4) m_d = m_{\tau d} o_X(4)$$

implies that $\tilde{A}_\Delta$ is compatible with this 1-periodicity. I.e. there is a graded algebra T such that $\tilde{A}_\Delta = \check{T}$ and (1.96) is induced by an inclusion

$$T \hookrightarrow A^{(4)}$$

By construction T is the subalgebra of $A^{(4)}$ generated by the elements in $A^{(4)}_1 = A_4$ whose images in $A^{(4)}/g$ lie in $\Gamma(Y, \mathcal{L} \otimes \sigma^* \mathcal{L} \otimes \sigma^{*2} \mathcal{L} \otimes \sigma^{*3} \mathcal{L}(-p))$. Taking into mind Remark 1.11.1 and Definition 1.11.2 we conclude

$$T \cong A^{(4)}(p)$$

Completely similar, replacing $\tilde{A}$ by $\tilde{A}'$ and Δ by $\Delta'(i) = (i, 2i)$, we obtain an equivalence of categories

$$\mathrm{QGr}(\tilde{A}') \cong \mathrm{QGr}(A'^{(3)}(q + r))$$

Hence it only remains to show that $T \cong A'^{(3)}(q+r)$. First note that

$$\delta^{-2i}(-)\delta^{2j} : (\tilde{A}_\Delta)_{i,j} = \tilde{A}_{(2i,i),(2j,j)} \to \tilde{A}_{(0,3i),(0,3j)} \cong A'_{3i,3j} \tag{1.97}$$

defines an inclusion $\check{T} = \tilde{A}_\Delta \hookrightarrow \widetilde{A'^{(3)}}$. We need to show that (up to replacing the $\delta_{(i,j)}$ by some scalar multiples) (1.97) is compatible with the 1-periodicity of $\widetilde{A'^{(3)}}$ and $\check{T}$ such that there is an induced inclusion $T \hookrightarrow A'^{(3)}$. To see this we need to show that the periodicity isomorphisms $\tilde{A}_{(0,3i),(0,3j)} \cong \tilde{A}_{(0,3i+3),(0,3j+3)}$ and $\tilde{A}_{(2i,i),(2j,j)} \cong \tilde{A}_{(2i+2,i+1),(2j+2,j+1)}$, induced by $A'^{(3)}$ and T respectively, are compatible in the sense that the following diagram commutes

$$\begin{tikzcd}
\tilde{A}_{(2i,i),(2j,j)} \arrow[r, "\delta^{-2i}(-)\delta^{2j}"] \arrow[d, "\cong"'] & \tilde{A}_{(0,3i),(0,3j)} \arrow[d, "\cong"] \\
\tilde{A}_{(2i+2,i+1),(2j+2,j+1)} \arrow[r, "\delta^{-2i-2}(-)\delta^{2j+2}"'] & \tilde{A}_{(0,3i+3),(0,3j+3)}
\end{tikzcd} \tag{1.98}$$

As all morphisms in the above diagram are compatible with the algebra structure of $\tilde{A}$ and as T and A' are generated in degree 1, it suffices to check that the above diagram commutes for $j = i+1$.

Recall that there is a central element $g' \in A'_3$ such that $A'/g'A' \cong B'$ where

$$B' = B(Y, \mathcal{G}, \psi) = B(Y, \mathcal{L} \otimes \sigma^* \mathcal{L}(-p), \psi)$$

Let g'_j be the corresponding element in $\check{A}'_{j,j+3}$. Then for each i there is an exact sequence

$$0 \longrightarrow kg'_{3i} \longrightarrow \tilde{A}_{(0,3i),(0,3i+3)} \longrightarrow \tilde{B}_{(0,3i),(0,3i+3)} \longrightarrow 0 \tag{1.99}$$

(With $\tilde{B}$ as in (1.65))

Similarly by Lemma 1.9.1 for each i there is an element $g_{(2i,i)} \in \tilde{A}_{(2i,i),(2i+2,i+1)}$ together with an exact sequence

$$0 \longrightarrow kg_{(2i,i)} \longrightarrow \tilde{A}_{(2i,i),(2i+2,i+1)} \longrightarrow \tilde{B}_{(2i,i),(2i+2,i+1)} \longrightarrow 0 \tag{1.100}$$

Now if we let $\overline{\delta}_{(i,j)}$ be the image of $\delta_{(i,j)}$ under $\tilde{A}_{(i,j),(i-1,j+1)} \to \tilde{B}_{(i,j),(i-1,j+1)}$ then we see that (1.97), (1.99) and (1.100) are compatible in the sense that there is a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & kg'_{3i} \arrow[r] \arrow[d, "\delta^{-2i}(-)\delta^{2j}"'] & \tilde{A}_{(0,3i),(0,3i+3)} \arrow[r] \arrow[d, "\delta^{-2i}(-)\delta^{2j}"'] & \tilde{B}_{(0,3i),(0,3i+3)} \arrow[r] \arrow[d, "\overline{\delta}^{-2i}(-)\overline{\delta}^{2j}"'] & 0 \\
0 \arrow[r] & kg_{(2i,i)} \arrow[r] & \tilde{A}_{(2i,i),(2i+2,i+1)} \arrow[r] & \tilde{B}_{(2i,i),(2i+2,i+1)} \arrow[r] & 0
\end{tikzcd}$$

Hence in order to prove commutativity of (1.98) it suffices to prove commutativity of the following two diagrams:

$$\begin{tikzcd}
\tilde{B}_{(2i,i),(2i+2,i+1)} \arrow[r, "\overline{\delta}^{-2i}(-)\overline{\delta}^{2i+2}"] \arrow[d, "\cong"'] & \tilde{B}_{(0,3i),(0,3i+3)} \arrow[d, "\cong"] \\
\tilde{B}_{(2i+2,i+1),(2j+2,j+1)} \arrow[r, "\overline{\delta}^{-2i-2}(-)\overline{\delta}^{2i+4}"'] & \tilde{B}_{(0,3i+3),(0,3i+6)}
\end{tikzcd} \tag{1.101}$$

and

$$\begin{tikzcd}
kg_{(2i,i)} \arrow[r, "\delta^{-2i}(-)\delta^{2i+2}"] \arrow[d, "\cong"'] & kg'_{3i} \arrow[d, "\cong"] \\
kg_{(2i+2,i+1)} \arrow[r, "\delta^{-2i-2}(-)\delta^{2i+4}"'] & kg'_{3i+3}
\end{tikzcd} \tag{1.102}$$

We first focus on (1.101). The left vertical arrow is given by

$$\tau^* = \sigma^{3*} : \Gamma(Y, \sigma^{*4i}\mathcal{L}\ldots\sigma^{*(4i+3)}\mathcal{L}(-\tau^{-i}p)) \to \Gamma(Y, \sigma^{*4i+4}\mathcal{L}\ldots\sigma^{*(4i+7)}\mathcal{L}(-\tau^{-i-1}p))$$

A closer investigation of the 1-periodicity of $\widetilde{B'^{(3)}}$ shows that the right vertical arrow in (1.101) factors as

$$\tilde{B}_{(0,3i),(0,3i+3)} \xrightarrow{\tau^*} \tilde{B}_{(2,3i+1),(2,3i+4)} \xrightarrow{\varphi} \tilde{B}_{(0,3i+3),(0,3i+6)} \tag{1.103}$$

where φ is given by multiplication by a nonzero section of

$$\left(\sigma^{*(6i+4)}\mathcal{L}\sigma^{*(6i+5)}\mathcal{L}(-\tau^{-3i-1}p - \tau^{-3i-2}p)\right)^{-1}\left(\sigma^{*(6i+11)}\mathcal{L}\sigma^{*(6i+12)}\mathcal{L}(-\tau^{-3i-4}p - \tau^{-3i-5}p)\right)$$

As $\overline{\delta}_{(m,n)}$ is a nonzero section of $\sigma^{*(m+2n)}\mathcal{L}(-\tau^{-n}p)$, we can (after inductively changing the $\delta_{(1,a)}$ by a scalar multiple) assume that the morphism

$$\left(\overline{\delta}_{(1,3i+2)}\right)^{-1}\left(\overline{\delta}_{(2,3i+1)}\right)^{-1}(-)\overline{\delta}_{(2,3i+4)}\ \overline{\delta}_{(1,3i+5)} : \tilde{B}_{(2,3i+1),(2,3i+4)} \to \tilde{B}_{(0,3i+3),(0,3i+6)}$$

coincides with φ for all i. Next note that τ^* maps $k\overline{\delta}_{(m,n)} = \tilde{B}_{(m,n),(m-1,n+1)}$ to $\tilde{B}_{(m+2,n+1),(m+1,n+2)} = k\overline{\delta}_{(m+2,n+1)}$. Hence (after changing the $\delta_{(1+2b,a+b)}$ and $\delta_{(2+2b,a+b)}$ by a scalar multiple by induction on b) we can assume

$$\tau^*\overline{\delta}_{(m,n)} = \overline{\delta}_{(m+2,n+1)} \tag{1.104}$$

In particular (1.101) commutes as it factors as follows:

$$\begin{tikzcd}
\tilde{B}_{(2i,i),(2i+2,i+1)} \arrow[r, "\overline{\delta}^{-2i}(-)\overline{\delta}^{2i+2}"] \arrow[d, "\cong"', "\tau^*"] & \tilde{B}_{(0,3i),(0,3i+3)} \arrow[d, "\tau^*"'] \arrow[dr, "\cong"] & \\
\tilde{B}_{(2i+2,i+1),(2j+2,j+1)} \arrow[r, "\overline{\delta}^{-2i}(-)\overline{\delta}^{2i+2}"'] & \tilde{B}_{(2,3i+1),(2,3i+4)} \arrow[r, "\overline{\delta}^{-2}(-)\overline{\delta}^{2} = \varphi"'] & \tilde{B}_{(0,3i+3),(0,3i+6)}
\end{tikzcd}$$

Next we focus on commutativity of (1.102). The right vertical arrow in (1.102) can be described as follows: write g'_{3i} as a product of elements in $\check{A}'_{3i,3i+1} = \check{B}'_{3i,3i+1}$, $\check{A}'_{3i+1,3i+2} = \check{B}'_{3i+1,3i+2}$ and $\check{A}'_{3i+2,3i+3} = \check{B}'_{3i+2,3i+3}$, apply $\varphi \circ \tau^* : \check{B}'_{(m,m+1)} \to \check{B}'_{(m+3,m+4)}$ on each of the 3 factors, then g'_{3i+3} is the product of these 3 new elements. This remark together with (1.104) shows that the commutativity of (1.102) reduces to the following claim:

if $g_{(2i,i)}\delta_{(2i,i)}\delta_{(2i-1,i+1)} = a_0 \cdot a_1 \cdot a_2$ for $a_n \in \tilde{A}_{(2i,i+n),(2i,i+n+1)} = \tilde{B}_{(2i,i+n),(2i,i+n+1)}$ then $g_{(2i+2,i+1)}\delta_{(2i+2,i+1)}\delta_{(2i+1,i)} = \tau^* a_0 \cdot \tau^* a_1 \cdot \tau^* a_2$.

As there are embeddings

$$\tilde{A}_{(i,j),(m,n)} \hookrightarrow \check{A}_{i+2j,m+2n} \tag{1.105}$$

it suffices to check the claim in $\check{A}$. To prove this, denote $\underline{x}$ for the image of some $x \in \tilde{A}$ under the embedding (1.105). Consider the equality

$$\underline{g_{(2i,i)}} \cdot \underline{\delta_{(2i,i)}} \cdot \underline{\delta_{(2i-1,i+1)}} = \underline{a_0} \cdot \underline{a_1} \cdot \underline{a_2} \tag{1.106}$$

As $\underline{\delta_{(2i,i)}}, \underline{\delta_{(2i-1,i+1)}}, \underline{a_0}, \underline{a_1}$ and $\underline{a_2}$ lie in $\check{B}$, the 4-periodicity morphism $\check{A}_{m,n} \to \check{A}_{m+4,n+4}$ sends them to $\tau^* \underline{\delta_{(2i,i)}}, \tau^* \underline{\delta_{(2i-1,i+1)}}, \tau^* \underline{a_0} = \underline{\tau^* a_0}, \tau^* \underline{a_1} = \underline{\tau^* a_1}$ and $\tau^* \underline{a_2} = \underline{\tau^* a_2}$. Moreover, by (1.104)

$$\tau^* \underline{\delta_{(i,j)}} = \underline{\tau^* \overline{\delta}_{(i,j)}} = \underline{\overline{\delta}_{(i+2,j+1)}} = \underline{\delta_{(i+2,j+1)}}$$

Finally notice that by construction $\underline{g_{(2i,i)}}$ is the element in $\check{A}_{4i,4i+4}$ corresponding to the central element $g \in A_4$. The morphism $\check{A}_{4i,4i+4} \to \check{A}_{4i+4,4i+8}$ sends $\underline{g_{(2i,i)}}$ to $\underline{g_{(2i+2,i+1)}}$. In particular the 4-periodicity of $\check{A}$ turns the equality (1.106) into

$$\underline{g_{(2i+2,i+1)}} \cdot \underline{\delta_{(2i+2,i+1)}} \cdot \underline{\delta_{(2i+1,i)}} = \underline{\tau^* a_0} \cdot \underline{\tau^* a_1} \cdot \underline{\tau^* a_2}$$

proving our claim and hence showing commutativity of (1.102) and hence of (1.98). As mentioned above this implies that the inclusion (1.97) induces an inclusion

$$T \hookrightarrow A'^{(3)}$$

Our goal is to show that this inclusion identifies T with $A'^{(3)}(-q-r)$. As T is generated in degree 1, in order to understand the image of T under the above inlcusion, it suffices to understand the image of

$$T_1 \cong \tilde{A}_{(0,0),(2,1)} \xrightarrow{\cdot\delta^2} \tilde{A}_{(0,0),(0,3)} \cong A'_3$$

As was mentioned above this image contains g' and the image of $T_1 \to A'_3 \to A'_3/g'$ is the same as the image of

$$\cdot\overline{\delta}_{(2,1)}\overline{\delta}_{(1,2)} : \tilde{B}_{(0,0),(2,1)} \longrightarrow \tilde{B}_{(0,0),(0,3)} \cong B'_3$$

Using (1.78), $\overline{\delta}_{(2,1)}\overline{\delta}_{(1,2)}$ is a nonzero global section of

$$\sigma^{4*}\mathcal{L}\sigma^{5*}\mathcal{L}(-\tau^{-1}p-\tau^{-2}p-q-r)$$

Hence T is the subalgebra of $A'^{(3)}$ generated by the elements of A'_3 whose image in

$$B'_3=\Gamma(Y,\mathcal{L}\dots\sigma^{5*}\mathcal{L}(-p-\tau^{-1}p-\tau^{-2}p))$$

is given by

$$\begin{aligned}&\Gamma(Y,\mathcal{L}\dots\sigma^{5*}\mathcal{L}(-p-\tau^{-1}p-\tau^{-2}p-q-r))\\&=\Gamma(Y,(\mathcal{L}\otimes\sigma^{*}\mathcal{L}(-p))\otimes(\sigma^{2*}\mathcal{L}\otimes\sigma^{3*}\mathcal{L}(-\tau^{-1}p))\otimes(\sigma^{4*}\mathcal{L}\otimes\sigma^{5*}\mathcal{L}(-\tau^{-2}p))(-q-r))\\&=\Gamma(Y,\mathcal{G}\otimes\psi^{*}\mathcal{G}\otimes\psi^{2*}\mathcal{G}(-q-r)))\end{aligned}$$

(where we used (1.20) and Lemma 1.3.6)

Hence T is isomorphic to the noncommutative blow up $A'^{(3)}(p'+q')$ as required. This finishes the proof of Theorem 1.11.3.

1.12 Noncommutative Cremona transformations revisited

In §1.6 we introduced a noncommutative version of the Cremona transform $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Such a noncommutative Cremona transform is given by an inclusion $\gamma:\check{A}'\hookrightarrow\check{A}^{(2)}$ where $A=A(Y,\mathcal{G},\psi)$ and $A'=A(Y,\mathcal{G}',\psi)$ are quadratic Sklyanin algebras. This inclusion is constructed starting from 3 points $p,q,r\in Y$ which are assumed to be non-collinear and to lie in different τ orbits where $\tau=\psi^3=\psi'^3$.

It was proved (Theorem 1.6.4) that γ factors as in the commutative case. I.e. there exists a cubic Sklyanin $\mathbb{Z}$-algebra A'' and inclusions $\gamma_1:\check{A}'\hookrightarrow A''^{(2)}$, $\gamma_2:A''\hookrightarrow\check{A}$ such that γ_1 is as in (1.37), γ_2 is as in (1.39) and $\gamma=\gamma_2\circ\gamma_1$. In particular γ_1 is constructed starting from p and γ_2 is constructed starting from q,r. We will use this factorization to show that γ satisfies similar properties as γ_1 and γ_2. Unfortunately the results from §1.11 do not lift immediately to noncommutative Cremona transforms. The problem lies in the fact that A'' is usually not 1-periodic, see for example Remark 1.5.2. Nevertheless, adapting the results of the previous sections, we can still conclude.

Theorem 1.12.1. *Let $\gamma:\check{A}'\hookrightarrow\check{A}^{(2)}$ be a noncommutative Cremona transform. Then*

i) γ is an invertible quadratic transform. Moreover, the inverse $\delta:\check{A}\hookrightarrow\check{A}'^{(2)}$ can be chosen as the noncommutative Cremona constructed starting from points p',q',r', which are defined by the following relations in Pic(Y)*:*

$$p+q+\tau r'\sim[\mathcal{G}],\quad p+r+\tau q'\sim[\mathcal{G}],\quad q+r+\tau p'\sim[\mathcal{G}] \tag{1.107}$$

ii) A and A' contain a common noncommutative blowup:

$$A^{(3)}(p+q+r)\cong A'^{(3)}(p'+q'+r')$$

Proof. *i)* The invertability of γ is guaranteed by Theorem 1.7.6. Moreover, the proof of Lemma 1.7.7 shows that an inverse of γ can be chosen as

$$\delta = \delta_1 \circ \delta_2$$

where δ_1 and δ_2 are inverses to γ_1 and γ_2 respectively. In particular $\delta_2 : \check{A} \hookrightarrow A''^{(2)}$ can be constructed as a noncommutative $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ starting from a point $p' \in Y$ given by the following relation in $\mathrm{Pic}(Y)$ (see (1.70))

$$q + r + \tau p' \sim [\mathcal{G}]$$

Similarly $\delta_1 : A'' \hookrightarrow \check{A}'$ can be constructed as a noncommutative $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ starting from two points $q', r' \in Y$ given by the following relations in $\mathrm{Pic}(Y)$ (see (1.76) and (1.43))

$$\begin{aligned} p + \tau r' &\sim [\mathcal{L}_0] = [\mathcal{G}] - q \\ p + q' &\sim [\mathcal{L}_1] = [\psi^* \mathcal{G}] - r \end{aligned}$$

The first equation is obviously equivalent to $p + q + \tau r' \sim [\mathcal{G}]$. For the second equation recall that there is an invertible sheaf $\mathcal{N}$ of degree zero (see for example (1.21) or [VdB11, Theorem 4.2.3]) such that

$$[\psi^* \mathcal{G}] = [\mathcal{G}] + 3[\mathcal{N}] \quad \text{and} \quad \tau q' \sim q - 3[\mathcal{N}] \tag{1.108}$$

Hence the condition $p + q' \sim [\psi^* \mathcal{G}] - r$ is equivalent to $p + r + \tau q' \sim [\mathcal{G}]$. We conclude that δ is a noncommutative Cremona transform constructed starting from points p', q', r' which satisfy (1.107).

ii) We first prove this under the condition that A'' is 1-periodic. Hence assume that $A'' = A(Y, \mathcal{L}, \sigma)$ and that γ_1 and γ_2 are of the form $\check{A}' \hookrightarrow \check{A}''^{(2)}$ and $\check{A}'' \hookrightarrow \check{A}$ respectively. Theorem 1.11.3 then gives rise to isomorphisms

$$A^{(3)}(q + r) \cong A''^{(4)}(p') \text{ and } A'^{(3)}(q' + r') \cong A''^{(4)}(p)$$

We can now use [RSS14, Proposition 5.4] to obtain

$$\begin{aligned} A^{(3)}(p + q + r) &= \left(A^{(3)}(q + r) \right)(p) \\ &\cong \left(A''^{(4)}(p') \right)(p) = \left(A''^{(4)}(p) \right)(p') \\ &\cong \left(A^{(3)}(q' + r') \right)(p') = A'^{(3)}(p' + q' + r') \end{aligned}$$

In order to prove the isomorphism $A^{(3)}(p + q + r) \cong A'^{(3)}(p' + q' + r')$ in case A'' is not 1-periodic, we have to work through a $\mathbb{Z}^2$-algebra $\tilde{A}$. We will do so below. $\square$

Let p, q, r be as above and denote $d = p+q+r$ for the associated divisor on Y. We define a $\mathbb{Z}^2$-algebra $\tilde{A}$ as follows:

$$\tilde{A}_{(i,j),(m,n)} := \begin{cases} \operatorname{Hom}_X(\mathcal{O}_X(-m-2n), \mathcal{O}_X(-i-2j)m_{\tau^{-j}d}\cdots m_{\tau^{-n+1}d}) & \text{if } n > j \\ \operatorname{Hom}_X(\mathcal{O}_X(-m-2n), \mathcal{O}_X(-i-2j)) & \text{if } n \leq j \end{cases} \tag{1.109}$$

This algebra satisfies similar conditions as its counterparts from §1.8:

Proposition 1.12.2. *$\tilde{A}$ as in* (1.109) *satisfies the following properties*

i) $\tilde{A}_{(i,0),(m,0)} \cong \check{A}_{i,m} \cong A_{m-i}$

ii) $\tilde{A}_{(0,j),(0,n)} \cong \check{A}'_{j,n} \cong A'_{n-j}$

iii) $\tilde{A}$ *contains no nontrivial zero divisors*

iv) $\dim_k \tilde{A}_{(i,j),(i+a,j+b)} = h(a+2b)$ *for* $b \leq 0$ *and* $h(n) = \frac{(n+1)(n+2)}{2}$ *the Hilbert series of* A *and* A'.

v) $\dim_k \tilde{A}_{(i,j),(i+a,j+b)} = \frac{a^2+4ab+b^2+3a+3b+2}{2}$ *for* $a, b \geq 0$

vi) $\dim_k \tilde{A}_{(i,j),(i+a,j+b)} = h(b+2a)$ *for* $a \leq 0$

Proof. *i)*, *ii)*, *iii)* and *iv)* are obvious from the definition of $\tilde{A}$.

v) As p, q, r lie in different τ-orbits, we can use Lemma 1.6.5 to see that the colength of $m_{\tau^{-j}d}\cdots m_{\tau^{-j-b+1}d}$ in o_X is three times the colength of $m_{\tau^{-j}p}\cdots m_{\tau^{-j-b+1}p}$ inside o_X, hence $3\frac{b(b+1)}{2}$ by [VdB01, Corollary 5.2.4]. Using Ext^1-vanishing as in Lemma 1.6.2 we hence find:

$$\begin{aligned} \dim_k \tilde{A}_{(i,j),(i+a,j+b)} &= \dim_k(A_{a+2b}) - 3\frac{b(b+1)}{2} \\ &= \frac{(a+2b+1)(a+2b+2)}{2} - 3\frac{b(b+1)}{2} \\ &= \frac{a^2+4ab+b^2+3a+3b+2}{2} \end{aligned}$$

vi) for $a = -1$ (which is the only case we actually need) the computation follows again by considering the colength of $m_{\tau^{-j}d}\cdots m_{\tau^{-j-b+1}d} \subset o_X$ and using Ext^1-vanishing. To get the result for all $a \leq -1$ we use an I-basis for A as in §1.13. We refer the interested reader to §1.13.2 for the proof for all negative a. □

As a corollary of Proposition 1.12.2, we know that $\tilde{A}_{(i,j),(i-1,j+2)}$ and $\tilde{A}_{(i,j),(i+2,j-1)}$ are one dimensional. Let $\delta_{i,j}$ and $\gamma_{i,j}$ be nonzero elements in these spaces. We can then visualize $\tilde{A}$ on a 2-dimensional square grid.

$$\begin{tikzcd}
(3,0) \arrow[r] & (3,1) \arrow[r] & (3,2) \arrow[r] & (3,3) \\
(2,0) \arrow[r] \arrow[u] \arrow[urr, dotted, leftarrow, "\delta_{3,0}"'] & (2,1) \arrow[r] \arrow[u] \arrow[urr, dotted, leftarrow, "\delta_{3,1}"'] & (2,2) \arrow[r] \arrow[u] & (2,3) \arrow[u] \\
(1,0) \arrow[r] \arrow[u] \arrow[urr, dotted, leftarrow, "\delta_{2,0}"'] & (1,1) \arrow[r] \arrow[u] \arrow[uul, dotted, "\gamma_{1,1}"] \arrow[urr, dotted, leftarrow, "\delta_{2,1}"'] & (1,2) \arrow[r] \arrow[u] \arrow[uul, dotted, "\gamma_{1,2}"] & (1,3) \arrow[u] \arrow[uul, dotted, "\gamma_{1,3}"] \\
(0,0) \arrow[r] \arrow[u] \arrow[urr, dotted, leftarrow, "\delta_{1,0}"'] & (0,1) \arrow[r] \arrow[u] \arrow[uul, dotted, "\gamma_{0,1}"] \arrow[urr, dotted, leftarrow, "\delta_{1,1}"'] & (0,2) \arrow[r] \arrow[u] \arrow[uul, dotted, "\gamma_{0,2}"] & (0,3) \arrow[u] \arrow[uul, dotted, "\gamma_{0,3}"]
\end{tikzcd}$$

All vertical and horizontal arrows represent three dimensional vector spaces whereas the dotted arrows represent one dimensional vector spaces.

Completely identical to §1.8.1 and §1.8.2, left and right multiplication with (inverses of) the elements $\delta_{i,j}$ induces an inclusion of algebras

$$\delta : \check{A} \hookrightarrow \check{A}'^{(2)} \tag{1.110}$$

We claim that this inclusion coincides with the noncommutative Cremona transform $\delta_1 \circ \delta_2$ as constructed in Theorem 1.12.1. As $\check{A}$ is generated in degree 1, it suffices to prove that

$$\begin{aligned} \delta(\check{A}_{i,i+1}) &= \Gamma(Y, \psi'^{2i*}\mathcal{L}' \otimes \psi'^{(2i+1)*}\mathcal{L}'(-\tau^{-i}(-p'-q'-r'))) \\ &\subset \Gamma(Y, \psi'^{2i*}\mathcal{L}' \otimes \psi'^{(2i+1)*}\mathcal{L}') = \check{A}'_{2i,2i+2} \end{aligned} \tag{1.111}$$

Similar to the computations in §1.8, we use $\check{A}_{i,i+1} = \check{B}_{i,i+1}$ and $\check{A}'_{2i,2i+2} = \check{B}'_{2i,2i+2}$ to conclude that we can check (1.111) at the level of the $\mathbb{Z}^2$-algebra $\tilde{B}$:

$$\tilde{B}_{(i,j),(m,n)} := \begin{cases} \Gamma(Y, \psi^{(i+2j)*}\mathcal{L} \dots \psi^{(m+2n-1)*}\mathcal{L}(-\tau^{-j}d - \dots - \tau^{-n+1}d)) & \text{if } n > j \\ \Gamma(Y, \psi^{(i+2j)*}\mathcal{L} \dots \psi^{(m+2n-1)*}\mathcal{L}) & \text{if } n \leq j \end{cases}$$

There is a canonical map $\tilde{A} \to \tilde{B}$ which is surjective in positive degrees. Let $\overline{\delta}_{i,j}$ be the image of $\delta_{i,j}$ under this canonical map. In particular

$$\overline{\delta}_{i,j} \in \tilde{B}_{(i,j),(i-1,j+2)} = \Gamma\big(Y, \psi^{(i+2j)*}\mathcal{L}\psi^{(i+2j+1)*}\mathcal{L}\psi^{(i+2j+2)*}\mathcal{L}(-\tau^{-j}d - \tau^{-j-1}d)\big)$$

and (1.111) follows if we can prove that $\overline{\delta}_{i,j}$ lies in the 1-dimensional space

$$\Gamma\big(Y, \psi^{(i+2j)*}\mathcal{L}\psi^{(i+2j+1)*}\mathcal{L}\psi^{(i+2j+2)*}\mathcal{L}(-\tau^{-j}d - \tau^{-j-1}d - \tau^{-i}(p'+q'+r'))\big) \subset \tilde{B}_{(i,j),(i-1,j+2)} \tag{1.112}$$

Contrary to the computations in §1.8, we no longer have $\tilde{B}_{(i,j),(i-1,j+2)} \cong \tilde{A}_{(i,j),(i-1,j+2)}$ because $\dim_k(\tilde{B}_{(i,j),(i-1,j+2)}) = 3 \neq 1 = \dim_k(\tilde{A}_{(i,j),(i-1,j+2)})$. Similarly the left hand side in (1.112) is 1-dimensional, whereas the right hand side is 3-dimensional. To solve this inconvenience, note that $\delta_{i,j}$ can be written as a product of three elements:

$$\partial_{i,j,1} \in \operatorname{Hom}_X(\mathcal{O}_X(-i-2j-1), \mathcal{O}_X(-i-2j)m_{\tau^{-j}p+\tau^{-j}q})$$
$$\partial_{i,j,2} \in \operatorname{Hom}_X(\mathcal{O}_X(-i-2j-2), \mathcal{O}_X(-i-2j-1)m_{\tau^{-j-1}p+\tau^{-j}r})$$
$$\partial_{i,j,3} \in \operatorname{Hom}_X(\mathcal{O}_X(-i-2j-3), \mathcal{O}_X(-i-2j-2)m_{\tau^{-j-1}q+\tau^{-j-1}r})$$

(here used that $m_{p+q+r} = m_p m_q m_r$ holds by Lemma 1.6.5)

In particular we can write $\overline{\delta}_{i,j} = \overline{\partial}_{i,j,1}\overline{\partial}_{i,j,2}\overline{\partial}_{i,j,3}$ where

$$\overline{\partial}_{i,j,1} \in \Gamma(Y, \psi^{(i+2j)*}\mathcal{G}(-\tau^{-j}p - \tau^{-j}q))$$
$$\overline{\partial}_{i,j,2} \in \Gamma(Y, \psi^{(i+2j+1)*}\mathcal{G}(-\tau^{-j-1}p - \tau^{-j}r))$$
$$\overline{\partial}_{i,j,3} \in \Gamma(Y, \psi^{(i+2j+2)*}\mathcal{G}(-\tau^{-j-1}q - \tau^{-j-1}r))$$

and (1.112) follows if we can prove

$$\overline{\partial}_{i,j,1} \in \Gamma(Y, \psi^{(i+2j)*}\mathcal{G}(-\tau^{-j}p - \tau^{-j}q - \tau^{-i}r'))$$
$$\overline{\partial}_{i,j,2} \in \Gamma(Y, \psi^{(i+2j+1)*}\mathcal{G}(-\tau^{-j-1}p - \tau^{-j}r - \tau^{-i}q'))$$
$$\overline{\partial}_{i,j,3} \in \Gamma(Y, \psi^{(i+2j+2)*}\mathcal{G}(-\tau^{-j-1}q - \tau^{-j-1}r - \tau^{-i}p'))$$

Without loss of generality we prove the first statement. Note that $\psi^{(i+2j)*}\mathcal{G}(-\tau^{-j}p - \tau^{-j}q)$ is a degree 1 line bundle on the elliptic curve Y, hence we only need to prove

$$[\psi^{(i+2j)*}\mathcal{G}] \sim \tau^{-j}p + \tau^{-j}q + \tau^{-i}r'$$

and this in turn follows from (1.107) and (1.108).

In particular δ as in (1.110) is the noncommutative Cremona transform constructed starting from p', q', r'.

A Hilbert series computation shows that there exist normalizing elements $g_{(i,j)} \in \tilde{A}_{(i,j),(i+1,j+1)}$ such that $g_{(i,j)}A = \ker(e_{(i,j)}\tilde{A}_+ \to e_{(i,j)}\tilde{B}_+)$. The methods in §1.9 and §1.10 can be adapted easily to prove that $\tilde{A}$ is noetherian. As such, we can apply Theorem 0.6.6 and conclude that there is an equivalence

$$\operatorname{QGr}(\tilde{A}) \cong \operatorname{QGr}(\tilde{A}_\Delta)$$

For each diagonal-like $\Delta : \mathbb{Z} \to \mathbb{Z}^2$. Now let $\Delta(i) = (i,i)$ then

$$\left(\tilde{A}_\Delta\right)_{i,j} = \operatorname{Hom}_X(\mathcal{O}_X(-3j), \mathcal{O}_X(-3i)m_{\tau^{-i}d}\cdots m_{\tau^{-j+1}d})$$

and we immediately find $\tilde{A}_\Delta \cong A^{(3)}(\widetilde{p+q+r})$. Computations similar to the ones in §1.11 (starting at (1.98)) show that $\tilde{A}_\Delta \cong A'^{(3)}(\widetilde{p'+q'+r'})$ holds as well. This finishes the proof of Theorem 1.12.1 as well as the following:

Corollary 1.12.3. *With the notations from this section, there is an equivalence of categories*

$$\mathrm{QGr}(\tilde{A}) \cong \mathrm{QGr}(T)$$

where

$$A^{(3)}(p+q+r) \cong T \cong A'^{(3)}(p'+q'+r')$$

1.13 I-bases for quadratic Sklyanin algebras

Throughout this section we assume $A = A(Y, \mathcal{L}, \psi)$ is a quadratic Sklyanin algebra with Hilbert series h. p, q and r are points lying in different τ-orbits with $\tau = \psi^3$. Our goal is to prove that for $a \le -2$:

$$\dim_k \left(\mathrm{Hom}(\mathcal{O}_X(-i-a-2j-2b), \mathcal{O}_X(-i-2j) \otimes m_{\tau^{-j}d} \dots m_{\tau^{-j-b}d})\right) = h(2a+b) = \begin{cases} \frac{(2a+b+1)(2a+b+2)}{2} & \text{if } 2a+b \ge 0 \\ 0 & \text{if } 2a+b < 0 \end{cases} \tag{1.113}$$

where $d = p+q+r$ and

$$\dim_k \left(\mathrm{Hom}(\mathcal{O}_X(-i-a-j-b), \mathcal{O}_X(-i-j) \otimes m_{d_j} \dots m_{d_{j+b-1}})\right) = h'(2a+b) = \begin{cases} (n+1)^2 & \text{if } 2a+b = 2n \ge 0 \\ (n+1)(n+2) & \text{if } 2a+b = 2n+1 > 0 \\ 0 & \text{if } 2a+b < 0 \end{cases} \tag{1.114}$$

where d_i is as in (1.40). Using $(\mathcal{O}_X(n) \otimes m_p)(m) = (\mathcal{O}_X \otimes m_{\psi^{-n}p})(n+m)$ (similar to (1.22)) and replacing p, q and r by $\psi^x p$, $\psi^y q$, $\psi^z r$ for the appropriate values of x, y and z this is equivalent to proving

$$\dim_k \left(\mathrm{Hom}(\mathcal{O}_X, (\mathcal{O}_X \otimes m_d \dots m_{\tau^{-b+1}d})(a+2b))\right) = h(2a+b) \tag{1.115}$$

and

$$\dim_k \left(\mathrm{Hom}(\mathcal{O}_X, (\mathcal{O}_X \otimes m_{d_0} \dots m_{d_{b-1}})(a+b))\right) = h'(2a+b) \tag{1.116}$$

We will prove this using I-bases.

1.13.1 I-bases

In this subsection we recall the definition and construction of an I-basis for a quadratic Sklyanin algebra. For a more thorough introduction to I-bases we refer the reader to [TV96].

Definition 1.13.1. Let $A = A(Y, \mathcal{L}, \psi)$ be a quadratic Sklyanin algebra and let G denote the monoid of monomials in x, y, z. Let G_n denote the subset of all degree n monomials. An I-basis for A is then given by a map $v : G \to A$ satisfying the following properties:

i) $v(G_n)$ is a k-basis for A_n

ii) for any $g \in G$ there are elements $x_g, y_g, z_g \in A_1$ such that

$$v(gx) = v(g)x_g, v(gy) = v(g)y_g \text{ and } v(gz) = v(g)z_g$$

Remark 1.13.2. *Note that $v(x) = x_1, v(y) = y_1, v(z) = z_1$. An I-basis can hence alternatively be given by a collection of $\{x_g, y_g, z_g\}_{g\in G}$ satisfying $x_g y_{xg} = y_g x_{yg}, x_g z_{xg} = z_g x_{zg}, y_g z_{yg} = z_g y_{zg}$*

In [TV96, §4] Tate and Van den Bergh give a construction for an I-basis for a Sklyanin algebra. In the case of a quadratic Sklyanin algebra this construction depends on the choice of a rational point $\overline{o} = (o_1, o_2, o_3) \in Y^3$. For each $g \in G$ one defines $\overline{og}$ by setting

$$\overline{ox} = (\psi o_1, \psi^{-2} o_2, \psi^{-2} o_3)$$
$$\overline{oy} = (\psi^{-2} o_1, \sigma o_2, \psi^{-2} o_3)$$
$$\overline{oz} = (\psi^{-2} o_1, \psi^{-2} o_2, \psi o_3)$$

such that if $g = x^\alpha y^\beta z^\lambda$ then $\overline{og} = (\psi^{\alpha-2\beta-2\lambda} o_1, \psi^{\beta-2\alpha-2\lambda} o_2, \psi^{\lambda-2\alpha-2\beta} o_3)$.

We then define $x_g, y_g, z_g \in A_1 = \Gamma(Y, \mathcal{L})$ (up to a scalar multiple) by setting

$$\begin{array}{lll} x_g((\overline{og})_1) \neq 0 & x_g((\overline{og})_2) = 0 & x_g((\overline{og})_3) = 0 \\ y_g((\overline{og})_1) = 0 & y_g((\overline{og})_2) \neq 0 & y_g((\overline{og})_3) = 0 \\ z_g((\overline{og})_1) = 0 & z_g((\overline{og})_2) = 0 & z_g((\overline{og})_3) \neq 0 \end{array}$$

(the scalar multiples are then chosen such that the relations in Remark 1.13.2 hold)

In particular

$$\begin{aligned} v(x) = x_1 \in \Gamma(Y, \mathcal{L}(-o_2 - o_3)) &= \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_X(1) \otimes m_{o_2+o_3}) \\ &= \mathrm{Hom}_X(\mathcal{O}_X, (\mathcal{O}_X \otimes m_{\sigma^{-1}o_2+\sigma^{-1}o_3})(1)) \end{aligned}$$

and analogously

$$v(y) \in \mathrm{Hom}_X(\mathcal{O}_X, (\mathcal{O}_X \otimes m_{\psi^{-1}o_1+\psi^{-1}o_3})(1))$$
$$v(z) \in \mathrm{Hom}_X(\mathcal{O}_X, (\mathcal{O}_X \otimes m_{\psi^{-1}o_1+\psi^{-1}o_2})(1))$$

Similar computations are possible for monomials of higher degree, for example: $v(xy) = x_1 \cdot y_x$ lies in the image of

$$\begin{array}{c} \mathrm{Hom}_X(\mathcal{O}_X, (\mathcal{O}_X \otimes m_{\psi^{-1}o_2 + \psi^{-1}o_3})(1)) \otimes \mathrm{Hom}_X(\mathcal{O}_X, (\mathcal{O}_X \otimes m_{\psi^{-1}\overline{ox}_1 + \psi^{-1}\overline{ox}_3})(1)) \\ \big\downarrow \\ A_1 \otimes A_1 \\ \big\downarrow \\ A_2 \end{array}$$

This image is given by

$$\begin{aligned} &\mathrm{Hom}\left(\mathcal{O}_X, \left(\mathcal{O}_X \otimes m_{\psi^{-1}o_2 + \psi^{-1}o_3} m_{\psi^{-1}o_1 + \psi^{-4}o_3}\right)(2)\right) \\ &\quad \cong \mathrm{Hom}\left(\mathcal{O}_X, \left(\mathcal{O}_X \otimes m_{\psi^{-1}o_1 + \psi^{-1}o_2 + \psi^{-1}o_3} m_{\tau^{-1}\psi^{-1}o_3}\right)(2)\right) \end{aligned}$$

Where we used the fact that o_1, o_2, o_3 lie in different τ-orbits. Inspired by the above results we make the following choices for o_1, o_2, o_3:

$$o_1 = \psi p \text{ and } o_2 = \psi q \text{ and } o_3 = \psi r \tag{1.117}$$

We can can inductively show the following: assume α, β, λ are nonnegative integers then:

$$\begin{aligned} v(x^\alpha y^\beta z^\lambda) \in \mathrm{Hom}\Big(\mathcal{O}_X, \big(\mathcal{O}_X \otimes & m_p m_{\tau^{-1}p} \dots m_{\tau^{-\beta-\lambda+1}p} \\ & m_q m_{\tau^{-1}q} \dots m_{\tau^{-\alpha-\lambda+1}q} \\ & m_r m_{\tau^{-1}r} \dots m_{\tau^{-\alpha-\beta+1}r}\big)(\alpha+\beta+\lambda)\Big) \end{aligned} \tag{1.118}$$

1.13.2 Proof of Lemma 1.8.4 and Proposition 1.12.2.vi)

We first prove Lemma 1.8.4. As mentioned above, this is equivalent to proving (1.116). Using the above language of I-bases the following proposition reduces the proof of (1.116) to a combinatorial problem

Proposition 1.13.3. *Let $\alpha, \beta, \gamma \geq 0$ and let d_i be as in* (1.40)*. Then*

$$\begin{array}{c} v(x^\alpha y^\beta z^\lambda) \in \mathrm{Hom}\Big(\mathcal{O}_X, \big(\mathcal{O}_X \otimes m_{d_0} \dots m_{d_{h-1}}\big)(\alpha+\beta+\lambda)\Big) \\ \Updownarrow \\ \left\lceil \dfrac{h}{2} \right\rceil \leq \alpha + \lambda \text{ and } \left\lfloor \dfrac{h}{2} \right\rfloor \leq \alpha + \beta \end{array}$$

Proof. $\Uparrow$ follows from (1.118). For $\Downarrow$ we need some more computations ... $\square$

Fix $h \in \mathbb{N}$ and define the following right submodules M and M' of A_A:

$$M_n = \operatorname{Hom}_X\left(\mathcal{O}_X, \left(\mathcal{O}_X \otimes m_{d_0} \ldots m_{d_{h-1}}\right)(n)\right)$$
$$M'_n = \operatorname{Span}\{v(x^\alpha y^\beta z^\lambda) \mid \alpha + \beta + \lambda = n, \lceil h/2 \rceil \le \alpha + \lambda \text{ and } \lfloor h/2 \rfloor \le \alpha + \beta\}$$

To see that M' is a right A-module, recall that for each $g = x^\alpha y^\beta z^\lambda$, the elements x_g, y_g, z_g give a k-basis for A_1 hence the image of

$$k \cdot v(x^\alpha y^\beta z^\lambda) \otimes A_1 \longrightarrow A_n \otimes A_1 \longrightarrow A_{n+1}$$

lies inside $\operatorname{Span}\{v(x^{\alpha+1} y^\beta z^\lambda), v(x^\alpha y^{\beta+1} z^\lambda), v(x^\alpha y^\beta z^{\lambda+1})\}$.
We then have the following lemmas.

Lemma 1.13.4. *Let M and M' be as above then for all n sufficiently large we have*

$$\dim_k(M_n) = \dim_k(M'_n)$$

Proof. We prove this for h even. The case h odd is completely similar. Let $h = 2a$, then we have for $n \ge 2a - 1$ we have

$$\dim_k(M_n) = \frac{(n+1)(n+2)}{2} - 2 \cdot \frac{a(a+1)}{2} = \dim_k(M'_n)$$

for M this follows from by a computation as in (1.45), using Lemma 1.5.4. For M' this follows from the fact that for $n \ge 2a - 1$ at most one of the inequalities $\alpha + \lambda \ge a, \beta + \lambda \ge a\}$ can fail when $\alpha + \beta + \lambda = n$. $\square$

Lemma 1.13.5. *Let N be a right A-modules such that $M' \subset N \subset A$ and suppose there is an $n_0 \in \mathbb{N}$ such that $M'_{n_0} \subsetneq N_{n_0}$ then we have $M'_n \subsetneq N_n$ for all $n \ge n_0$*

Proof. By induction it suffices to show $M'_{n_0+1} \subsetneq N_{n_0+1}$. For this choose some nonzero element in $N_{n_0} \setminus M'_{n_0}$. This element can be written as

$$\sum_{\substack{\alpha,\beta,\lambda \\ \alpha+\beta+\lambda=n}} t_{\alpha,\beta,\lambda} v(x^\alpha y^\beta z^\lambda).$$

Without loss of generality we can assume there is a $t_{\alpha_0,\beta_0,\lambda_0} \ne 0$ with $\beta_0 + \lambda_0 < \left\lceil \frac{h}{2} \right\rceil$. Choose such a t with α_0 maximal and let $g = x^{\alpha_0} y^{\beta_0} z^{\lambda_0}$ then

$$\left(\sum_{\substack{\alpha,\beta,\lambda \\ \alpha+\beta+\lambda=n}} t_{\alpha,\beta,\lambda} v(x^\alpha y^\beta z^\lambda) \right) x_g$$

can be written as

$$t_{\alpha_0,\beta_0,\lambda_0} v(x^{\alpha_0+1} y^{\beta_0} z^{\lambda_0}) + \sum_{\substack{\alpha',\beta',\lambda' \\ \alpha'+\beta'+\lambda'=n+1 \\ \alpha' \le \alpha_0}} t'_{\alpha',\beta',\lambda'} v(x^{\alpha'} y^{\beta'} z^{\lambda'})$$

which obviously is a nonzero element in $N_{n_0+1} \setminus M'_{n_0+1}$ $\square$

continuation of Proposition 1.13.3. We have already proven $\Downarrow$ which is equivalent to $M' \subset M$. It then immediately follows from Lemmas 1.13.4 and 1.13.5 that $M = M'$, finishing the proof of the proposition. □

We can now prove (1.116). By Proposition 1.13.3 it suffices to count the number of triples of natural numbers α, β, λ satisfying $\alpha + \beta + \lambda = a + b$ and $\lceil \frac{b}{2} \rceil \le \alpha + \lambda$ and $\lfloor \frac{b}{2} \rfloor \le \alpha + \beta$. The latter is equivalent to $\beta \le a + b - \lceil \frac{b}{2} \rceil, \lambda \le a + b - \lfloor \frac{b}{2} \rfloor$ such that our problem has turned into a combinatorial problem: we need to show

Lemma 1.13.6.

$$\begin{aligned} \#\Big\{ (\alpha, \beta, \lambda) \in \mathbb{N}^3 \mid \alpha + \beta + \lambda = a + b, \\ \alpha \le a + b - \left\lceil \frac{b}{2} \right\rceil, \beta \le a + b - \left\lfloor \frac{b}{2} \right\rfloor \Big\} = h'(2a + b) \end{aligned} \tag{1.119}$$

We first show that the left hand side equals zero when $2a + b < 0$. Note that in this case $a + b < \frac{b}{2} \le \lceil \frac{b}{2} \rceil$, hence the condition $\alpha \le a + b - \lceil \frac{b}{2} \rceil$ contradicts $\alpha \in \mathbb{N}$ such that the left hand side of (1.119) is zero. Hence from now on we can assume $2a + b \ge 0$.

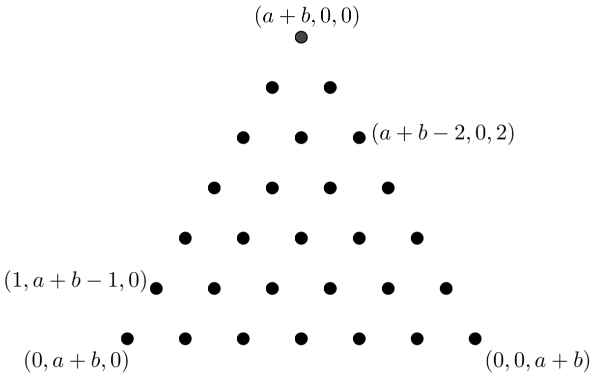

Figure 1.2: Combinatorial problem

This combinatorial problem has a graphical interpretation: it asks for counting the number of dots in Figure 1.2 whose coefficients (α, β, λ) satisfy the above inequalities.

In order to compute this number of dots we can rewrite (1.119)) as:

$$
\begin{aligned}
\#\left\{(\alpha,\beta,\lambda)\in\mathbb{N}^3 \mid \alpha+\beta+\lambda=a+b, \alpha\le a+b-\left\lceil\frac{b}{2}\right\rceil, \beta\le a+b-\left\lfloor\frac{b}{2}\right\rfloor\right\} &= \\
\#\left\{(\alpha,\beta,\lambda)\in\mathbb{N}^3 \mid \alpha+\beta+\lambda=a+b\right\}- \\
\#\left\{(\alpha,\beta,\lambda)\in\mathbb{N}^3 \mid \alpha+\beta+\lambda=a+b, \alpha\ge a+b+1-\left\lceil\frac{b}{2}\right\rceil\right\}- \\
\#\left\{(\alpha,\beta,\lambda)\in\mathbb{N}^3 \mid \alpha+\beta+\lambda=a+b, \beta\ge a+b+1-\left\lfloor\frac{b}{2}\right\rfloor\right\}+ \\
\#\left\{(\alpha,\beta,\lambda)\in\mathbb{N}^3 \mid \alpha+\beta+\lambda=a+b, \alpha\ge a+b+1-\left\lceil\frac{b}{2}\right\rceil, \beta\ge a+b+1-\left\lfloor\frac{b}{2}\right\rfloor\right\}
\end{aligned}
\tag{1.120}
$$

The green, red and blue numbers can be visualized as in Figure 1.3. (Where we are eventually interested in counting the number of dots that are not green, red or blue. Moreover the blue dots are actually the overlap of the green dots and the red dots)

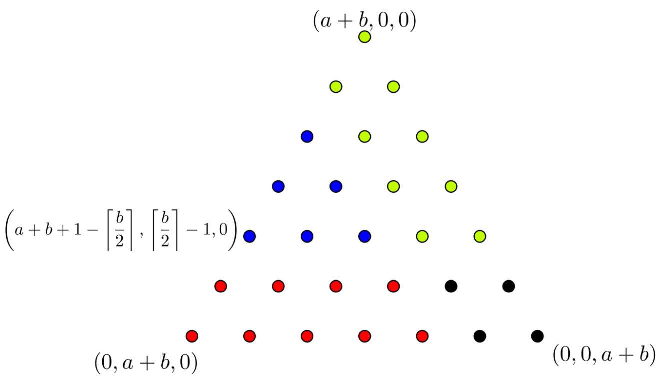

Figure 1.3: Visualizing formula (1.120)

The reason for writing our combinatorial problem as in (1.120) is the existence of the following bijection:

$$
\begin{aligned}
\{(\alpha,\beta,\lambda)\in\mathbb{N}^3 \mid \alpha+\beta+\lambda=n_1, \alpha\ge n_2\} \to \{(\alpha,\beta,\lambda)\in\mathbb{N}^3 \mid \alpha+\beta+\lambda=n_1-n_2\}: \\
(\alpha,\beta,\lambda)\mapsto(\alpha-n_2,\beta,\lambda)
\end{aligned}
\tag{1.121}
$$

Using similar bijections for the other sets we can write (1.120) as

$$\begin{aligned}\#\left\{(\alpha,\beta,\lambda)\in\mathbb{N}^3 \mid \alpha+\beta+\lambda=a+b, \alpha\le a+b-\left\lceil\frac{b}{2}\right\rceil, \beta\le a+b-\left\lfloor\frac{b}{2}\right\rfloor\right\} = \\ \#\left\{(\alpha,\beta,\lambda)\in\mathbb{N}^3 \mid \alpha+\beta+\lambda=a+b\right\}- \\ \#\left\{(\alpha,\beta,\lambda)\in\mathbb{N}^3 \mid \alpha+\beta+\lambda=\left\lceil\frac{b}{2}\right\rceil-1\right\}- \\ \#\left\{(\alpha,\beta,\lambda)\in\mathbb{N}^3 \mid \alpha+\beta+\lambda=\left\lfloor\frac{b}{2}\right\rfloor-1\right\}+ \\ \#\left\{(\alpha,\beta,\lambda)\in\mathbb{N}^3 \mid \alpha+\beta+\lambda=\left\lceil\frac{b}{2}\right\rceil+\left\lfloor\frac{b}{2}\right\rfloor-a-b-2=-a-2\right\}\end{aligned} \tag{1.122}$$

Now we can use the following: for all $n\ge 0$:

$$\begin{aligned}\#\{(\alpha,\beta,\lambda)\in\mathbb{N}^3 \mid \alpha+\beta+\lambda=n\} &= \sum_{i=0}^{n}\#\{(\alpha,\beta)\in\mathbb{N}^2 \mid \alpha+\beta=i\} \\ &= \sum_{i=0}^{n} i+1 \\ &= \frac{(n+2)(n+1)}{2}\end{aligned} \tag{1.123}$$

(recall that we assumed $a\le -2$ and $a+b\ge 0$ such that $-a-2, \left\lceil\frac{b}{2}\right\rceil-1, \left\lfloor\frac{b}{2}\right\rfloor-1\ge 0$) Hence combining (1.122) and (1.123) we find that

$$\#\left\{(\alpha,\beta,\lambda)\in\mathbb{N}^3 \mid \alpha+\beta+\lambda=a+b, \alpha\le a+b-\left\lceil\frac{b}{2}\right\rceil, \beta\le a+b-\left\lfloor\frac{b}{2}\right\rfloor\right\}$$

equals

$$\begin{aligned}\frac{(a+b+2)(a+b+1)}{2}-\frac{\lceil\frac{b}{2}\rceil(\lceil\frac{b}{2}\rceil+1)}{2}-\frac{\lfloor\frac{b}{2}\rfloor(\lfloor\frac{b}{2}\rfloor+1)}{2}+\frac{(-1-a)(-a)}{2} = \\ \frac{a^2+2ab+b^2+3a+3b+2-\lceil\frac{b}{2}\rceil^2-\lfloor\frac{b}{2}\rfloor^2-\lceil\frac{b}{2}\rceil-\lfloor\frac{b}{2}\rfloor+a^2+a}{2} = \\ \frac{2a^2+2ab+b^2+4a+2b+2-\lceil\frac{b}{2}\rceil^2-\lfloor\frac{b}{2}\rfloor^2}{2}\end{aligned} \tag{1.124}$$

We now treat the cases b even and b odd separately. First assume $b=2r$ for some $r\in\mathbb{N}$. Then (1.124) equals

$$\frac{2a^2+4ar+4r^2+4a+4r+2-2r^2}{2} = a^2+2ar+r^2+2a+2r+1=(a+r+1)^2$$

Next assume $b=2r+1$. Then (1.124) equals

$$\begin{aligned}\frac{2a^2+4ar+2a+4r^2+4r+1+4a+4r+2+2-(r+1)^2-r^2}{2} = \\ \frac{2a^2+4ar+2r^2+6r+6a+4}{2} = \\ a^2+2ar+r^2+3r+3a+2 = \\ (a+r+1)(a+r+2)\end{aligned}$$

Finally letting $n = a + r$ we see that this agrees with (1.114).

Next we turn our attention to Proposition 1.12.2.vi) As mentioned above, this is equivalent to proving

$$\dim_k \left(\mathrm{Hom}(\mathcal{O}_X, (\mathcal{O}_X \otimes m_d \dots m_{\tau^{-b+1}d})\,(a+2b))\right) = h(2a+b)$$

We turn this dimension problem into a combinatorial problem via the following:

Proposition 1.13.7. *Let $\alpha, \beta, \gamma \geq 0$ and let $d = p + q + r$. Then*

$$\begin{array}{rcl} v(x^\alpha y^\beta z^\lambda) & \in & \mathrm{Hom}\Big(\mathcal{O}_X, \big(\mathcal{O}_X \otimes m_d \dots m_{\tau^{-b+1}d}\big)(\alpha+\beta+\lambda)\Big) \\ & \Updownarrow & \\ b & \leq & \alpha+\beta, \alpha+\lambda, \beta+\lambda \end{array}$$

Proof. $\Uparrow$ follows from (1.118).

$\Downarrow$ is proved via the obvious generalizations of Lemma 1.13.4 and Lemma 1.13.5. □

As such, in order to prove (1.115), it suffices to prove

$$\#\Big\{(\alpha,\beta,\lambda) \in \mathbb{N}^3 \mid \alpha+\beta+\lambda = a+2b \text{ and } b \leq \alpha+\beta, \alpha+\lambda, \beta+\lambda\Big\} = h(2a+b) \qquad (1.125)$$

Both sides are equal to 0 for $2a+b<0$. For the right hand side this holds by definition. For the left hand side, the following shows that if there is a triple (α,β,λ) in this set, then necessarilly $2a+b \geq 0$:

$$3b \leq (\alpha+\beta)+(\alpha+\lambda)+(\beta+\lambda) = 2(\alpha+\beta+\lambda) = 2a+4b = 3b+(2a+b)$$

Hence from now on we assume $2a + b \geq 0$ in which case the result follows by the following chain of equalities

$$
\begin{aligned}
&\#\Big\{(\alpha,\beta,\lambda)\in\mathbb{N}^3 \mid \alpha+\beta+\lambda=a+2b \text{ and } b\le\alpha+\beta,\alpha+\lambda,\beta+\lambda\Big\} = \\
&\#\Big\{(\alpha,\beta,\lambda)\in\mathbb{N}^3 \mid \alpha+\beta+\lambda=a+2b \text{ and } a+b\ge\lambda,\beta,\alpha\Big\} = \\
&\#\Big\{(\alpha,\beta,\lambda)\in\mathbb{N}^3 \mid \alpha+\beta+\lambda=a+2b\} \\
&-3\cdot\#\Big\{(\alpha,\beta,\lambda)\in\mathbb{N}^3 \mid \alpha+\beta+\lambda=a+2b \text{ and } \alpha\ge a+b+1\} \\
&+3\cdot\#\Big\{(\alpha,\beta,\lambda)\in\mathbb{N}^3 \mid \alpha+\beta+\lambda=a+2b \text{ and } \alpha,\beta\ge a+b+1\} \\
&-\#\Big\{(\alpha,\beta,\lambda)\in\mathbb{N}^3 \mid \alpha+\beta+\lambda=a+2b \text{ and } \alpha,\beta,\lambda\ge a+b+1\} = \\
&\#\Big\{(\alpha,\beta,\lambda)\in\mathbb{N}^3 \mid \alpha+\beta+\lambda=a+2b\} \\
&-3\cdot\#\Big\{(\alpha,\beta,\lambda)\in\mathbb{N}^3 \mid \alpha+\beta+\lambda=b-1\} \\
&+3\cdot\#\Big\{(\alpha,\beta,\lambda)\in\mathbb{N}^3 \mid \alpha+\beta+\lambda=-a-2\} \\
&-\#\Big\{(\alpha,\beta,\lambda)\in\mathbb{N}^3 \mid \alpha+\beta+\lambda=-2a-b-3\} = \\
&\frac{(a+2b+1)(a+2b+2)}{2}-3\frac{b(b+1)}{2}+3\frac{(-a-1)(-a)}{2}-0 = \\
&\frac{(2a+b+1)(2a+b+2)}{2}
\end{aligned}
\tag{1.126}
$$

where the third equality is given by (1.121) and the fourth equality follows from (1.123) and the assumption $2a+b\ge 0$. Moreover these numbers are visualized in Figure 1.4. Here each green dot is also yellow and each yellow dot is also blue. As was mentioned above, there are no red dots and the goal is to count the number of blue dots that are not yellow (or green).

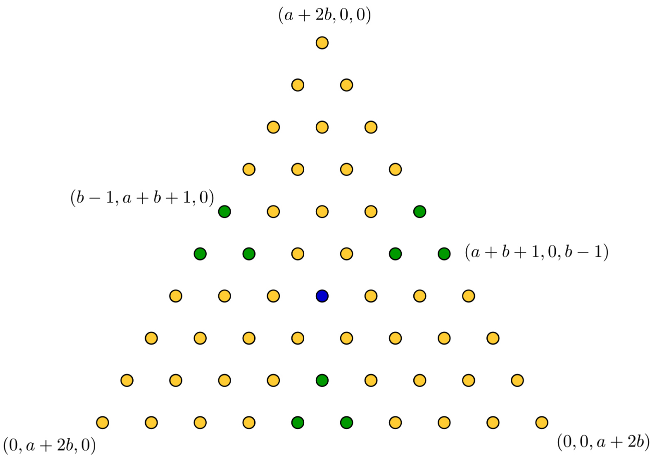

Figure 1.4: Visualizing formula (1.126)

Chapter 2

Some generalizations of Preprojective Algebras and their properties

Generalization is a natural human mental process, and many generalizations are true, in average. What often does promote evil behavior is the lazy, nasty habit of believing that generalizations have anything at all to do with individuals.

David Brin (Science Fiction author)(Glory Season 1993)

This chapter is based upon results by the author and Louis de Thanhoffer de Völcsey in [dTdVP17].

2.1 Introduction

Preprojective algebras were introduced almost 4 decades ago by Gelfand and Ponomarev in [GP79]. Since then these algebras have been studied extensively (e.g. [BGL87, DR80, Rin98b, Rin98a]), for example it is well known that their properties are very different whether the associated quiver is of Dynkin type or not. Recently deformations of such algebras have been studied in for example [BES11, CBH98]. The goal of this chapter is to look at generalizations of preprojective algebras in stead of deformations (see Remark 2.1.6 for more details on the difference in approach). As such we will construct a well behaved family of algebras which in the smooth case produces the classical preprojective algebra at the geometric generic fibre.

Our construction starts from the notion of a relative Frobenius pair of commutative rings S/R. To such a pair, we associate an $\mathbb{N}$-graded R-algebra $\Pi_R(S)$ which has a simple description and coincides with the preprojective algebra of a quiver with a single central node and several outgoing edges in the split case. If the rank of S over R is 4 and R is noetherian, we prove that $\Pi_R(S)$ is itself noetherian and finite over its center and that each $\Pi_R(S)_d$ is finitely generated projective. We also prove that $\Pi_R(S)$ is of finite global dimension if R and S are regular.

Although these algebras are interesting from a purely algebraic point of view, our actual motivation for studying these algebras comes from geometry: in Chapter 3 we will construct a class of noncommutative $\mathbb{P}^1$-bundles which can be described locally using the generalized preprojective algebras from this chapter.

2.1.1 Definitions

For the purpose of this section, we consider pairs of commutative rings R, S equipped with a map $R \to S$. We often refer to such a pair as S/R. Moreover we will always assume R is noetherian, although some of the results also hold in higher generality.

Definition 2.1.1. We say that S/R is *relative Frobenius* of rank n if:

- S is a free R-module of rank n.
- $\mathrm{Hom}_R(S, R)$ is isomorphic to S as S-module.

Remark 2.1.2. *i) It is clear that if R is a field, a relative Frobenius pair coincides with a finite dimensional Frobenius algebra in the classical sense.*

ii) Let $e_1, \ldots, e_n$ be any basis for S as an R-module. Then the second condition is equivalent to the existence of a $\lambda \in \mathrm{Hom}_R(S, R)$ such that the R-matrix $(\lambda(e_i e_j))_{i,j}$ is invertible.

iii) We may equally well assume that S/R is projective of rank n. However all results we prove may be reduced to the free case by suitably localizing R.

We shall need the following notation: for a relative Frobenius pair S/R, denote $M := {}_R S_S$. This R-S-bimodule can be viewed as an $R \oplus S$ bimodule by letting the R-component act on the left and the S-component on the right, the other actions being trivial. I.e.

$$(r, s) \cdot m = rm \text{ and } m \cdot (r, s) = ms$$

Similarly, we let $N := {}_S S_R$ and view it as an $R \oplus S$-bimodule by only letting the S-component act on the left and the R-component act on the right, the other actions again begin trivial.

We now define

$$\mathrm{T}(R, S) := \mathrm{T}_{R\oplus S}(M \oplus N) \tag{2.1}$$

Note that by construction, we have $M \otimes_{R\oplus S} M = N \otimes_{R\oplus S} N = 0$, hence

$$\mathrm{T}(R, S)_2 = (M_{R\oplus S} N) \oplus (N \otimes_{R\oplus S} M) = ({}_R S \otimes_S S_R) \oplus ({}_S S \otimes_R S_S)$$

The algebra we are interested in will be a quotient of $\mathrm{T}(R,S)$ as follows: let λ be a generator of $\mathrm{Hom}_R(S,R)$ as an S-module (this λ exists by the Frobenius condition we imposed in Definition 2.1.1). The R-bilinear form $\langle a,b\rangle := \lambda(ab)$ is nondegenerate and hence we can find dual R-bases $(e_i)_i$, $(f_j)_j$ satisfying

$$\lambda(e_i f_j) = \delta_{ij}$$

Definition 2.1.3. For a relative Frobenius pair, the *generalized preprojective algebra* $\Pi_R(S)$ is given by

$$\mathrm{T}(R,S)/I$$

where the ideal I is generated by relations in degree 2 given by

$$1 \otimes 1 \in {}_R S \otimes_S S_R$$

$$\sum_i e_i \otimes f_i \in {}_S S \otimes_R S_S$$

Remark 2.1.4. *The above construction is independent of choice of generator and dual basis. To see this, note that the isomorphism* $\mathrm{Hom}_R(S,R) \cong S$ *can be viewed as an element of* $\mathrm{Hom}_S(\mathrm{Hom}_R(S,R),S)$.

Under the identification $\mathrm{Hom}_R(\mathrm{Hom}_R(S,R),S) = {}_S S \otimes_R S_S$*, this elements corresponds to* $\sum_i e_i \otimes f_i \in {}_S S \otimes_R S_S$.

The name generalized preprojective algebra is motivated by the following:

Lemma 2.1.5. *Let S be the ring $R^{\oplus n}$. Then $\Pi_R(S)$ is isomorphic to the preprojective algebra over R associated to the quiver with one central vertex and n outgoing arrows.*

Proof. Let $e_1,\dots,e_n$ be the set of complete orthogonal idempotents in S and write $x_1,\dots,x_n$ (respectively $y_1,\dots,y_n$) $\in \Pi_R(S)_1$ for the corresponding elements in the bimodules N (respectively M). We can describe the tensor algebra $\mathrm{T}(R,S)$ as the free algebra $F := R\langle e_1,\dots,e_n,x_1,\dots,x_n,y_1,\dots,y_n\rangle$ subject to the relations

1. $e_ie_j = \delta_{ij}e_i$.

2. $e_ix_j = \delta_{ij}x_i$ and $y_ie_j = \delta_{ij}y_i$

3. $x_ie_j = e_iy_j = 0$

4. $x_ix_j = y_iy_j = 0$

The first relation defining $\Pi_R(S)$ is given by $1 \otimes 1 \in M \otimes_S N$. The first 1 is given by $1 = \sum x_i$ whereas the second $1 = \sum y_i$, we obtain

5. $y_1x_1 + \ldots + y_nx_n = 0$

To compute the second relation, we note that

$$\lambda : S \longrightarrow R : \sum_{i=1}^{n} r_i e_i \mapsto \sum_i r_i$$

is a generator of $\mathrm{Hom}_R(S, R)$ as an S-module and hence $(e_i)_i$ is a basis, self-dual for the associated form $\langle -, - \rangle$ introduced in the discussion preceding Definition 2.1.3. The relation on ${}_S S \otimes_R S_S$ now becomes

6. $x_1 y_1 + \ldots + x_n y_n = 0$

It now remains to show that F subject to the above 6 relations is isomorphic to the preprojective algebra of the quiver Q:

$$\begin{tikzcd}
\bullet & \bullet & \bullet \\
\bullet & \bullet \arrow[ul, "a_n"'] \arrow[u, "a_1"] \arrow[ur, "a_2"] \arrow[r, "a_3"'] \arrow[l, "a_{n-1}"] \arrow[dl] \arrow[dr] & \bullet \\
\bullet & \cdots & \bullet
\end{tikzcd}$$

We let $\overline{Q}$ denote the double quiver of Q and consider the map $F \to R\overline{Q}$ defined by

- sending e_i to the outer node n_i
- sending y_i to the arrow a_i and x_i to the formal inverse a_i^*

The first 4 relations now precisely describe the multiplication in the path algebra of $\overline{Q}$ and the relations (5) an (6) precisely map to the two relations defining a preprojective algebra $\sum a_i a_i^* = 0 = \sum a_i^* a_i$ □

Remark 2.1.6. *In [CBH98], Crawley-Boevey and Holland define so-called deformed preprojective algebras $\Pi^\lambda(Q)$ over a field k, where Q is a quiver with n vertices and $\lambda \in k^{\oplus n}$ is a parameter. In this way they obtain a $k[\lambda]$-family which at the special point $\lambda = 0$ produces the classical preprojective algebra on Q. Our construction on the other hand produces an R-family of algebras of the form $\Pi_R(S)$ which in the smooth case produces the classical preprojective algebra at the geometric generic fibre. Hence we consider generalizations of preprojective algebras instead of deformations. This allows for certain special fibres to have infinite global dimension (if S has infinite global dimension) but produces a nice family.*

2.1.2 Statement of the Results

Throughout this chapter we assume S/R is relative Frobenius of rank 4 (with the exception of Lemmas 2.3.1 and 2.3.2 which are stated in higher generality). Moreover R will always be a noetherian ring. We prove three basic properties of the algebra $\Pi_R(S)$ (under the above assumptions). Section 2.3 is dedicated to the following result:

Theorem (see Theorem 2.3.9). *The R-module $\Pi_R(S)_d$ is projective of rank*

$$\begin{cases} 5(d+1) & \text{if } d \text{ is even} \\ 4(d+1) & \text{if } d \text{ is odd} \end{cases}$$

In §2.4 we investigate the center of $\Pi_R(S)$. To this end, we use the short hand notation $\mathrm{Z}_d(R,S) := \big(\mathrm{Z}(\Pi_R(S))\big)_d$ and prove

Theorem (see Theorem 2.4.1, Theorem 2.4.2 and Theorem 2.4.3). *$\mathrm{Z}_d(R,S)$ is a split submodule of $\Pi_R(S)_d$ for each $d \in \mathbb{N}$, it is projective and its rank is given by:*

$$\mathrm{rk}(\mathrm{Z}_d(R,S)) = \begin{cases} \frac{d}{4}+1 & \text{if } d \equiv 0 \ (mod\ 4) \\ \frac{d-2}{4} & \text{if } d \equiv 2 (mod\ 4) \\ 0 & \text{else} \end{cases}$$

We deduce from it that $\mathrm{Z}(R,S)$ is compatible with base change.

