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Jeroen Wynen

DOCTORAL DISSERTATION

Semi-local normal forms of saddle connections and a study of non-elementary singularities

Promoter: Prof. Dr Peter De Maesschalck | UHasselt

Co-promoters: Prof. Dr Em Patrick Bonckaert | UHasselt
Prof. Dr Vincent Naudot | Florida Atlantic University, USA

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"Pure mathematics is in its way the poetry of logical ideas."

Albert Einstein

The beauty of pure mathematics lies in the undeniable consistency of a mathematical proof. Far too often it is stigmatized as being too abstract, notwithstanding the various applications in for instance astronomy (describing the motion of planets), art (determining the authenticity of paintings), internet security (RSA algorithm), epidemiology (modelling the spread of diseases), etc. My own journey as a PhD student encompasses only a fraction of the vast world of mathematics where I have had the privilege to indulge myself in the beautiful domain of dynamical systems. Therefore I want to express my sincere gratitude to Prof. dr. Peter De Maesschalck for introducing me in this fascinating domain and giving me the opportunity to develop myself in these years as a researcher. Thanks to his invaluable guidance, support and patience, I was able to present this contribution to the theory of smooth dynamical systems.

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Contents

Introduction

In this thesis, we consider planar dynamical systems. According to the dictionary, a dynamical (from Greek *dunamikos*: powerful) system describes the physical and moral forces that produce motion and changes in any field or system. In the mathematical setting we will study how a specific state, for example the position of a particle subjected to the gravitational forces of planets, evolves over the course of time. Often this time-dependence is expressed as a differential equation, a vector field (continuous system, see Section 1.1) or a map (discrete system). One of the first occurrences of dynamical systems is in Newtonian mechanics. Newton's second law states that inertia forces are proportional to mass times accelerations, which are time derivatives of displacements due to motion in a system. This motion can then be modelled by vector fields where the direction and speed of a moving particle is portrayed by a vector. The classical approach at that time to study the motions due to external forces was to compute the solutions explicitly. Unfortunately, these systems tended to have a complex behaviour and are not easily solvable and it is difficult to predict the long-term behaviour.

Henri Poincaré (1854–1912) is often regarded as the founder of the theory of dynamical systems. Some of his main contributions are the qualitative techniques developed for planar vector fields, i.e. ordinary differential equations depending on two variables, in his series of memoirs "Sur les courbes définies par une équation différentielle" (see [43]). In particular, Poincaré showed that the long-term behaviour of a typical bounded orbit in the plane are limit cycles, singularities or graphics, a result that has no equivalent in higher dimensions. With an eye set on the three-body problem in celestial mechanics he tackled questions of stability and behaviour of orbits near periodic orbits in [45], [46], [47] and [48]. He also introduced the notion of bifurcations and the study of families of dynamical systems depending on a parameter in the notion of his study of planetary figures of equilibrium and rotating fluid masses ([44]). A bifurcation occurs when a small change made to the parameter values of a system causes a qualitative or topological change in its behaviour. Since many dynamical systems describing real-life applications depend on parameters obtained from experimental data with some error margins, this is a very important subject to validate the empirical results. In the footsteps of Poincaré, a lot of brilliant mathematicians contributed to the domain of dynamical systems like Lyapunov, Birkhoff, van der Pol, Andronov, Smale, etc., but providing a full history of the theory of dynamical systems is beyond the scope of this thesis.

In 1900, at the International Congress of Mathematicians in Paris, David Hilbert proposed a list of 23 challenging problems for the 20th century (see [26]) of which the 16th problem is one of the most elusive. Whereas the first part of this problem focuses on properties of algebraic curves, we will focus on the second part in terms of dynamical systems. The question is as follows. Is the number of isolated periodic solutions (limit cycles) of a planar polynomial vector field finite? We give a more detailed description in Section 1.4. Due to the complexity of this problem, weakened version of the Hilbert 16th problem, like the infinitesimal problem or Smale's 13th problem concerning Lienard's equation, have been studied but the problem remains unsolved.

One important contribution to the study of Hilbert's 16th problem was made by Robert Roussarie (see [51]). Instead of finding a global bound for the number of limit cycles, he proposed to show that any limit periodic set appearing in an analytic family of vector fields can only perturb to a finite number of limit cycles. This shows the close relation of bifurcation theory to the study of Hilbert's 16th problem. Limit periodic sets include singular points, periodic orbits and polycycles or graphics which are a connected finite union of singular points and obrits between them. The study of bifurcations of singularities is already quite complex, as is illustrated by Bogdanov and Takens in the study of singular points with non-zero nilpotent linear part (see [4], [3] and [55]).

In this thesis, the main objective is to study the cyclicity of graphics consisting of a curve with exactly one singularity. When the singularity is a hyperbolic saddle, one can bound the cyclicity of this homoclinic saddle connection by transforming the vector field near the saddle to the Poincaré-Dulac normal form (see [52]). We elaborate on these techniques in Section 1.3. The typical tool to obtain cyclicity results is the Poincaré map. However, when passing near a singularity, the flow slows down drastically and different phenomena occur. Hence when the singularity is non-elementary, we need advanced techniques to study the transition near the singularity.

A first method consists of transforming the vector field by means of a coordinate change to a simpler form, called a normal form. Poincaré already discovered that the obstacle to transform a vector field to its linear part are so-called resonances. By formal transformations exploiting the linear part, Dulac showed that a vector field can be reduced to only these resonant terms [16], leading to the so-called Poincaré-Dulac normal form. However, when the singularities are nilpotent or even more degenerate, we need to use more information of the vector field than the linear part. Such results using a similar procedure for instance exploit the (quasi-)homogeneous principle part (see [35], [36] or [54]). These results are applicable in a neighbourhood of the singularity and do not give a direct result in the computation of the transition near the singularity.

Another approach is desingularizing the singularity using the blow-up technique introduced by Takens in [56]. The singularity is then replaced by a circle depicting the directions in which one can approach the singularity, called the blow-up locus. It is shown by Dumortier in [17] that every non-elementary singularity satisfying a Łojasiewicz condition can be decomposed to a connection between (semi-)hyperbolic singularities by means of a sequence of blow-ups. One can expect that semi-local normal forms are needed, since they need to be applicable in the vicinity of (a part of) the blow-up locus. Such semi-local normal forms are for instance obtained in [33] near normally hyperbolic invariant manifolds. However in the setting under consideration here, we lose normal hyperbolicity at some point indicating the need for new techniques for semi-local normal forms.

The aim of this thesis is to provide techniques to study the dynamics of smooth planar vector fields in the vicinity of a non-elementary singularity and to bound the cyclicity of a graphic containing this singularity. In many situations a quasi-homogeneous blow-up of the non-elementary singularity leads to two hyperbolic saddles having reciprocal saddle quantities on the blow-up locus. In order to determine the transition near the singularity, one needs to compute the transition near a saddle connection having the aforementioned symmetry. The original objective was to give cyclicity results for cuspidal loops as considered in [21]. However, as shown in the Bogdanov-Takens bifurcation, a perturbation can lead to different kinds of singularities. As a starting point, we only consider a part of the bifurcation diagram where the cusp singularity is preserved as is done in [41]. In this way of thinking, it is quite natural to fix some properties of the vector fields under consideration. We will not give cyclicity results for the full bifurcation, but we believe that this work can be a starting point to simplify computations as we will shortly discuss at the end of the thesis.

The vector fields appearing in the blow-up phase space are of the form

$$\begin{cases} \dot{x} = (1-x^2)\left(\frac{q}{2} + \mathrm{O}\big(1-x^2\big)\right) + \mathrm{O}\big(y\big), \\ \dot{y} = \big(px + \mathrm{O}\big(1-x^2\big)\big)\, y + \mathrm{O}\big(y^2\big), \end{cases}$$

where y denotes the radial direction of the blow-up and x takes on the role of the angular variable. The values p and q are directly related to the spectrum of the hyperbolic saddles situated at $(\pm 1, 0)$. Keeping the application to the blow-up of non-elementary singularities in mind, these eigenvalues will be considered to be fixed, since we only consider bifurcations preserving the nature of the singularity. We also assume that the separatrix connection is fixed, since in applications, we will often encounter the connection on the blow-up locus or at the boundary of a Poincaré disk.

According to the discussion in [24], having a connection between saddles with reciprocal saddle quantities is the most degenerate case in the study of 2-saddle cycles. Besides possible local resonances there is a supplementary resonance between the two saddles. Cyclicity results for these kinds of hyperbolic 2-polycycles with resonant saddles have already been proven in [23] however when studying specific cases one loses information near the fixed connection. The regular transition inbetween the saddles can be of importance, even more than the non-smooth transition close to the saddles themselves. For instance in [41], it can be seen that the regular transition will add a non-smooth contribution to the transition map which is dominant in the transition map. More non-degenerate cases consider connections between non-resonant saddles, i.e. where $p/q \notin \mathbb{Q}$, or situations where the anti-symmetry is dropped and therefore has no additional resonance between the saddles. Cyclicity results for these cases can be found in [38]. The thesis is constructed as follows.

Since we are interested in the transition map when passing near the saddle connection, we need to simplify the vector field in a neighbourhood of this connection in a smooth way. A topological conjugation of planar saddle connections has already been constructed in [57], however this will not be sufficient to obtain cyclicity results. In Chapter 2 we construct a smooth semi-local normal form to which the original vector field is smoothly equivalent. This equivalence is valid close to the connection. This result can be seen as a generalization of the smooth conjugacy of saddle connections in scalar vector fields obtained in [2]. We distinguish several cases depending on the local resonance of the saddles. This is done in a similar way as the Poincaré-Dulac normalization. First, a formal conjugacy is constructed near the separatrix. This inductive procedure provides either resonant terms, which also appear in the local normal forms, or connecting terms. The latter is necessary for gluing both local normal forms together in a smooth way. Then, a smooth change of coordinates is realized by applying a generalized version of Borel's Theorem (see Section 1.2). Using the homotopic method as explained in [18] and [32], one can locally remove the flat terms. We generalize these results to be applicable in a neighbourhood of the connection, which is not straightforward since the connection is not normally hyperbolic.

In [7], the authors show how to linearize the vector field in a neighbourhood of a resonant saddle using logarithmic expressions. This provides a straightforward method to compute the transition map near the saddle. Similarly in Chapter 3, we normally linearise the obtained normal forms. By this we mean that we eliminate the terms that are not linear in terms of y, which is the direction normal to the separatrix connection. Doing so, we lose smoothness of the transformation. However we can express this transformation as a smooth function of some finitely smooth tags of logarithmic form and terms with fractional powers. Since the normally linearized systems admits a straightforward first integral, we can define a constant of motion of the vector field in normal form. This invariant, in turn, leads to a method in computing the transition map through the saddle connection.

As can be seen in applications (Section 5.2), the ratio of eigenvalues can be parameter dependent whereas the symmetry is preserved. We generalize the techniques of Chapter 2 and Chapter 3 to also include these bifurcations in Chapter 4. Similar as in the case of a family of hyperbolic saddles (Section 1.3.2 and Section 1.3.3), we can only obtain finitely smooth normal forms and need to introduce compensators. The results presented here are not yet complete but demonstrate the difficulty of allowing perturbations in the eigenvalues of the saddles.

Eventually we apply these results to some examples, i.e. the cusp and the fake saddle. Cyclicity results for the cuspidal loop when the nature of the singularity is preserved have been obtained in [41]. In contrast to the complex analytic methods used in this paper, we get similar results using real techniques which should be generalizable to higher dimensions. The latter singularity has been studied in [13]. First the vector fields need to be adapted to be suitable for the blow-up procedure and the following normalization. We make use of the typical blow-up charts for a (quasi-) homogeneous blow-up as well construct a parabolic blow-up in order to fit both singularities in one chart. Here, the parametrization of the blow-up circle (except for one point) is induced by the parametrization of a parabola via stereographic projection. By applying the previous results, we obtain the transition maps near the singularities under investigation and acquire partial cyclicity results for graphics containing only one singularity, having the form under consideration. This will be done in Chapter 5.

Chapter 1

Preliminaries: Basic notions of Dynamical Systems and Analysis

In this chapter we introduce the terminology which will be used throughout this thesis. Some basic results from analysis will be presented and generalized to the situation in which we are interested. Finally we describe the well-known Poincaré-Dulac normal form theory and discuss the progress of research concerning Hilbert's 16th problem.

1.1 Dynamical Systems

In this section we introduce the basic notions of dynamical systems and key properties for their qualitative analysis. Since the main focus of this thesis is on planar vector fields, we restrict the definitions to the Euclidean plane $\mathbb{R}^2$. However this terminology can easily be generalized to a higher-dimensional setting. For a complete view on this subject, we refer to [18], [27] or [42].

1.1.1 Vector fields, flows and equivalences

We start by defining smooth vector fields in the plane. Essentially we assign to each point in a domain of definition $U \subset \mathbb{R}^2$ a vector $v \in \mathbb{R}^2$ which can be interpreted as the velocity (speed and direction) of the movement of a particle subjected to this vector field.

Definition 1.1.1. *Let $X : U \to \mathbb{R}^2$ be a C^k, respectively analytic map on an open set $U \subset \mathbb{R}^2$, where $k \in \mathbb{N}^* \cup \{\infty\}$. A* $\mathbf{C^k}$*, respectively* ***analytic vector field*** *$\mathcal{X}$ on U is defined by a map*

$$\mathcal{X} : U \to U \times \mathbb{R}^2 : (x, y) \mapsto ((x, y), X(x, y))\,.$$

Typically, we will refer to the map X as the vector field. Vector fields on an arbitrary smooth manifold M can be defined in a similar way, with the extra condition that the vector $X(p)$ in a point $p \in M$ should lie in the tangent space T_pM. Most of the vector fields encountered in this thesis will be at least C^∞ unless stated otherwise and will be called smooth.

Let $F : I \times U \to \mathbb{R}^2 : (t, x, y) \mapsto (f(t, x, y), g(t, x, y))$ be a smooth function where $I \subset \mathbb{R}$ and $U \subset \mathbb{R}^2$ are open. Consider the system of differential equations given by

$$\begin{cases} \dot{x} = f(t, x, y), \\ \dot{y} = g(t, x, y), \end{cases} \tag{1.1.1}$$

where $\dot{x}$ denotes the derivative $\frac{dx}{dt}$ with respect to time. We have the following theorem.

Theorem 1.1.2. ***(Picard-Lindelöf Theorem)*** *Consider the system* (1.1.1) *together with the initial values $t_0 \in I$ and $(x_0, y_0) \in U$. If F is Lipschitz continuous in the (x, y)-variable, i.e. $\exists L \geq 0 :$*

$$\|F(t, x_1, y_1) - F(t, x_2, y_2)\| \leq L \|(x_1, y_1) - (x_2, y_2)\|, \ (x_1, y_1), (x_2, y_2) \in U, \, t \in I,$$

then (1.1.1) *locally has a unique solution $(x(t), y(t))$ of the same smoothness as F in the (x, y)-variables with $(x(t_0), y(t_0)) = (x_0, y_0)$.*

Remark that the Lipschitz condition is satisfied when the first derivative of F with respect to (x, y) is bounded on U. For instance, when F is C^1 and I is a bounded interval, the function F is locally Lipschitz in the x-variable. If F is smooth, the solutions also depend smoothly on the initial values (t_0, x_0, y_0). An example of a vector field and some solutions is given in Figure 1.1. In this visualization we consider the smooth vector field

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -\sin(x) - \frac{y}{10}. \end{cases} \tag{1.1.2}$$

Each solution can be extended to a ***maximal solution***, i.e. where its domain $M_{(t_0, x_0, y_0)}$ is maximal.

Definition 1.1.3. *Consider the smooth set of differential equations given by* (1.1.1). *If F is a vector field and thus independent of t, we will call this system* ***autonomous***. *Let*

$$\varphi : M_{(t_0, x_0, y_0)} \times I \times U \to \mathbb{R}^2 : (t, t_0, x_0, y_0) \mapsto \varphi(t, t_0, x_0, y_0),$$

denote the unique solution of (1.1.1) *with $\varphi(t_0, t_0, x_0, y_0) = (x_0, y_0)$ and $M_{(t_0, x_0, y_0)}$ the maximal domain of existence for the initial values (t_0, x_0, y_0). The smooth map φ will be called the* ***flow*** *of the system.*

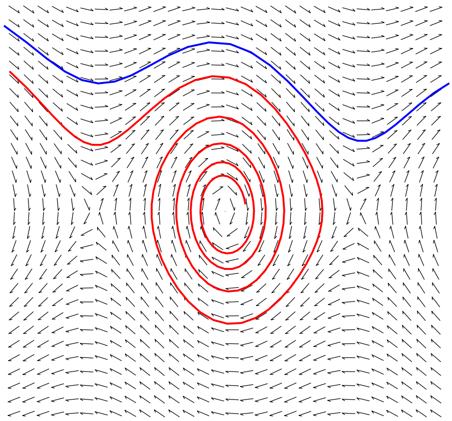


Figure 1.1: Solutions of (1.1.2) through $(0,2)$ (red) and $\left(0, \frac{5}{2}\right)$ (blue)

It is an easy exercise to check the so-called translation property of the flow, i.e.

$$\varphi\left(t, \varphi(s, t_0, x_0, y_0)\right) = \varphi(t+s, t_0, x_0, y_0), \quad \text{for } t, s, t+s \in M_{(t_0, x_0, y_0)}.$$

This allows us to consider (1.1.1) in a dynamic way. We shall assume that all systems are autonomous from now on.

Definition 1.1.4. *A **smooth (continuous) dynamical system** on $\mathbb{R}^2$ is a smooth function $\varphi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ where $\varphi(t,p) = \varphi_t(p)$ satisfies*

(1) $\varphi_0 : \mathbb{R}^2 \to \mathbb{R}^2$ is the identity function, i.e. $\varphi_0(p) = p$;

(2) $\varphi_t \circ \varphi_s = \varphi_{t+s}$ for each $t, s \in \mathbb{R}$.

Typically we assume $t_0 = 0$, hence we omit the t_0-dependence from the notation of the flow. Discrete dynamical systems are defined in a similar fashion where we restrict t to $\mathbb{N}$ or $\mathbb{Z}$. The ***orbit*** $\gamma(p_0)$ of a point $p_0 = (x_0, y_0)$ is defined as

$$\gamma(p_0) = \{\varphi(t, x_0, y_0) \mid t \in M_{(x_0, y_0)}\}.$$

Since this definition is independent of the parametrization, different vector fields can have the same set of orbits. The ***trajectory*** or ***integral curve*** of p_0 should then be understood as the solution $(t, \varphi(t, x_0, y_0))$ for all $t \in M_{(x_0, y_0)}$ parametrized by t.

Theorem 1.1.5. *Let φ be the flow associated to (1.1.1) and let $p_0 = (x_0, y_0) \in \mathbb{R}^2$. Then one of the following statements holds*

(1) $\varphi(\cdot, p_0) : M \to \mathbb{R}^2$ is a bijection onto its image.

(2) $\varphi(\cdot, p_0)$ is constant and defined on $\mathbb{R}$, i.e. $\gamma(p_0)$ is a point.

(3) $\varphi(\cdot, p_0)$ *is periodic and defined on* $\mathbb{R}$ *with minimal period* $T > 0$.

Points having a constant orbit, as in case (2) of the previous theorem, can have a rich dynamic around them and are interesting to study. We call them singularities.

Definition 1.1.6. *Consider a smooth autonomous dynamical system* (1.1.1)*. If the orbit of some* $p \in \mathbb{R}^2$ *is a point, then we call this point an* ***equilibrium*** *or a* ***singular point*** *of the system. Other points are called* ***regular****.*

The collection of all orbits, oriented in positive direction, is called the ***phase portrait*** of the vector field. When we apply a diffeomorphism on $\mathbb{R}^2$, we get a different phase portrait but the dynamics are essentially the same. For this purpose, we introduce the notions of equivalence and conjugacy of vector fields.

Definition 1.1.7. *Let* $F_1 : U_1 \to \mathbb{R}^2$ *and* $F_2 : U_2 \to \mathbb{R}^2$ *be two vector fields defined on open subsets of the plane and* $k \in \mathbb{N}\cup\{\infty\}$*. We say that* F_1 *and* F_2 *are* $\mathbf{C^k}$***-equivalent*** *if there exists a* C^k*-diffeomorphism* $H : U_1 \to U_2$ *sending orbits of* F_1 *to orbits of* F_2 *preserving the orientation.*

As mentioned before, equivalent vector fields have essentially the same phase portrait however the time needed to pass from one point to another subjected to F_1 can be different than for the corresponding points following the flow of F_2. Even more the transition time may depend in a non-continuous manner on the position. If we don't allow time-rescaling, we can impose a stronger condition:

Definition 1.1.8. *Let* $\varphi_1 : \Omega_1 \to \mathbb{R}^2$ *and* $\varphi_2 : \Omega_2 \to \mathbb{R}^2$ *be the respective flows of vector fields* $F_1 : U_1 \to \mathbb{R}^2$ *and* $F_2 : U_2 \to \mathbb{R}^2$ *and* $k \in \mathbb{N} \cup \{\infty\}$*. The vector fields* F_1 *and* F_2 *are said to be* $\mathbf{C^k}$***-conjugate*** *if there exists a* C^k*-diffeomorphism* $H : U_1 \to U_2$ *such that*

$$H(\varphi_1(t, p)) = \varphi_2(t, H(p)), \quad \textit{for every} \ \ (t, p) \in \Omega_1.$$

Observe that an equivalence maps singular points to singular points and periodic orbits to periodic orbits. A conjugacy also preserves the minimal period of a periodic orbit. One can check that a necessary and sufficient condition for H to be a conjugacy is given by

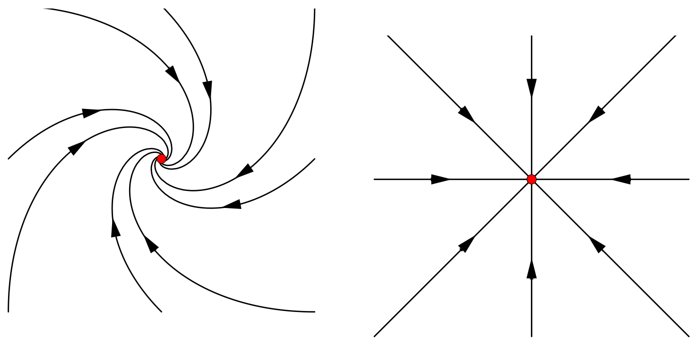
$$DH_pF_1(p) = F_2(H(p)), \quad \text{for every } p \in U_1.$$

A typical example of C^0-conjugate vector fields is given in Figure 1.2.

1.1.2 Singularities and invariant sets

When considering a singularity p, i.e. $\varphi(t, p) \equiv p$ for all t, one typically looks at the linearized system of (1.1.1) associated to p, i.e.

$$\dot{X} = JX, \tag{1.1.3}$$



(a) Spiral Sink (b) Diagonal Sink

Figure 1.2: Example of topological conjugation

where $J = DF(p)$ denotes the Jacobian of F evaluated at p. One can distinguish different types of behaviour depending on the spectrum of J.

Definition 1.1.9. *Consider a smooth autonomous dynamical system* (1.1.1) *with a singularity p. Let $\lambda_1, \lambda_2 \in \mathbb{C}$ denote the eigenvalues of the linearized system* (1.1.3) *at p. If both λ_1 and λ_2 have a non-zero real part, we call p a* ***hyperbolic singularity****. We distinguish different cases*

1. $Re(\lambda_1) < 0, Re(\lambda_2) < 0$*: p is called a* ***sink*** *(see Figure 1.2);*
2. $Re(\lambda_1) > 0, Re(\lambda_2) > 0$*: p is called a* ***source****;*
3. $Re(\lambda_1) \cdot Re(\lambda_2) < 0$*: p is called a* ***saddle*** *(see Figure 1.3), the value $-\lambda_1/\lambda_2$ is called the* ***saddle quantity****.*

If there is exactly one zero eigenvalue, p is said to be ***semi-hyperbolic****. If both eigenvalues are zero but $J \neq 0$, we will call p a* ***nilpotent singularity****. In general, singularities that are not hyperbolic nor semi-hyperbolic will be referred to as* ***non-elementary singular points****.*

Eigenvalues λ with non-zero real part and their corresponding eigenvectors are called ***stable*** if $\mathrm{Re}(\lambda) < 0$ and ***unstable*** if $\mathrm{Re}(\lambda) > 0$. Observe that the linearized system (1.1.3) is attracting in the directions of the stable eigenvectors and repelling in the directions of the unstable eigenvectors.

Suppose that the flow for $p \in \mathbb{R}^2$ is defined for all $t \in \mathbb{R}$. We define

$$\omega(p) = \{Y \in \mathbb{R}^2 \mid \exists (t_n)_{n\in\mathbb{N}} \subset \mathbb{R} : \lim_{n\to\infty} t_n = \infty, \lim_{n\to\infty} \varphi(t_n, p) = Y\},$$

and

$$\alpha(p) = \{Y \in \mathbb{R}^2 \mid \exists (t_n)_{n\in\mathbb{N}} \subset R : \lim_{n\to\infty} t_n = -\infty, \lim_{n\to\infty} \varphi(t_n, p) = Y\}.$$

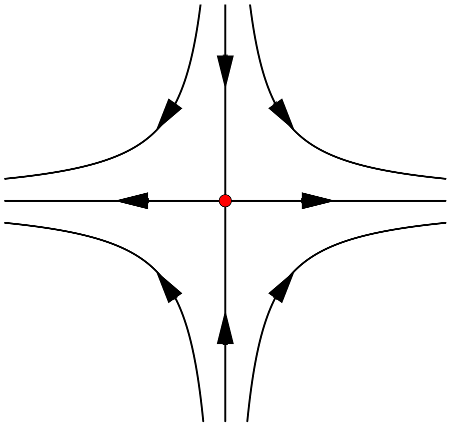


Figure 1.3: Hyperbolic saddle

Definition 1.1.10. *The sets $\omega(p)$ and $\alpha(p)$ are called the **ω-limit set**, respectively **α-limit set** of p.*

Observe that for equilibria and periodic solutions we have

$$\alpha(p) = \omega(p) = \gamma(p).$$

Denote by $\gamma^+(p) = \{\varphi(t,p) \mid t \geq 0\}$ the orbit of p in positive direction. If $\gamma^+(p)$ is contained in a compact subset of $\mathbb{R}^2$, one can show that $\omega(p)$ is a non-empty compact connected subset of $\mathbb{R}^2$. Moreover it has the property that it is an invariant set. Similar properties can be obtained for $\gamma^-(p) = \{\varphi(t,p) \mid t \leq 0\}$ and $\alpha(p)$.

Definition 1.1.11. *A subset $K \subset \mathbb{R}^2$ is called **invariant** for* (1.1.3) *if the orbit $\gamma(p)$ is contained in K for every $p \in K$. We call K **positively invariant**, respectively **negatively invariant**, if $\gamma^+(p) \subset K$, respectively $\gamma^-(p) \subset K$, for every $p \in K$.*

The following result only holds for $\mathbb{R}^2$.

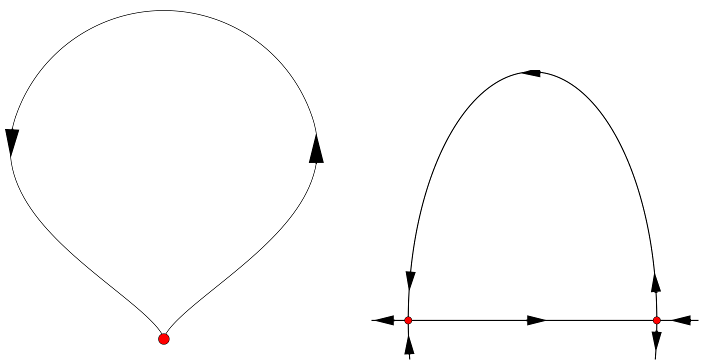
Theorem 1.1.12. ***(Poincaré-Bendixson Theorem)*** *Let $p \in \mathbb{R}^2$ and $\gamma^+(p)$ be its positive orbit associated to* (1.1.3)*. Assume that $\gamma^+(p)$ is contained in a compact subset K of the plane and that the vector field has only a finite number of singularities in K, then one of the following holds*

(1) If $\omega(p)$ contains only regular points, then $\omega(p)$ is a periodic orbit.