§2.5 is dedicated to constructing a map

$$\sigma_{R,S} : R[\mathrm{Z}_4(R,S)]^{\oplus n} \longrightarrow \Pi_R(S)$$

and we prove

Theorem (see Theorem 2.5.2 and Theorem 2.5.1). *$\sigma_{R,S}$ is surjective, in particular $\Pi_R(S)$ is noetherian and finite over its center.*

Finally §2.6 covers the global dimension of $\Pi_R(S)$, giving as main result:

Theorem (see Theorem 2.6.1). *If R and S have finite global dimension, then so does $\Pi_R(S)$. We have the following explicit upper bound:*

$$\mathrm{gr\,.\,gl.\,dim}(\Pi_R(S)) \leq max(\mathrm{gl.\,dim}(R), \mathrm{gl.\,dim}(S)) + 2$$

Before proving the main theorems of this chapter we explicitly describe the Frobenius algebras of rank 4 over an algebraically closed field and show that they are related by so-called Frobenius deformations (see Lemmas 2.2.1 and 2.2.4). The above theorems are then all proven using a common technique, namely we first prove them in case R is an algebraically closed field and S is extremal in the deformation graph (2.3), which yields two specific cases. Then we extend the results step by step, increasing the generality of R as follows (with references to the applied lemmas):

$$\begin{tikzcd}
\text{2 specific cases} \arrow[r, "2.2.4"] & \text{alg. closed field} \arrow[r] & \text{field} \arrow[rr, "{[Gro71, IV.4.4]}"] & & \begin{array}{c}\text{noetherian}\\ \text{local domain}\end{array} \arrow[d] \\
& & & & \begin{array}{c}\text{noetherian}\\ \text{domain}\end{array} \arrow[d, "2.3.8"] \\
\text{general ring} & \text{local ring} \arrow[l] & & & \text{local ring with } \overline{k} = k \arrow[lll, "2.3.7"']
\end{tikzcd} \tag{2.2}$$

2.2 Frobenius deformations

Throughout this chapter we will only consider relative Frobenius pairs S/R of rank 4. If R is a field, this implies that S is a Frobenius algebra of dimension 4. It is an easy exercise to describe all such algebras in the case of an algebraically closed field:

Lemma 2.2.1. *Let k be an algebraically closed field and F a commutative Frobenius algebra of dimension 4 over k. Then F is isomorphic to one of the following algebras:*

$$\begin{cases} k \oplus k \oplus k \oplus k \\ k[t]/(t^2) \oplus k \oplus k \\ k[s]/(s^2) \oplus k[t]/(t^2) \\ k[t]/(t^3) \oplus k \\ k[t]/(t^4) \\ k[s,t]/(s^2,t^2) \end{cases}$$

Proof. First recall that

1. a direct sum of Frobenius algebras is itself Frobenius.

2. a finite dimensional commutative local k-algebra is Frobenius if and only if it has a unique minimal ideal.

It follows immediately that $k[t]/(t^n)$ is Frobenius (of dimension n) over k as it has a unique minimal ideal (t^{n-1}) and $k[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2)$ is also Frobenius (of dimension 2^n) with unique minimal ideal $(x_1 \cdot \dots \cdot x_n)$. Thus the algebras in the above list are certainly Frobenius.

Now let F be Frobenius of dimension 4. Since F is Artinian, the structure theorem for Artinian rings [AM69, Theorem 8.7] states that F must (uniquely) decompose as a direct sum of local, Artinian k-algebras:

$$F \cong F_1 \oplus \ldots \oplus F_n$$

We can now use the classification of local k-algebras of small rank in [Poo08, Table 1].

If $n = 4$, then clearly $F = k \oplus k \oplus k \oplus k$.

If $n = 3$, then $F \cong A_1 \oplus k \oplus k$ where $\dim_k(A_1) = 2$, hence $A_1 \cong k[t]/(t^2)$ which is Frobenius.

If $n = 2$, then either F splits as a sum of 2-dimensional local k-algebras, in which case we obtain $F \cong k[s]/(s^2) \oplus k[t]/(t^2)$ or $F = A_1 \oplus k$ where $\dim_k(A_1) = 3$. This again yields 2 possibilities: either $A_1 \cong k[t]/(t^3)$, which is Frobenius, or $A_1 \cong k[s,t]/(s,t)^2$. The latter however cannot be Frobenius as it is not self-injective (the morphism $A_1 t \to A_1 : t \mapsto s$ cannot be lifted to $A_1 \to A_1$).

Finally, assume $n = 1$. In this case F is a local k-algebra of dimension 4 and by [Poo08] takes one of the five following forms:

$$\begin{cases} k[t]/(t^4) \\ k[s,t]/(s^2,t^2) \\ k[s,t]/(s^2,st,t^3) \\ k[s,t,u]/(s,t,u)^2 \\ k[s,t]/(s^2+t^2,st) \quad \text{(if char}(k)\text{=2)} \end{cases}$$

The first two algebras are Frobenius whereas the other three are not as they are not self-injective by a similar argument as above. □

The 6 Frobenius algebras listed in the above lemma are related to each other by a notion closely related to deformations. For this purpose, we introduce the following ad hoc notion:

Definition 2.2.2. Let F and G be Frobenius algebras over a field k.
A *Frobenius deformation* of F to G is a $k[[u]]$-algebra D such that D is relatively Frobenius over $k[[u]]$ and

i) $D/uD \cong F$ as a k-algebra

ii) $D_{(u)} \cong G \otimes_k k((u))$ as a $k((u))$-algebra

we write $F \overset{\text{def}}{\dashrightarrow} G$

Remark 2.2.3. *Instead of requiring that $D/k[[u]]$ is relative Frobenius we may equivalently require that D is free over $k[[u]]$ with rank equal to the dimension of F. The second condition in Definition 2.1.1 that $\mathrm{Hom}_{k[[u]]}(D, k[[u]])$ be isomorphic to D as a D-module is immediate by the corresponding condition on F/k. Indeed, the freeness of D together with (1) implies that $F \otimes_k k[[u] \cong D$.*

Lemma 2.2.4. *There is a diagram of Frobenius deformations*

$$\begin{tikzcd}
 & & k[t]/(t^2) \oplus k[s]/(s^2) \arrow[dr, dashed, "\mathrm{def}", "4"'] & & \\
k[s,t]/(s^2,t^2) \arrow[r, dashed, "\mathrm{def}", "1"'] & k[t]/(t^4) \arrow[ur, dashed, "\mathrm{def}", "2"'] \arrow[dr, dashed, "\mathrm{def}", "3"'] & & k[t]/(t^2) \oplus k \oplus k \arrow[r, dashed, "\mathrm{def}", "6"'] & k^{\oplus 4} \\
 & & k[t]/(t^3) \oplus k \arrow[ur, dashed, "\mathrm{def}", "5"'] & &
\end{tikzcd} \tag{2.3}$$

Proof. We first describe $F := k[s,t]/(s^2,t^2) \overset{\mathrm{def}}{\dashrightarrow} G := k[t]/(t^4)$. Let $R := k[[u]]$, $K := k((u))$ and define

$$D := R[s,t]/(us - t^2, s^2, t^4)$$

We claim that D defines a deformation from F to G. It is clear that $D/uD \cong F$ as a k-algebra and the map

$$D \longrightarrow K[t]/(t^4) : u \mapsto u, s \mapsto t^2/u, t \mapsto t$$

factors through an isomorphism

$$D_{(u)} \longrightarrow K[t]/(t^4) = G \otimes_k K$$

Hence by the above remark it suffices to check that D is a free R-module of rank 4. This is obviously the case with $e_1 = 1, e_2 = s, e_3 = t, e_4 = st$ providing an R-basis for D.

The other cases are similar. We first use the Chinese remainder theorem to find an alternate presentation for F of the forms $k[t]/(f(t))$. Then for each deformation $F \overset{\mathrm{def}}{\dashrightarrow} G$, we try to find an alternate presentation for $G \otimes_k K$ (again using the Chinese remainder theorem) of the form $K[t]/(g(t))$ in such a way that $g(t)|_{u=0} = f(t)$. We then exhibit an R-algebra $D := R[t]/(g(t))$. We leave the reader to check that in each of our choices, $(1, t, t^2, t^3)$ defines an R-basis.

$\overset{\text{def}}{\dashrightarrow}$	2	3	4	5	6*(char$(k) \neq 2$)
$g(t)$	$t^2(t-u)^2$	$t^3(t-u)$	$(t-1)^2 t(t-u)$	$t^2(t-1)(t-u)$	$(t^2-u^2)(t^2-1)$

* In case k has characteristic 2, one has to choose $D = R[t]/(t(t-u)) \oplus R^{\oplus 2}$ for the 6th deformation. In this case $(1,0,0), (t,0,0), (0,1,0), (0,0,1)$ provides an R-basis for D. □

2.3 Computing $\mathrm{rk}(\Pi_R(S)_d)$

The construction of $\Pi_R(S)$ (recall Definition 2.1.3) is compatible with base change in the following way:

Lemma 2.3.1 (Base Change for $\Pi_R(S)$)**.** *Let S/R be relative Frobenius of finite rank and $R \to R'$ a morphism of rings. Then*

i) $(R' \otimes_R S)/R'$ is relative Frobenius of rank n

ii) there is a canonical isomorphism

$$R' \otimes_R \Pi_R(S) \cong \Pi_{R'}(R' \otimes_R S)$$

Proof. Assume that S/R is relative Frobenius. We can pick an R-basis $\{e_1, \dots, e_n\}$ for S and a generator λ for the S-module $\mathrm{Hom}_R(S, R)$. It is then easy to see that $\{1 \otimes e_1, \ \dots, \ 1 \otimes e_n\}$ is an R'-basis for $R' \otimes_R S$ and that $1 \otimes \lambda$ is a generator for the $S' \otimes_R S$-module $\mathrm{Hom}_{R'}(R' \otimes_R S, R')$, proving the first point. With this data we can thus construct $\Pi_{R'}(R' \otimes_R S)$. Moreover,

$$R' \otimes_R ({}_R S_S \oplus {}_S S_R) \cong {}_{R'}(R' \otimes_R S)_{R' \otimes_R S} \oplus {}_{R' \otimes_R S}(R' \otimes_R S)_{R'}$$

as an $R' - R' \otimes_R S$-bimodule, and we obtain a canonical isomorphism

$$R' \otimes_R \mathrm{T}(R, S) \cong T(R', R' \otimes_R S)$$

which by our choice of basis preserves the relations, inducing an isomorphism

$$R' \otimes_R \Pi_R(S) \cong \Pi_{R'}(R' \otimes_R S)$$ □

To prove that the R-modules $\Pi_R(S)_d$ are projective and to compute their ranks, following the method of proof described by diagram (2.2), we first treat the case where R is an algebraically closed field. We have the following lemma relating these vector spaces under deformation:

Lemma 2.3.2. *Let F and G be Frobenius algebras over k and let $F \overset{\text{def}}{\dashrightarrow} G$ be a Frobenius deformation. Then for all d, we have*

$$\dim_k(\Pi_k(F)_d) \geq \dim_k(\Pi_k(G)_d)$$

Proof. Let $R = k[[u]]$ and $K = k((u))$.
Let $m = \dim_k(\Pi_k(F)_d)$. Assume that D is the R-algebra deforming F to G. Then since

$$\Pi_k(F) = \Pi_k(k \otimes_R D) = k \otimes_R \Pi_R(D)$$

by Lemma 2.3.1, Nakayama's lemma implies that a k-basis of length m for $\Pi_k(F)_d$ lifts to a set of generators for $\Pi_R(D)_d$. Moreover, as

$$K \otimes_k \Pi_k(G) = \Pi_K(K \otimes_k G) = \Pi_K(K \otimes_R D) = K \otimes_R (\Pi_R(D))$$

this set of generators contains a K-basis for $K \otimes \Pi_k(G)$. It follows that

$$\dim_K(K \otimes_k (\Pi_k(G)_d) = dim_k(\Pi_k(G)_d) \leq m$$ □

From now on we will only focus on the rank 4 case for the rest of the chapter. I.e. when using the notation S/R, we will always assume this is a relative Frobenius pair of rank 4. Similarly all upcoming Frobenius algebras F or G will have dimension 4 over k. We will now prove that under this assumption the above inequality is actually an equality. We first compute the ranks in two explicit cases:

Lemma 2.3.3. *We have*

$$\dim_k\left(\Pi_k\left(\frac{k[s,t]}{(s^2,t^2)}\right)_d\right) \leq \begin{cases} 5(d+1) & \text{if } d \text{ is even} \\ 4(d+1) & \text{if } d \text{ is odd} \end{cases}$$

Proof. This is proven in §2.7. □

Lemma 2.3.4. *Let k be an algebraically closed field, then*

$$\dim_k\left(\Pi_k(k^{\oplus 4})_d\right) = \begin{cases} 5(d+1) & \text{if } d \text{ is even} \\ 4(d+1) & \text{if } d \text{ is odd} \end{cases}$$

Proof. By Lemma 2.1.5, $\Pi_k(S)$ is the preprojective algebra over k associated to the extended Dynkin quiver of $Q = \widetilde{D_4}$.

Let $\overline{Q}$ denote the double quiver, let 0 denote the central vertex and 1, 2, 3, 4 the outer vertices. Then for each $d \in \mathbb{N}$ we consider the matrix $W_d \in \mathbb{N}^{5\times 5}$ where $(W_d)_{ij}$ is the number of paths of length d in $\overline{Q}$ starting at vertex i and ending at vertex j, modulo relations. Finally write $W(t) = \sum_{d=0}^{\infty} W_d t^d \in \mathbb{N}^{5\times 5}[[t]]$. Then by [EE07, Proposition 3.2.1] we have

$$W(t) = \frac{1}{1 - t\cdot C + t^2}$$

Where C is the adjacency matrix of $\overline{Q}$, i.e.

$$W(t) = \left(1 - t \cdot \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} + t^2 \right)^{-1}$$

$$= \frac{1}{(1-t^2)^2(1+t^2)} \cdot \begin{pmatrix} (1+t^2)^2 & t(1+t^2) & t(1+t^2) & t(1+t^2) & t(1+t^2) \\ t(1+t^2) & 1-t^2+t^4 & t^2 & t^2 & t^2 \\ t(1+t^2) & t^2 & 1-t^2+t^4 & t^2 & t^2 \\ t(1+t^2) & t^2 & t^2 & 1-t^2+t^4 & t^2 \\ t(1+t^2) & t^2 & t^2 & t^2 & 1-t^2+t^4 \end{pmatrix}$$

This gives the desired result as the Hilbert series of $\Pi_k(S)$ now becomes

$$\begin{aligned} h_{\Pi_k(S)}(t) &= \sum_{d=0}^{\infty} \left(\sum_{i,j=0}^{4} (W_d)_{i,j} \right) t^d \\ &= \sum_{i,j=0}^{4} \sum_{d=0}^{\infty} (W_d)_{i,j} t^d \\ &= \frac{(1+t^2)^2 + 8t(1+t^2) + 4(1-t^2+t^4) + 12t^2}{(1-t^2)^2(1+t^2)} \\ &= \frac{5+8t+5t^2}{(1-t^2)^2} \\ &= (5+8t+5t^2) \sum_{l=0}^{\infty} (l+1) t^{2l} \\ &= \sum_{l=0}^{\infty} (5l + 5(l+1)) t^{2l} + 8(l+1) t^{2l+1} \\ &= \sum_{l=0}^{\infty} 5(2l+1) t^{2l} + 4((2l+1)+1) t^{2l+1} \end{aligned}$$

□

Remark 2.3.5. *It is also possible to write down an explicit basis for $\Pi_k(k^{\oplus 4})_d$ and compute the dimensions in a similar way as is done in §2.7.1. Alternatively one could also use noncommutative Groebner bases.*

Corollary 2.3.6. *Let k be a field and F a Frobenius algebra (of rank 4) over k then:*

$$\dim_k (\Pi_k(F)_d) = \begin{cases} 5(d+1) & \textit{if } d \textit{ is even} \\ 4(d+1) & \textit{if } d \textit{ is odd} \end{cases}$$

Proof. By Lemma 2.3.1 we can reduce to the case where k is algebraically closed. The statement then follows as a combination of Lemmas 2.2.1, 2.2.4, 2.3.2, 2.3.3 and 2.3.4 □

To extend the result from fields to general rings we will need the following two lemmas. They essentially show that locally every relative Frobenius pair can be obtained through base change (following Lemma 2.3.1) from a relative Frobenius pair where the ground ring is a polynomial ring over the integers.

Lemma 2.3.7. *Let R be a local ring with residue field k with algebraic closure $\overline{k}$. Then there is a faithfully flat morphism $R \to \overline{R}$ where $\overline{R}$ is a local ring with residue field $\overline{k}$.*

Proof. This is an immediate application of [GD63, 10.3.1] □

Lemma 2.3.8. *Let R be a local ring with an algebraically closed residue field k. Let S/R be relative Frobenius of rank 4. Then there exists a domain $\tilde{R}$, together with a morphism $\tilde{R} \to R$ and a ring $\tilde{S}$ with $\tilde{S}/\tilde{R}$ relative Frobenius of rank 4 such that $\tilde{S} \otimes_{\tilde{R}} R \cong S$. Moreover $\tilde{R}$ can be chosen to be chosen of the form $\mathbb{Z}[x_1, \dots, x_m]_f$, the localization of a polynomial ring over $\mathbb{Z}$ at some non-zero element f. In particular we can always assume $\tilde{R}$ is noetherian.*

Proof. We prove the theorem in a specific case and quickly sketch the other cases, leaving some details to the reader. By Lemmas 2.3.1 and 2.2.4, $S \otimes_R k$ is one of the 6 Frobenius algebras listed in Lemma 2.2.1. Assume $S \otimes_R k = k[s,t]/(s^2, t^2)$ and let $\tilde{s}, \tilde{t} \in S$ be lifts of s and t. Since $(1, s, t, st)$ is a k-basis for S_k. Nakayama's Lemma implies $(1, \tilde{s}, \tilde{t}, \tilde{s}\tilde{t})$ forms a set of R-generators for S. In particular we can write:

$$\begin{aligned} \tilde{s}^2 &= a_1 + b_1\tilde{s} + c_1\tilde{t} + d_1\tilde{s}\tilde{t} \\ \tilde{t}^2 &= a_2 + b_2\tilde{s} + c_2\tilde{t} + d_2\tilde{s}\tilde{t} \end{aligned}$$

where $a_1, \dots, d_2$ all lie in the maximal ideal of R (because $s^2 = t^2 = 0$ in $S \otimes k$). We thus have a canonical morphism

$$\pi : R[\tilde{s}, \tilde{t}]/(a_1 + b_1\tilde{s} + c_1\tilde{t} + d_1\tilde{s}\tilde{t} - \tilde{s}^2, a_2 + b_2\tilde{s} + c_2\tilde{t} + d_2\tilde{s}\tilde{t} - \tilde{t}^2) \longrightarrow S$$

such that $\pi \otimes_R k$ is the identity morphism. It follows that π is surjective, moreover since S is free over R, we have $0 = \ker(\pi \otimes_R k) = \ker(\pi) \otimes_R k$ and $\ker(\pi) = 0$ using Nakayama's Lemma. π is thus an isomorphism.

There is a canonical morphism

$$A := \mathbb{Z}[a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2] \longrightarrow R$$

Let $f = 1 - d_1 d_2$ and denote $\tilde{R} = A_f$. Then as the image of f in R is invertible (because d_1, d_2 lie in the maximal ideal of R), the above morphism factors through a morphism $\tilde{R} \to R$. Finally set

$$\tilde{S} = \tilde{R}[\tilde{s}, \tilde{t}]/(a_1 + b_1\tilde{s} + c_1\tilde{t} + d_1\tilde{s}\tilde{t} - \tilde{s}^2, a_2 + b_2\tilde{s} + c_2\tilde{t} + d_2\tilde{s}\tilde{t} - \tilde{t}^2)$$

By construction we have

$$\tilde{S} \otimes_{\tilde{R}} R \cong R[\tilde{s}, \tilde{t}]/(a_1 + b_1\tilde{s} + c_1\tilde{t} + d_1\tilde{s}\tilde{t} - \tilde{s}^2, a_2 + b_2\tilde{s} + c_2\tilde{t} + d_2\tilde{s}\tilde{t} - \tilde{t}^2) \overset{\pi}{\cong} S$$

It hence suffice to prove $\tilde{S}/\tilde{R}$ is relative Frobenius of rank 4. For this note that $(e_i)_{i=1,\dots,4} := (1, \tilde{s}, \tilde{t}, \tilde{s}\tilde{t})$ is an $\tilde{R}$-basis for $\tilde{S}$. Moreover, if we let $\lambda \in \mathrm{Hom}_{\tilde{R}}(\tilde{S}, \tilde{R})$ denote the projection onto the component $\tilde{R}\tilde{s}\tilde{t}$, the matrix of $\lambda(e_i.e_j)$ is of the form

$$\Theta = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & d_1 & 1 & * \\ 0 & 1 & d_2 & * \\ 1 & * & * & * \end{bmatrix}$$

Hence Θ has determinant $1 - d_1 d_2$, which by construction is invertible in $\tilde{R}$, proving that $\tilde{S}$ is indeed Frobenius of rank 4 over $\tilde{R}$ by Remark 2.1.2.

In the 5 other cases from Lemma 2.2.1 we have $S \otimes k = k[t]/(t^4 + at^3 + bt^2 + ct + d)$ for some $a, b, c, d \in k$ and we can choose $\tilde{R}, \tilde{S}$ of the form $\tilde{R} := \mathbb{Z}[\alpha, \beta, \gamma, \delta]$ and $\tilde{S} := \tilde{R}[t]/(t^4 + \alpha t^3 + \beta t^2 + \gamma t + \delta)$. For each choice of $\alpha, \beta, \gamma, \delta$ we have that $\tilde{S}/\tilde{R}$ is relative Frobenius of rank 4, because the corresponding matrix Θ will have determinant exactly 1. We leave the details to the reader. □

We can now prove the main theorem of this section:

Theorem 2.3.9. *$\Pi_R(S)_d$ is projective of rank* $\begin{cases} 5(d+1) & \textit{if } d \textit{ is even} \\ 4(d+1) & \textit{if } d \textit{ is odd} \end{cases}$

Proof. First let R be a noetherian local domain with residue field k and field of fractions K. By Corollary 2.3.6 and Lemma 2.3.1 we have for each degree d:

$$\begin{aligned} \dim_K(K \otimes_R \Pi_R(S)_d) &= \dim_K(\Pi_K(K \otimes_R S)_d) \\ &= \dim_k(\Pi_k(k \otimes_R S)_d) \\ &= \dim_k(k \otimes_R \Pi_R(S)_d) \end{aligned}$$

such that by [Gro71, Corollaire IV.4.4] $\Pi_R(S)$ is free of the stated ranks.

Next, let R be any noetherian domain. Then for each $\mathfrak{p} \in \mathrm{Spec}(R)$,

$$R_{\mathfrak{p}} \otimes_R \Pi_R(S) \cong \Pi_{R_{\mathfrak{p}}}(R_{\mathfrak{p}} \otimes S)$$

is a generalized preprojective algebra over the noetherian local domain $R_{\mathfrak{p}}$ and hence in each degree is a free module of the stated rank. As these ranks do not depend on the choice of $\mathfrak{p}$, Serre's theorem (see for example [Ser55]) now implies that $\Pi_R(S)_d$ is projective of the stated rank.

Next, let R be a (possibly non-reduced) local ring with algebraically closed residue field. Then by Lemma 2.3.8 there is a noetherian domain $\tilde{R}$, a morphism $\tilde{R} \to R$ and a ring $\tilde{S}$ such that $\tilde{S}/\tilde{R}$ is relative Frobenius of rank 4 and $S \cong \tilde{S} \otimes_{\tilde{R}} R$. By the above $\Pi_{\tilde{R}}(\tilde{S})_d$ is a projective $\tilde{R}$-module of the given ranks and hence $\Pi_R(S)_d = \Pi_{\tilde{R}}(\tilde{S})_d \otimes R$ is a projective R-module of the above rank.

To extend the result to general local rings, we invoke Lemma 2.3.7 to find a faithfully flat morphism $R \to \overline{R}$. By the above $\Pi_{\overline{R}}(\overline{R} \otimes S)_d \cong \overline{R} \otimes \Pi_R(S)_d$ is a free $\overline{R}$-module of the stated rank. By the faithfully flatness of $R \to \overline{R}$, $\Pi_R(S)_d$ is itself a free R-module of the stated rank.

Finally we extend the statement from local rings to general commutative rings by again applying Serre's theorem [Ser55] . □

The following lemma is a slightly more technical variation of Theorem 2.3.9 which will be required in the final section of this chapter.

Lemma 2.3.10. *$(1_R \cdot \Pi_R(S))_d$ and $(1_S \cdot \Pi_R(S))_d$ are projective R-modules of ranks respectively*

$$\operatorname{rk}((1_R \cdot \Pi_R(S))_d = \begin{cases} d+1 & \text{if } d \text{ is even} \\ 2(d+1) & \text{if } d \text{ is odd} \end{cases}$$

and

$$\operatorname{rk}((1_S \cdot \Pi_R(S))_d = \begin{cases} 4(d+1) & \text{if } d \text{ is even} \\ 2(d+1) & \text{if } d \text{ is odd} \end{cases}$$

Proof. We can write $\Pi_R(S) = 1_R \cdot \Pi_R(S) \oplus 1_S \cdot \Pi_R(S)$. This immediately implies that both modules are projective by Theorem 2.3.9. Moreover, it is easy to see that this decomposition is preserved under both base change through a morphism $R \to R'$ and Frobenius deformations $F \overset{\text{def}}{\dashrightarrow} G$ in the obvious sense. From this, we can conclude, using an argument similar to the proof of Theorem 2.3.9 shows that it suffices to check the cases where $R = k$ and $S = k^{\oplus 4}$ or $S = k[s,t]/(s^2,t^2)$.

For the first case we notice that the values of the Hilbert series $h_{1_k \cdot \Pi_R(S)}(t)$ can be computed using the proof of Lemma 2.3.4 by adding the entries in the first column of $W(t)$, giving

$$\begin{aligned} h_{1_k \cdot \Pi_k(S)}(t) &= \frac{(1+t^2)^2 + 4 \cdot t(1+t^2)}{(1-t^2)^2(1+t^2)} \\ &= \frac{1+6t+t^2}{(1-t^2)^2} \\ &= (1+6t+t^2)\sum_{l=0}^{\infty}(l+1)t^{2l} \\ &= \sum_{l=0}^{\infty}(2l+1)t^{2l} + \sum_{l=0}^{\infty} 2((2l+1)+1)t^{2l+1} \end{aligned}$$

In a similar fashion, we find

$$h_{1_S \cdot \Pi_k(S)}(t) = \sum_{l=0}^{\infty} 4(2l+1)t^{2l} + \sum_{l=0}^{\infty} 2((2l+1)+1)t^{2l+1}$$

For the second case where we assume $S = k[s,t]/(s^2,t^2)$ this is a dreary calculation which follows the "Type I"-"Type II"-classification of the generators of $\Pi_k(S)$ found in §2.7. □

2.4 Base Change for $\mathrm{Z}(\Pi_R(S))$ and $\mathrm{rk}(\mathrm{Z}_d(R,S))$

In this section, we will prove some results describing the center of $\Pi_R(S)$. As above S will be relative Frobenius of rank 4 over a noetherian ring R. To ease notation, we will write $\mathrm{Z}_d(R,S)$ for the degree d-part of the center of $\Pi_R(S)$.

Theorem 2.4.1. $\mathrm{Z}_d(R,S)$ *is a split R-submodule of* $\Pi_R(S)_d$ *for each* $d \in \mathbb{N}$.

Theorem 2.4.2. *Let $R \to R'$ be a morphism of rings. Then the canonical morphism*

$$\mathrm{Z}(\Pi_R(S)) \otimes_R R' \longrightarrow \mathrm{Z}(\Pi_{R'}(S \otimes_R R'))$$

is an isomorphism.

Theorem 2.4.3. $\mathrm{Z}_d(R,S)$ *is a projective R-module of rank*

$$\begin{cases} \frac{d}{4}+1 & \textit{if } d \equiv 0 \ (\textit{mod } 4) \\ \frac{d-2}{4} & \textit{if } d \equiv 2 \ (\textit{mod } 4) \\ 0 & \textit{else} \end{cases}$$

The proofs of these theorems are heavily intertwined, we shall prove them according to the following diagram of implications:

$$\begin{gathered} \text{Theorem 2.4.3 when } R \text{ is a field} \\ \Downarrow \\ \text{Theorems 2.4.3 and 2.4.1 when } R \text{ is a noetherian local domain} \\ \Downarrow \\ \text{Theorem 2.4.1 for general } R \\ \Downarrow \\ \text{Theorem 2.4.2 for general } R \\ \Downarrow \\ \text{Theorem 2.4.3 for general } R \end{gathered}$$

In several of these steps we use the fact that in each degree the center $Z_d(R,S)$ can be obtained as kernel of a morphism between projective R-modules. For this recall from the discussion preceding Definition 2.1.3 that since $\Pi_R(S)_0 = R \oplus S$ and S is a free R-module of rank 4. there exist an R-basis $\{1_R := a_1^0 \ldots, a_5^0\}$ for $\Pi_R(S)_0$. Moreover, there exists an R-basis for $\Pi_R(S)_1$ of the form

$$(a_i^1)_{i=1,\ldots,8} := \{e_1, \ldots, e_4, f_1, \ldots, f_4\}$$

such that $\lambda(e_i f_j) = \delta_{ij}$ for a chosen generator λ of the S-module $\mathrm{Hom}_R(S,R)$. Now, since $\Pi_R(S)$ is generated in degrees 0 and 1, for each degree d there is a morphism

$$\phi_{R,S} : \Pi_R(S)_d \longrightarrow \Pi_R(S)_d^{\oplus 5} \oplus \Pi_R(S)_{d+1}^{\oplus 8} : x \mapsto \Big(\big([x, a_i^0]\big)_i, \big([x, a_j^1]\big)_j \Big) \tag{2.4}$$

whose kernel is precisely $\mathrm{Z}_d(R,S)$. I.e. there is a left-exact sequence

$$0 \longrightarrow \mathrm{Z}_d(R,S) \longrightarrow \Pi_R(S)_d \xrightarrow{\phi_{R,S}} \Pi_R(S)_d^{\oplus 5} \oplus \Pi_R(S)_{d+1}^{\oplus 8} \tag{2.5}$$

In particular we obtain the following special case of Theorem 2.4.2:

Lemma 2.4.4 (flat base change). *Let $R \to R'$ be a flat morphism of rings. Then the canonical map*

$$R' \otimes_R \mathrm{Z}_d(R,S) \longrightarrow \mathrm{Z}_d(R' \otimes_R S)$$

is an isomorphism for each $d \in \mathbb{N}$.

Proof. The construction of the morphism $\phi_{R,S}$ is compatible with base change by the proof of Lemma 2.3.1 and tensoring with flat modules preserves left exact sequences. Hence

$$R' \otimes \mathrm{Z}_d(R,S) = R' \otimes \ker(\phi_{R,S}) = \ker(R' \otimes \phi_{R,S}) = \ker(\phi_{R',R'\otimes S}) = \mathrm{Z}_d(R', R' \otimes S)$$

□

As stated in Theorem 2.4.2 we will show that in fact the above result holds for *arbitrary* morphisms. Following the technique of proof outlined in diagram (2.2) we shall first compute the dimension of $\mathrm{Z}_d(R,S)$ in two specific cases: In the case $S = k^{\oplus 4}$ we use the following lemma:

Lemma 2.4.5. *Let k be an algebraically closed field of characteristic different from 2 and $F = k^{\oplus 4}$, then $\Pi_k(F)$ is Morita equivalent to the skew group ring $k[x,y]\#BD_2$ where*

$$BD_2 = \langle a, b \mid a^4 = b^4 = 1, a^2 = b^2, ab = ba^3 \rangle$$

is the binary dihedral group of order 8 acting on $k[x,y]$ via

$$a \cdot x = ix, \;\; a \cdot y = -iy, \;\; b \cdot x = y, \;\; b \cdot y = x$$

Proof. Let Q be be the extended Dynkin quiver $\widetilde{D_4}$ and $\overline{Q}$ the double quiver. Then $\overline{Q}$ is the McKay-quiver of BD_2 acting on $k[x,y]$ through the rule described above. Now, by [BSW10, Corollary 4.2] (which was already announced in [RVdB89] or see [CBH98]) the (classical) preprojective algebra $\Pi_k(Q)$ is Morita equivalent to $k[x,y]\#BD_2$. The result now follows from Lemma 2.1.5. $\square$

Using this and the explicit presentation of generators and relations for the D_4-singularity (see for example [Rei16]), we obtain:

Lemma 2.4.6. *Let k be an algebraically closed field with* $\mathrm{char}(k) \neq 2$*, then there is an isomorphism of rings:*

$$\mathrm{Z}\left(\Pi_k(k^{\oplus 4})\right) \cong \frac{k[A,B,C]}{(C^2 - B(A^2 - 4B^2))}$$

Where A, B are homogeneous elements in degree 4 and C is a homogeneous element in degree 6.

Proof. By Lemma 2.4.5 and the fact that the center of a ring is invariant under Morita equivalence, we have

$$\mathrm{Z}\left(\Pi_k(k^{\oplus 4})\right) \cong \mathrm{Z}\left(k[x,y]\#BD_2\right) = k[x,y]^{BD_2}$$

Hence we need to find the invariants in $k[x,y]$ under the action of BD_2 where the generators a and b act on (x,y) through the rule

$$a \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } b \mapsto \begin{bmatrix} \xi_4 & 0 \\ 0 & \xi_4^3 \end{bmatrix}$$

where ξ_4 is a primitive 4th root of unity, i.e.

$$a \cdot x^m y^n = (-1)^n x^n y^m \quad \text{ and } \quad b \cdot x^m y^n = \xi_4^{m+3n} x^m y^n$$

Let $P(x,y) = \sum_{m,n} c_{m,n} x^m y^n \in k[x,y]$. Then $a \cdot P = P$ implies $c_{n,m} = (-1)^m c_{m,n}$ for all m, n and $b \cdot P = P$ implies $c_{m,n} = 0$ unless $m + 3n \equiv 0$ $(mod\ 4)$. In particular $k[x,y]\#BD_2$ has as a k-basis:

$$\begin{gathered}\{x^{4i-2j}y^{2j} + x^{2j}y^{4i-2j} \mid i,j \in \mathbb{N}, j \leq i\} \\ \cup \{x^{4i-2j+1}y^{2j+1} - x^{2j+1}y^{4i-2j+1} \mid i,j \in \mathbb{N}, j < i\}\end{gathered}$$

Hence, as a k-algebra it is generated by $A = x^4 + y^4$, $B = x^2y^2$, $C = x^5y - xy^5$ which satisfy the relation $C^2 - B(A^2 - 4B^2) = 0$ in degree 12.
By considering the degree in x and y separately we see that every other relation between A, B and C must be divisible by this relation. $\square$

Corollary 2.4.7. *Let k be an algebraically closed field, then*

$$\dim_k\left(\mathrm{Z}_d\left(k,k^{\oplus 4}\right)\right)=\begin{cases}\frac{d}{4}+1 & \textit{if } d\equiv 0 \ (\textit{mod } 4)\\ \frac{d-2}{4} & \textit{if } d\equiv 2 \ (\textit{mod } 4)\\ 0 & \textit{else}\end{cases}$$

Proof. Combining Lemma 2.1.5 with T. Schedler's result [Sch07, Theorem 10.1.1.] we may assume k has characteristic different from 2. In this case the result follows from an explicit computation using the presentation exhibited in Lemma 2.4.6. □

For the second specific case we have:

Lemma 2.4.8. *Let k be a field, then*

$$\mathrm{Z}\left(\Pi_k\left(k[s,t]/(s^2,t^2)\right)\right)\cong\frac{k[A,B,C]}{(C^2)}$$

Where A,B are homogeneous elements of degree 4 and C is a homogeneous element of degree 6. In particular the Hilbert series of $\mathrm{Z}\left(\Pi_k(k[s,t]/(s^2,t^2))\right)$ is the same as in Corollary 2.4.7.

Proof. This is proven in §2.7.2. □

We shall use the following lemma to compute $\dim_k(\mathrm{Z}_d(k,F))$ in the more general case where F is a Frobenius algebra of dimension 4 over a (possibly not algebraically closed) field k.

Lemma 2.4.9. *Let F and G be two Frobenius algebras over a field k such that $F\overset{\mathrm{def}}{\dashrightarrow}G$. Then for each $d\in\mathbb{N}$,*

$$\dim_k(\mathrm{Z}_d(k,F))\geq\dim_k(\mathrm{Z}_d(k,G))$$

Proof. Let D be the algebra deforming F to G provided by Definition 2.2.2 and denote $R=k[[u]]$, $K=k((u))$. As in the left exact sequence (2.4), we write $\mathrm{Z}_d(R,D)=\ker(\phi)$ and let Φ be the matrix corresponding to ϕ.
Let Φ_K denote the same matrix with coefficients viewed in the fraction field K and Φ_k denote the matrix with coefficients viewed in the residue field k. Then by construction,

$$\ker(\Phi_K)=\ker(K\otimes_R\phi)=\mathrm{Z}_d(K,K\otimes_R D)$$

and

$$\ker(\Phi_k)=\ker(k\otimes_R\phi)=\mathrm{Z}_d(k,k\otimes_R D)$$

Now,

$$\begin{aligned}\dim_k(\mathrm{Z}_d(k,G)) &= \dim_K(K \otimes_k (\mathrm{Z}_d(k,G)) \\ &= \dim_K(\mathrm{Z}_d(K, K \otimes_k G)) \\ &= \dim_K(\mathrm{Z}_d(K, K \otimes_R D)) \\ &= \dim_K(\ker(\Phi_K))\end{aligned}$$

Since clearly $\dim_k(\ker(\Phi_k)) \geq \dim_K(\ker(\Phi_K))$, the claim follows. □

Lemma 2.4.10. *Let F be a Frobenius algebra of dimension 4 over a field k. Then*

$$\dim_k(\mathrm{Z}_d(k,F)) = \begin{cases} \frac{d}{4}+1 & \textit{if } d \equiv 0 \ (mod\ 4) \\ \frac{d-2}{4} & \textit{if } d \equiv 2 \ (mod\ 4) \\ 0 & \textit{else} \end{cases} \tag{2.6}$$

Proof. First assume that k is algebraically closed. In this case all Frobenius algebras fit inside a directed diagram of deformations by Lemma 2.2.1 and Lemma 2.2.4. In particular, using Lemma 2.4.9, it suffices to prove the result for the extremal cases $F = k^{\oplus 4}$ and $F = k[s,t]/(s^2,t^2)$. These cases were covered by Corollary 2.4.7 and Lemma 2.4.8 respectively.
For arbitrary base fields k, we use the flat base change Lemma 2.4.4. □

Lemma 2.4.11. *Theorems 2.4.3 and 2.4.1 hold in when R is a noetherian local domain.*

Proof. Let $\phi_{R,S}$ be the morphism defining the left exact sequence (2.4). Then $\phi_{R,S}$ is a morphism between free R-modules of finite rank and hence can be represented by a matrix Φ with respect to some chosen basis for

$$V := \Pi_R(S)_d \text{ and } W := (\Pi_R(S)_d^{\oplus 5}) \bigoplus (\Pi_R(S)_{d+1}^{\oplus 8}).$$

Let Φ_k be the matrix obtained by replacing each entry of Φ by its corresponding class in $k = R/m$. Then Φ_k is a matrix representation for $k \otimes \phi$ using the induced k-basis for $k \otimes_R V$ and $k \otimes_R W$. Let $a = rk(\Phi_k)$, then there is an invertible $a \times a$ submatrix Ψ_k in Φ_k. The corresponding submatrix Ψ of Φ has a determinant which does not lie in $\mathfrak{m}$ and is thus itself invertible. By a suitable change of basis on V and W we can now rewrite Φ in the form:

$$\Phi = \left[\begin{array}{c|c} \mathrm{id}_{a\times a} & 0 \\ \hline 0 & \Psi' \end{array}\right]$$

where all entries of the submatrix Ψ' lie in $\mathfrak{m}$ (any entry not in $\mathfrak{m}$ would give rise to an invertible submatrix of rank $a+1$ by elementary row and column operations) .

Hence we can decompose V and W as a direct sum of free submodules $V = V_1 \oplus V_2$ and $W = W_1 \oplus W_2$ such that $\phi = \phi_1 \oplus \phi_2$ where $\phi_1 : V_1 \xrightarrow{\sim} W_1$ and $\phi_2 : V_2 \to W_2$ satisfies $k \otimes_R \phi_2' = 0$. In particular,

$$\mathrm{Z}_d(k, k \otimes_R S) = \ker(k \otimes \phi) = k \otimes V_2 \tag{2.7}$$

and hence V_2 is free of rank given by (2.6).

Now, we let K denote the fraction field of R. Then since by construction $\ker(\phi) \subset V_2$, we obtain $K \otimes_R \ker(\phi) \subset K \otimes V_2$. Hence since K is flat over R, Lemma 2.3.1 gives:

$$\dim_K(K \otimes_R \ker(\phi)) = \dim_K(K \otimes \mathrm{Z}_d(R, S)) = \dim_K(\mathrm{Z}_d(K, K \otimes_R S))$$

Which by Lemma 2.4.10 and the above equality (2.7) is equal to $\dim_K(K \otimes V_2)$. It follows that $\ker(\phi) = V_2$ from which we infer that $\phi_2 = 0$ and hence the monomorphism $\mathrm{Z}_d(R, S) \hookrightarrow \Pi_R(S)_d$ splits. It follows that $\mathrm{Z}_d(R, S)$ is projective and since the ranks can be computed after tensoring with a field, they must be given by (2.6). □

We can now finish the proofs of the main results of this section. This is done in a way similar to the proof of Theorem 2.3.9:

Proof of Theorem 2.4.1. Let $d \in \mathbb{N}$ and let $\iota_{R,S}$ denote the embedding

$$\iota : \mathrm{Z}_d(R, S) \hookrightarrow \Pi_R(S)_d$$

By Lemma 2.4.11 we already know that the result holds if R is a noetherian local domain and by the local nature of splitting (see for example [Lam07, Exercise 4.13, p.105]) it extends to the case where R is any noetherian domain.

Next let R be a local ring with algebraically closed residue field. Then by Lemma 2.3.8 S/R can be obtained as a base change of $\tilde{S}/\tilde{R}$ through a morphism $\tilde{R} \to R$ where $\tilde{R}$ is a noetherian domain. The result follows in this case as a split embedding remains split after base change.

Next, we assume R is any local ring. In this case we can consider the faithfully flat morphism $R \to \overline{R}$ provided by Lemma 2.3.7. As the residue field of $\overline{R}$ is algebraically closed, the monomorphism $\iota_{\overline{R}, S \otimes \overline{R}} = \iota_{R,S} \otimes \overline{R}$ is split. This implies that $\iota_{R,S}$ must be split itself by Lemma 2.4.12 below. Finally, again using the local nature of splitting, we have the result for any ring R. □

Proof of Theorem 2.4.2. This is an immediate consequence of Theorem 2.4.1 and the fact that the construction of $\phi_{R,S}$ in (2.5) is compatible with base change. □

Proof of Theorem 2.4.3. First let R be a noetherian local domain with residue field k and field of fractions K. Then by Lemma 2.4.10,

$$\begin{aligned}\dim_K(K \otimes_R (\mathrm{Z}_d(R,S)) &= \dim_K(\mathrm{Z}_d(K, K \otimes_R S)) \\ &= \dim_k(\mathrm{Z}_d(k, k \otimes_R S)) \\ &= \dim_k(k \otimes_R \mathrm{Z}_d(R,S))\end{aligned}$$

Hence by [Gro71, Corollaire IV.4.4], $\mathrm{Z}_d(R,S)$ is free of the expected rank.

If R is a noetherian domain, then for any $\mathfrak{p} \in \mathrm{Spec}(R)$, $R_\mathfrak{p}$ is a noetherian local domain such that $R_\mathfrak{p} \otimes_R \mathrm{Z}_d(R,S) = \mathrm{Z}_d(R_\mathfrak{p}, R_\mathfrak{p} \otimes_R S)$ is a free module of the stated rank. Serre's Theorem then proves that $\mathrm{Z}_d(R,S)$ is projective of the stated rank.

Next let R be a local ring with algebraically closed residue field and let $\tilde{S}/\tilde{R}$ the relative Frobenius ring provided by Lemma 2.3.8. By the above we know that $\mathrm{Z}_d(\tilde{R},\tilde{S})$ is projective over $\tilde{R}$ of the desired rank. Hence $\mathrm{Z}_d(R,S) = \mathrm{Z}_d(\tilde{R},\tilde{S}) \otimes R$ is free of the required rank over R.

To extend the statement to general local rings we just use Lemma 2.3.7.

Finally Serre's theorem extends the statement to general rings. □

We now prove the technical lemma used in the proof of Theorem 2.4.1.

Lemma 2.4.12. *Let R be a local ring and let $R \to \overline{R}$ be as in Lemma 2.3.7. Let $\iota : A \hookrightarrow B$ be an embedding of finitely generated R-modules where B is a projective R-module. Moreover assume $\iota \otimes \overline{R} : A \otimes \overline{R} \hookrightarrow B \otimes \overline{R}$ is split. Then ι is split as well.*

Proof. Let k be the residue field of R and $\overline{k}$ its algebraic closure, then there is a commutative diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & k \arrow[d] \\ \overline{R} \arrow[r] & \overline{k} \end{tikzcd}$$

As $\iota \otimes \overline{R}$ is split, $\iota \otimes \overline{k}$ is a monomorphism. The above commutative diagram (and the faithfully flatness of $k \to \overline{k}$) implies $\iota \otimes k$ is a monomorphism. Let $C = \mathrm{coker}(\iota)$, then we have a long exact sequence

$$\ldots \longrightarrow \mathrm{Tor}_1^R(B,k) \longrightarrow \mathrm{Tor}_1^R(C,k) \longrightarrow A \otimes k \xrightarrow{\iota \otimes k} B \otimes k \longrightarrow C \otimes k \longrightarrow 0$$

As B is a projective R-module and R is local, B is also flat, implying $\mathrm{Tor}_1^R(B,k) = 0$. From this it follows that $\mathrm{Tor}_1^R(C,k) = 0$ and, again because R is a local noetherian ring, this implies C is a projective R-module such that

$$0 \longrightarrow A \xrightarrow{\iota} B \longrightarrow C \longrightarrow 0$$

is a split exact sequence. □

2.5 $\Pi_R(S)$ is noetherian and finite over its center

Throughout this section, we keep the standing assumptions of before, namely S relative Frobenius of rank 4 over the noetherian ring R.

The main result of the section is the following:

Theorem 2.5.1. $\Pi_R(S)$ *is noetherian and finite over its center.*

As part of the proof, we construct a morphism

$$\sigma_{R,S} : R[\mathrm{Z}_4(R,S)]^{\oplus m} \longrightarrow \Pi_R(S)$$

as follows: first choose an R-basis (x,y,z,w) for S and let e be the element corresponding to $1_S \in N$ and f be the element corresponding to $1_S \in M$ (where we used the notation from the discussion preceding Definition 2.1.3). This choice yields an obvious morphism

$$\pi : R < x,y,z,w,e,f > \longrightarrow T_{R\oplus S}(M \oplus N)$$

where x,y,z,w have degree 0 and e,f have degree 1 in $R < x,y,z,w,e,f >$.
The R-module $\mathrm{T}(R,S)_0 = R \oplus S$ is generated by $(1_R, x, y, z, w)$ and these 5 elements are the images under π of the corresponding elements in $R < x,y,z,w,e,f >$ showing that π is surjective in degree 0. Moreover, $\mathrm{T}(R,S)_1 = {}_RS_S \oplus {}_SS_R$ is generated by $(xe, ye, ze, we, fx, fy, fz, fw)$ as an $R-R$-bimodule and hence π is also surjective in degree 1.

Finally since $\mathrm{T}(R,S)$ is a tensor algebra, it is generated in degree 0 and 1 and hence π is surjective in all degrees. Composing π with the canonical quotient map $\mathrm{T}(R,S) \twoheadrightarrow \Pi_R(S)$ yields a surjection

$$\chi : R < x,y,z,w,e,f > \longrightarrow\!\!\!\!\!\rightarrow \Pi_R(S) \tag{2.8}$$

Using the map χ we will construct a finite set of generators for $\Pi_R(S)$ as a module over its center. For this notice that the R-module $\Pi_R(S)_{\le 6}$ is generated by the image of the words of length at most 6 in $\{e,f\}$. This set of words is infinite, but we can reduce it to a finite set of generators for $\Pi_R(S)_{\le 6}$ using the following observations

1. since $\{1_R, x, y, z, w\}$ forms an R-basis for $\Pi_R(S)_0$, we can assume that any subword of degree zero is precisely a letter in this set
2. by the definition of the multiplication of $\Pi_R(S)$, we have $e^2 = f^2 = 0$

Hence if we let H be the finite set set of words in $\{x,y,z,w,e,f\}$ of length at most 6 in $\{e,f\}$ and such any two instances of x,y,z,w are separated by at least one e or f. We obtain $\chi(R \cdot H) = \Pi_R(S)_{\le 6}$. If we list this set as

$$H = \{a_1, \dots, a_n\}$$

we can define $\sigma_{R,S}$ as

$$\sigma_{R,S} : R[\mathrm{Z}_4(R,S)]^{\oplus n} \longrightarrow \Pi_R(S) : (z_i)_{i=1}^n \mapsto \sum_{i=1}^n z_i \chi(a_i) \tag{2.9}$$

We shall prove the following theorem

Theorem 2.5.2. *$\sigma_{R,S}$ is a surjective map. In particular $\Pi_R(S)$ is finite over its center.*

From this Theorem 2.5.1 will readily follow as $\mathrm{Z}_4(R,S)$ is finitely generated over R. We once again prove Theorem 2.5.2 by increasing the generality of the ring R.

First, we show that the construction of $\sigma_{R,S}$ commutes with base change: Let $R \to R'$ be any morphism of rings, then since both the construction of generalized preprojective algebras and taking their center commute with base change by Theorem 2.4.2 and Lemma 2.3.1, we have a diagram

$$\begin{tikzcd} R'[\mathrm{Z}_4(R', R' \otimes_R S)]^{\oplus n} \arrow[r, "\sigma_{R', R'\otimes_R S}"] & \Pi'_R(R' \otimes_R S) \\ R' \otimes_R R[\mathrm{Z}_4(R,S)]^{\oplus n} \arrow[u, "\zeta"] \arrow[r, "\mathrm{id}_{R'} \otimes_R (\sigma_{R,S})"'] & R' \otimes_R \Pi_R(S) \arrow[u, "\eta"'] \end{tikzcd} \tag{2.10}$$

where the vertical maps are isomorphisms by Lemma 2.4.11 and Lemma 2.3.1.

Lemma 2.5.3. *For any morphism $\varphi : R \to R'$, the diagram in* (2.10) *is commutative.*

Proof. Let S/R be relative Frobenius. Let basis $e_1, \ldots, e_n$ be an R-basis for S and λ a generator for the S-module $\mathrm{Hom}_R(S,R)$. Then as in the proof of Lemma 2.3.1, $(R' \otimes S)/R'$ is relative Frobenius with generator $1_{R'} \otimes \lambda$ and basis $1_{R'} \otimes e_1, \ldots, 1_{R'} \otimes e_n$. Following the successive steps in the construction of $\sigma_{R', R' \otimes_R S}$ we see that

$$\begin{cases} \Pi_{R'}(R' \otimes_R S) & = 1_{R'} \otimes \Pi_R(S) \\ \chi_{R'} & = 1_{R'} \otimes \chi_R \\ H_{R'} & = 1_{R'} \otimes H_R \end{cases}$$

Moreover if z_i is an element in $R[\mathrm{Z}_4(R,S)]$ (considered as the ith component of $R[\mathrm{Z}_4(R,S)]^{\oplus n}$), then

$$\begin{aligned} \eta \circ (1_{R'} \otimes_R (\sigma_{R,S}))\,(r' \otimes z_i) &= \eta\left(r' \otimes z_i \chi_R(a_i)\right) \\ &= r'(1 \otimes z_i \chi_R(a_i)) \\ &= r'(1 \otimes z_i)(1 \otimes \chi_R(a_i)) \\ &= r'(1 \otimes z_i)(\chi_{R'}(1 \otimes a_i)) \\ &= \sigma_{R', R' \otimes S}(r'(1 \otimes z_i)) \\ &= \sigma_{R', R' \otimes S} \circ \zeta(r' \otimes z_i) \end{aligned}$$

$\square$

Lemma 2.5.4. *Let F and G be Frobenius algebras over k such that $F \overset{\text{def}}{\dashrightarrow} G$. If $\sigma_{k,F}$ is surjective, then so is $\sigma_{k,G}$*

Proof. Let D be the algebra deforming F to G provided by Definition 2.2.2 and write $R := k[[u]]$ and $K := k((u))$.

Then by Lemma 2.5.3, $k \otimes_R \sigma_{R,D} = \sigma_{k,F}$ and Nakayama's lemma we know that $\sigma_{R,D}$ is surjective whenever $\sigma_{k,F}$ is. A second application of Lemma 2.5.3 shows that $K \otimes_k \sigma_{k,G} = K \otimes_R \sigma_{R,D}$, showing that $K \otimes_k \sigma_{k,G}$ is also surjective. Finally, $\sigma_{k,G}$ must be surjective as K is faithfully flat over k. □

Lemma 2.5.5. *Let $F := k[s,t]/(s^2,t^2)$. Then the map $\sigma_{k,F}$ is surjective*

Proof. This is proven in §2.7.3. □

Corollary 2.5.6. *Let F be Frobenius over a field k . Then $\sigma_{k,F}$ is surjective*

Proof. If k is algebraically closed, then any Frobenius algebra F over k can be obtained from $k[s,t]/(s^2,t^2)$ by a finite number of Frobenius deformations following the diagram (2.3). Hence the result follows from a combination of Lemma 2.5.5 and Lemma 2.5.4.
For a general field we use that $\overline{k}$ is faithfully flat over k. □

Proof of Theorem 2.5.2. If R is a local ring, then $k \otimes_R \sigma_{R,S} \cong \sigma_{k,k\otimes_R S}$ and the result follows by the above and Nakayama's Lemma.

If R is a non-local ring, for any $\mathfrak{p} \in \operatorname{Spec}(R)$, we have $R_\mathfrak{p} \otimes_R \sigma_{R,S} = \sigma_{R_\mathfrak{p},R_\mathfrak{p}\otimes_R S}$, which is a surjective morphism. As this holds for all $\mathfrak{p} \in \operatorname{Spec}(R)$, $\sigma_{R,S}$ is itself surjective. □

2.6 The global dimension of $\Pi_R(S)$

In this section we prove the following:

Theorem 2.6.1. *The global dimension of $\Pi_R(S)$ is bounded by the number*

$$\max(\operatorname{gl.dim}(R), \operatorname{gl.dim}(S)) + 2$$

We first bound the projective dimension of R and S as $\Pi_R(S)$-modules.

Lemma 2.6.2. *The R-module $R \oplus S$ admits a projective resolution of the following form:*

$$0 \to \Pi_R(S)(-2) \xrightarrow{\alpha_2} ({}_S S_R \oplus {}_R S_S) \otimes \Pi_R(S)(-1) \xrightarrow{\alpha_1} \Pi_R(S) \xrightarrow{\alpha_0} R \oplus S \to 0 \tag{2.11}$$

Proof. α_0 is the canonical projection with kernel $\Pi_R(S)_{\geq 1}$. This module is generated by $\Pi_R(S)_1 = {}_SS_R \oplus {}_RS_S$, hence $\mathrm{im}(\alpha_1) = \ker(\alpha_0)$. Since the relations of $\Pi_R(S)$ are generated in degree 2, we also have $\mathrm{im}(\alpha_2) = \ker(\alpha_1)$. Hence the injectivity of α_2 is the only nontrivial part of the claim.

The sequence (2.11) is a direct sum of the following two subsequences:

$$0 \to 1_R \cdot \Pi_R(S)(-2) \longrightarrow 1_S \cdot \Pi_R(S)(-1) \longrightarrow 1_R \cdot \Pi_R(S) \longrightarrow R \to 0 \tag{2.12}$$

$$0 \to 1_S \cdot \Pi_R(S)(-2) \longrightarrow \big(1_R \cdot \Pi_R(S)(-1)\big)^{\oplus 4} \longrightarrow 1_S \cdot \Pi_R(S) \longrightarrow S \to 0 \tag{2.13}$$

By Lemma 2.3.1 exactness can be checked after localization at each prime ideal of R, hence we may assume all terms in (2.12) and (2.13) are free R-modules of finite rank in each degree by Lemma 2.3.10. The claim reduces to the following relation on the Hilbert series: for each $d \in \mathbb{N}$ we must have

$$\begin{aligned} h_{d-2}(1_R \cdot \Pi_R(S)(-2)) - h_{d-1}(1_S \cdot \Pi_R(S)(-1)) + h_d(1_R \cdot \Pi_R(S)) - \delta_{d0} &= 0 \\ h_{d-2}(1_S \cdot \Pi_R(S)(-2)) - 4h_{d-1}(1_R \cdot \Pi_R(S)(-1)) + h_d(1_R \cdot \Pi_R(S)) - 4\delta_{d0} &= 0 \end{aligned}$$

(where $h_d(-)$ denotes the rank of the degree d-part as an R-module)
Using Lemma 2.3.10 we see that this these relations are satisfied. $\square$

Lemma 2.6.3. *A $\Pi_R(S)$-module is simple if and only if it is simple over R or simple over S.*

Proof. A $\Pi_R(S)$-module which is simple as an R- or S-module is clearly simple as a $\Pi_R(S)$-module. Conversely if M is a simple $\Pi_R(S)$-module, then $M = 1_RM$ or $M = 1_SM$ since $M = 1_RM \oplus 1_SM$. Moreover we claim that $\Pi_R(S)_{\geq 1}M = 0$ or equivalently $\Pi_R(S)_1M = 0$. For this assume (for example) that $M = 1_RM$. If $x \in {}_SS_R$ then

$$xM = (1_Sx)M = 1_S(xM) \subset 1_SM = 0$$

and if $x \in {}_RS_S$ then

$$xM = (x1_S)M = x(1_SM) = 0$$

Hence only the R-component in degree 0 acts non-trivially on M, it follows in particular that M is also a simple R-module. The case $M = 1_SM$ is completely similar. $\square$

Proof of Theorem 2.6.1. By [Bas68, Proposition III.6.7(a)] it suffices to check that if M is a simple $\Pi_R(S)$-module then:

$$pd_{\Pi_R(S)}(M) \leq \max(\mathrm{gl.dim}(R), \mathrm{gl.dim}(S)) + 2$$

By Lemma 2.6.3, M is a simple R-module or a simple S-module. We assume the former, the other case being completely similar. Let $P_\bullet \to M$ be a resolution of

M by projective R-modules of length $pd_R(M) \leq \mathrm{gl.dim}(R)$. Then for each i, by Lemma 2.6.2 we have

$$pd_{\Pi_R(S)}(P_i) \leq pd_{\Pi_R(S)}(R) \leq pd_{\Pi_R(S)}(R \oplus S) \leq 2$$

A standard long exact sequence-argument now gives the desired result. □

2.7 Explicit computations for $S = \dfrac{k[s,t]}{(s^2,t^2)}$

We describe $\Pi_k(S)$ through generators and relations:

- $\Pi_k(S)_0 = k \oplus S$. Let a denote $(1_k, 0)$ and $b = (0, 1_S)$ then since $a + b = 1$, $a, 1, s, t, st$ is a k-basis for $\Pi_k(S)_0$. It is clear that this set satisfies the relations

$$a^2 = a, as = sa = at = ta = 0$$

- $\Pi_k(S)_1 = {}_kS_S \oplus {}_SS_k$. Let f be $(1_S, 0)$ and $e = (0, 1_S)$, then we can write $\Pi_k(S)_1 = fS \oplus Se$. Hence $f, fs, ft, fst, e, se, te, ste$ is a k-basis for $\Pi_k(S)_1$. By construction, each generator $\neq 1$ of $\Pi_k(S)_0$ acts nontrivially on exactly one side of each component. Hence we have the relations

$$ea = e, af = f, ae = fa = 0, es = et = sf = tf = 0$$

  Note that this implies $e^2 = f^2 = 0$ since for example

$$e^2 = (ea)e = e(ae) = 0$$

- It is clear that the relation $1 \otimes 1 \in {}_kS_S \otimes {}_SS_k$ takes the form $fe = 0$. To compute the second relation, note that projection onto kst provides the duality isomorphism $\mathrm{Hom}_R(S, R) \cong S$ (see Lemma 2.2.4). It immediately follows that (e, se, te, ste) is dual to (fst, ft, fs, f) in the sense of Definition 2.1.3. The relation now takes the form

$$efst + seft + tefs + stef = 0 \tag{2.14}$$

To sumarize $\Pi_k(S)$ is a quotient of the free algebra $k < a, s, t, e, f >$ by the relations

$$\begin{cases} s^2 = t^2 = st - ts = 0 \\ a^2 = a, as = sa = at = ta = 0 \\ ea = e, af = f, ae = fa = 0, es = et = sf = tf = 0 \\ fe = efst + seft + tefs + stef = 0 \end{cases}$$

Note that $\Pi_k(S)$ is a graded algebra via $\deg(a) = \deg(s) = \deg(t) = 0$ and $\deg(e) = \deg(f) = 1$.

2.7.1 Proof of Lemma 2.3.3: a set of generators for each $(\Pi_k(S))_d$

In this subsection we give sets of generators in each degree, hence giving an upper bound for $\dim_k(\Pi_k(S)_d)$ and proving Lemma 2.3.3. More explicitly we prove that

$$\dim_k\left(\Pi_k\left(\frac{k[s,t]}{(s^2,t^2)}\right)_d\right) \le \begin{cases} 5(d+1) & \text{if } d \text{ is even} \\ 4(d+1) & \text{if } d \text{ is odd} \end{cases}$$

For this we make the following remarks:

- In each degree there are generators of two types:

  I) Elements of the form $f * ef * ef \dots * ef * e(f(*))$ where each $*$ is either s, t or st

  II) Elements of the form $(*)ef * ef \dots * ef * e(f(*))$ where each $*$ is either s, t or st

- Let $\mathcal{R}$ denote the relation (2.14), then $f\mathcal{R}e, t\mathcal{R}, s\mathcal{R}, st\mathcal{R}$ take the form

$$fsefte = -ftefse \tag{2.15}$$

$$steft = -tefst \tag{2.16}$$

$$stefs = -sefst \tag{2.17}$$

$$stefst = 0 \tag{2.18}$$

- As a consequence of the above equalities, we know that for any non-zero element there is at most one appearance of st. For example:

$$f\underline{st}efsef\underline{st} = fstef(sefst) = -fstef(stefs) = -f(stefst)efs = 0$$

We say any of the above elements is of bidegree (m,n) if there are m appearances of s and n appearances of t. It is easy to see that the above relations do not violate this bidegree and that it turns $\Pi_k(S)$ into a $\mathbb{Z}^2$-graded ring. Using the above remarks we create (minimal) sets of generators by a case-by-case study:

- <u>Case 1: d even and Type I</u>
  All words in this case take the form $(f*e)\dots(f*e)$. We can use relations (2.15), (2.16), (2.17) to write the element in the form $\pm(fse)^i(fste)^\varepsilon(fte)^j$ where $\varepsilon = 0,1$. For $\varepsilon = 0$ we have $\frac{d}{2}+1$ choices for i and j and for $\varepsilon = 1$ we have $\frac{d}{2}$ choices, giving a total $d+1$ generators.

- <u>Case 2: d even and Type II</u>
  These are elements of the form $(*)(ef*)\dots(ef*)ef(*)$ and since there is at most one occurrence of st the bidegree satisfies $\frac{d}{2}-1 \le m+n \le \frac{d}{2}+2$.

If $m+n = \frac{d}{2}-1$ the element can be written in the form $\pm(efs)^m(eft)^n ef$, giving $\frac{d}{2}$ choices. Similarly if $m+n = \frac{d}{2}+2$ the element can be written in the form $\pm(sef)^{m-1}st(eft)^{n-1}$. Giving $\frac{d}{2}+1$ choices.

Assume $m+n = \frac{d}{2}$. If $(n,m) = (\frac{d}{2},0)$ (or $(n,m) = (0,\frac{d}{2})$) we have 2 generators: $(sef)^{\frac{d}{2}}$ and $(efs)^{\frac{d}{2}}$ (or $(tef)^{\frac{d}{2}}$ and $(eft)^{\frac{d}{2}}$).
In all other cases we need 3 generators: $(sef)^m(tef)^n$, $(efs)^m(eft)^n$ and $(efs)^{m-1}efstef(tef)^{n-1}$. This gives a total of $\frac{3d}{2}+1$ generators for this subcase.

Finally assume $m+n = \frac{d}{2}+1$. If $(m,n) = (\frac{d}{2}+1,0)$ (or $(m,n) = (0,\frac{d}{2}+1)$) we have 1 generator: $(sef)^{\frac{d}{2}}s$ (or $(tef)^{\frac{d}{2}}t$).

In all other cases we need 3 generators: $(sef)^m(tef)^{n-1}t$, $(efs)^{m-1}efst(eft)^{n-1}$ and $(sef)^{m-1}stef(tef)^{n-1}$. This gives a total of $\frac{3d}{2}+1$ generators for this subcase.
For case 2 this results in $\dfrac{d}{2}+\left(\dfrac{3d}{2}+1\right)+\left(\dfrac{3d}{2}+2\right)+\left(\dfrac{d}{2}+1\right) = 4(d+1)$ generators.
Finally adding up the number of generators from Case 1 and Case 2 yields $5(d+1)$ generators in case d is even.

- Case 3: d odd and Type I
  All elements in this case take the form $(f*e)(f*e)\dots(f*e)f(*)$. By a completely similar argument as above, we conclude that generators can be chosen of the following forms: $(fse)^m(fte)^n f$, $(fse)^m(fte)^{n-1}ft$, $(fse)^{\frac{d-1}{2}}fs$, $(fse)^{n-1}(fte)^{m-1}fst$ and $fste(fse)^{n-1}(fte)^{m-1}f$. This gives a total of
  $$\frac{d+1}{2}+\frac{d+1}{2}+1+\frac{d+1}{2}+\frac{d-1}{2} = 2(d+1)$$
  generators

- Case 4: d odd and Type II
  Elements in this case are of the form $(*)e(f*e)(f*e)\dots(f*e)$. Note that any such word can be obtained by taking a word from Case 3, reading it from right to left and interchanging e and f. Applying this "procedure" to the generators of Case 3 yeilds a set of generators for the current case by symmetry. Hence in the current case we have $2(d+1)$ generators, adding up to $4(d+1)$ generators in case d is odd.