(2) If $\omega(p)$ does not contain regular points, then $\omega(p)$ is a singular point.

(3) If $\omega(p)$ contains both regular and singular points, then it consists of a finite number of singularities $P_1, \ldots, P_n$ and a finite number of orbits $\gamma_1, \ldots, \gamma_n$ such that $\alpha(\gamma_i) = P_i$, $\omega(\gamma_i) = P_{i+1}$ for $i = 1, \ldots, n-1$ and $\alpha(\gamma_n) = P_n$, $\omega(\gamma_n) = P_1$. Possible, some of the singular points P_i can coincide.

This third type of limit set is also known as a ***graphic***. Some examples are given in Figure 1.4.



(a) Cuspidal loop (b) 2-saddle cycle

Figure 1.4: Example of graphics

We can also construct other invariant sets using the attracting part of a hyperbolic point. Without loss of generality we can assume that the hyperbolic point coincides with the origin.

Theorem 1.1.13. *(**Stable manifold Theorem**) Let $U \subset \mathbb{R}^2$ be an open neighbourhood of the origin and $F : U \to \mathbb{R}^2$ a smooth vector field with flow φ. Suppose that the origin is a hyperbolic saddle. Then there exists a smooth one-dimensional manifold M_s, respectively M_u, tangent to the stable, respectively unstable, eigenvector at 0 such that M_s, respectively M_u is positively invariant, respectively negatively invariant, and for all $p \in M_s$, respectively $q \in M_u$ we have*

$$\lim_{t\to\infty} \varphi(t,p) = 0, \text{ respectively } \lim_{t\to-\infty} \varphi(t,q) = 0.$$

Definition 1.1.14. *The manifolds M_s and M_u of Theorem 1.1.13 are respectively called the **local stable** and **unstable manifold** of the saddle. The **global (un)stable** manifold is obtained by propagating the local manifolds using the flow in both time-directions.*

The existence of these manifolds depends completely on the presence of a stable or an unstable eigenvector. It is still applicable to the (un)stable direction when the singularity is only semi-hyperbolic and therefore implying the existence of a stable or unstable one-dimensional manifold at the semi-hyperbolic singularity.

1.1.3 Periodic orbits and bifurcations

Let $\gamma = \{\varphi(t, p_0) \mid t \in \mathbb{R}\}$ be a periodic orbit of a smooth vector field $X : U \to \mathbb{R}^2$ with period $T > 0$. We are interested in the behaviour of the orbits close to γ. For this, we introduce the following terminology.

Definition 1.1.15. *Let $I \subset \mathbb{R}$ be an interval. A C^k-map $f : I \to U$ is called a **transverse local section** of X if for every $x \in I$ the vectors $f'(x)$ and $X(f(x))$ are linearly independent. We call $\Sigma = f(I)$ a **transverse section** of X.*

Let Σ be a transverse section of X at p_0 (see Figure 1.5). Since γ is compact and the flow depends continuously on the initial value, there exists a neighbourhood $V \subset \Sigma$ of p_0 such that

$$\forall p_1 \in V, \exists T_1 > 0, \varphi(T_1, p_1) \in \Sigma.$$

We define a function $P : V \to \Sigma : p_1 \to P(p_1)$ such that there is a $T_1 > 0$ where $P(p_1) = \varphi(T_1, p_1)$ and $\varphi(t, p_1) \notin \Sigma$ for $t \in (0, T_1)$. This has the property that $P(p_0) = p_0$.

Definition 1.1.16. *The map $P : V \to \Sigma$ giving the first return map of the flow on Σ is called a **Poincaré map** (see Figure 1.5).*

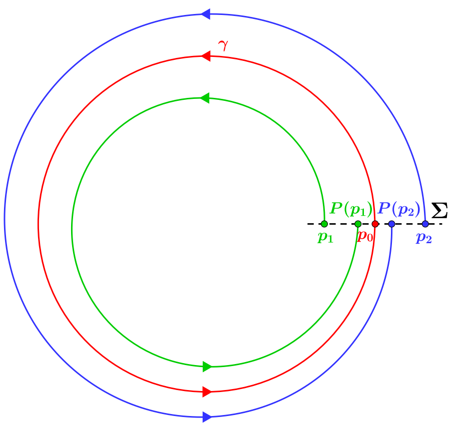


Figure 1.5: A limit cycle and a Poincaré map at a transverse section

A Poincaré map is also referred to as a ***first return map*** or the ***holonomy*** onto the section Σ. The map P reflects the asymptotic behaviour of the orbits of X close to γ. Other periodic orbits close to γ are represented by fixed points of the Poincaré map. If we parametrize the section Σ by $x \in I$, then these fixed points correspond to zeroes of the function

$$\Delta : I_0 \to I : x \mapsto P(x) - x,$$

where I_0 corresponds to V. We will call this map the ***displacement map***. If p_0 corresponds to an isolated zero of Δ, we have the following:

Definition 1.1.17. *A periodic orbit γ is called a **limit cycle** if there exists a neighbourhood W of γ such that there are no singularities or periodic orbits other than Γ in W.*

The asymptotic properties near a limit cycle γ can be categorized in three cases depending on the behaviour on a neighbourhood W of γ:

(1) γ is ***stable*** if $\omega(p_1) = \gamma$ for every $p_1 \in W$ (see Figure 1.5);

(2) γ is ***unstable*** if $\alpha(p_1) = \gamma$ for every $p_1 \in W$;

(3) γ is ***semi-stable*** if $\omega(p_1) = \gamma$ for some $p_1 \in W$ and $\alpha(p_2) = \gamma$ for some $p_2 \in W$.

In fact, it is an easy exercise to verify that (1-3) hold thanks to the Poincaré-Bendixson Theorem (Theorem 1.1.12). In case (3), the orbit splits the tubular neighbourhood W into 2 connected components: one converging towards the limit cycle and the other where points are repelled from the limit cycle. Stable and unstable periodic orbits are also referred to as ***hyperbolic limit cycles***.
Typically one studies the first derivative of the Poincaré map to classify the limit cycle, for instance when $P'(p_0) < 1$, respectively $P'(p_0) > 1$, then γ is stable, respectively unstable. In more degenerate cases, for instance when $P(X_0) = 1$, we look at the sign of the displacement map. Remark that the derivative of the Poincaré map is always positive due to the Jordan curve Theorem (remember we are working in the plane which is orientable) and uniqueness of solution. Orbits in the bounded region with boundary γ stay in the bounded region.

Consider families of vector fields (X_λ) of the form

$$\begin{cases} \dot{x} = f(x, y, \lambda), \\ \dot{y} = g(x, y, \lambda), \end{cases} \tag{1.1.4}$$

where λ typically lies in a compact space $\Lambda \subset \mathbb{R}^k$ for some $k \geq 1$. If the functions f and g depend smoothly on the parameters, then so does the flow. We are interested in changes of the phase portrait depending on the parameters. For this we need the notion of structurally stable vector fields. Consider the unit sphere $S^2 \subset \mathbb{R}^3$ which is the Alexandroff one-point compactification of the Euclidean plane $\mathbb{R}^2$ and denote by $\chi(S^2)$ the space of smooth vector fields on S^2.

Definition 1.1.18. *A vector field $X \in \chi(S^2)$ is **$\mathbf{C}^s$-structurally stable** for $1 \leq s$ if and only if there exists a neighbourhood U of X in $\chi(S^2)$ for the C^s-topology, such that any $Y \in U$ is topologically equivalent to X.*

We define a generic class of vector fields as follows:

Definition 1.1.19. *A vector field $X \in \chi(S^2)$ is* ***Morse-Smale*** *if and only if*

1. *All singular points and periodic orbits are hyperbolic.*

2. *There are no heteroclinic nor homoclinic saddle connections, i.e. orbits with either two different saddles as α- and ω-limit set, respectively the same saddle as α- and ω-limit set.*

These vector fields are stable when we perturb in a C^s-way, i.e.

Theorem 1.1.20. ***(Andronov-Pontryagin, Peixoto)*** *A vector field $X \in \chi(S^2)$ is C^s structurally stable if and only if it is Morse-Smale. Moreover the set of Morse-Smale vector fields is open and dense in $\chi(S^2)$.*

As a result, we know that the subset of Λ where X_λ is structurally stable is an open, dense set $U(X_\lambda)$. On each connected component, the phase portrait is topologically constant. However on its complement $\Sigma(X_\lambda)$, the phase portrait will change.

Definition 1.1.21. *We call $\Sigma(X_\lambda)$ the* ***bifurcation set*** *of the family (X_λ). If the parameter crosses the bifurcation set, we say that the vector field undergoes a* ***bifurcation****.*

Typical phenomena that can occur are for example the collision of two singularities which disappear (e.g. saddle-node bifurcation) or changes in the stability of a singularity (e.g. Hopf bifurcation). We are interested in the number of limit cycles that can originate from these bifurcations. This is done by looking at special invariant sets as we will explain below.

Definition 1.1.22. *A* ***limit periodic set*** *for (X_λ) is a compact non-empty invariant subset Γ of X_{λ_*} such that there exists a sequence $(\lambda_n)_{n\in\mathbb{N}} \to \lambda_*$ in Λ where X_{λ_n} has a limit cycle γ_{λ_n} for every n satisfying:*

$$\gamma_{\lambda_n} \to \Gamma, \quad \textit{for} \;\; n \to \infty,$$

where this convergence is defined in the Hausdorff topology.

A similar result as Theorem 1.1.12 holds for limit periodic sets Γ, i.e. Γ is either a singularity, a periodic orbit or a graphic. The maximal number of limit cycles that can bifurcate from a limit periodic set is called its cyclicity.

Definition 1.1.23. *Let Γ be a limit periodic set of a smooth family X_λ at some value $\lambda_* \in \Lambda$. We say that Γ has* ***cyclicity*** $\leq \mathbf{n}$ *if there exist neighbourhoods $U \subset \mathbb{R}^2$, respectively $V \subset \Lambda$ of Γ, respectively λ_* such that for every $\lambda \in V$, the vector field X_λ has at most n limit cycles in U. The minimal n for which this is true is called the* ***cyclicity*** *of Γ if it exists. Otherwise it is said to be infinite.*

One wants to bound the cyclicity of every limit periodic set that can occur in a family of smooth vector fields. In this way, we can show the finiteness of the number of limit cycles in this family, which is the objective of Hilbert's 16th problem (see Section 1.4).

Theorem 1.1.24. *(see [51]) Let (X_λ) be a C^1 family of vector fields defined on a compact surface K of genus 0 with a compact set of parameters Λ. Then there exists a uniform bound for the number of limit cycles of each vector field X_λ if and only if each limit periodic set Γ of (X_λ) has a finite cyclicity in (X_λ).*

The method of bounding the cyclicity of a particular graphic is typically done by a two-step procedure.
First one chooses a local transverse section $f : I = (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ for some $\varepsilon > 0$ near a regular point p of the graphic such that $f(0) = p$. Since the vector field depends smoothly on the parameters, this remains a transverse section for small perturbations in the vector field. Then one computes a parameter-dependent Poincaré map $P : V \subset \Sigma = f(I) \to \Sigma$. Depending on where a first return map exists, the interval V will be of the form $f((-\delta, \delta))$ or $f([0, \delta))$ for some $\delta > 0$. Computing P explicitly is not a straightforward task, since the vector field slows down near singularities. For instance, the transition near a saddle point, called a ***Dulac map*** D (see Figure 1.6), has a transition time that will diverge to infinity when the initial value approaches the stable manifold. In order to compute D, one simplifies the vector field locally by putting it in local normal form. This can be done by smooth equivalences since a time-reparametrization doesn't influence the Poincaré-map.

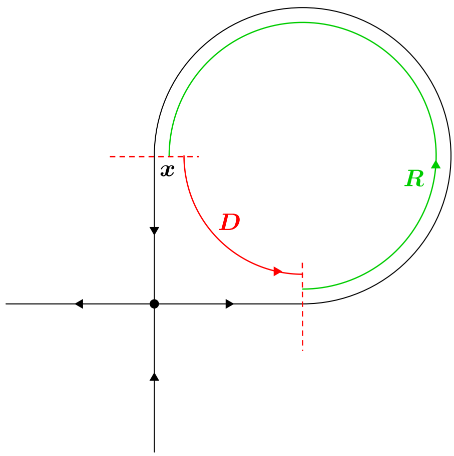


Figure 1.6: Poincaré map of homoclinic saddle connection as composition of the Dulac map D and a regular map R

Once the Poincaré map P is obtained, one studies the displacement map $\Delta = P - \mathrm{Id}$.

By using for instance a division-derivation algorithm, one can put a bound on the cyclicity as an application to Rolle's Theorem.
We present some of the analytic tools that we will need in the rest of the thesis in Section 1.2. Then we will illustrate this method of computing the Poincaré map for a hyperbolic saddle in Section 1.3.

1.2 Basic analytic tools and generalizations

In this section we briefly recollect some basic results in analysis. We state a generalization of Borel's Theorem to a closed interval as has already been done by Whitney. In order to make the text self-contained, we present a different version of the proof without imposing compatibility conditions. We end by listing the definitions and some properties of the Γ function and the hypergeometric function which will appear in applications (see Chapter 5).

1.2.1 Bounding cyclicity using Rolle's Theorem

As discussed in Section 1.1.3, we wish to bound the number of limit cycles that can bifurcate from a limit periodic set Γ. This is done by looking at the displacement map

$$\Delta : (-\varepsilon, \varepsilon) \times \Lambda \to \mathbb{R} : (x, \lambda) \mapsto \Delta(x, \lambda),$$

on a transversal section Σ for some compact set Λ and some $\varepsilon > 0$ (see Figure 1.6 where $\Delta = R \circ D - Id$). If the limit set occurs at λ_*, we assume that $\Delta(0, \lambda_*) = 0$ corresponds to the point on the graphic Γ. Hence the goal is to limit the number of zeroes of Δ for a neighbourhood of λ_* in Λ. A useful tool is Rolle's Theorem.

Theorem 1.2.1. ***(Rolle's Theorem)*** *If $f : U \subset \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b] \subset U$ and differentiable on (a, b) and $f(a) = f(b)$, then there exists at least one c in the open interval (a, b) such that $f'(c) = 0$.*

We can reverse the statement. If f' has at most n zeroes and f is smooth on its domain, then f can't have more than $n + 1$ zeroes. By inductively applying Theorem 1.2.1, we have the following corollary.

Corollary 1.2.2. *Let $f : U \subset \mathbb{R} \to \mathbb{R}$ be smooth on U. If there is a $c \in U$ such that $f^{(n)}(c) \neq 0$ for some $n \geq 1$, then there exists an open neighbourhood $V \subset U$ containing c such that f has at most $n + 1$ zeroes in V.*

1.2.2 Smooth realizations of formal transformations

When constructing normal forms, this is typically done in a formal way. By an induction procedure, one applies an infinite amount of near-identity transformations on the original system, which generically diverges. In order to deal with this problem, one can use the Borel Theorem to get smooth results. This result was first stated in [9]. An elegant proof can be found in [39].

Theorem 1.2.3. ***(Borel Theorem)*** *Let $n \geq 1$ and suppose we are given $c_\alpha \in \mathbb{R}$ for every n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. Then there exists a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ such that*

$$\frac{1}{\alpha!} D^\alpha f(0) = c_\alpha,$$

for every α. Hence the Taylor series of f is given by

$$j_\infty f(0) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha.$$

The formulation of this theorem uses the multi-index notation, i.e.

$$\alpha! = (\alpha_1 + \ldots + \alpha_n)!,\ D^\alpha = \frac{\partial^{\alpha_1 + \ldots + \alpha_n}}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}.$$

We will use $j_k f(0)$ for $k = 1, \ldots, \infty$ throughout this thesis and we shall therefore shortly explain the notation. Consider two smooth functions f and g defined in a neighbourhood of the origin. The germs of f and g at the origin are called ***k-jet equivalent*** for some $k \geq 0$ if they have the same value at 0 and all of their partial derivatives in 0 agree up to order k. The induced set of equivalence classes in $C^\infty(\mathbb{R}^n, \mathbb{R})$ is called the k-jet space $J_0^k(\mathbb{R}^n, \mathbb{R})$. The k-jet $j_k f(0)$ of a function f at 0 then refers to the equivalence class of f in $J_0^k(\mathbb{R}^n, \mathbb{R})$. As representative, one typically fixes a coordinate system and chooses the k-th Taylor polynomial of the function f at 0 which will therefore also be referred to as $j_k f(0)$.

There exist several generalizations of the Borel Theorem, most known is Whitney's extension Theorem (see [58]). Consider a closed subset $A \subset \mathbb{R}^n$. Let $f(x)$ be defined in A and let $m \geq 1$ be an arbitrary but fixed integer. For every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, denote

$$\sigma_\alpha = \alpha_1 + \cdots + \alpha_n.$$

The function $f(x) = f_0(x)$ is of ***class* $\mathbf{C^m}$ *in* $\mathbf{A}$ *in terms of some functions*** $\mathbf{f}_\alpha(\mathbf{x})$ ($\alpha \in \mathbb{R}^n$, $\sigma_\alpha \leq m$) if the functions $f_\alpha(x)$ are defined in A and

$$f_\alpha(y) = \sum_{\sigma_\alpha + \sigma_\beta \leq m} \frac{f_{\alpha+\beta}(x)}{\beta!} (y - x)^\beta + R_\alpha(y, x), \tag{1.2.1}$$

for each $f_\alpha(x)$, where $R_\alpha(y, x)$ has the following property. Given any point x_0 of A and any $\varepsilon > 0$, there is a $\delta > 0$ such that if x and y are any two points of A with $\|x - x_0\| < \delta$ and $\|y - x_0\| < \delta$, then

$$|R_\alpha(y, x)| \leq \|x - y\|^{m - \sigma_\alpha} \varepsilon.$$

Observe that this property is trivially satisfied at every isolated point of A.

Theorem 1.2.4. ***(Whitney extension Theorem, [58])*** *Let A be a closed subset of R^n and let $f(x) = f_0(x)$ be of class C^m in A in terms of the $f_\alpha(x)$ ($\sigma_\alpha \leq m$) in the sense of* (1.2.1)*. Then there is a function $F(x)$ of class C^m in $\mathbb{R}^n$ in the ordinary sense, such that*

1. *$F(x) = f(x)$ in A,*
2. *$D_\alpha F(x) = f_\alpha(x)$ in A ($\sigma_\alpha \leq m$),*
3. *$F(x)$ is analytic in $\mathbb{R}^n \backslash A$.*

Proof: A proof can be found in [58] or [39]. □

Since we are interested in a very specific situation, we do not want to impose the compatibility conditions (1.2.1). We provide a proof for the generalization of Borel's Theorem based upon a proof of [34], where a similar result is constructed for Banach spaces admitting smooth bump functions. It is similar to the proof of Borel in one dimension, but instead of constants we choose a sequence in the space $C^k([-1,1])$ for some $k = 1, \ldots, \infty$.

Theorem 1.2.5. *Let $f_n(x) \in C^k([-1,1])$ for $n \geq 0$ be a sequence of functions with $0 \leq k \leq \infty$. Then there exists a function $f(x,y)$ such that $\partial_x^i \partial_y^j f(x,y)$ is continuous on $[-1,1] \times \mathbb{R}$ for every $0 \leq i \leq k$ and $j \geq 0$ satisfying*

$$\partial_y^n f(x,0) = n! f_n(x),$$

for every $n \geq 0$ and thus

$$j_\infty f(x,0) = \sum_{n \geq 0} f_n(x) y^n.$$

Proof: Let $\rho : \mathbb{R} \to \mathbb{R}$ be a C^∞ bump function around zero, for instance satisfying

$$\rho : u \mapsto \begin{cases} 1 & |u| \leq \frac{1}{4}, \\ 0 & |u| \geq \frac{1}{2}. \end{cases}$$

For $\lambda > 0$, we can easily shrink the support of this function by dilating the variable x

$$\rho(\lambda u) = \begin{cases} 1 & |u| \leq \frac{1}{4\lambda}, \\ 0 & |u| \geq \frac{1}{2\lambda}. \end{cases}$$

Let $\mathcal{F}$ denote the functions on $[-1,1] \times \mathbb{R}$ which are at least continuous and have a compact support. We define a norm $\|\cdot\|$ on $\mathcal{F}$ by

$$\|f\| = \sup\{|f(x,y)| \mid (x,y) \in [-1,1] \times \mathbb{R}\}.$$

We define a sequence of positive numbers $1 \leq \lambda_0 < \lambda_1 < \ldots$, as follows:

$$\lambda_0 = \max\{2\|f_0\|, 1\}, \tag{1.2.2}$$

and

$$\lambda_n = 2\max\{\lambda_{n-1}, \left\| \frac{\partial^{l+m} g_n}{\partial x^l \partial y^m} \right\|^{\frac{1}{n-m-l}} \mid l+m < n, l \leq k\}. \tag{1.2.3}$$

Observe that $\lim_{n\to\infty} \lambda_n = \infty$. Denote

$$g_n(x,y) = f_n(x)\rho(y)y^n.$$

We show that the required function f is given by

$$f(x,y) = \sum_{n\geq 0} \frac{1}{\lambda_n^n} g_n(x, \lambda_n y) = \sum_{n\geq 0} f_n(x)\rho(\lambda_n y)y^n. \tag{1.2.4}$$

Since the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is strictly increasing and approaching infinity, we can easily see that f consists of a finite summation of functions having the required smoothness for every $y > 0$. Therefore it is sufficient to show the smoothness in the neighbourhood of $y = 0$, where f is given by $f(x, 0) = f_0(x)$.

First we show that differentiation of this function in a neighbourhood of $y = 0$ corresponds to term by term differentiation of the series (1.2.4). Therefore we have to show that for every $0 \leq i \leq k$ and $j \in \mathbb{N}$, the series

$$f^{i,j}(x,y) = \sum_{n\geq 0} \frac{1}{\lambda_n^n} \frac{\partial^{i+j} \left(g_n(x, \lambda_n y)\right)}{\partial x^i \partial y^j}, \tag{1.2.5}$$

converges uniformly for λ_n satisfying (1.2.2) and (1.2.3). A straightforward computation shows that due to (1.2.2) and (1.2.3) we have

$$\frac{1}{\lambda_n^{n-m-l}} \left\| \frac{\partial^{l+m} g_n}{\partial x^l \partial y^m} \right\| \leq \frac{1}{2^{n-m-l}}, \quad \text{for every } n \geq m+l. \tag{1.2.6}$$

Due to the general Leibniz rule, we have

$$\begin{aligned} \frac{\partial^{l+m} g_n}{\partial x^l \partial y^m}(x,y) &= \sum_{p=0}^{m} \binom{m}{p} \left(\frac{\partial^l}{\partial x^l} f_n(x)\right) \left(\frac{\partial^p}{\partial y^p} y^n\right) \left(\frac{\partial^{m-p}}{\partial y^{m-p}} \rho(y)\right), \\ &= \sum_{p=0}^{\min\{m,n+1\}} \binom{m}{p} \left(\frac{\partial^l}{\partial x^l} f_n(x)\right) \left(\frac{\partial^{m-p}}{\partial y^{m-p}} \rho(y)\right) (n)_p y^{n-p}, \end{aligned}$$

where $(n)_0 = 1$ and $(n)_p = n(n-1)\cdots(n-p+1)$ for $1 \leq p \leq n$. Since n is fixed, this is also a finite sum and if we confine ourselves to the open set $|y| < 1$, we get

$$\left\| \frac{\partial^{l+m} g_n}{\partial x^l \partial y^m} \right\| \leq \sum_{p=0}^{\min\{m,n+1\}} \binom{m}{p} (n)_p \left\| \frac{\partial^l f_n}{\partial x^l} \right\| \left\| \frac{\partial^{m-p} \rho}{\partial y^{m-p}} \right\|.$$

These are all bounded since these are continuous functions with a compact support. Observe that

$$f^{i,j}(x,y) = \sum_{n\geq 0} \frac{\lambda_n^j}{\lambda_n^n} \frac{\partial^{i+j} g_n}{\partial x^i \partial y^j}(x, \lambda_n y).$$

Thus

$$\begin{aligned}
\sum_{n\geq 0}\left\| \frac{1}{\lambda_n^n}\frac{\partial^{i+j}\left(g_n(x,\lambda_n y)\right)}{\partial x^i \partial y^j}\right\| &\leq \left(\sum_{n=0}^{i+j-1} + \sum_{n\geq i+j}\right)\frac{1}{\lambda_n^{n-j}}\left\|\frac{\partial^{i+j}g_n}{\partial x^i\partial y^j}\right\| \\
&\overset{\lambda_n\geq 1}{\leq} \sum_{n=0}^{i+j-1}\frac{1}{\lambda_n^{n-j}}\left\|\frac{\partial^{i+j}g_n}{\partial x^i\partial y^j}\right\| + \sum_{n\geq i+j}\frac{1}{\lambda_n^{n-j-i}}\left\|\frac{\partial^{i+j}g_n}{\partial x^i\partial y^j}\right\| \\
&\overset{(1.2.6)}{\leq} \sum_{n=0}^{i+j-1}\frac{1}{\lambda_n^{n-j}}\left\|\frac{\partial^{i+j}g_n}{\partial x^i\partial y^j}\right\| + \sum_{n\geq i+j}\frac{1}{2^{n-j-i}} \\
&= \sum_{n=0}^{i+j-1}\frac{1}{\lambda_n^{n-j}}\left\|\frac{\partial^{i+j}g_n}{\partial x^i\partial y^j}\right\| + \sum_{n\geq 0}\frac{1}{2^n},
\end{aligned}$$

which is a convergent series since the first summation is finite and the second a geometric series. Therefore we showed the uniform convergence of (1.2.5).
Hence we have constructed a function $f(x,y)$ of the required smoothness, such that for $i\leq k$, $j\in\mathbb{N}$ and $|y|<1$ we have

$$\frac{\partial^{i+j}f}{\partial x^i\partial y^j}(x,y) = f^{i,j}(x,y),$$

and thus

$$\frac{\partial^n f}{\partial y^n}(x,0) = f^{0,n}(x,0) = n!f_n(x)\rho^{(n)}(0) = n!f_n(x).$$

□

When we apply Theorem 1.2.3 or 1.2.5 to obtain smooth realizations of the constructed formal transformation, typically flat functions arise.

Definition 1.2.6. *A function $f:\mathbb{R}^n\times\mathbb{R}^k\to\mathbb{R}:(x,y)\mapsto f(x,y)$ will be called **(infinitely) flat in $\mathbf{x}$** at $a\in\mathbb{R}^n$ if the Taylor-series of f at x_0 seen as a function of x with parameters y vanishes, i.e.*

$$j_\infty f(x_0,y)\equiv 0.$$

A typical example of a flat function in $\mathbb{R}$ at 0 (see Figure 1.7) is given by

$$\chi_0:\mathbb{R}\to\mathbb{R}:x\mapsto\begin{cases} e^{-\frac{1}{x^2}} & \text{if } x>0,\\ 0 & \text{if } x\leq 0.\end{cases} \tag{1.2.7}$$

1.2.3 The Gamma function and the Hypergeometric function

In this section we list some properties of the Gamma function $\Gamma(\cdot)$ and the hypergeometric function Hypergeom $([\cdot,\cdot],[\cdot],\cdot)$. These will naturally occur when dealing with the normal forms and their applications. These definitions and more can be found in

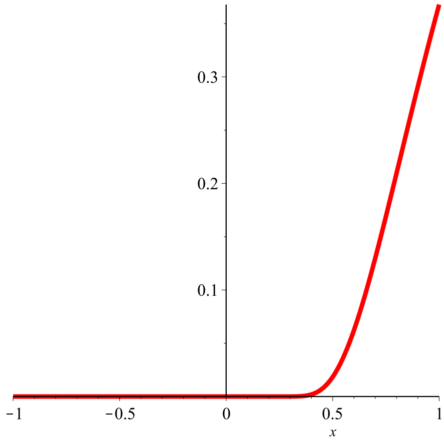


Figure 1.7: Graph of flat function $\chi_0(x)$ in $[-1, 1]$

[25].