2.7.2 Computation of $\mathrm{Z}(\Pi_k(S))$ and the proof of Lemma 2.4.8

Consider the elements

$$u := sef + efs + fse \text{ and } v := tef + eft + fte \tag{2.19}$$

It is easy to see that u is normalizing with respect to the automorphism σ on $\Pi_k(S)$ which sends t to $-t$ and is the identity on the other generators. As $\sigma^2 = Id$ we have as an immediate consequence that u^2 is central. A completely similar discussion yields that v^2 is central. Using the relations in $\Pi_k(S)$ we can write u^2 and v^2 as

$$A := sefsef + efsefs + fsefse \text{ and } B := teftef + efteft + ftefte$$

One also checks that the following element is central in degree 6:

$$C = sefsteftef + efsefsteft + fsefstefte$$

The proof of the lemma now follows from several technical steps

1. As any nonzero word in $\Pi_k(S)$ allows at most 1 appearance of st we have $C^2 = 0$. Any other relation in A, B and C can then be written as

$$p_1(A,B) + C \cdot p_2(A,B) = 0$$

for some polynomials p_1, p_2. This automatically implies $p_1 = p_2 = 0$ (consider bidegrees!). In particular there is an inclusion of rings

$$\zeta : \frac{k[A,B,C]}{(C^2)} \hookrightarrow \mathrm{Z}(\Pi_k(S))$$

We will prove that this inclusion is in fact an isomorphism. We do so by checking surjectivity in each degree separately.

2. <u>There are no homogeneous central elements of an odd degree.</u>
<u>In particular: ζ is surjective in each odd degree.</u>
Let x be a homogeneous element of odd degree. If $ex \neq 0$, then it is a linear combination of monomials starting with f and hence ending in f, fs, ft of fst, in particular such an element is never of the form ye. Hence the only way $ex = xe$ is possible, is when $xe = ex = 0$. Similarly for x to be cental we need $fx = xf = 0$. We claim that a non-trivial homogeneous element x of odd degree satisfying $ex = xe = fx = xf = 0$ does not exist. For this let x be such an element. Then x is of one of the following 4 forms:

(i) $x = \alpha s(efs)^m e + \beta(fse)^m fs$
or $x = \alpha t(eft)^n e + \beta(fte)^n ft$

(ii) $x = \alpha(efs)^m(eft)^n e + \beta(fse)^m(fte)^n f$

(iii) $x = \alpha(sef)^m(tef)^{n-1}te + \beta(fse)^m(fte)^{n-1}ft$
$+ \gamma(efs)^{m-1}efst(eft)^{n-1}e + \delta(fse)^{m-1}fste(fte)^{n-1}f$

(iv) $x = \alpha(sef)^{m-1}st(eft)^{n-1}e + \beta(fse)^{m-1}fst(eft)^{n-1}$

and for each of the 4 possibilities we check that $ex = xe = fx = xf = 0$ implies $\alpha = \beta(= \gamma = \delta) = 0$. However this is immediate because in each case $ex = 0$ implies $\beta(= \delta) = 0$ and $xf = 0$ implies $\alpha(= \gamma) = 0$.

3. In an even degree $d = 2l$ there are no central elements of bidgree (m, n) with $m + n = l - 1$.
   Such an element is necessarily of the form
   $$x = (efs)^m (eft)^n ef$$
   and does not commute with te as $tex = 0 \neq xte$.

4. In an even degree $d = 2l$ there are no central elements of bidgree (m, n) with $m + n = l + 2$.
   Such an element is necessarily of the form
   $$x = (sefs)^{m-1} st(eft)^{n-1}$$
   and does not commute with efs as $efs \cdot x = 0$ whereas $x \cdot efs = (-1)^n (sef)^m st(eft)^{n-1} \neq 0$.

5. If the degree is $d = 2l$ is even, then for each bidegree (m, n) with $m + n = l$ there is one central element if m and n are even, there are no central elements if one of them is odd.
   An element of the given degree and bidegree can be written as:
   $$\begin{aligned} x = &\alpha(sef)^m (tef)^n + \beta(efs)^m (eft)^n \\ + &\gamma(fse)^m (fte)^n + \delta(efs)^{m-1} efstef(tef)^{n-1} \end{aligned}$$
   $ex = xe$ implies $\gamma = \beta$ and $fx = xf$ implies $\alpha = \gamma$. Hence if x is central it can be written as
   $$x = \alpha x_1 + \delta' x_2$$
   where
   $$\begin{aligned} x_1 =&(sef)^m (tef)^n + (efs)^m (eft)^n \\ &+ (fse)^m (fte)^n + (efs)^{m-1} efstef(tef)^{n-1} \\ x_2 =&(efs)^{m-1} efstef(tef)^{n-1} \\ \delta' =&\delta - \alpha \end{aligned}$$
   Now let $m = 2m' + \epsilon_m$ and $n = 2n' + \epsilon_n$ with $\epsilon_m, \epsilon_n = 0, 1$, then if A, B, u, v are as above one sees
   $$x_1 = u^{\epsilon_m} A^{m'} B^{n'} v^{\epsilon_n}$$
   In particular x_1 commutes with se if $\epsilon_n = 0$ and anti-commutes with se if $\epsilon_n = 1$. On the other hand $se \cdot x_2 = 0 \neq x_2 \cdot se$ and $x_2 \cdot se$ is linearly independent from $x_1 \cdot se$. Hence in order for x to commute with se we need $\delta' = \epsilon_n = 0$. A similar

argument using te instead of se we see that $\epsilon_m = 0$. Hence m, n and l are even and setting $\alpha = 1$ gives

$$x = A^{m'} B^{n'}$$

which is central as A and B are central.

6. If the degree is $d = 2l$ is even, then for each bidegree (m, n) with $m + n = l + 1$ there is one central element if $m, n \geq 2$ are even and no central element in all other cases.
An element of bidegree $(l + 1, 0)$ is of the form

$$x = (sef)^l s$$

and does not commute with e. Similarly an element of bidegree $(0, l + 1)$ does not commute with e, hence we can assume $m, n \geq 1$ in which case

$$\begin{aligned} x =& \alpha(sef)^m (tef)^{n-1} t + \beta(sef)^{m-1} stef(tef)^{n-1} \\ &+ \gamma(efs)^{m-1} efst(eft)^{n-1} + \delta(fse)^{m-1} fste(fte)^{n-1} \end{aligned}$$

$ex = xe$ implies $\gamma = \delta$ and $\alpha = 0$. $fx = xf$ implies $\beta = \delta$. Hence if x is central it can (upto a scalar) be written as

$$x = (sef)^{m-1} stef(tef)^{n-1} + (efs)^{m-1} efst(eft)^{n-1} + (fse)^{m-1} fste(fte)^{n-1}$$

Now let $m = 2m' + \epsilon_m$ and $n = 2n' + \epsilon_n$ and A, B, C, u, v be as above, then

$$x = u^{\epsilon_m} A^{m'-1} C B^{n'-1} v^{\epsilon_n}$$

In order for x to commute with se we need $\epsilon_n = 0$. A similar argument using te instead of se we see that $\epsilon_m = 0$. Hence m and n are even such that

$$x = A^{m'-1} C B^{n'-1}$$

which is central as A, C and B are central.

7. ζ is surjective in all even degrees As we already have injectivity of ζ, it suffices to check that $\dfrac{k[A, B, C]}{(C^2)}$ and $\mathrm{Z}(\Pi_k(S))$ have the same dimension over k in each even degree d. Write $d = 2l$. By the above we find:
   - $\mathrm{Z}_d(k, S)$ is $\left(\frac{l}{2} + 1\right)$ - dimensional if l is even
   - $\mathrm{Z}_d(k, S)$ is $\left(\frac{l-1}{2}\right)$ - dimensional if l is odd

   By considering the Hilbert series, one sees that this agrees with

$$\dim_k\left(\left(\frac{k[A, B, C]}{(C^2)}\right)_d\right)$$

And surjectivity of ζ is proven.

2.7.3 Surjectivity of $\sigma_{k,S}$ (proof of Lemma 2.5.5)

Let u and v be the normalizing elements as in (2.19). Let $V \subset \Pi_k(S)_2$ be the k-vector space spanned by u and v. Let μ_3 be the multiplication morphism given by the composition

$$\mu_3 : V \otimes \Pi_k(S)_1 \longrightarrow \Pi_k(S)_2 \otimes \Pi_k(S)_1 \longrightarrow \Pi_k(S)_3$$

Then we use a brute force computation to show that μ_3 must be surjective. I.e. we show that any element of $\Pi_k(S)_3$ can be written as a linear combination of elements of the form $u \cdot x$ or $v \cdot x$ with $x \in \Pi_k(S)_1$. It suffices to check this for all (types of) generators of $\Pi_k(S)_3$:

I) : elements of the form $f * ef(*)$.
These can all be put into the form $fsef(*)$ or $ftef(*)$ where $*$ is either s, t or st. Now use $fsef(*) = u \cdot f(*)$ and similarly $ftef(*) = v \cdot f(*)$.

II) : elements of the form $(*)ef * e$

- $efse = u \cdot e$ and $efte = v \cdot e$
- $sefse = u \cdot se$ and $tefte = v \cdot te$
- $sefste = u \cdot ste$ and $tefste = v \cdot ste$
- $sefte = -tefse - eftse = v \cdot (-se)$
- $tefse = -sefte - efste = u \cdot (-te)$

Which shows that μ_3 is indeed surjective.

Now for each degree d we have a commutative diagram

$$\begin{tikzcd}
V \otimes \Pi_k(S)_{d+1} \arrow[rr] & & \Pi_k(S)_{d+3} \\
V \otimes \Pi_k(S)_1 \otimes \Pi_k(S)_d \arrow[u] \arrow[rr, "\mu_3 \otimes \Pi_k(S)_d"'] & & \Pi_k(S)_3 \otimes \Pi_k(S)_d \arrow[u]
\end{tikzcd}$$

where the top horizontal arrow must be a surjection as the other three are surjective. Hence by induction (and the fact that $V \otimes -$ is right exact) we have for each $n \in \mathbb{N}$ a surjection

$$\mu_{2n+\epsilon} : V^{\otimes n} \otimes \Pi_k(S)_\epsilon \longrightarrow V^{\otimes n-1} \otimes \Pi_k(S)_{2+\epsilon} \longrightarrow \ldots \longrightarrow \Pi_k(S)_{2n+\epsilon}$$

Next let W be the vector space spanned by u^2 and v^2, then for each n and $\omega = 1, 2$ there is a surjection

$$W^{\otimes n} \otimes V^{\otimes \omega} \twoheadrightarrow V^{\otimes 2n+\omega}$$

and we have a commutative diagram

$$\begin{tikzcd}
W^{\otimes n} \otimes V^{\otimes \omega} \otimes \Pi_k(S)_\epsilon \arrow[r] \arrow[d] & W^{\otimes n} \otimes \Pi_k(S)_{2\omega+\epsilon} \arrow[d, "\rho_{4n+2\omega+\epsilon}"] \\
V^{\otimes 2n+\omega} \otimes \Pi_k(S)_\epsilon \otimes \Pi_k(S)_d \arrow[r, "\mu_{4n+2\omega+\varepsilon}"'] & \Pi_k(S)_{4n+2\omega+\varepsilon}
\end{tikzcd}$$

where $\rho_{4n+2\omega+\epsilon}$ must be surjective because the other three morphisms are. Then using the commutative triangle

$$\begin{tikzcd}
W^{\otimes n} \otimes \Pi_k(S)_{2\omega+\epsilon} \arrow[rr, "\rho_{4n+2\omega+\epsilon}"] \arrow[dr] & & \Pi_k(S)_{4n+2\omega+\epsilon} \\
& k[\mathrm{Z}_4(k,S)]_n \otimes \Pi_k(S)_{2\omega+\epsilon} \arrow[ur, "\overline{\rho_{4n+2\omega+\epsilon}}"'] &
\end{tikzcd}$$

we must have that $\overline{\rho_{4n+2\omega+\epsilon}} : k[\mathrm{Z}_4(k,S)]_n \otimes \Pi_k(S)_{2\omega+\epsilon} \to \Pi_k(S)_{4n+2\omega+\epsilon}$ must be surjective. As $2\omega+\epsilon$ takes the values 3,4,5,6 we have an induced surjection:

$$\overline{\rho} : k[\mathrm{Z}_4(k,S)] \otimes \Pi_k(S)_{\leq 6} \longrightarrow\!\!\!\!\rightarrow \Pi_k(S)$$

(where we included $\Pi_k(S)_d$ for $d = 0,1,2$ on the left hand side to guarantee surjectivity in these three lowest degrees).
Now $\sigma_{k,S}$ (see (2.9) for the definition) factors as $\overline{\rho} \circ \varsigma$ where ς is the morphism:

$$\varsigma : k[\mathrm{Z}_4(k,S)]^{\oplus N} \longrightarrow k[\mathrm{Z}_4(k,S)] \otimes \Pi_k(S)_{\leq 6} : (z_i)_{i=1}^N \mapsto \sum_{i=1}^N z_i \otimes \chi(a_i)$$

By the choice of the a_i in H, ς is surjective and hence also $\sigma_{k,S}$ proving Lemma 2.5.5.

Chapter 3

Homological properties of a certain noncommutative del Pezzo Surface

Sometimes you have to shove all the surface stuff to the side in order to see what's underneath.

Beth Moore (Evangelist)

This chapter is based upon results by the author and Louis de Thanhoffer de Volsey in [dTdVP15].

3.1 Introduction

Exceptional collections are a convenient way to study derived categories in (noncommutative) algebraic geometry, for (the noncommutative analogues of) smooth projective varieties. The existence of a full and strong exceptional collection is a very strong property for a triangulated category, but when it is satisfied the study of the derived category and its invariants essentially becomes the study of a (directed) finite-dimensional algebra.

One thing which sets apart the algebras obtained in this way from the class of all algebras is the behavior of the Serre functor on the Grothendieck group: it can be shown that $(-1)^{\dim X}S$ is unipotent [BP94, lemma 3.1]. By [Oka11] the case of $\dim X = 1$ is trivial, the only finite-dimensional algebra that appears is the path algebra of the Kronecker quiver.

In [dTdV16] de Thanhoffer de Völcsey and Van den Bergh study the Serre functor in the case of a surface and determine some extra properties. In particular they show that if Λ is the numerical Grothendieck group of a smooth projective surface X and $\langle -,- \rangle \colon \Lambda \times \Lambda \to \mathbb{Z}$ is the Euler form, (up to a a technical condition) the following properties hold:

- there exists an $s \in \mathrm{Aut}(\Lambda)$ such that $\langle x, sy\rangle = \langle y, x\rangle$ for $x, y \in \Lambda$
- $(s - \mathrm{id}_\Lambda)$ is nilpotent
- $\mathrm{rk}(s - \mathrm{id}_\Lambda) = 2$

Where the automorphism s is induced by the Serre functor on X.

Inspired by these properties, any free abelian group Λ with a nondegenerate bilinear form $\langle -, -\rangle$ satisfying the above conditions is said to be of *surface** type. Hence, a classification of such lattices Λ together with an automorphism s can be thought of as a numerical classification of *noncommutative smooth projective surfaces* equipped with a Serre functor S. Moreover there is an action of the signed braid group on the collection of these lattices with automorphism s (corresponding to mutation of the exceptional collections, see §4.2). The goal is thus to classify the couples (Λ, s) up to this action.

For lattices of rank 3 it follows from the work of Markov [Mar79] that the only case is the Grothendieck group arising from $\mathbb{P}^2$ as the Beilinson quiver (and its mutations). The more interesting classification for rank 4 lattices was recently obtained by de Thanhoffer de Völcsey and Van den Bergh (independently, Kuznetsov developed a similar notions and obtained closely related results in [Kuz17].):

Theorem. *([dTdV16, Theorem A]) Let Λ be a rank 4 lattice of surface* type. Then Λ is isomorphic to $\mathbb{Z}^4$ where the matrix of the bilinear form is one of the following standard types:*

$$J = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad K_n = \begin{bmatrix} 1 & n & 2n & n \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.1}$$

for $n \in \mathbb{N}$

The first matrix corresponds to the Grothendieck group of the surface $\mathbb{P}^1 \times \mathbb{P}^1$ (and its noncommutative analogues, see [VdB11] and [Mor07, Lemmas 4.2 and 4.4]) where a basis of the Gram matrix is given by the standard exceptional collection:

$$\Big(\mathcal{O}(0,0), \mathcal{O}(1,0), \mathcal{O}(0,1), \mathcal{O}(1,1)\Big)$$

The second family has some familiar solutions for the values $n = 0$ and $n = 1$. In fact, K_0 corresponds to the trivial extension $X \cup \{\bullet\}$ where X is $\mathbb{P}^2$ or some noncommutative generalization ($\mathrm{qgr}(A)$ for A a quadratic Sklynanin algebra, see [AOU14, Theorem 7.1]). Using Orlov's blowup formula [Orl92, Theorem 4.3], one sees that K_1 corresponds to $\mathbb{F}_1 = \mathrm{Bl}_x(\mathbb{P}^2)$, the first Hirzebruch surface. For $n \geq 2$ however, the matrices above do not correspond to the Grothendieck group of a smooth projective surface.

Following the introduction in [dTdV16], de Thanhoffer de Völcsey and Van den Bergh associate to such a lattice of *surface* type* an invariant δ, the "degree", which in the case of a smooth projective surface coincides with the usual degree of the surface up to a sign. They compute that $\delta(K_n) = 9 - n^2$, so that the only remaining case which should geometrically correspond to a positive degree is that of $n = 2$. This is our motivation for restricting ourselves in this chapter to finding a noncommutative geometric model for K_2 which we will interpret as a *noncommutative del Pezzo surface.* Moreover it was shown in [dTdV16, §6] that K_2 is mutation equivalent to the following matrix:

$$\begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.2}$$

Our goal is to construct a noncommutative surface Z (i.e. a Grothendieck category satisfying suitable conditions, see Chapter 0) together with an exceptional sequence (E_1, E_2, E_3, E_4) for which the Gram matrix takes the form (3.2). As the top-left and bottom-right 2×2 submatrices, show that the sequences (E_1, E_2) and (E_3, E_4) are isomorphic to Beilinson's exceptional collection $(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1))$ on $K(\mathbb{P}^1)$, we heuristically conclude that Z should be equipped with 2 "maps" (in the noncommutative sense) $\Pi_0, \Pi_1 : Z \longrightarrow \mathbb{P}^1$ such that E is obtained by pulling back $(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1))$ along both maps, i.e.:

$$E = \left(\Pi_1^*(\mathcal{O}_{\mathbb{P}^1}), \Pi_1^*(\mathcal{O}_{\mathbb{P}^1}(1)), \Pi_0^*(\mathcal{O}_{\mathbb{P}^1}), \Pi_0^*(\mathcal{O}_{\mathbb{P}^1}(1)) \right) \tag{3.3}$$

The construction of this noncommutative surface is an adaptation of Van den Bergh's theory of noncommutative $\mathbb{P}^1$-bundles over a smooth base scheme X of finite type over k as developed in [VdB12]. In that paper Van den Bergh proposes a new construction which results in a sheafified notion of a $\mathbb{Z}$-algebra: more precisely, let $\mathcal{E}$ be a *coherent* X-bimodule (see Definition 3.2.3) which is locally free on both sides. Then there is an appropriate notion of left- (and right-) dual ${}^*\mathcal{E}$ (resp. $\mathcal{E}^*$) in this context (Lemma 3.2.10). Applying the construction indefinitely yields $\mathcal{E}^{*(-n)} := {}^{*n}\mathcal{E}$ resp. $\mathcal{E}^{*n}$, which by naturality comes with a unit morphism

$$i_n : \mathcal{O}_\Delta \longrightarrow \mathcal{E}^{*n} \otimes_X \mathcal{E}^{*(n+1)}$$

Van den Bergh then defines the *symmetric sheaf-$\mathbb{Z}$-algebra* $\mathbb{S}(\mathcal{E})$ as a quotient of the tensor sheaf-$\mathbb{Z}$-algebra $\mathrm{T}(\mathcal{E})$ with relations given by the images of the above unit morphisms.

I.e. (see Definition 3.2.5 for more details)

- $\mathbb{S}(\mathcal{E})_{n,n} = \mathcal{O}_X$
- $\mathbb{S}(\mathcal{E})_{n,n+1} = \mathcal{E}^{*n}$
- $\mathbb{S}(\mathcal{E})$ is freely generated by $\mathbb{S}(\mathcal{E})$ subject to the relations given by the images of the morphism i_n

There is an associated category of graded $\mathbb{S}(\mathcal{E})$-modules: $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$ which is Grothendieck (Theorem 3.2.19). The intuition behind the definition of $\mathbb{S}(\mathcal{E})$ comes from the fact that in the case where $\mathcal{E}$ is central of rank $(2,2)$, the definition coincides with the notion of a $\mathbb{P}^1$-bundle over X in the sense that there is an equivalence between their categories of graded modules. For the convenience of the reader, we provide an explicit proof of this in Corollary 3.2.24.

Pushing our heuristic intuition further,

$$\dim_k \left(\mathrm{Hom}_Z(\Pi_1^*(\mathcal{O}_{\mathbb{P}^1}(1)), \Pi_0^*(\mathcal{O}_{\mathbb{P}^1}(1))\right) = \left\langle \Pi_1^* \mathcal{O}_{\mathbb{P}^1}(1)), \Pi_0^*(\mathcal{O}_{\mathbb{P}^1}(1)\right\rangle = 4$$

seems to suggest that $\mathcal{E}$ should be built using a morphism of degree 4 on the left and similarly by the identity on the right. This leads one to adapt Van den Bergh's construction under the additional assumptions that the bimodule $\mathcal{E}$ is locally free of rank $(4,1)$ over a pair of base schemes X, Y. To construct the noncommutative scheme $\mathrm{Proj}(\mathbb{S}(\mathcal{E}))$ (or rather its representation $\mathrm{QGr}(\mathbb{S}(\mathcal{E}))$) and establish its properties, we shall first prove two facts in the setting: The first is a description of $\mathbb{S}(\mathcal{E})$ in the case where the base schemes X and Y are affine. More precisely, we relate $\mathbb{S}(\mathcal{E})$ to the generalized preprojective algebras introduced in Chapter 2 as follows:

Theorem. *(see Lemma 3.3.13 and Lemma 3.3.19 together with Lemma 3.3.22) Let $\mathcal{E}$ be a X-Y-bimodule of rank $(4,1)$. Then there is a finite affine open cover $U_i \subset X$ such that the category $\mathrm{Gr}\left(\mathbb{S}(\mathcal{E})|_{U_i}\right)$ identifies with a direct summand of $\mathrm{Gr}(\Pi_{R_i}(S_i))$ where $\Pi_{R_i}(S_i)$ is a generalized preprojective algebra associated to a relative Frobenius pair S_i/R_i as in Chapter 2.*

Second, we adapt the technique of point modules which was developed in [VdB12] and [Nym04a] for the rank $(2,2)$ to the rank $(4,1)$ case. This proves to be a substantial modification, requiring an adaptation of the very definition of a point module. We use this technique to prove that $\mathbb{S}(\mathcal{E})_{n,m}$ is a locally free bimodule in each degree. This, together with the previous result allows us to adapt the ideas of [Mor07] and [Nym04b] to obtain noetherianity of $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$.

We summarize:

Theorem. *(see Theorem 3.3.1, Corollary 3.3.2 and Corollary 3.4.5) Let $\mathcal{E}$ be a X-Y-bimodule of rank $(4,1)$. Then*

- $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$ *is locally noetherian.*
- *Each bimodule $\mathbb{S}(\mathcal{E})_{n,m}$ is locally free and the ranks can be computed explicitly. In particular, if $n-m$ is even, these ranks coincide with the "classical" case where $\mathcal{E}$ has rank $(2,2)$. (See Corollary 3.4.5). Moreover these ranks agree with the numbers obtained in [Nym15] where this case was studied for X, Y zero-dimensional schemes*

This allows us to consider the noncommutative scheme $Z = \mathrm{Proj}(\mathbb{S}(\mathcal{E}))$ in the language of [AZ94] . Z comes with a sequence of maps

$$\Pi_{2n} : Z \to X, \quad \Pi_{2n+1} : Z \to Y$$

(again in the sense of [AZ94]) given by taking the corresponding degree of the graded module. We describe these and show that Π_0 and Π_1 contain all the information on these maps in a certain sense. With these definitions, we finally prove

Theorem. *(See Lemma 3.3.6, Corollary 3.4.2, Theorem 3.4.19 and Remark 3.4.20) Let $\mathcal{E}$ be a $\mathbb{P}^1$-bimodule of rank $(4,1)$. Let $\mathbb{S}(\mathcal{E})$ be the associated symmetric sheaf-$\mathbb{Z}$-algebra and put $Z = \mathrm{Proj}(\mathbb{S}(\mathcal{E}))$. Then*

$$\Big(\Pi_1^*(\mathcal{O}_{\mathbb{P}^1}), \Pi_1^*(\mathcal{O}_{\mathbb{P}^1}(1)), \Pi_0^*(\mathcal{O}_{\mathbb{P}^1}), \Pi_0^*(\mathcal{O}_{\mathbb{P}^1}(1)\Big)$$

is an exceptional collection on Z. Moreover up to equivalence of categories we can assume $\mathcal{E} = {}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}$ (see Definition 3.2.4 for the appropriate definition of this bimodule), in which case the above exceptional collection is full and strong and the associated Gram matrix of the Euler form is given by

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3.2 Symmetric sheaf-$\mathbb{Z}$-algebras

3.2.1 Definitions and construction

We begin by giving a summary of the material needed to define symmetric sheaf-$\mathbb{Z}$-algebras following [VdB12].

Convention 3.2.1. *Throughout this chapter k denotes an algebraically closed field. W, X and Y will denote smooth varieties (that is smooth, integral, separated and of finite type over k).*

Remark 3.2.2. *One could leave out the integral-condition in this convention, leading to the more general setting of disjoint unions of varieties. We choose not to do this for purposes of clarity.*

Definition 3.2.3. A coherent $X-Y$ bimodule $\mathcal{E}$ is a coherent $\mathcal{O}_{X\times Y}$-module such that the support of $\mathcal{E}$ is finite over X and Y. We denote the corresponding category by $\operatorname{bimod}(X-Y)$. More generally an $X-Y$-bimodule is a quasi-coherent $\mathcal{O}_{X\times Y}$-module which is a filtered direct limit of objects in $\operatorname{bimod}(X-Y)$. The category of $X-Y$-bimodules is denoted $\operatorname{Bimod}(X-Y)$. Finally, a bimodule $\mathcal{E}$ is called locally free if $\pi_{X*}(\mathcal{E})$ and $\pi_{Y*}(\mathcal{E})$ are locally free, where π_X, π_Y denote the standard projections $X\times Y \longrightarrow X$ and $X\times Y\longrightarrow Y$ respectively. If $\operatorname{rk}(\pi_{X*}(\mathcal{E})) = m$ and $\operatorname{rk}(\pi_{Y*}(\mathcal{E})) = n$, we write $\operatorname{rk}\mathcal{E} = (m,n)$.

The tensor product of $\mathcal{O}_{W\times X\times Y}$-modules induces a tensor product

$$\operatorname{Bimod}(W-X)\otimes\operatorname{Bimod}(X-Y)\longrightarrow\operatorname{Bimod}(W-Y):(\mathcal{E},\mathcal{F})\mapsto\mathcal{E}\otimes_X\mathcal{F}$$

through the formula

$$\mathcal{E}\otimes\mathcal{F} := \pi_{W\times Y*}\big(\pi^*_{W\times X}(\mathcal{E})\otimes_{W\times X\times Y}\pi^*_{X\times Y}(\mathcal{F})\big)$$

For each $\mathcal{E}\in\operatorname{Bimod}(W-X)$ this defines a functor :

$$-\otimes_X\mathcal{E}:\operatorname{Qcoh}(W)\longrightarrow\operatorname{Qcoh}(X):\mathcal{M}\mapsto\mathcal{M}\otimes_X\mathcal{E}:=\pi_{X*}\big(\pi^*_W(\mathcal{M})\otimes_{W\times X}\mathcal{E}\big) \tag{3.4}$$

which is right exact in general and exact if $\mathcal{E}$ is locally free on the left. We mention that [VdB12, lemma 3.1.1.] shows that this functor determines the bimodule $\mathcal{E}$ uniquely. Using this, the category $\operatorname{Bimod}(W-X)$ is embedded in the more abstract categories $\operatorname{BiMod}(W-X)$ and $\operatorname{BIMOD}(W-X)$ as introduced in §1.2.

Definition 3.2.4. Consider morphisms $u:W\longrightarrow X$, $v:W\longrightarrow Y$. If $\mathcal{U}\in\operatorname{Qcoh}(W)$, then we denote $(u,v)_*\mathcal{U}\in\operatorname{Bimod}(X-Y)$ as ${}_u\mathcal{U}_v$. One easily checks:

$$-\otimes{}_u\mathcal{U}_v = v_*(u^*(-)\otimes_W\mathcal{U}) \tag{3.5}$$

A bimodule isomorphic to one of the form ${}_u\mathcal{U}_u\cong{}_{\mathrm{id}}(u_*\mathcal{U})_{\mathrm{id}}$ is called *central*.

Next we introduce the language of sheaf-$\mathbb{Z}$-algebras, a "sheafified version" of classical $\mathbb{Z}$-algebras which we introduced in §0.2.

Definition 3.2.5. Let $(X_i)_{i\in\mathbb{Z}}$ be a sequence of smooth varieties.
A sheaf-$\mathbb{Z}$-algebra $\mathcal{A}$, is a collection of $X_i - X_j$-bimodules $\mathcal{A}_{ij}$ equipped with multiplication - and identity maps

$$\mu_{i,j,k} : \mathcal{A}_{i,j} \otimes \mathcal{A}_{j,k} \longrightarrow \mathcal{A}_{i,k} \text{ and } u_i : \mathcal{O}_{X_i} \longrightarrow \mathcal{A}_{i,i}$$

such that the associativity and unit diagrams

$$\begin{tikzcd}
\mathcal{A}_{i,j} \otimes \mathcal{A}_{j,k} \otimes A_{k,l} \arrow[r, "\mu_{i,j,k} \otimes \mathrm{id}_{\mathcal{A}_{k,l}}"] \arrow[d, "\mathrm{id}_{\mathcal{A}_{i,j}} \otimes \mu_{j,k,l}"'] & \mathcal{A}_{i,k} \otimes \mathcal{A}_{k,l} \arrow[d, "\mu_{i,k,l}"] \\
\mathcal{A}_{i,j} \otimes \mathcal{A}_{j,l} \arrow[r, "\mu_{i,j,l}"'] & \mathcal{A}_{i,l}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{O}_{X_i} \otimes \mathcal{A}_{i,j} \arrow[rr, "u_i \otimes \mathrm{id}_{\mathcal{A}_{i,j}}"] \arrow[dr] & & \mathcal{A}_{i,i} \otimes \mathcal{A}_{i,j} \arrow[dl, "\mu_{i,i,j}"] \\
& \mathcal{A}_{i,j} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A}_{i,j} \otimes \mathcal{O}_{X_j} \arrow[rr, "\mathrm{id}_{\mathcal{A}_{i,j}} \otimes u_j"] \arrow[dr] & & \mathcal{A}_{i,j} \otimes \mathcal{A}_{j,j} \arrow[dl, "\mu_{i,j,j}"] \\
& \mathcal{A}_{i,j} &
\end{tikzcd}$$

commute.

In a similar vein, we introduce the notion of a graded (right) module over a sheaf-$\mathbb{Z}$-algebras:

Definition 3.2.6. Let $\mathcal{A}$ be a sheaf-$\mathbb{Z}$-algebra.
A graded $\mathcal{A}$-module is a sequence of quasi-coherent $\mathcal{O}_{X_i}$-modules $\mathcal{M}_i$ together with maps

$$\mu_{\mathcal{M},i,j} : \mathcal{M}_i \otimes \mathcal{A}_{i,j} \longrightarrow \mathcal{M}_j$$

compatible with the multiplication and identity maps on $\mathcal{A}$ in the usual sense. A morphism of graded $\mathcal{A}$-modules $f : \mathcal{M} \longrightarrow \mathcal{N}$ is a collection of X_i-module morphisms $f_i : \mathcal{M}_i \longrightarrow \mathcal{N}_i$ such that the diagram

$$\begin{tikzcd}
\mathcal{M}_i \otimes \mathcal{A}_{i,j} \arrow[r, "f_i \otimes 1"] \arrow[d, "\mu_{\mathcal{M},i,j}"'] & \mathcal{N} \otimes \mathcal{A}_{i,j} \arrow[d, "\mu_{\mathcal{N},i,j}"] \\
\mathcal{M}_j \arrow[r, "f_j"'] & \mathcal{N}_j
\end{tikzcd}$$

commutes. The associated category is denoted $\mathrm{Gr}(\mathcal{A})$.

Definition 3.2.7. An $\mathcal{A}$-module is right bounded if $\mathcal{M}_i = 0$ for $i \gg 0$. An $\mathcal{A}$-module is called torsion if it is a filtered colimit of right bounded modules. Let $\mathrm{Tors}(\mathcal{A})$ be the subcategory of $\mathrm{Gr}(\mathcal{A})$ consisting of torsion modules. If $\mathrm{Gr}(\mathcal{A})$ is a locally noetherian category[1], $\mathrm{Tors}(\mathcal{A})$ is a localizing subcategory and the corresponding quotient category is denoted by $\mathrm{QGr}(\mathcal{A})$.

[1] As mentioned in the introduction, this property is nontrivial and in fact one of the main results of this paper in case $\mathcal{A}$ is a *symmetric sheaf-$\mathbb{Z}$-algebra*.

This construction yields a projection functor $p : \mathrm{Gr}(\mathcal{A}) \longrightarrow \mathrm{QGr}(\mathcal{A})$ with right adjoint ω (see [Smi00]). Moreover, as in §1.2, it is customary to think of $\mathrm{QGr}(\mathcal{A})$ as representing a (non-existing) noncommutative scheme $\mathrm{Proj}(A)$.

Remark 3.2.8. *Let $\mathcal{R}$ be a sheaf of graded algebras on X, then $\mathcal{R}$ induces a sheaf-$\mathbb{Z}$-algebra $\check{\mathcal{R}}$ on $(X_i)_{i\in\mathbb{Z}}$ with $X_i = X$ for all i via*

$$\check{\mathcal{R}}_{i,j} = \mathcal{R}_{j-i}. \tag{3.6}$$

As in Remark 0.2.2, there are induced equivalences of categories

$$\mathrm{Gr}(\check{\mathcal{R}}) \cong \mathrm{Gr}(\mathcal{R}) \text{ and } \mathrm{QGr}(\check{\mathcal{R}}) \cong \mathrm{QGr}(\mathcal{R}). \tag{3.7}$$

Remark 3.2.9. *It an easy observation that $\mathrm{Gr}(\mathcal{A})$ is abelian and that all universal constructions are defined "degreewise"*

The fundamental example of a graded right $\mathcal{A}$-module is given by the collection $e_n\mathcal{A}$ satisfying

$$\left(e_n\mathcal{A}\right)_i = \mathcal{A}_{n,i} \tag{3.8}$$

The first crucial step in our construction is a certain duality between locally free bimodules. To this end, we recall that by Convention 3.2.1, X and Y denote smooth varieties over k.

Lemma 3.2.10. *(see [VdB12, §4]) Let $\mathcal{E} \in \mathrm{bimod}(X - Y)$ be a locally free coherent bimodule. Then there is a unique object $\mathcal{E}^* \in \mathrm{bimod}(Y - X)$ such that the functor*

$$- \otimes_Y \mathcal{E}^* : \mathrm{Qcoh}(Y) \longrightarrow \mathrm{Qcoh}(X)$$

(given by (3.4)) is right adjoint to the functor $- \otimes_X \mathcal{E}.$, i.e for $\mathcal{M} \in \mathrm{Qcoh}(X)$ and $\mathcal{N} \in \mathrm{Qcoh}(Y)$:

$$\mathrm{Hom}_Y(\mathcal{M} \otimes \mathcal{E}, \mathcal{N}) \cong \mathrm{Hom}_X(\mathcal{M}, \mathcal{N} \otimes \mathcal{E}^*)$$

Remark 3.2.11. *Van den Bergh also gives an explicit formula for these dual bimodules (see the discussion following Proposition 4.1.6 in [VdB12]): if $\mathcal{E} = {}_u\mathcal{U}_v$ then $\mathcal{E}^*$ is given by ${}_v\,\mathcal{H}om_W(\mathcal{U}, v^!\mathcal{O}_Y)_u$*

The dual notion leads to the left dual: an object ${}^*\mathcal{E}$ such that

$$\mathrm{Hom}_X(\mathcal{N} \otimes^* \mathcal{E}, \mathcal{M}) \cong \mathrm{Hom}_Y(\mathcal{N}, \mathcal{M} \otimes \mathcal{E})$$

By Yoneda's lemma we have

$$\mathcal{E} =^* (\mathcal{E}^*) = ({}^*\mathcal{E})^* \tag{3.9}$$

and repeated application of duals leads to the following notation:

$$\mathcal{E}^{*n} = \begin{cases} \mathcal{E}^{\overbrace{*\cdots*}^{n}} & n \geq 0 \\ {}^{\overbrace{*\cdots*}^{-n}}\mathcal{E} & n < 0 \end{cases}$$

In the sequel it will be convenient to invoke the following notation:

Convention 3.2.12. *For given varieties X and Y, we shall without further mention denote the sequence $(X_n)_{n\in\mathbb{Z}}$ defined as*

$$X_n = X \text{ if } n \text{ is even and } X_n = Y \text{ if } n \text{ is odd}$$

From the adjointness properties of the duals defined above, there are unit and counit morphisms:

$$\begin{aligned} i_n : \mathcal{O}_{X_n} &\longrightarrow \mathcal{E}^{*n} \otimes \mathcal{E}^{*n+1} \\ j_n : \mathcal{E}^{*n} \otimes \mathcal{E}^{*n-1} &\longrightarrow \mathcal{O}_{X_n} \end{aligned} \tag{3.10}$$

Example 3.2.13. *Let $f : Y \to X$ be a finite morphism, where Y is a smooth variety over* Spec k, *and let $\mathcal{E}$ be the bimodule ${}_f(\mathcal{O}_Y)_{\mathrm{id}}$. One can use the explicit formula for $\mathcal{E}^*$ as in Remark 3.2.11 to obtain $\mathcal{E}^* = {}_{\mathrm{id}}(\mathcal{O}_Y)_f$. Using* (3.5) *we find*

$$- \otimes_X \mathcal{E} = f^*(-) \text{ and } - \otimes_Y \mathcal{E}^* = f_*(-). \tag{3.11}$$

such that the adjunction $-\otimes_X \mathcal{E} \dashv -\otimes_Y \mathcal{E}^*$ is nothing but the usual adjunction $f^* \dashv f_*$. Using Grothendieck duality we know that f_* has a right adjoint given by $f^!(-) = f^*(-) \otimes \omega_{X/Y}$ because f is finite and flat (see for example [LN07]), where $\omega_{X/Y} \in$ Qcoh(Y) *is defined by*

$$f_*\omega_{X/Y} = \mathcal{H}om(f_*\mathcal{O}_Y, \mathcal{O}_X). \tag{3.12}$$

In particular we find

$$\mathcal{E}^{**} = {}_f(\omega_{X/Y})_{\mathrm{id}} = \mathcal{E} \otimes \omega_{X/Y} \tag{3.13}$$

with associated unit morphism

$${}_{\mathrm{id}}(\mathcal{O}_Y)_{\mathrm{id}} \to \mathcal{E}^* \otimes \mathcal{E} \otimes {}_{\mathrm{id}}(\omega_{X/Y})_{\mathrm{id}} \tag{3.14}$$

Moreover by induction we have that

$$\begin{aligned} \mathcal{E}^{*(2n)} &= {}_f(\omega_{X/Y}^{n})_{\mathrm{id}} \\ \mathcal{E}^{*(2n+1)} &= {}_{\mathrm{id}}(\omega_{X/Y}^{-n})_f \end{aligned} \tag{3.15}$$

Remark 3.2.14. *It was proved in [VdB12, Lemma 3.1.7.] that for any $X - Y$-bimodule $\mathcal{E}$ the following holds:*

$$\mathcal{E}^{**} \cong \omega_{X/k}^{-1} \otimes \mathcal{E} \otimes \omega_{Y/k}$$

This is compatible with (3.13) *as*

$$\omega_{X/k}^{-1} \otimes_X {}_f(\mathcal{O}_Y)_{\mathrm{id}} \otimes_Y \omega_{Y/k} = {}_f\big(f^*(\omega_{X/k}^{-1}) \otimes \omega_{Y/k}\big)_{\mathrm{id}}$$

and

$$\omega_{X/Y} = f^*(\omega_{X/k}^{-1}) \otimes \omega_{Y/k}$$

where the latter isomorphism is induced by [Kle80, Corollary 24].

Our next ingredient is that of a nondegenerate bimodule.

Definition 3.2.15. We say that $\mathcal{Q} \in \mathrm{bimod}(X - W)$ is *invertible* if there exists a bimodule $\mathcal{Q}^{-1} \in \mathrm{bimod}(W - X)$ such that

$$\mathcal{Q} \otimes_W \mathcal{Q}^{-1} \cong \mathcal{O}_X \text{ and } \mathcal{Q}^{-1} \otimes_X \mathcal{Q} \cong \mathcal{O}_W.$$

If there exist bimodules $\mathcal{E} \in \mathrm{bimod}(X - Y)$ and $\mathcal{F} \in \mathrm{bimod}(Y - W)$ such that $\mathcal{Q} \subset \mathcal{E} \otimes_Y \mathcal{F}$, then we say the inclusion is *nondegenerate* if the canonical composition

$$\mathcal{E}^* \otimes_X \mathcal{Q} \longrightarrow \mathcal{E}^* \otimes_X \otimes (\mathcal{E} \otimes_Y \mathcal{F}) \longrightarrow \mathcal{F}$$

is an isomorphism.

Definition 3.2.16. Let $(X_i)_{i \in \mathbb{Z}}$ be a sequence of smooth varieties over k and let $\mathcal{E}_i$ be locally free $X_i - X_{i+1}$-bimodules. Then the *tensor-sheaf-$\mathbb{Z}$-algebra* $\mathrm{T}(\{\mathcal{E}_i\})$ is the sheaf-$\mathbb{Z}$-algebra generated by the $\{\mathcal{E}_i\}$, more precisely

$$\mathrm{T}(\{\mathcal{E}_i\})_{m,n} = \begin{cases} 0 & n < m \\ {}_{\mathrm{id}}\big(\mathcal{O}_{X_m}\big)_{\mathrm{id}} & n = m \\ \mathcal{E}_m \otimes \ldots \otimes \mathcal{E}_{n-1} & n > m \end{cases}$$

If $\mathcal{E}$ is a locally free $X - Y$-bimodule, we define the standard tensor algebra

$$\mathrm{T}(\mathcal{E}) := \mathrm{T}(\{\mathcal{E}_i\})$$

as above where we follow Convention 3.2.12 and define

$$\mathcal{E}_i = \mathcal{E}^{*i}$$

We can now state the definition of the main object of study in this chapter: the symmetric sheaf-$\mathbb{Z}$-algebra.

Definition 3.2.17. Let $(X_i)_{i\in\mathbb{Z}}$ be a sequence of smooth varieties over k and let $\mathcal{E}_i$ be locally free $X_i - X_{i+1}$-bimodules. Suppose that for each i we are given a nondegenerate $X_i - X_{i+2}$-bimodule $\mathcal{Q}_i \subset \mathcal{E}_i \otimes \mathcal{E}_{i+1}$, then the *symmetric sheaf-$\mathbb{Z}$-algebra* $\mathbb{S}(\{\mathcal{E}_i\}, \{\mathcal{Q}_i\})$ is the quotient of $\mathrm{T}(\{\mathcal{E}_i\})$ by the relations $(\mathcal{Q}_i)_i$. More precisely, $\mathbb{S}(\{\mathcal{E}_i\}, \{\mathcal{Q}_i\})_{m,n}$ is defined as

$$\begin{cases} \mathrm{T}(\{\mathcal{E}_i\})_{m,n} & n \leq m+1 \\ \mathrm{T}(\{\mathcal{E}_i\})_{m,n} / \Big(\mathcal{Q}_m \otimes \ldots) + (\mathcal{E}_m \otimes Q_{m+1} \otimes \ldots) + \ldots + (\ldots \otimes Q_{n-2}) \Big) & n \geq m+2 \end{cases}$$

If X and Y are smooth varieties, and $\mathcal{E}$ an $X - Y$-bimodule, the standard symmetric sheaf-$\mathbb{Z}$-algebra $\mathbb{S}(\mathcal{E})$ is constructed by considering the standard tensor algebra $\mathrm{T}(\mathcal{E})$ and considering the following sequence of nondegenerate invertible bimodules:

$$\mathcal{Q}_n = i_n \left(\mathcal{O}_{X_n}\right) \subset \mathcal{E}^{*n} \otimes \mathcal{E}^{*(n+1)} \tag{3.16}$$

A fundamental operation in the context of sheaf-$\mathbb{Z}$-algebras is that of twisting by a sequence of invertible bimodules:

Theorem 3.2.18. *Let $(X_i)_i$ and $(Y_i)_i$ be sequences of smooth varieties over k and $\mathcal{A}$ a sheaf-$\mathbb{Z}$-algebra on $(X_i)_i$. Given a collection of invertible $X_i - Y_i$-bimodules $(\mathcal{T}_i)_i$, one can construct a sheaf-$\mathbb{Z}$-algebra $\mathcal{B}$ by*

$$\mathcal{B}_{ij} := \mathcal{T}_i^{\;-1} \otimes \mathcal{A}_{ij} \otimes \mathcal{T}_j \tag{3.17}$$

called the twist of $\mathcal{A}$ by $(\mathcal{T}_i)_i$.

There is an equivalence of categories given by the functor

$$\mathcal{T} : \mathrm{Gr}(\mathcal{A}) \cong \mathrm{Gr}(\mathcal{B}) : \mathcal{M}_i \longrightarrow \mathcal{M}_i \otimes \mathcal{T}_i$$

Moreover, every symmetric sheaf-$\mathbb{Z}$-algebra can be obtained from a standard symmetric one by a twist.

Proof. This is proven in [VdB12, §5.1] □

We also have the following important result concerning graded modules over symmetric sheaf-$\mathbb{Z}$-algebras:

Theorem 3.2.19. *Let $\mathcal{A}$ be a symmetric sheaf-$\mathbb{Z}$-algebra. Then the graded module category* $\mathrm{Gr}(\mathcal{A})$ *is Grothendieck.*

Proof. Let $(\mathcal{M}_i, f_{ij})$ be a direct system of graded $\mathcal{A}$-modules. In each degree d, we obtain a direct system of quasi-coherent X_d-modules $(\mathcal{M}^d, f^d_{ij})$. Since $\mathrm{Qcoh}(X_n)$ is Grothendieck, we can form the direct limit in each degree to obtain a sequence of

X_n-modules $\mathcal{L}_n := \varinjlim(\mathcal{M}_i^n, f_{ij}^n)$. If we fix a couple (n,m), the universality of the direct limit naturally defines a map

$$\mathcal{A}_{n,m} \otimes \mathcal{L}_n = \mathcal{A}_{n,m} \otimes \varinjlim(\mathcal{M}_i^n, f_{ij}^n) \longrightarrow \varinjlim(\mathcal{M}_i^m, f_{ij}^m) = \mathcal{L}_m$$

showing that $\mathcal{L}$ is in fact a graded $\mathcal{A}$-module. The fact that $\mathcal{L}$ is a direct limit and that the formation of $\mathcal{L}$ is exact is an easy consequence of the construction.
Next, for each i, let $\left\{\mathcal{G}_i^j \mid j \in J_i\right\}$ be a collection of generators for $\mathrm{Qcoh}(X_i)$. Then the collection

$$\{\mathcal{G}_i^j \otimes e_i\mathcal{A} \mid i \in \mathbb{Z}, j \in J_i\}$$

forms a set of generators for $\mathrm{Gr}(\mathcal{A})$. □

3.2.2 The rank $(2,2)$-case

In this section, we give a proof of the result that $\mathrm{Proj}(\mathbb{S}(\mathcal{E}))$ is Morita equivalent to a commutative scheme in the case where $\mathcal{E} = {}_{\mathrm{id}}\mathcal{V}_{\mathrm{id}}$ has rank $(2,2)$. As mentioned in the introduction however, our primary concern is the rank $(4,1)$ case. This section's sole purpose is to acquire a little geometric intuition in $\mathbb{S}(\mathcal{E})$. Therefor, it may be skipped by the reader without any trouble.

We first begin by "sheafifying" the $\mathbb{Z}$-graded-to-$\mathbb{Z}$-algebra construction from Notation 0.2.1 to the current setting:

Convention 3.2.20. *Let $\mathcal{G}$ be a $\mathbb{Z}$-graded algebra object in the monoidal category* $\mathrm{bimod}(X-X)$. *We denote by $\check{\mathcal{G}}$ the sheaf-$\mathbb{Z}$-algebra over X whose (i,j)-component is the X-bimodule $\mathcal{G}_{j-i}$*

Remark 3.2.21. *Similar to Remark 0.2.2, we see that in the above situation, taking the direct sum yields an equivalence:*

$$\mathrm{Gr}(\mathcal{G}) \xrightarrow{\simeq} \mathrm{Gr}(\check{\mathcal{G}}) : (\mathcal{M})_i \mapsto \bigoplus_i \mathcal{M}_i$$

The following lemma (which was already announced but not proven in [VdB12]) shows that symmetric sheaf-$\mathbb{Z}$-algebras over central bimodules rank (2,2) essentially coincide with sheaves of commutative graded algebras:

Lemma 3.2.22. *Let $\mathcal{V}$ be a locally free $\mathcal{O}_X$-module of rank 2. Then there is an equivalence of the form*

$$\mathrm{Gr}(\mathbb{S}({}_{\mathrm{id}}\mathcal{V}_{\mathrm{id}})) \xrightarrow{\mathcal{T}} \mathrm{Gr}\left(\widecheck{\mathrm{Sym}_{X\times X}({}_{\mathrm{id}}\mathcal{V}_{\mathrm{id}})}\right) \xrightarrow{\cong} \mathrm{Gr}(\mathrm{Sym}_X(\mathcal{V}))$$

where $\mathcal{T}$ is given by twisting through $\left(\left(\bigwedge^2 \mathcal{V}\right)^{\lfloor \frac{i}{2} \rfloor}\right)_{i\in\mathbb{Z}}$.

Proof. We first describe the second equivalence: by the Remark 3.2.21, we may remove the hat and simply consider the sheaf of graded algebras $\mathrm{Sym}_{X\times X}\,\mathrm{id}({}_{\mathrm{id}}\mathcal{V}_{\mathrm{id}})$. The second equivalence now follows tautologically from the definitions, since in each degree d, d',

$$\mathcal{M}_d \otimes \mathrm{Sym}_{X\times X}({}_{\mathrm{id}}\mathcal{V}_{\mathrm{id}})_{d'} = \mathcal{M}_d \otimes {}_{\mathrm{id}}\left(\mathrm{Sym}_X(\mathcal{V})\right)_{\mathrm{id}} \overset{(3.5)}{=} \mathcal{M}_d \otimes_X \mathrm{Sym}_X(\mathcal{V})_{d'}$$

implying that both multiplications coincide. We now explain the first equivalence: Let $\mathcal{E} = {}_{\mathrm{id}}\mathcal{V}_{\mathrm{id}}$. Using the explicit expression for the dual given in Remark 3.2.11, we obtain

$$\mathcal{E}^* = {}_{\mathrm{id}}\mathcal{H}om(\mathcal{V}, \mathrm{id}^!\,\mathcal{O}_X)_{\mathrm{id}} = {}_{\mathrm{id}}(\mathcal{V}^*)_{\mathrm{id}}$$

In particular the equalities $\mathcal{E}^{*2n} = \mathcal{E} = {}_{\mathrm{id}}(\mathcal{V})_{\mathrm{id}}$ and $\mathcal{E}^{*2n+1} = \mathcal{E}^* = {}_{\mathrm{id}}(\mathcal{V}^*)_{\mathrm{id}}$ hold for all n. Since the pairing $\mathcal{V}\otimes\mathcal{V} \longrightarrow \Lambda^2\mathcal{V}$ is perfect, there is an isomorphism

$$\mathcal{V}^* \otimes (\Lambda^2\mathcal{V}) \xrightarrow{\cong} \mathcal{V} \tag{3.18}$$

Let $(\mathcal{T}_i)_i = (\wedge^2\mathcal{V})^{\lfloor \frac{i}{2}\rfloor}$. It follows from the definition of $\mathrm{T}(\mathcal{E})$, that as sheaf-$\mathbb{Z}$-algebras, we have

$$\mathrm{T}(\mathcal{E}) = \widetilde{T_X(\mathcal{V})}$$

By Theorem 3.2.18 applying the twist by the sequence $(\mathcal{T}_i)$ yields an equivalence

$$\mathrm{Gr}(\mathrm{T}(\mathcal{E})) \to \mathrm{Gr}(\widetilde{T_X(\mathcal{V})}) : \left(\mathcal{M}_i\right)_i \mapsto \left(\mathcal{M}_i \otimes (\Lambda^2\mathcal{V})^{\lfloor \frac{i}{2}\rfloor}\right)_i$$

specifically in each component:

$$\mathrm{T}(\mathcal{E})_{m,n} \cong {}_{\mathrm{id}}\left((\Lambda^2\mathcal{V})^{\lfloor \frac{m}{2}\rfloor} \otimes T_X(\mathcal{V})_{n-m} \otimes (\Lambda^2\mathcal{V})^{-\lfloor \frac{n}{2}\rfloor}\right)_{\mathrm{id}} \tag{3.19}$$

We now claim that the twisting in (3.19) induces a twisting

$$\mathbb{S}(\mathcal{E})_{m,n} \cong {}_{\mathrm{id}}\left((\Lambda^2\mathcal{V})^{\lfloor \frac{m}{2}\rfloor} \otimes \mathrm{Sym}_X(\mathcal{V})_{n-m} \otimes (\Lambda^2\mathcal{V})^{-\lfloor \frac{n}{2}\rfloor}\right)_{\mathrm{id}}$$

and hence an equivalence of categories:

$$\mathrm{Gr}(\mathbb{S}(\mathcal{E})) \longrightarrow \mathrm{Gr}(\mathrm{Sym}_X(\mathcal{V})) : \left(\mathcal{M}_i\right)_i \mapsto \bigoplus_i \mathcal{M}_i \otimes (\Lambda^2\mathcal{V})^{\lfloor \frac{i}{2}\rfloor} \tag{3.20}$$

So we are left with proving the claim. We must hence understand what happens under (3.19) to the relations that define $\mathbb{S}(\mathcal{E})$ as a quotient of $\mathrm{T}(\mathcal{E})$.

As the relations are generated in degree 2 it suffices to consider $\mathbb{S}(\mathcal{E})_{m,m+2} \otimes {}_{\mathrm{id}}(\Lambda^2\mathcal{V})_{\mathrm{id}}$. This is the quotient of $\mathrm{T}(\mathcal{E})_{m,m+2} \otimes {}_{\mathrm{id}}(\Lambda^2\mathcal{V})_{\mathrm{id}} \cong {}_{\mathrm{id}}\left(T_X(\mathcal{V})_2\right)_{\mathrm{id}} = {}_{\mathrm{id}}\left(\mathcal{V}\otimes\mathcal{V}\right)_{\mathrm{id}}$ by the relation $i\left({}_{\mathrm{id}}(\mathcal{O}_X)_{\mathrm{id}}\right) \otimes {}_{\mathrm{id}}(\Lambda^2\mathcal{V})_{\mathrm{id}} \subset {}_{\mathrm{id}}\left(\mathcal{V}\otimes\mathcal{V}^*\otimes\Lambda^2\mathcal{V}\right)_{\mathrm{id}} \cong {}_{\mathrm{id}}\left(\mathcal{V}\otimes\mathcal{V}\right)_{\mathrm{id}}$. We have to check that this relation is exactly the one that defines $\mathrm{Sym}_X(\mathcal{V})$ as a quotient of $\mathrm{T}_X(\mathcal{V})$. The latter relation is defined locally, so it suffices to check on a trivializing open subset U for $\mathcal{V}$. If $\mathcal{V}|_U \cong \mathcal{O}_X|_U\, u \oplus \mathcal{O}_X|_U\, v$ then $i\left({}_{\mathrm{id}}(\mathcal{O}_X)_{\mathrm{id}}\right)$ is locally given by $u\otimes u^* + v\otimes v^*$. One checks that the isomorphism (3.18) maps $u^*\otimes(u\wedge v)$ to v and $v^*\otimes(u\wedge v)$ to $-u$, the induced relation in $\mathcal{V}\otimes\mathcal{V}$ is locally given by $u\otimes v - v\otimes u$, the defining relation of $\mathrm{Sym}_X(\mathcal{V})$. □

We have the following result:

Proposition 3.2.23. *Let $\mathcal{E}$ be any $X-Y$-bimodule of rank* $(2,2)$. *Then* $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$ *is a locally noetherian category.*

Proof. This is [VdB12, Theorem 1.2]. □

This proposition ensures that we can perform the QGr construction as in Definition 3.2.7 on $\mathbb{S}(\mathcal{E})$ if the rank of $\mathcal{E}$ is (2,2). The resulting noncommutative scheme is equivalent a projective bundle over X as follows:

Corollary 3.2.24. *Let $\mathcal{V}$ be locally free of rank 2 as above, then we have an induced equivalence:*

$$\Phi : \mathrm{QGr}(\mathbb{S}({}_{\mathrm{id}}(\mathcal{V})_{\mathrm{id}})) \xrightarrow{\cong} \mathrm{QGr}(\mathrm{Sym}_X(\mathcal{V})) \xrightarrow{\cong} \mathrm{Qcoh}(\mathbb{P}_X(\mathcal{V}))$$

Proof. The equivalence given in (3.20) obviously maps torsion modules onto torsion modules, hence it factors to yield an equivalence $\mathrm{QGr}(\mathbb{S}({}_{\mathrm{id}}(\mathcal{V})_{\mathrm{id}}) \xrightarrow{\cong} \mathrm{QGr}(\mathrm{Sym}_X(\mathcal{V}))$.

The second equivalence is a well known result from classical algebraic geometry and is given by the following pair of functors

$$\begin{tikzcd} \mathrm{QGr}(\mathrm{Sym}_X(\mathcal{V})) \arrow[rr, bend left, "\widetilde{(-)}"] & & \mathrm{Qcoh}(\mathbb{P}_X(\mathcal{V})) \arrow[ll, bend left, "p\circ\Gamma_* := p\left[\oplus_i \pi_*\left((-)(i)\right)\right]"] \end{tikzcd}$$

Where π is the canonical projection $\pi : \mathbb{P}_X(\mathcal{V}) \longrightarrow X$. □

3.2.3 Truncation functors and periodicity

Let $\mathcal{A}$ be a sheaf-$\mathbb{Z}$-algebra over a sequence of varieties $(X_i)_{i\in\mathbb{Z}}$. Then we can define a sequence of *truncation* functors as follows: for each $m \in \mathbb{Z}$, we can consider the functor

$$\mathrm{Gr}(\mathcal{A}) \xrightarrow{(-)_m} \mathrm{Qcoh}(X_m)$$

We shall need the following easy result on these functors:

Lemma 3.2.25. *Let $e_m\mathcal{A}$ be the right $\mathcal{A}$-module defined in* (3.8). *There is an adjoint pair*

$$- \otimes e_m\mathcal{A} \dashv (-)_m$$

Proof. The proof of this is standard and left to the reader. □

Our next result shows that there is a certain 2-periodic behavior among these functors. To this end, for $n \in \mathbb{Z}$, we denote by $\mathcal{A}(n)$ the sheaf-$\mathbb{Z}$-algebra

$$\mathcal{A}(n)_{i,j} = \mathcal{A}_{n+i,n+j} \tag{3.21}$$

Proposition 3.2.26. *Let $(X_i)_{i\in\mathbb{Z}}$ be a sequence of smooth varieties and $\mathcal{A}$ be a symmetric sheaf-$\mathbb{Z}$-algebra. Then there is an autoequivalence α on $\mathrm{Gr}(\mathcal{A})$ inducing a commutative diagram for each m*

$$\begin{tikzcd}
\mathrm{Gr}(\mathcal{A}) \arrow[r, "(-)_m"] \arrow[d, "\alpha"'] & \mathrm{Qcoh}(X_m) \arrow[d, "-\otimes\omega_{X_m}"] \\
\mathrm{Gr}(\mathcal{A}) \arrow[r, "(-)_{m+2}"'] & \mathrm{Qcoh}(X_m)
\end{tikzcd}$$

Proof. By Theorem 3.2.18 $\mathcal{A}$ is Morita equivalent to a symmetric sheaf-$\mathbb{Z}$-algebra $\mathbb{S}(\mathcal{E})$ in standard form with $\mathcal{E} \in \mathrm{bimod}(X-Y)$ (using the Convention 3.2.12). Moreover by [VdB12, Lemma 3.1.7.], we have

$$\mathcal{E}^{*2} \cong \omega_{X/k}^{-1} \otimes \mathcal{E} \otimes \omega_{Y/k}$$

Hence the twist by the sequence of line bundles $(\omega_{X_i})_{i\in\mathbb{Z}}$ yields an equivalence of categories

$$\mathcal{T} : \mathrm{Gr}(\mathbb{S}(\mathcal{E})) \xrightarrow{\cong} \mathrm{Gr}(\omega^{-1} \otimes \mathbb{S}(\mathcal{E}) \otimes \omega) \xrightarrow{\cong} \mathrm{Gr}(\mathbb{S}(\mathcal{E}^{*2}))$$

where we used the short-hand notation

$$\left(\omega^{-1} \otimes \mathbb{S}(\mathcal{E}) \otimes \omega\right)_{m,n} = \omega_{X_m/k}^{-1} \otimes \mathbb{S}(\mathcal{E})_{m,n} \otimes \omega_{X_n/k}$$

Next, the construction of a standard symmetric sheaf-$\mathbb{Z}$-algebra implies that there is an equivalence $\Psi : \mathrm{Gr}(\mathbb{S}(\mathcal{E})(2)) \longrightarrow \mathrm{Gr}(\mathbb{S}(\mathcal{E}^{*2}))$ (where we used the notation (3.21)). We now simply define

$$\alpha := (-2) \circ \Psi^{-1} \circ \mathcal{T} : \mathrm{Gr}(\mathbb{S}(\mathcal{E})) \longrightarrow \mathrm{Gr}(\mathbb{S}(\mathcal{E}^{*2})) \longrightarrow \mathrm{Gr}(\mathbb{S}(\mathcal{E})(2)) \longrightarrow \mathrm{Gr}(\mathbb{S}(\mathcal{E}))$$

□

In the commutative rank 2 case (discussed in Lemma 3.2.22) the 0^{th} truncation functor coincides with the pushforward functor in the following sense:

Theorem 3.2.27. *Let $\mathcal{V}$ be a vector bundle on X of rank 2 and consider the associated symmetric sheaf-$\mathbb{Z}$-algebra $\mathbb{S}({}_{\mathrm{id}}(\mathcal{V})_{\mathrm{id}})$. Let*

$$\Phi : \mathrm{QGr}(\mathbb{S}({}_{\mathrm{id}}(\mathcal{V})_{\mathrm{id}})) \longrightarrow \mathrm{Qcoh}(\mathbb{P}_X(\mathcal{V}))$$

be the equivalence provided by Corollary 3.2.24. Then the following diagram commutes

$$\begin{tikzcd}
 & \mathrm{Gr}(\mathbb{S}({}_{\mathrm{id}}(\mathcal{V})_{\mathrm{id}})) \arrow[dr, "(-)_0"] & \\
\mathrm{QGr}(\mathbb{S}({}_{\mathrm{id}}(\mathcal{V})_{\mathrm{id}})) \arrow[ur, "\omega"] \arrow[dr, "\Phi"'] & & \mathrm{Qcoh}(X) \\
 & \mathrm{Qcoh}(\mathbb{P}_X(\mathcal{V})) \arrow[ur, "\pi_*"'] &
\end{tikzcd}$$

Proof. Let $Z := \mathbb{P}_X(\mathcal{V})$ and $\mathcal{A} := \mathbb{S}({}_{\mathrm{id}}(\mathcal{V})_{\mathrm{id}})$. We have to prove

$$\pi_*\Big(\oplus_i \widetilde{(-)\otimes \mathcal{T}_i}\Big) \cong \big(\omega(-)\big)_0$$

where $\mathcal{T}_i = \big(\big(\bigwedge^2 \mathcal{V}\big)^{\lfloor \frac{i}{2} \rfloor}\big)_{i\in\mathbb{Z}}$ is given as in the statement of Lemma 3.2.22.
Now by Lemma 3.2.25 and the definition of ω, the functor $\big(\omega(-)\big)_0$ is right adjoint to $p\big((-)\otimes e_0\mathcal{A}\big)$. Another formal computation using Corollary 3.2.24 shows that $\pi_*\Big(\oplus_i \widetilde{(-)\otimes \mathcal{T}_i}\Big)$ is right adjoint to the functor $\mathcal{T}^{-1}\left[(p\circ\Gamma_*)\big(\pi^*(-)\big)\right]$. This functor is in turn equal to $p\left[\big(\pi_*\big(\pi^*(-)(i)\big)\otimes \mathcal{T}_i^{-1}\big)_i\right]$, which by the projection formula, simplifies to $p\Big(\big((-)\otimes \pi_*\mathcal{O}_Z(i)\otimes \mathcal{T}_i^{-1}\big)_i\Big)$. The unicity of adjoint functors thus reduces the claim to proving the isomorphism

$$\big((-)\otimes \pi_*\mathcal{O}_Z(i)\otimes \mathcal{T}_i\big)_i \cong (-)\otimes e_0\mathcal{A} \tag{3.22}$$

Since $\mathrm{rk}(\mathcal{E}) \geq 2$, [Har97, Proposition II.7.11.a] implies that there is an isomorphism $\pi_*\big(\mathcal{O}_Z(i)\big) = \mathrm{Sym}_X(\mathcal{V})_i$. Now, by the choice of $\mathcal{T}_i$, we have $\mathrm{Sym}_X(\mathcal{V})_i = \mathcal{A}_{0i}\otimes \mathcal{T}_i$. (3.22) thus becomes

$$\big((-)\otimes \pi_*\mathcal{O}_Z(i)\otimes \mathcal{T}_i^{-1}\big)_i = \big((-)\otimes \mathcal{A}_{0i}\otimes \mathcal{T}_i\otimes \mathcal{T}_i^{-1}\big)_i = \big((-)\otimes \mathcal{A}_{0i}\big)_i = (-)\otimes e_0\mathcal{A}$$

proving the claim. □

We also have 1-periodicity for the truncation functors in this case:

Proposition 3.2.28. *Let $\mathcal{V}$ be a locally free sheaf of rank 2 on X and $\mathbb{S}({}_{\mathrm{id}}\mathcal{V}_{\mathrm{id}})$ the associated symmetric sheaf-$\mathbb{Z}$-algebra. Then there is an auto-equivalence β of $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$ and for each n, a line bundle $\mathcal{L}_n$ on X making the following diagram commute:*

$$\begin{tikzcd}
\mathrm{Gr}(\mathbb{S}({}_{\mathrm{id}}\mathcal{V}_{\mathrm{id}})) \arrow[r, "(-)_n"] \arrow[d, "\beta"'] & \mathrm{Qcoh}(X) \arrow[d, "-\otimes \mathcal{L}_n"] \\
\mathrm{Gr}(\mathbb{S}({}_{\mathrm{id}}\mathcal{V}_{\mathrm{id}})) \arrow[r, "(-)_{n+1}"'] & \mathrm{Qcoh}(X)
\end{tikzcd}$$

Proof. By Lemma 3.2.22 there is a sequence of $X - X$-bimodules $\mathcal{T}_i$ such that the following is an equivalence of categories

$$\mathrm{Gr}(\mathbb{S}({}_{\mathrm{id}}\mathcal{V}_{\mathrm{id}})) \longrightarrow \mathrm{Gr}(\mathrm{Sym}_X(\mathcal{V})) : (\mathcal{M}_i)_i \mapsto \bigoplus_i \mathcal{M}_i \otimes \mathcal{T}_i$$

Let (-1) denote the inverse shift functor on $\mathrm{Gr}(\mathrm{Sym}_X(\mathcal{V}))$, i.e. $(\mathcal{M}(-1))_i = \mathcal{M}_{i-1}$ and define β as the autoequivalence making the diagram

$$\begin{tikzcd}
\mathrm{Gr}(\mathbb{S}({}_{\mathrm{id}}\mathcal{V}_{\mathrm{id}})) \arrow[r, "\mathcal{T}"] \arrow[d, "\beta"'] & \mathrm{Sym}_X(\mathcal{V}) \arrow[d, "(-1)"] \\
\mathrm{Gr}(\mathbb{S}({}_{\mathrm{id}}\mathcal{V}_{\mathrm{id}})) \arrow[r, "\mathcal{T}"'] & \mathrm{Sym}_X(\mathcal{V})
\end{tikzcd}$$

commute. Since we clearly have $(-)_{n+1} \circ (-1) = (-)_n$, we get the required result by choosing the line bundle $\mathcal{L}_n \coloneqq \mathcal{T}_n \otimes \mathcal{T}_{n+1}^{-1}$ with $\mathcal{T}_n$ as in the proof of Lemma 3.2.22. ☐

Remark 3.2.29. *the previous result of 1-periodicity clearly implies 2-periodicity after repeated application in the sense that*

$$(-)_{n+2} \circ \beta^2 = \big(\mathcal{L}_{n+1} \otimes \mathcal{L}_n\big) \otimes (-)_n$$

hence one can wonder whether this periodicity coincides with Proposition 3.2.26. This is not the case in general. Indeed, from the explicit form of $\mathcal{T}$ in Proposition 3.2.26 and β in Proposition 3.2.28, we obtain

$$\mathcal{L}_n = \big(\bigwedge^2 \mathcal{V}\big)^{\lfloor \frac{n}{2} \rfloor} \otimes \big(\bigwedge^2 \mathcal{V}\big)^{-\lfloor \frac{n+1}{2} \rfloor}$$

and $\mathcal{L}_{n+1} \otimes \mathcal{L}_n = \big(\wedge^2(\mathcal{V})\big)^{-1}$, which obviously does not coincide with $\omega_{X/S}$ in general.