The Gamma function Γ can be seen as an extension of the factorial function to all complex numbers except for the negative integers. In this thesis the Gamma function will only occur for positive real numbers, where it is defined via a convergent improper integral.

Definition 1.2.7. *The* ***Gamma function*** $\Gamma(z)$ *is defined for* $z > 0$ *by*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt.$$

Some easy results can be shown.

Lemma 1.2.8. *The function* $\Gamma(z)$ *has the following properties*

1. $\Gamma(z+1) = z\Gamma(z)$ *for* $z > 0$*;*

2. $\Gamma(n) = (n-1)!$ *for* $n \in \mathbb{N}^*$.

For $a > 0$, define

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)},$$

which corresponds to

$$(a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1) \quad \text{for } n = 1, 2, \ldots.$$

The ***hypergeometric function*** is seen as a solution of the differential equation

$$z(1-z)\frac{d^2u}{dz^2} + (c - (a+b+1)z)\frac{du}{dz} - abu = 0.$$

We will denote this solution as

$$\text{Hypergeom}\left([a,b],[c],z\right) = {}_2F_1(a,b;c;z).$$

One can then check that the following is a solution of the differential equation

$$\text{Hypergeom}\left([a,b],[c],z\right) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n, \tag{1.2.8}$$

where we impose that $-c \notin \mathbb{N}$.
This series is absolute convergent for $|z| < 1$. If moreover $\mathrm{Re}(a+b-c) < 0$, there is even absolute convergence for $|z| = 1$.

We can rewrite this function in an integral representation. If Re $c > 0$ and Re $b > 0$, then we have Euler's formula

$$\text{Hypergeom}\left([a,b],[c],z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt. \tag{1.2.9}$$

1.3 Dulac map and cyclicity of a homoclinic saddle connections

In this section we illustrate the well-known techniques in local normal form theory which we want to consider in a semi-local setting in the next chapters. First we describe the Poincaré-Dulac normalization in the special case of a resonant saddle as can be found in [18] in Section 1.3.1. This method of work will be generalized in Chapter 2 in a semi-local context. In Section 1.3.2, we construct a similar normal form when we consider perturbations of a resonant saddle. Then we use the Poincaré-Dulac normal form to construct an asymptotic expansion of the Dulac map near a resonant saddle, considered as an unperturbed vector field as well as part of a smooth family of vector fields in Section 1.3.3. This and further details can also be found in [49] or [50]. Even in the unperturbed system, computing the displacement map will prove to be a lot more complicated when dealing with a non-local displacement near a saddle connection as will be shown in Chapter 3. The unfoldings of saddle connections will then be considered in Chapter 4.

1.3.1 Poincaré-Dulac Normal Form

We display the typical Poincaré-Dulac normalization near a singularity. This method exploits the linear part of the vector field, which is interesting when we consider a hyperbolic or semi-hyperbolic singularity, as can be found in [18] and [32]. For non-elementary singularities, one typically resorts to other normalization methods.

First we prepare the linear part of the vector field. Suppose we have an autonomous system (1.1.1) with a saddle-singularity in the origin. A saddle is ***resonant*** if the associated saddle quantity is given by

$$\lambda = \frac{p}{q}, \quad \text{with } \; p, q \in \mathbb{N}^*, \, \gcd(p, q) = 1.$$

Without loss of generality we can assume, up to a time-reparametrization, that the eigenvalues are exactly given by $-p$ and q. The definition of resonance can be generalized to arbitrary singularities.

Definition 1.3.1. *Let $n \geq 2$. Consider a system $\dot{X} = F(X)$ in $\mathbb{R}^n$ with a singularity $P \in \mathbb{R}^n$ having $\{\lambda_1, \ldots, \lambda_n\}$ as the set of eigenvalues of the matrix $DF(P)$. The singularity is called*

a) ***resonant*** *if there exist relations among the numbers λ_j of the form $\lambda_j = \sum_{i=1}^n k_{i,j}\lambda_i$, where $k_{i,j} \in \mathbb{N}$, $i, j \in \{1, \ldots, n\}$ and $\sum_{i=1}^n k_{i,j} \geq 2$.*

b) ***non-resonant*** *if there exist no such relations.*

c) ***strongly one-resonant*** *if all the non-trivial resonance relations $\lambda_j = \sum_{i=1}^n k_{i,j}\lambda_i$ are generated by a single one $\sum_{i=1}^n k_i^*\lambda_i = 0$, i.e. $k_{i,j} = m\,(k_i^* - \delta_{ij})$ for some $m \in \mathbb{N}_0$, where δ_{ij} denotes the Kronecker delta function.*

It is easily seen that a resonant saddle is strongly one-resonant. If we display all the points (k_1^*, k_2^*) in the grid $\mathbb{N}^2$ for which we have an integral relation, then these will all lie on a line through the origin where the slope depends only on the saddle quantity (see Figure 1.8). As we will show below, these resonant points denote the exponents of monomials that occur in the normal form.

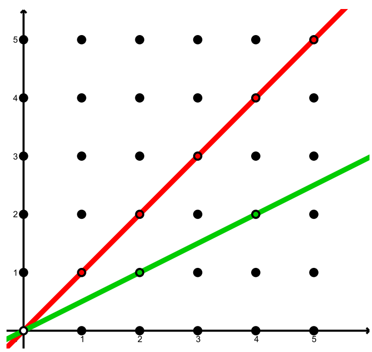


Figure 1.8: Resonant terms when $(\lambda_1, \lambda_2) = (1, -1)$ (red) and $(\lambda_1, \lambda_2) = (1, -2)$ (green)

If we transform the eigenvectors to the standard basis of $\mathbb{R}^2$ by means of a conjugation, then the linearized system $\dot{X} = AX$ is given by

$$\begin{cases} \dot{x} = qx, \\ \dot{y} = -py. \end{cases}$$

The total system can thus be written as

$$\begin{cases} \dot{x} = qx + f(x,y), \\ \dot{y} = -py + g(x,y), \end{cases} \tag{1.3.1}$$

where f and g are both $\mathrm{O}\big(\|(x,y)\|^2\big)$. Now we can apply an induction procedure given by the adjoint action of the linear part A of the vector field. This is a map

$$\mathrm{ad}_m A : H^m(\mathbb{R}^2) \to H^m(\mathbb{R}^2) : X \mapsto [A, X],$$

where $[A, X] = A \circ X - X \circ A$ is seen as a differential operator and $H^m(\mathbb{R}^2)$ denotes the set of polynomial vector fields on $\mathbb{R}^n$ of homogeneous degree m.

Theorem 1.3.2. ***(Poincaré-Dulac Formal Normal Form)*** *Let X be a smooth vector field in a neighbourhood of the origin where $X(0) = 0$, $DX(0) = A$. Then there is an analytic change of coordinates $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $Y = \Phi_*(X)$ is of the form*

$$Y(y) = Ay + g_2(y) + \ldots + g_r(y) + \mathrm{O}\big(\|y\|^{r+1}\big),$$

where $r \geq 1$ and $g_m \in G_m$ for $m = 2, \ldots, r$, where G_m is a complement for the image $B_m = ad_m A(H^m(\mathbb{R}^2))$ in $H^m(\mathbb{R}^2)$.

The monomials in G_i are called the ***resonant terms***. If we apply this theorem by induction for $r \to \infty$, we get a formal transformation leaving only the resonant terms which leaves us with the question of convergence of the transformation. If the convex hull of the eigenvalues in $\mathbb{C}$ does not contain the origin, this form is even polynomial and the reduction is real analytic when starting from an analytic vector field. The transformation is also analytic when the eigenvalues λ_1, λ_2 satisfy the small divisors condition for some $C > 0$, $\nu > 0$:

$$|m_1\lambda_1 + m_2\lambda_2 - \lambda_i| \geq \frac{C}{(m_1 + m_2)^\nu},$$

for all $m_1, m_2 \in \mathbb{N}$ satisfying $m_1 + m_2 \geq 2$ for $i = 1, 2$. When this condition is satisfied, Siegel's Theorem states that the system can be analytically linearized.

We describe the technique used to prove Theorem 1.3.2 (see [18]), applied to the resonant saddle case, by reducing (1.3.1) to its smooth orbital normal form, i.e. the optimal form to which we can reduce (1.3.1) by smooth equivalence. As a consequence of Theorem 1.1.13, there exist smooth graphs $y = \varphi(x)$ and $x = \psi(y)$ representing respectively the unstable and stable manifold of the saddle. By a transformation $\tilde{y} = y - \varphi(x)$, we can assume that the unstable manifold is given by $\tilde{y} = 0$. A similar process can be done for the stable manifold. This is called ***straightening the invariant manifolds***. If we rewrite the new variables again as (x, y), we get

$$\begin{cases} \dot{x} = qx(1 + f_1(x,y)), \\ \dot{y} = -py(1 + g_1(x,y)). \end{cases}$$

The factor $1 + f_1(x, y)$ is locally positive, so by reparametrizing time we arrive at

$$\begin{cases} \dot{x} = qx, \\ \dot{y} = y(-p + g_2(x,y)). \end{cases} \tag{1.3.2}$$

We consider g_2 as a formal power series and apply the transformation

$$(x, y) = (X, Y + \kappa X^m Y^{n+1}), \tag{1.3.3}$$

obtaining

$$\begin{cases} \dot{X} = qX, \\ \dot{Y} = Y\left(-p + g_2(X,Y) + \kappa(pn + qm)X^m Y^n + \mathrm{O}\left(\|(X,Y)\|^{n+m+1}\right)\right). \end{cases}$$

By an adequate choice of κ we can eliminate the term of degree (m, n) in g_2 if and only if

$$pn + qm \neq 0, \quad \text{or equivalently,} \quad n = qk,\ m = pk \text{ for some } k \in \mathbb{N}. \tag{1.3.4}$$

Remark that this corresponds to a point on the resonant line in Figure 1.8. By continuing by induction one can get a formal transformation

$$y = y_\infty \left(1 + \sum_{m+n\geq 2} \kappa_{m,n} x^m y_\infty^n\right), \tag{1.3.5}$$

transforming (1.3.2) to

$$\begin{cases} \dot{x} = qx, \\ \dot{y}_\infty = y_\infty\left(-p + \sum_{n\geq 1} \alpha_n x^{pn} y_\infty^{qn}\right). \end{cases}$$

Theorem 1.2.3 states that the transformation (1.3.5) can be realized as the Taylor series of a smooth function $y = F(x, z)$, such that in the new coordinates

$$\begin{cases} \dot{x} = qx, \\ \dot{z} = z\left(p + h(x^p z^q)\right) + zF(x,z), \end{cases} \tag{1.3.6}$$

for some smooth function h with $h(0) = 0$ and where F is flat at the origin. These flat terms can be removed by a smooth transformation using a theorem by Chen (see [10]).

Theorem 1.3.3. *Let $X = \sum a^i(x)\partial/\partial x^i$ and $Y = \sum b^i(x)\partial/\partial x^i$ be two C^∞ vector fields in $\mathbb{R}^n$ having* 0 *as a hyperbolic singular point. Denote by $\hat{a}^i(x)$ and $\hat{b}^i(x)$ the respective Taylor expansions of $a^i(x)$ and $b^i(x)$ in x. Then there exists a C^∞ transformation about* 0 *which carries X to Y if and only if there exists a formal transformation which carries the formal vector field $\sum \hat{a}^i(x)\partial/\partial x^i$ to $\sum \hat{b}^i(x)\partial/\partial x^i$.*

This is a generalization of a linearization theorem by Sternberg (see [53]).

Theorem 1.3.4. *If X is a smooth vector field with a hyperbolic equilibrium where there are no resonances, then it is smoothly equivalent to its linear part in the neighbourhood of the singularity.*

Following [49], we construct the transformation from Theorem 1.3.3 explicitly. The key element is the normal hyperbolicity of both invariant manifolds. First we decompose the flat terms such that they are flat in the normal direction of one of the invariant manifolds.

Lemma 1.3.5. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth functions in a neighbourhood V of the origin which is flat, i.e. $j_\infty f(0,0) = 0$. Then there is a smooth decomposition $f = g + h$ such that*

$$j_\infty g(x,0) = 0, \quad \textit{and} \quad j_\infty h(0,y) = 0, \ (x,0),(0,y) \in V.$$

These can separately be removed. For example, let X and Y be smooth vector fields such that $j_\infty(Y)(x,0) = 0$. Instead of directly constructing a smooth diffeomorphism φ with $j_\infty(\varphi - Id)(x,0) = 0$ that conjugates X and $X+Y$, one finds a path of transformations

$$\varphi : [0,1] \to \mathrm{Diff}_\infty(\mathbb{R}^2) : \tau \mapsto \varphi_\tau$$

which conjugate X and $X + \tau Y$. Denote $\varphi(x,y,\tau) = \varphi_\tau(x,y)$.

Lemma 1.3.6. *Let Z_τ be a τ-dependent vector field with $j_\infty(Z_\tau)(x,0) = 0$ such that*

$$[X + \tau Y, Z_\tau] = Y. \tag{1.3.7}$$

Then φ_τ determined by $\varphi_0 = \mathrm{Id}$ and

$$\frac{\partial \varphi}{\partial \tau}(x,y,\tau) = Z_\tau(\varphi(x,y,\tau))$$

is a smooth diffeomorphism, conjugating X and $X + \tau Y$ with $j_\infty(\varphi_\tau - Id)(x,0) = 0$.

The vector field from Lemma 1.3.6 can be constructed as follows. Define

$$Z_\tau(x,y) = -\int_0^\infty (F(\gamma(x,y,\tau,t)))^{-1}(Y(\gamma(x,y,\tau,t)))dt \ \text{ for } y \neq 0, \tag{1.3.8}$$

$$Z_\tau(x,y) = 0 \ \text{ for } y = 0,$$

where $F(\gamma(x,y,\tau,t))$ is the fundamental matrix solution along an orbit of the variational equation

$$\begin{aligned}&\frac{d}{dt}F(\gamma(x,y,\tau,t)) = [D_{(x,y)}(X+\tau Y)(\gamma(z,\tau,t))]F(\gamma(x,y,\tau,t)),\\&F(\gamma(x,y,\tau,0)) = I.\end{aligned}$$

It is easy to verify that this vector field Z_τ satisfies (1.3.7). The smoothness and convergence is assured due to the following proposition.

Proposition 1.3.7. *Consider the differential system*

$$\begin{cases}\dot{x} = f(x,y) + \tau Y_1,\\ \dot{y} = y\left(g(x,y) + \tau Y_2\right),\end{cases}$$

on some ε-neighbourhood V of $M = \{y = 0\}$, where f, g, Y_1 and Y_2 are C^∞ functions satisfying $j_\infty Y_1(x,0) = j_\infty Y_2(x,0) = 0$. Suppose there is a compact $C \subset M$ such that f is supported in $C \times \mathbb{R}$ and that $g(x,y) + \tau Y_2 < -\lambda$ for some $\lambda > 0$, $\tau \in [0,1]$ and $(x,y) \in V$. Then there exists a smooth vector field Z_τ satisfying (1.3.7) *and with $j_\infty(Z_\tau)(x,0) = 0$ for every x.*

Applying these previous results we have shown the existence of a smooth equivalence between (1.3.1) and

$$\begin{cases}\dot{x} = qx,\\ \dot{y} = y(-p + h(x^p y^q)),\end{cases} \tag{1.3.9}$$

for some smooth function h, $h(0) = 0$.

1.3.2 Normal form of a family of hyperbolic vector fields

When we perturb a vector field having a resonant saddle, we can do a similar process as in the previous section. However, since the saddle quantity becomes parameter dependent, we need to be careful since more resonant terms can occur. These results can also be found in [32].

Consider a C^∞ family of vector fields X_ε given by

$$\begin{cases}\dot{x} = f(x,y,\varepsilon),\\ \dot{y} = g(x,y,\varepsilon),\end{cases}$$

where $\varepsilon \in \mathbb{R}^k$ is considered in some neighbourhood V of the origin. Suppose that the origin is a resonant saddle for X_0 with $p : q$ resonance as in (1.3.2). By virtue of the implicit function Theorem, we can assume that X_ε has a singularity at the origin for ε in some neighbourhood of the origin. Indeed, due to the hyperbolicity the singularity persists and smoothly depends on the parameter. By a translation, this can be located in the origin. Then we want to prepare the linear part of the system for all parameter values such that we can repeat the Poincaré-Dulac normalization. This is possible since the transformation to Jordan normal form of the linear part depends smoothly on the parameters which is an immediate consequence of the following propositions.

Proposition 1.3.8. *Let A_ε be a real 2×2 matrix where the coefficients are C^∞ functions of a parameter $\varepsilon \in B(0,r)$ such that the eigenvalues have multiplicity 1. Then there exists an $\tilde{r} \in \mathbb{R}^+ \backslash \{0\}$ such that the eigenvalues of A_ε are C^∞ functions of ε for all $\varepsilon \in B(0,\tilde{r})$.*

The proof is essentially given by applying the implicit function Theorem to the characteristic polynomial of the matrix (see [40]). Using the requirement that the eigenvalues have multiplicity 1 we can also show:

Proposition 1.3.9. *Under the same conditions of Proposition 1.3.8, we have that the eigenvectors associated with the eigenvalues of A_ε are C^∞ functions of $\varepsilon \in B(0,\tilde{r})$.*

Observe that a small perturbation of a hyperbolic saddle remains a hyperbolic saddle due to Proposition 1.3.8. Following the transformation that puts the linear part in a diagonal form, we can again straighten the local stable and unstable manifolds since they depend smoothly on the parameters. Hence if we apply this to the case of a perturbation of a $p:q$ resonant saddle, we can assume that the family X_ε is of the form

$$\begin{cases} \dot{x} = qx(1 + f_1(x,y,\varepsilon)), \\ \dot{y} = (-p + \alpha(\varepsilon))y + yg_1(x,y,\varepsilon), \end{cases} \tag{1.3.10}$$

where α depends smoothly on ε and where f_1 and g_1 are $\mathrm{O}\big(\|(x,y)\|^2\big)$. For simplicity we omit the parameter dependence of α. In fact we can embed these parameters in a larger set such that α itself is considered as a parameter. By induction, we apply similar transformations as (1.3.3) but where the coefficient is parameter-dependent. However we have to take the small perturbance by α into account when we consider the resonance relation. Instead of a resonant line as in Figure 1.8, we now obtain a resonant sector as in Figure 1.9, where the angle of the sector reduces when α is restricted to a smaller neighbourhood of the origin. In such a way, we can make sure that no other resonant terms arise up to some homogeneous degree N. For example in Figure 1.9, the sector contains no other resonant terms up to degree 6.
Hence for an arbitrary $N > 1$, we can assume that the vector field (1.3.10) is equivalent to the following pre-normal form

$$\begin{cases} \dot{x} = qx, \\ \dot{y} = y\left(-p + \alpha(\varepsilon) + P_\varepsilon\left(x^p y^q\right) + R_\varepsilon(x,y)\right), \end{cases} \tag{1.3.11}$$

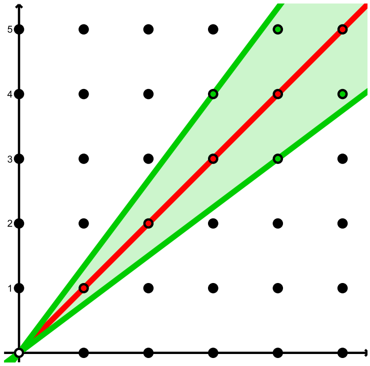


Figure 1.9: Resonant sector (green) when perturbing a $1-1$ resonant saddle (red)

where $R_\varepsilon(x,y) = \mathrm{O}\big(\|(x,y)\|^{N+1}\big)$ and $P_\varepsilon(u)$ is a polynomial in u where the coefficients depend smoothly on ε. In [32] it is shown that by means of a C^n-conjugacy for some n, we can eliminate the term $R_\varepsilon(x,y)$ if N is large enough, depending only on n, p, q. A similar result can also be obtained for diffeomorphisms as has been done in [6] where an explicit lower bound is given for N. Since there is a correspondence between diffeomorphisms and vector fields by means of the time one map (see [19]), we can translate this bound to the setting of vector fields. The method used in [32] is a finitely smooth version of Lemma 1.3.6 and Proposition 1.3.7. In fact the conjugating morphism is defined in the same way, but we loose smoothness since Y in (1.3.7) is only finitely flat.

Lemma 1.3.10. *(see [32]) Let $X_\varepsilon : \dot{x} = F(x,\varepsilon)$ and $Y_\varepsilon : \dot{x} = F(x,\varepsilon) + w(x,\varepsilon)$ be two local families of vector fields in $\mathbb{R}^{2+k}$. Assume that there is a C^k-smooth vector field $Z_{\varepsilon,\tau} : \dot{x} = h(x,\varepsilon,\tau)$ for $\tau \in [0,1]$ such that*

$$[F + \tau w, h] = w, \tag{1.3.12}$$

is satisfied where $h(0,\varepsilon,\tau) = 0$. The families X_ε and Y_ε are then C^k-conjugate.

The solution of the homological equation is of the form (1.3.8) and this expression converges for all derivatives up to order $k+1$ since for a hyperbolic family we have

$$dist\,(\Phi_t(x), M) \leq Ce^{-\lambda t}\, dist(x, M),\ t > 0,$$

where M is the unstable manifold. One needs that the divergence of the vector field F is bounded and since (1.3.8) is defined by an improper integral, we need to make sure that the orbits are complete, i.e. exist for all $t \in \mathbb{R}$. This can be acquired by a partition of unity using bump functions, a procedure called ***globalization*** of the vector field. We do not go in further details but refer to [32].

Proposition 1.3.11. *Let X_ε be a smooth family of vector fields with a saddle at the origin for $\varepsilon \in W$. There exists a function $N(n) : \mathbb{N} \to \mathbb{N}$ such that $N(n) \to \infty$ for $n \to \infty$, and such that if Y_ε is any germ of smooth family of vector fields along $0 \times W$ with the property*

$$j_{N(n)}(Y_\varepsilon - X_\varepsilon)(0) = 0,$$

then, the two family germs X_ε and Y_ε are C^n-conjugate.

If we apply Proposition 1.3.11 to the pre-normal form (1.3.11) for some sufficiently high order, we obtain the following result.

Theorem 1.3.12. *Let X_ε be a smooth deformation of a vector field X_0 having a resonant saddle singularity in the origin, where the saddle quantity is given by p/q, $gcd(p,q) = 1$. Let $n > 1$. There exists a C^n-transformation and a $N(n)$ depending on p, q and n such that X_ε is orbitally equivalent to the family*

$$\begin{cases} \dot{x} = qx, \\ \dot{y} = y\left(-p + \alpha(\varepsilon) + \displaystyle\sum_{i=1}^{N(n)} \alpha_i(\varepsilon) x^{pi} y^{qi}\right), \end{cases} \tag{1.3.13}$$

where ε is confined to some neighbourhood of the origin.

Remark 1.3.13. *The neighbourhood of ε obtained in Theorem 1.3.12 is dependent on the choice of n. Even more, the diameter of this neighbourhood shrinks to 0 when n grows to ∞.*

1.3.3 Dulac map

The normal form (1.3.9) allows us to express the Dulac map in terms of the coefficients of the resonant terms. Let $u = x^p y^q$ denote the resonant monomial. This decouples (1.3.9) in two independent differential equations

$$\begin{cases} \dot{x} = qx, \\ \dot{u} = quh(u) = \sum_{i \geq 1} \alpha_i u^{i+1}. \end{cases} \tag{1.3.14}$$

The solution of the second equation can formally be obtained by substituting

$$u(t, u_0) = \sum_{i=1}^{\infty} g_i(t) u_0^i,$$

where $g_1(0) = 1$ and $g_i(0) = 0$ for $i \geq 2$. A direct computation shows that

$$g_1(t) \equiv 1, \; g_{i+1}(t) = \alpha_i t + P_i(t), \quad \text{for } i \geq 1,$$

where P_i is a polynomial of degree i and with lowest degree 2 having coefficients which are monomials in $\alpha_1, \ldots, \alpha_{i-1}$. Let us compute the Dulac map

$$\tilde{D} : \sigma = [0,1) \times \{1\} \to \tau = \{1\} \times (-1,1) : (x_0, 1) \mapsto (1, D(x_0)),$$

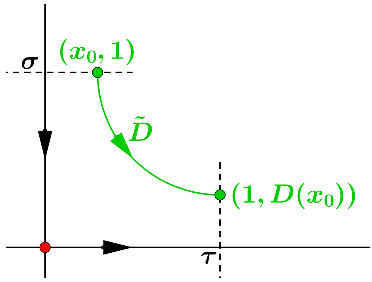


Figure 1.10: Dulac map near a saddle

as in Figure 1.10. The transition time of (1.3.14) is given by $T = -\frac{1}{q}\ln(x_0)$. Therefore we can explicitly express the Dulac map.

Proposition 1.3.14. *For any $n \in \mathbb{N}$ there exists a number $N(n)$ such that*

$$D(x_0)^q = \sum_{i=1}^{N(n)} g_i\left(-\frac{1}{q}\log(x_0)\right)x_0^{pi} + \psi_n(x_0),$$

where $\psi_n(x_0)$ is a C^{np} function, np-flat at $x_0 = 0$, i.e. $j_{np}\psi_n(0) = 0$.

Cyclicity results can only be found when considering unfoldings of saddles, i.e. where the eigenvalues can perturb as in Section 1.3.2. By virtue of Theorem 1.3.12, we can assume that the unfolding of the saddle is given by (1.3.13). In order to compute the Dulac map, we replace the logarithmic function by an adapted compensator known as the ***Ecalle-Roussarie compensators***

$$\omega(x_0, \alpha_0) = \begin{cases} \frac{x_0^{-\alpha_0} - 1}{\alpha_0} & \text{if } \alpha_0 \neq 0, \\ -\log(x_0) & \text{if } \alpha_0 = 0, \end{cases} \tag{1.3.15}$$

where $\alpha_0 = q\alpha(\varepsilon)$. Similar as in the unperturbed case, we can solve for an asymptotic solution.

Theorem 1.3.15. *The transition map D of (1.3.13) can be expanded as*

$$(D(x_0))^q = x_0^p + \alpha_0\left[x_0^p\omega + \ldots\right] + \alpha_1(\varepsilon)\left[x_0^{2p}\omega + \ldots\right] + \cdots \tag{1.3.16}$$

$$+ \alpha_{N-1}(\varepsilon)\left[x_0^{Np}\omega + \ldots\right] + \psi_n(x_0, \varepsilon), \tag{1.3.17}$$

for any $n \in \mathbb{N}$ and $N = N(n)$ is defined in Proposition 1.3.11. The $\ldots$ contain terms $x_0^i\omega^j$ $(i \geq j)$ of higher order and have coefficients which are polynomials in α_i and ψ_n is np-flat at $x_0 = 0$.

The terms $x_0^i\omega^j$ are non-smooth, but have some important properties. First of all, one defines an order on these monomials given by

$$x_0^i\omega^j \prec x_0^k\omega^l \Leftrightarrow (i < k) \text{ or } (i = k \text{ and } j > l).$$

This defines an asymptotic scale, which is used to determine cyclicity.

Definition 1.3.16. *An **asymptotic scale of functions** is a collection $\mathcal{F} = \{f_i\}_{i\in\mathbb{N}}$ of continuous functions $f_i : [0,\varepsilon) \to \mathbb{R}$ where $\varepsilon > 0$ such that $f_0 \equiv 1$, f_1 is strictly monotonic on $(0,\varepsilon)$, and $f_{i+1}(u)/f_i(u) \to 0$ as $u \to 0^+$. We denote this by $f_{i+1} \succ f_i$).*

Secondly these monomials are of Mourtada type which has been introduced in [37].