3.3 $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$ is locally noetherian.

As explained in the introduction, it is the case of a bimodule $\mathcal{E}$ of rank $(4,1)$ that we are particularly interested in. This section is dedicated to proving one of the important geometric properties of $\mathbb{S}(\mathcal{E})$ in this setting:

Theorem 3.3.1. *Let X and Y be smooth varieties over k and $\mathcal{E} \in \mathrm{bimod}(X - Y)$ be locally free of rank (4,1). Then the category $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$ is locally noetherian.*

Corollary 3.3.2. *With the assumptions as in Theorem 3.3.1: $\mathrm{QGr}(\mathbb{S}(\mathcal{E}))$ is locally noetherian.*

Proof. $\pi : \mathrm{Gr}(\mathbb{S}(\mathcal{E})) \longrightarrow \mathrm{QGr}(\mathbb{S}(\mathcal{E}))$ sends noetherian generators to noetherian generators by [Smi00, Lemma 14.19]. ☐

Convention 3.3.3. *Throughout this section we will always assume that X, Y and $\mathcal{E} \in \operatorname{bimod}(X - Y)$ satisfy the conditions in Theorem 3.3.1.*

The next lemma shows that under these assumptions, the bimodule $\mathcal{E}$ can written in a convenient form using a line bundle on Y and a finite map f of degree 4.

Lemma 3.3.4. *Assume that X, Y are smooth varieties of finite type and $\mathcal{E}$ is a locally free $X - Y$-bimodule of rank $(n, 1)$. Then there is a line bundle $\mathcal{L}$ on Y and a finite surjective morphism*[2] *$f : Y \longrightarrow X$ of degree n such that $\mathcal{E} \cong {}_f\mathcal{L}_{\mathrm{id}}$ (see Definition 3.2.4).*

Proof. Let $W \subset X \times Y$ be the scheme theoretic support of $\mathcal{E}$ and denote the projections $W \longrightarrow X$, $W \longrightarrow Y$ by g, h respectively:

$$\begin{tikzcd} & \operatorname{Supp}(\mathcal{E}) = W \arrow[d, hook, "\iota"] \arrow[ddl, bend right, "g"'] \arrow[ddr, bend left, "h"] & \\ & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

By definition g, h are finite morphisms. Furthermore $\mathcal{E} \cong {}_g(\mathcal{F})_h$ for $\mathcal{F} \in \operatorname{coh}(W)$ such that $\operatorname{Supp}\mathcal{F} = W$. By Lemma 3.3.5 below we conclude that h is an isomorphism and that $\mathcal{F}$ is a line bundle on W. Put $\mathcal{L} = h_*\mathcal{F}$, $f = gh^{-1}$. Then $\mathcal{E} \cong {}_f(\mathcal{L})_{\mathrm{id}}$. Since $\mathcal{L}$ is a line bundle, $f_*\mathcal{L}$ and $f_*\mathcal{O}_Y$ are locally isomorphic (e.g. by Lemma 5.3.8 below). So $f_*\mathcal{O}_Y$ is locally free of rank n as well and therefore f is flat of degree n. □

Lemma 3.3.5. *Assume that $h : W \longrightarrow Y$ is a finite morphism between smooth varieties, $\mathcal{F}$ is a coherent sheaf on W whose scheme theoretic support is W and $h_*\mathcal{F}$ is locally free of rank one. Then h is an isomorphism and $\mathcal{F}$ is a line bundle on W.*

Proof. Since h is finite, it is affine and we may restrict to an affine setting where $Y = \operatorname{Spec} R$, $W = \operatorname{Spec} S$ and $\mathcal{F} = \tilde{F}$ for F an S-module which is invertible as R-module. The composition of

$$R \xrightarrow{h} S \xrightarrow{s \mapsto (f \mapsto sf)} \operatorname{End}_R(F) \cong R$$

is the identity and the middle map is injective since W is the scheme-theoretic support of $\mathcal{F}$. It follows that all maps are isomorphisms. The claim follows. □

The following lemma shows that we can simplify the data describing $\operatorname{Gr}(\mathbb{S}(\mathcal{E}))$ even further

[2] note that f is automatically flat here [GD65, Proposition 6.1.5.]

Lemma 3.3.6. *Let X, Y are smooth varieties of finite type, let $f : Y \longrightarrow X$ be a finite surjective morphism of degree n and let $\mathcal{L}$ be a line bundle on Y. Then the following hold*

i) The category $\mathrm{Gr}(\mathbb{S}({}_f(\mathcal{L})_{\mathrm{id}})$ *does not depend on the choice of $\mathcal{L}$, i.e. there is an equivalence of categories*

$$\mathrm{Gr}(\mathbb{S}({}_f(\mathcal{L})_{\mathrm{id}}) \cong \mathrm{Gr}(\mathbb{S}({}_f(\mathcal{O}_Y)_{\mathrm{id}})$$

ii) The category $\mathrm{Gr}(\mathbb{S}({}_f(\mathcal{L})_{\mathrm{id}})$ *is invariant under the action of* $\mathrm{Aut}(X)$ *and* $\mathrm{Aut}(Y)$ *on $\{f : Y \to X$ surjective, finite, degree $n\}$. I.e.*

$$\mathrm{Gr}(\mathbb{S}({}_f(\mathcal{L})_{\mathrm{id}})) \cong \mathrm{Gr}(\mathbb{S}({}_{\varphi\circ f\circ\psi}(\mathcal{L})_{\mathrm{id}}))$$

Proof. Following Theorem 3.2.18 it suffices to show that $\mathbb{S}({}_f(\mathcal{L})_{\mathrm{id}})$, $\mathbb{S}({}_f(\mathcal{O}_Y)_{\mathrm{id}})$ and $\mathbb{S}({}_{\varphi\circ f\circ\psi}(\mathcal{O}_Y)_{\mathrm{id}})$ are twists of each other. We will show that this is the case for the associated tensor-sheaf-$\mathbb{Z}$-algebras and leave it as an exercise to the reader to check that this is compatible with the quadratic relations defining the symmetric sheaf-$\mathbb{Z}$-algebras.

i) Let $\mathcal{E}$ be any $X-Y$-bimodule. We have

$$\left(\mathcal{E} \otimes_Y {}_{\mathrm{id}}(\mathcal{L})_{\mathrm{id}}\right)^* = {}_{\mathrm{id}}(\mathcal{L}^{-1})_{\mathrm{id}} \otimes_Y \mathcal{E}^*$$

and

$$\left({}_{\mathrm{id}}(\mathcal{L}^{-1})_{\mathrm{id}} \otimes_Y \mathcal{E}^*\right)^* = \mathcal{E}^{**} \otimes_Y {}_{\mathrm{id}}(\mathcal{L})_{\mathrm{id}}$$

In particular, using the fact that ${}_f(\mathcal{L})_{\mathrm{id}} = {}_f(\mathcal{O}_Y)_{\mathrm{id}} \otimes_Y {}_{\mathrm{id}}(\mathcal{L})_{\mathrm{id}}$, it immediately follows that

$$\mathrm{T}\left({}_f(\mathcal{L})_{\mathrm{id}}\right)_{i,j} = \mathcal{T}_i^{-1} \otimes \mathrm{T}\left({}_f(\mathcal{O}_Y)_{\mathrm{id}}\right)_{i,j} \otimes \mathcal{T}_j$$

for

$$\mathcal{T}_n = \begin{cases} {}_{\mathrm{id}}(\mathcal{O}_X)_{\mathrm{id}} & n \text{ even} \\ {}_{\mathrm{id}}(\mathcal{L})_{\mathrm{id}} & n \text{ odd} \end{cases}$$

ii) We claim that

$$(\mathrm{T}({}_{\varphi\circ f\circ\psi}(\mathcal{O}_Y)_{\mathrm{id}}))_{i,j} \cong \mathcal{T}_i^{-1} \otimes (\mathrm{T}({}_f(\mathcal{O}_Y)_{\mathrm{id}}))_{i,j} \otimes \mathcal{T}_j. \tag{3.23}$$

holds for all i and j if we choose

$$\mathcal{T}_i := \begin{cases} {}_{\mathrm{id}}(\mathcal{O}_Y)_{\varphi} & \text{if } i \text{ is even} \\ {}_{\psi}(\mathcal{O}_Y)_{\mathrm{id}} & \text{if } i \text{ is odd} \end{cases}. \tag{3.24}$$

Indeed, it suffices to notice that

$${}_{\varphi\circ f\circ\psi}(\mathcal{O}_Y)_{\mathrm{id}} = {}_{\varphi}(\mathcal{O}_Y)_{\mathrm{id}} \otimes {}_f(\mathcal{O}_Y)_{\mathrm{id}} \otimes {}_{\psi}(\mathcal{O}_Y)_{\mathrm{id}}$$

and

$$({}_{\varphi}(\mathcal{O}_Y)_{\mathrm{id}})^{-1} = ({}_{\varphi}(\mathcal{O}_Y)_{\mathrm{id}})^* = {}_{\mathrm{id}}(\mathcal{O}_Y)_{\varphi}$$

□

Moreover we expect the following stronger result to hold:

Conjecture 3.3.7. *Let X, Y are smooth varieties of finite type and let $f, f' : Y \to X$ be finite surjective morphism of degree n such that*

$$\mathrm{Gr}(\mathbb{S}({}_f(\mathcal{O}_Y)_{\mathrm{id}}) \cong \mathrm{Gr}(\mathbb{S}({}_{f'}(\mathcal{O}_Y)_{\mathrm{id}}).$$

Then there exist $\psi \in \mathrm{Aut}(Y), \varphi \in \mathrm{Aut}(X)$ such that $f' = \varphi \circ f \circ \psi$.

Convention 3.3.8. *From now on we will only consider bimodules of rank $(4,1)$. Following the above lemmas, we shall assume these bimodules are of the form $\mathcal{E} = {}_f(\mathcal{O}_Y)_{\mathrm{id}}$ for some finite flat morphism $f : Y \longrightarrow X$ of degree 4.*

3.3.1 Restricting to an open subset

The first step in the proof of Theorem 3.3.1 is showing that there is an appropriate notion of restricting a sheaf-$\mathbb{Z}$-algebra to an open subset and that the statement of Theorem 3.3.1 can be reduced to an open cover in this sense.

To this end, we let $\mathcal{A}$ denote a sheaf-$\mathbb{Z}$-algebra over a sequence of smooth varieties X_i and $\mathcal{U} = (U^i)_{i\in\mathbb{Z}}$ be a sequence of affine open subsets $U^i \subset X_i$. For a bimodule $\mathcal{F} \in \mathrm{bimod}(X_m - X_{m+1})$, and a graded $\mathcal{A}$-module $\mathcal{M}$ we will use the notation $|_{\mathcal{U}}$ to denote the restriction to the corresponding open subset. I.e.

$$\begin{aligned} \mathcal{F}|_{\mathcal{U}} &:= \mathcal{F}|_{U^m \times U^{m+1}} \\ \left(\mathcal{A}|_{\mathcal{U}}\right)_{m,n} &:= (\mathcal{A}_{m,n})|_{\mathcal{U}} = (\mathcal{A}_{m,n})|_{U^m \times U^n} \\ \left(\mathcal{M}|_{\mathcal{U}}\right)_m &:= (\mathcal{M}_m)|_{U^m} \end{aligned} \tag{3.25}$$

To ensure that the restrictions of $\mathcal{A}$ to an open subset remains a sheaf-$\mathbb{Z}$-algebra, we need the following technical condition:

Lemma 3.3.9. *Let $\mathcal{A}$ be a sheaf-$\mathbb{Z}$-algebra and $\mathcal{U}$ as above such that $m, n \in \mathbb{Z}$:*

$$\mathrm{Supp}(\,(\mathcal{A}_{m,n})|_{U^m \times X_n}) \subset U^m \times U^n \text{ and } \mathrm{Supp}(\,(\mathcal{A}_{m,n})|_{X_m \times U^n}) \subset U^m \times U^n$$

then

i) $\mathcal{A}|_U$ has an induced algebra structure.

ii) Restriction of modules to U defines a functor $|_U : \mathrm{Gr}(\mathcal{A}) \to \mathrm{Gr}(\mathcal{A}|_U)$

Proof. *i)* We must show that for all $l, m, n \in \mathbb{Z}$ there are multiplication morphisms $\mathcal{A}_{l,m}|_U \otimes \mathcal{A}_{m,n}|_U \to \mathcal{A}_{l,n}|_U$ induced by the morphisms $\mathcal{A}_{l,m} \otimes \mathcal{A}_{m,n} \to \mathcal{A}_{l,n}$.

It is evident that the latter induces a morphism of $U^l - U^n$-bimodules as follows:

$$(\mathcal{A}_{l,m} \otimes \mathcal{A}_{m,n})|_U \longrightarrow \mathcal{A}_{l,n}|_U$$

Now the claim follows from the following chain of isomorphisms:

$$
\begin{aligned}
&(\mathcal{A}_{l,m} \otimes \mathcal{A}_{m,n})\big|_U \\
&\quad = \left(\pi_{X_l,X_n *}\left(\pi^*_{X_l,X_m}(\mathcal{A}_{l,m}) \otimes_{X_l\times X_m\times X_n} \pi^*_{X_m,X_n}(\mathcal{A}_{m,n})\right)\right)\Big|_{U^l\times U^n} \\
&\quad = \pi_{U^l,U^n *}\left(\left(\pi^*_{X_l,X_m}(\mathcal{A}_{l,m}) \otimes_{X_l\times X_m\times X_n} \pi^*_{X_m,X_n}(\mathcal{A}_{m,n})\right)\Big|_{U^l\times X_m\times U^n}\right) \\
&\quad = \pi_{U^l,U^n *}\left(\pi^*_{X_l,X_m}(\mathcal{A}_{l,m})\big|_{U^l\times X_m\times U^n} \otimes \pi^*_{X_m,X_n}(\mathcal{A}_{m,n})\big|_{U^l\times X_m\times U^n}\right) \\
&\quad = \pi_{U^l,U^n *}\left(\pi^*_{U^l,X_m}(\mathcal{A}_{l,m}|_{U^l\times X_m}) \otimes_{U^l\times X_m\times U^n} \pi^*_{X_m,U^n}(\mathcal{A}_{m,n}|_{X_m\times U^n})\right) \\
&\quad = \pi_{U^l,U^n *}\left(\pi^*_{U^l,U^m}(\mathcal{A}_{l,m}|_{U^l\times U^m}) \otimes_{U^l\times U^m\times U^n} \pi^*_{U^m,U^n}(\mathcal{A}_{m,n}|_{U^m\times U^n})\right) \\
&\quad = \mathcal{A}_{l,m}|_U \otimes \mathcal{A}_{m,n}|_U
\end{aligned}
$$

where π_{U^l,X_m} and π_{U^l,U^m} are the projections $\pi_{U^l,X_m} : U^l \times X_m \times U^n \to U^l \times X_m$ and $\pi_{U^l,U^m} : U^l \times U^m \times U^n \to U^l \times U^m$, with similar definitions for π_{X_m,U^n} and π_{U^m,U^n}.

The first equality is the definition of tensor product of bimodules

$$\operatorname{bimod}(X_l - X_m) \times \operatorname{bimod}(X_m - X_n) \to \operatorname{bimod}(X_l - X_n)$$

The second equality follows from the commutation of pushforward and restriction of sheaves. The third equality follows from the commutation of tensor product of sheaves and restriction. The fourth equality follows from the commutation of pullback and restriction of sheaves. The fifth equality follows the assumption of the lemma. The last equality is the definition of multiplication

$$\operatorname{bimod}(U^l - U^m) \times \operatorname{bimod}(U^m - U^n) \to \operatorname{bimod}(U^l - U^n)$$

ii) This essentially reduces to showing $(\mathcal{M}_i \otimes \mathcal{A}_{i,j})|_{U_j} = \left(\mathcal{M}|_U\right)_i \otimes \left(\mathcal{A}|_U\right)_{i,j}$ which is completely similar to *i)*. □

Our main motivation to study restriction of sheaf-$\mathbb{Z}$-algebra lies in the following result whose proof is straightforward:

Lemma 3.3.10. *Let $\mathcal{U}_\alpha$ be a finite set of sequences such that*

$$\forall i \in \mathbb{Z} : \bigcup_\alpha \left(U^i\right)_\alpha = X_i.$$

Assume that $\mathcal{A}$ is a sheaf-$\mathbb{Z}$-algebra such that the conditions in Lemma 3.3.9 are satisfied for all $\mathcal{U}_\alpha$, then

$$\left(\forall \alpha : \mathcal{M}|_{\mathcal{U}_\alpha} \in \operatorname{Gr}(\mathcal{A}|_{\mathcal{U}_\alpha}) \textit{ is noetherian }\right) \Longrightarrow \mathcal{M} \in \operatorname{Gr}(\mathcal{A}) \textit{ is noetherian}$$

Proof. Suppose we are given an ascending chain of sub-objects of $\mathcal{M}^n \subset \mathcal{M}$ in $\operatorname{Gr}(\mathcal{A})$ such that the restriction of this chain to all of the sequence $\mathcal{U}_\alpha$ stabilizes. As there are only finitely many $\mathcal{U}_\alpha$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$ and for all α: $(\mathcal{M}^n)|_{\mathcal{U}_\alpha} = (\mathcal{M}^{n+1})\big|_{\mathcal{U}_\alpha}$. The graded modules $\mathcal{M}^n$ and $\mathcal{M}^{n+1}$ must coincide. □

Following Convention 3.3.8, we now consider the case where $\mathcal{A} = \mathbb{S}(\mathcal{E})$ for $\mathcal{E} = {}_f(\mathcal{O}_Y)_{\mathrm{id}}$. Then, for an affine open subset $U \subset X$ we define the associated sequence $\mathcal{U}$ by $U^m \subset X_m$ as follows:

$$U^i = \begin{cases} U & \text{if } i \text{ is even} \\ f^{-1}(U) & \text{if } i \text{ is odd} \end{cases}$$

Note that U^i is indeed an affine open subset because f is a finite morphism. The results of Lemma 3.3.9 in this context can be stated as follows:

Corollary 3.3.11. *For any $U \subset X$,*

i) $\mathbb{S}(\mathcal{E})|_U$ *has an algebra structure induced by* $\mathbb{S}(\mathcal{E})$

ii) There is a functor $|_U : \mathrm{Gr}(\mathbb{S}(\mathcal{E})) \to \mathrm{Gr}(\mathbb{S}(\mathcal{E})|_U)$

iii) There is an isomorphism of symmetric sheaf-$\mathbb{Z}$-algebras: $\mathbb{S}(\mathcal{E})|_U \cong \mathbb{S}(\mathcal{E}|_U)$

Proof. i+ii) As $\mathcal{E}$ is given as ${}_f(\mathcal{O}_Y)_{\mathrm{id}}$ following Convention 3.3.8, the conditions in Lemma 3.3.9 are trivially satisfied for $\mathcal{A} = \mathbb{S}(\mathcal{E})$. For *iii)* we first show that for all $m \in \mathbb{N}$ there is a natural isomorphism

$$\theta_{\mathcal{E}} : (\mathcal{E}^{*m})\big|_U = \left(\mathcal{E}|_U\right)^{*m} \tag{3.26}$$

Using Example 3.2.13 we know that for each $m \in \mathbb{Z}$:

$$\begin{aligned} \mathcal{E}^{*(2m)} &= {}_f(\omega^{m}_{Y/X})_{\mathrm{id}} \\ \mathcal{E}^{*(2m+1)} &= {}_{\mathrm{id}}(\omega^{-m}_{Y/X})_f \end{aligned} \tag{3.27}$$

In particular the existence of $\theta_{\mathcal{E}}$ follows immediately from $(\omega_{Y/X})\big|_U = \omega_{f^{-1}U/U}$.

Finally, the naturality of $\theta_{\mathcal{E}}$ immediately implies that the restricted unit morphisms $i_m|_U$ coincides with

$${}_{\mathrm{id}}\left(\mathcal{O}_{U^m}\right)_{\mathrm{id}} \longrightarrow \left(\mathcal{E}|_U\right)^{*m} \otimes \left(\mathcal{E}|_U\right)^{*m+1}$$

Implying in particular that $\theta_{\mathcal{E}}$ induces an isomorphism

$$i_m\left({}_{\mathrm{id}}\left(\mathcal{O}_{U^m}\right)_{\mathrm{id}}\right) \cong i_m\left({}_{\mathrm{id}}\left(\mathcal{O}_{X^m}\right)_{\mathrm{id}}\right)\Big|_{U^m}$$

and we can extend $\theta_{\mathcal{E}}$ to an isomorphism

$$\theta : \mathbb{S}(\mathcal{E})|_U \cong \mathbb{S}(\mathcal{E}|_U)$$

□

3.3.2 Covering by relative Frobenius pairs

Lemma 3.3.10 shows that proving that a given set of generators is in fact a set of noetherian generators can be done locally. In this subsection we construct an open cover $X = \bigcup_l U_l$ for which the sections satisfy a relative version of the Frobenius property as introduced in Chapter 2 (see Lemma 3.3.13 *iv)* and Definition 2.1.1).

Throughout, we shall make use of the following lemma, well-known to experts:

Lemma 3.3.12. *Let $f : Y \longrightarrow X$ be a finite morphism of smooth varieties. Let $\mathcal{L}$ be a line bundle on Y and $p \in X$. Then there is an open subset $U \subset X$ containing p, such that $\mathcal{L}|_{f^{-1}(U)} \cong \mathcal{O}_{f^{-1}(U)}$.*

Proof. Since affine open subsets form a base for the topology on X and f is affine (as it is finite), we can reduce to the case where $X = \mathrm{Spec}(R)$, $Y = \mathrm{Spec}(S)$ are affine varieties over k and S is finitely generated over R and $\mathcal{L} = \tilde{L}$ for some invertible S-module L. Let $\mathfrak{p}$ be the prime ideal in $\mathrm{Spec}(R)$ corresponding to $f(p) \in X$, then $S_{\mathfrak{p}} := S \otimes_R R_{\mathfrak{p}}$ is a semi-local ring, hence every finitely generated projective $S_{\mathfrak{p}}$-module of constant rank is free and in particular the Picard group of $S_{\mathfrak{p}}$ is trivial. Consequently, there exists an $l \in L$ such that

$$S_{\mathfrak{p}} \xrightarrow{\cdot l} L_{\mathfrak{p}}$$

is an isomorphism.

Now consider the morphism $S \xrightarrow{\cdot l} L$ with kernel K and cokernel C. Then there is an exact sequence

$$0 \longrightarrow K \longrightarrow S \xrightarrow{\cdot l} L \longrightarrow C \longrightarrow 0 \tag{3.28}$$

K is a finitely generated R-submodule of S by the noetherianity of R. L is finitely generated over R, being an invertible S-module. It follows that C is finitely generated over R as a quotient of L.

Now let $\alpha_1, \ldots, \alpha_n$ be a set of generators for K, then as $K \otimes R_{\mathfrak{p}} = 0$ there exist elements $x_1, \ldots, x_n \in R\backslash\mathfrak{p}$ such that $\alpha_1 x_1 = \ldots = \alpha_n x_n = 0$. Set $x := x_1 \cdot \ldots \cdot x_n \in R\backslash\mathfrak{p}$, then $\alpha \cdot x = 0$ for all $\alpha \in K$. Similarly there is a $x' \in R\backslash\mathfrak{p}$ such that $\beta \cdot x' = 0$ for all $\beta \in C$. Now define $z = x \cdot x'$, then $K \otimes R_z = C \otimes R_z = 0$ implying that $\cdot l$ defines an isomorphism

$$S \otimes R_z \xrightarrow{\cong} L \otimes R_z$$

$U = \mathrm{Spec}(R_z)$ then is the desired open subset. □

We can now prove the main lemma of this subsection, which yields a cover over which many desirable geometric properties are satisfied:

Lemma 3.3.13. *Write $\mathcal{E} = {}_f(\mathcal{L})_{\mathrm{id}}$ as in Lemma 3.3.4 . There is a finite cover $X = \bigcup_l U_l$ by affine open subsets $U_l = \mathrm{Spec}(R_l)$ such that:*

i) $\mathcal{L}|_{f^{-1}(U_l)}$ *is a trivial* $\mathcal{O}_{f^{-1}(U_l)}$*-module*

ii) $\omega_Y|_{f^{-1}(U_l)}$ *is a trivial* $\mathcal{O}_{f^{-1}(U_l)}$*-module*

iii) $\omega_X|_{U_l}$ *is a trivial* $\mathcal{O}_{U_l}$*-module*

iv) $f^{-1}(U_l) = \mathrm{Spec}(S_l)$ *where* S_l/R_l *is relative Frobenius of rank 4.*

Proof. We first note the following two facts:

- Let $\mathrm{Spec}(R)$ be an affine open subset on which i), ii), iii) or iv) holds. Then the same statement holds for any standard open $\mathrm{Spec}(R_f) \subset \mathrm{Spec}(R)$. This is obvious for i), ii) and iii). For iv) it follows from Lemma 2.3.1.

- Let $\mathrm{Spec}(R)$ and $\mathrm{Spec}(R')$ be affine open subsets of X, then their intersection is covered by open subsets which are simultaneously distinguished in each space, in other words subsets of the form $\mathrm{Spec}(R_f) = \mathrm{Spec}(R'_g)$

By these two facts it suffices to find affine open covers for i), ii), iii) and iv) separately. For i) and ii) such a cover exists by Lemma 3.3.12 and the fact that ω_Y is a line bundle on the smooth variety Y. The existence of a cover satisfying iii) is immediate from the fact that ω_X is a line bundle. We have reduced the claim to exhibiting a cover satisfying iv).

Now by Lemma 3.3.12: $f^!\omega_X$ is completely determined by $f_*\left(f^!\omega_X\right)$ and we have an isomorphism of $f_*\mathcal{O}_Y$-modules

$$f_*\left(f^!\omega_X\right) := \mathcal{H}om_X(f_*\mathcal{O}_Y, \omega_X) \cong f_*\omega_Y \tag{3.29}$$

As moreover f is also surjective and flat, there is a cover $X = \bigcup_l U_l$ with $U_l = \mathrm{Spec}(R_l)$ and $f^{-1}(U_l) = \mathrm{Spec}(S_l)$ where S_l is a free R_l-module of rank 4 for each l. By the previous arguments we can assume that ii) and iii) are also satisfied on this cover. In this case, replacing f by its restriction $f^{-1}(U_l) \longrightarrow U_l$, (3.29) reads

$$f_*\left(f^!\mathcal{O}_{U_l}\right) := \mathcal{H}om_{U_l}(f_*\mathcal{O}_{f^{-1}(U_l)}, \mathcal{O}_{U_l}) \cong f_*\mathcal{O}_{f^{-1}(U_l)}$$

and taking sections yields the required isomorphism of S_l-modules:

$$\mathrm{Hom}_{R_l}(S_l, R_l) \cong S_l$$

□

Remark 3.3.14. *Recall that Lemma 3.3.6 showed that up to equivalence of categories one can always trivialize the line bundle $\mathcal{L}$ is a noncommutative $\mathbb{P}^1$-bundle $\mathrm{QGr}(\mathbb{S}({}_f(\mathcal{L})_{\mathrm{id}}))$. The above lemma independently shows that affine locally we can obtain this trivialization without the use of an equivalence of categories.*

3.3.3 From periodic $\mathbb{Z}$-algebras to graded algebras

The previous section showed how we can reduce the statement of Theorem 3.3.1 to the case where X and Y are affine, and satisfy some convenient geometric properties (see Lemma 3.3.13). In this section, we provide a second technical tool which allows us to reduce to the case where the $\mathbb{Z}$-algebra comes from a graded algebra. The $\widetilde{(-)}$-construction (see Notation 0.2.1) assigns a (1-periodic) $\mathbb{Z}$-algebra to a graded algebra. In this section, we consider the converse problem. More precisely, we show that an n-periodic $\mathbb{Z}$-algebras A gives rise to a graded algebra $\overline{A}$ such that $\mathrm{Gr}(A)$ is a direct summand of the category $\mathrm{Gr}(\overline{A})$. We start by describing the following slight generalization of $\mathbb{Z}$-algebras in order to be able to easily apply the result in our required setting:

Definition 3.3.15. Let $(R_i)_{i\in\mathbb{Z}}$ be a sequence of commutative rings. A *bimodule $\mathbb{Z}$-algebra* over $(R_i)_{i\in\mathbb{Z}}$ is a collection of $R_i - R_j$-bimodules $A_{i,j}$ together with multiplication maps

$$A_{i,j} \otimes_{R_j} A_{j,l} \longrightarrow A_{i,l}$$

and R_i-linear unit maps $R_i \longrightarrow A_{i,i}$ satisfying the usual $\mathbb{Z}$-algebra axioms.

If $\forall i:\ R_i = R$, then A is called a bimodule $\mathbb{Z}$-algebra over R.

Definition 3.3.16. Let A be a $\mathbb{Z}$-algebra over $(R_i)_{i\in\mathbb{Z}}$ and $d > 0$ an integer. Assume that for each i, we have $R_{i+d} = R_i$. We say A is d-periodic if there is an isomorphism of $\mathbb{Z}$-algebras $\varphi : A \xrightarrow{\sim} A(d)$. I.e. there is a collection of $R_i - R_j$-bimodule isomorphisms $\{\varphi_{ij} : A_{i,j} \xrightarrow{\sim} A_{i+d,j+d}\}_{i,j}$ compatible with the multiplication and unit maps.

Let A be d-periodic and let $R := \bigoplus_{i=0}^{d-1} R_i$. We construct a graded R-algebra $\overline{A}$ as follows: let $\overline{A}_n$ be a $d \times d$-matrix with entries:

$$\left(\overline{A}_n\right)_{i,j} = \begin{cases} A_{i,i+n} & \text{if } j - i \equiv n \pmod{d} \\ 0 & \text{else} \end{cases} \tag{3.30}$$

(Where we use the convention that the numbering of rows and columns of the matrix starts at 0 instead of 1.)

By way of example,

$$\overline{A}_1 = \begin{pmatrix} 0 & A_{0,1} & 0 & \dots & 0 \\ 0 & 0 & A_{1,2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_{d-2,d-1} \\ A_{d-1,d} & 0 & 0 & \dots & 0 \end{pmatrix}$$

Each $\overline{A}_n$ is naturally a left (resp. right) R-module by letting a d-tuple $(r_0, \dots r_{d-1})$ act as a diagonal matrix D with entries $D_{ii} := r_i$ on the left (resp. right).
Moreover, there is a canonical multiplication map

$$\overline{A}_n \otimes_R \overline{A}_m \longrightarrow \overline{A}_{n+m}$$

given by the ordinary matrix multiplication and applying the periodicity isomorphisms ϕ_{ij} whenever necessary. The $(R_i)_{i\in\mathbb{Z}}$-linearity of the $\mathbb{Z}$-algebra multiplication implies that the above maps are indeed R-bilinear.

Lemma 3.3.17. *Suppose A is d-periodic, then the above maps define a graded (unital) R-algebra structure on the R-module $\overline{A} := \oplus_{i\in\mathbb{Z}}\overline{A}_i$*

Proof. The reader checks that the compatibility of the periodicity isomorphisms with the $\mathbb{Z}$-algebra multiplication maps implies that the multiplication is associative. The algebra has a unit given by

$$1 = \begin{pmatrix} e_0 & 0 & \dots & 0 \\ 0 & e_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e_{d-1} \end{pmatrix} \in \overline{A}_0$$

where e_i is the unit in A_{ii}. □

There is a convenient description of the category of graded right $\overline{A}$-modules as follows: let $M \in \mathrm{Gr}(\overline{A})$. Then by definition we have a decomposition $M = \bigoplus_{i\in\mathbb{Z}} M_i$. Moreover, each R-module M_i in turn has a direct sum decomposition given by $M_i = \oplus_{j=0}^{d-1} e_j M_i$. We define $M_i^j := e_j M_i$. This decomposition allows us to give a description of the $\overline{A}$-module structure of M. For a matrix $\overline{a} \in \overline{A}_m$, $e_j.\overline{a}$ only has one nonzero entry at position $(j, j+m)$. It follows from the right R-structure on A_m that $e_j\overline{a} = \overline{a}.e_{j+m}$ (where we consider $j+m$ mod d following (3.30)). Thus the right action of $\overline{A}_m$ on M_i^j becomes a map of the form $M_i^j \otimes A_{j,j+m} \longrightarrow M_{i+m}^{j+m}$ or equivalently for $l = j+m$,

$$M_i^j \otimes A_{j,l} \longrightarrow M_{i+l-j}^l$$

Lemma 3.3.18. *Suppose A is d-periodic and let $\mathscr{C}$ be the category defined as follows:*

- $\mathrm{Ob}(\mathscr{C})$ *consists of collection of R-modules $(M_i^j)_{i\in\mathbb{Z},0\le j\le d-1}$, such that M_i^j is an R_j-module together with multiplication maps*

$$\mu_{i,j,l}^M : M_i^j \otimes A_{j,l} \longrightarrow M_{i+l-j}^l$$

for each i, j, l (where l and $i+l-j$ should be interpreted modulo d) satisfying the obvious compatibility condition for multiplication and unit.

- *a morphism $M \to N$ in $\mathscr{C}$ is a collection $f_{i,j}$ of R_j- linear maps $M_i^j \to N_i^j$ such that*
$$f_{i+l-j,l} \circ \mu^M_{i,j,l} = \mu^N_{i,j,l} \circ (f_{i,j} \otimes A_{j,l})$$

Then there is a canonical isomorphism of categories $\mathscr{C} \cong \mathrm{Gr}(\overline{A})$

Proof. The above discussion shows that the assignment $M \to (M_l e_i)_{l\in\mathbb{Z},0\le i\le n-1}$ is well defined and essentially surjective. A morphism of graded modules $f : M \longrightarrow N$ will satisfy $f(M_i e_j) \subset N_i e_j$ and we can define $f_{i,j}$ as the restriction to these submodules. The A-linearity guarantees that $(f_{i,j})_{i,j}$ indeed defines a morphism in $\mathcal{C}$ and since $\oplus M_i e_j = M$ it is clear that this assignment is faithful. Since any collection of maps $f_{i,j}$ satisfying the above compatibility with the multiplication will sum up to an $\overline{A}$-linear map, the assignment is also full. □

Lemma 3.3.19. *There is a decomposition*

$$\mathscr{C} = \mathscr{C}_0 \oplus \ldots \oplus \mathscr{C}_{d-1}$$

where $\mathscr{C}_n$ is the full subcategory of $\mathscr{C}$ whose objects are collections of R-modules $(M_i^j)_{i\in\mathbb{Z},0\le j\le d-1}$ where $M_i^j = 0$ unless $j - i \equiv n$ (mod d).

Proof. This follows immediately from the construction of $\mathscr{C}$ and the fact that $j - i = l - (l + i - j)$. Hence, if $(\mathcal{M}_i^j)_{ij}$ is a non-zero object in $\mathscr{C}_n$, then so is $(\mathcal{M}^l_{l+i-j})_{i,j}$ for all l. □

Proposition 3.3.20. *There is an exact embedding of categories*

$$\overline{(-)} : \mathrm{Gr}(A) \hookrightarrow \mathrm{Gr}(\overline{A})$$

moreover the essential image is a direct summand of $\mathrm{Gr}(\overline{A})$.

Proof. Let M be an A-module with multiplication maps $\mu_{i,m} : M_i \otimes_R A_m \longrightarrow M_{i+m}$ and let $\mathcal{C}$ be as above. We define an object $\overline{M}$ in $\mathscr{C}$ by

$$\overline{M}_i^j = \begin{cases} M_i & \text{if } j \equiv i \bmod d \\ 0 & \text{else} \end{cases}$$

where the multiplication is given by

$$\overline{\mu}_{i,j,l} = \begin{cases} \mu_{i,l-j} & \text{if } j \equiv i \bmod d \\ 0 & \text{else} \end{cases}$$

This assignment clearly defines an exact embedding

$$\mathrm{Gr}(A) \xrightarrow{\simeq} \mathscr{C}_0 \hookrightarrow \mathscr{C}$$

Which finishes the proof by Lemmas 3.3.18 and 3.3.19. □

3.3.4 A local description of $\mathbb{S}(\mathcal{E})$

In this final step in the preparation of the proof of Theorem 3.3.1, we complete the local description of $\mathbb{S}(\mathcal{E})$. By Lemma 3.3.13, we have reduced the claim to the case where X and Y are affine. By our hypothesis on X and $\mathcal{E}$ (see Conventions 3.2.1 and 3.3.8), we assume that $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$ are affine varieties over k such that S/R is relative Frobenius of rank 4 with induced morphism $f : Y \longrightarrow X$, that $\mathcal{E} = {}_f(\mathcal{O}_Y)_{\mathrm{id}}$, $\omega_X \cong \mathcal{O}_X$ and $\omega_Y \cong \mathcal{O}_Y$. After applying the global section functor, we obtain a bimodule $\mathbb{Z}$-algebra in the sense of Definition 3.3.15 which is 2-periodic. The graded algebra associated to this $\mathbb{Z}$-algebra by the construction in §3.3.3 is precisely the generalized preprojective algebra defined in Definition 2.1.3 and studied in Chapter 2.

We start by introducing some auxiliary notations. Recall Convention 3.2.12 and let $\mathcal{A}$ be a sheaf-$\mathbb{Z}$-algebra over X_i. There is a $\mathbb{Z}$-algebra over k, $\Gamma(\mathcal{A})$ defined in each component by

$$\Gamma(\mathcal{A})_{i,j} := \Gamma(X_i \times X_j, \mathcal{A}_{i,j})$$

since each component $\Gamma(\mathcal{A})_{i,j}$ is an $R-S$, $R-R$, $S-S$ or $S-R$ bimodule depending on the parity of the indices, $\Gamma(\mathcal{A})$ is in fact a $\mathbb{Z}$-algebra over commutative groundring $R \oplus S$. The equivalence between quasi-coherent sheaves over an affine scheme and modules over the ring of global sections can easily be adapted to our setting to yield an equivalence:

$$\Gamma : \mathrm{Gr}(\mathcal{A}) \xrightarrow{\cong} \mathrm{Gr}(\Gamma(\mathcal{A})) : \{\mathcal{M}_n\}_{n\in\mathbb{Z}} \mapsto \{\Gamma(X_n, \mathcal{M}_n)\}_{n\in\mathbb{Z}}$$

The following is an immediate consequence of the assumptions of this section:

Lemma 3.3.21. *The $\mathbb{Z}$-algebra $\Gamma(\mathbb{S}(\mathcal{E}))$ is 2-periodic in the sense that*

$$\Gamma(\mathbb{S}(\mathcal{E}))_{i,j} = \Gamma(\mathbb{S}(\mathcal{E}))_{i+2,j+2}$$

Proof. By Proposition 3.2.26, there are isomorphisms $\mathbb{S}(\mathcal{E}))_{i+2,j+2} \cong \omega_i^{-1} \otimes \mathbb{S}(\mathcal{E})) \otimes \omega_j$. By the assumptions in the beginning of this section, both canonical bundles are trivial, implying that $\mathbb{S}(\mathcal{E})_{i,j} = \mathbb{S}(\mathcal{E})_{i+2,j+2}$. The result follows after applying $\Gamma(-)$. □

Using Lemma 3.3.17, the 2-periodic $\mathbb{Z}$-algebra $\Gamma(\mathbb{S}(\mathcal{E}))$ gives rise to a graded algebra $\overline{\Gamma(\mathbb{S}(\mathcal{E}))}$. We now prove that this graded algebra is in fact a generalized preprojective algebra as in Chapter 2:

Lemma 3.3.22. *Let $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$ be smooth affine varieties such that S/R is relative Frobenius of rank 4. Let $f : Y \to X$ be the induced morphism and $\mathcal{E} = {}_f(\mathcal{O}_Y)_{\mathrm{id}}$. Then $\overline{\Gamma(\mathbb{S}(\mathcal{E}))} \cong \Pi_R(S)$.*

Proof. Consider the quotient map

$$\mathrm{T}(\mathcal{E}) \twoheadrightarrow \mathbb{S}(\mathcal{E})$$

Taking global sections in each component $\Gamma(X_i \times X_j, (-)_{i,j})$ yields a surjection

$$\Gamma(\mathrm{T}(\mathcal{E})) \twoheadrightarrow \Gamma(\mathbb{S}(\mathcal{E})).$$

as $X_i \times X_j$ is affine.
Since the functor $\overline{(-)}$ preserves surjectivity (see Proposition 3.3.20), we obtain a map

$$\pi : \overline{\Gamma(\mathrm{T}(\mathcal{E}))} \twoheadrightarrow \overline{\Gamma(\mathbb{S}(\mathcal{E}))}.$$

We first show that there is a canonical isomorphism of $R \oplus S$-modules

$$\overline{\Gamma(\mathrm{T}(\mathcal{E}))} \cong T(R,S) \tag{3.31}$$

For this, (as $\Gamma(\mathbb{S}(\mathcal{E}))$ is clearly generated in degrees 0 and 1) it suffices to show the following three facts:

- $\overline{\Gamma(\mathrm{T}(\mathcal{E}))}_0 \cong T(R,S)_0 = R \oplus S$ as rings
- $\overline{\Gamma(\mathrm{T}(\mathcal{E}))}_1 \cong T(R,S)_1 \cong {}_R S_S \oplus {}_S S_R$ as $R \oplus S$-modules
- the multiplication map yields isomorphisms

$$\overline{\Gamma(\mathrm{T}(\mathcal{E}))}_1 \otimes \overline{\Gamma(\mathrm{T}(\mathcal{E}))}_n \xrightarrow{\cong} \overline{\Gamma(\mathrm{T}(\mathcal{E}))}_{n+1}$$

For the first statement, we compute:

$$\begin{aligned}
\overline{\Gamma(\mathrm{T}(\mathcal{E}))}_0 &= \begin{pmatrix} \Gamma(\mathrm{T}(\mathcal{E}))_{0,0} & 0 \\ 0 & \Gamma(\mathrm{T}(\mathcal{E}))_{1,1} \end{pmatrix} \\
&= \begin{pmatrix} \Gamma\left(X \times X, {}_{\mathrm{id}}(\mathcal{O}_X)_{\mathrm{id}}\right) & 0 \\ 0 & \Gamma\left(Y \times Y, {}_{\mathrm{id}}(\mathcal{O}_Y)_{\mathrm{id}}\right) \end{pmatrix}
\end{aligned}$$

moreover, we have

$$\begin{aligned}
\Gamma\left(X \times X, {}_{\mathrm{id}}(\mathcal{O}_X)_{\mathrm{id}}\right) &= \mathrm{Hom}\left(\mathcal{O}_{X \times X}, \Delta_*\left(\mathcal{O}_X\right)\right) \\
&= \mathrm{Hom}\left(\Delta^*\left(\mathcal{O}_{X \times X}\right), \mathcal{O}_X\right) \\
&= \mathrm{Hom}\left(\mathcal{O}_X, \mathcal{O}_X\right) \\
&\cong R
\end{aligned}$$

And similarly $\Gamma\left(Y \times Y, {}_{\mathrm{id}}(\mathcal{O}_Y)_{\mathrm{id}}\right) \cong S$. combining these calculations yields

$$\overline{\Gamma(\mathrm{T}(\mathcal{E}))}_0 \cong \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} \cong R \oplus S$$

In a completely similar fashion, we check the second condition:

$$\begin{aligned}
\overline{\Gamma(\mathrm{T}(\mathcal{E}))}_1 &= \begin{pmatrix} 0 & \Gamma(\mathrm{T}(\mathcal{E}))_{0,1} \\ \Gamma(\mathrm{T}(\mathcal{E}))_{1,2} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \Gamma\left(X \times Y, \mathcal{E}\right) \\ \Gamma\left(Y \times X, \mathcal{E}^*\right) & \end{pmatrix} \\
&= \begin{pmatrix} 0 & \Gamma\left(X \times Y, {}_f(\mathcal{O}_Y)_{\mathrm{id}}\right) \\ \Gamma\left(Y \times X, {}_{\mathrm{id}}(\mathcal{O}_Y)_f\right) & 0 \end{pmatrix} \\
&\cong \begin{pmatrix} 0 & {}_RS_S \\ {}_SS_R & 0 \end{pmatrix} \cong {}_RS_S \oplus {}_SS_R
\end{aligned}$$

To check the final condition, we have the isomorphisms

$$\mathrm{T}(\mathcal{E})_{i,i+1} \otimes \mathrm{T}(\mathcal{E})_{i+1,i+n+1} \longrightarrow \mathrm{T}(\mathcal{E})_{i,i+n+1}$$

We now apply the functor $\Gamma(X_i \times X_{i+n+1}, -)$ and note that since all varieties are affine, the tensor product and $\Gamma(-)$ commute, resulting in an isomorphism

$$\Gamma(\mathrm{T}(\mathcal{E}))_{i,i+1} \otimes \Gamma(\mathrm{T}(\mathcal{E}))_{i+1,i+n+1} \longrightarrow \Gamma(\mathrm{T}(\mathcal{E}))_{i,i+n+1}$$

application of the functor $\overline{(-)}$ yields

$$\overline{\Gamma(\mathrm{T}(\mathcal{E}))}_1 \otimes \overline{\Gamma(\mathrm{T}(\mathcal{E}))}_n \xrightarrow{\simeq} \overline{\Gamma(\mathrm{T}(\mathcal{E}))}_{n+1}$$

we have thus constructed the required isomorphism (3.31). Finally, we prove that the relations defining $\Pi_R S$ coincide with the kernel of π, i.e. there is a commutative diagram:

$$\begin{tikzcd}
\overline{\Gamma(\mathrm{T}(\mathcal{E}))} \arrow[r, "\pi"] \arrow[d, "\cong"'] & \overline{\Gamma(\mathbb{S}(\mathcal{E}))} \arrow[d, "\cong"] \\
T(R,S) \arrow[r, "\overline{\pi}"'] & \Pi_R(S)
\end{tikzcd}$$

The isomorphisms in the previous step yield isomorphisms:

$$\zeta_0 : \mathrm{Hom}_{X\times X}\left({}_{\mathrm{id}}\left(\mathcal{O}_X\right)_{\mathrm{id}}, \mathcal{E} \otimes \mathcal{E}^*\right) \xrightarrow{\simeq} \mathrm{Hom}_R(R, {}_RS_S \otimes_S S_R)$$

$$\zeta_1 : \mathrm{Hom}_{Y\times Y}\left({}_{\mathrm{id}}\left(\mathcal{O}_Y\right)_{\mathrm{id}}, \mathcal{E}^* \otimes \mathcal{E}\right) \xrightarrow{\simeq} \mathrm{Hom}_S(S, {}_SS_R \otimes_R S_S)$$

Recall that $\mathbb{S}(\mathcal{E})$ is defined as a quotient of $\mathrm{T}(\mathcal{E})$ by the relations given by the unit morphisms

$$i_0 \in \mathrm{Hom}_{X\times X}\left({}_{\mathrm{id}}\left(\mathcal{O}_X\right)_{\mathrm{id}}, \mathcal{E} \otimes \mathcal{E}^*\right)$$
$$i_1 \in \mathrm{Hom}_{Y\times Y}\left({}_{\mathrm{id}}\left(\mathcal{O}_Y\right)_{\mathrm{id}}, \mathcal{E}^* \otimes \mathcal{E}\right)$$

described in (3.10). Similarly $\Pi_R(S)$ is defined as a quotient of $T_R(S)$ by elements

$$\eta_0 \in \mathrm{Hom}_R(R, {}_RS_S \otimes_S S_R)$$
$$\eta_1 \in \mathrm{Hom}_S(S, {}_SS_R \otimes_R S_S)$$

Hence we must prove $\zeta_0(i_0) = \eta_0$ and $\zeta_1(i_1) = \eta_1$. To this end, note that there is a commutative diagram of isomorphisms

$$\begin{tikzcd}
\mathrm{Hom}_{X\times Y}(\mathcal{E},\mathcal{E}) \arrow[r] \arrow[d] & \mathrm{Hom}_{X\times X}({}_{\mathrm{id}}(\mathcal{O}_X)_{\mathrm{id}}, \mathcal{E}\otimes\mathcal{E}^*) \arrow[d, "\zeta_0"] \\
\mathrm{Hom}_{R\otimes S}({}_RS_S, {}_RS_S) \arrow[r, "\varphi_0"] & \mathrm{Hom}_R(R, {}_RS_S \otimes_S S_R)
\end{tikzcd}$$

where φ_0 is given by the adjunction

$$(- \otimes_R S_S) \dashv (- \otimes_S S_R) = (-)_R.$$

Hence $\zeta_0(i_0) = \varphi_0(\mathrm{id}_{{}_RS_S}) : 1_R \mapsto 1_S \otimes 1_S$ and this morphism indeed coincides with η_0. Similarly the existence of the dual bases $(e_i)_i$, $(f_j)_j$ implies there is an adjunction

$$- \otimes_S S_R = (-)_R \dashv (-) \otimes_R S_S$$

given by

$$\varphi_1 : \mathrm{Hom}_R(M \otimes_S S_R, N) \to \mathrm{Hom}_S(M, N \otimes_R S_S) : \psi \mapsto \left(\psi' : m \mapsto \sum_i \psi(me_i) \otimes f_i\right) \tag{3.32}$$

Where we used Lemma 3.3.23 below to show that the morphisms in the image of φ_1 indeed have an S-module structure. A commutative diagram as above shows that $\zeta_1(i_1) = \varphi_1(\mathrm{id}_{{}_SS_R}) : 1_S \mapsto \sum_i e_i \otimes f_i$ which coincides with η_1. □

Lemma 3.3.23. *$\sum_i e_i \otimes f_i$ is central in the S-bimodule $S \otimes_R S$. I.e. for all $a \in S$ we have*

$$\sum_i ae_i \otimes f_i = \sum_i e_i \otimes f_i a$$

Proof. It is sufficient to prove that for all j, k we have

$$\sum_i \lambda(ae_i f_j)\lambda(f_i e_k) = \sum_i \lambda(e_i f_j)\lambda(f_i a e_k)$$

which is clear since both sides are equal to $\lambda(ae_k f_j)$. □

3.3.5 Proof of Theorem 3.3.1: $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$ is locally noetherian.

We will now combine the results in this section to prove Theorem 3.3.1. As X and Y are noetherian we know that $\mathrm{Qcoh}(X)$ and $\mathrm{Qcoh}(Y)$ are locally noetherian categories and hence there exist collections of noetherian generating objects for these categories, say $\mathcal{N}^X := \{\mathcal{N}^X_\alpha\}$ and $\mathcal{N}^Y := \{\mathcal{N}^Y_\beta\}_{j\in J}$. For each $n \in \mathbb{Z}$ we define $\mathcal{N}^n$ in $\mathrm{Qcoh}(X_n)$ as:

$$\mathcal{N}^n = \begin{cases} \mathcal{N}^X & \text{if } n \text{ is even} \\ \mathcal{N}^Y & \text{if } n \text{ is odd} \end{cases}$$

We shall prove that the collection

$$\{\mathcal{N} \otimes e_n\mathbb{S}(\mathcal{E}) \mid n \in \mathbb{Z}, \mathcal{N} \in \mathcal{N}^n\} \tag{3.33}$$

forms a set of noetherian generators for $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$. Note that the collection is easily seen to generate as for each $\mathcal{M} \in \mathrm{Gr}(\mathcal{A})$ there is a surjective morphism

$$\oplus_{n\in\mathbb{Z}}\mathcal{M}_n \otimes e_n\mathcal{A} \twoheadrightarrow \mathcal{M}$$

and for each $n \in \mathbb{Z}$ there is a surjective morphism

$$\oplus_\alpha(\mathcal{N}^n_\alpha)^{m_\alpha} \twoheadrightarrow \mathcal{M}_n$$

where $\mathcal{N}^n_\alpha \in \mathcal{N}^n$. Hence we only need to show that the elements of (3.33) are noetherian objects in $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$. By Lemma 3.3.10 and Corollary 3.3.11 this can be checked locally for any open cover $X = \bigcup_l U_l$. By Lemma 3.3.13 we may hence assume that $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$ are smooth affine varieties such that

i) $\mathcal{O}_Y \cong \omega_Y$

ii) $\omega_X \cong \mathcal{O}_X$

iii) S/R is relative Frobenius of rank 4.

With these assumptions there are functors

$$\begin{tikzcd}
\mathrm{Gr}(\mathbb{S}(\mathcal{E})) \arrow[d, "\Gamma(-)", "\cong"'] \\
\mathrm{Gr}(\Gamma(\mathbb{S}(\mathcal{E}))) \arrow[d, hook, "\text{Proposition 3.3.20}"] \\
\mathrm{Gr}\left(\overline{\Gamma(\mathbb{S}(\mathcal{E}))}\right) \arrow[d, "\text{Lemma 3.3.22}", "\cong"'] \\
\mathrm{Gr}(\Pi_R(S))
\end{tikzcd} \tag{3.34}$$

Let $F : \mathrm{Gr}(\mathbb{S}(\mathcal{E})) \longrightarrow \mathrm{Gr}(\Pi_R(S))$ be the composition. Then the above diagram shows that F is an exact embedding of categories. Hence $\mathcal{N} \otimes e_n\mathbb{S}(\mathcal{E})$ is a noetherian object in

$\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$ if $F(\mathcal{N} \otimes e_n\mathbb{S}(\mathcal{E}))$ is a noetherian object in $\mathrm{Gr}(\Pi_R(S))$. On the other hand, as $\mathcal{N}$ is noetherian in $\mathrm{Qcoh}(X_n)$ there is an $m \in \mathbb{N}$ and a surjection $\mathcal{O}_{X_n}^{\oplus m} \twoheadrightarrow \mathcal{N}$ giving rise to an surjection

$$F(\mathcal{O}_{X_n} \otimes e_n\mathbb{S}(\mathcal{E}))^{\oplus m} \twoheadrightarrow F(\mathcal{N} \otimes e_n\mathbb{S}(\mathcal{E}))$$

Hence it suffices to show that $F(\mathcal{O}_{X_n} \otimes e_n\mathbb{S}(\mathcal{E}))$ is a noetherian object in $\Pi_R(S)$. This is however obvious as

$$F(\mathcal{O}_{X_n} \otimes e_n\mathbb{S}(\mathcal{E})) = \begin{cases} R \cdot \Pi_R(S)(-n) & \text{if } n \text{ is even} \\ S \cdot \Pi_R(S)(-n) & \text{if } n \text{ is odd} \end{cases}$$

As both $R{\cdot}\Pi_R(S)$ and $S{\cdot}\Pi_R(S)$ are direct summands of $\Pi_R(S)$, which is a noetherian ring by Theorem 2.5.1, we have proven the theorem. □

3.4 The homological properties of $\mathbb{S}(\mathcal{E})$

3.4.1 A formula for Ext-groups

As before, $\mathcal{E} = {}_f(\mathcal{O}_Y)_{\mathrm{id}}$ will be a locally free $X-Y$-bimodule of rank $(4,1)$ throughout this section and we let $\mathcal{A} := \mathbb{S}(\mathcal{E})$ denote the associated symmetric sheaf-$\mathbb{Z}$-algebra in standard form (see Convention 3.3.3). This section is dedicated to adapting the results in [VdB12], [Nym04c], [Nym04b] and [Mor07] to obtain a formula for the Ext-groups of pulled back sheaves on $\mathrm{QGr}(\mathcal{A})$. To keep the geometric intuition we denote the truncation functors $(\omega(-))_m : \mathrm{QGr}(\mathcal{A}) \longrightarrow \mathrm{Qcoh}(X_m)$ by Π_{m*} (compare with Theorem 3.2.27). The left adjoints, which are given explicitly by $p((-) \otimes e_m\mathcal{A})$ following (3.8) and Definition 3.2.7, are in turn denoted by Π_m^*. We shall use the notations X_n and Q_n as in Convention 3.2.12 and (3.16).

If $\mathcal{E} \in \mathrm{bimod}(X-X)$ is locally free of rank (2,2) and $\mathcal{A} = \mathbb{S}(\mathcal{E})$, [Mor07] computes the Euler characteristics $\langle \Pi_m^*\mathcal{F}, \Pi_n^*\mathcal{G} \rangle$ for two locally free sheaves $\mathcal{F}$ and $\mathcal{G}$ on X. In this section, we perform an analogous calculation in our setting where the bimodule $\mathcal{E} \in \mathrm{bimod}(X-Y)$ is of rank (4,1). Motivated by Proposition 3.2.26 our focus lies on $\langle \Pi_m^*\mathcal{F}, \Pi_n^*\mathcal{G} \rangle$ with $|n-m| \leq 1$. This section is dedicated to proving the following slightly more general statement:

Theorem 3.4.1. *Let $\mathcal{E} \in \mathrm{bimod}(X,Y)$ be locally free of rank (4,1). Let $\mathcal{F}$ and $\mathcal{G}$ be locally free sheaves on X_m respectively X_n for $m,n \in \mathbb{Z}$ such that $m \geq n-1$. Then*

$$\mathrm{Ext}^i_{\mathrm{QGr}(\mathcal{A})}\left(\Pi_m^*\mathcal{F}, \Pi_n^*\mathcal{G}\right) \cong \mathrm{Ext}^i_{X_m}\left(\mathcal{F}, \mathcal{G} \otimes \mathbb{S}(\mathcal{E})_{n,m}\right)$$

for all $i \geq 0$.

This formula implies the following facts:

Corollary 3.4.2. *With the above assumptions, one has*

- $\langle \Pi_m^* \mathcal{F}, \Pi_n^* \mathcal{G}\rangle = \langle \mathcal{F}, \mathcal{G} \otimes \mathbb{S}(\mathcal{E})_{n,m}\rangle$
- *Let* $\{\mathcal{F}_1, \dots, \mathcal{F}_\alpha\}$ *and* $\{\mathcal{G}_1, \dots, \mathcal{G}_\beta\}$ *be exceptional sequences (see Definition 3.4.21 for the definition of an exceptional sequence) of locally free sheaves on* X_n *and* X_{n+1} *respectively. Then* $\Pi_{n+1}^*\mathcal{G}_1, \dots \Pi_{n+1}^*\mathcal{G}_b, \Pi_n^*\mathcal{F}_1, \dots, \Pi_n^*\mathcal{F}_b$ *is an exceptional sequence on* $\mathrm{QGr}(\mathcal{A})$.

The proof of Theorem 3.4.1 is based on the existence of an exact sequence (see (3.35) below). To this end, we consider Θ_m defined by

$$(\Theta_m)_n = \begin{cases} 0 & m \neq n \\ \mathcal{O}_{X_m} & n = m \end{cases}$$

Remark 3.4.3. *Note that* Θ_m *is a right* $\mathcal{A}$*-module using* $\mathcal{A}_{i,i} = \mathcal{O}_{X_i}$

Theorem 3.4.4. *For each* m*, there is an exact sequence of locally free* $(\mathcal{O}_{X_m} - \mathcal{A})$*-bimodules*[3].

$$0 \to \mathcal{Q}_m \otimes e_{m+2}\mathcal{A} \longrightarrow \mathcal{E}^{*m} \otimes e_{m+1}\mathcal{A} \longrightarrow e_m\mathcal{A} \longrightarrow \Theta_m \to 0 \tag{3.35}$$

Proof. By the nature of the relations this sequence is known to be right exact. The proof of the left exactness uses so-called "point modules" and is deferred to §3.4.2. □

As an immediate corollary of this theorem and its proof we find:

Corollary 3.4.5. *for each* $i, j \in \mathbb{Z}$*, the bimodule* $\mathcal{A}_{i,j}$ *is locally free both on the left and on the right. The ranks are given by*

$$\mathrm{rk}(\mathcal{A})_{i,j} := \begin{cases} (j-i+1, j-i+1) & i \equiv j \, mod \, 2 \\ \left(\frac{j-i+1}{2}, 2(j-i+1)\right) & i \, odd, \, j \, even \\ \left(2(j-i+1), \frac{j-i+1}{2}\right) & i \, even, \, j \, odd \end{cases}$$

Proof. We have $\mathrm{rk}(\mathcal{E}) = (4,1)$ and $\mathrm{rk}(\mathcal{E}^*) = (1,4)$, $\mathrm{rk}(\mathcal{Q}_m) = (1,1)$. Since the rank is additive on short exact sequence, one can now verify the claim by induction in the three cases on n using the sequences in (3.35). □

This result in turn implies the following convenient fact

Lemma 3.4.6. *For each* $m \in \mathbb{Z}$*, the functor* $\Pi_m^* : \mathrm{Qcoh}(X_m) \to \mathrm{QGr}(\mathcal{A})$ *is exact.*

Proof. For each $n \geq m$, $\mathcal{A}_{m,n}$ is locally free by Corollary 3.4.5, hence the functor $- \otimes \mathcal{A}_{m,n} : \mathrm{Qcoh}(X_m) \longrightarrow \mathrm{Qcoh}(X_n)$ is exact. □

[3] See [VdB12, Section 3.2.] for the definition of this category

As an example application of the above lemma, we mention the following adjunction formula:

Lemma 3.4.7. *There is a natural isomorphism for all* $\mathcal{F} \in \mathrm{Qcoh}(X_m)$ *and for all* $\mathcal{M} \in \mathcal{D}^+(\mathrm{QGr}(\mathcal{A}))$*:*

$$\mathbf{R}\mathrm{Hom}_{\mathrm{QGr}(\mathcal{A})}(\Pi_m^*\mathcal{F}, \mathcal{M}) \cong \mathbf{R}\mathrm{Hom}_{X_m}(\mathcal{F}, \mathbf{R}\Pi_{m*}\mathcal{M})$$

Proof. Since Π_m^* is an exact left adjoint to $\Pi_{m,*}$, the latter must preserve injective objects and the result follows. □

For the purposes of proving Theorem 3.4.1 we are especially interested in the case where $\mathcal{M} = \Pi_n^*\mathcal{G}$ for a locally free sheaf $\mathcal{G}$ on X_n. It follows that we need to understand complexes of the form $\mathbf{R}\Pi_{m*}(\Pi_n^*\mathcal{G})$. The strategy for computing the homology of this complex is as follows: by Lemma 3.4.9 below, it suffices to give a description the derived functors of the torsion functor τ. These in turn follow from the derived functors of an internal Hom-functor $\underline{\mathcal{H}om}$ constructed in [Nym04b] (Lemma 3.4.11).

Lemma 3.4.8. *We have the following facts for the derived functors of the torsion functor* $\tau : \mathrm{Gr}(\mathcal{A}) \longrightarrow \mathrm{Tors}(\mathcal{A})$*:*

i) for $i \geq 1$*, there is an isomorphism of functors*

$$\mathbf{R}^{i+1}\tau \cong (\mathbf{R}^i\omega) \circ p$$

ii) For each $\mathcal{M} \in \mathrm{Gr}(\mathcal{A})$ *there is an exact sequence:*

$$0 \longrightarrow \tau(\mathcal{M}) \longrightarrow \mathcal{M} \longrightarrow \omega(p(\mathcal{M})) \longrightarrow \mathbf{R}^1\tau(\mathcal{M}) \longrightarrow 0$$

Proof. By Theorem 3.3.1, $\mathrm{Gr}(\mathcal{A})$ is a locally noetherian category, by [Nym04c, Lemma 2.12], any essential extension of a torsion module remains a torsion module. In particular, the category $\mathrm{Tors}(\mathcal{A})$ is closed under injective envelopes, the result now follows from [Smi00, Theorem 2.14.15]. □

Lemma 3.4.9. *For* $i \geq 1$*, and* $\mathcal{F} \in \mathrm{Qcoh}(X_m)$ *there is an isomorphism*

$$\mathbf{R}^i\Pi_{m*}(\Pi_n^*\mathcal{F}) \cong \mathbf{R}^{i+1}\tau(\mathcal{F} \otimes e_n\mathcal{A})_m$$

Proof. As the functors p and $(-)_m$ are exact there is a functorial isomorphism

$$(\mathbf{R}^i\Pi_{m*})(p)(-) \cong \mathbf{R}^i\omega(p(-))_m \tag{3.36}$$

Combining this isomorphism with the one in Lemma 3.4.8 we obtain for each $i \geq 1$:

$$\mathbf{R}^i\Pi_{m*}(\Pi_n^*\mathcal{F}) := \mathbf{R}^i\Pi_{m*}(p(\mathcal{F} \otimes e_n\mathcal{A})) \cong \mathbf{R}^i\omega(p(\mathcal{F} \otimes e_n\mathcal{A}))_m \cong \mathbf{R}^{i+1}\tau(\mathcal{F} \otimes e_n\mathcal{A})_m$$ □

The following is based on [Nym04b, Section 3.2]:
Let $\mathrm{Bimod}(\mathcal{A}-\mathcal{A})$ denote the category whose objects are of the form

$$\{\mathcal{B}_{m,n} \in \mathrm{Bimod}(X_m - X_n)\}_{m,n}$$

such that the left and right multiplications

$$\mathcal{A}_{l,m} \otimes \mathcal{B}_{m,n} \longrightarrow \mathcal{B}_{l,n} \quad \text{and} \quad \mathcal{B}_{m,n} \otimes \mathcal{A}_{n,l} \longrightarrow \mathcal{B}_{m,l}$$

are compatible in the obvious sense. We denote by $\mathbb{B}$ for the subcategory for which all $\mathcal{B}_{m,n}$ are coherent and locally free. Finally for each m we define $\mathbb{G}\mathrm{r}_m(\mathcal{A})$ to be the subcategory of $\mathbb{B}$ for which $\mathcal{B}_{j,n} = 0$ unless $j = m$. In particular $\mathbb{G}\mathrm{r}_m(\mathcal{A})$ is a subcategory of $\mathrm{BiMod}(\mathcal{O}_{X_m} - \mathcal{A})$. By Corollary 3.4.5 we know that $e_m\mathcal{A} \in \mathbb{G}\mathrm{r}_m(A)$ for each m. If $\mathcal{Q} \in \mathrm{coh}(X_m)$ is locally free, then ${}_{\mathrm{id}}(\mathcal{Q})_{\mathrm{id}} \otimes e_m\mathcal{A} \in \mathbb{G}\mathrm{r}_m(A)$ as well. We will often use the short hand notation $\mathcal{Q} \otimes e_m\mathcal{A}$ for this object.