Definition 1.3.17. *A function $f : [0,\varepsilon) \times U \to \mathbb{R}$ is a **Mourtada type function** if f is smooth for $x > 0$ and if for all integer $k \geq 0$,*

$$\lim_{x\to 0} x^k \frac{\partial^k f}{\partial x^k}(x,y) = 0,$$

uniformly in y.

Exploiting this definition, we can rewrite Theorem 1.3.15 in a weaker form which is easier to use in applications.

Theorem 1.3.18. *The transition map D of (1.3.13) is given by*

$$D(x_0) = x_0^{\frac{p+\alpha(\varepsilon)}{q}} \left(A(\varepsilon) + F(x_0,\varepsilon)\right), \tag{1.3.18}$$

where $A(\varepsilon)$ is a C^∞ positive function and F is of Mourtada type.

If the saddle has a homoclinic connection, i.e. the stable and unstable manifold intersect, one can compute a Poincaré map near this graphic as displayed in Figure 1.6. The dominant term in the Dulac map D is non-smooth, i.e. of the form $x^{p/q}$, and can never be compensated by the regular part R of the displacement map near the connection unless $p = q$. Hence using Theorem 1.2.1 it is easy to show the following result.

Proposition 1.3.19. *(see [1]) Let Γ be a saddle connection with saddle quantity different from 1, then*

$$Cycl(X_\lambda, \Gamma) \leq 1.$$

When the saddle quantity is one, the bound on the cyclicity will depend on the parameters α_i of the initial vector field X_{λ_0}. More precise a higher order of the Dulac map in (1.3.16) or of the regular map R will be dominant. We refer to the literature for these results.

1.4 Hilbert's 16th Problem

At the International Congress of Mathematicians (Paris) in 1900, David Hilbert posed a list of 23 challenging problems in mathematics (see [26]) of which we are interested in the second part of his 16th problem:

In connection with this purely algebraic problem, I wish to bring forward a question which, it seems to me, may be attacked by the same method of continuous variation of coefficients, and whose answer is of corresponding value for the topology of families of curves defined by differential equations. This is the question as to the maximum number and position of Poincaré's limit cycles for a differential equation of the first order and degree of the form

$$\frac{dy}{dx} = \frac{Y}{X},$$

where X and Y are rational integral functions of the n-th degree in x and y. Written homogeneously, this is

$$X\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right) + Y\left(z\frac{dx}{dt} - x\frac{dz}{dt}\right) + Z\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = 0,$$

where X, Y, and Z are rational integral homogeneous functions of the n-th degree in x, y, z, and the latter are to be determined as functions of the parameter t. David Hilbert

We will give a brief overview of results concerning this problem. More details can be found in [29]. In this problem we consider planar polynomial vector fields

$$\begin{cases} \dot{x} = P_n(x, y), \\ \dot{y} = Q_n(x, y), \end{cases}$$

of degree n. Typically the problem is split into three sub-problems, each of subsequently higher complexity:

1. Does a planar polynomial vector field have a finite number of limit cycles? (Dulac problem)

2. Is the number of limit cycles in a planar polynomial vector field of degree n bounded by a constant $H(n)$ depending only on the degree n? (Existential problem)

3. Obtain an explicit expression for $H(n)$.

The number $H(n)$ is known as the ***Hilbert number***. It wasn't until 1923 until there was a first proof of the Dulac problem in its generality by Dulac himself ([16]). However in 1981 a huge gap was found in Dulac's proof ([28], [30]). Ten years later, Ilyashenko ([31]) and Écalle ([22]) provided simultaneously but independently an alternative proof for the existential problem.

Concerning the other sub-problems, no global results have been obtained until now. There exist a lot of results providing a lower bound on the cyclicity. For instance in

[11] the authors construct polynomial vector fields of degree n with $Cn^2 \log(n)$ limit cycles for some constant C, leading to the lower bound $H(n) \geq Cn^2 \log(n)$. Another important contribution to the study of Hilbert's 16th problem is the program connected to the finite cyclicity conjecture posed by Roussarie ([51]). If one can obtain an upper bound of the cyclicity of any limit periodic set in an analytic family of vector fields on the Riemann sphere, then one can bound the number of limit cycles in polynomial vector fields (see Theorem 1.1.24). For instance in [20] the authors reduce the problem of finding a uniform bound on the number of limit cycles in quadratic vector fields to the study of 121 graphics. This thesis can thus be situated as part of this program, where we study graphics containing only one singularity which is non-elementary.

Chapter 2

Semi-local Normal Form Theory near a fixed symmetric saddle connection

In this chapter we study vector fields of the form

$$\begin{cases} \dot{x} = (1-x^2)\left(\frac{q}{2} + \mathrm{O}\big(1-x^2\big)\right) + \mathrm{O}\big(y\big), \\ \dot{y} = \big(px + \mathrm{O}\big(1-x^2\big)\big)\, y + \mathrm{O}\big(y^2\big), \end{cases}$$

which contain a separatrix connection between hyperbolic saddles with symmetric eigenvalues where the connection is fixed. Since we are mainly interested in the case of resonant saddles, we shall denote the eigenvalues at the saddles by p and q. For completeness, we state a result for non-resonant saddles where we shall not change the notation for the sake of convenience but merely write $p/q \notin \mathbb{Q}$. Smooth semi-local normal forms are provided in vicinity of the connection, both in the resonant and non-resonant case. First, a formal conjugacy is constructed near the separatrix. Then, a smooth change of coordinates is realized by generalizing known local results near the hyperbolic points. The results presented here are as discussed in [59].

2.1 Normal forms near saddle connections

This chapter deals with normal forms of (families of) vector fields that are valid in a local neighbourhood of a separatrix connection. The separatrix connection is assumed to be persistent. We will assume furthermore that the eigenvalues of the linear part at the saddle p_1 are the opposite of those of the linear part at the saddle p_2, see Figure 2.1.

While both the persistence of a saddle connection and the opposite set of eigenvalues are clearly non-generic properties that reduce the set of vector fields to which our results are applicable, both assumptions appear naturally in the qualitative treatment

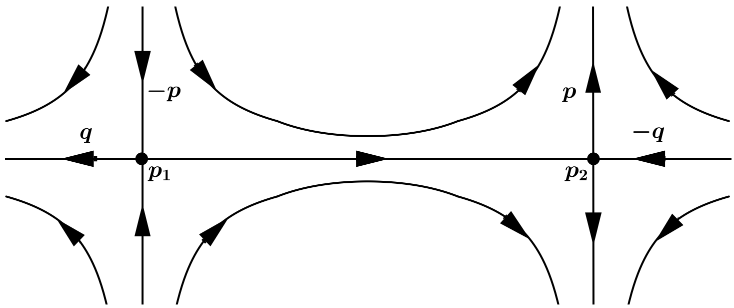


Figure 2.1: Orbits in the neighbourhood of a saddle connection

of planar vector fields. The semi-local situation that is described here is for example quite often found after compactification of the phase space by means of a Poincaré compactification; the separatrix connection is then typically a part of the circle at infinity. Similarly, saddle connections appear in the local study of non-elementary singular points such as points with a nilpotent linear part by blowing up the singularity. We refer to Chapter 5 for applications.

As a preliminary step, we can always smoothly straighten the saddle connection (joint stable and unstable manifolds), and position the two saddles at chosen coordinates, leading to the following smooth vector field that we use as main equation for which a normal form is sought:

$$\begin{cases} \dot{x} &= \left(\frac{q}{2} + O(1-x^2)\right)(1-x^2) + O(y) \\ \dot{y} &= \left(px + O(1-x^2)\right) y + O(y^2), \end{cases} \tag{2.1.1}$$

where we assume that $(\pm 1, 0)$ are the only singularities on the segment $[-1, 1] \times \{0\}$ and that $p > 0$ and $q > 0$ are relatively prime.

Remark 2.1.1. *The results presented here are applicable to families of vector fields of the form* (2.1.1) *where the perturbations only occur in the* $\mathrm{O}\big(1-x^2\big), \mathrm{O}\big(y\big)$ *and* $\mathrm{O}\big(y^2\big)$ *terms. For simplicity we omit this parameter-dependence from the notation. Perturbations where the spectrum at the saddles is perturbed in a symmetric way are considered in Chapter 4.*

The method of work follows the two-step procedure that is also employed in the study of local normal forms near singularities (see Section 1.3.1): we first establish a "formal normal form", and later eliminate the flat terms after applying the generalized Borel Theorem (Theorem 1.2.5). The normal forms that we envisage should be able to encompass all resonance information coming from both saddles. We therefore use semi-local ***resonant monomials*** of the form

$$\left((1-x^2)^p y^q\right)^n \quad \text{and} \quad x\left((1-x^2)^p y^q\right)^n . \tag{2.1.2}$$

Fixing n, a linear combination of both monomials

$$\omega_n := a_n \left((1-x^2)^p y^q\right)^n + b_n x \left((1-x^2)^p y^q\right)^n,$$

has the following asymptotic expansion about $x = \pm 1$:

$$\begin{aligned} \omega_n &= 2^{pn}(a_n + b_n)(1-x)^{pn} y^{qn}(1 + O(1-x)), \qquad && x \to 1 \\ \omega_n &= 2^{pn}(a_n - b_n)(1+x)^{pn} y^{qn}(1 + O(1+x)), \qquad && x \to -1, \end{aligned}$$

showing that any a priori arbitrary combination of resonant monomials in the saddle p_1 and the saddle p_2 can be realised by choosing a_n and b_n properly. We will show the following theorem:

Theorem 2.1.2. *Consider the vector field (2.1.1) with $p \in \mathbb{N}^*$, $q = 1$ (and assuming $(\pm 1, 0)$ are the only two singular points). Then there is a smooth coordinate transformation, defined on a neighbourhood of $[-1, 1] \times \{0\}$, such that the system is orbitally equivalent to*

$$\begin{cases} \dot{x} = \frac{1}{2}(1 - x^2), \\ \dot{y} = y\left[px + g_0((1-x^2)^p y) + x g_1((1-x^2)^p y)\right], \end{cases} \tag{2.1.3}$$

for some smooth functions g_0, g_1 with $g_0(0) = g_1(0) = 0$.

The choice of resonant monomials in (2.1.2) is non-unique, and in fact it seems that other candidates are sometimes more appropriate, we refer to Section 2.3.2 for alternatives. Here, the function g_1 could be seen as the symmetric resonant part, and g_0 as the asymmetric resonant part. This terminology follows from the fact that the transformation $(x, t) \mapsto (-x, -t)$ leaves the symmetric terms unchanged, but changes the asymmetric terms. Intuitively, this means that the asymmetric terms have a dominant effect on the transition map through the connection as we will show in Chapter 3. One of the benefits of using the above method is that the asymmetric resonant monomials satisfy some semi-group property, implying that we can further reduce the asymmetric part to

$$g_0((1-x^2)^p y) = \pm\left[(1-x^2)^{pn} y^n + \alpha(1-x^2)^{2pn} y^{2n}\right].$$

using techniques similar to those in the reduction of hyperbolic saddles (see [32] for example). Here, the coefficient α is a formal invariant as well as the order of the first non-zero non-linear term n. Linearity here should be considered as linear in the direction normal to the connection, i.e. in the y-variable.

The asymmetric part of the normal form has extra integrability properties, just like the Poincaré normal form of a single resonant saddle: writing $w = (1-x^2)^p y$, equation

(2.1.3) reduces to

$$\begin{cases} \dot{x} &= \frac{1}{2}(1-x^2), \\ \dot{w} &= w(g_0(w) + xg_1(w)). \end{cases} \tag{2.1.4}$$

When $g_1 = 0$, there is separation of variables.

When $q \neq 1$, the normal form contains so-called connecting terms besides the resonant monomials. These terms are necessary to glue together the local normal form around the two saddles to form a semi-local normal form along the separatrix connection. Therefore we define the smooth symmetric bump function χ by

$$\chi(x) = \chi_0(1-x^2), \tag{2.1.5}$$

where χ_0 is defined in (1.2.7).

Theorem 2.1.3. *Consider the vector field (2.1.1) with $p, q \in \mathbb{N}^*$ and $\gcd(p,q) = 1$, $q > 1$ (and assuming $(\pm 1, 0)$ are the only two singular points). Then there is a smooth coordinate transformation such that the system is orbitally equivalent to*

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{y} = y\left[px + g_0((1-x^2)^p y^q) + xg_1((1-x^2)^p y^q) + \chi(x)g_2(y)\right], \end{cases} \tag{2.1.6}$$

for some smooth functions g_0, g_1, g_2 with $g_0(0) = g_1(0) = g_2(0) = 0$, where χ is given by (2.1.5).

We will refer to the terms $\chi(x)g_2(y)$ as the ***connecting terms***. Similar connecting terms appear when the ratio of the eigenvalues at the saddles (in this notation given by p/q) is irrational.

Theorem 2.1.4. *Consider the vector field (2.1.1) with $p/q \in \mathbb{R} \setminus \mathbb{Q}$ (and assuming $(\pm 1, 0)$ are the only two singular points). Then there is a smooth coordinate transformation such that the system is orbitally equivalent to*

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{y} = y\left(px + \chi(x)g(y)\right), \end{cases} \tag{2.1.7}$$

for some smooth function g with $g(0) = 0$, where χ is given by (2.1.5).

The kernel function $\chi(x)$ that is used to represent the connectivity is quite arbitrary; in fact what is important is the symmetry with respect to $x \mapsto -x$ and flatness at the saddles. Due to this symmetry we also expect these terms to have a significant effect on the transition map. Alternatively, one could use a bump function that really vanishes near $x = \pm 1$. The choice of this function χ will only affect the connecting terms itself, without changing the resonant terms (if they are present). Other choices of kernel function χ will be discussed in Section 2.3.3.

The chapter is composed as follows. First we describe some results on scalar vector fields which are crucial for the normal form in 2.2. We provide a different but similar proof than described in [2]. In Section 2.3 we elaborate on a formal inductive process to achieve the normal form. Section 2.4 is devoted to removing the flat terms that appear after using Borel's Theorem (Theorem 1.2.5). We conclude by describing the transformation to return to local normal form in Section 2.5.

2.2 Scalar vector fields

In this section, we are interested in the smooth conjugacy classes of vector fields in $\mathbb{R}$ with the following properties. Let $v : \mathbb{R} \to \mathbb{R}$ be a C^∞ complete vector field with exactly two singular points x_1, x_2 (assume $x_1 < x_2$) such that

$$\lambda_1 := v'(x_1) > 0 \text{ and } \lambda_2 := v'(x_2) < 0. \tag{2.2.1}$$

The results in this section can be found in [2]. Two scalar vector fields v and w are (locally) conjugate, if there is a smooth diffeomorphism $\Phi : U \to V$ such that

$$w(x) = (\Phi'(x))^{-1} v(\Phi(x)) \text{ for every } x \in U.$$

A basic theorem in the theory of vector fields is the linearisation of hyperbolic points.

Theorem 2.2.1. *A smooth vector field v in a neighbourhood of a hyperbolic singular point $x_0 \in \mathbb{R}$ is locally smoothly conjugate with the linear vector field $\tilde{v}(x) = \alpha(x - x_0)$ where $\alpha = v'(x_0)$.*

If there is no other singularity than the hyperbolic point, it is globally attracting or repelling. Thus we can pull back every point in its stable/unstable manifold to a sufficiently small neighbourhood of the singular point. This leads to the following global result.

Lemma 2.2.2. *Let $v : \mathbb{R} \to \mathbb{R}$ be a C^∞ complete vector field with a unique hyperbolic singular point $x_0 \in \mathbb{R}$. The smooth linearisation $\varphi : U \to V$, obtained from Theorem 2.2.1, is then global, i.e. $U = V = \mathbb{R}$.*

Proof: Without loss of generality, we can suppose that $x_0 = 0$ is a sink. The proof for a source is similar and can be obtained from the proof for a sink by time reversal. Let $\varphi_0 : U \to V$ be the local linearisation from Theorem 2.2.1, where U, V are neighbourhoods of the origin.
Denote by $F(t, x), G(t, x)$ the flows associated to respectively the original vector field v and the linearized vector field $\tilde{v}$. By definition of conjugation, we have that

$$\varphi_0(F(t, x)) = G(t, \varphi_0(x)), \quad \text{when } F(t, x), x \in U.$$

We now extend φ_0 to a function $\varphi : \mathbb{R} \to \mathbb{R}$ as follows

$$\varphi(x) = G(-T, \varphi_0(F(T, x))),$$

where $T \geq 0$ is chosen such that $F(T, x) \in U$. First we prove that this is indeed a conjugation. Then we prove that this definition is independent of the choice of T. Let $(t, x) \in \mathbb{R} \times \mathbb{R}$ be arbitrary and let $T \geq 0$ be chosen in such a way that

$$F(T, F(t, x)) = F(T + t, x) \in U, \text{ and } F(T, x) \in U.$$

A straightforward computation shows

$$\begin{aligned}
\varphi(F(t, x)) &= G(-T, \varphi_0(F(T + t, x))) \\
&= G(-T, G(t, \varphi_0(F(T, x)))) \\
&= G(t, G(-T, \varphi_0(F(T, x)))) = G(t, \varphi(x)).
\end{aligned}$$

It remains to show that this definition is independent of the choice of T. Let $x \in \mathbb{R}$ be arbitrary and $T_1, T_2 \geq 0$ be such that $F(T_1, x)$ and $F(T_2, x)$ are in U. Suppose that $T_2 \geq T_1$ and let $\tilde{T} = T_2 - T_1$. Using the properties of a flow, we see

$$G(-T_2, \varphi_0(F(T_2, x))) = G(-T_2, G(\tilde{T}, \varphi_0(F(T_1, x)))) = G(-T_1, \varphi_0(F(T_1, x))),$$

which concludes the proof. □

By composing the global linearizations of the previous lemma, we can easily deduce the following result.

Corollary 2.2.3. *Let $v, w : \mathbb{R} \to \mathbb{R}$ be two complete smooth vector fields with a unique hyperbolic point x_1, respectively x_2 such that $v'(x_1) = w'(x_2)$. Then these vector fields are globally smoothly conjugated.*

Remark 2.2.4. *These results remain valid for real analytic vector fields.*

Now return to the complete vector field given in (2.2.1) with 2 singular points x_1, x_2. Denote

$$U_1 = (-\infty, x_2) \text{ and } U_2 = (x_1, \infty)$$

and let F^t be the flow associated to v. From Lemma 2.2.2 we find orientation preserving smooth diffeomorphisms $\Psi_i : U_i \to \mathbb{R}$ as in Figure 2.2 mapping x_i to 0 such that

$$F^t|_{U_i} = \Psi_i^{-1} \Lambda_i^t \Psi_i \text{ for } i = 1, 2,$$

where

$$\Lambda_i^t(x) = \Lambda_i(t, x) = e^{\lambda_i t} x.$$

Define the *gluing morphism* $G := \Psi_2 \circ \Psi_1^{-1} : \mathbb{R}_+ \to \mathbb{R}_-$. This satisfies

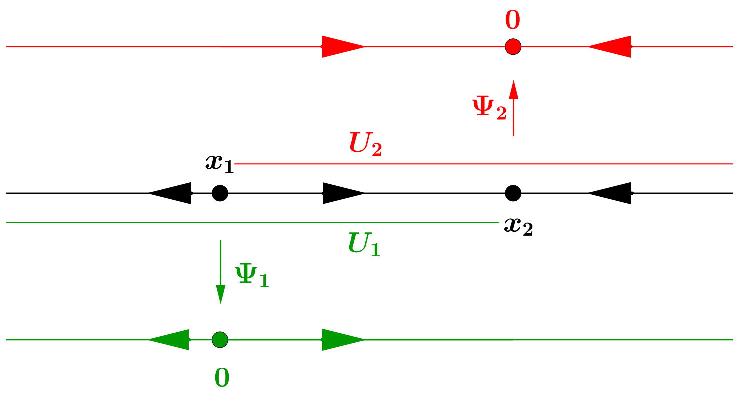


Figure 2.2: Local conjugation to linear vector field

$$G(e^{\lambda_1 t}x) = e^{\lambda_2 t}G(x),\ t \in \mathbb{R}, x \in \mathbb{R}_+.$$

If we differentiate this equality with respect to t and evaluate at $x = 1$, we get

$$G'(e^{\lambda_1 t})\lambda_1 e^{\lambda_1 t} = \lambda_2 e^{\lambda_2 t}G(1) = \lambda_2 G(e^{\lambda_1 t}).$$

However, since t is arbitrary and the origin is hyperbolic, we have a bijection between $\mathbb{R}_+$ and the image of $\Lambda_1(t,1)$, hence

$$xG'(x) = \mu G(x), \quad \text{where } \mu = \frac{\lambda_2}{\lambda_1} \ \text{ for every } x \in \mathbb{R}_+.$$

The general solution of this ODE is given by

$$G(x) = -Cx^{\mu},$$

where C is an arbitrary constant. Since G is orientation preserving, it follows that $C > 0$.

Denote by $C_0(\lambda_1,\lambda_2)$ the set of all smooth vector fields w on $\mathbb{R}$ with exactly two singular points $y_1 < y_2$ such that

$$w'(y_i) = \lambda_i \ \text{ for } i = 1, 2.$$

Theorem 2.2.5. *All complete vector fields $v \in C_0(\lambda_1,\lambda_2)$ are conjugate.*

Proof: Let v and $\tilde{v}$ be complete vector fields in $C_0(\lambda_1,\lambda_2)$ and let F^t and $\tilde{F}^t$ be their corresponding flows. Let $G(x) = -cx^{\mu}$, respectively $\tilde{G}(x) = -\tilde{c}x^{\mu}$, be the associated gluing diffeomorphisms, where Ψ_i, respectively $\tilde{\Psi}_i$ are the corresponding linearising transformations on U_i for $i = 1, 2$ obtained from Lemma 2.2.2. We choose numbers $d_1, d_2 > 0$ such that $d_2 = \frac{c}{\tilde{c}}d_1^{\mu}$ to define

$$\Phi(x) = \begin{cases} \Psi_1^{-1}(d_1\tilde{\Psi}_1(x)) & \text{if } x \in U_1 \\ \Psi_2^{-1}(d_2\tilde{\Psi}_2(x)) & \text{if } x \in U_2 \end{cases}.$$

Observe that for $x \in \mathbb{R}_+$

$$d_2\tilde{G}(x) = -cd_1^\mu x^\mu = G(d_1 x),$$

so the transformation Φ is well-defined. It is straightforward to see that Φ conjugates F^t and $\tilde{F}^t$, i.e. for $x \in U_i$ we have

$$\begin{aligned} F^t(\Phi(x)) &= \Psi_i^{-1}\Lambda_i^t\Psi_i\Psi_i^{-1}(d_i\tilde{\Psi}_i(x)) \\ &= \Psi_i^{-1}(d_i\Lambda_i^t\tilde{\Psi}_i(x)) \\ &= \Phi(\tilde{F}^t(x)). \end{aligned}$$

□

By Theorem 2.2.5, conjugacy between two complete scalar vector fields with exactly two hyperbolic singularities is reduced to comparing the respective eigenvalues. Now we want to find a representative in this class of conjugate vector fields. Therefore we construct a vector field v, with exactly two singular points in -1 and 1, such that

$$v'(-1) = \alpha,\ v'(1) = -\beta, \quad \text{where } \alpha, \beta > 0. \tag{2.2.2}$$

We can consider the polynomial

$$P(x) = \frac{1}{8}(1-x^2)\left((\beta-\alpha)(x+1)^2 + 4\alpha\right).$$

This induces a complete vector field given by

$$v_P(x) = \frac{P(x)}{1+P^2(x)}. \tag{2.2.3}$$

Thus every complete vector field v with exactly two singular points, satisfying equation (2.2.2), is conjugate with the vector field v_P.

However, we are only interested in the conjugation in a neighbourhood of the connection $[x_1, x_2]$ of the hyperbolic singularities x_1, x_2 of a vector field $v \in C_0(\lambda_1, \lambda_2)$. Therefore we will show in Theorem 2.2.8 that we can reduce to a vector field with a simpler form which is not necessarily complete. Since completeness is an essential requirement in Theorem 2.2.5, we construct a complete vector field in $C_0(\lambda_1, \lambda_2)$ which coincides with some incomplete vector field having a similar connection between two singular points. We make this more precise in the following lemma.

Lemma 2.2.6. *Let $v : \mathbb{R} \to \mathbb{R}$ be a C^∞ vector field such that there exist $x_1, x_2 \in \mathbb{R}$ ($x_1 < x_2$) such that*

$$v(x_1) = v(x_2) = 0,\ v'(x_1) = \lambda_1 > 0,\ v'(x_2) = \lambda_2 < 0,$$

and where $v^{-1}(\{0\}) \cap (x_1, x_2) = \emptyset$. *Then there exists a connected neighbourhood* U *of* $[x_1, x_2]$ *and a complete* C^∞ *vector field* $w : \mathbb{R} \to \mathbb{R}$ *such that*

$$v \equiv w \text{ on } U, \quad \text{and } w^{-1}(\{0\}) = \{x_1, x_2\}.$$

Proof: Let W be a connected neighbourhood of $[x_1, x_2]$ such that

$$v^{-1}(\{0\}) \cap W = \{x_1, x_2\}.$$

This is possible since hyperbolic singularities are isolated. Now choose a connected neighbourhood $U \subset W$ of $[x_1, x_2]$ such that $\overline{U} \subset W$. Denote by $\rho : \mathbb{R} \to [0, 1]$ a C^∞ bump function such that

$$\rho \equiv 1 \text{ on } \overline{U} \text{ and } \rho \equiv 0 \text{ on } W^c.$$

Define the vector field w by

$$w(x) = \rho(x)v(x) + z(x),$$

where the C^∞ vector field $z : \mathbb{R} \to \mathbb{R}$ is given by

$$z(x) = \begin{cases} \lambda_1(x - x_1)(1 - \rho(x)) & x < x_1, \\ \lambda_2(x - x_2)(1 - \rho(x)) & x > x_2, \\ 0 & \text{elsewhere.} \end{cases}$$

It is straightforward to see that w is a smooth vector field with only two equilibria x_1, x_2 and $v \equiv w$ on U. It remains to prove the completeness of w. Therefore let $x_0 \in \mathbb{R}$ be arbitrary and denote the flow of w by φ.

Suppose $x_0 \in \{x_1, x_2\}$. In this case it is trivial to conclude that $\varphi(\cdot, x_0)$ is defined for all $t \in \mathbb{R}$.
Suppose $x_0 \in (x_1, x_2)$. There exist neighbourhoods O_1, O_2 of respectively x_1, x_2 such that w is conjugate to the associated linear vector field due to Theorem 2.2.1. Let $T_2 > 0$ be chosen such that $y_2 = \varphi(T_2, x_0) \in O_2$. By the existence of the local conjugacy in O_2, we know that $\varphi(\cdot, y_2)$ is defined for all $t \in [0, \infty)$. Due to the translation property of a flow, this implies that $\varphi(\cdot, x_0)$ is defined for all $t \in [0, \infty)$. Similarly there exists a $T_1 < 0$ such that $y_1 = \varphi(T_1, x_0) \in O_1$. Since $\varphi(\cdot, y_1)$ is defined for all $t \in (-\infty, 0]$, the same is true for $\varphi(\cdot, x_0)$ and thus $\varphi(\cdot, x_0)$ is defined for all $t \in \mathbb{R}$.
Finally suppose $x_0 < x_1$. The case $x_0 > x_2$ can be treated similarly. As in the previous case there exists a $T_1 < 0$ such that $\varphi(T_1, x_0) \in O_1$ and analogously we have that $\varphi(\cdot, x_0)$ is defined for all $t \in (-\infty, 0]$. Now let $T > 0$ be such that $y = \varphi(T, x_0) \in W^c$. This is possible since the singularity x_1 is repelling on $(-\infty, x_2)$. Since $\varphi(t, y) < y$ for all $t > 0$, we know that $\varphi(t, y) \in W^c$ for all positive t where $\varphi(\cdot, y)$ is defined. In

W^C, the flow is the solution of a linear vector field and therefore we know that $\varphi(\cdot, y)$ is defined for all $t \in [0, \infty)$ and so is $\varphi(\cdot, x_0)$. Hence $\varphi(\cdot, x_0)$ is defined for all $t \in \mathbb{R}$. $\square$

Remark 2.2.7. *Observe that v and w in Lemma 2.2.6 are conjugate in neighbourhoods of $[x_1, x_2]$.*

Theorem 2.2.8. *Let $v : (a, b) \to \mathbb{R}$ be a smooth vector field with exactly 2 singular points $x_1, x_2 \in (a, b)$ such that*

$$v'(x_1) = \alpha > 0 \quad \text{and} \quad v'(x_2) = -\beta < 0.$$

Then there exist neighbourhoods U and V of respectively $[x_1, x_2]$ and $[-1, 1]$ and a smooth transformation $y = \varphi(x) : U \to V$ such that the equation $\frac{dx}{dt} = v(x)$ is transformed into

$$\frac{dy}{dt} = (Ay + B)(1 - y^2), \tag{2.2.4}$$

where $A = \frac{\beta - \alpha}{4}$ and $B = \frac{\beta + \alpha}{4}$.