There are Hom-functors

$$\underline{\mathcal{H}om} : \mathbb{B}^{op} \times \mathrm{Gr}(\mathcal{A}) \longrightarrow \mathrm{Gr}(\mathcal{A}) \quad \text{and}$$
$$\mathcal{H}om : \mathbb{G}\mathrm{r}_m(\mathcal{A}) \times \mathrm{Gr}(\mathcal{A}) \longrightarrow \mathrm{Qcoh}(X_m)$$

satisfying the following properties:

Proposition 3.4.10. *i)* $\underline{\mathcal{H}om}(\mathcal{B},\mathcal{M})_m = \mathcal{H}om(e_m\mathcal{B},\mathcal{M})$ *for all* $\mathcal{B} \in \mathbb{B}$ *and* $\mathcal{M} \in \mathrm{Gr}(\mathcal{A})$

ii) $\underline{\mathcal{H}om} : \mathbb{B}^{op} \times \mathrm{Gr}(\mathcal{A}) \to \mathrm{Gr}(\mathcal{A})$ *is a bifunctor, left exact in both its arguments*

iii) $\mathcal{H}om : \mathbb{G}r_m(\mathcal{A}) \times \mathrm{Gr}(\mathcal{A}) \to \mathrm{Qcoh}(X_m)$ *is a bifunctor, left exact in both its arguments*

iv) $\mathcal{H}om(\mathcal{Q}\otimes e_m\mathcal{A},\mathcal{M}) \cong \mathcal{M}_m \otimes \mathcal{Q}^*$ *for all* $\mathcal{M} \in \mathrm{Gr}(\mathcal{A})$ *and locally free* X_m*-bimodules* $\mathcal{Q}$

Proof. $i)$ This follows immediately by checking the definitions in [Nym04b, §3.2]

$ii)$ see [Nym04b, Proposition 3.11, Theorem 3.16(1)]

$iii)$ see [Nym04b, Theorem 3.16(3)]

$iv)$ see [Nym04b, Theorem 3.16(4)] □

By *ii)* and *iii)* in the above proposition one can define the right derived functors $\underline{\mathcal{E}xt}^i$ and $\mathcal{E}xt^i$ for all $i \geq 0$. Moreover we use the notation $\mathcal{A}_{\geq l}$ to denote the object in $\mathbb{B}$ given by

$$(\mathcal{A}_{\geq l})_{m,n} = \begin{cases} \mathcal{A}_{m,n} & \text{if } n - m \geq l \\ 0 & \text{else} \end{cases}$$

and $\mathcal{A}_0 := \mathcal{A}/\mathcal{A}_{\geq 1}$. Then we have the following relation between the derived functors of τ and the $\underline{\mathcal{E}xt}^i$:

Lemma 3.4.11. $\mathbf{R}^i\tau(-) \cong \lim_{l\to\infty} \underline{\mathcal{E}xt}^i_{\mathrm{Gr}(\mathcal{A})}(\mathcal{A}/\mathcal{A}_{\geq l}, -)$

Proof. By [Nym04c, Proposition 3.19], we have an isomorphism of functors

$$\tau \cong \lim_{l\to\infty} \underline{\mathcal{H}om}_{\mathrm{Gr}(\mathcal{A})}(\mathcal{A}/\mathcal{A}_{\geq l}, -)$$

Applying this to the injective resolution and subsequently taking homology yields the required result ☐

Lemma 3.4.12. *Let $\mathcal{B} \in \mathbb{B}$ be concentrated in degree $l \geq 0$ (i.e. $\mathcal{B}_{m,n} = 0$ whenever $m + l \neq n$) and $\mathcal{V}$ a locally free sheaf. Then for $n - l - 1 \leq m$ and for all $i \geq 0$:*

$$\underline{\mathcal{E}xt}^i(\mathcal{B}, \mathcal{V} \otimes e_n\mathcal{A})_m = 0$$

Proof. By [Nym04b, cor. 4.6], there is an isomorphism

$$\underline{\mathcal{E}xt}^i(\mathcal{B}, \mathcal{V} \otimes e_n\mathcal{A})_m \cong \underline{\mathcal{E}xt}^i(\mathcal{A}_0, \mathcal{V} \otimes e_n\mathcal{A})_{m+l} \otimes \mathcal{B}^*_{m,m+l}$$

which easily reduces the proof to the case $\mathcal{B} = \mathcal{A}_0$ and in particular $l = 0$.
By Proposition 3.4.10*(iv)* we see that the exact sequence from Theorem 3.4.4 forms a resolution of $e_m\mathcal{A}_0 = \Theta_m$ through $\mathcal{H}om(-, \mathcal{V} \otimes e_n\mathcal{A})$-acyclic sheaves. In particular we can calculate $\underline{\mathcal{E}xt}^i(\mathcal{A}_0, \mathcal{V} \otimes e_n\mathcal{A})_m = \mathcal{E}xt^i(e_m\mathcal{A}_0, \mathcal{V} \otimes e_n\mathcal{A})$ by taking homology of the complex

$$\begin{aligned} 0 \longrightarrow \mathcal{H}om(e_m\mathcal{A}, \mathcal{V} \otimes e_n\mathcal{A}) &\xrightarrow{d_0} \mathcal{H}om(\mathcal{E}^{*m} \otimes e_{m+1}\mathcal{A}, \mathcal{V} \otimes e_n\mathcal{A}) \\ &\xrightarrow{d_1} \mathcal{H}om(\mathcal{Q}_m \otimes e_{m+2}\mathcal{A}, \mathcal{V} \otimes e_n\mathcal{A}) \longrightarrow 0 \end{aligned}$$

using Proposition 3.4.10*(iv)*, this complex becomes

$$0 \to \mathcal{V} \otimes \mathcal{A}_{n,m} \xrightarrow{d_0} \mathcal{V} \otimes \mathcal{A}_{n,m+1} \otimes \mathcal{E}^{*m+1} \xrightarrow{d_1} \mathcal{V} \otimes \mathcal{A}_{n,m+2} \otimes \mathcal{Q}^*_m \to 0 \qquad (3.37)$$

Hence we have

- $\mathcal{E}xt^0(e_m\mathcal{A}_0, \mathcal{V} \otimes e_n\mathcal{A}) = \ker(d_0)$
- $\mathcal{E}xt^1(e_m\mathcal{A}_0, \mathcal{V} \otimes e_n\mathcal{A}) = \ker(d_1)/\operatorname{im}(d_0)$
- $\mathcal{E}xt^2(e_m\mathcal{A}_0, \mathcal{V} \otimes e_n\mathcal{A}) = \operatorname{coker}(d_1)$
- $\mathcal{E}xt^i(e_m\mathcal{A}_0, \mathcal{V} \otimes e_n\mathcal{A}) = 0$ for all $i \geq 3$

To show the exactness of (3.37), we first note that the explicit nature of the isomorphisms in [Nym04b] yield that (3.37) is obtained from the sequence

$$0 \longrightarrow \mathcal{A}_{n,m} \longrightarrow \mathcal{A}_{n,m+1} \otimes \mathcal{E}^{*m+1} \longrightarrow \mathcal{A}_{n,m+2} \otimes Q_m^* \longrightarrow 0 \tag{3.38}$$

by tensoring with $\mathcal{V}$. Since $\mathcal{V}$ is locally free, it preserves exactness and it suffices to verify that (3.38) is exact. Next, we tensor with the invertible bimodule Q_m to obtain

$$0 \longrightarrow \mathcal{A}_{n,m} \otimes Q_m \xrightarrow{d_0} \mathcal{A}_{n,m+1} \otimes \mathcal{E}^{*m+1} \otimes Q_m \xrightarrow{d_1} \mathcal{A}_{n,m+2} \longrightarrow 0 \tag{3.39}$$

We can replace the middle term in (3.39) to obtain:

$$0 \longrightarrow \mathcal{A}_{n,m} \otimes Q_m \xrightarrow{d_0} \mathcal{A}_{n,m+1} \otimes \mathcal{E}^{*m+1} \xrightarrow{d_1} \mathcal{A}_{n,m+2} \longrightarrow 0 \tag{3.40}$$

A similar but tedious computation as in [Nym04b, §7.5] shows that this sequence coincides with the exact sequence in Theorem 3.4.4 in degree n for left modules. We conclude the result by the same argument as for Theorem 3.4.4. □

Lemma 3.4.13. $\underline{\mathcal{E}xt}^i(\mathcal{A}/\mathcal{A}_{\geq l}, \mathcal{V} \otimes e_n\mathcal{A})_m = 0$ *for* $m \geq n-1$ *and* $i \geq 0$

Proof. Consider the short exact sequence

$$0 \longrightarrow \mathcal{A}_{\geq l}/\mathcal{A}_{\geq l+1} \longrightarrow \mathcal{A}/\mathcal{A}_{\geq l+1} \longrightarrow \mathcal{A}/\mathcal{A}_{\geq l} \longrightarrow 0$$

Applying $\underline{\mathcal{H}om}(-, \mathcal{V} \otimes e_n\mathcal{A})$ gives rise to a long exact sequence for each $m \geq n-1$

$$\begin{aligned} \dots \to \underline{\mathcal{E}xt}^i(\mathcal{A}_{\geq l}/\mathcal{A}_{\geq l+1}, \mathcal{V} \otimes e_n\mathcal{A})_m &\longrightarrow \underline{\mathcal{E}xt}^i(\mathcal{A}/\mathcal{A}_{\geq l+1}, \mathcal{V} \otimes e_n\mathcal{A})_m \\ \longrightarrow \quad \underline{\mathcal{E}xt}^i(\mathcal{A}/\mathcal{A}_{\geq l}, \mathcal{V} \otimes e_n\mathcal{A})_m &\longrightarrow \underline{\mathcal{E}xt}^{i+1}(\mathcal{A}_{\geq l}/\mathcal{A}_{\geq l+1}, \mathcal{V} \otimes e_n\mathcal{A})_m \to \dots \end{aligned}$$

As $m \geq n-1$ it follows from Lemma 3.4.12 that for each $i \geq 0$ we have an exact sequence

$$0 \longrightarrow \underline{\mathcal{E}xt}^i(\mathcal{A}/\mathcal{A}_{\geq l+1}, \mathcal{V} \otimes e_n\mathcal{A})_m \longrightarrow \underline{\mathcal{E}xt}^i(\mathcal{A}/\mathcal{A}_{\geq l}, \mathcal{V} \otimes e_n\mathcal{A})_m \longrightarrow 0$$

Hence

$$\underline{\mathcal{E}xt}^i(\mathcal{A}/\mathcal{A}_{\geq l}, \mathcal{V} \otimes e_n\mathcal{A})_m \cong \underline{\mathcal{E}xt}^i(\mathcal{A}/\mathcal{A}_{\geq 0}, \mathcal{V} \otimes e_n\mathcal{A})_m = \underline{\mathcal{E}xt}^i(0, \mathcal{V} \otimes e_n\mathcal{A})_m = 0$$ □

We can now finish the proof of Theorem 3.4.1.

Proof. of Theorem 3.4.1
Take $m, n \in \mathbb{Z}$ with $m \geq n-1$. Let $\mathcal{F}$ be locally free on X_m and $\mathcal{G}$ locally free on X_n, then by Lemma 3.4.7:

$$\begin{aligned} \operatorname{Ext}^i_{\mathrm{QGr}(\mathcal{A})}\left(\Pi_m^*\mathcal{F}, \Pi_n^*\mathcal{G}\right) &= \mathrm{H}^i\left(\mathbf{R}\mathrm{Hom}_{\mathrm{QGr}(\mathcal{A})}\left(\Pi_m^*\mathcal{F}, \Pi_n^*\mathcal{G}\right)\right) \\ &\cong \mathrm{H}^i\left(\mathbf{R}\mathrm{Hom}_{X_m}\left(\mathcal{F}, \mathbf{R}\Pi_{m*}\Pi_n^*\mathcal{G}\right)\right) \end{aligned}$$

Now for $i \geq 1$ we have

$$\mathbf{R}^i\Pi_{m*}\Pi_n^*\mathcal{G} \cong \mathbf{R}^{i+1}\tau(\mathcal{G}\otimes e_n\mathcal{A})_m \cong \lim_{l\to\infty}\underline{\mathcal{E}xt}^{i+1}(\mathcal{A}/\mathcal{A}_{\geq l},\mathcal{G}\otimes e_n\mathcal{A})_m = 0$$

by Lemmas 3.4.9, 3.4.11 and 3.4.13 respectively.
In particular the complex $\mathbf{R}\Pi_{m*}\Pi_n^*\mathcal{G}$ is quasi-isomorphic to the complex that is equal to $\Pi_{m*}\Pi_n^*\mathcal{G}$ concentrated in position zero. Finally we can conclude by noticing that $\Pi_{m*}\Pi_n^*\mathcal{G} = (\omega p(\mathcal{G}\otimes e_n\mathcal{A}))_m$ and by Lemma 3.4.8 there is an exact sequence

$$0 = \tau(\mathcal{G}\otimes e_n\mathcal{A})_m \longrightarrow \mathcal{G}\otimes\mathcal{A}_{n,m} \xrightarrow{\cong} \omega(p(\mathcal{G}\otimes e_n\mathcal{A}))_m \longrightarrow \mathbf{R}^1\tau(\mathcal{G}\otimes e_n\mathcal{A})_m = 0$$

where the first term equals zero because $\mathcal{G}\otimes e_n\mathcal{A}$ is torsion free and the last term is zero because $\mathbf{R}^1\tau(\mathcal{G}\otimes e_n\mathcal{A})_m \cong \lim_{l\to\infty}\underline{\mathcal{E}xt}^{\,1}(\mathcal{A}/\mathcal{A}_{\geq l},\mathcal{G}\otimes e_n\mathcal{A})_m = 0$.
Hence we can conclude that for $m \geq n-1$ we have

$$\begin{aligned}\mathrm{Ext}^i_{\mathrm{QGr}(\mathcal{A})}\left(\Pi_m^*\mathcal{F},\Pi_n^*\mathcal{G}\right) &\cong h^i\left(\mathbf{R}\mathrm{Hom}_{X_m}\left(\mathcal{F},\mathbf{R}\Pi_{m*}\Pi_n^*\mathcal{G}\right)\right)\\ &\cong h^i\left(\mathbf{R}\mathrm{Hom}_{X_m}\left(\mathcal{F},\mathcal{G}\otimes\mathcal{A}_{n,m}\right)\right)\\ &= \mathrm{Ext}^i_{X_m}\left(\mathcal{F},\mathcal{G}\otimes\mathcal{A}_{n,m}\right)\end{aligned}$$

□

3.4.2 Point modules in the rank (4, 1)-case

We remain in the setting where $\mathcal{A} = \mathbb{S}(\mathcal{E})$ denotes a symmetric sheaf-$\mathbb{Z}$-algebra in standard form with $\mathcal{E} \in \mathrm{bimod}(X-Y)$ locally free of rank (4,1), given in the form of $\mathcal{E} = {}_f(\mathcal{O}_Y)_{\mathrm{id}}$ for a finite (flat) morphism $f : Y \longrightarrow X$ as in Lemma 3.3.4. Denote by $\alpha : X \longrightarrow \mathrm{Spec}(k)$ and $\beta : Y \longrightarrow \mathrm{Spec}(k)$ the structure morphisms. Extending our Convention 3.2.12 we will write

$$(X_n,\alpha_n) = \begin{cases}(X,\alpha) & \text{if } n \text{ is even}\\ (Y,\beta) & \text{if } n \text{ is odd}\end{cases}$$

The goal of this section is to introduce a sheafified version of the notion of point modules (see Definition 0.1.2).

We say $P_n \in \mathrm{coh}(X_n)$ is locally free over k of rank l if the support of P_n is finite over k and $\dim_k(\alpha_{n,*}P_n) = l$.

A module $P \in \mathrm{Gr}(\mathcal{A})$ is said to be generated in degree m if $P_n = 0$ for all $n < m$ and $P_m \otimes \mathcal{A}_{m,n} \to P_n$ is surjective for all $n \geq m$. As $\mathcal{A}$ is generated in degree one as an algebra, we have surjectivity of $P_{n_1}\otimes\mathcal{A}_{n_1,n_2} \to P_{n_2}$ for all $n_2 \geq n_1 \geq m$ by the following commutative diagram

$$\begin{tikzcd} P_m\otimes\mathcal{A}_{m,n_1}\otimes\mathcal{A}_{n_1,n_2} \arrow[r, two heads] \arrow[d, two heads] & P_{n_1}\otimes\mathcal{A}_{n_1,n_2} \arrow[d] \\ P_m\otimes\mathcal{A}_{m,n_2} \arrow[r, two heads] & P_{n_2} \end{tikzcd}$$

Remark 3.4.14. *An obvious example of a module generated in degree m is $e_m\mathcal{A}$. The above diagram implies that the maps $\mathcal{A}_{m,n} \otimes e_n\mathcal{A} \to e_m\mathcal{A}$ are surjective for all $m \geq n$.*

An m-shifted point-module over $\mathcal{A}$ is defined in [VdB12] and [Nym04a] as an object $P \in \mathrm{Gr}(\mathcal{A})$ such that P is generated in degree m and for which P_n is locally free of rank one over k for all $n \geq m$. As the next lemma shows, this concept is not very useful in our setting:

Lemma 3.4.15. *Let $i \in \mathbb{Z}$ and $P \in \mathrm{Gr}(\mathcal{A})$ generated in degree $2i$ such that P_{2i} and P_{2i+1} are locally free of rank one over k. Then $P_n = 0$ for all $n \geq 2i+2$.*

Proof. Recall that the following composition

$$P_{2i} \longrightarrow P_{2i} \otimes \mathcal{E}^{*(2i)} \otimes \mathcal{E}^{*(2i+1)} \longrightarrow P_{2i+1} \otimes \mathcal{E}^{*(2i+1)} \longrightarrow P_{2i+2}$$

must be zero as it represents the action of $\mathcal{Q}_{2i}$. By [VdB12, lemma 4.3.2.] this composition equals

$$P_{2i} \xrightarrow{\varphi_{2i}^*} P_{2i+1} \otimes \mathcal{E}^{*(2i+1)} \xrightarrow{\varphi_{2i+1}} P_{2i+2}$$

where φ_{2i}^* is obtained by adjointness from $\varphi_{2i} : P_{2i} \otimes \mathcal{E}^{*2i} \longrightarrow P_{2i+1}$ and $\mathcal{E}^{*2i+1}$ has rank $(1,4)$. Since P_{2i} and $P_{2i+1} \otimes \mathcal{E}^{*(2i+1)}$ are locally all free of rank one over k we obtain that φ_{2i}^* is either an isomorphism or the zero morphism. Similarly φ_{2i+1} is either injective or zero. Hence the only way the composition can be zero is if $\varphi_{2i}^* = 0$ or $\varphi_{2i+1} = 0$. The first doesn't occur as $\varphi_{2i} \neq 0$ (because P is generated in degree $2i$ and $P_{2i+1} \neq 0$). Hence we have $\varphi_{2i+1} = 0$. However φ_{2i+1} is surjective (because P is generated in degree $2i$), implying that $P_{2i+2} = 0$. Using surjectivity of $P_{2i+2} \otimes \mathcal{A}_{2i+2,n} \to P_n$ for all $n \geq 2i+2$ the result follows. □

We thus propose the following variation of the above definition, better suited to our needs:

Definition 3.4.16. A shifted point module is an object $P \in \mathrm{Gr}(\mathcal{A})$ which is generated in degree $2i$ for some integer i and such that for all $n \geq 2i$, P_n is locally free over k of rank one if n is even and rank two if n is odd. We will often use the short hand notation $\dim_k(P_n) = \dim_k(\alpha_{n,*}(P_n))$ whenever the latter is finite. So we could say P is a shifted point module if is generated in degree $2i$ and:

$$\dim_k(P_n) = \begin{cases} 0 & \text{if } n < 2i \\ 1 & \text{if } n \geq 2i \text{ is even} \\ 2 & \text{if } n > 2i \text{ is odd} \end{cases}$$

The following lemma shows that this new definition of point modules is better behaved than the naive one:

Lemma 3.4.17. *Let $P \in \mathrm{Gr}(\mathcal{A})$ be a graded module and $i \in \mathbb{Z}$ such that:*

- *P is generated in degree $2i$*
- $\dim_k(P_{2i}) = 1$
- $\dim_k(P_{2i+1}) = 2$

Then for all $n \geq 2i+2$ fixed, we have

$$\dim_k(P_n) \leq \begin{cases} 1 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd} \end{cases} \tag{3.41}$$

Moreover if equality holds in (3.41)*, then P_n is characterized up to unique isomorphism by the data $\varphi_{2i} : P_{2i} \otimes \mathcal{E}^{*2i} \to P_{2i+1}$.*
If on the other hand (3.41) *is a strict inequality for some n, then $P_l = 0$ for all $l > n$.*

Proof. We prove all facts by induction on n. So suppose (3.41) and the subsequent claims hold for $n = 2i, \dots, m$. We distinguish several cases depending on whether the inequalities are in fact equalities or not.
Case 1: Equality holds in (3.41) **for $n = 2i, \dots, m$.**
The following composition is zero:

$$P_{m-1} \xrightarrow{\varphi^*_{m-1}} P_m \otimes \mathcal{E}^{*m} \xrightarrow{\varphi_m} P_{m+1}$$

φ_m is surjective, using the fact that the ranks are $(4,1)$ or $(1,4)$ depending on the parity of m, on easily verifies that (3.41) holds for $n = m+1$ if φ^*_{m-1} is injective. Moreover the same reasoning shows that if the equality holds for $\dim_k(P_{m+1})$, then $P_{m+1} \cong \mathrm{coker}(\varphi^*_{m-1})$ and is hence defined up to unique isomorphism.
Case 1a: m is odd

We have $\dim_k(P_{m-1}) = 1$ and the claim reduces to $\varphi^*_{m-1} \neq 0$ which holds because $\varphi_{m-1} \neq 0$
Case 1b: m is even

If φ^*_{m-1} is not injective, then there exists a $W \subset P_{m-1}$ with $\dim_k(W) = 1$ such that the composition

$$W \hookrightarrow P_{m-1} \xrightarrow{\varphi^*_{m-1}} P_m \otimes \mathcal{E}^{*m}$$

or equivalently the composition

$$W \otimes \mathcal{E}^{*m-1} \hookrightarrow P_{m-1} \otimes \mathcal{E}^{*m-1} \longrightarrow P_m$$

is zero. This implies that there is a $\overline{W} \in \mathrm{Gr}(\mathcal{A})$ given by $\overline{W}_{m-1} = W$ and $\overline{W}_l = 0$ for $l \neq m-1$ such that there is an embedding $\chi : \overline{W} \hookrightarrow P_{\geq m-2}$. Let $C = \mathrm{coker}(\chi)_{\geq m-2}$. Then C is generated in degree $m-2$ (which is even!) and $\deg_k(C_{m-2}) = \deg_k(C_{m-1}) = \deg_k(C_m) = 1$ contradicting Lemma 3.4.15.

Case 2: There is an integer $n \in \{2i+2, \dots, m\}$ such that there is a strict inequality for $\dim_k(P_n)$ in (3.41)

Let n_0 be the smallest such n. We have to show $P_l = 0$ for all $l > n_0$. Assume that $P_{n_0} = 0$, then $P_l = 0$ by surjectivity of $P_{n_0} \otimes \mathcal{A}_{n_0,l} \longrightarrow P_l$. The only nontrivial case is when n_0 is odd and $\dim_k(P_{n_0}) = 1$. In this case $\dim_k(P_{n_0-1}) = 1$ as well and the result follows from Lemma 3.4.15. ☐

Remark 3.4.18. *The proof of the above lemma also shows that any data*

$$\varphi_{2i} : P_{2i} \otimes \mathcal{E}^{*2i} \twoheadrightarrow P_{2i+1}$$

with $\dim_k(P_{2i}) = 1$ *and* $\dim_k(P_{2i+1}) = 2$ *can be extended to a shifted point module which is unique up to unique isomorphism.*

From now on we use the following short hand notation:

$$L_{n,p} := \mathcal{O}_p \otimes e_n\mathcal{A} \tag{3.42}$$

where p is any point on X_n.

Proof. of Theorem 3.4.4
Exactness of the sequence (3.35) can be checked for each degree n separately:

$$0 \longrightarrow \mathcal{Q}_m \otimes \mathcal{A}_{m+2,n} \longrightarrow \mathcal{E}^{*m} \otimes \mathcal{A}_{m+1,n} \longrightarrow \mathcal{A}_{m,n} \longrightarrow 0 \tag{3.43}$$

As all terms in this sequence are elements of $\operatorname{bimod}(X_m - X_n)$, applying the pushforward of the projection $\pi_m : X_m \times X_n \longrightarrow X_m$, yields a sequence of coherent sheaves on X_m:

$$0 \to \pi_{m,*}(\mathcal{Q}_m \otimes \mathcal{A}_{m+2,n}) \longrightarrow \pi_{m,*}(\mathcal{E}^{*m} \otimes \mathcal{A}_{m+1,n}) \longrightarrow \pi_{m,*}(\mathcal{A}_{m,n}) \to 0 \tag{3.44}$$

and (3.44) is exact if and only if (3.43) is since the support of these bimodules is finite. The structure of the relations on $\mathcal{A}$ implies that (3.35) and hence also (3.43) and (3.44) are right exact. Now for any point $p \in X_m$ the following complex will be right exact as well:

$$\begin{aligned} 0 \to \mathcal{O}_p \otimes \pi_{m,*}(\mathcal{Q}_m \otimes \mathcal{A}_{m+2,n}) \longrightarrow \mathcal{O}_p \otimes \pi_{m,*}(\mathcal{E}^{*m} \otimes \mathcal{A}_{m+1,n}) \longrightarrow \\ \longrightarrow \mathcal{O}_p \otimes \pi_{m,*}(\mathcal{A}_{m,n}) \to 0 \end{aligned} \tag{3.45}$$

As all terms (3.45) are locally free over k, its left exactness can be checked numerically. Hence in order to prove the lemma we show that the terms in (3.45) have the "correct" constant dimension (see (3.50)) for each point p. From this it follows that (3.43) is exact and its terms are locally free on the left. The locally freeness on the right then follows from [VdB12, Proposition 3.1.6].)

So we are left with finding the length of the objects in (3.45). Any object in $\mathrm{bimod}(X_m - X_n)$ is of the form ${}_u\mathcal{U}_v$ for finite maps u and v. As taking the direct image through a finite morphism preserves the length, we have for such a bimodule:

$$\begin{aligned}\dim_k(\mathcal{O}_p \otimes \pi_{m,*}({}_u\mathcal{U}_v)) &= \dim_k(\mathcal{O}_p \otimes u_*\mathcal{U}) \\ &= \dim_k(u_*(u^*(\mathcal{O}_p) \otimes \mathcal{U})) \\ &= \dim_k(u^*(\mathcal{O}_p) \otimes \mathcal{U}) \\ &= \dim_k(v_*(u^*(\mathcal{O}_p) \otimes \mathcal{U})) \\ &= \dim_k(\mathcal{O}_p \otimes {}_u\mathcal{U}_v))\end{aligned}$$

Hence the length of the terms in (3.45) can be calculated from

$$0 \to \mathcal{O}_p \otimes \mathcal{Q}_m \otimes \mathcal{A}_{m+2,n} \to \mathcal{O}_p \otimes \mathcal{E}^{*m} \otimes \mathcal{A}_{m+1,n} \to \mathcal{O}_p \otimes \mathcal{A}_{m,n} \to 0 \tag{3.46}$$

In the case where $m = 2i-1$, the fact that $\dim_k\left(\mathcal{O}_p \otimes \mathcal{E}^{*(2i-1)}\right) = 1$, implies that there must be a point $q \in X_{2i}$ such that $\mathcal{O}_p \otimes \mathcal{E}^{*(2i-1)} = \mathcal{O}_q$. Similarly, in the case where $m = 2i$, we have $\dim_k\left(\mathcal{O}_p \otimes \mathcal{E}^{*(2i)}\right) = 4$, and there must be points $\widetilde{q^a} \in X_{2i+1}$, $a = 1, \ldots, 4$ such that $\mathcal{O}_p \otimes \mathcal{E}^{*(2i)}$ is an extension of the $\mathcal{O}_{\widetilde{q^a}}$. Put

$$M_{2i+1,p} = \mathcal{O}_p \otimes_{X_{2i}} \mathcal{E}^{*(2i)} \otimes_{X_{2i+1}} e_{2i+1}\mathcal{A}.$$

Then $M_{2i+1,p}$ is an extension of the $L_{2i+1,\widetilde{q^a}}$. The sequence (3.46) now gives rise to the following right exact sequences

$$L_{2i+1,p} \longrightarrow L_{2i,q} \longrightarrow L_{2i-1,p} \longrightarrow 0 \tag{3.47}$$

$$L_{2i+2,p} \longrightarrow M_{2i+1,p} \longrightarrow L_{2i,p} \longrightarrow 0 \tag{3.48}$$

Finally there also is a right exact sequence:

$$L_{2i+1,p'} \longrightarrow L_{2i-1,p} \longrightarrow P_p \longrightarrow 0 \tag{3.49}$$

where the morphism $L_{2i+1,p'} \to L_{2i-1,p}$ comes from the fact that

$$\dim_k(\mathcal{O}_p \otimes \mathcal{A}_{2i-1,2i+1}) = 3 > 0$$

so that there is a $p' \in X_{2i+1}$ with a nonzero morphism $\mathcal{O}_{p'} \to \mathcal{O}_p \otimes \mathcal{A}_{2i-1,2i+1}$. P_p is defined as the cokernel of this morphism.

We now prove the following by induction on j (simultaneously for all points p and all $i \in \mathbb{Z}$):

$$\begin{gathered}\dim_k((P_p)_{2i+2j}) = 1 \text{ and } \dim_k((P_p)_{2i+2j+1}) = 2 \\ \dim_k((L_{2i,p})_{2i+2j}) = 2j+1 \text{ and } \dim_k((L_{2i,p})_{2i+2j+1}) = 4j+4 \\ \dim_k((L_{2i-1,p})_{2i+2j}) = j+1 \text{ and } \dim_k((L_{2i-1,p})_{2i+2j+1}) = 2j+3\end{gathered} \tag{3.50}$$

It is easy to see that these claims hold for $j = 0$. So by induction we suppose they hold for $j = 0, \dots, l$, for all p and for all $i \in \mathbb{Z}$. We prove that the claims also hold for $j = l+1$.

By (3.48) we see:

$$\begin{aligned} \dim_k((L_{2i,p})_{2i+2l+2}) &\geq \dim_k((M_{2i+1,p})_{2i+2l+2}) - \dim_k((L_{2i+1,p})_{2i+2l+2}) \\ &= \sum_{a=1}^{4} \dim_k((L_{2i+2,\widetilde{q^a_{2i+1}}})_{2i+2l+2}) - \dim_k((L_{(2i+2,p})_{2i+2l+2}) \\ &= 4 \cdot (l+1) - (2l+1) \\ &= 2l+3 \end{aligned}$$

where the last equality follows from the induction hypothesis. This can be written schematically as:

$$\begin{array}{cccccccccc} 0 & & & & & & & & & \\ \uparrow & & & & & & & & & \\ L_{2i,p} & 0 & 1 & 4 & 3 & \dots & 2l+1 & 4l+4 & \underline{2l+3} & \underline{4l+8} \\ \uparrow & & & & & & & & & \\ M_{2i+1,p} & 0 & 0 & 4 & 4 & \dots & 4l & 8l+4 & 4l+4 & 8l+12 \\ \uparrow & & & & & & & & & \\ L_{2i+2,\widetilde{p}} & 0 & 0 & 0 & 1 & \dots & 2l-1 & 4l & 2l+1 & 4l+4 \end{array} \tag{3.51}$$

Where the numbers on the right of a module signifies $\dim_k((-)_x)$ for $x = 2i-1, \dots, 2i+2l+3$ and an underlined number implies a lower bound for $\dim_k$. Similarly we write $\overline{N}$ to denote an upperbound for a certain $\dim_k$.

Now consider the module $P_{p,\geq 2i+2l}$. It is generated in degree $2i+2l$ because P_p is a quotient of $L_{2i-1,p}$. Moreover $\dim_k((P_p)_{2i+2l}) = 1$ and $\dim_k((P_p)_{2i+2l+1}) = 2$, so Lemma 3.4.17 implies $\dim_k((P_p)_{2i+2l+2}) \leq 1$ and $\dim_k((P_p)_{2i+2l+3}) \leq 2$.

Together with the right exact sequence (3.49) this gives us the following upper bounds:

$$\begin{array}{ccccccccc} 0 & & & & & & & & \\ \uparrow & & & & & & & & \\ P & 1 & 1 & 2 & \dots & 1 & 2 & \overline{1} & \overline{2} \\ \uparrow & & & & & & & & \\ L_{2i-1,p} & 1 & 1 & 3 & \dots & l+1 & 2l+3 & \overline{l+2} & \overline{2l+5} \\ \uparrow & & & & & & & & \\ L_{2i+1,p'} & 0 & 0 & 1 & \dots & l & 2l+1 & l+1 & 2l+3 \end{array} \tag{3.52}$$

Combining the bounds found in (3.51) and (3.52) and using (3.47) we have:

$$\begin{array}{cccccccc} 0 & & & & & & & \\ \uparrow & & & & & & & \\ L_{2i-1,p} & 1 & 1 & 4 & \dots & l+1 & 2l+3 & \overline{l+2} & \overline{2l+5} \\ \uparrow & & & & & & & \\ L_{2i,q} & 0 & 1 & 4 & \dots & 2l+1 & 4l+4 & \underline{2l+3} & \underline{4l+8} \\ \uparrow & & & & & & & \\ L_{2i+1,\widetilde{p}} & 0 & 0 & 1 & \dots & l-1 & 2l+1 & l+1 & 2l+3 \end{array} \tag{3.53}$$

Right exactness of (3.47) implies that the bounds in (3.53) are in fact equalities, because for example we find the upper bound

$$\begin{aligned} \dim_k((L_{2i,q})_{2i+2l+2}) &\leq \dim_k((L_{2i-1,p})_{2i+2l+2}) + \dim_k((L_{2i+1,\widetilde{p}})_{2i+2l+2}) \\ &\leq l+2+l+1 \\ &= 2l+3 \end{aligned}$$

which equals the already known lower bound for $\dim_k((L_{2i,q})_{2i+2l+2})$. Hence we have found exact values for $\dim_k(L_{2i+1,q})$. A priori the above right exact sequence only gives those exact value for the points $q \in X_{2i}$ for which there is a $p \in X_{2i-1}$ such that $\mathcal{O}_p \otimes \mathcal{E}^{*2i-1} = \mathcal{O}_q$. But as $\mathcal{E}^{*2i-1}$ is of the form ${}_{\mathrm{id}}(\omega_{Y/X}^{-i+1})_f$ as in (3.27) we have $q = f(p)$ and surjectivity of f implies that q runs through all points of X_{2i} as p runs through all points of X_{2i-1}. With the same reasoning we now obtain from (3.53) the exact values for $\dim_k((L_{2i-1})_{2i+2l+2})$ and $\dim_k((L_{2i-1})_{2i+2l+3})$.

Hence we have proven (3.50) for all $i, j \in \mathbb{Z}$ and for all points p. As these values do not depend on p and X is a smooth variety, it follows from [Har97, ex. II, §5, no.8] that the terms in (3.44) are locally free on the left (and hence also on the right). Filling in these values for (3.45), the theorem follows. □

3.4.3 The full exceptional sequence

This section is dedicated to the proof of the following theorem:

Theorem 3.4.19. *Let $\mathcal{E} = {}_f(\mathcal{O})_{\mathrm{id}}$ be a $\mathbb{P}^1$-bimodule of rank $(4,1)$. Let $\mathbb{S}(\mathcal{E})$ be the associated symmetric sheaf-$\mathbb{Z}$-algebra and put $Z = \mathrm{Proj}(\mathbb{S}(\mathcal{E}))$. Let $\mathbf{D}$ denote the triangulated subcategory of objects in $\mathbf{D}(Z)$ with bounded noetherian cohomology. Then $\mathbf{D}$ is* Ext*-finite and*

$$\left(\Pi_1^*(\mathcal{O}_{\mathbb{P}^1}), \Pi_1^*(\mathcal{O}_{\mathbb{P}^1}(1)), \Pi_0^*(\mathcal{O}_{\mathbb{P}^1}), \Pi_0^*(\mathcal{O}_{\mathbb{P}^1}(1))\right) \tag{3.54}$$

is a full and strong exceptional sequence in $\mathbf{D}$.

The Gram matrix of the Euler form for this exceptional sequence is given by

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Remark 3.4.20. *The (sub)category* $\mathbf{D}$ *has a more down-to-earth definition: by [LVdB06, Proposition 2.14] (together with Theorem 3.3.1 and Corollary 3.3.2) we know that* $\mathbf{D} = \mathbf{D}^{\mathrm{b}}(\mathrm{qgr}(\mathbb{S}(\mathcal{E})))$*. Here* $\mathrm{gr}(\mathbb{S}(\mathcal{E}))$ *is the subcategory of noetherian objects in* $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$ *and* $\mathrm{qgr}(\mathbb{S}(\mathcal{E}))$ *the associated Serre quotient.*

We will prove this Theorem 3.4.19 through a series of lemmas. We first introduce the relevant definitions and exhibit some technical results required to prove Theorem 3.4.19.

Definition 3.4.21. Let $\mathcal{T}$ be a k-linear triangulated category with shift functor $[1]$. We say that $E \in \mathcal{T}$ is an *exceptional object* if

$$\mathrm{Hom}_{\mathcal{T}}(E, E[m]) \cong \begin{cases} k & m = 0 \\ 0 & m \neq 0 \end{cases}. \tag{3.55}$$

A sequence $(E_1, \dots, E_n)$ of objects $E_i \in \mathcal{T}$ is an *exceptional sequence* if each E_i is exceptional, and $\mathrm{Hom}_{\mathcal{T}}(E_i, E_j[m]) = 0$ for all $i > j$ and m. It is said to be *strong* if moreover $\mathrm{Hom}_{\mathcal{T}}(E_i, E_j[m]) = 0$ for all $m \neq 0$. It is said to be *full* if it generates the category $\mathcal{T}$.

Remark 3.4.22. *It is a well known fact from tilting theory that a full and strong exceptional collection* $E_1, \dots, E_n$ *induces a tilting object*

$$T = E_1 \oplus \ldots \oplus E_n$$

This in turn gives rise to an equivalence of categories $\mathcal{T} \cong \mathbf{D}(\mathrm{End}(T))$*. Hence if moreover* $\mathrm{Hom}(E_i, E_j)$ *is finite dimensional for all* i *and* j*, then* $\mathcal{T}$ *is equivalent to* $\mathbf{D}(kQ/\mathcal{R})$ *for a quiver* Q *with relations* $\mathcal{R}$*. We will use this fact in Proposition 3.4.28.*

Lemma 3.4.23. *Let* $\mathcal{T}$ *be a* k*-linear triangulated category. Assume that* $E_1, \dots, E_n$ *is a collection of objects in* $\mathcal{T}$ *such that*

(a) $\sum_j \dim \mathrm{Hom}^j_{\mathcal{T}}(E_i, T) < \infty$ *for all* i *and for all* $T \in \mathrm{Ob}(\mathcal{T})$*.*

(b) $(E_i)_i$ *satisfies the conditions for an exceptional sequence, except that we do not require* Hom*-finiteness of* $\mathcal{T}$*.*

(c) we have

$$((E_m)_m)^{\perp} := \{Y \in \mathcal{T} \mid \forall m : Hom^i(E_m, Y) = 0\} = \{0\}$$

Then

1. $E_1, \ldots, E_n$ *generate* $\mathcal{T}$ *as a triangulated category and*
2. $\mathcal{T}$ *is* Ext-*finite.*

Proof. Let $T \in \mathcal{T}$. We have to prove that T is in the triangulated subcategory of $\mathcal{T}$ generated by $E_1, \ldots, E_n$. We put $T_n = T$ and define T_{i-1} inductively by $L_{E_i} T_i$ for $i = n, n-1, \ldots, 1$, i.e.

$$T_{i-1} = \operatorname{cone}(\operatorname{Hom}^{\bullet}_{\mathcal{T}}(E_i, T_i) \otimes_k E_i \longrightarrow T_i)$$

Then T_i is in the triangulated subcategory of $\mathcal{T}$ generated by T_{i-1} and E_i. Furthermore

$$\operatorname{Hom}^{\bullet}_{\mathcal{T}}(E_j, T_i) = 0 \text{ for } j > i$$

It follows that $T_0 = 0$. Hence we are done.

For (2) we have to prove that if $T_1, T_2 \in \mathcal{T}$ then $\sum_j \dim \operatorname{Hom}^j_{\mathcal{T}}(T_1, T_2) < \infty$. Since T_1 is in the triangulated category generated by $(E_i)_i$ we may assume $T_1 = E_i$ for some i. But then the claim is part of the hypotheses. □

Lemma 3.4.24. *Let X, Y be smooth varieties over k and let $\mathcal{E}$ be a locally free $X-Y$-bimodule of rank $(4,1)$. Then for all $m \in \mathbb{Z}$ one has*

i) The cohomological dimension of $\Pi_{m,*}$ satisfies

$$\operatorname{cd} \Pi_{m,*} \leq 1.$$

ii) If $\mathcal{F}$ is a noetherian object then $\mathbf{R}^i \Pi_{m,*}\mathcal{F}$ is a coherent sheaf for all i.

Remark 3.4.25. *It is easy to see that the cohomological dimension in i) is exactly one, but we do not need it and leave it out for clarity.*

Proof. i) By (3.36), we have

$$\mathbf{R}^i \Pi_{m,*}(p(-)) = \mathbf{R}^i \omega(p(-))_m$$

which reduces the claim to $\operatorname{cd} \omega = 1$. From Lemma 3.4.8 we in turn obtain

$$\mathbf{R}^i \omega(p(-)) \cong \mathbf{R}^{i+1} \tau$$

and the claim now reduces to $\operatorname{cd} \tau = 2$. This is proved as in [Nym04b, Corollary 4.10] using the exact sequence (3.35) instead of the exact sequence (4.1) in op. cit.

ii) Since $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$ is locally noetherian we may construct a left resolution of $\mathcal{F}$ by objects which are finite direct sums of objects of the form

$$p(\mathcal{G} \otimes_{\mathcal{O}_{X_n}} e_n \mathbb{S}(\mathcal{E})) = \Pi_n^*(\mathcal{G})$$

for $\mathcal{G} \in \mathrm{coh}(X_n)$. Using that $\Pi_{m,*}$ has finite cohomological dimension we reduce to the case $\mathcal{F} = \Pi_n^*(\mathcal{G})$.

Tensoring (3.35) (with m replaced by n) on the left with $\mathcal{G} \in \mathrm{coh}(X_n)$ we obtain exact sequences in $Z = \mathrm{Proj}(\mathbb{S}(\mathcal{E}))$

$$0 \longrightarrow \Pi_{n+2}^*(\mathcal{G}) \longrightarrow \Pi_{n+1}^*(\mathcal{G} \otimes_{X_n} \mathcal{E}^{*n}) \longrightarrow \Pi_n^*(\mathcal{G}) \longrightarrow 0 \tag{3.56}$$

Hence repeatedly using such exact sequences we may reduce to the case $\mathcal{F} = \Pi_n^*(\mathcal{G})$ for $n \leq m$. When $n \leq m$ it is shown in the proof of theorem 5.4.1 that

$$\mathbf{R}^i \Pi_{m,*} \Pi_n^* \mathcal{G} = \begin{cases} \mathcal{G} \otimes_{X_n} \mathbb{S}(\mathcal{E})_{n,m} & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

This is indeed coherent. □

Lemma 3.4.26. *Let $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be a morphism of degree 4. Put $\mathcal{E} = {}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}$. Then the right orthogonal to the subcategory generated by*

$$E = (\Pi_1^*(\mathcal{O}_{\mathbb{P}^1}), \Pi_1^*(\mathcal{O}_{\mathbb{P}^1}(1)), \Pi_0^*(\mathcal{O}_{\mathbb{P}^1}), \Pi_0^*(\mathcal{O}_{\mathbb{P}^1}(1)))$$

in $\mathbf{D}(\mathrm{QGr}(\mathbb{S}(\mathcal{E})))$ is zero.

Proof. Assume that $A \in \mathrm{QGr}(\mathbb{S}(\mathcal{E}))$ is right orthogonal to E. Using the exact sequences

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(a) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(a+1)^{\oplus 2} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(a+2) \longrightarrow 0$$

and the exactness of Π_m^* (Lemma 3.4.6) we find that A is in fact right orthogonal to $\Pi_m^*(\mathcal{O}_{\mathbb{P}^1}(a))$ for $m = 0, 1$ and all a.

From (3.56) we obtain exact sequences in $\mathrm{QGr}(\mathbb{S}(\mathcal{E}))$

$$0 \to \Pi_{m+2}^*(\mathcal{O}_{\mathbb{P}^1}(a)) \longrightarrow \Pi_{m+1}^*(\mathcal{O}_{\mathbb{P}^1}(a) \otimes_{X_m} \otimes \mathcal{E}^{*m}) \longrightarrow \Pi_m^*(\mathcal{O}_{\mathbb{P}^1}(a)) \to 0$$

Since $\mathcal{O}_{\mathbb{P}^1}(a) \otimes_{X_m} \mathcal{E}^{*m}$, being locally free, is isomorphic to a sum of $\mathcal{O}_{\mathbb{P}}(b)$ we conclude by induction that A is right orthogonal to $\Pi_m^*(\mathcal{O}_{\mathbb{P}^1}(a))$ for all m, a.

Now the collection $(\Pi_m^*(\mathcal{O}_{\mathbb{P}^1}(a)))_{m,a}$ generates $\mathrm{QGr}(\mathbb{S}(\mathcal{E}))$ as a Grothendieck category. From this it is easy to see that the right orthogonal to $(\Pi_m^*(\mathcal{O}_{\mathbb{P}^1}(a)))_{m,a}$ in $\mathbf{D}(\mathrm{QGr}(\mathbb{S}(\mathcal{E})))$ is zero. This finishes the proof. □

Proof. of Theorem 3.4.19. The computation of the Gram matrix, the strongness and exceptionality is an immediate application of Theorem 3.4.1:

$$\mathrm{Ext}^i_Z\left(\Pi_n^*\mathcal{F}, \Pi_n^*\mathcal{G}\right) = \mathrm{Ext}^i_{\mathbb{P}^1}\left(\mathcal{F}, \mathcal{G}\otimes \mathbb{S}(\mathcal{E})_{n,n}\right) = \mathrm{Ext}^i_{\mathbb{P}^1}\left(\mathcal{F}, \mathcal{G}\right)$$

proving the claim for the subsequences

$$\left(\Pi_1^*(\mathcal{O}_{\mathbb{P}^1}), \Pi_1^*(\mathcal{O}_{\mathbb{P}^1}(1))\right) \text{ and } \left(\Pi_0^*(\mathcal{O}_{\mathbb{P}^1}), \Pi_0^*(\mathcal{O}_{\mathbb{P}^1}(1)\right)$$

There are no backward Hom's by Theorem 3.4.1 once again.

There are four remaining cases. Since they are all very similar, we pick one out and leave the other three to the reader:

$$\begin{aligned}\mathrm{Ext}^i_Z\left(\Pi_1^*(\mathcal{O}_{\mathbb{P}^1}), \Pi_0^*(\mathcal{O}_{\mathbb{P}^1}(1))\right) &= \mathrm{Ext}^i_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1)\otimes \mathbb{S}(\mathcal{E})_{0,1})\\
&= \mathrm{Ext}^i_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1)\otimes {}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}})\\
&= \mathrm{H}^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)\otimes {}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}})\\
(\text{see } (3.5)) &\Downarrow\\
&= \mathrm{H}^i(\mathbb{P}^1, f^*\mathcal{O}_{\mathbb{P}^1}(1))\\
&= \mathrm{H}^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4))\end{aligned}$$

which is indeed only nonzero for $i \neq 0$, in which case it is 5-dimensional over k.

To show that the sequence is full, we have to verify conditions (a)(b) and (c) of Lemma 3.4.23. Condition (a) follows from Lemma 3.4.24, which implies$\mathbf{R}\Pi_{m,*}\mathcal{G}$ lives in $D^b_{\mathrm{coh}}(\mathrm{Qcoh}(X_m))$, combined with the fact that by Lemma 3.4.7, we have

$$\mathrm{Ext}^i_{\mathrm{QGr}\,\mathcal{A}}(\Pi_m^*(\mathcal{O}_{\mathbb{P}^1}(a)), \mathcal{G}) = \mathrm{Ext}^i_{X_m}(\mathcal{O}_{\mathbb{P}^1}(a), \mathbf{R}\Pi_{m,*}\mathcal{G})$$

Condition (b) is proven above. Finally, condition (c) follows from Lemma 3.4.26. □

Remark 3.4.27. *Although Lemma 3.3.6 showed that up to equivalence of categories* $\mathrm{QGr}(\mathbb{S}({}_f(\mathcal{L})_{\mathrm{id}}))$ *does not depend on the choice of* $\mathcal{L}$*, the functors* Π_0^*, Π_1^* *and hence the exceptional collection in* (3.54) *heavily depend on the choice of* $\mathcal{L}$*.*

For example one easily checks that the sequence is no longer strong if we choose $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(-l)$ *for* $l \geq 2$ *as*

$$\begin{aligned}\mathrm{Ext}^1_Z\left(\Pi_1^*(\mathcal{O}_{\mathbb{P}^1}), \Pi_0^*(\mathcal{O}_{\mathbb{P}^1})\right) &= \mathrm{Ext}^1_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}\otimes {}_f(\mathcal{O}_{\mathbb{P}^1}(-l))_{\mathrm{id}})\\
&= \mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-l))\\
&\neq 0\end{aligned}$$

We conclude this chapter by computing the associated finite-dimensional algebra of the sequence provided by tilting theory: As the exceptional collection (3.54) is full and strong we can prove the following:

Proposition 3.4.28. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$, be given by $[x : y] \mapsto [x^4 : y^4]$. Let $\mathcal{E} = {}_f (\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}$ and $\mathbf{D}$ be as in Theorem 3.4.19, then there is an equivalence*

$$\mathbf{D} \cong \mathbf{D}(kQ/\mathcal{R})$$

where Q is the quiver

$$\begin{tikzcd}
\bullet \arrow[r, Rightarrow, "\alpha", "\boxed{2}" description] \arrow[d, "\omega"'] \arrow[dr, Rightarrow, "\boxed{5}" description, "\gamma"'] & \bullet \arrow[d, Rightarrow, "\boxed{4}" description, "\beta"] \\
\bullet \arrow[r, Rightarrow, "\boxed{2}" description, "\delta"'] & \bullet
\end{tikzcd} \tag{3.57}$$

and the relations $\mathcal{R}$ are given by

$$\begin{cases} \alpha_i \beta_j = \gamma_{i+j} & 0 \le i \le 1, 0 \le j \le 3 \\ \omega \delta_i = \gamma_{4i} & 0 \le i \le 1 \end{cases} \tag{3.58}$$

Proof. As the exceptional collection (3.54) is full and strong, there is a tilting object

$$\mathcal{T} := \Pi_1^*(\mathcal{O}) \oplus \Pi_1^*(\mathcal{O}(1)) \oplus \Pi_0^*(\mathcal{O}(1)) \oplus \Pi_0^*(\mathcal{O})$$

showing that $\mathbf{D} \cong \mathbf{D}(\mathrm{End}(\mathcal{T}))$.

Now, $\mathrm{End}(\mathcal{T})$ is obviously isomorphic to $kQ/\mathcal{R}$ where Q is given by the quiver

$$\begin{tikzcd}
\Pi_1^*(\mathcal{O}) \arrow[r, Rightarrow, "\alpha", "\boxed{2}" description] \arrow[d, "\omega"'] \arrow[dr, Rightarrow, "\boxed{5}" description, "\gamma"'] & \Pi_1^*(\mathcal{O}(1)) \arrow[d, Rightarrow, "\boxed{4}" description, "\beta"] \\
\Pi_0^*(\mathcal{O}) \arrow[r, Rightarrow, "\boxed{2}" description, "\delta"'] & \Pi_0^*(\mathcal{O}(1))
\end{tikzcd}$$

and the relations in $\mathcal{R}$ are induced by the composition of the Hom-sets in the exceptional collection (3.54). To check that these relations actually coincides with the conditions in (3.58), note that the above quiver can be identified with:

$$\begin{tikzcd}
k[x,y]_0 \arrow[r, Rightarrow, "\alpha", "\boxed{2}" description] \arrow[d, "\mathrm{id}"'] \arrow[dr, Rightarrow, "\boxed{5}" description, "\gamma"'] & k[x,y]_1 \arrow[d, Rightarrow, "\boxed{4}" description, "\beta"] \\
k[x,y]_0 \arrow[r, Rightarrow, "\boxed{2}" description, "\delta"'] & k[x,y]_4
\end{tikzcd}$$

Where α_i, β_j, γ_m and δ_n denote multiplication by $x^i y^{1-i}$, $x^j y^{3-j}$, $x^m y^{4-m}$ and $x^{4n} y^{4-4n}$ respectively. The result now follows. $\square$

Remark 3.4.29. *The result in the above proposition can be generalized as follows: let $f : \mathbb{P}^1 \to \mathbb{P}^1$ is any degree 4 morphism and let $\mathcal{E} = {}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}$ and $\mathbf{D}$ be as in Theorem 3.4.19. There is an equivalence of categories*

$$\mathbf{D} \cong \mathbf{D}(kQ/\mathcal{R})$$

where Q is a quiver as in (3.57) and the relations $\mathcal{R}$ are as follows

- *There is a 5-dimensional space of "diagonal" morphisms γ_n.*
- *Each γ_n can be written as a linear combination of the $\alpha_i\beta_j$.*
- *δ_0, δ_1 are homogeneous degree 4 polynomials defining f:*

$$f : \mathbb{P}^1 \to \mathbb{P}^1 : [x : y] \mapsto [\delta_0(x : y) : \delta_1(x : y)]$$

- *$\omega\delta_0$ and $\omega\delta_1$ are linearly independent.*

Corollary 3.4.30. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a degree 4 morphism. The associated noncommutative $\mathbb{P}^1$-bundle $\mathrm{qgr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}))$ has finite global dimension.*

Proof. By Lemma 3.4.6 we have that Π_m^* is an exact functor, whilst Lemma 3.4.24 shows that $\Pi_{m,*}$ has cohomological dimension 1. The Grothendieck spectral sequence for Hom in $\mathrm{qgr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}))$ and $\Pi_{m,*}$ then shows that the cohomological dimension of $\Pi_m^*(\mathcal{O}_{\mathbb{P}^1}(i))$ is 2, i.e. that $\mathrm{Ext}^n(\Pi_m^*(\mathcal{O}_{\mathbb{P}^1}(i)), \mathcal{M}) = 0$ for $n \geq 3$, using Lemma 3.4.7.

From Theorem 3.4.19 and Proposition 3.4.28 we have that there exists a full and strong exceptional collection in $\mathbf{D}^{\mathrm{b}}(\mathrm{qgr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}))$ consisting of objects of cohomological dimension 2. Because the global dimension of this endomorphism algebra is finite (as it is the path algebra of an acyclic quiver modulo an ideal of relations), and the cohomological dimension of the functor realising the equivalence is 2, we can compute Ext^n in qgr using uniformly bounded complexes in $\mathbf{D}^{\mathrm{b}}(kQ/\mathcal{R})$, and we can conclude that the global dimension is indeed finite. ☐

Remark 3.4.31. *We conjecture that this result must hold for all noncommutative $\mathbb{P}^1$-bundle $\mathrm{qgr}(\mathbb{S}({}_f(\mathcal{O}_Y)_{\mathrm{id}}))$ associated to a finite, degree 4 morphism $f : Y \to X$ between nonsingular curves. Moreover we conjecture that the global dimension is exactly equal to 2. In the special case where $X \cong Y \cong \mathbb{P}^1$, it follows from Corollary 5.5.4 that $\mathbf{D}^{\mathrm{b}}(\mathrm{qgr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}))$ is equivalent to a category $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}(p^*\mathcal{S}))$, where $p^*\mathcal{S}$ is a sheaf of maximal orders on $\mathbb{F}_1$ and hence has global dimension 2 (Theorem 4.5.7).*

Remark 3.4.32. *The total number of degrees of freedom is 3 (in choosing f, or in choosing such a quiver). This follows for example from the classification of pencils of binary quartics as in [Wal98, Table 3], where the generic type (M) has 3 degrees of freedom. Alternatively, this can be seen intuitively as follows:*

- *We first fix bases for the 4 vertices*
- *There are* $8 = 2 \cdot 4$ *compositions of* $\alpha_i\beta_j$. *As these generate the 5-dimensional space of diagonal morphisms, there are 3 relations between them. The degrees of freedom for these choices is given by the dimension of* $Gras(3,8)$ *which is* $3(8-3) = 15$
- *The* $\omega\delta_m$ *should be expressed in the 5-dimensional space of* γ_n. *The amount of ways this can be done is given by the amount of morphisms from a 2-dimensional vectorspace to a 5-dimensional one: hence 10 ways.*
- *Now we can base change each of the 4 vertices, giving an action of* $GL(1) \times GL(2) \times GL(2) \times GL(4)$, *which is 1+4+4+16=25-dimensional. But we should mod out this group by all scalar multiplications by* a, b, c, d *respectively which satisfy* $ab = cd$. *So there is an action by a 22-dimensional group.*
- *An action of 22-dimensional group on a 25 dimensional vector space gives a 3-dimensional moduli space.*

Chapter 4

Maximal orders on $\mathbb{F}_1$ as noncommutative surfaces with exceptional collections of length 4

We adore chaos because we love to produce order.

Maurits C. Escher (Artist)

This chapter is based upon results by the author and Pieter Belmans in [BP17].

4.1 Introduction

In Chapter 3 we constructed a noncommutative del Pezzo surface which was suggested numerically by the matrix K_2 in [dTdV16, Theorem A], see (3.1). More precisely, [dTdV16] defines so called lattices of *surface** type and shows that for rank 4 such a lattice is isomorphic to $\mathbb{Z}^4$ where the matrix of the bilinear form is one of the following standard types:

$$J = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad K_n = \begin{bmatrix} 1 & n & 2n & n \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

There are commutative models for the first matrix, as well as for K_0 and K_1. For all K_m with $m \geq 2$ there exist no commutative models and using an invariant δ one shows that K_n has nonpositive degree for $n \geq 3$. (see §3.1 for more details).

The goal of this chapter is to construct for each n a noncommutative surfaces Z_n (i.e. a Grothendieck category satisfying suitable conditions, see Chapter 0 and §3.1) together with an exceptional sequence (E_1, E_2, E_3, E_4) for which the Gram matrix takes the form K_n as described above. Before giving this construction, we recall in §4.2 how mutation and shifting of exceptional collections gives an action of the signed braid group $\Sigma\mathrm{B}_n$ on the set of exceptional collections of $\mathbf{D}(Z_n)$. There is an induced action on the set of Gram matrices, see [dTdV16, §4]. Using this action, we show (Proposition 4.2.4) that the matrices K_n are mutation equivalent to

$$K'_n = \begin{bmatrix} 1 & 3 & 6 & n \\ 0 & 1 & 3 & n \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

As the top-left 3×3 submatrix, shows that the sequence (E_1, E_2, E_3) is isomorphic to Beilinson's collection ([Bl78], see also Example 4.4.2)

$$(\mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2))$$

for $X = \mathrm{coh}(\mathbb{P}^2)$, we heuristically conclude that Z_n should be equipped with a map (in the noncommutative sense) $p : Z_n \to \mathbb{P}^2$ such that $E_i = p^*(\mathcal{O}_X(i-1))$ for $i = 1, 2, 3$. Moreover [AOU14, Theorem 7.1] (see also Example 4.4.3) shows that we can (and will) replace X by its noncommutative analogue $\mathrm{qgr}(A)$ for A a quadratic Sklyanin algebra. Finally by Orlov's blowup formula ([Orl92, Theorem 4.3]) we expect p to be a blowup morphism where E_4 corresponds to some exceptional rank n object on the exceptional divisor. In fact, we will construct $E_4 = p^*\mathcal{F}$ out of a *fat point module* F. For this we recapitulate some facts on fat point modules in §4.3.

The main result of this Chapter (Theorem 4.5.7) is proved in §4.5. For any $m \in \mathbb{N}$ we can construct a noncommutative surface $Z_m \coloneqq \mathrm{coh}(p^*\mathcal{S})$ together with a full and strong exceptional sequence

$$(E_1, E_2, E_3, E_4) = (p^*\mathcal{S}_0, p^*\mathcal{S}_1, p^*\mathcal{S}_2, p^*\mathcal{F})$$

whose images form a basis of $\mathrm{K}_0(Z_m) \cong \mathbb{Z}^4$ for which the Gram matrix is given by K'_m.

Here $\mathcal{S}$ is a sheaf of maximal orders on $\mathbb{P}^2$ such that $\mathrm{coh}(\mathcal{S}) \cong \mathrm{qgr}(A)$ for a quadratic Sklyanin algebra A (Lemma 4.5.1 and Lemma 4.5.3) and $p : \mathbb{F}_1 \to \mathbb{P}^2$ is the blowup morphism for some point $x \in \mathbb{P}^2$ outside of the ramification divisor of $\mathcal{S}$. In particular, we provide an actual geometric construction for the numerical blowups of [dTdV16]. The main technique for understanding $\mathbf{D}(\mathrm{coh}(p^*\mathcal{S}))$ is a noncommutative generalisation of Orlov's blowup formula (Theorem 4.5.11), which is probably of independent interest. For this theorem we need the notion of semi-orthogonal decompositions, which we introduce in §4.4.

Finally in §4.6 we give some properties of the orders we have constructed in the context of the minimal model program for orders. Especially for $m = 2$ they turn out to be interesting, as we get interesting new examples of so called *half ruled orders*. Moreover we also show (Proposition 4.6.4) that Z_m is a del Pezzo order if and only if $m \leq 2$. This obviously agrees with the positivity of the degree δ as in [dTdV16].

4.2 Mutation

We quickly recall the theory of mutations of exceptional sequences.

Definition 4.2.1. Let $\mathcal{T}$ be an Ext-finite triangulated category and let (E, F) be an exceptional pair of objects in $\mathcal{T}$ (see Definition 3.4.21). We define the *left mutation* $\mathrm{L}_E\, F$ as the cone of the morphism

$$\mathrm{Hom}_{\mathcal{T}}(E, F) \otimes E \longrightarrow F \tag{4.1}$$

I.e. we extend (4.1) to a triangle

$$\mathrm{Hom}_{\mathcal{T}}(E, F) \otimes E \longrightarrow F \longrightarrow \mathrm{L}_E\, F \tag{4.2}$$

(since the pair is exceptional, this defines $\mathrm{L}_E\, F$ up to unique isomorphism).

If $\mathbb{E} \coloneqq (E_1, \dots E_n)$ is an exceptional collection in $\mathcal{T}$ we define the *mutation at* i to be the exceptional collection $(E_1, \dots, \mathrm{L}_{E_i} E_{i+1}, E_i, \dots, E_n)$.

These mutations can be interpreted as an action of the braid group on n strings, denoted B_n, on the set of all exceptional collections. To see this, recall that B_n is generated by standard braids $\sigma_1, \dots, \sigma_{n-1}$ subject to the relations

$$\begin{aligned} \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j &= \sigma_j \sigma_i \text{ for all } |i - j| \neq 1 \end{aligned} \tag{4.3}$$

B_n then acts on an exceptional collection by letting σ_i act via mutation at i, i.e.

$$\sigma_i(\mathbb{E}) \coloneqq (E_1, \dots, \mathrm{L}_{E_i} E_{i+1}, E_i, \dots, E_n). \tag{4.4}$$

It is a tedious but straightforward computation to check that the construction of cones is compatible with the relations (4.3).

Remark 4.2.2. *By a celebrated theorem by Kuleshov and Orlov [KO94] we know that for a del Pezzo surface X the braid group B_m (where $m = \mathrm{rk}\, \mathrm{K}_0(X)$) acts transitively on the set of exceptional collections in $\mathbf{D}^{\mathrm{b}}(X)$.*

Inspired by [dTdV16] we also consider the action of the *signed* braid group, which also takes shifting into account.

Definition 4.2.3. The *signed braid group* $\Sigma\mathrm{B}_n$ is the semidirect product $\mathrm{B}_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$, where $(\mathbb{Z}/2\mathbb{Z})^n$ acts on B_n by considering the quotient $\mathrm{B}_n \twoheadrightarrow \mathrm{Sym}_n$. As such, the signed braid group has $2n-1$ generators:

- $n-1$ generators $\sigma_1,\ldots,\sigma_{n-1}$, as for the braid group B_n,

- n generators $\epsilon_1,\ldots,\epsilon_n$, as for $(\mathbb{Z}/2\mathbb{Z})^n$.

These generators satisfy the usual braid relations (4.3) as well as

$$\begin{aligned} \epsilon_i^2 &= 1 \\ \epsilon_i\epsilon_j &= \epsilon_j\epsilon_i \\ \epsilon_i\sigma_i\epsilon_{i+1} &= \sigma_i \end{aligned} \tag{4.5}$$

In [dTdV16, §4] the rules for computing the action of $\Sigma\mathrm{B}_n$ on a bilinear form $\langle -,-\rangle$ are given. These rules naturally generalize the induced action of B_n on the Euler form on the Grothendieck group $\mathrm{K}_0(\mathcal{T})$. We use this action in the following:

Proposition 4.2.4. *The matrices K_m (introduced in the introduction and* (3.1)*) are mutation equivalent to the matrices*

$$K'_m = \begin{bmatrix} 1 & 3 & 6 & m \\ 0 & 1 & 3 & m \\ 0 & 0 & 1 & m \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.6}$$

for all $m \in \mathbb{N}$.

Proof. We have that $\epsilon_1\epsilon_3\sigma_3\sigma_1\sigma_2\sigma_3$ sends K'_m to K_m. The mutations $\sigma_1\sigma_2\sigma_3$ provide a so-called *shift in the helix* (see [dTdV16, §4]), whilst the mutation σ_3 at that point corresponds to the mutation that sends $(\mathcal{O}_{\mathbb{P}^2},\mathcal{O}_{\mathbb{P}^2}(1),\mathcal{O}_{\mathbb{P}^2}(2))$ to $(\mathcal{O}_{\mathbb{P}^2},\mathrm{T}_{\mathbb{P}^2}(-1),\mathcal{O}_{\mathbb{P}^2}(1))$. Finally the mutation ϵ_i corresponds to $E_i \mapsto E_i[1]$ (taking into account that [2] is invisible for the Euler form).

For the interested reader we provide all intermediate steps below

$$\sigma_3 K'_m = \begin{bmatrix} 1 & 3 & -5m & 6 \\ 0 & 1 & -2m & 3 \\ 0 & 0 & 1 & -m \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sigma_2\sigma_3 K'_m = \begin{bmatrix} 1 & m & 3 & 6 \\ 0 & 1 & 2m & 5m \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sigma_1\sigma_2\sigma_3 K'_m = \begin{bmatrix} 1 & -m & -m & -m \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sigma_3\sigma_1\sigma_2\sigma_3 K'_m = \begin{bmatrix} 1 & -m & -2m & -m \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\epsilon_3\sigma_3\sigma_1\sigma_2\sigma_3 K'_m = \begin{bmatrix} 1 & -m & -2m & -m \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

□

As a result of Proposition 4.2.4 it is equivalent to find a noncommutative surface Z_m together with an exceptional sequence $\mathbb{E} \coloneqq (E_1, E_2, E_3, E_4)$ for which the Gram matrix takes the form K_m as in (3.1) or K'_m as in (4.6). One simply goes from one model to the other by applying the above action of the signed braid group $\Sigma\mathrm{B}_4$. This is the reason why we restrict ourself in Theorem 4.5.7 to finding a model for K'_m.

4.3 Fat point modules

Recall that point modules were introduced in Definition 0.1.2. A well known generalization are so called *fat point modules*. We introduce these in this section and recapitulate some important properties.

According to [Smi94, §7] we take:

Definition 4.3.1. A *fat point module* for A is a graded module F satisfying the following properties:

i) F is generated by F_0

ii) The Hilbert function $\dim_k F_n$ is a constant ≥ 2, which is called the *multiplicity*.

iii) F has no nonzero finite-dimensional submodules.

iv) $\pi F \in \operatorname{qgr} A$ is simple

A *fat point* is the isomorphism class of a fat point module in $\operatorname{qgr} A$.

Remark 4.3.2. *Note that conditions iii. and iv. are automatically true for (non-fat) point modules. For multiplicity ≥ 2 these conditions are however necessary. For example iv. allows to distinguish fat point modules from direct sums of point modules.*

The following result tells us that fat point modules for quadratic Artin–Schelter regular algebras which are finite over their center behave particularly nice. By the classification of such algebras which are finite over their center (see Theorem 0.1.3) we know that such an algebra is described by a triple $(Y, \mathcal{L}, \sigma)$ where $\sigma \in \operatorname{Aut}(Y)$ is of finite order. We will denote

$$s \coloneqq \min\{k \mid \sigma^{k,*}(\mathcal{L}) \cong \mathcal{L}\}. \tag{4.7}$$

It is this integer, and not the order of σ (which will be denoted n) that is the important invariant of the triple $(E, \sigma, \mathcal{L})$. Observe that we have $s \mid n$. The following proposition then describes the exact value of s, which depends on the behaviour of the normal element g which lives in degree 3.

Proposition 4.3.3. *If $(E, \sigma, \mathcal{L})$ is the regular triple associated to an Artin–Schelter regular algebra A which is finite over its center, then the multiplicity of the fat point modules of A is given by s as in (4.7), moreover*

$$s = \begin{cases} n & \gcd(n,3) = 1 \\ n/3 & \gcd(n,3) = 3. \end{cases} \tag{4.8}$$

Proof. In [ATVdB91, Theorem 7.3] it is shown that $A[g^{-1}]$ is Azumaya of degree s. Now by [ACdJL, Lemma 5.5.5(i)] we have that s is the order of the automorphism η introduced in [ATVdB91, §5]. By [ACdJL, Theorem 5.3.6] we have that $\eta = \sigma^3$, hence s is n or $n/3$ depending on $\gcd(n,3)$.

Moreover, by [ACdJL, Lemma 5.5.5(ii)] we have that all fat point modules are of multiplicity s. □

Convention 4.3.4. *For the rest of the chapter A will always denote a quadratic AS-regular algebra which is finite over its center and for which all fat point modules have multiplicity s. Note that Proposition 4.3.3 implies that we can find such an A (which even is a Sklyanin algebra) for each s, simply by choosing Y to be an elliptic curve and $\sigma \in \operatorname{Aut}(Y)$ a translation of order $3s$ (or s when s is not divisble by 3). Moreover we will always assume $s \geq 2$. The reason for this lies in the fact that it can be shown that a Sklyanin algebra associated to a translation of order 3 has the property that $\operatorname{qgr} A \cong \operatorname{coh} \mathbb{P}^2$. As such our construction for $s = 1$ will give $Z_1 = \operatorname{coh}(\mathbb{F}_1)$, a commutative model which is already well known.*

Remark 4.3.5. *It is well known that a quadratic Sklyanin algebra can be written as the quotient of $k\langle x, y, z\rangle$ by the ideal generated by*

$$\begin{cases} axy + byx + cz^2 = 0 \\ ayz + bzy + cx^2 = 0 \\ azx + bxz + cy^2 = 0 \end{cases} \tag{4.9}$$

where $[a : b : c] \in \mathbb{P}^2 \smallsetminus S$ and S is the following well-known finite set of 12 points:

$$S = \{(1:0:0), (0:1:0), (0:0:1)\} \cup \{(a:b:c) \mid a^3 = b^3 = c^3\}$$

Remark 4.3.6. *In Convention 4.3.4 we observed that for each $s \geq 2$ there is quadratic Sklyanin algebra for which all fat points have multiplicity s. Recall that a (non-linear) quadratic AS-regular algebra A is described by a triple $(Y, \mathcal{L}, \sigma)$ where $Y \subset \mathbb{P}^2$ is a cubic curve. Such cubic curves as well as their automorphism groups were classified in [BP94, Table1]. Out of the 9 possible cubic curves they listed, only 5 of them allowed non-trivial, finite order automorphisms for which $\sigma^{s*}\mathcal{L} \cong \mathcal{L}$. These are the elliptic curves, the nodal cubic, the triangle of lines, a conic and line in general position or three lines intersecting in one point. The latter however does not give rise to an AS-regular algebra (see [VG02, Proposition 4.2]). As such there are only 4 possibilities for Y we will encounter in this chapter, see Figure 4.1. For an alternative approach using the classification of maximal orders, see Remark 4.5.2.*

We will need the following two facts about fat point modules.

Proposition 4.3.7. *Let A be a quadratic AS-regular algebra of finite order and let F be a fat point module as above. Let $g \in A_3$ be the normalizing element for which $A/gA \cong B(Y, \mathcal{L}, \sigma)$. Then F is g-torsion free.*

Proof. Let M be a simple graded A-module. Then there exists an $n \in \mathbb{Z}$ such that $M_i = 0$ for all $i \neq n$. To see this, note that if $M_i \neq 0$ and $M_n \neq 0$ for some $i > n$, then the truncation $M_{\geq n+1}$ is a non-trivial submodule of M.

In particular, let F be a fat point module and $M \subset F$ a simple graded submodule. By the above and Definition 4.3.1ii. M is finite dimensional, implying $M = 0$ by Definition 4.3.1iii.

Hence F has a trivial socle. As such we can apply [ATVdB91, Proposition 7.7(ii)] from which the lemma follows because F cannot be an extension of point modules as we assumed $\pi F \in \operatorname{qgr} A$ to be simple. $\square$

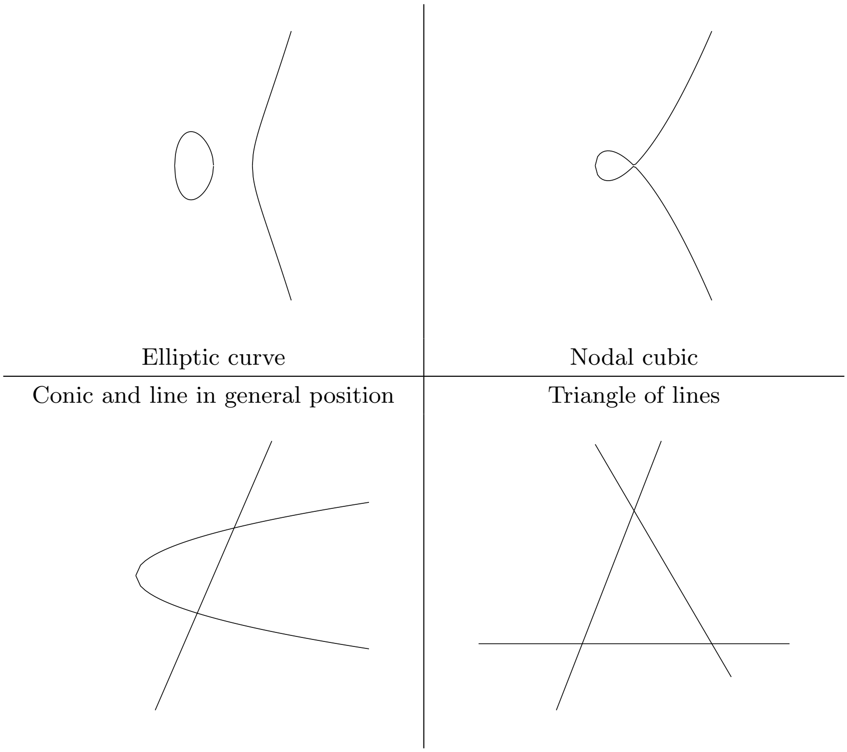

Figure 4.1: Four possible cubics

Lemma 4.3.8. *The fat point module F is invariant under triple degree shifting, i.e. there exists an isomorphism in* $\operatorname{gr} A$*:*

$$F \cong (F(3))_{\geq 0}\,. \tag{4.10}$$

Proof. This is direct corollary of Proposition 4.3.7: the isomorphism is given by multiplication by g. $\square$

4.4 Semiorthogonal decompositions

We will also use the notion of semiorthogonal decomposition and exceptional sequences. We will denote $\mathcal{T}$ a k-linear triangulated category, and all constructions are k-linear. For more details one is referred to [Orl16].

Definition 4.4.1. A *semiorthogonal decomposition* of $\mathcal{T}$ is a sequence $(\mathcal{S}_1, \dots, \mathcal{S}_n)$ of full triangulated subcategories of $\mathcal{T}$, such that there exists a filtration

$$0 = \mathcal{T}_0 \subseteq \mathcal{T}_1 \subseteq \dots \subseteq \mathcal{T}_n = \mathcal{T} \tag{4.11}$$

where $\mathcal{T}_i \subseteq \mathcal{T}$ is a (left) admissible subcategory such that $\mathcal{S}_i \cong \mathcal{T}_i / \mathcal{T}_{i-1}$. We will denote this by

$$\mathcal{T} = \langle \mathcal{S}_1, \dots, \mathcal{S}_n \rangle . \tag{4.12}$$

A special case of a semiorthogonal decomposition is provided by a full exceptional sequence as in Definition 3.4.21. In this case we have $\mathcal{S}_i = \langle E_i \rangle \cong \mathbf{D}(k)$ for all i.

The main property of semiorthogonal decompositions that we will use in this paper is that they are sent to direct sums by so called additive invariants. In particular for the Grothendieck group we get in the situation of (4.12) that

$$\mathrm{K}_0(\mathcal{T}) \cong \bigoplus_{i=1}^{n} \mathrm{K}_0(\mathcal{S}_i). \tag{4.13}$$

In particular, if $\mathcal{T}$ has a full exceptional collection of length n, then

$$\mathrm{K}_0(\mathcal{T}) \cong \mathbb{Z}^{\oplus n}. \tag{4.14}$$

The main example of a full and strong exceptional collection, and also the example that motivated the construction in §4.5 is Beilinson's collection on $\mathbb{P}^2$ [Bl78].

Example 4.4.2. *The derived category of $\mathbb{P}^2$ has a full and strong exceptional collection*

$$\mathbf{D}^{\mathrm{b}}(\mathbb{P}^2) = \langle \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2) \rangle \tag{4.15}$$

whose quiver is

$$\begin{tikzcd} \circ \arrow[r, bend left, "x_0"] \arrow[r, "y_0"] \arrow[r, bend right, "z_0"'] & \circ \arrow[r, bend left, "x_1"] \arrow[r, "y_1"] \arrow[r, bend right, "z_1"'] & \circ \end{tikzcd}, \tag{4.16}$$

and the relations are

$$\begin{cases} x_0 y_1 = y_0 x_1 \\ x_0 z_1 = z_0 x_1 \\ y_0 z_1 = z_0 y_1 . \end{cases} \tag{4.17}$$

In particular, this means that $\mathrm{K}_0(\mathbf{D}^{\mathrm{b}}(\mathbb{P}^2)) \cong \mathbb{Z}^{\oplus 3}$*, and we can read off that the Gram (or Cartan) matrix is*

$$M = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.18}$$

whilst the Coxeter matrix is

$$C = \begin{bmatrix} -10 & -6 & -3 \\ 15 & 8 & 3 \\ -6 & -3 & -1 \end{bmatrix}. \tag{4.19}$$

This example can be generalised to noncommutative $\mathbb{P}^2$'s in the following way.

Example 4.4.3. *The derived category of* $\operatorname{qgr} A$*, where* A *is a quadratic 3-dimensional Artin–Schelter regular algebra, has a well-known full and strong exceptional collection mimicing that of Beilinson for* $\mathbb{P}^2$*, given by*

$$\mathbf{D}^{\mathrm{b}}(\operatorname{qgr} A) = \langle \pi A, \pi A(1), \pi A(2) \rangle \tag{4.20}$$

where π *denotes the quotient functor* $\pi\colon \operatorname{gr} A \to \operatorname{qgr} A$*, and* $A(i)$ *denotes the grading shift of* A*. More details can be found in [AOU14, theorem 7.1].*

The quiver has the same shape as (4.16)*, and the relations can be read off from the presentation of* A *as a quotient of* $k\langle x, y, z\rangle$ *by 3 quadratic relations as in* (4.17)*, see also Lemma 4.5.4.*

For instance in the case of the Sklyanin algebra of Remark 4.3.5 they are

$$\begin{cases} ax_0y_1 + by_0x_1 + cz_0z_1 = 0 \\ ay_0z_1 + bz_0y_1 + cx_0x_1 = 0 \\ az_0x_1 + bx_0z_1 + cy_0y_1 = 0 \end{cases} \tag{4.21}$$

The Cartan and Coxeter matrices A *and* C *describing the Euler form and the Serre functor only depend on the structure of the quiver with relations, and this stays the same, so we obtain the matrices from* (4.18) *and* (4.19)*. In [dTdV16] it is explained how up to the signed braid group action introduction in Section 4.2 this is the only solution of rank 3.*

4.5 Construction

4.5.1 Noncommutative planes finite over their center

As in Convention 4.3.4, consider a 3-dimensional quadratic Artin–Schelter-regular algebra A which is finite over its center $\mathrm{Z}(A)$ and let s denote the multiplicity of the fat point modules.

In this case we can consider

$$X' \coloneqq \operatorname{Proj} \mathrm{Z}(A), \tag{4.22}$$

and the sheafification $\mathcal{R}$ of A over X'. This is a sheaf of noncommutative $\mathcal{O}_{X'}$-algebras, coherent as $\mathcal{O}_{X'}$-module.

Often, but not always, we have that $X' \cong \mathbb{P}^2$ ([Art92, Theorem 5.2]). It is possible to improve this situation by considering a finite cover of X'.

The center $\mathrm{Z}(\mathcal{R})$, which is not necessarily $\mathcal{O}_{X'}$, is a (coherent) sheaf of commutative $\mathcal{O}_{X'}$-algebras, hence we can consider

$$f\colon X \coloneqq \underline{\operatorname{Spec}}_{X'} \mathrm{Z}(\mathcal{R}) \to X'. \tag{4.23}$$

Because $\mathrm{Z}(\mathcal{R})$ is coherent as an $\mathcal{O}_{X'}$-module the projection map f is finite.

The main result about X, for any Artin–Schelter regular algebra finite over its center, is that X is isomorphic to $\mathbb{P}^2$. This was proved:

1. by Artin for Sklyanin algebras associated to points of order coprime to 3, where $X \cong X'$, as mentioned before,
2. by Smith–Tate for all Sklyanin algebras [ST94],
3. by Mori for algebras of type S_1 [Mor98] (these have a triangle of $\mathbb{P}^1$'s as their point scheme),
4. and finally by Van Gastel in complete generality [VG02], with an analogous proof in [ACdJL, Theorem 5.3.7].

We will denote by $\mathcal{S}$ the sheaf of algebras on X induced by $\mathcal{R}$, so the situation is described as follows.

$$\begin{tikzcd} \mathcal{S} \arrow[d, dash, dashed] & \mathcal{R} = f_*(\mathcal{S}) \arrow[d, dash, dashed] \\ \mathbb{P}^2 \cong X \coloneqq \underline{\operatorname{Spec}}_{X'} \mathrm{Z}(\mathcal{R}) \arrow[r, "f"] & X'. \end{tikzcd} \tag{4.24}$$

The sheaf of algebras $\mathcal{S}$ has many pleasant properties and will be used in the construction.

Lemma 4.5.1. *$\mathcal{S}$ is a sheaf of maximal orders on $\mathbb{P}^2$ of rank s^2, with s as in Proposition 4.3.3.*

Proof. By the discussion above we have $X \cong \mathbb{P}^2$, so that $\mathcal{S}$ is a maximal order follows from [LB95, proposition 1]. Observe that the notation in the statement of loc. cit. is somewhat unfortunate, and should be taken as in (4.24).

It is locally free because it is a reflexive sheaf over a regular scheme of dimension 2, and the statement on the rank follows from [ATVdB91, theorem 7.3]. □

Using this we can decompose $\mathbb{P}^2$ into a *ramification divisor* C and its complement, the *Azumaya locus.*

Remark 4.5.2. *It is also possible to classify the curves that can appear as ramification divisors for a maximal order on $\mathbb{P}^2$ using the Artin–Mumford sequence [AM72], as explained in [VdB97, Lemma 1.1(2)]. This gives the same result as Remark 4.3.6, taking care of the distinction between the point scheme and the ramification divisor.*

The algebra A induces an abelian category $\mathrm{qgr}(A)$ which is a representation of the noncommutative plane $\mathrm{Proj}(A)$. In the case where A is finite over its center we have a second interpretation for this category, namely as the category of coherent $\mathcal{R}$-and $\mathcal{S}$-modules.

Lemma 4.5.3. *There are equivalences of categories*

$$\mathrm{qgr}(A) \cong \mathrm{coh}(\mathcal{R}) \cong \mathrm{coh}(\mathcal{S}). \tag{4.25}$$

Proof. The equivalence $\mathrm{QGr}(A) \cong \mathrm{Qcoh}\,\mathcal{R}$ is given by the restriction of the equivalence $\widetilde{(-)} : \mathrm{QGr}(\mathrm{Z}(A)) \to \mathrm{Qcoh}(X')$. Similarly there is an equivalence of categories $\mathrm{Qcoh}(\mathrm{Z}(\mathcal{R})) \cong \mathrm{Qcoh}(X)$, and one easily checks that this restricts to $\mathrm{Qcoh}(\mathcal{R}) \cong \mathrm{Qcoh}(\mathcal{S})$ (see for example [BDG17, Proposition 3.5]). This equivalence also restricts to noetherian objects. ☐

4.5.2 Description of the exceptional sequence

We will use the notation $\mathcal{S}_i \in \mathrm{coh}(\mathcal{S})$ for the images of $\pi A(i) \in \mathrm{qgr}(A)$ under the above equivalences. Similarly we fix a fat point module F and let $\mathcal{F} \in \mathrm{coh}(\mathcal{S})$ be its image. The collection

$$(\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \mathcal{F})$$

in $\mathbf{D}^{\mathrm{b}}(\mathcal{S}) \coloneqq \mathbf{D}^{\mathrm{b}}(\mathrm{coh}(\mathcal{S}))$ is the noncommutative analogue of

$$(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2), k(x))$$

where $k(x)$ is the skyscraper in a closed point x. This is not an exceptional collection for $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^2)$: we have that $\mathrm{Ext}^2(k(x), \mathcal{O}_{\mathbb{P}^2}(i)) \neq 0$ for all i, but it will become one after blowing up at p.

The point we wish to blow up is the support of the fat point module, considered as an object in $\mathrm{coh}(\mathcal{S})$. This corresponds precisely with a point of $\mathbb{P}^2 \setminus C$, where C is the ramification divisor of $\mathcal{S}$. This will give us a new exceptional object, with the appropriate number of morphisms towards it.

We can perform the analogues construction in the noncommutative situation. Let $x \in \mathbb{P}^2 \setminus C$ be the unique closed point in the support of $\mathcal{F}$, where C is the ramification locus of $\mathcal{S}$.

Consider the blowup square

$$
\begin{tikzcd}
E = \mathbb{P}^1 \arrow[r, "j"] \arrow[d, "q"'] & Z = \mathbb{F}_1 \arrow[d, "p"] \\
x \arrow[r, "i"'] & X = \mathbb{P}^2
\end{tikzcd}
$$

As in §4.5.3 we will use the notation $p_{\mathcal{S}}^*$ for the inverse image functor obtained from the morphism of ringed spaces $(\mathbb{F}_1, p^*(\mathcal{S})) \to (\mathbb{P}^2, \mathcal{S})$.

As explained in Example 4.4.3 the structure of $\mathbf{D}^{\mathrm{b}}(\mathcal{S})$ is obtained by changing the relations in the quiver according to the generators and relations for the Artin–Schelter regular algebra. This gives the following well known result, and a noncommutative analogue of Serre's description of the sheaf cohomology of $\mathcal{O}_{\mathbb{P}^n}(i)$:

Lemma 4.5.4. *Let A be a quadratic 3-dimensional Artin–Schelter regular algebra. Then for each i there is a full and strong exceptional collection*

$$\mathbf{D}^{\mathrm{b}}(\mathrm{qgr}(A)) = \langle \pi A(i), \pi A(i+1), \pi A(i+2) \rangle, \tag{4.26}$$

such that

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathrm{qgr}(A))}(\pi A(i), \pi A(i+1)) \cong \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathrm{qgr}(A))}(\pi A(i+1), \pi A(i+2)) \cong A_1 \\ \text{and} \qquad \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathrm{qgr}(A))}(\pi A(i), \pi A(i+2)) \cong A_2\end{aligned}$$

and the composition law in the quiver is given by the multiplication law

$$A_1 \otimes_k A_1 \twoheadrightarrow A_2$$

Proof. By [AZ94, theorem 8.1] we have

$$\mathrm{Ext}^m_{\mathrm{qgr}(A)}(\pi A, \pi A(j-i)) \cong \begin{cases} A_{j-i} & m = 0 \\ A^{\vee}_{i-j-3} & m = 2 \\ 0 & m \neq 0, 2 \end{cases}. \tag{4.27}$$

Moreover by assumption the algebra is generated in degree 1, hence we have a complete description of the structure of the exceptional collection. That it is full is proved in [AOU14, Theorem 7.1]. □

Using Lemma 4.5.3 this gives a description for the derived category $\mathcal{S}$ as

$$\mathbf{D}^{\mathrm{b}}(\mathcal{S}) = \langle \mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2 \rangle. \tag{4.28}$$

Lemma 4.5.5. *Let F be a fat point module and let s be as in Proposition 4.3.3. Then*

$$\dim_k \left(\mathrm{Hom}_{\mathrm{qgr}(A)}(\pi A(j), \pi F)\right) = s, \tag{4.29}$$

and

$$\mathrm{Ext}^k_{\mathrm{qgr}(A)}(\pi A(j), \pi F) = 0 \tag{4.30}$$

for $k \geq 1$.

Proof. Using the identities

$$\begin{aligned} \mathrm{Hom}_{\mathrm{qgr}(A)}(\pi A(j), \pi F) &\cong \mathrm{Hom}_{\mathrm{qgr}(A)}(\pi A, \pi F(-j)) \\ F(-j) &\cong F(-j+3) \qquad \text{(see Lemma 4.3.8)} \end{aligned}$$

we can assume, without loss of generality, that $j \leq 0$.

Recall that

$$\mathrm{Hom}_{\mathrm{qgr}(A)}(\pi A, \pi F(-j)) \cong \lim_{i\to\infty} \mathrm{Hom}_{\mathrm{gr}(A)}(A_{\geq i}, F(-j)).$$

We now claim

$$\lim_{i\to\infty} \mathrm{Hom}_{\mathrm{gr}(A)}(A_{\geq i}, F(-j)) \cong \mathrm{Hom}_{\mathrm{gr}(A)}(A, F(-j)) \cong F_{-j}.$$

The lemma follows by combining this claim with Proposition 4.3.3.

To prove the claim, note that one can compute the limit by restricting to the directed subsystem $3\mathbb{N}$. It then suffices to prove that the natural map

$$\alpha_i \colon \mathrm{Hom}_{\mathrm{gr}(A)}(A_{\geq 3i}, F(-j)) \longrightarrow \mathrm{Hom}_{\mathrm{gr}(A)}(A_{\geq 3i+3}, F(-j))$$

is an isomorphism for all $i \geq 0$. But this follows as its inverse is given by

$$\beta_i \colon \mathrm{Hom}_{\mathrm{gr}(A)}(A_{\geq 3i+3}, F(-j)) \longrightarrow \mathrm{Hom}_{\mathrm{gr}(A)}(A_{\geq 3i}, F(-j)) : \beta_i(\varphi)(x) = g^{-1}\varphi(gx)$$

where $g^{-1} \colon F_n \to F_{n-3}$ is the inverse of the isomorphism as in (4.10).

To see that the Ext vanish, we use that F (resp. $F(-j)$) is finitely generated (by the degree zero part) and has Gelfand–Kirillov dimension 1.

By [ATVdB91, Theorem 4.1] we can conclude

$$\mathrm{Ext}^i_{\mathrm{gr}(A)}(F, A(j)) = 0 \text{ for all } j \text{ and } i = 0, 1 \tag{4.31}$$

and [AZ94, Theorem 8.1] implies

$$\mathrm{Ext}^i_{\mathrm{qgr}(A)}(\pi F, \pi A(j)) = \mathrm{Ext}^i_{\mathrm{gr}(A)}(F, A(j)) = 0 \text{ for all } j \text{ and } i = 0, 1 \tag{4.32}$$

It was proved in [dNVdB04, Theorem 2.9.1] that $\mathrm{qgr}(A)$ satisfies the following version of noncommutative Serre duality:

$$\mathrm{Ext}^i_{\mathrm{qgr}(A)}(\pi F, \pi A(j)) \cong \mathrm{Ext}^{2-i}_{\mathrm{qgr}(A)}(\pi A(j), \pi F(-3))^\vee. \tag{4.33}$$

But the latter is isomorphic to $\mathrm{Ext}^{2-i}_{\mathrm{qgr}(A)}(\pi A(j), \pi F)^\vee$ by Lemma 4.3.8, which implies the vanishing of Ext. $\square$

We will also use the following lemma in checking that the exceptional collection is indeed strong, and of the prescribed form.

Lemma 4.5.6. *There exists an isomorphism*

$$\mathbf{R}p_* \circ p^*(\mathcal{F}) \cong \mathcal{F}. \tag{4.34}$$

Proof. Consider the divisor short exact sequence

$$0 \to \mathcal{O}_{\mathbb{F}_1}(-E) \to \mathcal{O}_{\mathbb{F}_1} \to j_*(\mathcal{O}_E) \to 0. \tag{4.35}$$

Applying the exact functor $p^*(\mathcal{S}) \otimes_{\mathcal{O}_{\mathbb{F}_1}} -$ to it we get a short exact sequene of left $p^*(\mathcal{S})$-modules

$$0 \to p^*(\mathcal{S}) \otimes_{\mathcal{O}_{\mathbb{F}_1}} \mathcal{O}_{\mathbb{F}_1}(-E) \to p^*(\mathcal{S}) \to p^*(\mathcal{S}) \otimes_{\mathcal{O}_{\mathbb{F}_1}} j_*(\mathcal{O}_E) \to 0. \tag{4.36}$$

We have the chain of isomorphisms

$$\begin{aligned} p^*(\mathcal{S}) \otimes_{\mathcal{O}_{\mathbb{F}_1}} j_*(\mathcal{O}_E) &\cong j_* \circ j^* \circ p^*(\mathcal{S}) && \text{projection formula} \\ &\cong j_* \circ q^* \circ i^*(\mathcal{S}) && \text{functoriality} \\ &\cong p^* \circ i_* \circ i^*(\mathcal{S}) && \text{base change for affine morphisms} \\ &\cong p^*(\mathcal{F}^{\oplus n}) \end{aligned} \tag{4.37}$$

where the last step uses that $i^*(\mathcal{S}) \cong \mathrm{Mat}_n(k)$ is the direct sum of the n-dimensional representation corresponding to the fat point F.

Because the first two terms in (4.36) are p_*-acyclic, so is the third and its direct summands $p^*(\mathcal{F})$. Hence we can use the projection formula

$$p_*(p^*(\mathcal{S}) \otimes_{p^*(\mathcal{S})} p^*(\mathcal{F})) \cong p_* \circ p^*(\mathcal{S}) \otimes_{\mathcal{S}} \mathcal{F} \cong \mathcal{F}. \qquad \square$$

We can now prove the main theorem of this paper, which gives a construction of noncommutative surfaces with prescribed Grothendieck group. It uses a semiorthogonal decomposition that generalises Orlov's blowup formula, and which is proved in some generality in §4.5.3.

Theorem 4.5.7. *Let A be a quadratic 3-dimensional Artin–Schelter regular algebra, finite over its center. Let F be a fat point module of A with multiplicity s as in* (4.8)*. Let $p : \mathrm{Bl}_x \mathbb{P}^2 \to \mathbb{P}^2$ be the blowup in the point x which is the support of the $\mathcal{S}$-module $\mathcal{F}$ as a sheaf on $\mathbb{P}^2$. Then $p^*\mathcal{S}$ is a maximal order on $\mathrm{Bl}_x \mathbb{P}^2 \cong \mathbb{F}_1$ and*

$$\mathbf{D}^{\mathrm{b}}(p^*\mathcal{S}) = \left\langle \mathbf{L}p_{\mathcal{S}}^*\mathcal{S}_0, \mathbf{L}p_{\mathcal{S}}^*\mathcal{S}_1, \mathbf{L}p_{\mathcal{S}}^*\mathcal{S}_2, p_{\mathcal{S}}^*\mathcal{F} \right\rangle \tag{4.38}$$

is a full and strong exceptional collection, whose Gram matrix is of type K'_m as in (4.6)*, with $m = s$.*

Proof. By the Auslander-Goldman criterion [AG60] one can check that $p^*\mathcal{S}$ is a maximal order locally on $\mathrm{Bl}_x \mathbb{P}^2 \cong \mathbb{F}_1$. For a point z outside of the exceptional divisor we simply have $(p^*\mathcal{S})_z \cong \mathcal{S}_{p(z)}$. For a point on the exceptional divisor $(p^*\mathcal{S})_z$ is a base change of the Azumaya order $\mathcal{S}_x$, thus it is itself a Azumaya (hence maximal) order.

By Theorem 4.5.11 below we obtain that the collection is indeed a full and strong exceptional collection, where we use that $\mathcal{F}$ can be considered as the (noncommutative) skyscraper sheaf for $\mathcal{S}$, because we are in the Azumaya locus.

The structure of $\langle \mathbf{L}p_{\mathcal{S}}^*\mathcal{S}_0, \mathbf{L}p_{\mathcal{S}}^*\mathcal{S}_1, \mathbf{L}p_{\mathcal{S}}^*\mathcal{S}_2 \rangle$ is described in Lemma 4.5.4 using the fully faithfulness of $\mathbf{L}p_{\mathcal{S}}^*$ from Lemma 4.5.14 below. Using Lemma 4.5.6, we get that

$$\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(p^*\mathcal{S})}(\mathbf{L}p_{\mathcal{S}}^*\mathcal{S}_i, p^*\mathcal{F}) \cong \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathcal{S})}(\mathcal{S}_i, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{S}}(\mathcal{S}_i, \mathcal{F}) \tag{4.39}$$

which is s-dimensional by Lemma 4.5.5, and similarly we get that there are no forward Ext's, so we conclude that the collection is indeed strong. □

Remark 4.5.8. *There are three degrees of freedom in this construction: generically the point scheme is an elliptic curve for which we have the j-line as moduli space, for each curve there are only finitely many torsion automorphisms, and then there is the choice of a point in $\mathbb{P}^2 \setminus C$. There are only finitely many automorphisms of $\mathbb{P}^2$ that preserve C, so we get three degrees of freedom. This is the expected number, using [Bel17], where a formula for $\dim_k \mathrm{HH}^2 - \dim_k \mathrm{HH}^1$ is given in terms of the number of exceptional objects.*

Moreover, when focusing on fat point modules of multiplicity 2, we find the same number of degrees of freedom. This follows from the fact that the degree of freedom provided by the j-invariant can now be replaced by one degree of freedom coming from the parameter c as in Remark 4.6.6 below.

We do not know whether we have described all noncommutative surfaces of rank 4 in this way.

Remark 4.5.9. *Similar to Proposition 3.4.28 there is an equivalence of categories*

$$\mathbf{D}^{\mathrm{b}}(p^*\mathcal{S}) \cong \mathbf{D}^{\mathrm{b}}(kQ/\mathcal{R})$$

Here Q is the quiver

$$\begin{tikzcd}
\bullet \arrow[rr, Rightarrow, "\alpha", "\boxed{3}" description] \arrow[drr, Rightarrow, "\gamma"', "\boxed{s}" description] & & \bullet \arrow[rr, Rightarrow, "\beta", "\boxed{3}" description] \arrow[d, Rightarrow, "\delta"', "\boxed{s}" description] & & \bullet \arrow[dll, Rightarrow, "\varepsilon", "\boxed{s}" description] \\
& & \bullet & &
\end{tikzcd} \tag{4.40}$$

The arrows $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3$ correspond to k-vector space bases for A_1. Similarly $\gamma_1, \dots, \gamma_s$ gives a basis for F_1, $\delta_1, \dots, \delta_s$ gives a basis for F_2 and $\epsilon_1, \dots, \epsilon_s$ gives a basis for F_3.

The relations $\mathcal{R}$ correspond to the (kernels of the) multiplication maps:

$$A_1 \otimes_k A_1 \longrightarrow A_2$$
$$F_i \otimes_k A_1 \longrightarrow F_{i+1}$$

Remark 4.5.10. *Using [Orl16] and the full and strong exceptional collection from Theorem 4.5.7 there exists an embedding of $\mathbf{D}^{\mathrm{b}}(p^*\mathcal{S})$ into the derived category of a smooth projective variety. Now in the spirit of [Orl15, BR16, KS15, FK16] it is an interesting question whether there exists a natural embedding, i.e. where the smooth projective variety is associated to $p^*\mathcal{S}$ in a natural way. An obvious candidate would be the Brauer–Severi scheme of the maximal order, and indeed in the case where the automorphism is of order 2 there exists a fully faithful embedding into $\mathbf{D}^{\mathrm{b}}(\mathrm{BS}(p^*\mathcal{S}))$ by [CI12, §6] and [Kuz08], as the maximal order is the even part of a sheaf of Clifford algebras. What happens for the more general case and the study of the derived category of the Brauer–Severi scheme in this situation, is not known yet.*

4.5.3 Orlov's blowup formula for orders

The main ingredient in the construction of Theorem 4.5.7 is the observation that it is possible to generalise Orlov's blowup formula [Orl92, Theorem 4.3] to a sufficiently nice noncommutative setting where we blow up a point on the underlying variety, and pull back the sheaf of algebras to the blown up variety. This is a result of independent interest.

Theorem 4.5.11. *Let X be a smooth quasiprojective variety. Let $\mathcal{A}$ be a locally free sheaf of orders of degree n on X such that $\operatorname{gl.dim}\mathcal{A} < +\infty$. Let Y be a smooth closed subvariety such that it does not meet the ramification locus of $\mathcal{A}$. Assume moreover that $\mathcal{A}|_Y \cong \mathrm{Mat}_n(\mathcal{O}_Y)$. Consider the blowup square*

$$\begin{tikzcd} E \coloneqq \mathbb{P}(\mathrm{N}_Y X) \arrow[r, "j"] \arrow[d, "q"] & Z \coloneqq \mathrm{Bl}_Y X \arrow[d, "p"] \\ Y \arrow[r, "i"'] & X \end{tikzcd}$$

and its noncommutative analogue obtained by pulling back the sheaf of algebras $\mathcal{A}$

$$\begin{tikzcd} (E, \mathrm{Mat}_n(\mathcal{O}_E)) \arrow[r, "j_{\mathcal{A}}"] \arrow[d, "q_{\mathcal{A}}"] & (Z, p^*\mathcal{A}) \arrow[d, "p_{\mathcal{A}}"] \\ (Y, \mathrm{Mat}_n(\mathcal{O}_Y)) \arrow[r, "i_{\mathcal{A}}"'] & (X, \mathcal{A}) \end{tikzcd}$$

Then we have a semiorthogonal decomposition

$$\mathbf{D}^{\mathrm{b}}(Z, p^*(\mathcal{A})) = \left\langle \mathbf{D}^{\mathrm{b}}(X, \mathcal{A}), \mathbf{D}^{\mathrm{b}}(Y), \ldots, \mathbf{D}^{\mathrm{b}}(Y) \right\rangle \tag{4.41}$$

where the first component is embedded using the functor $\mathbf{L}p_{\mathcal{A}}^*$, *and the subsequent components by* $j_{\mathcal{A},*}(q_{\mathcal{A}}^*(-) \otimes \mathcal{O}_E(kE))$, *for* $k = 0, \ldots, \operatorname{codim}_X(Y) - 2$.

Remark 4.5.12. *It is immediate that in the above theorem* $p^*\mathcal{A}$ *is of finite global dimension.*

Remark 4.5.13. *The case where* $\mathcal{A} = \mathcal{O}_X$ *is Orlov's blowup formula. Observe that his proof works verbatim for a smooth quasiprojective variety as all morphisms are projective, so the bounded derived category is preserved throughout. We will use this in the proof of Theorem 4.5.11.*

We can prove Theorem 4.5.11 by bootstrapping the original proof. To do so we will need generalisations of some standard results in algebraic geometry such as the adjunction between (derived) pullback and direct image, or the projection formula. A reference for these in the setting of Azumaya algebras can be found in [Kuz06, §10]. We will only need results that do not depend on the algebras being Azumaya, hence in the words of Remark 10.5 of op. cit. we are working with noncommutative finite flat (and not étale) coverings.

Lemma 4.5.14. *The functor* $\mathbf{L}p_{\mathcal{A}}^*$ *is fully faithful.*

Proof. In the commutative setting this is proved using the derived projection formula and the fact that $\mathbf{R}p_* \circ \mathbf{L}p^*(\mathcal{O}_X) \cong \mathcal{O}_X$. In the noncommutative setting the appropriate projection formula is given as the first isomorphism in [Kuz06, Lemma 10.12] taking into account that we have a bimodule structure, whilst the isomorphism

$$\mathbf{R}p_{\mathcal{A},*} \circ \mathbf{L}p_{\mathcal{A}}^*(\mathcal{A}) \cong \mathcal{A} \tag{4.42}$$

follows from the third isomorphism in loc. cit. □

By the assumption that $\mathcal{A}|_Y \cong \operatorname{Mat}_n(\mathcal{O}_Y)$ and the projection formula as given in (4.42) we have that [Orl92, lemma 4.2] goes through as stated. In particular we only need to check that the semiorthogonal decomposition is indeed full.

Proof of Theorem 4.5.11. We can mimic the proof of the first part of [Orl92, Theorem 4.3]. Consider an object in the right orthogonal of (4.41), in particular it is right orthogonal to $\mathbf{L}p_{\mathcal{A}}^*(\mathbf{D}^{\mathrm{b}}(X, \mathcal{A}))$. As Serre duality for sheaves of maximal orders takes on the expected form by [VdBVG84, Corollary 2] we get that $\mathbf{R}p_*$ of this object is indeed zero, and therefore that it is contained in the minimal full subcategory containing the image of $\mathbf{D}(E, \operatorname{Mat}_n(\mathcal{O}_E))$.

Now use that blowups commute with flat base change, in particular we can take an étale neighbourhood of the exceptional divisor that splits $\mathcal{A}$ and such that its image in X and the ramification divisor are disjoint. Then we are in the usual setting of Orlov's blowup formula (up to Morita equivalence), and we can use the usual proof to conclude that the object is indeed zero. □

Two remarks are in order, which are already important in the case of blowing up a point on a surface.

Remark 4.5.15. *If we were to blow up a point* on the ramification divisor*, then the resulting algebra is not necessarily of finite global dimension. Considering the case of a Sklyanin algebra associated to a point of order 2, we have that the complete local structure of this algebra at the point on the intersection of the exceptional divisor and the ramification locus is given by*

$$\begin{bmatrix} R & R \\ (xy) & R \end{bmatrix}$$

where $R = k[[x,y]]$*. One then checks that the module* $\left(\begin{smallmatrix} 0 \\ R/(x) \end{smallmatrix}\right)$ *has a periodic minimal projective resolution of the form*

$$\ldots \to \begin{bmatrix} (xy) \\ (xy) \end{bmatrix} \oplus \begin{bmatrix} (x) \\ (x^2y) \end{bmatrix} \xrightarrow{\psi'} \begin{bmatrix} (x) \\ (x) \end{bmatrix} \oplus \begin{bmatrix} R \\ (xy) \end{bmatrix} \xrightarrow{\varphi} \begin{bmatrix} R \\ R \end{bmatrix} \xrightarrow{\psi} \begin{bmatrix} 0 \\ R/(x) \end{bmatrix} \to 0$$

To see that this resolution is in fact periodic, note that $\ker(\psi') \cong \ker(\psi)$ *as*

$$\ker(\psi) = \begin{bmatrix} R \\ (x) \end{bmatrix}$$
$$\text{and } \ker(\psi') \cong \begin{bmatrix} (xy) \\ (x^2y) \end{bmatrix} = xy \begin{bmatrix} R \\ (x) \end{bmatrix}.$$

This description also shows that the result is no longer a maximal order, and there is a choice of embedding, as explained in [CI05, §4]. Without the embedding in a maximal order one does not expect a meaningful semiorthogonal decomposition. In our setting we do not encounter this problem as the pullback is already maximal itself.

Remark 4.5.16. *In the construction of [VdB01] a point* on the point scheme *is blown up. For an algebra finite over its center this is not the same as the ramification curve, these two curves are only isogeneous. In the context of the previous remark, the difference is measured by the choice of a maximal order containing the pullback.*

4.6 Properties of the maximal orders

In the minimal model program for orders on surfaces as studied in [CI05, ACdJL] there exists the notion of del Pezzo orders, and (half-)ruled orders. The orders we have constructed are obviously not minimal, but they give rise to interesting examples in the study of maximal orders.

In the commutative case the surface $\mathrm{Bl}_x \mathbb{P}^2 = \mathbb{F}_1$ is both del Pezzo and ruled. We are considering orders *on* this surface, and for the value of $m = 2$ in the classification we obtain that it is both *del Pezzo* and *half ruled*, as explained in Proposition 4.6.4 and Proposition 4.6.10.

This latter notion is introduced by Artin, to describe a class of orders which is not ruled, but whose cohomological properties mimic those of ruled orders.

4.6.1 The case $m = 2$ is del Pezzo

In this section we quickly recall the notion of del Pezzo order, and show that the intuition from numerical blowups as in [dTdV16, §5.3] agrees with the a priori independent notion of del Pezzo order, introduced in [CK03, §3].

Throughout we let $\mathcal{A}$ be a maximal order on a smooth projective surface S.

Definition 4.6.1. The *canonical sheaf of* $\mathcal{A}$ is the $\mathcal{A}$-bimodule

$$\omega_{\mathcal{A}} := \mathcal{H}\mathrm{om}_{\mathcal{O}_S}(\mathcal{A}, \omega_S). \tag{4.43}$$

Now denote $\omega_{\mathcal{A}}^* := \mathcal{H}\mathrm{om}_{\mathcal{A}}(\omega_{\mathcal{A}}, \mathcal{A})$, whereas $\mathcal{F}^\vee$ is used for $\mathcal{H}\mathrm{om}_{\mathcal{O}_S}(\mathcal{F}, \mathcal{O}_S)$, hence the reflexive hull is denoted $\mathcal{F}^{\vee\vee}$.

Definition 4.6.2. Let $\mathcal{L}$ be an invertible $\mathcal{A}$-bimodule, which is moreover $\mathbb{Q}$-Cartier, i.e. $(\mathcal{L}^{\otimes n})^{\vee\vee}$ is again invertible for some n. Then $\mathcal{L}$ is *ample* if

$$\mathrm{R}^q \operatorname{Hom}_{\mathcal{A}}(\mathcal{A}, (\mathcal{L}^{\otimes k})^{\vee\vee} \otimes \mathcal{F}) \cong \mathrm{H}^q(S, (\mathcal{L}^{\otimes k})^{\vee\vee} \otimes \mathcal{F}) \tag{4.44}$$

is zero for $q \geq 1$ and $k \gg 0$, where the isomorphism is induced by applying the forgetful functor.

Then analogous to the commutative situation we define

Definition 4.6.3. The maximal order $\mathcal{A}$ is *del Pezzo* if $\omega_{\mathcal{A}}^\vee$ is ample.

In particular, by [CK03, lemma 8] it suffices to check that the divisor

$$\mathrm{K}_{\mathcal{A}} = \mathrm{K}_S + \sum_{i=1}^n \left(1 - \frac{1}{e_i}\right) C_i \tag{4.45}$$

is anti-ample: the del Pezzo-property only depends on the center and the ramification data.

Proposition 4.6.4. *Let $\mathcal{A}$ be the pullback of a maximal order of degree m on $\mathbb{P}^2$ ramified on a cubic curve along the blowup $\mathbb{F}_1 \to \mathbb{P}^2$ in a point outside the ramification locus. Then $\mathcal{A}$ is del Pezzo if and only if $m = 2$.*

Proof. If we denote $\operatorname{Pic}\mathbb{F}_1 = \mathbb{Z}H \oplus \mathbb{Z}E$, such that $H^2 = 1$, $H \cdot E = 0$ and $E^2 = -1$, then

$$\begin{aligned} \mathrm{K}_{\mathcal{A}} &= -3p^*(H) + E + \left(1 - \frac{1}{m}\right) 3p^*(H) \\ &= \frac{-3}{m} p^*(H) + E \end{aligned} \tag{4.46}$$

because the ramification data for the pullback is the pullback of the ramification data, which is a cyclic cover of degree ≥ 2 of an elliptic curve.

By the Kleiman criterion for ampleness we need to check that $-\mathrm{K}_{\mathcal{A}} \cdot C \geq 0$, for C in the Mori–Kleiman cone of $\mathbb{F}_1$. This cone is spanned by a fibre f and a section C_0 of the projection $\mathbb{F}_1 \to \mathbb{P}^1$. We have that $p^*(H) = C_0 + f$ and $E = C_0$ in the translation between the canonical bases for $\operatorname{Pic}(\mathbb{F}_1)$ and $\operatorname{Pic}(\operatorname{Bl}_x \mathbb{P}^2)$.

Using the description of the intersection form on a ruled surface we obtain

$$\begin{aligned} -\mathrm{K}_{\mathcal{A}} \cdot f &= \frac{3}{m} - 1, \\ -\mathrm{K}_{\mathcal{A}} \cdot C_0 &= 1. \end{aligned} \tag{4.47}$$

The first intersection number is positive if and only if $m = 2$. The second intersection number is always positive. ☐

Remark 4.6.5. *The computation for the del Pezzo-property of the numerical blowup reduces to the same equation (up to multiplication by m^2), giving rise to the "degree" δ as introduced in §3.1 and §4.1.*

Remark 4.6.6. *For the case $m = 2$ the equations for a Sklyanin algebra from Remark 4.3.5 take on a particularly easy form*

$$\begin{cases} xy + yx + cz^2 = 0 \\ yz + zy + cx^2 = 0 \\ zx + xz + cy^2 = 0 \end{cases} \tag{4.48}$$

where $c^3 \neq 0, 1, -8$, [DL15a, theorem 3.1]. The cases $c = 0$ and $c^3 = -8$ define AS-regular algebras and the corresponding order on $\mathbb{P}^2$ has a triangle of $\mathbb{P}^1$'s as ramification curve. For $c^3 = 1$ the algebra is no longer AS-regular, see for example [Wal09, Proposition 1.5], hence it cannot be used in the current construction. For the interested reader we mention that, as in [DL15b], their exist quotients of this degenerate algebra which induce maximal orders on $\mathbb{P}^2$.

4.6.2 The case $m = 2$ is half ruled

In the following definition, the curve of genus 0 will be a curve over the function field of the base curve of a ruled surface, i.e. if we consider $\pi: S \to C$ over the field k, then K will be the function field of $\mathbb{P}^1_{k(C)}$.

Definition 4.6.7. Let $\mathcal{A}$ be a maximal order in a central simple algebra of degree 2 over the function field K of a curve X of genus 0. If $\mathcal{A}$ is ramified in 3 points, with ramification degree 2 in each point, then we say that $\mathcal{A}$ is *half ruled.*

Remark 4.6.8. *The case where the ramification is of type* (e, e) *is the* ruled *case.*

It is shown in [ACdJL, Proposition 4.2.4] that being (half-)ruled is equivalent to the Euler characteristic $\chi(X, \mathcal{A})$ of the coherent sheaf $\mathcal{A}$ being positive.

The following definition seems to be missing as such from the literature, but it is used implicitly in [ACdJL, CC15].

Definition 4.6.9. Let $\mathcal{A}$ be a maximal order on a ruled surface $S \to C$. Then we say that it is *half ruled* if the fiber of the order over the generic point of C is half ruled.

Proposition 4.6.10. *The sheaf of maximal orders constructed for the case* $m = 2$ *is half ruled.*

Proof. The ramification divisor on $\mathbb{P}^2$ being a cubic curve we get that the generic intersection of the fibre of the ruling with the inverse image of the ramification divisor in $\mathbb{F}_1$ is 3, which proves the claim. □

4.6.3 The case $m = 3$ is elliptic

We will reuse the notation of §4.6.2.

Definition 4.6.11. Let $\mathcal{A}$ be a maximal order in a central simple algebra of degree 3 over the function field K of a curve X of genus 0. If $\mathcal{A}$ is ramified in 3 points, with ramification degree 3 in each point, then we say that $\mathcal{A}$ is *elliptic.*

The following proposition is proved in the same way as Proposition 4.6.10.

Proposition 4.6.12. *The sheaf of maximal orders constructed for the case* $m = 3$ *is elliptic.*

Chapter 5

Comparison of two constructions of noncommutative del Pezzo surfaces with exceptional collections of length 4

Happiness is not an absolute value. It is a state of comparison.

Zadie Smith (novelist)

This chapter is based upon results by the author, Pieter Belmans and Michel Van den Bergh in [BPVdB17].

5.1 Introduction

In the previous two chapters we focused on constructing models for the matrices appearing in the numerical classification of *noncommutative smooth projective surfaces of rank 4* as appearing in [dTdV16]. Recall that by a *noncommutative smooth projective surfaces of rank 4* we mean a Grothendieck category with a full and strong exceptional collection of length 4 satisfying suitable conditions (see Chapter 0 and §3.1). The Gram matrices for the associated Euler forms were classified in [dTdV16] (see (3.1)). This classification results in one matrix J and a family K_m depending on a natural number m. In Chapter 3 we constructed a noncommutative surface of type K_2 using noncommutative $\mathbb{P}^1$-bundles. In Chapter 4 we found a different construction for all K_m with $m \geq 2$, using maximal orders on $\mathrm{Bl}_x \mathbb{P}^2$. Moreover both construction for $m = 2$ have equally many degrees of freedom, namely three (see Remarks 3.4.32 and 4.5.8).

The goal of this chapter is to compare these constructions, i.e. we compare the categories arising from half ruled del Pezzo orders on $\mathbb{F}_1$ (see §4.6.2 for the notion of half ruled orders) with noncommutative $\mathbb{P}^1$-bundles on $\mathbb{P}^1$. This can be seen as a noncommutative instance of the isomorphism $\mathbb{F}_1 \coloneqq \underline{\mathrm{Proj}}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(1)) \cong \mathrm{Bl}_x \mathbb{P}^2$.

$$\begin{tikzcd} & \mathbb{F}_1 \cong \mathrm{Bl}_x \mathbb{P}^2 \arrow[dl, "p"'] \arrow[dr, "\pi"] & \\ \mathbb{P}^2 & & \mathbb{P}^1 \end{tikzcd} \tag{5.1}$$

It turns out that both categories can be explicitly compared using the geometry of the linear systems appearing in their construction. The idea is to write both categories in terms of (relative) Clifford algebras, which in turn reduces the problem to comparing the data defining these algebras. This allows us to explicitly relate the input for the noncommutative $\mathbb{P}^1$-bundle to the input for the maximal order construction.

In §5.2 we recall various notions of Clifford algebra which are needed for the results in this chapter. Most of these are well-known, but we also introduce a comparison result (Proposition 5.2.11) which might be of independent interest, relating the Clifford algebra with values in a relatively ample line bundle to the total Clifford algebra.

In §5.3 we consider the noncommutative $\mathbb{P}^1$-bundles constructed in Chapter 3, and explain how we can use the formalism of generalized preprojective algebras as in Chapter 2 to describe these abelian categories using Clifford algebras. For this we use the following three facts:

- Graded algebras with symmetric relations have a quotient which is a Clifford algebra (Lemma 5.3.2, Proposition 5.3.3)
- Noncommutative $\mathbb{P}^1$-bundles can be described locally using the theory of generalized preprojective algebras (Lemmas 3.3.13 and 3.3.22)
- Generalized preprojective algebras have graded subalgebras with symmetric relations and the "correct" Hilbert series (Lemma 5.3.6 and Proposition 5.3.7)

In §5.4 we obtain a similar description of the blowups of maximal orders on $\mathbb{P}^2$ constructed in Chapter 4 for the special case where the fat point module (see §4.3) corresponding to the blown up point in $\mathbb{P}^2$ has multiplicity 2. This description is based upon the following three facts:

- By Propositions 4.3.3 and 5.4.2 and Lemma 5.4.1 we know that up to a Zhang twist the underlying quadratic AS-regular algebra is a graded Clifford algebra.
- By Lemmas 5.4.8 and 5.4.9 the induced orders $\mathcal{S}$ on $\mathbb{P}^2$ and $p^*\mathcal{S}$ on $\mathbb{F}_1$ can be described as sheaves of Clifford algebras with values in ample line bundles.
- The comparison result obtained in Proposition 5.2.11 relates the order $p^*\mathcal{S}$ to a sheaf of Clifford algebras on $\mathbb{P}^1$.

Finally this allows us to compare these constructions in §5.5. The results in the previous sections allow us to conclude that both the noncommutative $\mathbb{P}^1$-bundles $\operatorname{qgr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}))$ and the half-ruled del Pezzo orders $\operatorname{coh}(p^*\mathcal{S})$ are equivalent to a category of the form

$$\operatorname{qgr}\Big(\mathrm{C}\ell_{\operatorname{Sym}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1))}\left(E\otimes_k \operatorname{Sym}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1)),q\right)\Big)$$

The latter category is determined by a quadruple (E,V,q,ξ) where E and V are 3-dimensional vector spaces, $q:\operatorname{Sym}^2 E\to V$ a net of conics and $\xi:V\to k$ is a map corresponding to a point in $\mathbb{P}(V^\vee)\cong\mathbb{P}^2$. As such the comparison of the constructions from Chapters 3 and 4 reduces to comparing the quadruples that arise in both constructions. This is done in §5.5 where we prove that both construction give rise to exactly the same quadruples: so-called algebraic quadruples (Definition 5.5.1 and Theorems 5.5.2 and 5.5.3). From this we prove the main result of this chapter:

Theorem (Corollary 5.5.4)**.** *Every noncommutative* $\mathbb{P}^1$*-bundle* $\operatorname{qgr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}))$ *is equivalent to a quaternionic order* $\operatorname{coh}(p^*\mathcal{S})$ *and vice versa.*

Finally in §5.6 we relate this comparison to the classification of pencils of binary quartics and nets of conics as in [Wal77, Wal98]. As such we find a complete correspondence of the types appearing in both classifications (see Table 5.4). Moreover the degrees of freedom agree in each type for both models.

Convention 5.1.1. *The more general parts of this chapter work over any scheme where 2 is invertible. For the actual comparison we will work with varieties over an algebraically closed field k not of characteristic 2 or 3.*

5.2 Clifford algebras

The main tool in comparing the constructions from Chapters 3 and 4 is the formalism of Clifford algebras. It turns that it is possible to write both abelian categories as a category associated to a certain Clifford algebra. This is done in §5.3 and §5.4. It then becomes possible to compare the linear algebra data describing the quadratic forms. In this way we can set up an explicit correspondence between the two models, and explicitly relate the input data for the $\mathbb{P}^1$-bundle model to the input data for the blowup model. This is done in §5.5.

In §5.2.1 we recall the notion of the Clifford algebra associated to a quadratic form on a scheme. This is a coherent $\mathbb{Z}/2\mathbb{Z}$-graded sheaf of algebras. In §5.2.2 we recall the more general version where the quadratic form is allowed to take values in a line bundle [BK94, CVO93]. In this case we cannot obtain a $\mathbb{Z}/2\mathbb{Z}$-algebra, but the even part of the Clifford algebra is nevertheless well-defined.

In §5.2.3 we recall the notion of the graded Clifford algebra associated to a linear system of quadrics [LB95]. Provided the linear system is basepoint-free we get an Artin–Schelter regular algebra which is finite over its center (Proposition 5.2.9).

Finally, we can generalise the notion of a graded Clifford algebra to the relative setting for projective morphisms. This is done in §5.2.4, by combining §5.2.2 and §5.2.3. The classical case of a graded Clifford algebra then corresponds to the morphism $\mathrm{Proj}(\mathrm{Sym}_k(E)) \to \mathrm{Spec}(k)$, where E is the vector space spanned by the elements of degree 1.

5.2.1 Clifford algebras

Classically the Clifford algebra is a finite-dimensional k-algebra constructed with respect to a quadratic form $q\colon E \to k$ on a vector space E. It is defined as the quotient of the tensor algebra $\mathrm{T}(E)$ by the two-sided ideal generated by

$$v \otimes v - q(v)$$

for $v \in E$. We will use the notation $\mathrm{C}\ell_k(E,q)$ for the Clifford algebra. Under the standing assumption that the characteristic of k is not 2 we can also consider the symmetric bilinear form b_q associated to q, and rewrite the relation as

$$u \otimes v + v \otimes u - 2b_q(u,v).$$

The correspondence between symmetric bilinear forms and quadratic forms is an instance of the isomorphism

$$\mathrm{Sym}^2 E \cong \mathrm{Sym}_2 E \tag{5.2}$$

Here $\mathrm{Sym}^2 E$ is the quotient of $E\otimes E$ by all relations of the form $u\otimes v - v\otimes u$. Conversely, $\mathrm{Sym}_2 E \subset E \otimes E$ is the subspace of all symmetric tensors, i.e. as a vectorspace it is generated by elements of the form $u \otimes v + v \otimes u$. The isomorphism in (5.2) is then given by

$$\overline{u \otimes v} \mapsto \frac{u \otimes v + v \otimes u}{2}$$

This isomorphism will be used in a more general form on schemes later.

One can similarly define a Clifford algebras over arbitrary commutative rings

Definition 5.2.1. Let A be a commutative algebra and E a finitely generated projective A-module. Moreover let $q\colon E \to A$ be a quadratic form, i.e. (up to the identification of q and b_q) q is a symmetric A-linear bilinear form

$$q\colon \mathrm{Sym}^2_A(E) \longrightarrow A$$

Then the Clifford algebra $\mathrm{C}\ell_A(E,q)$ is the quotient of $\mathrm{T}_A(E)$ by the relations

$$v \otimes w + w \otimes v - q(vw)$$

Observe that the obvious $\mathbb{Z}$-grading on $\mathrm{T}(V)$ equips $\mathrm{C}\ell(V,q)$ with a $\mathbb{Z}/2\mathbb{Z}$-grading. The even degree part forms a subalgebra which we will denote $\mathrm{C}\ell(V,q)_0$, and the odd degree part forms the $\mathrm{C}\ell(V,q)_0$-bimodule $\mathrm{C}\ell(V,q)_1$.

We want to globalise these constructions to quadratic forms over schemes. First we have to say what we mean by a quadratic form on a scheme X.

Definition 5.2.2. Let X be a scheme such that 2 is invertible on X. A *quadratic form on* X is a pair $(\mathcal{E},q)$ where $\mathcal{E}$ is a locally free sheaf and $q\colon \mathrm{Sym}_2(\mathcal{E}) \to \mathcal{O}_X$ is a morphism of $\mathcal{O}_X$-modules, and $\mathrm{Sym}_2\,\mathcal{E}$ denotes the submodule of symmetric tensors inside $\mathrm{T}^2(\mathcal{E})$.

Let us denote the symmetric square as $\mathrm{Sym}^2\,\mathcal{E}$, i.e. this is the quotient of $\mathrm{T}^2\,\mathcal{E}$ by the relation $v\otimes w - w\otimes v$. As in (5.2), using the assumption that 2 is invertible on X, there exists an isomorphism $\mathrm{Sym}_2\,\mathcal{E} \cong \mathrm{Sym}^2\,\mathcal{E}$, relating quadratic forms to symmetric bilinear forms. We will from now on identify both sheaves and consider quadratic forms as morphisms from $\mathrm{Sym}^2\,\mathcal{E}$.

We can then define what we mean by a Clifford algebra associated to such a quadratic form.

Definition 5.2.3. Let X be a scheme. Let $(\mathcal{E},q)$ be a quadratic form over X. The *Clifford algebra* $\mathrm{C}\ell_X(\mathcal{E},q)$ is the sheafification of

$$U \mapsto \mathrm{T}_{\mathcal{O}_X(U)}(\mathcal{E}(U))/(v\otimes v - q(v)1 \mid v\in \mathrm{H}^0(U,\mathcal{E})).$$

From the grading on the tensor algebra we see that $\mathrm{C}\ell_X(\mathcal{E},q)$ is a $\mathbb{Z}/2\mathbb{Z}$-graded algebra, as was the case for $\mathrm{C}\ell(V,q)$. In order to be compatible with the notation from §5.2.2 we will sometimes denote this Clifford algebra as $\mathrm{C}\ell_X(\mathcal{E},q,\mathcal{O}_X)$.

Remark 5.2.4. *In the case that* q *is nondegenerate we get that* $\mathrm{C}\ell_X(\mathcal{E},q)$ *is* strongly *$\mathbb{Z}/2\mathbb{Z}$-graded, and* $\mathrm{C}\ell_X(\mathcal{E},q)_1$ *is an invertible sheaf of* $\mathrm{C}\ell_X(\mathcal{E},q)_0$*-bimodules. See also [Haz16, Example 1.1.24].*

Similar to Definition 5.2.1 we can extend this definition by replacing $\mathcal{O}_X$ by a sheaf of commutative algebras $\mathcal{A}$ as follows.

Definition 5.2.5. Let X be a scheme and $\mathcal{A}$ a sheaf of commutative algebras on X. Let $\mathcal{E}$ be a locally free $\mathcal{A}$-module and let $q\colon \mathrm{Sym}^2_{\mathcal{A}}(\mathcal{E}) \to \mathcal{A}$ be a morphism of $\mathcal{A}$-modules. The *Clifford algebra* $\mathrm{C}\ell_{\mathcal{A}}(\mathcal{E},q)$ is the sheafification of

$$U \mapsto \mathrm{T}_{\mathcal{A}(U)}(\mathcal{E}(U))/(v\otimes_{\mathcal{A}(U)} v - q(v)1 \mid v\in \mathrm{H}^0(U,\mathcal{E})).$$

We will use this construction in Remark 5.2.10 and section 5.2.4, where $\mathcal{A}$ will be a sheaf of *graded* algebras concentrated in even degree. In this case we give $\mathcal{E}$ degree 1, and then $\mathrm{C}\ell_{\mathcal{A}}(\mathcal{E},q)$ will be a $\mathbb{Z}$-graded algebra.

5.2.2 Clifford algebras with values in line bundles

We also want to consider morphisms of the form $\mathrm{Sym}^2(\mathcal{E}) \to \mathcal{L}$, where $\mathcal{L} \ncong \mathcal{O}_X$ is a line bundle on X. In this case we cannot mimic the construction of §5.2.1 to produce a $\mathbb{Z}/2\mathbb{Z}$-graded Clifford algebra for this more general situation. But an analogue of the even part of the Clifford algebra *can* be defined, together with a bimodule over this algebra [BK94, CVO93] which generalizes the odd part of the usual Clifford algebra.

The generalization of quadratic forms taking values in a line bundle is the following.

Definition 5.2.6. Let X be a scheme such that 2 is invertible on X. Let $\mathcal{L}$ be an invertible sheaf. An *$\mathcal{L}$-valued quadratic form on X* is a triple $(\mathcal{E}, q, \mathcal{L})$ where $\mathcal{E}$ is a locally free sheaf and $q\colon \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{L}$ is a morphism of $\mathcal{O}_X$-modules.

Let X be a scheme. Let $(\mathcal{E}, q, \mathcal{L})$ be an $\mathcal{L}$-valued quadratic form on X. We can consider the total space of $\mathcal{L}$ as

$$p\colon Y \coloneqq \underline{\mathrm{Spec}}_X \left(\bigoplus_{n\in\mathbb{N}} \mathcal{L}^{\otimes n} \right) \to X. \tag{5.3}$$

Because $p^*\mathcal{L} \cong \mathcal{O}_Y$ the quadratic form $(p^*\mathcal{E}, p^*q)$ takes values in $\mathcal{O}_Y$, so we are in the situation of the previous section.

Definition 5.2.7. The sheaf of algebras $p_*(\mathrm{C}\ell_Y(p^*\mathcal{E}, p^*q))$ on X is called the *generalized Clifford algebra* or *total Clifford algebra*. As p is an affine morphism, p_* is just the forgetful functor, and the structure as $\mathcal{O}_Y$-module induces a $\mathbb{Z}$-grading on the total Clifford algebra.

The *even Clifford algebra* $\mathrm{C}\ell_X(\mathcal{E}, q, \mathcal{L})_0$ is the degree 0 subalgebra of $p_*(\mathrm{C}\ell_Y(p^*\mathcal{E}, p^*q))$.

The *Clifford module* $\mathrm{C}\ell_X(\mathcal{E}, q, \mathcal{L})_1$ is the degree 1 submodule of the total Clifford algebra considered as a bimodule over $\mathrm{C}\ell_X(\mathcal{E}, q, \mathcal{L})_0$.

It is not possible to combine these two pieces into a $\mathbb{Z}/2\mathbb{Z}$-graded algebra. Rather we get that the bimodule structure gives rise to a multiplication map

$$\mathrm{C}\ell_X(\mathcal{E}, q, \mathcal{L})_1 \otimes_{\mathrm{C}\ell_X(\mathcal{E}, q, \mathcal{L})_0} \mathrm{C}\ell_X(\mathcal{E}, q, \mathcal{L})_1 \to \mathrm{C}\ell(\mathcal{E}, q, \mathcal{L})_0 \otimes_{\mathcal{O}_X} \mathcal{L}.$$

These construction satisfy the same pleasant properties as the Clifford algebra from §5.2.1. In particular we will use that it is compatible with base change [BK94, Lemma 3.4].

5.2.3 Graded Clifford algebras

In [BK94, CVO93] there is a construction of connected graded algebras based on linear algebra input. For sufficiently general choices it gives rise to an interesting class of Artin–Schelter regular algebras of arbitrary dimension which are finite over their center.

Definition 5.2.8. Let $M_1, \ldots, M_n$ be symmetric matrices in $\mathrm{Mat}_n(k)$. The *graded Clifford algebra* associated to $(M_i)_{i=1}^n$ is the quotient of the graded free k-algebra $k\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle$, where $|x_i| = 1$ and $|y_i| = 2$, by the homogeneous relations

- $x_i x_j + x_j x_i = \sum_{m=1}^n (M_m)_{i,j} y_m$, where $i, j = 1, \ldots, n$
- $[x_i, y_j] = 0$ and $[y_i, y_j] = 0$, where $i, j = 1, \ldots, n$, i.e. y_i is central.

To understand the properties of this graded Clifford algebra, we have to interpret the matrices $M_1, \ldots, M_n$ as quadratic forms Q_i (in the variables y_i). In this way we obtain n quadric hypersurfaces in $\mathbb{P}^{n-1}$, which span a linear system. There are many different ways in which the geometry of the linear system of quadrics influences the algebraic and homological properties of these Clifford algebras.

In particular, the following results were obtained in [LB95]

Proposition 5.2.9. *[LB95, Propositions 7 and 9] The graded Clifford algebra is Artin–Schelter regular if and only if the linear system of quadrics is basepoint-free. Moreover there is an explicit description of the (fat) point modules.*

We can consider the matrix $M = \sum_{i=1}^n M_i y_i$, and at a point $p \in \mathbb{P}^{n-1}_{y_1,\ldots,y_n}$ we can consider the rank of the matrix $M(p)$. Generically it is of full rank, and the point modules correspond to those points for which the rank is 1 or 2. We are interested in the case where $n = 3$, so the point modules are given by the determinant of M, which describes a cubic curve inside $\mathbb{P}^2$. We get back to this in §5.4.2.

Remark 5.2.10. *Although Definition 5.2.8 is at first sight quite different from Definition 5.2.1, we would like to mention that every graded Clifford algebra is in fact a Clifford algebra. To see this, let $\mathrm{C}\ell(M)$ be the graded Clifford algebra defined by n symmetric $n \times n$ matrices M_m. Then there is an isomorphism*

$$\mathrm{C}\ell(M) \cong \mathrm{C}\ell_A(E, q) \tag{5.4}$$

by setting

$$\begin{aligned} A &= k[y_1, \ldots, y_n] \\ F &= kx_1 \oplus \ldots \oplus kx_n \\ E &= F \otimes_k A \end{aligned} \tag{5.5}$$

and considering the quadratic form

$$\begin{aligned} q_F \colon \mathrm{Sym}^2_k F &\to ky_1 \oplus \ldots \oplus ky_n \\ x_i x_j + x_j x_i &\mapsto \sum_{m=1}^n (M_m)_{i,j} y_m \end{aligned} \tag{5.6}$$

and extending it as

$$q\colon \mathrm{Sym}^2_A E = \mathrm{Sym}^2_k F \otimes_k A \xrightarrow{q_F \otimes \mathrm{id}_A} (ky_1 \oplus \ldots \oplus ky_n) \otimes_k A = A_2 \otimes_k A \to A$$

In the next section we continue this idea of interpreting Clifford algebras in two ways.

5.2.4 Clifford algebras with values in ample line bundles

Consider a projective morphism $f\colon Y \to X$, and let $\mathcal{L}$ be an f-relatively ample line bundle on Y. In this case we can consider the graded $\mathcal{O}_X$-algebra

$$\mathcal{A} \coloneqq \bigoplus_{n\geq 0} f_*(\mathcal{L}^{\otimes n}) \tag{5.7}$$

and we have that the relative Proj recovers Y, i.e. $Y \cong \underline{\mathrm{Proj}}_X \mathcal{A}$.

Let $\mathcal{E}$ be a vector bundle on X. Consider a quadratic form

$$q\colon \mathrm{Sym}^2_Y(f^*\mathcal{E}) \to \mathcal{L} \tag{5.8}$$

We can associate two Clifford algebras to this quadratic form:

1. Using §5.2.2 there is the sheaf of even Clifford algebras $\mathrm{C}\ell_0(f^*(\mathcal{E}, q, \mathcal{L}))$, which is a coherent sheaf of algebras on Y.

2. On the other hand, we can use the algebra $\mathcal{A}$ from (5.7) and consider it with a doubled grading, i.e.

$$\mathcal{A}_n = \begin{cases} f_*(\mathcal{L}^{\otimes m}) & n = 2m \\ 0 & n = 2m+1. \end{cases} \tag{5.9}$$

   Changing the grading in this way does not change the property that Y is the relative Proj of $\mathcal{A}$.

   Because f^* is monoidal and using the adjunction $f^* \dashv f_*$ we can consider the quadratic form q as a morphism $\mathrm{Sym}^2_X \mathcal{E} \to f_*(\mathcal{L}) = \mathcal{A}_2$. We can then extend this morphism to the quadratic form

$$Q\colon \mathrm{Sym}^2_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{A}) \to \mathcal{A}. \tag{5.10}$$

   Using §5.2.1 we can define the Clifford algebra $\mathrm{C}\ell_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{A}, Q)$, which we can consider as a sheaf of graded $\mathcal{A}$-algebras on X.

We can compare these two constructions as follows.

Proposition 5.2.11. *With the notation from above, the coherent sheaf of algebras on Y associated to $\mathrm{C}\ell_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{A}, Q)$ is isomorphic to the sheaf of even Clifford algebras $\mathrm{C}\ell_0(f^*\mathcal{E}, q, \mathcal{L})$.*

Moreover, the coherent sheaf on Y associated to $\mathrm{C}\ell_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{A}, Q)(1)$ is isomorphic to the Clifford bimodule $\mathrm{C}\ell_1(f^*\mathcal{E}, q, \mathcal{L})$.

Proof. We have by the relative version of Serre's theorem that $\operatorname{Qcoh} Y \cong \operatorname{QGr}_X \mathcal{A}$. Then $f^*\mathcal{E}$ on Y is given by the graded $\mathcal{A}$-module $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{A}$. It now suffices to observe that by the characterisation of relatively ample line bundles [Sta17, tag 01VJ] we can compare the two constructions of Clifford algebras, using [BK94, Lemmas 3.1 and 3.2].