Proof: The vector field given by (2.2.4) only has singularities at $y = \pm 1$ in $[-1, 1]$. Indeed, the only other singularity that can occur is $y = -\frac{B}{A}$ if $A \neq 0$ and this is not an element of $[-1, 1]$ since $\alpha > 0$ and $\beta > 0$. The result then follows directly from Theorem 2.2.5 and Lemma 2.2.6. $\square$

When $\beta = \alpha$ in Theorem 2.2.8, equation (2.2.4) reduces to

$$\frac{dy}{dt} = \frac{\alpha}{2}(1 - y^2).$$

Remark 2.2.9. *In [2], an analogue of Theorem 2.2.5 has been obtained for any finite number of hyperbolic points.*

2.3 Formal transformation

In this section we construct by an induction procedure the formal normal forms leading to Theorems 2.1.2, 2.1.3 and 2.1.4.

2.3.1 Preliminary reduction and the induction procedure

Similar as in Section 1.3.1, we need to prepare the linear part before we can apply an induction procedure. This is summarized in the following lemma.

Lemma 2.3.1. *Consider the smooth vector field (2.1.1) (and assume $(\pm 1, 0)$ are the only two singular points). Then there are neighbourhoods U, V of $[-1, 1] \times \{0\}$ and* *a smooth coordinate transformation $\Phi : U \to V$ such that under this transformation, the system is transformed to a system of the form*

$$\begin{cases} \dot{x} = (1-x^2)\left(\frac{q}{2} + yf(x,y)\right), \\ \dot{y} = y\left(px + yg(x,y)\right), \end{cases} \tag{2.3.1}$$

where f and g are smooth functions of (x, y).

Proof: Following Belitskii's result on normal forms of one-dimensional systems (see Theorem 2.2.8), we can normalize the system reduced to the invariant line $\{y = 0\}$ to $\dot{x} = \frac{q}{2}(1-x^2)$. It allows to write (2.1.1) as

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2) + yf(x,y), \\ \dot{y} = yg(x,y). \end{cases}$$

Since the eigenvalues of the two saddles have not changed, we have $g_0(-1) = -p$ and $g_0(1) = p$ where $g(x, y) = g_0(x) + O(y)$. It means that we can write

$$g_0(x) = px + (1-x^2)h(x),$$

for some smooth function $h(x)$.
By the invariant manifold Theorem, the stable manifold of $(-1, 0)$ can locally be seen as the graph of $x = \varphi_1(y)$. Similarly, the unstable manifold of $(1, 0)$ is given locally by the graph of $x = \varphi_2(y)$. These smooth functions φ_1, φ_2 are defined in a neighbourhood of $y = 0$ and can be approximated by

$$\varphi_1(y) = -1 + \mathrm{O}\big(y\big) \text{ and } \varphi_2(y) = 1 + \mathrm{O}\big(y\big).$$

By applying the transformation

$$(x,y) = \left(\frac{1}{2}\left((1-\bar{x})\varphi_1(\bar{y}) + (1+\bar{x})\varphi_2(\bar{y})\right), \bar{y}\right),$$

we have a 1-1 correspondence between points of the form $(\varphi_1(y), y)$, respectively $(\varphi_2(y), y)$, and $(-1, \bar{y})$, respectively $(1, \bar{y})$. Also remark that this is a well-defined transformation, since the old first coordinate is of the form $\bar{x} + \mathrm{O}\big(\bar{y}\big)$ and this is a near-identity transformation near $\bar{y} = 0$. So by now, we have arrived at the following vector field

$$\begin{cases} \dot{x} = (1-x^2)\left(\frac{q}{2} + yf(x,y)\right), \\ \dot{y} = pxy + (1-x^2)h(x)y + y^2 g(x,y). \end{cases} \tag{2.3.2}$$

For the sake of convenience, we omitted the bars after applying the transformation. In order to get the form (2.3.1), we still have to eliminate the function $h(x)$. Applying the transformation

$$(x,y) = (\bar{x}, \bar{y} + l(\bar{x})\bar{y})$$

for some smooth $l(x)$ defined on a neighbourhood of $[-1,1]$, results to a vector field of the form

$$\begin{cases} \dot{x} = (1-x^2)\left(\frac{q}{2} + yf(x,y) + \mathrm{O}(y)\right), \\ \dot{y} = pxy + \frac{(1-x^2)}{(1+l(x))}\left[-\frac{q}{2}\frac{dl(x)}{dx} + h(x)(1+l(x))\right]y + y^2 g(x,y) + \mathrm{O}(y^2). \end{cases}$$

Again we have omitted the bars. If we denote by $H(x)$ a primitive function of $h(x)$, we can take

$$l(x) = e^{\frac{2H(x)}{q}} - 1$$

in order to get the factor between square brackets zero. Notice that $l(x) \neq -1$ for all values x so that, by the inverse function Theorem, we have indeed a well-defined coordinate transformation in some neighbourhood of $y=0$. □

The form of (2.3.1) is stable upon performing additional changes of coordinates

$$(x,y) = (\bar{x} + (1-\bar{x}^2)\bar{y}A(\bar{x},\bar{y}), \bar{y} + \bar{y}^2 B(\bar{x},\bar{y})).$$

In particular, we will prove two lemmas below that form the basis for the next subsections where a formal transformation

$$(x,y) = \left(\bar{x} + (1-\bar{x}^2)\bar{y}\sum_{n=0}^{\infty} A_n(\bar{x})\bar{y}^n, \bar{y} + \bar{y}^2\sum_{n=0}^{\infty} B_n(\bar{x})\bar{y}^n\right) \tag{2.3.3}$$

is defined, bringing the coefficients with y^n in (2.3.1) in reduced form by induction on n. Transformations like (2.3.3) are formal in the sense that they are formal power series in $\bar{y}$; however the coefficient functions are typically smooth in $\bar{x}$ on a neighbourhood of $[-1,1]$.

Lemma 2.3.2. *Given $n \geq 1$. The effect of the near-identity transformation of the form*

$$(x,y) = (\bar{x}, \bar{y} + h_n(\bar{x})\bar{y}^{n+1})$$

on (2.3.1) is that $yf(x,y)$ is replaced by $\bar{y}f(\bar{x},\bar{y}) + O(\bar{y}^{n+1})$ and that $yg(x,y)$ is replaced by

$$\bar{y}g(\bar{x},\bar{y}) + \left(\frac{-q}{2}(1-\bar{x}^2)h_n'(\bar{x}) - pn\,\bar{x}\,h_n(\bar{x})\right)\bar{y}^n + O(\bar{y}^{n+1}). \tag{2.3.4}$$

Proof: Direct computation. □

Lemma 2.3.3. *Given $n \geq 1$. The effect of the near-identity transformation of the form*

$$(x,y) = (\bar{x} + (1-\bar{x}^2)\ell_n(\bar{x})\bar{y}^n, \bar{y})$$

on (2.3.1) is that $yg(x,y)$ is replaced by $\bar{y}g(\bar{x},\bar{y}) + O(\bar{y}^n)$ and that $yf(x,y)$ is replaced by

$$\bar{y}f(\bar{x},\bar{y}) + \left(-\frac{q}{2}(1-\bar{x}^2)\ell_n'(\bar{x}) - pn\,\bar{x}\,\ell_n(\bar{x})\right)\bar{y}^n + O(\bar{y}^{n+1}). \tag{2.3.5}$$

Proof: Direct computation. □

In contrast to the local normal form theory, we need to consider solutions of a differential equation instead of solving an algebraic equation. We should split in two separate cases depending on whether n is divisible by q (Section 2.3.2) or not (Section 2.3.3).

Remark 2.3.4. *Observe that the perturbations in the functions f and g in Lemma 2.3.2 and Lemma 2.3.3 only arise in terms of higher order which allows us to perform an induction procedure.*

2.3.2 Resonant Terms

The homological equation that we need to consider in order to define the transformations in Lemma 2.3.2 and Lemma 2.3.3 can be smoothly solved as described by the following lemma.

Lemma 2.3.5. *Let $k \geq 1$ be an integer and f an arbitrary smooth function. Then for every $0 \leq l \leq k$, there exist numbers α and β such that the ODE*

$$-\frac{1}{2}(1-x^2)\frac{dy(x)}{dx} - kxy(x) + f(x) = \alpha x(1-x^2)^k + \beta(1-x^2)^l$$

has a smooth solution in a neighbourhood of $[-1,1]$.

Before tackling the proof of Lemma 2.3.5, we need some auxiliary results. Smooth functions $f(x)$ on a neighbourhood of $[-1,1]$ can be decomposed as follows

$$f(x) = \sum_{n=0}^{N}(A_n x + B_n)(1-x^2)^n + (1-x^2)^{N+1}F(x), \tag{2.3.6}$$

for some constants A_n, B_n, where F is a smooth function. Indeed, for a smooth function f, define

$$A_0 = \frac{f(1)-f(-1)}{2},\ B_0 = \frac{f(1)+f(-1)}{2},$$

and let F_1 be the smooth solution satisfying

$$F_1(x) = \frac{f(x)-(A_0x+B_0)}{1-x^2},\ \text{ for } x \neq 1.$$

It is straightforward to see that

$$f(x) = A_0x + B_0 + (1-x^2)F_1(x).$$

By repeating this procedure on F_1 we get (2.3.6) for $N = 2$. By induction we can show (2.3.6) for arbitrary N. When solving the differential equation defined in Lemma 2.3.5, the function f divided by $(1-x^2)^k$ will appear as integrands. Therefore we need a bit of information on the partial fraction decomposition that appears at that point.

Remark 2.3.6. *When dealing with families of vector fields, we need a version of* (2.3.6) *for a family of functions. From the construction it is clear that when f depends smoothly on a parameter, we get the same result as* (2.3.6)*, where A_n, B_n and F depend smoothly on the parameter.*

Lemma 2.3.7. *Let $n \geq 1$. Denote by λ_k, μ_k for $k = 1, \ldots, n$ the coefficients of the partial fraction decomposition of $(1-x^2)^{-n}$, i.e.*

$$\frac{1}{(1-x^2)^n} = \frac{\lambda_1}{(1-x)} + \ldots + \frac{\lambda_n}{(1-x)^n} + \frac{\mu_1}{(1+x)} + \ldots + \frac{\mu_n}{(1+x)^n}.$$

Then we have the following

$$\lambda_1 = \mu_1 = \binom{2n-2}{n-1} 2^{-2n+1}.$$

Proof: We will consider the function $f(z) = (1-z^2)^{-n}$ as a complex analytic function. Denote the functions

$$g(z) = (1+z)^{-n} \text{ and } h(z) = (1-z)^{-n}.$$

Then we can write

$$f(z) = \frac{g(z)}{(1-z)^n} = \frac{h(z)}{(1+z)^n}.$$

This implies that both 1 and -1 are poles of order n since g and h are analytic near respectively 1 and -1. We can look at the Laurent series expansions

$$\sum_{k=-n}^{\infty} a_k (z-1)^k \text{ and } \sum_{k=-n}^{\infty} b_k (z+1)^k$$

of f near respectively 1 and -1. It is easy to see that the first n terms in the partial fraction decomposition of f cover the principal part of the Laurent series around 1 and the same holds for the last n terms and the series around -1. Therefore we have

$$\lambda_1 = -a_{-1} = -\text{Res}(f, 1) \text{ and } \mu_1 = b_{-1} = \text{Res}(f, -1).$$

Clearly,

$$\text{Res}(f, 1) = (-1)^n \frac{g^{(n-1)}(1)}{(n-1)!}.$$

A simple induction argument shows that

$$g^{(n-1)}(z) = (-1)^{n-1} \frac{(2n-2)!}{(n-1)!} (z+1)^{-2n+1},$$

which gives us the expression of λ_1 as in the lemma. Similarly one can find the expression for μ_1. □

Now we are able to give a constructive proof of Lemma 2.3.5.

Proof: *(proof of Lemma 2.3.5)* Variation of constants predicts a solution of the form

$$y(x) = C(x)(1-x^2)^k, \tag{2.3.7}$$

where

$$\frac{dC(x)}{dx} = \frac{2}{(1-x^2)^{k+1}}\left(f(x) - \alpha x(1-x^2)^k - \beta(1-x^2)^l\right).$$

We smoothly decompose f as

$$f(x) = \sum_{n=0}^{k}(A_n x + B_n)(1-x^2)^n + (1-x^2)^{k+1}\tilde{f}(x).$$

This leads to

$$\frac{1}{2}\frac{dC(x)}{dx} = \sum_{n=0}^{k}\frac{A_n x + B_n}{(1-x^2)^{k+1-n}} - \frac{\alpha x}{1-x^2} - \frac{\beta}{(1-x^2)^{k+1-l}} + \tilde{f}(x). \tag{2.3.8}$$

The $\tilde{f}$-term is of course smoothly integrable, so we only focus on the first terms. In order to find a primitive function of this term, we have to decompose this rational function in partial fractions. The process of partial fractions is linear, so we first focus on the partial fraction decomposition of the summation in (2.3.8): Suppose this is given by

$$\frac{\gamma_1}{1-x} + \ldots + \frac{\gamma_{k+1}}{(1-x)^{k+1}} + \frac{\delta_1}{1+x} + \ldots + \frac{\delta_{k+1}}{(1+x)^{k+1}}. \tag{2.3.9}$$

The integration of terms $1/(1+x)^n$ for $1 < n \leq k+1$ contributes, up to a constant, to a term $1/(1+x)^{n-1}$ in $C(x)$. This corresponds, by equation (2.3.7), to a term

$$(x-1)^k(x+1)^{k+1-n}$$

in $y(x)$ and this will clearly be a smooth function on a neighbourhood of $[-1,1]$. However the terms $1/(1+x)$ and $1/(1-x)$ will produce a logarithm which is not defined on a neighbourhood of $[-1,1]$. We therefore aim at compensating the contribution of δ_1 and γ_1 by choosing α and β adequately. Indeed it suffices to show that the partial fraction decomposition of the terms

$$\frac{1}{(1-x^2)^{k+1}}\left(\alpha x(1-x^2)^k + \beta(1-x^2)^l\right)$$

can generate any combination of coefficients in the terms $1/(1+x)$ and $1/(1-x)$, upon varying α and β. First we see that

$$\frac{\alpha x(1-x^2)^k}{(1-x^2)^{k+1}} = \frac{\alpha}{2}\left(\frac{1}{1-x} - \frac{1}{1+x}\right).$$

Therefore it suffices to show that in the decomposition

$$\frac{\beta(1-x^2)^l}{(1-x^2)^{k+1}} = \frac{\lambda_1}{1-x} + \ldots + \frac{\lambda_{k+1-l}}{(1-x)^{k+1-l}} + \frac{\mu_1}{1+x} + \ldots + \frac{\mu_{k+1-l}}{(1+x)^{k+1-l}}$$

we have $\lambda_1 \neq -\mu_1$. This follows from Lemma 2.3.7. □

Now we are able to describe an induction process, putting (2.3.1) into formal normal form when q divides pk for any $k \in \mathbb{N}$, i.e. $q = 1$.

Theorem 2.3.8. *Given (2.3.1) with $q = 1$ and $p \in \mathbb{N}^*$, there exists a formal change of coordinates of the form (2.3.3), formally bringing (2.3.1) in the form*

$$\begin{cases} \dot{x} = (1-x^2)\left[\frac{1}{2} + (f_0((1-x^2)^p y) + x f_1((1-x^2)^p y)\right], \\ \dot{y} = y\left[px + g_0((1-x^2)^p y) + x g_1((1-x^2)^p y)\right], \end{cases} \tag{2.3.10}$$

for some formal series f_0, f_1, g_0, g_1 with $f_0(0) = f_1(0) = g_0(0) = g_1(0) = 0$.

Proof: We assume by induction that the vector field (2.3.1) is such that

$$\begin{aligned} yf(x,y) &= f_0((1-x^2)^p y) + x f_1((1-x^2)^p y) + O(y^n), \\ yg(x,y) &= g_0((1-x^2)^p y) + x g_1((1-x^2)^p y) + O(y^n), \end{aligned}$$

for some polynomials f_0, f_1, g_0, g_1 that vanish at the origin. The original vector field satisfies this claim trivially for $n = 1$, so it suffices to prove the induction step. We first apply Lemma 2.3.3 to replace $yf(x,y)$ by the function in (2.3.5), and keeping $yg(x,y)$ as stated in the induction hypothesis. The term in $\bar{y}^n$ of (2.3.5) is given by

$$f_n(\bar{x}) + \left(-\frac{1}{2}(1-\bar{x}^2)\ell_n'(\bar{x}) - pn\,\bar{x}\,\ell_n(\bar{x})\right),$$

where $f_n(x)$ is the term with order y^n of $yf(x,y)$. We claim that a well-chosen function $\ell_n(\bar{x})$ reduces this term to $(c_n + d_n\bar{x})(1-\bar{x}^2)^{pn}$. In fact, this is the topic of Lemma 2.3.5. This normalizes the function $yf(x,y)$ up to $O(y^{n+1})$.
For the next step, we forget about the bars in the variables and may assume that

$$\begin{aligned} yf(x,y) &= f_0((1-x^2)^p y) + x f_1((1-x^2)^p y) + O(y^{n+1}), \\ yg(x,y) &= g_0((1-x^2)^p y) + x g_1((1-x^2)^p y) + O(y^n), \end{aligned}$$

for some series f_0, f_1, g_0, g_1 that vanish at the origin. We then apply Lemma 2.3.2 to keep $yf(x,y)$ according to the above hypothesis and to replace $yg(x,y)$ by the formula presented in (2.3.4). An identical application of Lemma 2.3.5 shows that we may assume that the coefficient with order y^n in $yg(x,y)$ is of the form $(a_n + b_n x)(1-\bar{x}^2)^{pn}$. This proves the theorem. □

As Lemma 2.3.5 indicates, there is a certain degree of freedom when solving the homological equation at degree y^n. To illustrate this, we prove the following:

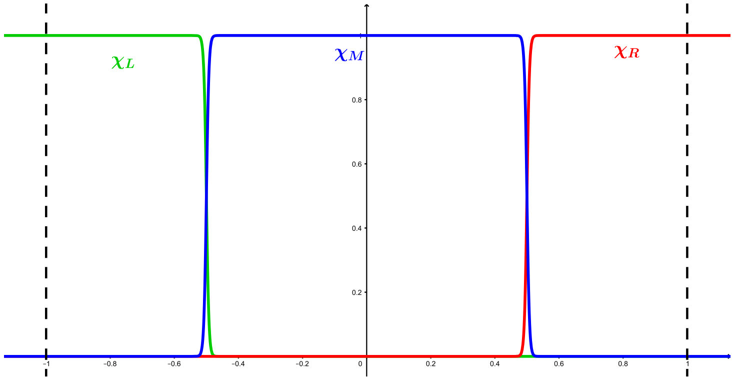


Figure 2.3: Smooth partition of unity and the functions χ_L, χ_M, χ_R.

Theorem 2.3.9. *Given (2.3.1) with $q = 1$ and $p \in \mathbb{N}^*$, there exists a formal change of coordinates of the form (2.3.3), formally bringing (2.3.1) in the form*

$$\begin{cases} \dot{x} = (1 - x^2)\left[\frac{1}{2} + f_0(y) + x f_1((1-x^2)^p y)\right], \\ \dot{y} = y\left[px + g_0(y) + x g_1((1-x^2)^p y)\right], \end{cases} \tag{2.3.11}$$

for some formal power series f_0, f_1, g_0, g_1 that vanish at the origin.

Proof: The proof is completely analogous to that of Theorem 2.3.8, this time replacing the coefficients with y^n of $yf(x,y)$ and $yg(x,y)$ in the induction step by a linear combination of $x(1-x^2)^{pn}$ and $(1-x^2)^0 = 1$ (instead of $x(1-x^2)^{pn}$ and $(1-x^2)^{pn}$). Solving the homological equation at this level is possible, as has been explained in Lemma 2.3.5. $\square$

We believe that in some cases it could be beneficial that the semi-local normal form around the saddle separatrix reduces to the Poincaré normal form of the individual saddles if one restricts to a neighbourhood. This can be achieved using bump functions and a partition of unity. Denote by $\chi_0 : \mathbb{R} \to \mathbb{R}$ the flat function defined in (1.2.7). This function allows us to construct a smooth function which is locally zero near 1 and locally 1 near -1, and vice-versa, i.e.

$$\chi_L : \mathbb{R} \to \mathbb{R} : x \mapsto \frac{\chi_0\left(-\frac{1}{4} - x\right)}{\chi_0\left(-\frac{1}{4} - x\right) + \chi_0\left(x + \frac{3}{4}\right)}, \qquad \chi_R(x) = \chi_L(-x). \tag{2.3.12}$$

The partition of unity is completed by defining $\chi_M(x) = 1 - \chi_L(x) - \chi_R(x)$, which is a function that vanishes near the saddles, see Fig. 2.3.

Theorem 2.3.10. *Given (2.3.1) with $q = 1$ and $p \in \mathbb{N}^*$, there exists a formal change* *of coordinates of the form (2.3.3), formally bringing (2.3.1) in the form*

$$\begin{cases} \dot{x} = (1-x^2)\left[\frac{1}{2} + f_0((1+x)^p y)\chi_L(x) + f_1((1-x)^p y)\chi_R(x)\right], \\ \dot{y} = y\left[px + g_0((1+x)^p y)\chi_L(x) + g_1((1-x)^p y)\chi_R(x)\right], \end{cases} \tag{2.3.13}$$

for some formal power series f_0, f_1, g_0, g_1 that vanish at the origin and where χ_L, χ_R are defined in (2.3.12).

As the proof follows the same guidelines as before, we do not give details here; we only present the lemma that replaces Lemma 2.3.5 for this choice of normal form, allowing to inductively solve the homological equation as before:

Lemma 2.3.11. *Let $k \geq 1$ be an integer, and f an arbitrary smooth function. Then there exist constants α and β such that there exists a smooth solution of*

$$-\frac{1}{2}(1-x^2)\frac{dy(x)}{dx} - kxy(x) + f(x) = \alpha(1+x)^k\chi_L(x) + \beta(1-x)^k\chi_R(x)$$

in a neighbourhood of $[-1,1]$.

Proof: As before, we can decompose

$$f(x) = \sum_{n=0}^{k}(A_n x + B_n)(1-x^2)^n + (1-x^2)^{k+1}\tilde{f}(x),$$

where $\tilde{f}$ is a smooth function. As in the proof of Lemma 2.3.5, we can reduce to finding a smooth solution of

$$\begin{aligned} -\frac{1}{2}(1-x^2)\frac{dy(x)}{dx} - kxy(x) + \sum_{n=0}^{k}(A_n x + B_n)(1-x^2)^n =& \alpha(1+x)^k\chi_L(x) \\ &+ \beta(1-x)^k\chi_R(x), \end{aligned}$$

since the term involving $\tilde{f}(x)$ automatically gives a smooth contribution. For the remaining part, we have to cancel the non-smooth "resonant" terms in

$$\frac{dC(x)}{dx} = \sum_{n=0}^{k}\frac{A_n x + B_n}{(1-x^2)^{k+1-n}} - \frac{\alpha(1+x)^k\chi_L(x)}{(1-x^2)^{k+1}} - \frac{\beta(1-x)^k\chi_R(x)}{(1-x^2)^{k+1}}.$$

Again we rewrite the first term in a partial fraction decomposition

$$\frac{\gamma_1}{1-x} + \ldots + \frac{\gamma_{k+1}}{(1-x)^{k+1}} + \frac{\delta_1}{1+x} + \ldots + \frac{\delta_{k+1}}{(1+x)^{k+1}}.$$

We can show now that there exists an α such that

$$\frac{\delta_1}{1+x} - \alpha\frac{(1+x)^k}{(1-x^2)^{k+1}}\chi_L(x) = \frac{\delta_1}{1+x} - \alpha\frac{1}{(1+x)(1-x)^{k+1}}\chi_L(x)$$

has a smooth integral. First we need a certain partial fraction decomposition. Denote

$$I_n = \frac{1}{(1+x)(1-x)^n}.$$

We can find the following recurrence relation

$$\begin{aligned} I_n &= \frac{1}{(1-x^2)(1-x)^{n-1}} \\ &= \frac{1}{2}\left(\frac{1}{(1-x)} + \frac{1}{(1+x)}\right)\frac{1}{(1-x)^{n-1}} \\ &= \frac{1}{2}\frac{1}{(1-x)^n} + \frac{1}{2}I_{n-1}. \end{aligned}$$

An easy calculation shows

$$I_n = \left(\frac{1}{2}\right)^n \frac{1}{(1+x)} + \sum_{l=1}^{n}\left(\frac{1}{2}\right)^{n-l+1}\frac{1}{(1-x)^l}.$$

On the other hand, if we rewrite

$$\frac{\delta_1}{1+x} = \frac{\delta_1\chi_L(x)}{1+x} + \frac{\delta_1\chi_M(x)}{1+x} + \frac{\delta_1\chi_R(x)}{1+x}$$

we see that the last two terms are smooth functions and thus have smooth integrals. However if we take $\alpha = 2^{k+1}\delta_1$, we see that this first term will be cancelled by

$$-\alpha\frac{1}{(1+x)(1-x)^{k+1}}\chi_L(x) = -\frac{\delta_1\chi_L(x)}{1+x} - \alpha\chi_L(x)\sum_{l=1}^{k+1}\left(\frac{1}{2}\right)^{k-l+2}\frac{1}{(1-x)^l}.$$

Again these last terms will have a smooth integral, since χ_L is locally zero at $x = 1$. A similar construction can be made for β to cancel the non-smooth part of $\frac{\gamma_1}{1-x}$. □

It immediately follows that the coefficients of the formal normal form of f_0 and g_0, respectively f_1 and g_1 in Theorem 2.3.10 correspond to the coefficients of the resonant monomials of the local normal form in respectively the saddle at $(-1,0)$ and $(1,0)$. However for the normal form (2.3.10) it is not clear how to obtain the local normal form coefficients coefficients from f_0, f_1, g_0, g_1. We will describe shortly how to return to the local normal forms in Section 2.5.

2.3.3 Connecting Terms

Contrary to what may be thought, the general $q : -p$ case of (2.3.1) with $q \neq 1$ is not a simple generalization of the $q = 1$ case; its treatment contains an additional difficulty, namely the presence of connecting terms. This is due to a possible lack of compatibility between the Poincaré normal forms at both saddles. We capture these connecting terms by using a smooth symmetric bump function χ as defined in (2.1.5).

Theorem 2.3.12. *Given (2.3.1) with $p, q \in \mathbb{N}_0$, $\gcd(p,q) = 1$, $q > 1$. There exists a formal change of coordinates of the form (2.3.3), formally bringing (2.3.1) in the form*

$$\begin{cases} \dot{x} = (1-x^2)\left[\frac{q}{2} + f_0\left((1-x^2)^p y^q\right) + x f_1\left((1-x^2)^p y^q\right) + \chi(x) f_2(y)\right], \\ \dot{y} = y\left[px + g_0\left((1-x^2)^p y^q\right) + x g_1((1-x^2)^p y^q) + \chi(x) g_2(y)\right], \end{cases} \qquad (2.3.14)$$

for some formal power series f_0, f_1, f_2, g_0, g_1, g_2 *that vanish at the origin and where* χ *is defined in* (2.1.5).

As we can see, the resonant terms will be contained in f_0, f_1 and g_0, g_1. In order to smoothly connect the local normal forms, which are encompassed in the resonant terms, we need to allow some extra terms f_2, g_2, but only in the degrees of y where there is no resonance. These terms will be called the ***connecting terms***. We devote the rest of this section to prove this theorem.

A direct generalization of the method used in the $q = 1$ case leads us to consider again an induction method, iteratively applying Lemma 2.3.2 and 2.3.3, where we need to examine the following differential equation in each step:

$$\frac{-q}{2}\frac{dh(x)}{dx}(1-x^2) - pkxh(x) + F(x) = G(x), \tag{2.3.15}$$

where $F(x)$ is a function coming from the initial vector field that we ideally want to replace by a simple function $G(x)$. Variation of constants for this equation suggests solutions of the form

$$h(x) = C(x)(1-x^2)^{\frac{pk}{q}},$$

which potentially have limited smoothness (in contrast to the suggested solutions that appear in the proof of Lemma 2.3.5 for the $q = 1$ case). We can however safely state that Lemma 2.3.5 generalizes directly to the case where $q|pk$.
So, suppose there is a natural number ℓ such that $k = \ell q$. If we divide (2.3.15) by q, we get a differential equation as in Lemma 2.3.5. Therefore we can take G to be

$$G(x) = (Ax + B)(1-x^2)^{\ell p}, \tag{2.3.16}$$

(or G can take any of the other forms from the formulation of that lemma.)
If on the other hand q does not divide k, there may not always be a smooth solution of the requested form; it is to reduce such coefficients that connecting terms are included. We will prove the following

Lemma 2.3.13. *Let* $p, k, q \in \mathbb{N}_0$ *such that* p *and* q *are relatively prime and suppose that* q *does not divide* k*. Then, for every smooth function* F *defined on a neighbourhood of* $[-1, 1]$*, there exists a number* $C \in \mathbb{R}$ *such that the differential equation*

$$\frac{-q}{2}\frac{dh(x)}{dx}(1-x^2) - pkxh(x) + F(x) = -C\chi(x) \tag{2.3.17}$$

has a smooth solution.

Proof: If there is a smooth solution, then its graph $y = h(x)$ will be tangent to the vector field

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{y} = -pkxy + F(x) + C\chi(x). \end{cases} \tag{2.3.18}$$

This vector field has two singular points, namely $Q_1 = \left(-1, -\frac{F(-1)}{pk}\right)$ and $Q_2 = \left(1, \frac{F(1)}{pk}\right)$. It is easy to verify that both these points are nodes, with respective eigenvalues (q, pk) and $(-q, -pk)$. Recall that a node is resonant if and only if the ratio of eigenvalues belongs to $\mathbb{N} \cup \mathbb{N}^{-1}$. Therefore, due to the conditions of the theorem, these nodes are non-resonant except for the case $p = 1$ and $q = nk$ for some $n \in \mathbb{N}$. In this case, the Poincaré-Dulac normal form at -1 or 1 admits only one resonant term of the form $y^{nk}\frac{\partial}{\partial x}$. However, since the first equation of (2.3.18) is independent of y, this resonant term will not appear and therefore (2.3.18) is locally smoothly linearizable (as is also the case if the ratio of eigenvalues does not belong to $\mathbb{N} \cup \mathbb{N}^{-1}$). The curve $y = 0$ is a smooth separatrix of the linearized system and induces the local existence of a smooth separatrix of (2.3.18). Hence we can locally find smooth functions $\tilde{\varphi}_C(x), \tilde{\psi}_C(x)$ near respectively Q_1, Q_2 whose graphs correspond to the invariant manifolds associated to the eigenvalues q and $-q$.

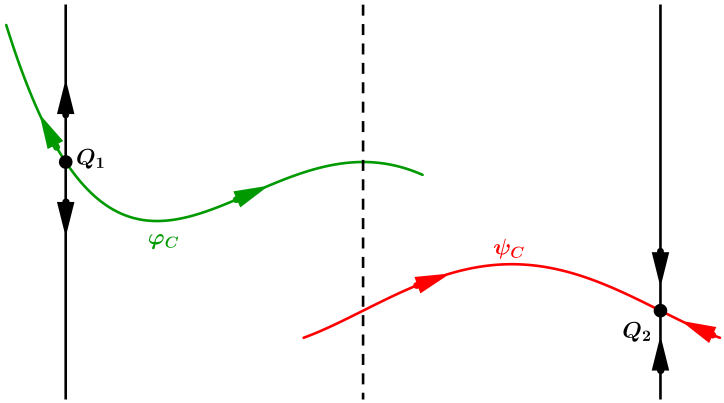


Figure 2.4: Possible phase portrait of (2.3.18)

Observe that $\dot{x} > 0$ for $x \in (-1, 1)$. This implies that we can extend $\tilde{\varphi}_C$ and $\tilde{\psi}_C$ to smooth functions

$$\varphi_C : (-1-\delta, 1) \to \mathbb{R} \text{ and } \psi_C : (-1, 1+\delta) \to \mathbb{R},$$

for some $\delta > 0$ with graphs tangent to the vector field (2.3.18) as in Figure 2.4. Indeed, on any interval $x \in [-1+\varepsilon, 1-\varepsilon]$ where $0 < \varepsilon << 1$, we can bound the derivative

$$\left|\frac{dy}{dx}\right| = \left|\frac{\dot{y}}{\dot{x}}\right| \leq Ay + B,$$

for some constants A and B. By applying the Lemma of Gronwall on this interval, we see that y remains bounded as a function of x. Another way of seeing this, is by examining the line $y_\infty = 0$ at infinity, where $y = \frac{1}{y_\infty}$. In this case we get the system

$$\begin{cases} \dot{x} = \frac{q}{2}(1 - x^2), \\ \dot{y}_\infty = pkxy_\infty - (F(x) + C\chi(x))y_\infty^2, \end{cases}$$

which has a saddle connection formed by $x = \pm 1$ and $z = 0$. Therefore no other solutions of (2.3.18) than the invariant curves $x = \pm 1$ can escape to infinity.

We claim that there exists a $C_0 \in \mathbb{R}$ such that $\varphi_{C_0}(x) = \psi_{C_0}(x)$ for $x \in (-1, 1)$. This will prove the lemma, since the function

$$h : (-1 - \delta, 1 + \delta) \to \mathbb{R} : x \mapsto \begin{cases} \varphi_{C_0}(x) & x \in (-1 - \delta, 1) \\ \psi_{C_0}(x) & x \in (-1, 1 + \delta). \end{cases}$$

will then be a smooth solution of (2.3.17). The connection of both separatrices is the only possibility to obtain a smooth graph (as can be easily verified). By uniqueness of solution in $(-1, 1)$, it suffices to prove that there is a C_0 such that $\varphi_{C_0}(0) = \psi_{C_0}(0)$.

It remains to prove the claim. Let $\Delta_C(x) = \varphi_C(x) - \psi_C(-x)$. This is a smooth invariant curve of

$$\begin{cases} \dot{x} = \frac{q}{2}(1 - x^2), \\ \dot{\Delta} = -pkx\Delta + F(x) + F(-x) + 2C\chi(x), \end{cases}$$

which is defined on $(-1 - \delta, 1)$. Therefore $\Delta_C(x)$ will correspond to the smooth invariant manifold of the above vector field at $(-1, \frac{F(-1)+F(1)}{pk})$. We see that Δ_C depends smoothly on C. Even more, we know that it depends on C in an affine way. Indeed, by using the variational equations, we can see that

$$\frac{\partial^j \Delta_C}{\partial C^j} = 0, \quad \text{for any } j \geq 2.$$

Indeed, $z = \frac{\partial^j \Delta_C}{\partial C^j}(x)$ is a smooth separatrix of the system

$$\begin{cases} \dot{x} = \frac{q}{2}(1 - x^2), \\ \dot{z} = -pkxz, \end{cases}$$

whereas this system only possess two smooth separatrices at $x = \pm 1$, namely $y = 0$ and $x = \pm 1$. Hence it suffices to show $\frac{\partial \Delta}{\partial C}(0) \neq 0$, leading to $C = -\Delta_0(0)/\frac{\partial \Delta}{\partial C}(0)$. Let $w(x) = \frac{\partial \Delta}{\partial C}(x)$. Then, by the variational equation, this is a smooth invariant curve of the vector field

$$\begin{cases} \dot{x} = \frac{q}{2}(1 - x^2), \\ \dot{w} = -pkxw + 2\chi(x). \end{cases}$$

Since we are interested at the value at 0, we can restrict our attention to the interval $[-1,0]$. There we can apply the smooth transformation $x=-\sqrt{1-u}$. In this new coordinate, we get that $w(u)$ is an invariant curve of

$$\begin{cases} \dot{u} = -qu, \\ \dot{w} = -pkw - 2\frac{\chi\left(-\sqrt{1-u}\right)}{\sqrt{1-u}}. \end{cases}$$

Let

$$w_0 = w(u)\big|_{u=1} = w(x)\big|_{x=0}.$$

The general solution for an arbitrary initial value w_0 is given by

$$\begin{aligned} w(u) &= w_0 u^\lambda - \int_u^1 \frac{u^\lambda}{s^\lambda} \frac{2\chi\left(-\sqrt{1-s}\right)}{s\sqrt{1-s}}\,\mathrm{d}s \\ &= \underbrace{u^\lambda\left(w_0 - \int_0^1 \frac{2\chi\left(-\sqrt{1-s}\right)}{s^{\lambda+1}\sqrt{1-s}}\,\mathrm{d}s\right)}_{P_1} + u^\lambda \underbrace{\int_0^u \frac{2\chi\left(-\sqrt{1-s}\right)}{s^{\lambda+1}\sqrt{1-s}}\,\mathrm{d}s}_{P_2}, \end{aligned} \tag{2.3.19}$$

where $\lambda = \frac{pk}{q}$. Now we are interested for which w_0 we get a smooth solution. First notice that the integrand of P_2 is a smooth function, which is flat at $s=0$. Thus the integral will still be a smooth function, flat at $u=0$. This makes $u^\lambda\, P_2$ a smooth function of u. Hence if $P_1 = 0$ we have a smooth graph $w(u)$ near 0. Therefore we take the initial value to be

$$w_0 = \int_0^1 \frac{2\chi\left(-\sqrt{1-s}\right)}{s^{\lambda+1}\sqrt{1-s}}\,\mathrm{d}s.$$

Now remark that the integrand is a positive function, and $w_0 > 0$ is a finite value since

$$\int_0^1 \frac{2\chi\left(-\sqrt{1-s}\right)}{s^{\lambda+1}\sqrt{1-s}}\,\mathrm{d}s \leq A \int_0^1 \frac{1}{\sqrt{1-s}}\,\mathrm{d}s = 2A,$$

with $A = \sup_{s\in[0,1]} \frac{2\chi\left(-\sqrt{1-s}\right)}{s^{\lambda+1}}$ which is finite since this is a smooth function on a compact interval. □

Remark 2.3.14. *The lemma above remains valid if we substitute $\chi(x)$ by $\chi_M(x)$ defined in Figure 2.3.*

This lemma replaces Lemma 2.3.5 and finishes the proof of Theorem 2.3.12 (as well as the theorem states below). As observed in the previous section, we have some freedom in which form we choose for the right-hand side of (2.3.16). If we take for the resonant terms solutions as in Lemma 2.3.11, we get the following normal form.

Theorem 2.3.15. *Given (2.3.1) with $p, q \in \mathbb{N}_0$, $\gcd(p,q) = 1$, $q > 1$. There exists a formal change of coordinates of the form (2.3.3), formally bringing (2.3.1) in the form*

$$\begin{cases} \dot{x} = (1-x^2)\left[\frac{q}{2} + f_0\left((1+x)^p y^q\right)\chi_L(x) + f_1\left((1-x)^p y^q\right)\chi_R(x) + f_2(y)\chi_M(x)\right], \\ \dot{y} = y\left[px + g_0\left((1+x)^p y^q\right)\chi_L(x) + g_1\left((1-x)^p y^q\right)\chi_R(x) + g_2(y)\chi_M(x)\right], \end{cases} \tag{2.3.20}$$

for some formal power series f_0, f_1, f_2, g_0, g_1, g_2 that vanish at the origin. Moreover, when $q = 1$ we can assume $f_2 = g_2 = 0$.

Finally, we consider non-resonant saddles. This means that we assume that the ratio p/q of eigenvalues is an irrational number. It should be clear that Lemma 2.3.13 applies for all k, directly showing the following theorem:

Theorem 2.3.16. *Given (2.3.1) with $p/q \in \mathbb{R} \setminus \mathbb{Q}$. There exists a formal change of coordinates of the form (2.3.3), formally bringing (2.3.1) in the form*

$$\begin{cases} \dot{x} = (1-x^2)\left(\frac{q}{2} + \chi_M(x) f(y)\right), \\ \dot{y} = y\left(px + \chi_M(x) g(y)\right), \end{cases}$$

for some formal power series f, g that vanish at the origin.

Although the use of flat functions underlines that these connecting terms have no influence on the local normal form, one may prefer to work with monomials such that one gets a polynomial normal form. Replacing Lemma 2.3.13, we can show the following lemma.

Lemma 2.3.17. *Let $p, k, q \in \mathbb{N}_0$ such that p and q are relatively prime and suppose that q does not divide k, i.e. $\lambda = \frac{pk}{q} \notin \mathbb{N}$. For every smooth function f defined on a neighbourhood of $[-1, 1]$, there exists a number $C \in \mathbb{R}$ such that the differential equation*

$$\frac{-q}{2}\frac{dh(x)}{dx}(1-x^2) - pkxh(x) + f(x) = -C(1-x^2)^{N+1} \tag{2.3.21}$$

has a smooth solution, where $N = \lfloor \lambda \rfloor$.

Proof: If there is a smooth solution, then its graph $y = h(x)$ will be tangent to the vector field

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{y} = -pkxy + f(x) + C(1-x^2)^{N+1}. \end{cases} \tag{2.3.22}$$

This vector field contains two nodes at $(-1, -\frac{f(-1)}{pk})$ and $(1, \frac{f(1)}{pk})$. Similar as in the proof of Lemma 2.3.13, we consider the local smooth solutions $y = \varphi_C(x)$ and $y = \psi_C(x)$ where their domains of definition contain $(-1, 1)$.

Consider $\Delta_C(x) = \varphi_C(x) - \psi_C(-x)$ (in further calculations we omit the subscript C). By uniqueness of solution, it suffices to prove that $\Delta(0) = 0$ for some well-chosen C such that the smooth solutions coincide on $(-1, 1)$ and this gives the solution of (2.3.21).
By equation (2.3.22), we know that $\Delta(x)$ is tangent to the vector field

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{\Delta} = -pkx\Delta + f(x) + f(-x) + 2C(1-x^2)^{N+1}. \end{cases} \tag{2.3.23}$$

By use of the variational equations, we get that

$$\frac{\partial^m \Delta_C}{\partial C^m} = 0,$$

for $m \geq 2$ since Δ_C smoothly depends on C. Again it suffices to show $\frac{\partial \Delta}{\partial C}(0) \neq 0$. Let $w(x) = \frac{\partial \Delta}{\partial C}(x)$. Applying the variational technique to (2.3.23) gives

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{w} = -pkxw + 2(1-x^2)^{N+1}. \end{cases} \tag{2.3.24}$$

By putting $w(x) = (1-x^2)^N z(x)$, we get that $z = z(x)$ is a smooth graph tangent to

$$\begin{cases} \dot{x} = \frac{1}{2}(1-x^2), \\ \dot{z} = -\alpha x z + \frac{2}{q}(1-x^2), \end{cases} \tag{2.3.25}$$

where $\alpha = \lambda - N \in (0, 1)$. Since we want to investigate the smoothness of the solution near $x = -1$, we apply the transformation

$$x = x_1 - 1,$$

resulting in

$$\begin{cases} \dot{x_1} = x_1 - \frac{x_1^2}{2}, \\ \dot{z} = \alpha z - \alpha x_1 z + \frac{2}{q}(2x_1 - x_1^2). \end{cases} \tag{2.3.26}$$

A straightforward calculation show that the associated differential equation has

$$z(x_1) = x_1^\alpha (2 - x_1)^\alpha C(x_1)$$

as solution, where $C(x_1)$ is a solution of

$$C'(x_1) = \frac{4}{q x_1^\alpha (2 - x_1)^\alpha}. \tag{2.3.27}$$

By rewriting this, we get

$$\begin{aligned}
C(x_1) &= D + \frac{4}{q}\int_0^{x_1} u^{-\alpha}(2-u)^{-\alpha}du,\\
&\overset{u=tx_1}{=} D + \frac{4}{q}x_1^{-\alpha+1}\int_0^1 t^{-\alpha}(2-tx_1)^{-\alpha}dt,\\
&= D + \frac{2^{2-\alpha}}{q}x_1^{-\alpha+1}\int_0^1 t^{-\alpha}(1-t\frac{x_1}{2})^{-\alpha}dt,\\
&= D + \frac{2^{2-\alpha}}{q(1-\alpha)}x_1^{-\alpha+1}\frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)\Gamma(1)}\int_0^1 t^{-\alpha}(1-t\frac{x_1}{2})^{-\alpha}dt,\\
&= D + \frac{2^{2-\alpha}}{q(1-\alpha)}x_1^{-\alpha+1}\,\mathrm{Hypergeom}\left([\alpha,1-\alpha],[2-\alpha],\frac{x_1}{2}\right),
\end{aligned}$$

where the last equality follows from (1.2.9). In order to get a smooth function near $x_1 = 0$, we choose $D = 0$ such that

$$z(x_1) = \frac{2^{2-\alpha}}{q(1-\alpha)}x_1(2-x_1)^{\alpha}\,\mathrm{Hypergeom}\left([\alpha,1-\alpha],[2-\alpha],\frac{x_1}{2}\right), \tag{2.3.28}$$

which is smooth for $|x_1| < 2$. In order to prove the lemma, it suffices to show that

$$w(x)\mid_{x=0} = z(x_1)\mid_{x_1=1} \neq 0.$$

By (2.4.18), this value corresponds with

$$\frac{2^{2-\alpha}}{q(1-\alpha)}\,\mathrm{Hypergeom}\left([\alpha,1-\alpha],[2-\alpha],\frac{1}{2}\right). \tag{2.3.29}$$

Since

$$\mathrm{Hypergeom}\left([\alpha,1-\alpha],[2-\alpha],\frac{1}{2}\right) = \sum_{n=0}^{\infty}\frac{(1-\alpha)_n(\alpha)_n}{(2-\alpha)_n n!2^n},$$

where

$$(a)_0 = 1,\ (a)_n = a(a+1)\cdots(a+n-1),\quad \text{for } n = 1,2,\ldots,$$

we see that (2.3.29) only contains positive terms and thus is different from zero. □

The direct equivalent of Theorem 2.3.12 with polynomial connecting terms is given by

Theorem 2.3.18. *Given (2.3.1) with* $p, q \in \mathbb{N}_0$, $\gcd(p,q) = 1$, $q > 1$. *There exists a formal change of coordinates of the form (2.3.3), formally bringing (2.3.1) in the form*

$$\begin{cases}
\dot{x} = (1-x^2)\left[\frac{q}{2} + xf_1((1-x^2)^p y^q) + \sum_{k=1}^{\infty}\alpha_k(1-x^2)^{\lceil\frac{pk}{q}\rceil}y^k\right],\\
\dot{y} = y\left[px + xg_1((1-x^2)^p y^q) + \sum_{k=1}^{\infty}\beta_k(1-x^2)^{\lceil\frac{pk}{q}\rceil}y^k\right],
\end{cases} \tag{2.3.30}$$

for some formal power series f_1, g_1 *that vanish at the origin and some real coefficients* α_k, β_k

Remark 2.3.19. *The asymmetric resonant part is contained in the description of the connecting terms in Theorem 2.3.18.*

2.4 Smooth Realization

The formal transformations of Theorems 2.3.8, 2.3.9, 2.3.10, 2.3.12, 2.3.15, 2.3.16 and 2.3.18 can be realized as a smooth equivalence thanks to the generalized Borel Theorem (Theorem 1.2.5). However, since it originates from an induction process on the degree of y, flat terms in y come forward similar as in the local normal form (see (1.3.6)). We can summarize these forms after the smooth transformation as

$$\begin{cases} \dot{x} = (1-x^2)\left[\frac{q}{2} + R_1(x,y) + F_1(x,y)\right], \\ \dot{y} = y\left[px + R_2(x,y) + F_2(x,y)\right], \end{cases} \tag{2.4.1}$$

where $R_1(x,y)$ and $R_2(x,y)$ contain the resonant and the connecting terms and $F_1(x,y)$ and $F_2(x,y)$ denote the flat terms, i.e.

$$R_1(x,0) = R_2(x,0) = j_\infty F_1(x,0) = j_\infty F_2(x,0) = 0.$$

The topic of study is to eliminate the flat remainders F_1 and F_2. For the sake of convenience, we reduce the question to a question on orbital equivalence of vector fields, allowing us to reparametrize time, i.e. divide both equations by the strictly positive function $1 + \frac{2}{q}(R_1(x,y) + F_1(x,y))$, leading us to study

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{y} = y\left[px + R(x,y) + F(x,y)\right], \end{cases} \tag{2.4.2}$$

where $R(x,y)$ contains the resonant and the connecting terms and where $F(x,y)$ is infinitely flat with respect to y at $y = 0$. In applications, we shall apply a time reparametrization to the system in pre-normal form (Lemma 2.3.1) to get the first equation of (2.4.2) and then proceed to the formal normal form by relying only on the transformations of Lemma 2.3.2. In this section we will prove Theorem 2.4.1, hence finishing the proofs of the theorems announced in Section 2.1.

Theorem 2.4.1. *Let (2.4.2) arise from applying one of the formal normal form theorems (Theorem 2.3.8, 2.3.9, 2.3.10, 2.3.12, 2.3.15, 2.3.16,2.3.18).*
Let $k \geq 1$ be arbitrary. There exists a C^k-smooth equivalence Φ defined in a neighbourhood of $[-1,1]$ between (2.4.2) *and*

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{y} = y\left[px + R(x,y)\right], \end{cases} \tag{2.4.3}$$

where $R(x,y)$ is identical to the function in the original system (2.4.2).

Proof: Here we give a cursory description of the proof. Details are provided throughout the following sections. Section 2.4.1 establishes the required conjugacy near $(-1, 0)$ (and also separately near $(+1, 0)$). An argument is shown that the conjugacy near $(-1, 0)$ can be extended to a conjugacy valid along almost half of the separatrix. This extension can even be extended arbitrarily close to the other hyperbolic point as will be proven in Section 2.4.2. Let $k \geq 1$ and $x_0 = 1 - \varepsilon$, where ε is small enough such that x_0 lies in the domain of the normalizing transformation (2.4.11) near the hyperbolic saddle $(+1, 0)$. By Lemma 2.4.2, there exist $\gamma, \kappa > 0$ and smooth functions

$$\Phi_1 : (-1 - \gamma, x_0 + \gamma) \times (-\kappa, \kappa) \to \mathbb{R}, \text{ and } \Phi_2 : (x_0 - \gamma, 1 + \gamma) \times (-\kappa, \kappa) \to \mathbb{R},$$

such that the transformations $(x, y) = (\tilde{x}, \Phi_i(\tilde{x}, \tilde{y}))$ remove the flat terms on their domains for $i = 1, 2$. Consider $\Delta_0 = \Phi_1 - \Phi_2$, which is a smooth function defined at the cut $x = x_0$. We extend this to a C^k-function

$$\Delta : (x_0 - \gamma, 1) \times (-\kappa, \kappa) \to \mathbb{R},$$

which is k-flat for $x \to 1$ as proved in Lemma 2.4.5. Finally, the transformation $(x, y) = (\tilde{x}, \Phi(\tilde{x}, \tilde{y}))$, where

$$\Phi(x, y) = \begin{cases} \Phi_1(x, y) & \text{if } x \in (-1 - \gamma, x_0), \\ \Phi_2(x, y) + \Delta(x, y) & \text{if } x \in (x_0, 1), \\ \Phi_2(x, y) & \text{if } x \in [1, 1 + \gamma), \end{cases} \tag{2.4.4}$$

provides the required equivalence. □

2.4.1 Local results near the saddles

Following Chen (Theorem 1.3.3), system (2.4.2) is smoothly conjugate to (2.4.3) near the saddle $(-1, 0)$, since they are formally conjugate (they are even formally equal). Despite the fact that $\dot{x}$ in both equations are identical, it is not directly clear from Chen's method that the transformation leaves x invariant, i.e. is of the form

$$(x, y) \mapsto (x, y + \sigma(x, y)), \tag{2.4.5}$$

with σ flat at $y = 0$. (In fact, examining Chen's method, it seems very unlikely that their transformation is of the form (2.4.5).) To get a diffeomorphism that leaves x invariant, we use the homotopic method from [18] as explained in Section 1.3.1.

The benefit of the homotopic method is that extra structure can be captured. If we for example use the method to conjugate (2.4.2) to (2.4.3) whose $\dot{x}$-equations are identical, then the $\dot{x}$-equation in the difference field Y is zero, implying from

(1.3.7) that the $\dot{x}$-equation in Z_τ is zero as well. Following the way φ_τ is defined in Section 1.3.1, it is clear that the x-component of $\varphi - \mathrm{Id}$ should be zero, implying that the obtained φ_τ is of the form (2.4.5).

Proposition 1.3.7 states that in order to remove the flat terms in (2.4.2), it suffices to use bump functions to give the vector field compact support near the saddle at $(-1, 0)$ and to reduce to a neighbourhood where the hyperbolicity in the y-direction is uniformly bounded away from 0. This is possible in any neighbourhood inside the compact set

$$[-1-\delta, 0-\delta] \times [0, \delta'],$$

with δ and δ' arbitrarily small. Similarly, we can remove the flat terms in any neighbourhood in the compact set

$$[0+\delta, 1+\delta] \times [0, \delta'].$$

Clearly, this method cannot be used to present a normal form that is defined on $[-1, 1] \times [0, \delta']$, because there is a change of stability along the separatrix connecting both saddles.

2.4.2 Extending local conjugacies

In this section, we extend the domain of validity of the conjugacy defined near the saddle at $(-1, 0)$, to conjugate (2.4.2) to its counterpart without flat terms. As an ansatz to conjugate (2.4.2) to (2.4.3) we propose a transformation

$$(x, y) = (u, \varphi(u, v)).$$

The unknown function φ satisfies the following PDE:

$$\frac{q}{2}(1-u^2)\varphi_u + v\left[pu + R(u, v)\right]\varphi_v = \varphi\left[pu + R(u, \varphi) + F(u, \varphi)\right]. \tag{2.4.6}$$

Clearly, Section 2.4.1 explains the existence of a solution of the form

$$\varphi(u, v) = v + \psi(u, v) \text{ with } j_\infty \psi(u, 0) = 0, \tag{2.4.7}$$

near $(u, v) = (-1, 0)$. The aim in this section is to extend the domain of the solution φ to a section arbitrarily close to $(+1, 0)$. The next section will then deal with the local passage near the saddle at $(+1, 0)$.