The statement about the Clifford module follows from the fact that $\mathcal{A}$ is concentrated in even degree, and [BK94, Lemma 3.1]. □

Example 5.2.12. *As a special case of this construction we recover the graded Clifford algebra from §5.2.3. To see this, we consider the graded Clifford algebra as an algebra over $k[y_1, \ldots, y_n]$. If we denote V the vector space spanned by the y_i's and E the vector space spanned by the x_i's, we have a morphism $\mathbb{P}(V) \to \operatorname{Spec} k$, and the graded Clifford algebra is nothing but the Clifford algebra associated to $(E \otimes_k \mathcal{O}_{\mathbb{P}(V)}, q, \mathcal{O}_{\mathbb{P}(V)}(1))$, where q is as in* (5.6)*.*

5.3 Noncommutative $\mathbb{P}^1$-bundles as Clifford algebras

As mentioned in the introduction the goal of this paper is to compare two constructions of noncommutative surfaces of (numerical) type K_2 as in (3.1) above. We will prove that both categories are equivalent to (a Serre quotient of) the module category over a sheaf of graded Clifford algebras on $\mathbb{P}^1$. In this section we describe how the noncommutative $\mathbb{P}^1$-bundles obtained in Chapter 3 are equivalent to a category of the form

$$\operatorname{qgr}\Big(\mathrm{C}\ell_{\operatorname{Sym}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))}\big(E \otimes_k \operatorname{Sym}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)), q\big)\Big)$$

In §5.3.1 we quickly recall the construction (from Chapter 3) of a noncommutative $\mathbb{P}^1$-bundle $\operatorname{qgr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}))$.

In §5.3.4 we recall the notion of *twisting* for sheaf-$\mathbb{Z}$-algebras, an operation which induces equivalences of categories at the level of gr and qgr. We use this twist operation to show that $\operatorname{qgr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}))$ is equivalent to $\operatorname{qgr}(\mathrm{H}(\mathbb{P}^1/\mathbb{P}^1))$ for a sheaf of graded algebras $\mathrm{H}(\mathbb{P}^1/\mathbb{P}^1)$ associated to a finite morphism $f\colon \mathbb{P}^1 \to \mathbb{P}^1$.

In §5.3.5 we then show that $\mathrm{H}(\mathbb{P}^1/\mathbb{P}^1)$ is a *symmetric sheaf of graded algebras*. In the specific case that f has degree 4, this implies (using Lemma 5.3.17) that $\mathrm{H}(\mathbb{P}^1/\mathbb{P}^1)$ is isomorphic to a sheaf of graded Clifford algebras on $\mathbb{P}^1$.

The proofs of both Lemma 5.3.15 and Lemma 5.3.17 are based on local computations: recall from §3.3.2 that noncommutative $\mathbb{P}^1$-bundles can be studied locally using the theory of generalized preprojective algebras $\Pi_C(D)$ associated to a relative Frobenius pair D/C of finite rank as in Chapter 2. We recall this theory in §5.3.3. In this section we also show that $\operatorname{qgr}(\Pi_C(D))$ is equivalent to $\operatorname{qgr}(\mathrm{H}(D/C))$ where $\mathrm{H}(D/C)$ is a graded algebra with symmetric relations. The theory in §5.3.2

then shows that $\mathrm{H}(D/C)$ maps surjectively onto a graded Clifford algebra $\mathrm{C}\ell_A(E,q)$. In the particular case of a relative Frobenius pair D/C of rank 4 (these are the only pairs we encounter locally when f has degree 4) this map is in fact an isomorphism by Proposition 5.3.7.

We can now summarize the categorical comparison between noncommutative $\mathbb{P}^1$-bundles and Clifford algebras in the following chain of equivalences of categories:

$$\begin{array}{cr}
\operatorname{qgr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}})) & \\
\downarrow & \text{Lemma 5.3.10} \\
\operatorname{qgr}(\Pi(Y/X)) & \\
\downarrow & \\
\operatorname{qgr}(\Pi(Y/X)^{(2)}) & \\
\downarrow & (5.32) \\
\operatorname{qgr}(\mathrm{H}(Y/X)) & \\
\downarrow & \text{Corollary 5.3.22} \\
\operatorname{qgr}\left(\mathrm{C}\ell_{\mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1))}\left(\mathcal{O}_{\mathbb{P}^1}^{\oplus 3}\otimes_{\mathcal{O}_{\mathbb{P}^1}}\mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1)),q\right)\right) & \\
\downarrow & \text{Remark 5.3.23} \\
\operatorname{qgr}\left(\mathrm{C}\ell_{\mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1))}\left(E\otimes_k\mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1)),q\right)\right) &
\end{array}$$

5.3.1 Construction of the surface as a noncommutative bundle

In [VdB12] a notion of noncommutative $\mathbb{P}^1$-bundles was introduced, in order to describe noncommutative Hirzebruch surfaces. This is done by defining a suitable notion of a noncommutative symmetric algebra for a locally free sheaf, where the left and right structures do not necessarily agree but where the rank is 2 on both sides.

In Chapter 3 this construction was modified to define a noncommutative symmetric algebra where the rank is 1 on the left and 4 on the right. In particular it provides noncommutative $\mathbb{P}^1$-bundles on $\mathbb{P}^1$ as categories $\operatorname{QGr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}))$ where

1. $f:\mathbb{P}^1\to\mathbb{P}^1$ is a finite morphism of degree 4,
2. $\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}})$ is the *symmetric sheaf-$\mathbb{Z}$-algebra* associated to the $\mathbb{P}^1$-bimodule ${}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}$ (see Definition 3.2.17)

Such a noncommutative $\mathbb{P}^1$-bundle on $\mathbb{P}^1$ has "morphisms" to two copies of $\mathbb{P}^1$. This means that there are functors

$$\Pi_{0,*},\Pi_{1,*}:\operatorname{QGr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}))\to\operatorname{Qcoh}(\mathbb{P}^1)$$

with left adjoints denoted by Π_0^*,Π_1^* (see §3.4.1 and Lemma 3.4.7). It was shown in Theorem 3.4.19 that one can use these functors to lift the Beilinson exceptional sequence on $\mathbb{P}^1$ to a full and strong exceptional sequence

$$\mathbf{D}^{\mathrm{b}}(\operatorname{qgr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}})))=\left\langle\Pi_1^*\mathcal{O}_{\mathbb{P}^1},\Pi_1^*\mathcal{O}_{\mathbb{P}^1}(1),\Pi_0^*\mathcal{O}_{\mathbb{P}^1},\Pi_0^*\mathcal{O}_{\mathbb{P}^1}(1)\right\rangle \tag{5.11}$$

in $\mathbf{D}^{\mathrm{b}}(\mathrm{qgr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}))$ such that the Gram matrix for this exceptional sequence is given by

$$\begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5.12}$$

Moreover, recall from Chapter 3 that this matrix is mutation equivalent to K_2 ([dTdV16, §6]).

5.3.2 Graded algebras with symmetric relations

Let C be a commutative ring in which 2 is invertible. Let F be a finitely generated, projective C-module and let R be a direct summand of $\mathrm{Sym}^2_C(F)$. As we assumed $2 \in C$ to be invertible, we can use the isomorphism in (5.2) to view R as a submodule of $\mathrm{Sym}_2(F) \subset \mathrm{T}^2_C(F)$. As such we can consider the algebra $\mathrm{T}_C\, F/(R)$. The goal of this section is to show that this algebra is isomorphic to a Clifford algebra.

We construct the Clifford algebra $\mathrm{C}\ell_A(E,q)$ as in Definition 5.2.3. For this let $Q = \mathrm{Sym}^2_C(F)/R$, $A = \mathrm{Sym}_C\, Q$ and $E = F \otimes_C A$. The quotient map

$$\tilde{q}\colon \mathrm{Sym}^2_C(F) \to Q$$

induces a map:

$$q\colon \mathrm{Sym}^2_A(E) = \mathrm{Sym}^2_C(F) \otimes_C A \to Q \otimes_C A \xrightarrow{\mu} A$$

where μ is given by multiplication in A. Moreover A can be considered as a $\mathbb{Z}$-graded algebra by giving Q degree 2.

Remark 5.3.1. *Giving A degree 0 and E degree 1 equips $\mathrm{C}\ell_A(E,q)$ with a filtration such that*

$$\mathrm{gr}^\bullet\, \mathrm{C}\ell_A(E,q) \cong \textstyle\bigwedge^\bullet E \tag{5.13}$$

as graded A-modules.

Lemma 5.3.2. *With the above notation there exists a morphism*

$$\varphi\colon \mathrm{T}_C(F)/(R) \to \mathrm{C}\ell_A(E,q). \tag{5.14}$$

Proof. The inclusions $C \hookrightarrow A$ and $F \hookrightarrow E$ induce a morphism

$$\mathrm{T}_C(F) \to \mathrm{T}_A(E) \to \mathrm{C}\ell_A(E,q). \tag{5.15}$$

It hence suffices to show that this morphism factors through $\mathrm{T}_C(F)/(R)$, i.e. that the above map sends R to 0. For this note that R is generated by elements of the form $v \otimes w + w \otimes v$. The image of such an element in $\mathrm{C}\ell_A(E,q)$ is given by $q(vw)$. And $q(vw) = 0$ for $v \otimes w + w \otimes v \in R$ because q was obtained as an A-linear extension of the quotient map $\tilde{q}$. □

Proposition 5.3.3. *The morphism* (5.14) *is always an epimorphism.*

Proof. It suffices to show that $A \subset \mathrm{C}\ell_A(E,q)$ and $E \subset \mathrm{C}\ell_A(E,q)$ lie in the image of φ. By construction C and F lie in the image of φ and $E = F \otimes_C A$. In order to prove surjectivity of φ, we hence only need to show that A lies in the image of φ. Finally, as A is $\mathbb{N}$-graded and generated in degree 2, it suffices to show that A_0 and A_2 lie in the image of φ. For A_0 this is obvious as $A_0 = C$. For A_2 this follows by noticing that $A_2 = q(F \otimes F)$. $\square$

5.3.3 Generalized preprojective algebras have symmetric relations

In this section we recall the construction generalized preprojective algebras as in Chapter 2, and prove that they have symmetric relations.

Recall that a generalized preprojective algebra $\Pi_C(D)$ is constructed starting from a relative Frobenius pair D/C. Such a pair consists of commutative rings, such that D is a C algebra, which has finite rank as a C-module, and such that there is an isomorphism $\varphi : \mathrm{Hom}_C(D,C) \to D$ of D-(bi)modules.

The generalized preprojective algebra $\Pi_C(D)$ is defined as the quotient of the tensor algebra $\mathrm{T}(C,D) \coloneqq \mathrm{T}_{C\oplus D}({}_CD_D \oplus {}_DD_C)$ with the (quadratic) relations given by the images of the following morphisms

- the structure morphism

$$i: C \to {}_CD_C = {}_CD_D \otimes_D D_C \tag{5.16}$$

- the D-bimodule morphism

$$r: D \to {}_DD_C \otimes_C D_D : 1_D \mapsto r_\varphi \tag{5.17}$$

  where r_φ is the element in ${}_DD_C \otimes_C D_D$ corresponding to the isomorphism

$$\varphi \in \mathrm{Hom}_D(\mathrm{Hom}_C(D,C),D) \tag{5.18}$$

  under the identification

$$\mathrm{Hom}_D(\mathrm{Hom}_C(D,C),D) \cong {}_DD_C \otimes_C D_D.$$

Remark 5.3.4. *It was mentioned in Chapter 2 that the morphism r as above can also be described in a different way. For this one fixes a morphism $\Lambda: C \to D$ which generates* $\mathrm{Hom}_C(D,C)$ *as a D-module, i.e.* $\mathrm{Hom}_C(D,C) = \Lambda D$. *Every basis $\{e_1,\dots,e_n\}$ for D as a C-module admits a dual basis $\{f_1,\dots,f_n\}$ for D in the sense that*

$$\Lambda(e_i f_j) = \delta_{i,j}. \tag{5.19}$$

The element $r_\varphi \in {}_DD_C \otimes_C D_D$ is then given by $\sum_{i=1}^n e_i \otimes f_i$. The fact that this defines a D-bimodule morphism follows from Lemma 3.3.23

Definition 5.3.5. Let D/C be a relative Frobenius extension of rank n and let $\Pi_C(D)$ be as above. Then we define

$$\mathrm{H}(D/C) \coloneqq ((1_C, 0)\Pi_C(D)(1_C, 0))^{(2)} \tag{5.20}$$

It is immediate that $\mathrm{H}(D/C)$ can be written as a quotient of $\mathrm{T}_C(D/i(C))$.

Lemma 5.3.6. *There exists a Clifford algebra $\mathrm{C}\ell_A(E,q)$ and a surjective morphism*

$$\varphi: \mathrm{H}(D/C) \to \mathrm{C}\ell_A(E,q). \tag{5.21}$$

Proof. By Proposition 5.3.3 it suffices to prove that the relations in $\mathrm{H}(D/C)$ as a quotient of $\mathrm{T}_C(D/i(C))$ are symmetric. Note that

$$\mathrm{H}(D/C) = (\mathrm{T}_C(D/i(C)))/(R) \tag{5.22}$$

where R is the image of

$$D \xrightarrow{r} D \otimes_C D \to D/i(C) \otimes_C D/i(C) \tag{5.23}$$

We now make the following three observations:

- The image of r is described by r_φ, which can be written as
$$r_\varphi = \sum_{i=1}^{n} e_i \otimes f_i$$
where $\{e_1, \ldots, e_n\}$ is a C-module-basis for D and $\{f_1, \ldots, f_n\}$ is its dual basis.

- As r_φ can be defined without the choice of a basis for D as C-module (see (5.17) and (5.18)), it does not depend on the choice of basis $\{e_1, \ldots, e_n\}$.

- If $\Lambda(e_i f_j) = \delta_{i,j}$ then $\Lambda(f_i e_j) = \delta_{i,j}$ as well.
In particular if $\{f_1, \ldots, f_n\}$ is the dual basis for $\{e_1, \ldots, e_n\}$, then the dual basis for $\{f_1, \ldots, f_n\}$ is given by $\{e_1, \ldots, e_n\}$.

These 3 observations together imply

$$\sum_{i=1}^{n} e_i \otimes f_i = r_\varphi = \sum_{i=1}^{n} f_i \otimes e_i \tag{5.24}$$

such that $R \subset \mathrm{Sym}_{C,2}(D/i(C)) \cong \mathrm{Sym}^2_C(D/i(C))$ as required. $\square$

Moreover we have the following:

Proposition 5.3.7. *The morphism in* (5.21) *is an isomorphism if* $n = 4$.

Proof. Surjectivity is immediate by the construction as in Proposition 5.3.3. Hence it suffices to prove injectivity.

We now claim that $\mathrm{H}(D/C)$ and $\mathrm{C}\ell_A(E,q)$ have a structure as graded C-algebra such that γ becomes a graded C-algebra morphism and such that $\mathrm{H}(D/C)_n$ and $(\mathrm{C}\ell_A(E,q))_n$ are free C-modules of the same rank for each n. Obviously the proposition follows from the claim hence we only need to prove the claim.

As $(1_C\Pi_C(D))^{(2)} = (1_C\Pi_C(D)1_C)^{(2)} = \mathrm{H}(D/C)$ we can use Lemma 2.3.10 to conclude that $\mathrm{H}(D/C)$ is a graded C-algebra such that $\mathrm{H}(D/C)_n$ is a free C-module of rank $n+1$.

Next note that A has a structure of a graded C-algebra. Moreover, as before, we let $Q \subset A$ have degree 2. Hence letting $F \subset E$ have degree 1, the relations

$$u \otimes v + v \otimes u = q(uv)$$

become homogeneous of degree 2 for each $u, v \in F$ and there is a unique induced grading on $\mathrm{C}\ell_A(E,q)$. Using Remark 5.3.1 we find isomorphisms of C-modules:

$$\mathrm{C}\ell_A(E,q)_{2n} \cong A_{2n} \oplus (E \wedge E)_{2n-2}, \mathrm{C}\ell_A(E,q)_{2n+1} \quad \cong E_{2n} \oplus (E \wedge E \wedge E)_{2n-2}.$$

As E, $E \wedge E$ and $E \wedge E \wedge E$ are free A-modules of ranks 3, 3 and 1 respectively, we can calculate the rank of $\mathrm{C}\ell_A(E,q)_{2n(+1)}$ using the rank of A_n.

Recall that $A = \mathrm{Sym}_C(Q)$ and that Q is a free C-module of rank 2. As such

$$h_A(2n) = n+1.$$

We hence find that $\mathrm{C}\ell_A(E,q)_{2n}$ is a free C-module of rank $n+1+3n = 4n+1 = 2(2n)+1$ and that $\mathrm{C}\ell_A(E,q)_{2n+1}$ is a free C-module of rank $3(n+1)+n = 4n+3 = 2(2n+1)+1$.

Finally it is immediate from the construction that $\gamma\colon \mathrm{H}(D/C) \to \mathrm{C}\ell_A(E,q)$ is a graded C-algebra morphism. The claim and the result follow. □

For use below we also introduce the following $\mathbb{Z}$-algebra:

Definition 5.3.8. Let D/C be a relative Frobenius extension of rank n. We define the tensor-$\mathbb{Z}$-algebra $\mathrm{T}_{\mathbb{Z}}(D/C)$ via

$$\mathrm{T}_{\mathbb{Z}}(D/C)_{(j,j+1)} = \begin{cases} {}_C D_D & \text{if } j \text{ is even} \\ {}_D D_C & \text{if } j \text{ is odd} \end{cases} \tag{5.25}$$

and we denote $\Pi_{\mathbb{Z}}(D/C)$ for the quotient of $\mathrm{T}_{\mathbb{Z}}(D/C)$ with the relations given by the images of i and r as in (5.16) and (5.17).

Remark 5.3.9. *The generalized preprojective algebra* $\Pi_D(C)$ *is related to the* $\mathbb{Z}$*-algebra* $\Pi_{\mathbb{Z}}(D/C)$ *as follows:*

- $\Pi_D(C) = \overline{\Pi_{\mathbb{Z}}(D/C)}$ *using the notation of §3.3.3.*

- $\mathrm{H}(\widetilde{D/C}) \coloneqq \big((1_C, 0)\Pi_C(\widetilde{D})(1_C, 0)\big)^{(2)} \cong \Pi_{\mathbb{Z}}(D/C)^{(2)}$

5.3.4 Bimodule $\mathbb{Z}$-algebras and twisting

In §3.2 the notions of bimodules, $\mathbb{Z}$-algebras and symmetric algebras were generalized to the level of sheaves. We refer the reader to §3.2.1 for all relevant definitions. For the current applications, it suffices to realize that if $\mathcal{U}$ is a locally free sheaf of finite rank on a variety W and if $u : W \to X$, $v : W \to Y$ are finite morphisms to varieties X, Y, then ${}_u\mathcal{U}_v$ is an $X-Y$-bimodule and

$$\mathcal{F} \otimes {}_u\mathcal{U}_v = v_*(u^*(\mathcal{F}) \otimes_W \mathcal{U})$$

holds for all sheaves $\mathcal{F}$ on X. A sheaf-$\mathbb{Z}$-algebra is nothing but a collection of bimodules with a multiplicative structure.

A fundamental operation in the context of sheaf-$\mathbb{Z}$-algebras is that of twisting by a sequence of invertible bimodules (Theorem 3.2.18). This operation allows to *twist* a sheaf-$\mathbb{Z}$-algebra $\mathcal{A}$ into a sheaf-$\mathbb{Z}$-algebra $\mathcal{B}$ which is equivalent at the level of qgr. Recall that invertible bimodules were defined in Definition 3.2.15, but for our current applications it suffices to realize that bimodules of the form ${}_{\mathrm{id}}\mathcal{L}_{\mathrm{id}}$ (with $\mathcal{L}$ a line bundle) are invertible.

Our main application of this twisting operation is the following:

Lemma 5.3.10. *Let* $f{:}\, Y \to X$ *be a finite morphism between smooth projective curves and let* $\mathbb{S}({}_f(\mathcal{O}_Y)_{\mathrm{id}})$ *be the associated symmetric sheaf-*$\mathbb{Z}$*-algebra. Moreover denote* $\Pi_{\mathbb{Z}}(Y/X)$ *for the sheaf-*$\mathbb{Z}$*-algebra (over the collection* $(X_i)_{i\in\mathbb{Z}}$ *with* $X_{2j} = X$ *and* $X_{2j+1} = Y$*) defined as follows:*

- $\Pi_{\mathbb{Z}}(Y/X)_{m,n} = 0$ *whenever* $m > n$*;*

- $\Pi_{\mathbb{Z}}(Y/X)_{2j,2j} = {}_{\mathrm{id}}(\mathcal{O}_X)_{\mathrm{id}}$ *for all* j*;*

- $\Pi_{\mathbb{Z}}(Y/X)_{2j+1,2j+1} = {}_{\mathrm{id}}(\mathcal{O}_Y)_{\mathrm{id}}$ *for all* j*;*

- $\Pi_{\mathbb{Z}}(Y/X)_{2j,2j+1} = {}_f(\mathcal{O}_Y)_{\mathrm{id}}$ *for all* j*;*

- $\Pi_{\mathbb{Z}}(Y/X)_{2j+1,2j+2} = {}_{\mathrm{id}}(\mathcal{O}_Y)_f$ *for all* j*;*

- $\Pi_{\mathbb{Z}}(Y/X)$ *is freely generated by the* $\Pi_{\mathbb{Z}}(Y/X)_{n,n+1}$ *subject to the relations*

$$\begin{aligned}{}_{\mathrm{id}}(\mathcal{O}_X)_{\mathrm{id}} &\subset {}_{\mathrm{id}}(f_*\mathcal{O}_Y)_{\mathrm{id}}\\ &= {}_f(\mathcal{O}_X)_f \\ &= \Pi_{\mathbb{Z}}(Y/X)_{2j,2j+1} \otimes_Y \Pi_{\mathbb{Z}}(Y/X)_{2j+1,2j+2}\end{aligned} \tag{5.26}$$

and

$$r({}_{\mathrm{id}}(\omega_f^{-1})_{\mathrm{id}}) \subset \Pi_{\mathbb{Z}}(Y/X)_{2j-1,2j} \otimes_Y \Pi_{\mathbb{Z}}(Y/X)_{2j,2j+1} \tag{5.27}$$

where

$$r\colon {}_{\mathrm{id}}(\omega_f^{-1})_{\mathrm{id}} \to {}_{\mathrm{id}}(\mathcal{O}_Y)_f \otimes {}_f(\mathcal{O}_Y)_{\mathrm{id}} \tag{5.28}$$

is induced by $f_* \dashv f^! = f^* \otimes \omega_{X/Y}$ *as in* (3.14).

Then $\Pi_{\mathbb{Z}}(Y/X)$ *is a twist of* $\mathbb{S}({}_f(\mathcal{O}_Y)_{\mathrm{id}})$.

Proof. Using (3.15) it follows immediately that

$$(\mathbb{S}({}_f(\mathcal{O}_Y)_{\mathrm{id}}))_{2j,2j+1} = (\Pi_{\mathbb{Z}}(Y/X))_{2j,2j+1} \otimes {}_{\mathrm{id}}(\omega^j_{X/Y})_{\mathrm{id}}$$

and

$$(\mathbb{S}({}_f(\mathcal{O}_Y)_{\mathrm{id}}))_{2j+1,2j+2} = {}_{\mathrm{id}}(\omega^{-j}_{X/Y})_{\mathrm{id}} \otimes (\Pi_{\mathbb{Z}}(Y/X))_{2j+1,2j+2}.$$

Now let $\mathcal{T}_i$ be defined by

$$\mathcal{T}_i := \begin{cases} {}_{\mathrm{id}}(\mathcal{O}_X)_{\mathrm{id}} & i = 2j \\ {}_{\mathrm{id}}(\omega^j_{X/Y})_{\mathrm{id}} & i = 2j+1. \end{cases}$$

We then claim that

$$(\mathbb{S}({}_f(\mathcal{O}_Y)_{\mathrm{id}}))_{m,n} \cong \mathcal{T}_m^{-1} \otimes \Pi_{\mathbb{Z}}(Y/X)_{m,n} \otimes \mathcal{T}_n$$

holds for all m, n.

As both algebras are generated in degree 1 and have quadratic relations, it suffices to check the claim for $n - m = 0, 1, 2$. By the above the claim holds for $n - m = 0, 1$.

For $n = m + 2 = 2j + 2$ we have

$$(\mathbb{S}({}_f(\mathcal{O}_Y)_{\mathrm{id}}))_{2j,2j+2} = {}_{\mathrm{id}}\left(f_*\mathcal{O}_Y/\mathcal{O}_X\right)_{\mathrm{id}} = (\Pi_{\mathbb{Z}}(Y/X))_{2j,2j+2}$$

To get the claim for $n = m + 2$, $m = 2j + 1$ it suffices to check that

$$\begin{aligned} i_{2j+1}({}_{\mathrm{id}}(\mathcal{O}_Y)_{\mathrm{id}}) &= {}_{\mathrm{id}}(\omega^{-j}_{X/Y})_{\mathrm{id}} \otimes r({}_{\mathrm{id}}(\omega_f^{-1})_{\mathrm{id}}) \otimes {}_{\mathrm{id}}(\omega^{j+1}_{X/Y})_{\mathrm{id}} \\ &= {}_{\mathrm{id}}(\omega^{-j}_{X/Y})_{\mathrm{id}} \otimes r({}_{\mathrm{id}}(\omega_f^{-1})_{\mathrm{id}}) \otimes {}_{\mathrm{id}}(\omega_{X/Y})_{\mathrm{id}} \otimes {}_{\mathrm{id}}(\omega^j_{X/Y})_{\mathrm{id}} \end{aligned}$$

This in turn follows by using the fact that i_{2j+1} and r are defined as unit maps, the fact that the bimodules underlying these unit maps are twists of each other and the fact that

$$\left({}_{\mathrm{id}}(\omega^{-j}_{X/Y})_{\mathrm{id}} \otimes {}_{\mathrm{id}}(\mathcal{O}_Y)_f\right)^* = ({}_{\mathrm{id}}(\mathcal{O}_Y)_f)^* \otimes {}_{\mathrm{id}}(\omega^j_{X/Y})_{\mathrm{id}}.$$ □

Remark 5.3.11. *The sheaf-$\mathbb{Z}$-algebra $\Pi_{\mathbb{Z}}(Y/X)$ as introduced above is 2-periodic, i.e. for each i and j there is an isomorphism*

$$(\Pi_{\mathbb{Z}}(Y/X))_{i,j} \cong (\Pi_{\mathbb{Z}}(Y/X))_{i+2,j+2} \tag{5.29}$$

and these isomorphisms are compatible with the multiplication in $\Pi_{\mathbb{Z}}(Y/X)$.

In particular let $\mathrm{H}(Y/X)$ be the graded sheaf of algebras on X defined as follows:

- $\mathrm{H}(Y/X)_0 = \mathcal{O}_X$*;*
- $\mathrm{H}(Y/X)_1 = f_*\mathcal{O}_Y/\mathcal{O}_X$*;*
- $\mathrm{H}(Y/X)$ *is generated by* $\mathrm{H}(Y/X)_1$ *subject to the relation*

$$f_*\omega_{X/Y}^{-1} \subset \mathrm{H}(Y/X)_1 \otimes \mathrm{H}(Y/X)_1 = f_*\mathcal{O}_Y/\mathcal{O}_X \otimes f_*\mathcal{O}_Y/\mathcal{O}_X \tag{5.30}$$

which is induced by

$$\begin{aligned} {}_{\mathrm{id}}(f_*\omega_{X/Y}^{-1})_{\mathrm{id}} &= {}_f(\omega_{X/Y})_f \\ &= {}_f(\mathcal{O}_Y)_{\mathrm{id}} \otimes {}_{\mathrm{id}}(\omega_{X/Y}^{-1})_{\mathrm{id}} \otimes {}_{\mathrm{id}}(\mathcal{O}_Y)_f \\ &\xrightarrow{r} {}_f(\mathcal{O}_Y)_{\mathrm{id}} \otimes {}_{\mathrm{id}}(\mathcal{O}_Y)_f \otimes {}_f(\mathcal{O}_Y)_{\mathrm{id}} \otimes {}_{\mathrm{id}}(\mathcal{O}_Y)_f \\ &= {}_{\mathrm{id}}(f_*\mathcal{O}_Y \otimes f_*\mathcal{O}_Y)_{\mathrm{id}} \\ &\to {}_{\mathrm{id}}\big(f_*\mathcal{O}_Y/\mathcal{O}_X \otimes f_*\mathcal{O}_Y/\mathcal{O}_X\big)_{\mathrm{id}}; \end{aligned} \tag{5.31}$$

then there is an isomorphism of sheaf-$\mathbb{Z}$-algebras:

$$\Pi_{\mathbb{Z}}(Y/X)^{(2)} \cong \check{H}(Y/X) \tag{5.32}$$

Using Theorem 3.2.18, Lemma 5.3.10, Remark 5.3.11 and Remark 3.2.8 we have equivalences of categories

$$\begin{aligned} \mathrm{QGr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}})) &\cong \mathrm{QGr}(\Pi_{\mathbb{Z}}(\mathbb{P}^1/\mathbb{P}^1)) \\ &\cong \mathrm{QGr}(\mathrm{H}(\mathbb{P}^1/\mathbb{P}^1)). \end{aligned} \tag{5.33}$$

5.3.5 Symmetric sheaves of graded algebras

This section is devoted to generalizing Lemmas 5.3.2 and 5.3.6 and Proposition 5.3.3 to the level of sheaves of algebras.

Convention 5.3.12. *Throughout this section, all schemes are assumed to have 2 invertible, and for ease of statement we will also assume that they are irreducible.*

Definition 5.3.13. Let $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \ldots$ be a graded sheaf of algebras on a scheme X. We say $\mathcal{H}$ is a *symmetric sheaf of graded algebras* if the following conditions hold:

- $\mathcal{H}_1$ is a locally free sheaf on X;
- there is a surjective morphism of sheaves of graded algebras
$$\psi : \mathrm{T}_{\mathcal{O}_X}(\mathcal{H}_1) \twoheadrightarrow \mathcal{H};$$
- ψ is an isomorphism in degree 0 and 1;
- $\ker(\psi) \cong \mathcal{R} \otimes_{\mathcal{O}_X} \mathrm{T}_{\mathcal{O}_X}(\mathcal{H}_1)$
where $\mathcal{R}$ is a direct summand of $(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{H}_1))_2 \subset (\mathrm{T}_{\mathcal{O}_X}(\mathcal{H}_1))_2$.

The following is immediate from the local nature of the construction of $\mathrm{Sym}_{\mathcal{O}_X}(-)$ and $\mathrm{T}_{\mathcal{O}_X}(-)$.

Lemma 5.3.14. *Let $\mathcal{H}$ be a sheaf of graded algebras on X. Then the following are equivalent:*

i) $\mathcal{H}$ is a symmetric sheaf of graded algebras on X.

ii) for each point $p \in X$ we have that $\mathcal{H}_p$ is a graded $\mathcal{O}_{X,p}$-algebra with symmetric relations as in §5.3.2.

We use this lemma to prove the following.

Lemma 5.3.15. *Let X and Y be smooth varieties over a field k of characteristic different from 2 and let $f\colon Y \to X$ be a finite morphism of degree n. Let* $\mathrm{H}(Y/X)$ *be the associated sheaf of graded algebras as in Remark 5.3.11. Then* $\mathrm{H}(Y/X)$ *is a symmetric sheaf of graded algebras on X*

Proof. Using Lemma 5.3.14 it suffices to prove that $\mathrm{H}(Y/X)_p$ is a graded algebra with symmetric relations as in §5.3.2. For this we first use Lemma 3.3.13 which (among other things) shows that $(f_*\mathcal{O}_Y)_p/\mathcal{O}_{X,p}$ is relative Frobenius of rank n. For example one has an isomorphism of $(f_*\mathcal{O}_Y)_p$-modules

$$(f_*\mathcal{O}_Y)_p \cong (\omega_{X/Y})_p \cong \mathrm{Hom}_{\mathcal{O}_{X,p}}((f_*\mathcal{O}_Y)_p/\mathcal{O}_{X,p}) \tag{5.34}$$

We now claim that

$$\mathrm{H}(Y/X)_p = \mathrm{H}\big((f_*\mathcal{O}_Y)_p/\mathcal{O}_{X,p}\big)$$

where the right hand side was defined in (5.20). The lemma follows from this claim and the fact that $\mathrm{H}((f_*\mathcal{O}_Y)_p/\mathcal{O}_{X,p})$ is a graded algebra with symmetric relations (see for example the proof of Lemma 5.3.6). By comparing (5.22) with Remark 5.3.11 we see that the claim holds in degree 0 and 1: both algebras can be written as quotients of $\mathrm{T}_{\mathcal{O}_{X,p}}((f_*\mathcal{O}_Y)_p/\mathcal{O}_{X,p})$ subject to quadratic relations.

Recall that the quadratic relations in $\mathrm{H}((f_*\mathcal{O}_Y)_p/\mathcal{O}_{X,p})$ were obtained by composing the morphisms r in (5.23) with the obvious morphism

$$\pi\colon (f_*\mathcal{O}_Y)_p \otimes_{\mathcal{O}_{X,p}} (f_*\mathcal{O}_Y)_p \to (f_*\mathcal{O}_Y)_p/\mathcal{O}_{X,p} \otimes_{\mathcal{O}_{X,p}} (f_*\mathcal{O}_Y)_p/\mathcal{O}_{X,p} \tag{5.35}$$

A closer examination of the morphisms r in (5.23), see for example (3.32), shows that the forgetful functor $\mathrm{Mod}((f_*\mathcal{O}_Y)_p) \to \mathrm{Mod}(\mathcal{O}_{X,p})$ is left adjoint to the functor $-\otimes_{\mathcal{O}_{X,p}} (f_*\mathcal{O}_Y)_p : \mathrm{Mod}(\mathcal{O}_{X,p}) \to \mathrm{Mod}((f_*\mathcal{O}_Y)_p)$ and that r coincides with the associated unit morphism

$$(f_*\mathcal{O}_Y)_p \to (f_*\mathcal{O}_Y)_p \otimes_{\mathcal{O}_{X,p}} (f_*\mathcal{O}_Y)_p \tag{5.36}$$

Similarly the quadratic relations in $\mathrm{H}(Y/X)_p$ were obtained as the image of $\pi\circ r'$ where π is as in (5.35) and r' is obtained by localizing the unit morphism in (5.28). Hence in order to prove the claim it suffices to show that the localization of the unit morphism in (5.28) coincides with the unit morphism in (5.36). For this we use the isomorphism $(f_*\mathcal{O}_Y)_p \cong (\omega_{X/Y})_p$ as in (5.34) and the commutativity of the following diagrams

$$\begin{tikzcd}
\mathrm{Qcoh}(\mathcal{O}_Y) \arrow[r, "f_* = -\otimes_{\mathrm{id}} (\mathcal{O}_Y)_f"] \arrow[d, "(f_*(-))_p"'] & \mathrm{Qcoh}(\mathcal{O}_X) \arrow[d, "(-)_p"] \\
\mathrm{Mod}((f_*\mathcal{O}_Y)_p) \arrow[r, "\text{forget}"'] & \mathrm{Mod}(\mathcal{O}_{X,p})
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Qcoh}(\mathcal{O}_X) \arrow[r, "f^* = -\otimes_f (\mathcal{O}_Y)_{\mathrm{id}}"] \arrow[d, "(-)_p"'] & \mathrm{Qcoh}(\mathcal{O}_Y) \arrow[d, "(f_*(-))_p"] \\
\mathrm{Mod}(\mathcal{O}_{X,p}) \arrow[r, "-\otimes_{\mathcal{O}_{X,p}} (f_*\mathcal{O}_Y)_p"'] & \mathrm{Mod}((f_*\mathcal{O}_Y)_p)
\end{tikzcd}$$

where for the second diagram, commutativity follows from the projection formula. $\square$

The local nature of the constructions in Definition 5.2.5 allow us to generalize Lemma 5.3.2 and Proposition 5.3.3 as follows.

Lemma 5.3.16. *Let $\mathcal{H} = \mathrm{T}_{\mathcal{O}_X}(\mathcal{H}_1)/\mathcal{R}$ be a symmetric sheaf of graded algebras on X. Let $\mathcal{Q} = \mathrm{Sym}^2_{\mathcal{O}_X}(\mathcal{H}_1))/\mathcal{R}$, $\mathcal{A} = \mathrm{Sym}_{\mathcal{O}_X}\mathcal{Q}$ and $\mathcal{E} = \mathcal{H}_1 \otimes_{\mathcal{O}_X} \mathcal{A}$.*

Let $q\colon \mathrm{Sym}^2_{\mathcal{A}}(\mathcal{E}) \to \mathcal{A}$ be induced by the quotient map $\tilde{q}\colon \mathrm{Sym}^2_{\mathcal{O}_X}(\mathcal{H}_1)) \to \mathcal{Q}$; i.e.

$$q\colon \mathrm{Sym}^2_{\mathcal{A}}(\mathcal{E}) = \mathrm{Sym}^2_{\mathcal{O}_X}(\mathcal{H}_1) \otimes_{\mathcal{O}_X} \mathcal{A} \to \mathcal{Q} \otimes_{\mathcal{O}_X} \mathcal{A} \xrightarrow{\mu} \mathcal{A} \tag{5.37}$$

where μ is given by multiplication in $\mathcal{A}$.

Then the inclusions $\mathcal{O}_X \hookrightarrow \mathcal{A}$ and $\mathcal{H}_1 \hookrightarrow \mathcal{E}$ induce a epimorphism

$$\mathcal{H} \longrightarrow\!\!\!\!\rightarrow \mathrm{C}\ell_{\mathcal{A}}(\mathcal{E}, q).$$

We now combine Lemma 5.3.16 with Lemma 5.3.15 to obtain the following.

Lemma 5.3.17. *Let X and Y be smooth varieties over a field k of characteristic different from 2 and let $f: Y \to X$ be a finite morphism of degree 4. Let $\mathrm{H}(Y/X)$ and $\mathrm{C}\ell_{\mathcal{A}}(\mathcal{E}, q)$ be as above. Then there is an isomorphism*

$$\mathrm{H}(Y/X) \cong \mathrm{C}\ell_{\mathcal{A}}(\mathcal{E}, q). \tag{5.38}$$

Proof. Combining Lemma 5.3.16 and Lemma 5.3.15 we already know that there is an epimorphism

$$\psi: \mathrm{H}(Y/X) \twoheadrightarrow \mathrm{C}\ell_{\mathcal{A}}(\mathcal{E}, q).$$

It hence suffices to show that ψ is in fact an isomorphism, and this can be checked locally. Hence let p be any point in X. We must show that

$$\psi_p: \mathrm{H}(Y/X)_p = \mathrm{H}((f_*\mathcal{O}_Y)_p/\mathcal{O}_{X,p}) \twoheadrightarrow \left(\mathrm{C}\ell_{\mathcal{A}}(\mathcal{E}, q)\right)_p = \mathrm{C}\ell_{\mathcal{A}_p}(\mathcal{E}_p, q_p)$$

is an isomorphism. This follows from Proposition 5.3.7 by noticing that ψ_p coincides with (5.21). ☐

We are particularly interested in the case where $Y = X = \mathbb{P}^1$. Hence from now on fix a finite, degree 4 morphism $f: \mathbb{P}^1 \to \mathbb{P}^1$. We continue this section by describing the sheaf of algebras $\mathcal{A}$ and the $\mathcal{A}$-module $\mathcal{E}$ in (5.38). We will use the following easy lemmas.

Lemma 5.3.18. *We have for $n \in \mathbb{Z}$ and $i = 0, 1, 2, 3$ that*

$$f_*(\mathcal{O}_{\mathbb{P}^1}(4n + i)) \cong \mathcal{O}_{\mathbb{P}^1}(n)^{\oplus i+1} \oplus \mathcal{O}_{\mathbb{P}^1}(n-1)^{3-i}. \tag{5.39}$$

Proof. Because f is a finite morphism between regular schemes we have that it is necessarily flat [GD65, Proposition 6.1.5]. Hence $f_*(\mathcal{O}_{\mathbb{P}^1}(4n+i))$ must be locally free. By Grothendieck's splitting theorem it must split as a direct sum of line bundles. The exact splitting can be found using that

$$\begin{aligned} \mathrm{Hom}_{\mathbb{P}^1}\left(f_*\left(\mathcal{O}_{\mathbb{P}^1}(4n+i)\right), \mathcal{O}_{\mathbb{P}^1}(a)\right) &\cong \mathrm{Hom}_{\mathbb{P}^1}\left(\mathcal{O}_{\mathbb{P}^1}(4n+i), f^*\left(\mathcal{O}_{\mathbb{P}^1}(a)\right)\right) \\ &\cong \mathrm{Hom}_{\mathbb{P}^1}\left(\mathcal{O}_{\mathbb{P}^1}(4n+i), \mathcal{O}_{\mathbb{P}^1}(4a)\right) \end{aligned}$$

which is $\max\{0, 4a - 4n - i\}$-dimensional. ☐

Lemma 5.3.19. *We have that*

$$\mathcal{F}_{Y/X} \coloneqq \mathrm{H}(\mathbb{P}^1/\mathbb{P}^1)_1 \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3}.$$

Proof. By Lemma 5.3.18 we have that $f_*(\mathcal{O}_{\mathbb{P}^1}) \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3}$. As the structure morphism $\mathcal{O}_{\mathbb{P}^1} \to f_*(\mathcal{O}_{\mathbb{P}^1})$ is nonzero we have that it is necessarily an isomorphism between $\mathcal{O}_{\mathbb{P}^1}$ and the summand $\mathcal{O}_{\mathbb{P}^1}$ of $f_*(\mathcal{O}_{\mathbb{P}^1})$. ☐

Lemma 5.3.20. *We have that*

$$\omega_{X/Y} \cong \mathcal{O}_{\mathbb{P}^1}(6).$$

Proof. By construction

$$\begin{aligned} f_*\omega_{X/Y} &\cong \mathcal{H}om(f_*\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}) \\ &\cong \mathcal{H}om(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3}, \mathcal{O}_{\mathbb{P}^1}) \\ &\cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3} \\ &\cong f_*(\mathcal{O}_{\mathbb{P}^1}(6)) \end{aligned}$$

where we applied Lemma 5.3.18 twice. The result now follows as $\mathcal{I}$ is an invertible sheaf and f_* is faithful (because f is affine). ☐

We can use these lemmas to obtain the following.

Proposition 5.3.21. *Let* $\mathcal{Q} := \mathrm{Sym}^2_{\mathbb{P}^1}\left(\mathcal{F}_{Y/X}/f_*(\omega_{X/Y}^{-1})\right)$ *as above.*

We have that

$$\mathcal{Q} \cong \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

Proof. We wish to prove that

i) the morphism $r\colon f_*(\omega_{X/Y}^{-1}) \to \mathrm{Sym}^2_{\mathbb{P}^1}(\mathrm{H}(\mathbb{P}^1/\mathbb{P}^1)_1)$ as in (5.31) is injective;

ii) the quotient $\mathcal{Q}$ is a locally free sheaf.

Assuming that these hold there is a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(-3) \to \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus 6} \to \mathcal{Q} \to 0 \tag{5.40}$$

where we used Lemma 5.3.19 to obtain $\mathrm{Sym}^2_{\mathbb{P}^1}(\mathcal{F}) \cong \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus 6}$ and Lemma 5.3.20 together with Lemma 5.3.18 to find $f_*(\omega_{X/Y}^{-1}) \cong f_*(\mathcal{O}_{\mathbb{P}^1}(-6)) \cong \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$. If $\mathcal{Q}$ is locally free, it splits as a sum of two line bundles $\mathcal{O}_{\mathbb{P}^1}(i) \oplus \mathcal{O}_{\mathbb{P}^1}(j)$ with $i \geq j \geq -2$. By [Har97, exercise II.5.16.d] we find

$$\mathcal{O}_{\mathbb{P}^1}(-12) \cong \mathcal{O}_{\mathbb{P}^1}(-9) \otimes \mathcal{O}_{\mathbb{P}^1}(i+j) \tag{5.41}$$

with unique solution $i = -1, j = -2$. It hence remains to prove that r is injective and that is cokernel is locally free.

The injectivity is checked locally. As in the proof of Lemma 5.3.15, r is locally given by

$$D \to {}_D D_C \otimes_C D_D : 1_D \mapsto \sum_{i=1}^{4} e_i \otimes f_i$$

where $D = (f_*\mathcal{O}_Y)_p$, $C = \mathcal{O}_{X,p}$ and e_1, e_2, e_3, e_4 and f_1, f_2, f_3, f_4 form dual bases for D as C-module. As $\mathrm{coker}(r) = (0, 1_D)\left((\Pi_C(D))_2\right)(0, 1_D)$ is a free C-module of rank 12 by Lemma 2.3.10, r must necessarily be injective.

That the cokernel of r is locally free can also be checked locally. Hence let $Q = \mathrm{Sym}^2_C(D/C)/\operatorname{im}(r)$ for a relative Frobenius pair D/C. A computation similar to the one carried out in §2.3 reduces to showing that $\dim_k(Q)$ does not depend on D in the case where C is an algebraically closed field k. This in turn follows from the equality

$$\dim_k(Q) = \dim_k(\mathrm{Sym}^2_k(D/k)) - \dim_k(D) = 6 - 4 = 2.$$

□

We can now conclude the description in the following corollary.

Corollary 5.3.22. *There exists an isomorphism*

$$\mathrm{H}(\mathbb{P}^1/\mathbb{P}^1) \cong \mathrm{C}\ell_{\mathrm{Sym}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(-2)\oplus\mathcal{O}_{\mathbb{P}^1}(-1))}\left(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3} \otimes \mathrm{Sym}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)), q\right)$$

We can moreover twist everything in the construction by the appropriate $\mathcal{O}_{\mathbb{P}^1}(i)$'s to find and equivalence of categories

$$\begin{gathered}\mathrm{qgr}\left(\mathrm{C}\ell_{\mathrm{Sym}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(-2)\oplus\mathcal{O}_{\mathbb{P}^1}(-1))}\left(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3} \otimes \mathrm{Sym}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)), q\right)\right) \\ \downarrow \cong \\ \mathrm{qgr}\left(\mathrm{C}\ell_{\mathrm{Sym}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1))}\left(\mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \otimes \mathrm{Sym}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)), q\right)\right)\end{gathered}$$

Remark 5.3.23. *The morphism* $q\colon \mathrm{Sym}^2(\mathcal{O}_{\mathbb{P}^1}^{\oplus 3}) \to \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ *in* (5.21) *is obtained from a surjective map* $q\colon \mathrm{Sym}^2(E) \to V$ *where* $E \coloneqq \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^{\oplus 3})$ *and* $V \coloneqq \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ *because these sheaves are generated by their global sections, and the dimensions of the* Hom*'s between the sheaves is the same as the dimension of the* Hom*'s between their global sections.*

5.4 Quaternion orders on $\mathbb{F}_1$ as Clifford algebras

We will now write the construction from Chapter 4 for the case giving rise to numerical type K_2 in (3.1) as a Clifford algebra. This allows us to compare it to the Clifford algebra obtained in §5.3, which is done in §5.5.

In §5.4.1 we quickly recall the construction of Chapter 4.

In §5.4.2 we show that up to a Zhang-twist the underlying Artin–Schelter regular algebras are in fact graded Clifford algebras associated to basepoint-free nets of conics.

In §5.4.3 we can then use the description as a graded Clifford algebra before blowing up to give a description as a Clifford algebra with values in a line bundle after blowing up, which allows us to write the associated abelian category in the same way as was obtained in §5.3.

This can be summarized in the following chain of equivalences of categories.

$$\begin{array}{cr}
\operatorname{coh}(p^*\mathcal{S}) & \\
\big\downarrow & \text{Lemma 5.4.8} \\
\operatorname{coh}\Big(p^*\big(\,\mathrm{C}\ell_{\mathbb{P}^2}(\mathrm{Sym}^2(E)\otimes_k \mathcal{O}_{\mathbb{P}^2}, q, \mathcal{O}_{\mathbb{P}^2}(1))_0\big)\Big) & \\
\big\downarrow & \S 5.4.3 \\
\operatorname{coh}\big(\mathrm{C}\ell_{\mathbb{F}_1}(\mathrm{Sym}^2(E)\otimes_k \mathcal{O}_{\mathbb{F}_1}, q, \mathcal{O}_{\mathbb{F}_1}(1))_0\big) & \\
\big\downarrow & \S 5.2.2 \\
\operatorname{qgr}\Big(\mathrm{C}\ell_{\mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1))}\big(E\otimes_k \mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1)), q\big)\Big) &
\end{array}$$

5.4.1 Construction of the surface as a blowup

In this section we quickly recapitulate the construction from Chapter 4. Recall that every quadratic 3-dimensional Artin–Schelter regular algebra which is finite over its center induces a maximal order $\mathcal{S}$ on $\mathbb{P}^2$ (§4.5.1), moreover the module category over this order is completely determined by the underlying algebra (Lemma 4.5.3). The points in $\mathbb{P}^2$ which do not lie on the ramificiation locus of this order, correspond to fat point modules over the underlying algebra and the multiplicities of these fat point modules are explicitely known (Proposition 4.3.3). We can then blow up such a point to obtain a maximal order $p^*\mathcal{S}$ on $\mathrm{Bl}_x\,\mathbb{P}^2$. Using a generalisation of Orlov's blowup formula (Theorem 4.5.11) we can show (Theorem 4.5.7) that the derived categoriy $\mathbf{D}^{\mathrm{b}}(p^*\mathcal{S})$ allows a full and strong exceptional collection

$$p^*\mathcal{S}_0, p^*\mathcal{S}_1, p^*\mathcal{S}_2, p^*\mathcal{F}$$

(where the $\mathcal{S}_i$ correspond to appropriate shifts of the underlying algebra and $\mathcal{F}$ corresponds to the fat point module) for which the Gram matrix takes the form

$$K'_m = \begin{bmatrix} 1 & 3 & 6 & m \\ 0 & 1 & 3 & m \\ 0 & 0 & 1 & m \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

One easily sees (Proposition 4.2.4) that such a matrix can be mutated to the matrix K_m appearing in (3.1). For the purpose of this paper we are only interested in the case $m = 2$, where we wish to show that this construction is equivalent to the construction using noncommutative $\mathbb{P}^1$-bundles.

5.4.2 Quaternionic noncommutative planes

We wish to understand all quadratic AS-regular algebras $A(C, \mathcal{L}, \sigma)$ for which the Gram matrix appearing in Theorem 4.5.7 takes the form K'_2. As we are only interested in the associated qgr-category, we want to understand these algebras up to

Zhang-twist. By Proposition 4.3.3 we know that we need only consider quadratic AS-regular algebras $A(C,\mathcal{L},\sigma)$ for which the order of $\sigma \in \mathrm{Aut}(C)$ is either 6 or 2. The following proposition simplifies matters even more:

Lemma 5.4.1. *Let $A = A(C,\sigma,\mathcal{L})$ be a an Artin–Schelter regular algebra such that the order of σ is 6. Then A is the Zhang twist of an Artin–Schelter regular algebra $A' = A(C,\mathcal{L},\sigma')$ where the order of σ' is 2.*

Proof. We have that $\sigma = \sigma^3 \circ \sigma^{-2}$, where σ^{-2} is of order 3. Moreover σ^{-2} commutes with σ, and it extends to an automorphism of the ambient $\mathbb{P}^2$ in which C is embedded using $\mathcal{L}$. Hence up to a Zhang twist we can also work with the elliptic triple $(C,\mathcal{L},\sigma^3)$. □

In particular we can, and will, assume that the order of the automorphism σ in $A = A(C,\sigma,\mathcal{L})$ has order 2. In this case we will say that $A(C,\sigma,\mathcal{L})$ (respectively $\mathrm{qgr}(A(C,\sigma,\mathcal{L}))$) is *quaternionic.*

We will now show that all quaternionic noncommutative planes can be written as a graded Clifford algebra, up to Zhang twist. This result does not appear as such in the literature, but it follows almost immediately by combining the classification of elliptic triples and the description of "Clifford quantum $\mathbb{P}^2$'s" in [SV07, corollary 4.8]. These are precisely the 3-dimensional graded Clifford algebras which are Artin–Schelter regular, i.e. which are associated to a basepoint-free net of conics. The following proposition can be seen as a converse to this result. (The condition that the characteristic of k is not 3 is important here.)

Proposition 5.4.2. *Let $A = A(C,\sigma,\mathcal{L})$ be a quaternionic Artin–Schelter regular algebra. Then A is the Zhang twist of a graded Clifford algebra.*

Proof. Recall from [Sie11, Theorem 1.2] that A is a Zhang twist of A' if and only if their associated $\mathbb{Z}$-algebras $\check{A}$ and $\check{A}'$ are isomorphic. Now $\check{A} = A(C,\mathcal{L},\sigma^*\mathcal{L})$ is a quadratic Artin–Schelter regular $\mathbb{Z}$-algebra in the sense of §0.3. It is shown in [VdB11, Theorem 3.5] (based upon the results in [ATVdB91]) that there is a morphism of algebraic groups $\eta\colon \mathrm{Pic}_0\, C \to \mathrm{Aut}\, C$ where $\mathrm{Pic}_0\, C$ consists of all line bundles having degree 0 on each component of C. Moreover we have by [VdB11, Theorem 4.2.2] that every quadratic Artin–Schelter regular $\mathbb{Z}$-algebra $A(C,\mathcal{L}_0,\mathcal{L}_1)$ is of the form $\check{A}'$ where $A' = A(C,\mathcal{L}_0,\sigma')$ and $\sigma' \in \mathrm{im}(\eta)$. In particular A is a Zhang twist of a quadratic Artin–Schelter regular algebra A' whose associated C-automorphism lies in $\mathrm{im}(\eta)$ (It is customary to refer to such a quadratic Artin–Schelter regular algebra $A = A(C,\mathcal{L},\sigma)$ as a *translation algebra.*). We claim that A' is a graded Clifford algebra.

A closer investigation of [ATVdB91, §5] shows that for every $\mathcal{G} \in \operatorname{Pic}_0 C$ we have that $\eta(\mathcal{G})$ is uniquely defined by the following property:

$$\mathcal{O}_C(\eta(\mathcal{G})(p)) \cong \mathcal{G}(p) \tag{5.42}$$

if p is a nonsingular point of C and

$$\eta(\mathcal{G})(p) = p \tag{5.43}$$

if p is a singular point of C. In particular if all points of C are singular (which only happens when C is a triple line), the only finite order automorphism in $\operatorname{im}(\eta)$ is id_C. However in this case A' does not allow fat point modules. This is a contradiction, because A allows fat point modules of multiplicity 2 and the existence (and multiplicity) of fat point modules is an invariant of the category $\operatorname{QGr}(A)$ and by construction there are equivalences of categories

$$\operatorname{QGr}(A) \cong \operatorname{QGr}(\check{A}) \cong \operatorname{QGr}(\check{A}') \cong \operatorname{QGr}(A').$$

Hence we can assume that C contains a nonsingular point p. Now assume that $\eta(\mathcal{G})^n = \mathrm{id}_C$, then by (5.42) we find

$$\mathcal{O}_C(p) \cong \mathcal{G}^{\otimes n}(p) \tag{5.44}$$

and hence

$$\mathcal{O}_C \cong \mathcal{G}^{\otimes n}. \tag{5.45}$$

This implies that η preserves the order of an element in $\operatorname{Pic}_0 C$. By [ATVdB90, Theorem 2.ii] we know that $\mathcal{L}^{-1} \otimes \sigma^* \mathcal{L}$ has order 2 in $\operatorname{Pic}_0 C$, and hence $\sigma' = \eta(\mathcal{L}^{-1} \otimes \sigma^* \mathcal{L})$ has order 2 as well. By the description of $\operatorname{Pic}_0 C$ as e.g. in [BP94, Table 1], we see that $\operatorname{Pic}_0 C$ only admits order 2 elements when C is an elliptic curve, a nodal cubic, the union of a conic and a line in general position or a triangle of lines.

In the case of an elliptic curve, A' is a quaternionic Sklyanin algebra and it is shown in Example 5.4.5 that such an A' is a graded Clifford algebra. For the other 3 options for the point scheme C there is a unique order 2 element in $\operatorname{Pic}_0 C$ and the associated algebra A' is the quotient of $k\langle x, y, z\rangle$ by the relations

$$\begin{cases} xy + yx = 0 \\ yz + zy = c_1 x^2 \\ zx + xz = c_2 y^2 \end{cases}, \tag{5.46}$$

where $(c_1, c_2) = (1,1), (0,1), (0,0)$ for the 3 cases respectively. These algebras are graded Clifford algebras by Example 5.4.6. □

Remark 5.4.3. *The condition that σ is of order 2 gives a restriction on the curves which can appear in a regular triple: out of 9 possible point schemes appearing in [BP94, Table 1], only 4 remain. These are (see also Remarks 4.3.6 and 4.5.2 and Figure 4.1)*

- *the elliptic curves,*
- *the nodal cubic,*
- *a conic and line in general position,*
- *a triangle of lines.*

Remark 5.4.4. *The algebra A' constructed in the above proposition is a graded k-algebra with symmetric relations. As such Lemma 5.3.2 and Proposition 5.3.3 show the existence of an epimorphism $\varphi: A' \to \mathrm{C}\ell_k(E,q)$. Both A' and the Clifford algebra $\mathrm{C}\ell_k(E,q)$ are graded k-algebras with Hilbert series $1/(1-t)^3$, hence φ is an isomorphism. This gives an alternative proof for showing that A' is a Clifford algebra, but it does not highlight the fact that $\mathrm{C}\ell_k(E,q)$ is a graded Clifford algebra.*

Recall from §5.2.3 that every graded Clifford algebra $\mathrm{C}\ell((M_i)_{i=1}^n)$ is constructed with respect to n matrices $M_i \in \mathrm{Mat}_n(k)$ and that we can interpret these matrices as quadric hypersurfaces in $\mathbb{P}^{n-1}$. Moreover one easily sees that changing the basis of this linear system of hypersurface corresponds to changing the generators y_i for the graded Clifford algebra. In particular, up to isomorphism, the Clifford algebra only depends on the linear system and not on the choice of the M_i. As we only encounter graded Clifford algebra $\mathrm{C}\ell((M_i)_{i=1}^n)$ where $n=3$, we are hence particularly interested in nets of conics. Moreover by Proposition 5.2.9 we are only interested in basepoint-free nets of conics. (Note that Proposition 5.4.2 can be seen as a converse to Proposition 5.2.9.)

Below we give a few examples of such nets of conics, for a more thorough introduction on nets of conics, we refer to §5.6.

Example 5.4.5 (Quaternionic Sklyanin algebras)**.** *3-dimensional Sklyanin algebras associated to points of order 2 are an interesting class of graded Clifford algebras. As in Remark 4.6.6, such an algebra has a presentation of the form*

$$\begin{cases} xy + yx = cz^2 \\ yz + zy = cx^2 \\ zx + xz = cy^2 \end{cases},$$

where $c \neq 0$, $c^3 \neq 8$ and $c^3 \neq -1$.

If we identify $x = x_1$, $y = x_2$ and $z = x_3$ these algebras are Clifford algebras associated to the net of conics given by

$$M_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & c \\ 0 & c & 0 \end{bmatrix}, \; M_2 = \begin{bmatrix} 0 & 0 & c \\ 0 & 2 & 0 \\ c & 0 & 0 \end{bmatrix}, \; M_3 = \begin{bmatrix} 0 & c & 0 \\ c & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

or equivalently the symmetric matrix

$$M = \begin{bmatrix} 2x^2 & cz^2 & cy^2 \\ cz^2 & 2y^2 & cx^2 \\ cy^2 & cx^2 & 2z^2 \end{bmatrix}$$

(I.e. the net is spanned by the conics $x^2 + cyz, y^2 + cxz, z^2 + cxy$)
In this case the central elements y_i are x^2, y^2 and z^2 which are in turn the norms of the elements x, y, z in degree 1. Computing the determinant of this matrix to describe the point modules we get the discriminant curve

$$(2c^3 + 8)x^2y^2z^2 - 2c^2(x^6 + y^6 + z^6).$$

Example 5.4.6 (Special quaternionic algebras)**.** *We have seen in Proposition 5.4.2 that there are three other isomorphism classes of graded Clifford algebras which are relevant to us, besides the generic case of a quaternionic Sklyanin algebra. These are described in , and their corresponding nets of conics are given by*

$$M^{\mathrm{B}} = \begin{bmatrix} 2x^2 & 0 & y^2 \\ 0 & 2y & x^2 \\ y^2 & x^2 & 2z^2 \end{bmatrix},$$
$$M^{\mathrm{D}} = \begin{bmatrix} 2x^2 & 0 & 0 \\ 0 & 2y^2 & x^2 \\ 0 & x^2 & 2z^2 \end{bmatrix},$$
$$M^{\mathrm{E}} = \begin{bmatrix} 2x^2 & 0 & 0 \\ 0 & 2y^2 & 0 \\ 0 & 0 & 2z^2 \end{bmatrix}$$

where B, D and E refer to the classification of Table 5.2 in §5.6. The particular choice of basis is made to be compatible with [SV07, corollary 4.8]. The nets of conics are in turn given by

$$N^{\mathrm{B}} = \langle x^2 + yz, y^2 + xz, z^2 \rangle,$$
$$N^{\mathrm{D}} = \langle x^2 + yz, y^2, z^2 \rangle,$$
$$N^{\mathrm{E}} = \langle x^2, y^2, z^2 \rangle.$$

Example 5.4.7. *Not every quaternionic Artin–Schelter regular algebra is a graded Clifford algebra on the nose. By the proof of Proposition 5.4.2 we can also say that not every Artin–Schelter algebra is a translation algebra. As an example we can consider*

$$k\langle x,y,z\rangle/(xz-zx, yz-zy, xy+yx).$$

The point scheme C is the triangle of lines defined by xyz, and the automorphism is given by rescaling by -1 in one component, and exchanging the two others. In a translation algebra the automorphism needs to preserve the components and rescale them in the same way. The prescribed Zhang twist in this case is induced from the automorphism of the degree 1 part which has $z \mapsto -z$, and we obtain the Clifford algebra associated to the net N^{E} in (5.4.6).

5.4.3 Blowing up Clifford algebras

Fix an AS-regular algebra $A(C,\mathcal{L},\sigma)$ and a point $x \in \mathbb{P}^2$ for which the corresponding order $p^*\mathcal{S}$ on $\mathrm{Bl}_x \mathbb{P}^2$ is of numerical type K_2 as in (3.1). Using Proposition 5.4.2 we get that (up to Zhang twist, which is invisible in qgr) we can write $A(C,\mathcal{L},\sigma)$ as a graded Clifford algebra $\mathrm{C}\ell((M_i)_{i=1}^3)$. As was noted in §5.2.3, the matrices M_1, M_2, M_3 define a net of conics in $\mathbb{P}^2_{x_1,x_2,x_3}$. As in Examples 5.4.5 and 5.4.6 one can read of these nets of conics from the entries of the M_i. Conversely, as in Remark 5.2.10 we can write

$$\mathrm{C}\ell((M_i)_{i=1}^3) \cong \mathrm{C}\ell_{k[y_1,y_2,y_3]}(E \otimes_k k[y_1,y_2,y_3], q)$$

by identifying

$$E = kx_1 \oplus kx_2 \oplus kx_3$$

and by $k[y_1,y_2,y_3]$-linearly extending

$$q : \mathrm{Sym}_k^2 E \to ky_1 \oplus ky_2 \oplus ky_3 : x_i x_j + x_j x_i \mapsto \sum_{m=1}^{3} (M_m)_{i,j} y_m \tag{5.47}$$

Let $q^\vee : ky_1^\vee \oplus ky_2^\vee \oplus ky_3^\vee \to \mathrm{Sym}_k^2 E^\vee$ be the dual map. Then $q^\vee(y_m^\vee) = \sum_{1=i\leq j}^3 (M_m)_{i,j}$, i.e. the net of conics in $\mathbb{P}(E)$, which was defined via the entries of M_i above, is in fact spanned by $q^\vee(y_1^\vee), q^\vee(y_2^\vee), q^\vee(y_3^\vee)$. In particular, it makes sense to consider *the net of conics defined by q* whenever E and V are 3-dimensional vector spaces and $q : \mathrm{Sym}_k^2 E \to V$ is a surjective map. We will do so for the remainder of this chapter.

Lemma 5.4.8. *Let $A(C,\mathcal{L},\sigma), \mathcal{S}, E$ and q be as above. There exists an isomorphism*

$$\mathcal{S} \cong \mathrm{C}\ell_{\mathbb{P}^2}(E \otimes_k \mathcal{O}_{\mathbb{P}^2}, q, \mathcal{O}_{\mathbb{P}^2}(1))_0. \tag{5.48}$$

Proof. This follows from the above description and Example 5.2.12. ☐

In particular we also have that

$$p^*\mathcal{S} \cong p^* \mathrm{C}\ell_{\mathbb{P}^2}(E \otimes_k \mathcal{O}_{\mathbb{P}^2}, q, \mathcal{O}_{\mathbb{P}^2}(1))_0.$$

By functoriality of the Clifford algebra construction we then get the following description.

Lemma 5.4.9. *There exists an isomorphism*

$$p^*\mathcal{S} \cong \mathrm{C}\ell_{\mathrm{Bl}_x \mathbb{P}^2}(E \otimes_k \mathcal{O}_{\mathrm{Bl}_x \mathbb{P}^2}, q, \mathcal{O}_{\mathrm{Bl}_x \mathbb{P}^2}(1))_0.$$

Proof. This follows from the isomorphism

$$p^* \mathrm{C}\ell_{\mathbb{P}^2}(E \otimes_k \mathcal{O}_{\mathbb{P}^2}, q, \mathcal{O}_{\mathbb{P}^2}(1))_0 \cong \mathrm{C}\ell_{\mathrm{Bl}_x \mathbb{P}^2}(E \otimes_k \mathcal{O}_{\mathrm{Bl}_x \mathbb{P}^2}, q, \mathcal{O}_{\mathrm{Bl}_x \mathbb{P}^2}(1))_0.$$

(which itself is a consequence of the fact that Clifford algebras are compatible with base change) ☐

Then by Proposition 5.2.11 we get the following description of the Clifford algebra, which puts the category of coherent sheaves over $p^*\mathcal{S}$ on the same footing as that of the noncommutative $\mathbb{P}^1$-bundle as in Corollary 5.3.22. This will allow us to compare the two constructions in §5.5.

Corollary 5.4.10. *There exists exists an equivalence of categories*

$$\operatorname{coh}(p^*\mathcal{S}) \cong \operatorname{qgr}_{\mathbb{P}^1} \mathrm{C}\ell_{\mathrm{Sym}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))}(E \otimes_k \mathrm{Sym}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)), q). \tag{5.49}$$

Proof. In (5.1) we have seen the classical isomorphism

$$\mathrm{Bl}_x \mathbb{P}^2 \cong \underline{\mathrm{Proj}}\, \mathrm{Sym}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)).$$

We apply Proposition 5.2.11 to the morphism $\pi\colon \mathrm{Bl}_x \mathbb{P}^2 \cong \mathbb{F}_1 \to \mathbb{P}^1$, where

$$\mathcal{E} \coloneqq E \otimes_k \mathcal{O}_{\mathbb{P}^2},$$
$$\mathcal{L} \coloneqq p^*(\mathcal{O}_{\mathbb{P}^2}(1)),$$

and therefore

$$\mathcal{A} \cong \mathrm{Sym}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$$

using the identification

$$\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) = \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \tag{5.50}$$

In this situation we have that $p^*(\mathcal{O}_{\mathbb{P}^2}(1))$ corresponds to the shift by 1 in the sheaf of graded algebras $\mathrm{Sym}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ on $\mathbb{P}^1$. The result follows. ☐

5.5 Comparing the two constructions

By §5.3 and §5.4 the categories $\mathrm{qgr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1}(1))_{\mathrm{id}}))$ and $\mathrm{coh}(p^*\mathcal{S})$ are equivalent to a category of the form

$$\mathrm{qgr}\Big(\mathrm{C}\ell_{\mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1))}\big(E\otimes_k \mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1)),q\big)\Big) \tag{5.51}$$

where q is obtained from a surjective map $q' : \mathrm{Sym}^2(E) \twoheadrightarrow V$. Moreover the identification $V = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1))$ provides us with a map $\xi : V \twoheadrightarrow \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}) = k$ which is unique up to scalar multiple. Conversely, in order to define q, we need to specify both q' and ξ. Hence the category (5.51) is completely defined by (E, V, q', ξ).

In what follows we will often write q for q'. Hence, whenever we interpret q as a map of sheaves, we are actually combining q with an extra piece of information given by ξ.

The goal of this section is to compare the quadruples appearing in both constructions

5.5.1 Categorical comparison using algebraic quadruples

Recall from the first paragraph in §5.4.3 that the data E, V, q defines a net of conics $V^\vee$ (or rather $\mathbb{P}(V^\vee)$) in $\mathbb{P}(E) \cong \mathbb{P}^2$. Similarly the map $\xi : V \twoheadrightarrow k$ induces an algebra morphism $\mathrm{Sym}_k(V) \twoheadrightarrow \mathrm{Sym}_k(k)$ and hence defines a point $x \in \mathbb{P}(V^\vee)$ (i.e. it defines a conic in the net). Inspired by these fact, we make the following definition:

Definition 5.5.1. A *pre-algebraic quadruple* is a tuple (E, V, q, ξ), where E and V are vector spaces of dimension 3, $q\colon \mathrm{Sym}^2 E \to V$ is a surjective morphism and $\xi\colon V \to k$ is a non-zero morphism. If moreover q defines a basepoint-free net of conics and ξ defines a nonsingular conic in this net, the quadruple is called an *algebraic quadruple*.