Lemma 2.4.2. *The solution φ of (2.4.6) determining the local conjugacy between (2.4.2) and (2.4.3) and defined in Section 2.4.1 near $(-1, 0)$, can be extended to $[-1, U_{max}] \times [0, \delta'']$, for any $U_{max} < 1$, provided δ'' is taken small enough (U_{max}-dependent). In the extended domain, $\varphi(u, v) - v$ is still (uniformly) flat with respect to v.*

Proof: Following the methods of characteristics, the function $\varphi(u,v)$ corresponds to a graph $\Psi = \varphi(u,v)$ in (u,v,Ψ)-space that is tangent to the flow of the vector field

$$\begin{cases} \dot{u} = \frac{q}{2}(1-u^2), \\ \dot{v} = v\left[pu + R(u,v)\right], \\ \dot{\Psi} = \Psi\left[pu + R(u,\Psi) + F(u,\Psi)\right]. \end{cases}$$

Following the results in Section 2.4.1, we are aware of the existence of such an invariant graph for $x < -\delta$ and $y < \delta'$. As a basis for using the method of characteristics, we may hence use the following initial condition

$$u_s(0) = -2\delta, \qquad v_s(0) = s, \qquad \Psi_s(0) = \varphi(-2\delta, s),$$

where we have used the solution obtained from the previous section on $x = -2\delta$ and where s is a parameter. Since the solution is known to be identity + flat terms, it is clear that also $\Psi_s(0) = s +$ flat in s. In particular, $\Psi_0(0) = 0$. As a consequence, we can integrate the characteristic corresponding to $s = 0$ explicitly:

$$u_0(t) = \tanh\left(\operatorname{arctanh}(-2\delta) + \frac{q}{2}t\right), \qquad v_0(t) = 0, \qquad \Psi_0(t) = 0.$$

It is defined for all $t \geq 0$. By semi-continuity of the maximal domain of existence of orbits, given any $T > 0$, the orbit $(u_s(t), v_s(t), \Psi_s(t))$ is defined for all $s \leq \delta'$, $t \leq T$. We can invert $t \mapsto u_0(t)$ to a map $\tau(u)$, so that

$$u_s(\tau(u)) \equiv u, \forall u \leq U_{max},$$

for any given $U_{max} < 1$ (since T is arbitrarily large). From the variational equation and from $\frac{\partial}{\partial s} v_s(0) = 1$, we know that $C(t) := \frac{\partial}{\partial s} v_s(t)|_{s=0}$ is uniformly bounded away from 0 and strictly positive for all $t \in [0,T]$. Therefore, $v_s(\tau(u))$ is strictly monotonously increasing in s, and we can find a $\sigma(u,v)$ such that

$$v_{\sigma(u,v)}(\tau(u)) = v,$$

for s sufficiently close to 0. As a consequence, the required invariant graph is given by

$$\varphi(u,v) = \Phi_{\sigma(u,v)}(\tau(u)),$$

following the method of characteristics.

Let us now explain the flatness of $\varphi(u,v) - v$. Then $\Delta = \tilde{\varphi}(u,v)$, with $\tilde{\varphi}(u,v) := \varphi(u,v) - v$, is an invariant graph of

$$\begin{cases} \dot{u} &= \frac{q}{2}(1-u^2), \\ \dot{v} &= v\left[pu + R(u,v)\right], \\ \dot{\Delta} &= (\Delta + v)\left[pu + R(u,\Delta+v) + F(u,\Delta+v)\right] - v\left[pu + R(u,v)\right]. \\ &= vF(u,v) + O(\Delta). \end{cases}$$

and we use

$$u_s(0) = -2\delta, \qquad v_s(0) = s, \qquad \Delta_s(0) = \varphi(-2\delta, s) - s,$$

Since $v_0(t) = 0$ for $t \leq T$ we have $\sigma(u, 0) = 0$ for all u implying that $\sigma(u, v) = O(v)$ uniformly for $u \leq U_{max}$. This means that during the integration of the orbit, $\dot{\Delta} = O(\Delta) +$ flat in s. Since the initial condition for Δ is also flat in s, an application of Gronwall's Lemma shows that $\Delta_s(t)$ is uniformly flat in s for all $t \leq T$. As a consequence, and keeping in mind that $\sigma(u, v) = O(v)$,

$$\tilde{\varphi}(u, v) := \Delta_{\sigma(u,v)}(\tau(u))$$

is flat in v, uniformly for $u \leq U_{max}$. □

2.4.3 Conjugacy near the second saddle

As observed before, the techniques of the previous subsection can not be applied in a straightforward manner near the hyperbolic point at $(+1, 0)$. However we can use the local normal forms to overcome the problem of a displacement time that grows to infinity.

Let $\varepsilon > 0$ be arbitrary but small. We will put some restraints on ε in the following discussion.

By the results in the previous sections, we already have smooth solutions φ_1 and φ_2 of (2.4.6), defined on neighbourhoods N_1 and N_2 of respectively $[-1, 1 - \frac{\varepsilon}{2}]$ and $[1 - \frac{3\varepsilon}{2}, 1]$ as part of the x-axis.

Now we look at the difference of these transformations, more precise

$$\Delta(x, y) = \varphi_1(x, y) - \varphi_2(x, y).$$

We investigate how this difference, for instance at the transverse section $x = 1 - \varepsilon$, propagates toward the second hyperbolic point. The aim is to show that this difference goes to zero in a finitely flat manner when it approaches the unstable manifold of the hyperbolic point, where the order of flatness can be arbitrarily chosen.

Since both the transformations φ_1 and φ_2 are of the form (2.4.7), we see that at the section the function

$$\Delta_0(s) = \varphi_1(1 - \varepsilon, s) - \varphi_2(1 - \varepsilon, s),$$

is infinitely flat near $s = 0$.

If we denote by

$$h(x, y) = y\left(R(x, y) + F(x, y)\right),$$

we see that Δ needs to satisfy

$$\frac{q}{2}(1 - x^2)\Delta_x + y\left[px + R(x, y)\right]\Delta_y = \Delta\left[px + H(x, y, \Delta)\right], \tag{2.4.8}$$

in the neighbourhood N_2, where

$$H(x, y, \Delta) = \int_0^1 \frac{\partial h}{\partial y}(x, \varphi_2(x, y) + z\Delta)\, dz.$$

By the method of characteristics, we examine the propagation of the difference satisfying (2.4.8). This satisfies the ODE

$$\begin{cases} \dot{x} = \frac{q}{2}(1 - x^2), \\ \dot{y} = y\left[px + R(x, y)\right], \\ \dot{\Delta} = \Delta\left[px + H(x, y, \Delta)\right], \end{cases} \tag{2.4.9}$$

with initial values

$$x(0) = 1 - \varepsilon,\ y(0) = s,\ \Delta(0) = \Delta_0(s),$$

which represents a parametrization of our transversal section.

Since we are interested about the dynamics near the second hyperbolic point, we apply the translation

$$u = x - 1,$$

such that we get the new system

$$\begin{cases} \dot{u} = u\left(-q - \frac{u}{2}\right), \\ \dot{y} = y\left[p + pu + R(u + 1, y)\right], \\ \dot{\Delta} = \Delta\left[p + pu + H(u + 1, y, \Delta)\right], \end{cases} \tag{2.4.10}$$

with initial values

$$u(0) = -\varepsilon,\ y(0) = s,\ \Delta(0) = \Delta_0(s).$$

The hyperbolic point is now situated in the origin. We can put this system in a normal form by a smooth near-identity orientation-preserving transformation

$$(x_1, y_1, \Delta_1) = (\psi(u), \chi(u, y), \omega(u, y, \Delta)), \tag{2.4.11}$$

such that we get

$$\begin{cases} \dot{x}_1 = -qx_1, \\ \dot{y}_1 = y_1 \left[p + f(x_1^p y_1^q) \right], \\ \dot{\Delta}_1 = \Delta_1 \left[p + g(x_1^p y_1^q, x_1^p \Delta_1^q) \right], \end{cases} \tag{2.4.12}$$

where f and g are some smooth functions representing the resonant terms near the hyperbolic point. Now we can put a condition on ε since we like $(-\varepsilon, s)$ to be in the domain of the coordinate transformation (2.4.11). Hence it suffices to search for a solution of (2.4.12) with initial values

$$\begin{aligned} &x_1(0) = \psi(-\varepsilon),\ y_1(0) = \chi(-\varepsilon, s) = s(1 + \text{ h.o.t.}), \\ &\Delta_1(0) = \omega(-\varepsilon, s, \Delta_0(s)) = \Delta_0(s)(1 + \text{ h.o.t.}). \end{aligned} \tag{2.4.13}$$

Furthermore, if we introduce the new variable

$$z_1 = x_1^p y_1^q$$

we get

$$\begin{cases} \dot{x}_1 = -qx_1, \\ \dot{z}_1 = qz_1 \left[f(z_1) \right], \\ \dot{\Delta}_1 = \Delta_1 \left[p + g(z_1, x_1^p \Delta_1^q) \right], \end{cases} \tag{2.4.14}$$

with new initial value $z_1(0) = \psi(-\varepsilon)^p \chi(-\varepsilon, s)^q$.

Suppose we wish to define Δ at a point $(1 - x_E, y_E)$ with $1 - \varepsilon < 1 - x_E < 1$ and y_E small enough. In order to do this, it suffices to define Δ_1 at the point

$$(x_{1,E}, y_{1,E}) = (\psi(-x_E), \chi(-x_E, y_E)).$$

Introduce the following parameters

$$\mu = \psi(-\varepsilon) \text{ and } \lambda = -\frac{1}{q} \ln \left(\frac{x_{1,E}}{\mu} \right).$$

Remark that $\lambda > 0$ since ψ is orientation-preserving and thus $\mu < x_{1,E} < 0$. Therefore we can do a time rescaling of the previous system (2.4.14) to get

$$\begin{cases} \dot{x}_1 = -q\lambda x_1, \\ \dot{z}_1 = \lambda q z_1 f(z_1), \\ \dot{\Delta}_1 = \lambda \Delta_1 \left[p + g(z_1, x_1^p \Delta_1^q) \right], \end{cases} \tag{2.4.15}$$

where the initial values remain unchanged. It is a straightforward calculation that the solution $x_1(t)$ reaches $x_{1,E}$ after $t = 1$. However it is still not clear for which initial value $y_1(0)$ we reach y_E at $t = 1$.
This obstacle can be overcome by looking at the same system, more precisely the first

two equations of (2.4.15), but in reversed time, with initial values $(x_{1,E}, y_{1,E})$.

For this, represent the Taylor series of f as

$$xf(x) = \sum_{i\geq 2} \alpha_i x^i.$$

By results of the Dulac map obtained in Proposition 1.3.14, we can show that for an arbitrary $k \in \mathbb{N}$:

$$\begin{aligned} s_{1,E}^q := y_1(0)^q &= \frac{1}{\mu^p}\left[x_{1,E}^p y_{1,E}^q + \alpha_1 \ln\left(\frac{x_{1,E}}{\mu}\right) x_{1,E}^{2p} y_{1,E}^{2q} + \ldots\right] \\ &= \frac{1}{\mu^p}\left[\sum_{i=1}^{N(k)} x_{1,E}^{pi} y_{1,E}^{qi} P_i\left(\ln\left(\frac{x_{1,E}}{\mu}\right)\right) + \psi_k(x_{1,E}^p y_{1,E}^q)\right] \end{aligned}$$

where P_i are polynomials of degree $i-1$ with coefficients in the ideal generated by $\langle \alpha_1, \ldots, \alpha_{i-1}\rangle$ and where ψ_k is C^{k+1} and $(k+1)$-flat. For further calculations, we remark the following. We can rewrite the above expression as

$$\begin{aligned} s_{1,E}^q &= x_{1,E}^p y_{1,E}^q \cdot \frac{1}{\mu^p}\left[\sum_{i=1}^{K(k)} x_{1,E}^{p(i-1)} y_{1,E}^{q(i-1)} P_i\left(\ln\left(\frac{x_{1,E}}{\mu}\right)\right) + \frac{1}{x_{1,E}^p y_{1,E}^q}\psi_k(x_{1,E}^p y_{1,E}^q)\right] \\ &= x_{1,E}^p y_{1,E}^q \rho_k(x_{1,E}; y_{1,E}; x_{1,E}\ln(x_{1,E})), \end{aligned} \tag{2.4.16}$$

where ρ_k is a C^k-function in its arguments, which goes to μ^{-p} as $x_{1,E}$ goes to zero. This implies that $s_{1,E}$ approaches zero in a continuous way for $x_{1,E} \to 0$ and thus by applying the inverse transformation of (2.4.11) and using the fact that this is a near-identity transformation, we see that the corresponding $s_E = y(0)$ goes to zero. Let us return to system (2.4.15). It is easy to get smooth solutions

$$x_1(t) = \mu e^{-q\lambda t}, \tag{2.4.17}$$

and the solution of z_1 can be represented as a series

$$\begin{aligned} z_1(t) &= \sum_{i\geq 1} \mu^{pi} s_{1,E}^{qi} P_i(q\lambda t), \\ &= x_{1,E}^p y_{1,E}^q \left(\sum_{i\geq 1} \mu^{pi} x_{1,E}^{p(i-1)} y_{1,E}^{q(i-1)} \rho_k^i(x_{1,E}; y_{1,E}; x_{1,E}\ln(x_{1,E})) P_i(q\lambda t)\right), \\ &= x_{1,E}\tilde{z}(x_{1,E}; x_{1,E}\lambda t), \end{aligned} \tag{2.4.18}$$

where the P_i are identical to those of (2.4.16) and $\tilde{z}$ is C^k. Since we are only interested in the $x_{1,E}$-dependence, we omit $y_{1,E}$ from the notation. Although $P_i(q\lambda t)$ blows up when $x_{1,E}$ goes to zero, we see that the factor $x_{1,E}^{p(i-1)}$ will compensate such that

$$x_{1,E}^{p(i-1)} P_i(q\lambda t) \to 0 \text{ when } x_{1,E} \to 0,$$

for every $t \in [0,1]$. Thus it remains to focus on the differential equation

$$\dot{\Delta}_1 = \lambda \Delta_1 \left[p + g(z_1, x_1^p \Delta_1^q) \right],$$

with initial value

$$\Delta_{1,0} := \Delta_1(0) = \omega(-\varepsilon, s_E, \Delta_0(s_E)) = \Delta_0(s_E) \cdot (\text{smooth function in } s_E). \quad (2.4.19)$$

Using (2.4.17) and the notation

$$\Delta_1(t) = \Delta_{1,0} e^{p\lambda t} \Gamma(t),$$

we are focused on solving

$$\dot{\Gamma} = \lambda \Gamma\, g\left(z_1, \mu^p \Delta_{1,0}^q \Gamma^q\right), \text{ with } \Gamma(0) = 1. \quad (2.4.20)$$

The desired expression $\Delta_1(x_{1,E}, y_{1,E})$ is then given by

$$\Delta_{1,0} e^{p\lambda} \Gamma(1) = \Delta_{1,0} \Gamma(1) \left(\frac{\mu}{x_{1,E}} \right)^{p/q}. \quad (2.4.21)$$

Now since $\Delta_{1,0}$ is a flat function of s_E as in (2.4.19) it follows from (2.4.13) and (2.4.16) that for every $N \geq 1$, we can find a C^k-function $\Delta_{1,N}(x_{1,E}; y_{1,E})$ which is flat for $x_{1,E} \to 0$ such that

$$\Delta_{1,0} = x_{1,E}^N \Delta_{1,N}(x_{1,E}; y_{1,E}). \quad (2.4.22)$$

So by using an appropriate N, we can see directly that (2.4.21) is k-flat if we can prove the following lemma.

Lemma 2.4.3. *Let $k \geq 1$ be arbitrary. The C^k-function*

$$\Theta_k(x_{1,E}, y_{1,E}) = x_{1,E}^k \Gamma(1)$$

has bounded partial derivatives, with respect to $x_{1,E}$ up to order k for $x_{1,E}$ close to zero.

Proof: First we look at the differential equation of $\Gamma(t)$ given by (2.4.20). This can be rewritten as

$$\dot{\Gamma} = \lambda \Gamma g\left(x_{1,E} \tilde{z}(x_{1,E}; x_{1,E} \lambda t), x_{1,E}^q \mu^p \Delta_{1,1}^q \Gamma^q\right).$$

Since $g(0,0) = 0$, this is the same as

$$\begin{aligned} \dot{\Gamma} &= x_{1,E} \lambda \Gamma \tilde{g}(x_{1,E}; x_{1,E} \ln(x_{1,E}); t), \\ &= \Gamma G(x_{1,E}; x_{1,E} \ln(x_{1,E}); t) \end{aligned}$$

where $\tilde{g}$ and G are C^k-functions. Therefore this equation admits a unique C^k solution, which is C^k dependent on the parameters $x_{1,E}$ and $x_{1,E} \ln(x_{1,E})$. Denote this solution by $\Gamma(x_{1,E}; x_{1,E}\ln(x_{1,E}); t)$. By the chain rule, we can see that the partial derivatives of

$$\Theta_k(x_{1,E}, y_{1,E}) = x_{1,E}^k \Gamma(x_{1,E}; x_{1,E}\ln(x_{1,E}); 1)$$

up to order k with respect to $x_{1,E}$ are smooth functions of $x_{1,E}$ and $x_{1,E}\ln(x_{1,E})$ and thus are bounded. □

Remark 2.4.4. *A similar observation as in* (2.4.22) *can be made for $y_{1,E}$. Therefore we see that expression* (2.4.21) *for $\Delta_1(x_{1,E}, y_{1,E})$ is infinitely flat for $y_{1,E} \to 0$ when $x_{1,E} < 0$.*

These results hence lead to the following lemma.

Lemma 2.4.5. *Let $k \geq 1$ be arbitrary. There exists a C^k solution $\Delta(x_E, y_E)$ for* (2.4.8) *which is k-flat for $x_E \to 0$.*

Proof: It suffices to prove this for the normal form coordinates. In these coordinates we know by (2.4.21), (2.4.22) and Lemma 2.4.3, that this can be expressed by

$$\Delta_1(x_{1,E}, y_{1,E}) = \Delta_{1,2k+N}(x_{1,E}, y_{1,E})\Theta_k(x_{1,E}, y_{1,E})\mu^{p/q} x_{1,E}^{k+N-p/q},$$

where $N > \frac{p}{q}$. Since $\Delta_{1,2k+N}(x_{1,E}, y_{1,E})$ is k-flat at $x_{1,E} = 0$ and the derivatives of the other functions with respect to $x_{1,E}$ up to order k remain bounded, the product rule implies the desired result. □

2.5 From semi-local to local normal form

Up to smooth equivalence, the vector field (2.1.1) for $p, q \in \mathbb{N}^*$, $gcd(p, q) = 1$, can be put in the orbital normal form given by

$$\begin{cases} \dot{x} = \frac{q}{2}(1 - x^2), \\ \dot{y} = y\left[px + g_0((1+x)^p y^q)\chi_L(x) + g_1((1-x)^p y^q)\chi_R(x) + \chi_M(x) g_2(y)\right], \end{cases} \tag{2.5.1}$$

where f_0, f_1, f_2, g_0, g_1 and g_2 are smooth functions with a zero in the origin and the functions χ_R, χ_L and χ_M are as depicted in Figure 2.3. This semi-local normal form is locally equal to the local normal form up to some low-order terms. For the normal forms from Theorem 2.1.3 however, it is not straightforward to pass to the local normal forms. We show briefly how this can be done in a neighbourhood of the saddle $(-1, 0)$.

First observe that we can neglect the flat terms, since they can locally be removed due to a similar result as Proposition 1.3.7 for $M = \{x = 1\}$. This transformation is of the form

$$(x, y) = (\bar{x}, \bar{y} + F(\bar{x})\bar{y}^k + \mathrm{O}(\bar{y}^{k+1}),$$

where k is the exponent of first non-zero connecting term, and $F(\bar{x})$ is a locally smooth function which is flat at $\bar{x} = -1$. Formally we consider

$$\begin{cases} \dot{x} = \frac{q}{2}(1 - x^2), \\ \dot{y} = y \left(px + \sum_{n \geq 1} (A_n x + B_n)(1 - x^2)^{pn} y^{qn} \right), \end{cases} \tag{2.5.2}$$

where A_n and B_n denote the Taylor coefficients of respectively g_1 and g_0. After applying

$$(x, y) = \left(\frac{u-1}{u+1}, (u+1)^{\frac{2p}{q}} z \right),$$

the vector field (2.5.2) is transformed to

$$\begin{cases} \dot{u} = qu, \\ \dot{z} = z \left(-p + \sum_{n \geq 1} \left(D_n + E_n \frac{u}{u+1} \right) u^{pn} z^{qn} \right), \end{cases} \tag{2.5.3}$$

where $D_n = 4^{pn}(B_n - A_n)$ and $E_n = 2^{2pn+1} A_n$. We show that there is a smooth change of coordinates putting (2.5.3) in the same form, but with $E_n = 0$ for all n. The coefficients D_n remain unchanged and thus correspond to the coefficients of the local normal form. This is shown by an induction principle followed by applying Theorem 1.2.3.

Lemma 2.5.1. *There exists a transformation of the form*

$$z = y + g(u) y^{qk+1},$$

where $g(u)$ is locally smooth, which puts

$$\begin{cases} \dot{u} = qu, \\ \dot{z} = -pz + \sum_{n \geq 1} D_n u^{pn} z^{qn+1} + \sum_{n \geq k} f_n(u) u^{pn+1} z^{qn+1}, \end{cases}$$

where the $f_n(u)$ are locally smooth functions, in the form

$$\begin{cases} \dot{u} = qu, \\ \dot{y} = -py + \sum_{n \geq 1} D_n u^{pn} y^{qn+1} + \sum_{n \geq k+1} \tilde{f}_n(u) u^{pn+1} y^{qn+1}, \end{cases}$$

where the $\tilde{f}_n(u)$ are also locally smooth.

Proof: Let $g(u)$ be a smooth solution of

$$-qug'(u) + pqkg(u) + f_k(u)u^{pk+1} = 0,$$

i.e.

$$g(u) = C(u)u^{pk}, \quad \text{with } C'(u) = \frac{1}{q} f_k(u),$$

thus

$$g(u) = \frac{1}{q} u^{pk} \int_0^u f_k(s)ds = u^{pk+1} h(u),$$

where $h(u)$ is a locally smooth function. If we apply the transformation as defined in the lemma, the second equation in the new variable y will be given by

$$\begin{aligned}\dot{y} = -py &+ \lambda \sum_{n \geq 1} D_n u^{pn} y^{qn+1} (1 + h(u)u^{pk+1} y^{qk})^{qn+1} \\ &+ \lambda f_k(u) u^{pk+1} y^{qk+1} \sum_{n=1}^{qk+1} \binom{qk+1}{n} h(u)^n u^{(pk+1)n} y^{qkn} \\ &+ \lambda \sum_{n \geq k+1} f_n(u) u^{pn+1} y^{qn+1} (1 + h(u)u^{pk+1} y^{qk})^{qn+1}.\end{aligned}$$

By comparing the degrees of the u and y in each term, we can see that the only place where a term with $\deg(u) \leq pn$ can arise, is where the exponent of y is equal to $qn+1$ and these are the resonant terms $D_n u^{pn}$. Therefore this can be rewritten in the form as described in the lemma. □

If for example qk is the first degree of y where E_k is non-zero, it is a simple computation that this total transformation is of the form

$$z = y + \frac{E_k}{q} u^{pk} \ln(u+1) y^{qk+1} + \mathrm{O}\big(y^{2qk+1}\big).$$

The local normal form at $(-1, 0)$ of (2.5.2) is of the form

$$\begin{cases} \dot{x} = qx, \\ \dot{y} = y\left(-p + \sum_{n \geq 1} 4^{pn}(B_n - A_n)x^{pn}y^{qn}\right). \end{cases}$$

Similarly the local normal form

$$\begin{cases} \dot{x} = -qx, \\ \dot{y} = y\left(p + \sum_{n \geq 1} 4^{pn}(B_n + A_n)x^{pn}y^{qn}\right), \end{cases}$$

of (2.5.2) can be obtained near $(1, 0)$. Observe that only the resonant terms of (2.1.6) are of importance when we consider the dynamics close to the saddles. However the transition in between the saddles is contained in the connecting terms and thus they need to be considered in the full transition map as we will see in the next chapter.

Chapter 3

Transition map near symmetric saddle connections using normal linearization

Using finitely smooth transformations, we normally linearize the normal form of Chapter 2. Moreover, we find an expression as a C^∞-function of some finitely smooth functions called tags. As a consequence of this procedure, we define an invariant of the system in terms of these tags and we can use this to obtain an expression for the transition map through the saddle connection. These results are part of a paper which is accepted for publication in Journal of Differential Equations ([12]).

3.1 Introduction

As before we consider C^∞ vector fields in the plane with two hyperbolic saddles P_1 and P_2 having a heteroclinic connection with reciprocal saddle quantities p/q and q/p. We do not consider unfoldings, i.e. here we do not consider families of vector fields in which the parameters either break the saddle connection and/or perturb the ratios of eigenvalues.

In Chapter 2 (Theorem 2.1.3) we established a C^∞ normal form (up to time rescaling) near the connection:

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{y} = y\left(px + w^n f(w) + xw^n g(w) + \chi(x)h(y)\right), \end{cases} \tag{3.1.1}$$

where $w = (1-x^2)^p y^q$, $|f(0)| + |g(0)| \neq 0$, and χ, defined in (2.1.5), is infinitely flat at $x = \pm 1$, $n \geq 1$ and all occurring functions are C^∞. Here we highlight the first non-zero resonant terms in contrast to (2.1.6). The expression $xw^n g(w)$ represents the part of the normal form where P_2 behaves truly reversible with respect to P_1; it is the symmetric part. The expression $w^n f(w)$ represents the anti-symmetric part.

The function $\chi(x)h(y)$ contains the connecting terms and is only present when $q \neq 1$. We will see that these terms may have an effect that is distinguishably different from the effect of the resonant terms on the dynamics near the connection.

The goal of this chapter is to establish a transition map along the connection. This allows us to apply it to cyclicity problems, counting the number of limit cycles nearby a given limit periodic set. Following the idea of linearizing individual saddles using logarithmic expressions (see e.g. [7]), we show in Section 3.2 that we can normally linearise (3.1.1) in terms of the y-variable in a similar way using the local logarithmic expressions $\log(1-x)$ and $\log(1+x)$. This is obtained by a near-identity coordinate transformation $(x, y) \mapsto (x, z) = (x, z(1 + \psi(x, z)))$ which is C^∞ in these logarithmic expressions. The resulting normally linearized equation is given by

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{z} = pxz. \end{cases} \tag{3.1.2}$$

A precise statement is given in Theorem 3.2.1. Clearly, this model can be integrated since $(1-x^2)^p z^q$ is a first integral of the system. Moreover the map $\Sigma_{\text{in}}^{\text{lin}} \to \Sigma_{\text{out}}^{\text{lin}}$ is trivially given by $x_0 \mapsto -x_0$, where

$$\begin{aligned} \Sigma_{\text{in}}^{\text{lin}} &=]-1, -1+\delta[\times \{z_0\}, \\ \Sigma_{\text{out}}^{\text{lin}} &=]1-\delta, 1[\times \{z_0\}, \end{aligned}$$

for any given $z_0 > 0$ and $\delta \in]0, 1[$. Using the normal form transformation we can then specify a constant of motion for the original system (3.1.1) and obtain qualitative information on the map $\Sigma_{\text{in}} \to \Sigma_{\text{out}}$, where

$$\begin{aligned} \Sigma_{\text{in}} &=]-1, -1+\delta[\times \{y_0\}, \\ \Sigma_{\text{out}} &=]1-\delta, 1[\times \{y_0\}. \end{aligned} \tag{3.1.3}$$

A precise statement for the asymptotics of the transition map is given in Theorem 3.3.2.

3.2 Normal linearization using finitely smooth transformations

This section is devoted to proving the following theorem.