The following proposition shows that algebraic quadruples are closely related to the quaternionic orders $\mathrm{coh}(p^*\mathcal{S})$ on $\mathbb{F}_1$:

Theorem 5.5.2. *Let $\mathrm{coh}(p^*\mathcal{S})$ be a quaternionic order on $\mathbb{F}_1$ associated to an AS-regular algebra $A(C,\mathcal{L},\sigma)$ and a blowup morphism $p : \mathbb{F}_1 \to \mathbb{P}^2$. Let (E, V, q, ξ) be the associated pre-algebraic quadruple as in §5.4.3. Then (E, V, q, ξ) is an algebraic quadruple. Conversely, every algebraic quadruple can be obtained from a quaternionic order on $\mathbb{F}_1$.*

Proof. It follows immediately from Proposition 5.2.9 that q defines a basepoint-free net of conics. Now recall that by construction $p : \mathbb{F}_1 \to \mathbb{P}^2$ is a blowup morphism and ξ was constructed as a nonzero morphism $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1)) \to \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1})$. It is then a standard exercise in commutative algebraic geometry to check that the isomorphism $\underline{\mathrm{Proj}}_{\mathbb{P}^1}(\mathcal{O}\oplus\mathcal{O}(1)) \cong \mathrm{Bl}_x \mathbb{P}^2$ in (5.1) identifies the point in $\mathbb{P}^2$ defined by ξ with the point blown up by the map p. It hence suffices to check that all points on $\mathbb{P}(V^\vee) \cong \mathbb{P}^2$ which do not lie on the ramification curve of $\mathcal{S}$ correspond to nonsingular conics in the net $V^\vee$. This in turn follows from [LB95, Proposition 8].

Conversely if (E, V, q, ξ) is an algebraic quadruple, then it defines a Clifford algebra $\mathrm{C}\ell_{\mathrm{Sym}_k(V)}(E \otimes_k \mathrm{Sym}_k(V), q)$ which is AS-regular by Proposition 5.2.9. In particular it defines an order $\mathcal{S}$ on $\mathbb{P}(V^\vee)$ and again using [LB95, Proposition 8] the point defined by ξ does not lie on the ramification curve of this order. □

Our goal is to show that a similar result holds for noncommutative $\mathbb{P}^1$-bundles, i.e. we want to prove the following

Theorem 5.5.3. *Let* $\mathrm{qgr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}))$ *be a noncommutative* $\mathbb{P}^1$*-bundle associated to a degree 4 morphism* $f : \mathbb{P}^1 \to \mathbb{P}^1$ *as in Chapter 3 and let* (E, V, q, ξ) *be the associated quadruple as in §5.3. Then* (E, V, q, ξ) *is an algebraic quadruple. Moreover every algebraic quadruple can be obtained in this way.*

The proof of Theorem 5.5.3 is considerably more advanced than the one for Theorem 5.5.2. We first highlight the importance of these theorems:

Corollary 5.5.4. *Every noncommutative* $\mathbb{P}^1$*-bundle* $\mathrm{qgr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}))$ *is equivalent to a quaternionic order* $\mathrm{coh}(p^*\mathcal{S})$ *and vice versa.*

Proof of Corollary 5.5.4. Using the techniques in §5.3 and §5.4 we know that all noncommutative $\mathbb{P}^1$-bundles $\mathrm{qgr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}))$, as well as all quaternionic orders $\mathrm{coh}(p^*\mathcal{S})$ are equivalent to a category of the form

$$\mathrm{qgr}\,\mathrm{C}\ell_{\mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))}\big(E \otimes_k \mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)), q\big) \tag{5.52}$$

This category is completely determined by a pre-algebraic quadruple (E, V, q, ξ). The result now follows as Theorem 5.5.3 and Theorem 5.5.2 show that all quadruples we can, and will, encounter in (5.52) starting from either one of the two models are the algebraic quadruples. □

We will now devote the rest of this section to proving Theorem 5.5.3. In §5.5.2 we prove the first statement from Theorem 5.5.3 (noncommutative $\mathbb{P}^1$-bundles give rise to algebraic quadruples). The proof of the second statement (every algebraic quadruple can be obtained) is based on some computations for a graded Frobenius algebra and will be covered in §5.5.3.

5.5.2 $\mathbb{P}^1$-bundles and basepoint-free nets of conics

We quickly recall some results from §5.3. Starting from a degree 4 morphism $f: Y \to X$ with $X \cong Y \cong \mathbb{P}^1$, one constructs a quadruple as follows:

$$\begin{aligned} \mathcal{F}_{Y/X} &:= f_*\mathcal{O}_{\mathbb{P}^1}/\mathcal{O}_{\mathbb{P}^1} \\ \mathcal{Q}_{Y/X} &:= \operatorname{coker}(\omega_{Y/X}^{-1} \to \operatorname{Sym}^2 \mathcal{F}_{Y/X}) \\ E_{Y/X} &:= \mathrm{H}^0(X, \mathcal{F}_{Y/X}(1)) \\ V_{Y/X} &:= \mathrm{H}^0(X, \mathcal{Q}_{Y/X}(2)). \end{aligned} \tag{5.53}$$

Using Lemma 5.3.19 and Proposition 5.3.21 we see that

$$\dim_k E_{Y/X} = \dim_k V_{Y/X} = 3,$$

and the quotient morphism $\operatorname{Sym}^2 \mathcal{F}_{Y/X} \twoheadrightarrow \mathcal{Q}_{Y/X}$ defines a net of conics

$$q_{Y/X} \colon \operatorname{Sym}^2 E_{Y/X} \longrightarrow\!\!\!\!\!\rightarrow V_{Y/X} \tag{5.54}$$

because the twist in the definition of $E_{Y/X}$ and $V_{Y/X}$ makes the higher sheaf cohomology vanish in the definition of $\mathcal{Q}_{Y/X}$.

Recall from §5.3.5 that

$$\mathcal{F}_{Y/X}(1) = \mathcal{O}_{\mathbb{P}^1}^{\oplus 3},\ f_*\omega_{Y/X}^{-1}(2) = \mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \text{ and } \mathcal{Q}_{Y/X}(2) \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1).$$

In particular there is a nonzero morphism

$$\chi_{Y/X} \colon \mathcal{Q}_{Y/X}(2) \longrightarrow \mathcal{O}_{\mathbb{P}^1} \tag{5.55}$$

which is unique up to scalar multiple. Let $\xi_{Y/X}$ be the induced morphism at the level of global sections. I.e. this is a surjective morphism

$$\xi_{Y/X} \colon V \longrightarrow\!\!\!\!\!\rightarrow k \tag{5.56}$$

Similarly there is a nonzero morphism

$$\theta \colon f_*\omega_{Y/X}^{-1}(2) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \tag{5.57}$$

which is unique up to scalar multiple.

If x is not a branch point of f, then $f^{-1}(x) = \{y_1, y_2, y_3, y_4\}$ and

$$(f_*\omega_{Y/X}^{-1}(2))_x \otimes k(x) \cong (f_*\mathcal{O}_Y)_x \otimes k(x) \cong k(y_1) \oplus k(y_2) \oplus k(y_3) \oplus k(y_4) \tag{5.58}$$

Lemma 5.5.5. *For a generic $x \in X$ the map $\theta_x \otimes k(x)$ does not vanish on any of the $k(y_i)$ in (5.58).*

Proof. As in §3.3.1 we can restrict to an affine open $\mathrm{Spec}(C) \subset X$ such that

$$f^{-1}(\mathrm{Spec}(C)) = \mathrm{Spec}(D),$$

with D/C is relative Frobenius of rank 4 and

$$f_*\omega_{Y/X}^{-1}(2)\big|_{\mathrm{Spec}(C)} \cong f_*(\mathcal{O}_D)$$

In particular θ is given by a morphism $D \to C$. Moreover as f has only finitely many branch points, we can choose C such that $f^{-1}(x)$ consists of 4 points y_1, y_2, y_3, y_4 for all $x \in \mathrm{Spec}(C)$. We will now prove that the lemma holds for all but a finite number of points of $\mathrm{Spec}(C)$.

Note that if x is any point in $\mathrm{Spec}(C)$, we can use Lemma 2.3.1 to conclude that

$$(f_*\mathcal{O}_Y)_x \otimes k(x) \cong D \otimes_C k(x) \cong k(y_1) \oplus k(y_2) \oplus k(y_3) \oplus k(y_4)$$

is relative Frobenius of rank 4 over $k(x)$.

This implies that $k(y_i) \cong k(x)$ for $i = 1,2,3,4$. Let $e_{x,1}, e_{x,2}, e_{x,3}, e_{x,4}$ denote the idempotents in

$$k(y_1) \oplus k(y_2) \oplus k(y_3) \oplus k(y_4) \cong k(x)^{\oplus 4}$$

We now need to prove (for generic x) that $(\theta_x \otimes k(x))(e_{x,i}) \neq 0$. For this consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_C(D,C) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}_{k(x)}(D \otimes_C k(x), k(x)) \arrow[rr, "\mathrm{eval}(e_{x,i})"] \arrow[d, "\cong"] & & k(x) \arrow[d, "\cong"] \\
D \arrow[r, "- \otimes_C k(x)"] & D \otimes_C k(x) \arrow[r, "\cong"'] & k(x)^{\oplus 4} \arrow[r, "\pi_i"'] & k(x) \cong D_{y_i} \otimes k(y_i)
\end{tikzcd} \tag{5.59}$$

Commutativity of the left square follows from the fact that the construction of relative Frobenius pairs is compatible with base change. Commutativity of the right square follows from the fact that the middle vertical isomorphism (or rather its inverse) is given by

$$k(x)^{\oplus 4} \xrightarrow{\cong} \mathrm{Hom}_{k(x)}(k(x)^{\oplus 4}, k(x)) : (a_1, a_2, a_3, a_4) \mapsto ((b_1, b_2, b_3, b_4) \mapsto \sum_{i=1}^{4} a_i b_i)$$

Now recall that every element in $d \in D$ defines a function on $\mathrm{Spec}(D)$ by sending $y_i \in \mathrm{Spec}(D)$ to the image of d under the bottom horizontal composition in (5.59). In particular $\theta \in \mathrm{Hom}_C(D,C)$ defines an element in D through the left vertical isomorphism in (5.59) and hence function $\tilde{\theta}$ on $\mathrm{Spec}(D)$. By construction we have $\tilde{\theta}(y_i) \neq 0 \iff (\theta_x \otimes k(x))(e_{x,i}) \neq 0$. The lemma now follows by noticing that $\tilde{\theta}$ can only have finitely many zeroes on $\mathrm{Spec}(D)$. □

Proof of first statement in Theorem 5.5.3: We need to show that $q_{Y/X}$ (as in (5.54)) defines a basepoint-free net of conics. Recall that $q_{Y/X}$ was constructed via the following short exact sequence

$$0 \to \mathrm{H}^0(X, f_*\omega_{Y/X}^{-1}(2)) \xrightarrow{i_{Y/X}} \mathrm{Sym}^2(\mathrm{H}^0(X, \mathcal{F}_{Y/X}(1))) \xrightarrow{q_{Y/X}} \mathrm{H}^0(X, \mathcal{Q}_{Y/X}(2)) \to 0$$

In particular it suffices to prove that

$$\mathrm{im}(i_{Y/X}) \cap \left\{e \cdot e \mid e \in \mathrm{H}^0(X, \mathcal{F}_{Y/X}(1))\right\} = \{0\} \tag{5.60}$$

To see this, note that $\mathrm{im}(i_{Y/X}) = \ker(q_{Y/X})$ and

$$\begin{aligned}\ker(q_{Y/X})) &= \left\{\sum_{1=i\leq j}^{3} \beta_{i,j}x_ix_j \mid q_{Y/X}\left(\sum_{1=i\leq j}^{3} \beta_{i,j}x_ix_j\right) = 0\right\} \\ &= \left\{\sum_{1=i\leq j}^{3} \beta_{i,j}x_ix_j \mid v_n^\vee([\beta_{i,j}]_{i,j}) = 0 \text{ for } n = 1,2,3\right\}\end{aligned}$$

Where $v_1^\vee, v_2^\vee, v_3^\vee$ span the net $V_{Y/X}^\vee$ and are interpreted as functions on $\mathbb{P}(\mathrm{Sym}^2 E_{Y/X}) \cong \mathbb{P}^5$ via $q^\vee : V_{Y/X}^\vee \to (\mathrm{Sym}^2 E_{Y/X})^\vee$.

The condition that the net spanned by $v_1^\vee, v_2^\vee, v_3^\vee$ is basepoint-free is equivalent to requiring that the points $[\beta_{1,1} : \beta_{2,2} : \beta_{3,3} : \beta_{1,2} : \beta_{1,3} : \beta_{2,3}]$ for which $\sum_{1=i\leq j}^{3} \beta_{i,j}x_ix_j \in \ker(q)$, do not lie in the image of the Veronese embedding $\mathbb{P}(E_{Y/X}) \cong \mathbb{P}^2 \hookrightarrow \mathbb{P}(\mathrm{Sym}^2 E_{Y/X})$. And finally the latter is equivalent to $\sum_{1=i\leq j}^{3} \beta_{i,j}x_ix_j$ not lying in the image of

$$E_{Y/X} \longrightarrow \mathrm{Sym}^2 E_{Y/X} : \sum_{i=1}^{3} \gamma_i x_i \mapsto (\gamma_i x_i)^2 = \sum_{1=i\leq j}^{3} \gamma_i\gamma_j x_i x_j$$

In particular in order to prove the proposition, it suffices to prove (5.60).

Let θ be as in (5.57) and let $j_{Y/X} : \ker(\theta) \to f_*\omega_{Y/X}^{-1}(2) \to \mathrm{Sym}^2(\mathcal{F}_{Y/X}(1))$ be the induced composition. Then, using the fact that $\mathcal{O}_{\mathbb{P}^1}(-1)$ has no global sections, we see that $\mathrm{H}^0(X, j_{Y/X}) = i_{Y/X}$. Moreover as both $\ker(\theta)$ and $\mathrm{Sym}^2(\mathcal{F}_{Y/X}(1))$ are given by direct sums of copies of the structure sheaf we have for each point $x \in X$:

$$j_{Y/X,x} \otimes_{\mathcal{O}_{X,x}} k(x) = \mathrm{H}^0(X, j_{Y/X}) \otimes_k k(x) = i_{Y/X} \otimes_k k(x)$$

and

$$\{(e \cdot e) \otimes_k 1_{k(x)} \mid e \in \mathrm{H}^0(X, \mathcal{F}(1))\} = \{e_x \cdot e_x \mid e_x \in \mathcal{F}_x \otimes k(x)\}$$

As such it suffices to prove that

$$\mathrm{im}\left(j_{Y/X,x} \otimes_{\mathcal{O}_{X,x}} k(x)\right) \cap \{(e_x \cdot e_x) \otimes_{\mathcal{O}_{X,p}} 1_{k(x)} \mid e_x \in \mathcal{F}_x\} = \{0\}$$

holds for some point $x \in X$. For this let x be a generic point as in Lemma 5.5.5

Then as in the proof of Lemma 5.3.15 (using the fact that the e_i form a self-dual basis (Remark 5.3.4)) we see that $(j_{Y/X})_x \otimes_{\mathcal{O}_{X,x}} k(x)$ is given by composing the inclusion $\ker(\theta_x) \otimes_{\mathcal{O}_{X,x}} k(x) \subset k(x)^{\oplus 4}$ with

$$k(x)^{\oplus 4} \to \mathrm{Sym}^2(k(x)^{\oplus 4}) \to \mathrm{Sym}^2((k(x)^{\oplus 4})/k(x)) : e_i \mapsto e_i \cdot e_i$$

Note that $\sum_{i,j=1,i\leq j}^{4} a_{i,j} \frac{e_i \cdot e_j + e_j \cdot e_i}{2}$ is of the form $e \cdot e$ if and only if $\alpha_{i,i}\alpha_{j,j} = 4\alpha_{i,j}^2$ holds for all $i < j$. Such an element lies in the image of $(j_{Y/X})_x \otimes_{\mathcal{O}_{X,x}} k(x)$ if and only if there is 1 i such that $\alpha_{j,k} = 0$ whenever $(j,k) \neq (i,i)$. Hence it suffices to show that $\ker(\theta_x) \otimes_{\mathcal{O}_{X,x}} k(x)$ does not contain an element of the form λe_i. This is however guaranteed by the fact that $\theta_x \otimes_{\mathcal{O}_{X,x}} k(x)$ does not vanish on any of the $k(y_i)$ in (5.58).

We now need to show that the point $x \in \mathbb{P}(V_{Y/X}^\vee)$ corresponding to $\xi_{Y/X}$ is a nonsingular conic. Assume on the contrary that x is contained in the discriminant of the net. As we have already proved that $q : \mathrm{Sym}^2 E_{Y/X} \twoheadrightarrow V_{Y/X}$ defines a basepoint-free net of conics, it induces a maximal order $\mathcal{S}$ on $\mathbb{P}(V_{Y/X}^\vee)$. Then blowing up at x yields a sheaf of algebras on $\mathrm{Bl}_x \mathbb{P}^2$ which is not of finite global dimension (and which is not a maximal order), using the local computation from Remark 4.5.15. But this contradicts Corollary 3.4.30, as these two categories are equivalent. □

5.5.3 Branched coverings and nets of conics

The goal of this section is to prove the second statement in Theorem 5.5.3, i.e. to prove that every algebraic quadruple comes from a noncommutative $\mathbb{P}^1$-bundle $\mathrm{qgr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}))$. For this we first give a procedure which produces degree 4 morphisms $f : \mathbb{P}^1 \to \mathbb{P}^1$ out of an algebraic quadruple. Hence fix some algebraic quadruple (E, V, q, ξ).

Consider

$$Z := \mathbb{P}(E) \cong \mathbb{P}^2 \text{ and } T := \mathbb{P}(V) \cong \mathbb{P}^2 \tag{5.61}$$

From $q\colon \mathrm{Sym}^2 E \twoheadrightarrow V$ we get a corresponding morphism $g\colon Z \to T$.

Lemma 5.5.6. *The morphism $g\colon Z \to T$ sends a point to the pencil of conics for which it is a basepoint.*

Proof. Note that every element in $\mathrm{Sym}^2 E^\vee$ (or rather $(\mathrm{Sym}^2 E^\vee)/k^*$) defines a conic in $\mathbb{P}(E)$. In particular $q^\vee\colon V^\vee \hookrightarrow \mathrm{Sym}^2 E^\vee$ identifies $V^\vee$ (or rather $V^\vee/k^*$) as a net of conics. More concretely, fix bases x_1, x_2, x_3 and v_1, v_2, v_3 for E and V respectively and write $q(x_i x_j) = \alpha_{1,i,j} v_1 + \alpha_{2,i,j} v_2 + \alpha_{3,i,j} v_3$. Then $V^\vee$ is the net of conics spanned by

$$v_1^\vee = \sum_{1=i\leq j}^{3} \alpha_{1,i,j} x_i^\vee x_j^\vee,\ v_2^\vee = \sum_{1=i\leq j}^{3} \alpha_{2,i,j} x_i^\vee x_j^\vee \text{ and } v_3^\vee = \sum_{1=i\leq j}^{3} \alpha_{3,i,j} x_i^\vee x_j^\vee$$

An element $[t_1 : t_2 : t_3] \in T = \mathbb{P}(V)$ corresponds to the following one dimensional subspace of $V^\vee/k^*$ (i.e. the pencil of conics):

$$\{a_1 v_1^\vee + a_2 v_2^\vee + a_3 v_3^\vee \mid a_1 t_1 + a_2 t_2 + a_3 t_3 = 0\}/k^* \tag{5.62}$$

Now $q^\vee$ induces a map $g: Z \to T$ as follows:

$$Z = \mathbb{P}(E) \xrightarrow{\text{Veronese embedding}} \mathbb{P}(\operatorname{Sym}^2 E) \xrightarrow{\operatorname{Proj}(\operatorname{Sym}(q^\vee))} \mathbb{P}(V) \tag{5.63}$$

In particular g acts as follows

$$\begin{aligned}[z_1 : z_2 : z_3] &\mapsto [z_1^2 : z_2^2 : z_3^2 : z_1 z_2 : z_1 z_3 : z_2 z_3] \\ &\mapsto [v_1^\vee(z_1, z_2, z_3) : v_2^\vee(z_1, z_2, z_3) : v_3^\vee(z_1, z_2, z_3)]\end{aligned} \tag{5.64}$$

The lemma follows by combining (5.62) and (5.64). □

Remark 5.5.7. *Note that $g([z_1 : z_2 : z_3])$ is not well defined when $[z_1 : z_2 : z_3]$ is a basepoint of the net of conics induced by q. As we started from an algebraic quadruple, this situation cannot occur. (For more general pre-algebraic quadruples however, this situation can occur.)*

From Lemma 5.5.6 we get the following corollary, which is also contained in [EH, §11.4.4].

Corollary 5.5.8. *With the notation as above as above, let $\xi \in V^\vee$, let $t_\xi \in \mathbb{P}(V^\vee)$ be the associated point and $X_\xi \subset \mathbb{P}(V)$ the associated line. Conversely let $Y_\xi \subset \mathbb{P}(E)$ be the conic defined by ξ. Then $g^{-1}(X_\xi) = Y_\xi$.*

Proof. Write $\xi = a_1 v_1^\vee + a_2 v_2^\vee + a_3 v_3^\vee$, then X_ξ is the line in the coordinates $[v_1 : v_2 : v_3]$ defined by $a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$. In particular

$$g^{-1}(X_\xi) = \{[x_1 : x_2 : x_3] \mid \sum_{i=1}^{3} a_i v_i^\vee([x_1 : x_2 : x_3]) = 0$$

This is exactly equal to Y_ξ. □

From the algebraic quadruple (E, V, q, ξ) we can define a morphism $f: \mathbb{P}^1 \to \mathbb{P}^1$ of degree 4 as follows: let $g: Z \to T$ be as in Lemma 5.5.6. From $\xi \in \mathbb{P}(V^\vee)$ we obtain a line $X = X_\xi$ inside $\mathbb{P}(V)$. Then define $f: Y \to X$ using the fibre product

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, "f"] & Z \arrow[d, "g"] \\ X \arrow[r, hook, "i"] & T. \end{tikzcd} \tag{5.65}$$

As explained in Corollary 5.5.8, we can identify Y with the conic parametrized by the point $\xi \in \mathbb{P}(V^\vee)$ in the net of conics. Hence if ξ is taken outside the discriminant locus in $\mathbb{P}(V^\vee)$, Y is a smooth conic. Hence we get a morphism $f: \mathbb{P}^1 \to \mathbb{P}^1$ of degree 4. The goal of the current section is to show that the algebraic quadruple $(E_{Y/X}, V_{Y/X}, q_{Y/X}, \xi_{Y/X})$ induced by this f is isomorphic to the original quadruple (where an isomorphism of quadruple consists of isomorphisms of the E's and V's compatible with the q's and ξ's.).

This is done in Theorem 5.5.16, based on some computations for graded Frobenius algebras, which we will do below. (For the notion of a (graded) Frobenius algebra we refer to [Smi96, §3].)

Define

$$\begin{aligned} S &\coloneqq \operatorname{Sym}(E^\vee), \\ R &\coloneqq \operatorname{Sym}(V^\vee) \end{aligned} \tag{5.66}$$

and consider S as a graded R-algebra via $q^\vee : V^\vee \to \operatorname{Sym}^2 E^\vee$. In particular S is $\mathbb{Z}$-graded and R is $2\mathbb{Z}$-graded. Moreover we define

$$A \coloneqq S/SR_{\geq 1}. \tag{5.67}$$

Lemma 5.5.9. *A is a graded Frobenius algebra with Hilbert series* $1, 3, 3, 1$.

Proof. Because the net of conics is basepoint-free, any choice of basis for the net gives a regular sequence in S. The Hilbert series of the associated complete intersection can then be easily computed, e.g. using [Sta78, Corollary 3.2], and it equals

$$(1-t^2)^3 \cdot h_S(t) = \frac{(1-t^2)^3}{(1-t)^3} = (1+t)^3$$

Likewise by basepoint-freeness we have that A is Frobenius because it defines a zero-dimensional complete intersection (hence it is Gorenstein). □

If $x \in S$, then we will denote $\overline{x}$ for the induced element in A. If $\overline{x} \in A_0$ or A_1, then we will also use the notation $\overline{x}$ to indicate the corresponding element in S_0 or S_1 depending on the context.

There exists a "trace" morphism

$$\operatorname{tr}: S \to S/(RS_0 \oplus RS_1 \oplus RS_2) \tag{5.68}$$

which defines a non-degenerate pairing

$$S \otimes_R S \to R(-3) \otimes A_3 : s \otimes t \mapsto \operatorname{tr}(st), \tag{5.69}$$

where we have used the isomorphism

$$S/(RS_0 \oplus RS_1 \oplus RS_2) \cong R(-3) \otimes A_3.$$

We will use dual R-bases $\{e_0, \dots, e_7\}$ and $\{f_0, \dots, f_7\}$ for S, where the degrees of the elements e_i, f_j are

$$|e_i| = |f_{7-i}| = \begin{cases} 0 & i = 0 \\ 1 & i = 1,2,3 \\ 2 & i = 4,5,6 \\ 3 & i = 7. \end{cases} \tag{5.70}$$

Moreover we can and will assume that $e_0 = f_7 = 1$ (and therefore $\overline{e}_7 = \overline{f}_0$), $f_4 = e_1$, $f_5 = e_2$, $f_6 = e_3$, and $\overline{f}_1 = \overline{e}_4$, $\overline{f}_2 = \overline{e}_5$, $\overline{f}_3 = \overline{e}_6$. If we introduce the structure constants

$$\overline{e}_i\overline{e}_j = \sum_{l=1}^{3} c_{i,j,l}\overline{e}_{l+3} \tag{5.71}$$

for the multiplication in A, where $i, j = 1,2,3$, then we have the following lemma.

Lemma 5.5.10. *The structure constants $c_{i,j,l}$ are invariant under permutation of i, j and l.*

Proof. We evaluate $\operatorname{tr}(\overline{e_a e_b e_d})$ for $a, b, d = 1,2,3$ in A.

$$\begin{aligned} \operatorname{tr}(\overline{e_a e_b e_d}) &= \sum_{i=1}^{3} c_{a,b,i} \operatorname{tr}(\overline{e_{i+3} e_d}) \\ &= \sum_{i=1}^{3} c_{a,b,i} \operatorname{tr}(\overline{f_i e_d}) \\ &= \sum_{d} c_{a,b,i}\delta_{i,d} \\ &= c_{a,b,d} \end{aligned}$$

□

By construction we have the identification

$$\begin{aligned} A_1 &= E^\vee, \\ A_2 &= \operatorname{Sym}^2 E^\vee / V^\vee, \\ A_3 &= \operatorname{Sym}^3 E^\vee / E^\vee V^\vee \end{aligned} \tag{5.72}$$

and the multiplication

$$\text{mult}\colon A_1 \otimes_k A_1 \to A_2$$

is nothing but the quotient map $E^\vee \otimes E^\vee \to \operatorname{Sym}^2 E^\vee \to \operatorname{Sym}^2 E^\vee / V^\vee$, with the inclusion of $V^\vee$ into $\operatorname{Sym}^2 E^\vee$ given by $q^\vee$.

Because A is graded Frobenius by Lemma 5.5.9, the multiplication in the algebra provides us with a duality in the form of a perfect pairing

$$A_1 \otimes A_2 \to A_3$$

which yields the identifications

$$\begin{aligned} A_1^\vee &= A_2 \otimes A_3^{-1}, \\ A_2^\vee &= A_1 \otimes A_3^{-1}. \end{aligned} \tag{5.73}$$

If we define

$$\alpha\colon A_1 \otimes A_3^{-1} \to \operatorname{Sym}^2 A_2 \otimes A_3^{-2} : e_i \otimes \overline{e}_7^{-1} \mapsto \sum_{j=1}^{3} \overline{e}_i\overline{e}_j \cdot \overline{f}_j \otimes \overline{e}_7^{-2} \tag{5.74}$$

and

$$\beta\colon \operatorname{Sym}^2 A_2 \otimes A_3^{-2} \to V : \overline{e}_{i+3} \cdot \overline{e}_{j+3} \otimes \overline{e}_7^{-2} \mapsto q(e_i^\vee \cdot e_j^\vee) \tag{5.75}$$

then these morphisms are compatible with the algebra structure on A in the following way using Lemma 5.5.10.

Lemma 5.5.11. *The diagram*

$$\begin{tikzcd} 0 \arrow[r] & A_1 \otimes A_3^{\otimes -1} \arrow[r, "\alpha"] \arrow[d, equal] & \operatorname{Sym}^2 A_2 \otimes A_3^{\otimes -2} \arrow[r, "\beta"] \arrow[d, equal] & V \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A_2^\vee \arrow[r, "\mathrm{mult}^\vee"] & \operatorname{Sym}^2 A_1^\vee \arrow[r, "q"] & V \arrow[r] & 0 \end{tikzcd} \tag{5.76}$$

where the vertical arrows are the identifications obtained from the graded Frobenius structure, is commutative.

Proof. We first discuss the square involving α. Note that

$$\begin{aligned} \alpha(e_i \otimes \overline{e}_7^{-1}) &= \sum_{j=1}^{3} \overline{e_i e_j} \cdot \overline{e}_{j+3} \otimes \overline{e}_7^{-2} \\ &= \sum_{l,j=1}^{3} c_{ijl}\overline{e}_{l+3} \cdot \overline{e}_{j+3} \otimes \overline{e}_7^{-2} \end{aligned}$$

whereas the lower composition in this square is given by

$$\begin{aligned} e_i \otimes \overline{e}_7^{-1} &\mapsto \overline{e}_{i+3}^\vee \\ &\mapsto \sum_{p,q=1}^{3} c_{pqi}\overline{e}_p^\vee \cdot \overline{e}_q^\vee \\ &\mapsto \sum_{p,q=1}^{3} c_{pqi}\overline{e}_{p+3} \cdot \overline{e}_{q+3} \otimes \overline{e}_7^{-2} \end{aligned}$$

It now suffices to use the symmetry of the structure constants as in Lemma 5.5.10.

The fact that the square involving β commutes, follows immediately from the definition of β and the fact that the middle vertical identification is given by

$$\overline{e_{i+3}} \cdot \overline{e_{j+3}} \otimes \overline{e_7}^{-2} \mapsto \overline{e_i}^\vee \overline{e_j}^\vee$$

∎

The trace pairing from (5.69) gives an identification of graded S-modules:

$$S^\vee \coloneqq \operatorname{Hom}_R(S,R) \cong S(3) \otimes A_3^{\otimes -1} : e_i^\vee \mapsto f_i \otimes e_7^{-1} \tag{5.77}$$

Using this identification, together with the R-dualised multiplication $S \otimes_R S \to S$ we get a copairing

$$\delta\colon S(-3) \otimes A_3 \longrightarrow S \otimes_R S \tag{5.78}$$

such that

$$\delta(u \otimes \overline{e}_7) = \sum_{i=0}^{7} ue_i \otimes f_i.$$

Similar to (5.24) and Lemma 3.3.23 we see that $\sum_{i=0}^7 e_i \otimes f_i = \sum_{i=0}^7 f_i \otimes e_i$ is a central element in $S \otimes_R S$.

For use below, consider the decomposition $S = S_{\mathrm{even}} \oplus S_{\mathrm{odd}}$ of graded R-modules into the part of even and odd grading. Then there exists a canonical identification

$$S_{\mathrm{even}}/R = (R \otimes A_2)(-2).$$

We moreover define a graded R-module Ω as the cokernel in the leftmost morphism of the Koszul sequence, i.e.

$$0 \longrightarrow R(-6) \xrightarrow{\kappa} V \otimes R(-4) \longrightarrow \Omega \longrightarrow 0. \tag{5.79}$$

Finally consider the morphism

$$\Phi\colon S_{\mathrm{odd}}(-3) \otimes A_3 \longrightarrow \operatorname{Sym}^2 A_2 \otimes_k R(-4) \tag{5.80}$$

which is the composition of the inclusion into $S(-3) \otimes A_3$, the morphism δ from (5.78), the projection $S \twoheadrightarrow S_{\mathrm{even}}$, the quotient $S_{\mathrm{even}} \twoheadrightarrow S_{\mathrm{even}}/R \cong A_2 \otimes_k R(-2)$ in both factors of the tensor product, and the quotient $A_2 \otimes_k A_2 \to \operatorname{Sym}^2 A_2$.

Proposition 5.5.12. *There exists an isomorphism*

$$\operatorname{coker} \Phi \cong \Omega \otimes A_3^{\otimes 2}.$$

More precisely, the induced morphism

$$\operatorname{Sym}^2 A_2 \otimes_k R(-4) \longrightarrow \operatorname{coker} \Phi \cong \Omega \otimes A_3^{\otimes 2}$$

is the composition of $\beta \otimes \mathrm{id}_{R(-4)}$, with β as in (5.75)*, and the quotient map in* (5.79)*.*

(By [Eis95, §17.5] we have that Ω induces $\mathrm{T}_{\mathbb{P}(V)}(-3)$ after sheafification, which in turn is isomorphic to $\Omega^1_{\mathbb{P}(V)}$, hence the use of the notation Ω in this setting is compatible with the usual interpretation in terms of (co)tangent bundles.)

Proof. S_{odd} contains canonically a graded R-submodule $A_1 \otimes R(-1)$. We first describe how Φ acts on $A_1 \otimes A_3 \otimes R(-4)$. By definition of Φ, this map is given by

$$\overline{e}_i \otimes \overline{e}_7 \overset{\delta}{\mapsto} \sum_{j=0}^{7} e_i e_j \cdot f_j \mapsto \sum_{j=0}^{7} \overline{e_i e_j} \cdot \overline{f}_j = \sum_{j=1}^{3} \overline{e_i e_j} \cdot \overline{f}_j$$

for $i = 1,2,3$ (where we used the fact that $e_i e_j \notin S_{\mathrm{even}}$ for $j = 0,4,5,6$ and $e_i e_7 \in R$)

In other words, up to tensoring with A_3^{-2}, this is precisely the morphism α as in Lemma 5.5.11. More precisely, it is given by the unique map α' making the following diagram commute

$$\begin{tikzcd}
A_1 \otimes A_3^{\otimes -1} \otimes R(-4) \arrow[r, "\alpha \otimes \mathrm{id}_{R(-4)}"] \arrow[d, "\cong"'] & \mathrm{Sym}^2 A_2 \otimes A_3^{\otimes -2} \otimes R(-4) \arrow[d, "\cong"] \\
A_1 \otimes A_3 \otimes R(-4) \arrow[r, "\alpha'"] & \mathrm{Sym}^2 A_2 \otimes R(-4)
\end{tikzcd}$$

Next consider the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 & & & \\
 & R(-6) \otimes A_3 \otimes A_3 \arrow[u] & 0 & 0 & \\
 & S_{\mathrm{odd}}(-3) \otimes A_3 \arrow[u] \arrow[r, "\Phi"] & \mathrm{Sym}^2 A_2 \otimes R(-4) \arrow[u] \arrow[r] & \mathrm{coker}\,\Phi \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & A_1 \otimes R(-4) \arrow[u] \arrow[r, "\alpha'"] & \mathrm{Sym}^2 A_2 \otimes R(-4) \arrow[u, "="'] \arrow[r, "\beta"] & V \otimes R(-4) \otimes A_3^{\otimes 2} \arrow[u, "\gamma"'] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

In particular $\beta \circ \Phi$ induces a map $R(-6) \otimes A_3^2 \longrightarrow V \otimes R(-4) \otimes A_3^2$.

We claim that this map is trivial on $A_3^{\otimes 2}$.

$$\begin{aligned}
\overline{e}_7 \otimes \overline{e}_7 &\overset{\Phi}{\mapsto} \sum_{i=4}^{6} \overline{e_7 f_i} \otimes \overline{e_i} \\
&= \sum_{i,j=4}^{6} \mathrm{tr}(e_7 f_i f_j) \overline{e_j} \otimes \overline{e_i} \\
&\overset{\beta}{\mapsto} \sum_{i,j=4}^{6} \mathrm{tr}(e_7 f_i f_j) q(\overline{e}_{j-3}^{\vee} \cdot e_{i-3}^{\vee}) \otimes e_7^2 \\
&= \sum_{i,j=4}^{6} \mathrm{tr}(e_7 f_i f_j) q(f_j^{\vee} \cdot f_i^{\vee}) \otimes \overline{e}_7^2
\end{aligned}$$

The right hand side should be considered as an element of $R_2 \otimes V \otimes A_3^{\otimes 2}$.

Now with the notation of Lemma 5.5.13 (which is an easy result from linear algebra) applied to $\alpha = \mathrm{tr}(e_7-) : S^2E^\vee \to R_2 = V^\vee$, $\beta = q^\vee : V^\vee \to S^2E^\vee$ and $(a_l)_l = (f_i \cdot f_j)_{i,j=4,5,6}$ we find

$$\sum_{i,j=4}^{6} \mathrm{tr}(e_7 f_i f_j) \otimes q(f_j^\vee \cdot f_i^\vee) \otimes \overline{e}_7^2 = \mathrm{id}_V \otimes \overline{e}_7^2$$

with id_V considered as an element of $V^\vee \otimes V$. As such, by the construction of the Koszul sequence we have that $\beta \circ \Phi : R(-6) \otimes A_3^2 \to V \otimes R(-4) \otimes A_3^2$ can be decomposed as $\kappa \otimes \mathrm{id}_{A_3^{\otimes 2}}$ with κ as in (5.79).

In particular $\mathrm{coker}(\beta \circ \Phi) \cong \mathrm{coker}(\kappa) = \Omega$. Conversely the image of $\beta \circ \Phi$ is given by $\ker(\gamma)$, such that $\mathrm{coker}(\beta \circ \Phi) \cong \mathrm{im}(\gamma) = \mathrm{coker}(\Phi)$, proving the lemma. □

Lemma 5.5.13. *Let V_1, V_2, V_3 be finite-dimensional vector spaces. Let $\alpha: V_1 \to V_2$ and $\beta: V_3 \to V_1$ be linear maps. Let $v_1, \dots, v_n$ be a basis for V_1. Then*

$$\sum_{i=1}^{n} \alpha(v_i) \otimes \beta^\vee(v_i^\vee)$$

is equal to

$$\alpha \circ \beta: V_3 \to V_2$$

considered as elements of $V_3^\vee \otimes_k V_2$.

We will now use these algebraic results, to explicitly describe the morphism $g: \mathbb{P}(E) \to \mathbb{P}(V)$ from Lemma 5.5.6, as well as the induced degree 4 morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ as in (5.65). For this we use the shorthand notation

$$Z = \mathbb{P}(E),\ T = \mathbb{P}(V)$$

as in (5.61).

Dualizing the multiplication

$$g_*\mathcal{O}_Z \otimes_{\mathcal{O}_T} g_*\mathcal{O}_Z \to g_*\mathcal{O}_Z,$$

we obtain a copairing

$$g_*\omega_{Z/T} \to g_*\omega_{Z/T} \otimes_{\mathcal{O}_T} g_*\omega_{Z/T}.$$

and tensoring it over $g_*\mathcal{O}_Z$ on the left and right with $g_*\omega_{Z/T}^{-1}$ we get an equivalent copairing

$$\delta_{Z/T}: g_*\omega_{Z/T}^{-1} \to g_*\mathcal{O}_Z \otimes_{\mathcal{O}_Z} g_*\mathcal{O}_Z. \tag{5.81}$$

This will be the geometric incarnation of (5.78). This, and other identifications between sheaves and the corresponding graded modules are given in the following lemma.

Lemma 5.5.14. *There exist isomorphisms*

$$g_*\mathcal{O}_Z \cong \widetilde{S_{\mathrm{even}}},$$

and

$$g_*\omega_{Z/T}^{-1} \cong \widetilde{S_{\mathrm{odd}}(-3)} \otimes_k A_3.$$

Using these identifications the morphism (5.81) *is the sheafification of the composition*

$$S_{\mathrm{odd}}(-3) \otimes A_3 \hookrightarrow S(-3) \otimes A_3 \xrightarrow{\delta} S \otimes_R S \twoheadrightarrow S_{\mathrm{even}} \otimes_R S_{\mathrm{even}}. \tag{5.82}$$

Proof. The identification for $g_*\mathcal{O}_Z$ follows from the fact that R is concentrated in even degrees.

The identification for $g_*\omega_{Z/T}^{-1}$ is obtained by restricting (5.77) to the even part, which gives

$$g_*\omega_{Z/T} = \widetilde{S_{\mathrm{odd}}(3)} \otimes A_3^{\otimes -1},$$

and using

$$(S_{\mathrm{odd}}(-3) \otimes_k A_3) \otimes_{S_{\mathrm{even}}} (S_{\mathrm{odd}}(3) \otimes_k A_3^{-1}) \cong S_{\mathrm{even}}$$

we get the desired identification.

Finally the identification in (5.82) follows from the explicit form of duality for a finite flat morphism. □

We will now define sheaves on T, which will induce the sheaves $\mathcal{F}$ and $\mathcal{Q}$ on $X \cong \mathbb{P}^1$, as in §5.3.5. These are

$$\begin{aligned} \overline{\mathcal{F}} &\coloneqq g_*\mathcal{O}_Z/\mathcal{O}_T, \\ \overline{\mathcal{Q}} &\coloneqq \operatorname{coker}(\delta' \colon \omega_{Z/T}^{-1} \to \operatorname{Sym}^2 \overline{\mathcal{F}}) \end{aligned} \tag{5.83}$$

where δ' is the composition of $\delta_{Z/T}$ with the projection to $\operatorname{Sym}^2 \overline{\mathcal{F}}$. Then we have the following result.

Proposition 5.5.15. *There exist isomorphisms*

$$\begin{aligned} \overline{\mathcal{F}} &\cong A_2 \otimes_k \mathcal{O}_T(-1), \\ \overline{\mathcal{Q}} &\cong \Omega_{T/k} \otimes_k A_3^{\otimes 2}. \end{aligned}$$

Moreover we have that the quotient map

$$\operatorname{Sym}^2 \overline{\mathcal{F}} \twoheadrightarrow \overline{\mathcal{Q}}$$

is the composition of the sheafification of the morphism β from (5.75)*:*

$$\operatorname{Sym}^2 A_2 \otimes_k \mathcal{O}_T(-2) \longrightarrow V \otimes_k \mathcal{O}_T(-2) \otimes A_3^{\otimes 2}$$

with the quotient map

$$V \otimes_k \mathcal{O}_T(-2) \otimes A_3^{\otimes 2} \longrightarrow \Omega_{T/k} \otimes A_3^{\otimes 2}.$$

Proof. This follows directly from the construction together with Proposition 5.5.12 and Lemma 5.5.14. ☐

We can now prove the main result of this section, from which the second statement in Theorem 5.5.3 (every algebraic quadruple comes from a noncommutative $\mathbb{P}^1$-bundle.) immediately follows:

Theorem 5.5.16. *Let* (E,V,q,ξ) *be an algebraic quadruple and let* $f : Y \to X$ *with* $X \cong Y \cong \mathbb{P}^1$ *be the associated morphism as in* (5.65) *and Lemma 5.5.6. Let* $(E_{Y/X}, V_{Y/X}, q_{Y/X}, \xi_{Y/X})$ *be the algebraic quadruple induced by* f *as in* (5.53), (5.54) *and* (5.56). *Then*

$$(E,V,q,\xi) \cong (E_{Y/X}, V_{Y/X}, q_{Y/X}, \xi_{Y/X}).$$

Proof. Using Lemma 5.3.19, Propositions 5.3.21 and 5.5.15 and the fact that X is a curve of degree 1 inside T we have a commutative diagram

$$\begin{tikzcd} \operatorname{Sym}^2 \mathcal{F}_{Y/X} \arrow[r, two heads] \arrow[d, equal] & \mathcal{Q}_{Y/X} \arrow[r, "\cong"] \arrow[d, equal] & \mathcal{O}_X(-2) \oplus \mathcal{O}_X(-1) \\ \operatorname{Sym}^2(i^*\overline{\mathcal{F}}) \arrow[r, two heads] & i^*\overline{\mathcal{Q}} \arrow[r, "\cong"] & i^*\Omega^1_{T/k} \arrow[u] \end{tikzcd} \tag{5.84}$$

Because $\Omega^1_{T/k}(2)$ sits inside the twist of the Euler exact sequence

$$0 \to \mathcal{O}_T(-1) \xrightarrow{\gamma} V \otimes_k \mathcal{O}_T \to \Omega^1_{T/k}(2) \to 0$$

we get that $\mathrm{H}^0(\Omega^1_{T/k}(2)) = V$. From this we get isomorphisms

$$V \cong V_{Y/X}$$

similarly

$$\begin{aligned} E_{Y/X} &\coloneqq \mathrm{H}^0(\mathcal{F}_{Y/X}(1)) \\ &\Downarrow && (5.84) \\ &\cong \mathrm{H}^0(i^*(\overline{\mathcal{F}}(1))) \\ &\Downarrow && \text{Proposition 5.5.15} \\ &\cong \mathrm{H}^0(i^*(A_2 \otimes_k \mathcal{O}_T)) \\ &\cong \mathrm{H}^0(A_2 \otimes_k \mathcal{O}_X) \\ &\cong A_2 \\ &\Downarrow && (5.72) \text{ and } (5.73) \\ &\cong E. \end{aligned}$$

We also get that $q = q_{Y/X}$ by (5.84).

Finally to show that $\xi_{Y/X} = \xi$ we need to show that ξ equals the composition

$$V = \mathrm{H}^0(T, \Omega_T(2)) \to \mathrm{H}^0(X, i^*\Omega_T(2)) = \mathrm{H}^0(X, \mathcal{O}_X \oplus \mathcal{O}_X(1)) \to \mathrm{H}^0(X, \mathcal{O}_X) = k.$$

Recall that, similarly to (5.79), $\Omega_T(2)$ is defined by the exact sequence

$$0 \to \mathcal{O}_T(-1) \xrightarrow{\gamma} V \otimes_k \mathcal{O}_T \to \Omega_T(2) \to 0.$$

The fiber in $p \in \mathbb{P}(V)$ of the inclusion is (up to scalar) the morphism $p\colon k \to V$. If we pull back the short exact sequence along the closed immersion i we see that the image of the fibers of all pullbacks are contained in $\ker \xi$, hence we have a complex

$$\mathcal{O}_X(-1) \xrightarrow{i^*\gamma} V \otimes_k \mathcal{O}_X \xrightarrow{\xi \otimes \mathrm{id}_{\mathcal{O}_X}} \mathcal{O}_X.$$

Up to scalars this must be the unique non-zero morphism

$$i^*\Omega_T \to \mathcal{O}_X.$$

By taking global sections we see that $\xi_{Y/X} = \xi$. □

We have now fully proven Theorem 5.5.3. In particular we know that all noncommutative $\mathbb{P}^1$-bundles

$$\mathrm{qgr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}))$$

are equivalent to a category of the form

$$\mathrm{qgr}_{\mathbb{P}^1}\, \mathrm{C}\ell_{\mathrm{Sym}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))}(E \otimes_k \mathrm{Sym}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)), q)$$

which is completely determined by an algebraic quadruple (E, V, q, ξ). Moreover (up to isomorphism of quadruples, which induces an isomorphism of categories), we know that all algebraic quadruples can be obtained from a degree 4 morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$.

Conversely we know that every degree 4 morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ can be constructed from a quadruple (E, V, q, ξ) up to the equivalence relation given by

$$f \sim f' \Leftrightarrow (\mathrm{qgr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}})) \cong \mathrm{qgr}(\mathbb{S}({}_{f'}(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}})))$$

A priori, there could hence be morphisms f and f' giving rise to an equivalent noncommutative $\mathbb{P}^1$-bundle, but where f and f' cannot be obtained from each other via the base change action of PGL(2). Moreover one could start from a morphism f, create an algebraic quadruple from it and create a new morphism f' from this quadruple. Using Theorem 5.5.16 we know that (up to the action of PGL(2)) $(f')' = f'$. In some sense this would imply that there is a collection of “minimal” degree 4 morphisms $\mathbb{P}^1 \to \mathbb{P}^1$. In the light of Conjecture 3.3.7, we expect that these inconveniences cannot occur. I.e. we conjecture that there is a 1-1-correspondence between degree 4 maps f (up to base change) and algebraic quadruples (up to isomorphism). In the light of §5.6 this is a 1-1-correspondence between pencils of binary quartics (up to base change) and nets of conics with the choice of a nonsingular conic (up to base change).

5.6 Comparison of linear systems

The goal of this section is to understand which quaternionic orders on $\mathbb{F}_1$ correspond to which noncommutative $\mathbb{P}^1$-bundles. For this we use that both categories are constructed with respect to geometric data: respectively nets of conics together with a fixed nonsingular conic and pencils of binary quartics. These linear systems have been classified in [Wal77] and [Wal98]. As such, we will explain how our main result (Corollary 5.5.4) relates these two classifications. Before we go into this, we first recall the relevant classifications and related facts.

Convention 5.6.1. *The classifications in [Wal77] and [Wal98] are done for the complex numbers. Probably, these results hold for sufficiently high characteristic as well. However, in order to avoid any problems in finite characteristic, we assume we are working over an algebraically closed field of characteristic 0.*

Pencils of conics

As a tool in setting up the explicit correspondence, we need to say a few words on pencils of conics. These are 2-dimensional subspaces of $\mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$. Their classification, up to the action of $\mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}(3)$, is completely classical. It is also the first (non-trivial) case of Segre's classification of pencils of quadrics using Segre symbols (see for example [HP52, Chapter XIII, §11, Theorem II]). In Table 5.1 we list pencils of conics for which there are only finitely many singular fibres (i.e. finitely many singular conics in the pencil. In the original classification, 3 Segre symbols correspond to pencils where each conic is singular.). The main observation here is that each Segre symbol can be described both by its base locus (i.e. the points all conics in the pencil have in common, counted with multiplicity) and by the types of singular conics which appear in the pencil.

Segre symbol	base locus	# singular fibres	# double lines
$[1,1,1]$	$(1,1,1,1)$	3	0
$[2,1]$	$(2,1,1)$	2	0
$[3]$	$(3,1)$	2	0
$[(1,1),1]$	$(2,2)$	2	1
$[(2,1)]$	(4)	1	1

Table 5.1: Pencils of conics

Nets of conics

As above, we will consider a net of conics as a surjective morphism

$$q\colon \mathrm{Sym}^2 E \to V,$$

where E and V are 3-dimensional vector spaces. Then the net itself is $\mathbb{P}(V^\vee)$, whilst the conics live in $\mathbb{P}(E)$.

[Wal77] gives a complete classification of nets of conics. We are only interested in basepoint-free nets of conics, otherwise the associated graded Clifford algebra is not Artin-Schelter regular. The classification in this case can be summarized as in Table 5.2, with the labeling from op. cit.

Recall that the *discriminant* is the locus in $\mathbb{P}(V^\vee)$ of the singular conics in the net. (It is also known as the *Hessian curve*.) Such a singular conic is either two lines intersecting in a point or a double line. The double lines in the net of conics correspond precisely to the singularities of the discriminant for the basepoint-free nets. In Table 5.2 the situation is summarized for the types which are relevant to us, for a complete classification one is referred to [Wal77, table 2].

type	discriminant	number of double lines
A	elliptic curve	0
B	nodal cubic	1
D	conic and line in general position	2
E	triangle of lines	3

Table 5.2: Base-point free nets of conics

A net of conics gives rise to many more curves in different ambient spaces:

- The *Jacobian* is the union of the singular points of each conic in $\mathbb{P}(E)$. Similar to the discriminant, it is a cubic curve. But the Jacobian and discriminant live in different spaces and they are not necessarily of the same type.

- The *branch curve* inside $\mathbb{P}(V)$ is the locus of points whose fibres do not have 4 distinct points, Generically the branch curve is the dual of the discriminant, hence it is of degree 6, but if the discriminant is zero the degree is lower.

- The *ramification curve* inside $\mathbb{P}(E)$ are those points which are multiple points of a fiber. Using Table 5.1 this also means that the associated subpencil of conics has at most two singular points, hence the ramification curve is equal to the Jacobian.

In Table 5.3 we give an overview of all the curves associated to a net of conics. Observe that the point scheme for an Artin–Schelter regular Clifford algebra is a double cover of the ramification curve, and hence coincides with the double cover of the discriminant constructed by considering the two lines in $\mathbb{P}(E)$ associated to the singular conic parametrised by a point on the discriminant.

	curve	ambient space	degree
Δ	discriminant	$\mathbb{P}(V^\vee)$	3
J	Jacobian	$\mathbb{P}(E)$	3
R	ramification curve	$\mathbb{P}(E)$	3
B	branch curve	$\mathbb{P}(V)$	≤ 6
C	point scheme	$\mathbb{P}(E^\vee)$	3
$\widetilde{\Delta}$	double cover of discriminant	$\mathbb{P}(E^\vee)$	3
	conic in the net	$\mathbb{P}(E)$	2

Table 5.3: Curves associated to a net of conics

We now give an example of a net of conics. It will be important in understanding a special element in the classification of noncommutative $\mathbb{P}^1$-bundles, see also Example 5.6.3 and section 5.7.

Example 5.6.2. *Consider the net of conics given by $x^2, y^2, z^2 + 2xy$. It is (up to base change) the only net of conics of type D. Its discriminant is a conic and line in general position, and the two singularities of the discriminant correspond to the double lines x^2 and y^2. To see this, note that the discriminant is given by those $[a : b : c] \in \mathbb{P}(V^\vee)$ for which $ax^2 + by^2 + c(z^2 + 2xy)$ is a singular conic. I.e. we need to find the $[a : b : c] \in \mathbb{P}(V^\vee)$ for which there is a nonzero solution (x, y, z) to the system of equations*

$$\begin{cases} \dfrac{\partial(ax^2 + by^2 + c(z^2 + 2xy))}{\partial x} = 2ax + 2cy = 0 \\ \dfrac{\partial(ax^2 + by^2 + c(z^2 + 2xy))}{\partial y} = 2by + 2cx = 0 \\ \dfrac{\partial(ax^2 + by^2 + c(z^2 + 2xy))}{\partial z} = 2cz = 0 \end{cases}$$

One hence easily finds that the discriminant is the union of the line $c = 0$ with the conic $ab = c^2$. The Jacobian turns out to be a triangle of lines, but we will not use this.

Pencils of binary quartics

The only input needed for the construction of a noncommutative $\mathbb{P}^1$-bundle in the sense of Chapter 3 is a finite morphism $f\colon\mathbb{P}^1\to\mathbb{P}^1$ of degree 4. This data is equivalent to the data of a basepoint-free pencil of binary quartics, i.e. a 2-dimensional subspace of $\mathrm{H}^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(4))$. Indeed, if $\langle f_1,f_2\rangle$ is such a pencil, then $[x:y]\mapsto[f_1(x,y):f_2(x,y)]$ is a well-defined finite morphism $f\colon\mathbb{P}^1\to\mathbb{P}^1$. These pencils have been studied in detail in [Wal98], and we will quickly recall the relevant results here.

As for nets of conics, we can associate the *Jacobian* and the *discriminant* to a pencil of binary quartics. In this case the Jacobian represents the branch points of the associated morphism, and the discriminant are the images of these. We can moreover describe the branching behaviour of the associated morphism using its *symbol*: algebraically it is obtained by considering the roots of the discriminant and listing the multiplicities of the remaining factors. Geometrically it describes the cycle type of the monodromy around each branch point. Its notation is not to be confused with that of Segre symbols as in Table 5.1.

The classification is given in Table 5.4, where we have already included the comparison to the net of conics, from Theorem 5.5.16.

Example 5.6.3. *The "most degenerate" morphism $f\colon\mathbb{P}^1\to\mathbb{P}^1$ is given by $[x^4:y^4]$. It is ramified in $[0:1]$ and $[1:0]$ only, where 4 branches come together, hence its symbol is $[(4)(4)]$. It corresponds to type A in the classification of Table 5.4. Moreover, up to the base chang action by $\mathrm{PGL}(2)$, this is the only morphism of type A.*

Remark 5.6.4. *In [Wal98, §2] it is remarked that there is no pencil of binary quartics whose symbol is $[(2,2)(2,2)(3,1)]$, which is excluded by considering the monodromy of the would-be corresponding morphism $f\colon\mathbb{P}^1\to\mathbb{P}^1$ of degree 4. Using the correspondence we have set up between nets of conics and pencils of binary quartics we can also argue that this case cannot occur using the geometry of cubics: there is no cubic curve with 2 singularities and 1 inflection point.*

Explicit correspondence

We will now relate the classification of pencils of binary quartics from [Wal98] to algebraic quadruples, i.e. to the classification of nets of conics from [Wal77] and the choice of a smooth conic inside the net. We do this by giving for every element in the classification of pencils of binary quartics the discriminant of the net of conics and the position of the point which is blown up, corresponding to the choice of the smooth conic inside the net.

To understand which type of f we obtain from an algebraic quadruple, we need to describe the inverse image of each point on X_ξ, i.e. describe the ramification behaviour of f. For this, recall that f was constructed by restricting $g : \mathbb{P}(E) \to \mathbb{P}(V)$ to (the inverse image of) a line $X_\xi \subset \mathbb{P}(V)$. By construction each point $x \in X_\xi \subset \mathbb{P}(V)$ corresponds to a pencil of conics $P_x \subset \mathbb{P}(V^\vee)$ and $f^{-1}(x) = g^{-1}(x)$ is given by the base locus of P_x (Lemma 5.5.6). Recall from Table 5.1 that describing the base locus of a pencil of conics is equivalent to describing the types of singular conics appearing in the pencil. Moreover, recall that the singular conics in $\mathbb{P}(V^\vee)$ are given by points on the discriminant, where the singularities of the discriminant describe double lines.

Hence in order to describe the inverse image $f^{-1}(x)$ of a point $x \in X_\xi \subset \mathbb{P}(V)$, one can equivalently describe the intersection of the pencil P_x with the discriminant Δ. This correspondence is explicitly given as follows:

1. The Segre symbol $[1,1,1]$ (using the notation from Table 5.1) is the generic case. It corresponds to a generic line intersecting the discriminant cubic in 3 points, i.e. a pencil of conics P_x with 3 singular fibers. If $x \in X_\xi \subset \mathbb{P}(V)$ is the corresponding point, $f^{-1}(x)$ consists of 4 points. As such, this case is not relevant for understanding the construction as each degree 4 morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ is generically 4-to-1.

2. The Segre symbol $[2,1]$ corresponds to ramification of type $(2,1,1)$, and to realise it we must have a tangent line to the discriminant cubic intersecting the discriminant cubic in another (smooth) point: we have two singular fibres of rank 2, one of which is counted twice.

3. The Segre symbol $[3]$ corresponds to ramification of type $(3,1)$, and to realize it we must have a tangent at an inflection point: we have one singular fibre of rank 2 which is counted thrice.

4. The Segre symbol $[(1,1),1]$ corresponds to ramification of type $(2,2)$, and to realize it we must have a line through a node intersecting the discriminant cubic in another point.

5. The Segre symbol $[(2,1)]$ corresponds to ramification of type (4). To realize it we need a line through a node intersecting the discriminant only in this node.

Using this it is straightforward to set up the correspondence given in Table 5.4. There are 13 types in the classification of the pencils of binary quartics, and for each of these types we can choose an algebraic quadruple (i.e. a net of conics and a nonsingular conic in this net) giving rise to this type. Moreover, whenever there are degrees of freedom within a certain class, the degrees of freedom agree with the degrees of freedom in choosing the net and the conic.

label	normal form location of the point	discriminant cubic	moduli
A	$[(4)(4)]$ intersection of tangent lines at conic in singularities	conic and line	0
B	$[(4)(2,2)(2,1,1)]$ intersection of tangent line at conic in singularity and generic line	conic and line	0
C	$[(4)(3,1)(2,1,1)]$ intersection of tangent line at singularity and tangent line of inflection point	nodal cubic	0
D	$[(4)(2,1,1)(2,1,1)(2,1,1)]$ intersection of tangent line at singularity and tangent line at non-inflection point	nodal cubic	1
E	$[(2,2)(2,2)(2,2)]$ any point	triangle of lines	0
F	$[(2,2)(2,2)(2,1,1)(2,1,1)$ intersection of tangent lines at two smooth points on conic	conic and line	1
G	$[(2,2)(3,1)(3,1)]$ intersection of tangent lines at inflection points	nodal cubic	0
H	$[(2,2)(3,1)(2,1,1)(2,1,1)]$ intersection of non-tangent line through singularity and tangent line at inflection point	nodal cubic	1
I	$[(3,1)(3,1)(3,1)]$ intersection of three concurrent tangent lines at inflection points	elliptic curve, $j = 0$	0
J	$[(3,1)(3,1)(2,1,1)(2,1,1)]$ generic intersection of two tangent lines at inflection points	elliptic curve	1
K	$[(2,2)(2,1,1)(2,1,1)(2,1,1)(2,1,1)]$ intersection of two generic tangent lines	nodal cubic	2
L	$[(3,1)(2,1,1)(2,1,1)(2,1,1)(2,1,1)]$ generic intersection of tangent line at inflection point and another tangent line	elliptic curve	2
M	$[(2,1,1)(2,1,1)(2,1,1)(2,1,1)(2,1,1)(2,1,1)]$ generic point	elliptic curve	3

Table 5.4: Correspondence between pencils of binary quartics and algebraic quadruples

Example 5.6.5. *Consider the "most degenerate" (type A) morphism* $f : \mathbb{P}^1 \to \mathbb{P}^1$ *given by* $[x^4 : y^4]$ *as in Example 5.6.3. In order to realize this pencil, we need a net of conics for which the discriminant cubic has 2 singularities for which the tangent lines (to one of the irreducible components passing through this singularity) intersect in a point which does not lie on the discriminant. Obviously this is only possible when the discriminant is the union of a conic and a line in general position, i.e. we need to consider a net of conics of type D as in Table 5.2. In Example 5.6.2 we saw that such a net is given by* $x^2, y^2, z^2 + 2xy$ *with discriminant given by* $c(ab - c^2) = 0$*. The singularities of this discriminant are given by* $[0:1:0]$ *and* $[1:0:0]$ *and the tangent lines in these singularities (to the conic) intersect in the point* $\xi = [0:0:1]$*. Now let* $[x:y:z]$ *denote coordinates for* $\mathbb{P}(E) \cong \mathbb{P}^2$ *and* $[\alpha:\beta:\gamma]$ *denote coordinates for* $\mathbb{P}(V)$*, then using Lemma 5.5.6 and Corollary 5.5.8 we get that* $X_\xi \subset \mathbb{P}(V)$ *is the line* $\gamma = 0$ *and* $Y_\xi \subset \mathbb{P}(E)$ *is the conic* $z^2 + 2xy = 0$*. Moreover the induced map* f *is given by*

$$f : Y_\xi \to X_\xi : [x:y:z] \mapsto [x^2 : y^2 : z^2 + 2xy] = [x^2 : y^2 : 0],$$

and using the isomorphisms

$$\varphi_1 \colon \mathbb{P}^1 \xrightarrow{\cong} Y_\xi : [u:v] \mapsto [u^2 : v^2 : \sqrt{2}uv]$$
$$\varphi_2 \colon X_\xi \xrightarrow{\cong} \mathbb{P}^1 : [\alpha:\beta:0] \mapsto [\alpha:\beta]$$

we see that the (actual induced) map is given by

$$\varphi_2 \circ f \circ \varphi_1 : \mathbb{P}^1 \longrightarrow \mathbb{P}^1 : [u:v] \mapsto [u^4 : v^4]$$

as expected.

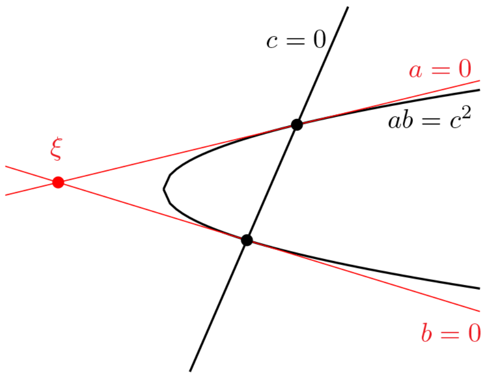

Figure 5.1: Illustration to Example 5.6.5

Example 5.6.6. *The pencils of binary quartics which are probably the hardest to realize are those of type I, i.e. for which the corresponding morphism* $f : \mathbb{P}^1 \to \mathbb{P}^1$ *has ramification type* $[(3,1)(3,1)(3,1)]$. *(An example of such a pencil is given by* $x^3(x-2y), y^3(2x+y)$, *the corresponding morphism* $\mathbb{P}^1 \to \mathbb{P}^1$ *is ramified over the points* $[0:1], [1:1]$ *and* $[1:0]$.*)*

In order to realize such a pencil in our correspondence, we need a net of conics for which the discriminant cubic has a set of 3 inflections points whose tangent lines are concurrent. Obviously the only candidates for the discriminant cubic are elliptic curves or nodal cubics. The latter however will not suffice as on a nodal cubic there are exactly 3 inflections points but their tangent lines are not concurrent. On an elliptic curve there are 9 inflections points in total. For $j = 0$ *there exist sets of 3 inflection points whose tangent lines* are *concurrent, if* $j \neq 0$ *this is not possible. This follows from the proof of [AD09, Lemma 1], where the inflection tangents are concurrent if* $c = 0$, *which gives the elliptic curve defined by* $x^3 + y^3 + z^3$, *which indeed has* j*-invariant 0.*

5.7 Final remarks

There is an important difference between the construction of Chapter 4 and the construction of a blowup as in [VdB01] or [RSS14]. In the latter, a point on the point scheme on an arbitrary noncommutative surface is constructed resulting in a noncommutative $\mathrm{Bl}_x \mathbb{P}^2$, whereas the former only works for sheaves of orders where a point outside (an isogeny of) the point scheme is blown up. Hence these situations have no overlap. In other terms, the latter construction gives a deformation of the commutative $\mathrm{Bl}_x \mathbb{P}^2$, whereas there is no commutative counterpart for the former.

Likewise there is an important difference between the notion of a noncommutative $\mathbb{P}^1$-bundle of type $(4,1)$ as in Chapter 3 and that of a noncommutative $\mathbb{P}^1$-bundle of type $(2,2)$ [VdB12]. The latter can be used to construct the noncommutative Hirzebruch surface $\mathbb{F}_1$. It is expected that this construction gives the same surfaces as the blowup from [VdB01], but no proof has been found so far.

We can sum up the situation in Table 5.5, where the rows give the same noncommutative surfaces (albeit conjecturally for the first row), whilst the columns are constructions of an analogous nature. This chapter shows that the bottom row gives isomorphic noncommutative surfaces.

It is possible to degenerate the situation in the bottom row. If one were to blow up a point on the ramification of the maximal order on $\mathbb{P}^2$ and pull back the sheaf of algebras along this morphism in the setting of §5.4, the resulting sheaf of orders is no longer a maximal order, nor does it have finite global dimension. But there exist two maximal orders in which the pullback can be embedded[1], and these sheaves of algebras describe the construction of [VdB01] in the special case where everything is finite over the center. These two maximal orders correspond to the two inverse images under the isogeny $C \to C/\tau$, where C is the point scheme and C/τ the ramification divisor.

The method of comparison we obtained in this chapter suggests that there exists a construction of a noncommutative $\mathbb{P}^1$-bundle which does not use a finite morphism $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ of degree 4, but rather a finite morphism $f\colon C \to \mathbb{P}^1$ of degree 4 where C is the *singular* conic in the net of conics associated to the point which is being blown up. Then the formalism from Chapter 3 yields an abelian category of infinite global dimension, which should be the same as the category associated to the non-maximal order of infinite global dimension by methods similar to the ones used in this paper.

Now the choice of a maximal order (of finite global dimension) containing the pullback gives the construction from the top row in Table 5.5. In the construction of the noncommutative $\mathbb{P}^1$-bundle this seems to correspond to a resolution of singularities of the singular conic[2] C degenerating the degree 4 morphism $f\colon C \to \mathbb{P}^1$ to two degree 2 morphisms $f_i\colon \mathbb{P}^1 \to \mathbb{P}^1$. These then degenerate the noncommutative $\mathbb{P}^1$-bundle of type $(4,1)$ to two different noncommutative $\mathbb{P}^1$-bundles of type $(2,2)$. The details for this comparison can definitely serve for future work.

noncommutative $\mathbb{P}^1$-bundle of type $(2,2)$	abstract blowup on point scheme
noncommutative $\mathbb{P}^1$-bundle of type $(4,1)$	pullback along blowup outside ramification outside ramification

Table 5.5: Four constructions of a noncommutative $\mathbb{F}_1 = \mathrm{Bl}_x \mathbb{P}^2$

[1] At least when the point which is blown up is a smooth point on the ramification. If one blows up a singular point, there is only 1 preimage under the isogeny and the maximal order is unique.

[2] If C is a double line there will be only one morphism $\mathbb{P}^1 \to \mathbb{P}^1$ of degree 2, which should correspond to there only being one choice of maximal order.

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