Theorem 3.2.1. *Consider the C^∞ vector field (3.1.1) with $p, q \in \mathbb{N}_*$ and gcd(p, q)= 1. There exists a near-identity coordinate change*

$$(x, y) \mapsto (x, z) = (x, y\,(1 + \psi\,(x, y)))\,,$$

preserving $y = 0$ and bringing (3.1.1) *in the form* (3.1.2). *Moreover ψ is of the form*

$$\psi(x, y) = \Psi(x, y, w^n \log(1 + x), w^n \log(1 - x), (1 - x^2)^{1/q}),$$

where Ψ is C^∞ near $[-1, 1] \times \{(0, 0, 0)\} \times [0, 1]$ and $w = (1 - x^2)^p y^q$.

Proof: The proof of Theorem 3.2.1 is a subsequent application of Theorem 3.2.7 and Theorem 3.2.9 (only if $q > 1$) presented in the next sections. □

First we eliminate the resonant terms by an induction process (Section 3.2.1) before removing the connecting terms (Section 3.2.2). The connecting terms are absent when $q = 1$ so in this case Theorem 3.2.1 can be replaced by Theorem 3.2.7.

3.2.1 Reduction of the resonant part

Consider (3.1.1) and recall that $w = (1 - x^2)^p y^q$, so

$$\dot{w} = qw^{n+1} f(w) + qxw^{n+1} g(w) + qw\chi(x)h(y). \tag{3.2.1}$$

For the moment we focus on the resonant part of (3.1.1), i.e. we neglect $\chi(x)h(y)$. A simple manipulation of the functions $f + xg$ leads to

$$\begin{cases} \dot{x} = \frac{q}{2}(1 - x^2), \\ \dot{w} = w^{n+1}(1 - x)F_L(w) + w^{n+1}F_R(w)(1 + x), \end{cases} \tag{3.2.2}$$

where F_L and F_R are a linear combination of the original f and g, more precisely

$$F_L(w) = \frac{q}{2}(f(w) - g(w)), \quad \text{and} \quad F_R(w) = \frac{q}{2}(f(w) + g(w)).$$

Later we will see the effect of our manipulations on the full system. Our intention is to increase the order of w in the equation for $\dot{w}$ step by step using changes of coordinates in w. In the easier setting where one locally works around a single saddle, it is possible to remove the resonant terms using finitely smooth expressions involving logarithms (see e.g. [8]). Here we will extend this idea and therefore introduce the notion of tags.

Tags

In this paragraph we will introduce a series of tags which are functions of x, defined on $(-1, 1)$, by recursion. First we define T_L and T_R as the tags of order 1 satisfying $T_L(0) = T_R(0) = 0$ where we impose that their time-derivatives, denoted as $\dot{T}_L$ and $\dot{T}_R$, should satisfy

$$\dot{T}_L = (1 - x), \quad \text{and} \quad \dot{T}_R = (1 + x).$$

The time dependence of x is expressed in the first line of (3.2.2). A direct computation shows that

$$T_L(x) = \frac{2}{q} \log(1 + x), \quad \text{and} \quad T_R(x) = -\frac{2}{q} \log(1 - x). \tag{3.2.3}$$

are the unique solutions satisfying the requirements. We recursively define T_* for any word $*$ composed of the alphabet $\{L, R\}$ as solutions of

$$\dot{T}_{*L} = (1-x)T_*,\ T_{*L}(0) = 0,\ \dot{T}_{*R} = (1+x)T_*,\ T_{*R}(0) = 0,$$

more specifically

$$T_{*L}(x) = \int_0^x \frac{2}{q}\frac{T_*(s)}{1+s}ds,\ T_{*R}(x) = \int_0^x \frac{2}{q}\frac{T_*(s)}{1-s}ds. \tag{3.2.4}$$

A similar approach, using iterated integrals in terms of words, has been applied in the study of the Abel equation in order to determine the number of limit cycles in the center-focusproblem (see [14] and [15]). Unlike in the case [8], the tags do not easily admit a closed expression: tags of order 2 may already contain dilogarithm expressions and order 3 tags may even be more complicated. We do however show the following proposition:

Proposition 3.2.2. *Let $k \geq 1$. The tags T_* of order k (i.e. of word length k in $*$) are of the form*

$$T_*(x) = P_*(x, T_L(x)) + Q_*(x, T_R(x)),$$

where $P_*(x, u)$, respectively $Q_*(x, u)$, is polynomial in u with C^∞ coefficients in x of degree $L(*)$, respectively $R(*)$, corresponding to the number of times the letter L, respectively R, appears in the word $*$.

Proof: For $k = 1$ this is obviously true. Suppose it is true for $k \geq 1$. This means that for words $*$ of length k we have

$$T_*(x) = \sum_{i=1}^{L(*)} f_*^i(x)T_L^i + \sum_{j=1}^{R(*)} g_*^j(x)T_R^j, \tag{3.2.5}$$

where the functions f_*^i and g_*^j are C^∞. We show that the expression for T_{*R} is similar to (3.2.5) but the second summation is expanded to $R(*)+1$. The case T_{*L} is treated similarly.

By the recursive definition (3.2.4), it suffices to show that for every positive integer k and C^∞ functions f and g,

$$\int_0^x f(s)\frac{\log^k(1-s)}{1-s}ds = \sum_{i=0}^{k+1} F_i(x)\log^i(1-x), \tag{3.2.6}$$

and

$$\int_0^x g(s)\frac{\log^k(1+s)}{1-s}ds = G(x)\log(1-x) + \sum_{i=0}^{k} H_i(x)\log^i(1+x), \tag{3.2.7}$$

for some C^∞ functions F_i, G and H_i. Observe that for any C^∞ function f, we have

$$\int_0^x \frac{f(s)}{1-s}ds = -f(1)\log(1-x) + G(x),$$

for some C^∞ function G. Similarly, by partial integration we have

$$\int_0^x f(s)\log(1-s)ds = F(x)\log(1-x)+\int_0^x \frac{F(s)}{1-s}ds = (F(x)-F(1))\log(1-x)+G(x),$$

for some C^∞ functions F and G. By induction on $n \in \mathbb{N}$ it follows:

$$\int_0^x f(s)\log^n(1-s)ds = \sum_{i=0}^{n} F_i^n(x)\log^i(1-x), \tag{3.2.8}$$

for some C^∞ functions F_i^n $(i = 0, \ldots, n)$ since by partial integration

$$\int_0^x f(s)\log^n(1-s)ds = (F(x)-F(1))\log^n(1-x) - \int_0^x G(s)\log^{n-1}(1-s)ds,$$

where G is C^∞ and F is a C^∞ primitive function of f. From these observations (3.2.6) immediately follows since

$$\int_0^x f(s)\frac{\log^k(1-s)}{1-s}ds = f(1)\int_0^x \frac{\log^k(1-s)}{1-s}ds + \int_0^x g(x)\log^k(1-s)ds.$$

To deal with (3.2.7), we now define C^∞ bump functions $\chi_L(x)$ and $\chi_R(x) = \chi_L(-x)$ such that $\chi_L(x) + \chi_R(x) = 1$, and χ_L is locally 1, respectively 0, near $x = -1$, respectively $x = 1$. The integral in (3.2.7) can be separated in

$$\begin{aligned}\int_0^x g(s)\frac{\log^k(1+s)}{1-s}ds &= \int_0^x \left(\frac{g(s)}{1-s}\chi_L(s)\right)\log^k(1+s)ds \\ &\quad + \int_0^x \left(g(s)\log^k(1+s)\chi_R(s)\right)\frac{1}{1-s}ds.\end{aligned}$$

The expressions between brackets in each of the integrals are now C^∞ functions and since a similar result as (3.2.8) holds for $\log(1+x)$, (3.2.7) follows from all of the above. $\square$

Formal reduction of the resonant part using tags

We show that we can formally eliminate the resonant terms in (3.2.2). Normal linearization amounts to finding a perturbation w_∞ of $w = (1-x^2)^p y^q$ for which $\dot{w}_\infty = 0$, in other words we seek a first integral of the form $w_\infty = w + w^2\bar{\psi}(x, w)$. The new coordinate $Y(x, y)$ is chosen such that $w_\infty = (1-x^2)^p Y^q$, i.e.

$$Y = y(1 + (1-x^2)^p y^q \bar{\psi}(x, (1-x^2)^p y^q))^{1/q}, \tag{3.2.9}$$

which will give the required normal linear form, eliminating completely the resonant part. Denote by $\mathcal{W}$ the set of words with alphabet $\{L, R\}$ and define for every $k \in \mathbb{N}_0$ the set $\mathcal{W}_k$ of words with length k.

Theorem 3.2.3. *There exists a formal transformation*

$$w_\infty = w - \sum_{k=1}^{\infty} w^{kn+1} \sum_{*\in\mathcal{W}_k} F_*(w) T_*,$$

where F_* are C^∞ functions and the tags T_* are defined by (3.2.3) *and* (3.2.4)*, such that* (3.2.2) *transforms to*

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{w}_\infty = 0. \end{cases}$$

Proof: Let $w_0 = w$. We claim that by appropriately choosing

$$w_{k+1} = w_k - w^{(k+1)n+1} \sum_{*\in\mathcal{W}_{k+1}} F_*(w) T_*(w), \qquad k \geq 0, \tag{3.2.10}$$

for some C^∞ functions F_*, we can ensure that

$$\dot{w}_k = w^{(k+1)n+1} \left(\sum_{*\in\mathcal{W}_k} F_{*L}(w)(1-x)T_* + F_{*R}(w)(1+x)T_* \right), \tag{3.2.11}$$

for some C^∞ functions F_{*L} and F_{*R}, $* \in \mathcal{W}_k$. We show how these are defined in the induction step below. Since the order in w of the words of length k increases with k, it will imply that $\dot{w}_k$ becomes flatter with growing k. The limit w_∞ of this transformation is of the desired form and due to the growing flatness will satisfy $\dot{w}_\infty = 0$. The claim is obviously true for $k = 0$. Let us now proceed under the induction hypothesis that the claim is correct up to order k, i.e. (3.2.11) holds. Define

$$w_{k+1} = w_k - w^{(k+1)n+1} \left(\sum_{*\in\mathcal{W}_k} F_{*L}(w) T_{*L} + F_{*R}(w) T_{*R} \right).$$

A simple computation shows that

$$\begin{aligned} \dot{w}_{k+1} = {} & \sum_{*\in\mathcal{W}_k} \underbrace{\left(-w^{n+1} F_L(w) \frac{d\left(w^{(k+1)n+1} F_{*L}(w)\right)}{dw} \right)}_{w^{(k+2)n+1} F_{*LL}(w)} (1-x) T_{*L} \\ & + \sum_{*\in\mathcal{W}_k} \underbrace{\left(-w^{n+1} F_R(w) \frac{d\left(w^{(k+1)n+1} F_{*L}(w)\right)}{dw} \right)}_{w^{(k+2)n+1} F_{*LR}(w)} (1+x) T_{*L} \\ & + \sum_{*\in\mathcal{W}_k} \underbrace{\left(-w^{n+1} F_L(w) \frac{d\left(w^{(k+1)n+1} F_{*R}(w)\right)}{dw} \right)}_{w^{(k+2)n+1} F_{*RL}(w)} (1-x) T_{*R} \end{aligned}$$

$$+\sum_{*\in\mathcal{W}_k}\underbrace{\left(-w^{n+1}F_R(w)\frac{d\left(w^{(k+1)n+1}F_{*R}(w)\right)}{dw}\right)}_{w^{(k+2)n+1}F_{*RR}(w)}(1+x)T_{*R},$$

which is equivalent to (3.2.11) for $k+1$. □

When we compute the transition map in the next section, it may be beneficial to first delete a finite part of the symmetric resonant terms corresponding to the function g in (3.2.1) since we expect these to have a non-dominant effect on the transition. Indeed the linearized system (3.1.2) is invariant under the symmetry $(x,t)\mapsto(-x,-t)$ and so is the system (3.1.1) when $f=h=0$. The transition map will in these cases be given by $(x_0,1)\mapsto(-x_0,1)$. Therefore one can desire to use tags highlighting the symmetry with respect to $x=0$ other than the definition in (3.2.3) and (3.2.4).

We define the tags $T_E(x)$ and $T_O(x)$ of order 1 in the alphabet $\{E,O\}$ as the functions, smooth for $x\in(-1,1)$, satisfying $T_E(0)=T_O(0)=0$ and where their time-derivative satisfies

$$\dot{T}_E=x,\quad\text{and}\quad\dot{T}_O=1,$$

in contrast to (3.2.3). Concretely a direct computation shows that

$$T_E(x)=-\frac{1}{q}\log\left(1-x^2\right),\quad\text{and}\quad T_O(x)=\frac{1}{q}\log\left(\frac{1+x}{1-x}\right),\tag{3.2.12}$$

are the unique solutions satisfying the requirements. Observe that these can easily be related to the previous defined tags (and vice versa) since

$$T_E=\frac{1}{2}\left(T_R-T_L\right),\quad\text{and}\ T_O=\frac{1}{2}\left(T_R+T_L\right).\tag{3.2.13}$$

By induction we define

$$T_{*E}(x)=\int_0^x\frac{2}{q}\frac{sT_*(s)}{1-s^2}ds,\ T_{*O}(x)=\int_0^x\frac{2}{q}\frac{T_*(s)}{1-s^2}ds,\tag{3.2.14}$$

for any $*\in\mathcal{W}^{sym}$ consisting of words in the alphabet $\{E,O\}$. Remark that T_{*E} preserves the symmetric behaviour of T_* while T_{*O} inverses the symmetry of T_* with respect to $x=0$. Therefore we know that a tag T_* is odd whenever the number of times that O appears in $*$ is odd, otherwise T_* is even. Due to the relation to the previous tags in (3.2.13) and the result of Proposition 3.2.2, it is easy to see that the higher-order terms (3.2.14) can be expressed in terms of the tags (3.2.13) in a similar way as in (3.2.5). Indeed this is a straightforward consequence of the following lemma. Denote by $\mathcal{W}_k^{sym}$ the words of length k in $\mathcal{W}^{sym}$.

Lemma 3.2.4. *Let $\diamond \in \mathcal{W}^{sym}$ be arbitrary and denote the length of $\diamond$ by $k \geq 1$. The tag $T_\diamond$ can be written as a linear combination of the tags T_* with $* \in \mathcal{W}_k$, i.e.*

$$T_\diamond = \sum_{* \in \mathcal{W}_k} c_*^\diamond T_*,$$

for some $c_*^\diamond \in \mathbb{R}$. Similarly, any tag T_* with $* \in \mathcal{W}_k$ can be written as a linear combination of tags $T_\diamond$ with $\diamond \in \mathcal{W}_k^{sym}$.

Proof: We proof this by induction on the length k. For words of length 1, this is expressed in (3.2.13). Suppose it is true for all $\diamond \in \mathcal{W}_k^{sym}$ for some $k \geq 1$. Due to the recursive definition of the tags (3.2.14), it suffices to show that

$$\int_0^x \frac{sT_*(s)}{1-s^2} ds, \text{ and } \int_0^x \frac{T_*(s)}{1-s^2} ds,$$

can be written as a linear combination of tags $T_\circ$ with $\circ \in \mathcal{W}_{k+1}$ for any $* \in \mathcal{W}_k$. This follows immediately from the recursive definition (3.2.4) and the decompositions

$$\frac{s}{1-s^2} = \frac{1}{2}\left(\frac{1}{1-s} - \frac{1}{1+s}\right), \quad \frac{1}{1-s^2} = \frac{1}{2}\left(\frac{1}{1-s} + \frac{1}{1+s}\right).$$

The proof for expressing T_* with $* \in \mathcal{W}_k$ as a linear combination of $T_\diamond$ with $\diamond \in \mathcal{W}_k^{sym}$ can be done similarly using the above decompositions. $\square$

Another direct consequence of Lemma 3.2.4 is an adaptation of Theorem 3.2.3 in terms of the tags T_* with $* \in \mathcal{W}^{sym}$. Instead of (3.2.2) we consider

$$\dot{w} = qw^{n+1} f(w) + qxw^{n+1} g(w). \tag{3.2.15}$$

Theorem 3.2.5. *There exists a formal transformation*

$$w_\infty = w - \sum_{k=1}^{\infty} w^{kn+1} \sum_{* \in \mathcal{W}_k^{sym}} F_*(w) T_*,$$

where F_* are C^∞ functions and the tags T_* are defined by (3.2.12) *and* (3.2.14)*, such that* (3.2.15) *transforms to*

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{w}_\infty = 0. \end{cases}$$

Proof: Let $w_0 = w$ and denote $F_E(w) = qg(w)$, $F_0(w) = qf(w)$. We show by induction that there exist smooth functions $F_*(w)$ for $* \in \mathcal{W}^{sym}$ such that

$$\dot{w}_k = w^{(k+1)n+1} \left(\sum_{* \in \mathcal{W}_k^{sym}} F_{*E}(w) x T_* + F_{*O}(w) T_* \right), \tag{3.2.16}$$

where these variables w_k are defined by

$$w_{k+1} = w_k - w^{(k+1)n+1} \sum_{* \in \mathcal{W}_{k+1}^{sym}} F_*(w) T_*(w), \tag{3.2.17}$$

for $k \geq 0$. Obviously (3.2.16) is true for $k = 0$, so suppose it is true for some $k \geq 0$ arbitrary. Define

$$w_{k+1} = w_k - w^{(k+1)n+1} \left(\sum_{* \in \mathcal{W}_k^{sym}} F_{*E}(w) T_{*E} + F_{*O}(w) T_{*O} \right),$$

which is of the form (3.2.17). A straightforward computation shows

$$\begin{aligned}
\dot{w}_{k+1} = & \sum_{* \in \mathcal{W}_k^{sym}} \underbrace{\left(-w^{n+1} F_E(w) \frac{d\left(w^{(k+1)n+1} F_{*E}(w)\right)}{dw} \right)}_{w^{(k+2)n+1} F_{*EE}(w)} x T_{*E} \\
& + \sum_{* \in \mathcal{W}_k^{sym}} \underbrace{\left(-w^{n+1} F_E(w) \frac{d\left(w^{(k+1)n+1} F_{*O}(w)\right)}{dw} \right)}_{w^{(k+2)n+1} F_{*OE}(w)} x T_{*O} \\
& + \sum_{* \in \mathcal{W}_k^{sym}} \underbrace{\left(-w^{n+1} F_O(w) \frac{d\left(w^{(k+1)n+1} F_{*E}(w)\right)}{dw} \right)}_{w^{(k+2)n+1} F_{*EO}(w)} T_{*E} \\
& + \sum_{* \in \mathcal{W}_k^{sym}} \underbrace{\left(-w^{n+1} F_O(w) \frac{d\left(w^{(k+1)n+1} F_{*O}(w)\right)}{dw} \right)}_{w^{(k+2)n+1} F_{*OO}(w)} T_{*O},
\end{aligned}$$

which concludes the proof since it is of the form (3.2.16). □

We can now explicitly prove that the first non-zero term of f in (3.2.15) provides the dominant asymmetric term in the linearization transformation of Theorem 3.2.5. For this we consider (3.2.15) in the form

$$\dot{w} = q w^{m+1} f(w) + q x w^{l+1} g(w), \tag{3.2.18}$$

where $f(0) \neq 0$ and $g(0) \neq 0$ and $m, l \geq 1$. For simplicity of notation we preserved the notation f and g, but these can be different than the ones from (3.2.15). The exponent n then corresponds to $\min\{m, l\}$.

Lemma 3.2.6. *The transformation obtained in Theorem 3.2.3 and Theorem 3.2.5 applied to* (3.2.18) *where* $f(0)g(0) \neq 0$ *can be decomposed as*

$$w_\infty = w - w^{m+1} F_{asym}(x, w) - w^{j+1} F_{sym}(x, w),$$

where

$$F_{asym}(-x,w) = -F_{asym}(x,w), \quad \textit{and} \;\; F_{sym}(-x,w) = F_{sym}(x,w),$$

and $j \geq min\{m,l\}$. *The functions* F_{asym} *and* F_{sym} *should be considered as formal series in* w *where the coefficients are given by linear combinations of the tags* T_* *where* $* \in \mathcal{W}$ *or* $* \in \mathcal{W}^{sym}$. *In particular*

$$F_{asym}(x,0) = qf(0)T_O(x).$$

Proof: The first step in the proof of Theorem 3.2.5 applied to (3.2.18) is given by

$$w_1 = w - qw^{m+1}f(w)T_O - qw^{l+1}g(w)T_E,$$

where the tags T_O and T_E are given by (3.2.12). By a straightforward computation, we get the new equation

$$\begin{aligned}\dot{w}_1 = &-q^2w^{m+1}f(w)\frac{d\left(w^{m+1}f(w)\right)}{dw}T_O - q^2w^{l+1}g(w)\frac{d\left(w^{m+1}f(w)\right)}{dw}xT_O \\ &- q^2w^{m+1}f(w)\frac{d\left(w^{l+1}g(w)\right)}{dw}T_E - q^2w^{l+1}g(w)\frac{d\left(w^{l+1}g(w)\right)}{dw}xT_E.\end{aligned}$$

All these terms are of order at least $m+2$ in w (in fact of order $2m+1$ or $m+l+1$) except for the last term when $2l+1 \leq m+1$. However this term is odd in x and can therefore only be removed by a symmetric function. More precise in the next step this is eliminated by a term $\tilde{g}(w)T_{EE}$ in the procedure of Theorem 3.2.5. Again after this step, the only term which is of lower degree than $m+2$ is $w^{3l+1}G(w)xT_{EE}$, when $3l+1 \leq m+1$. Hence for any $i \geq 1$ where $i \cdot l + 1 \leq m+1$, equation (3.2.16) is of the form

$$\dot{w}_i = w^{(i+1)l+1}G_i(w)xT_{\underbrace{E\ldots E}_{i \text{ times}}} + \mathrm{O}\left(w^{m+2}\right),$$

for a smooth function $G_i(w)$ where $i \geq 1$. Hence by completing the procedure of Theorem 3.2.5, we get the result stated in the lemma since we can expand the transformation in terms of linear combinations of the tags T_* for $* \in \mathcal{W}^{sym}$ in each degree of w and such a tag is odd if and only if the number of times the letter O occurs in $*$ is odd. $\square$

Finitely smooth reduction of the resonant part

The functions $w^n \log(1-x)$ and $w^n \log(1+x)$ are of Logarithmic Mourtada type (LMT) near respectively 1 and -1 (see [8] or [37]) and C^∞ in $]-1,1[$. The loss of smoothness is thus located at the points $x = \pm 1$. Hence we can prove the following.

Theorem 3.2.7. *Consider the C^∞ vector field* (3.1.1) *with $p, q \in \mathbb{N}^*$ and $gcd(p, q) = 1$. There exists a finitely smooth near-identity coordinate change*

$$(x, y) \mapsto (x, Y) = (x, y(1 + \varphi(x, y))),$$

bringing (3.1.1) *to the smooth vector field*

$$\begin{cases} \dot{x} = \frac{q}{2}(1 - x^2), \\ \dot{Y} = Y(px + \chi(x)\tilde{h}(x, Y)). \end{cases} \tag{3.2.19}$$

for some $\tilde{h}$ which is C^∞ in (x, Y) for $x \in (-1, 1)$ which can be expressed as a smooth function of the variables

$$\big(x, Y, (1 - x^2)^n \log(1 + x), (1 - x^2)^n \log(1 - x)\big).$$

Moreover φ is of the form

$$\varphi(x, y) = \Phi\left(x, y, w^n \log(1 + x), w^n \log(1 - x)\right),$$

where Φ is C^∞ near $[-1, 1] \times \{(0, 0, 0)\}$ and $w = (1 - x^2)^p y^q$.

Remark 3.2.8. *Since $(1-x^2)^n \log(1+x)$ and $(1-x^2)^n \log(1-x)$ are of Mourtada type near $x = -1$ and $x = 1$, it follows immediately that $(1 - x^2)\tilde{h}(x, Y)$ is of Mourtada type near $x = -1$ and $x = 1$. The flatness of χ at $x = \pm 1$ therefore induces the smoothness of* (3.2.19).

Proof: From Theorem 3.2.3 and the discussion before (3.2.9), there exists a formal transformation

$$Y = y\left(1 - \sum_{k=1}^{\infty} w^{kn} \sum_{* \in \mathcal{W}_k} F_*(w) T_*\right)^{1/q},$$

which removes the resonant terms in (3.1.1). Thanks to Proposition 3.2.2, we can express this formal transformation as a function of $(x, y, w^n T_L, w^n T_R)$. Using Borel's Theorem (Theorem 1.2.5), there exists a coordinate change $Y = y(1 + \psi(x, y))$ of the form

$$\psi(x, y) = \Psi(x, y, w^n T_L, w^n T_R),$$

where Ψ is C^∞ near $[-1, 1] \times \{(0, 0, 0)\}$, transforming (3.1.1) to

$$\begin{cases} \dot{x} = \frac{q}{2}(1 - x^2), \\ \dot{Y} = Y(px + \chi(x)\tilde{h}(x, Y)) + F(x, Y), \end{cases}$$

where F is a smooth function satisfying $j_\infty F(x, 0) = 0$. Using the technique in Section 2.4, it is possible to adapt ψ by a flat function such that the new transformation $Y = y(1 + \varphi(x, y))$ transforms (3.1.1) to (3.2.19). $\square$

3.2.2 Removing the connecting terms

Consider the vector field (3.2.19). When $q = 1$, we know from Theorem 2.1.2 that connecting terms do not occur in the original vector field (3.1.1), i.e. $h = 0$. In this case Theorem 3.2.1 reduces to Theorem 3.2.7. When $q > 1$ is arbitrary, we transform vector field (3.2.19) to the case without connecting terms using a non-smooth transformation in x. This amounts to the following theorem.

Theorem 3.2.9. *Consider the vector field* (3.2.19) *with* $q > 1$. *There exists a near-identity coordinate change*

$$(x, Y) \mapsto (x, z) = (x, Y(1 + \varphi(x, Y)),$$

bringing (3.2.19) *in the form* (3.1.2). *Moreover* φ *is of the form*

$$\varphi(x, Y) = \Phi\left(x, Y, (1 - x^2)^{1/q}\right),$$

where Φ *is* C^∞ *near* $[-1, 1] \times \{0\} \times [0, 1]$.

Proof: Write $1 - x^2 = (1 - X^2)^q$ and therefore

$$x = X\,\Omega(X), \quad \text{where} \quad \Omega(X) = \sqrt{\frac{1 - (1 - X^2)^q}{X^2}}. \tag{3.2.20}$$

Observe that $\Omega(X)$ is a C^∞ strictly positive function for $X \in (-\sqrt{2}, \sqrt{2})$. This change of coordinates maps $[-1, 1]$ to itself, although in a finitely smooth way at the boundary. After division by $\Omega(X)$ the effect of (3.2.20) on system (3.2.19) is given by:

$$\begin{cases} \dot{X} = \frac{1}{2}(1 - X^2), \\ \dot{Y} = Y\left(pX + \chi\left(X\Omega(X)\right)\frac{\tilde{h}(X\Omega(X), Y)}{\Omega(X)}\right). \end{cases}$$

Since the transformation fixes $x = \pm 1$, the second term remains flat and therefore $\dot{y}$ can be written as

$$\dot{Y} = Y(pX + H(X, Y)) \tag{3.2.21}$$

for some C^∞ function H that is flat at $X = \pm 1$. Since this system has a saddle connection with ratios of eigenvalues $-p : 1$ and $p : -1$, its normal form has no connecting terms (see Theorem 2.1.2); it even has no resonant terms due to the flatness of H. This is true since the inductive process of Chapter 2 does not create non-flat terms. Indeed applying a smooth transformation of the form

$$Y = y + g(X)y^{k+1},$$

satisfying

$$-\frac{1}{2}(1 - X^2)g'(X) - pkXg(X) + F(X) = 0,$$

where F is a smooth function, infinitely flat at $x = \pm 1$, to (3.2.19) amounts to

$$\dot{y} = pXy + \chi(X)y\tilde{h}(X, y + g(X)y^{k+1}) - F(X)y^{k+1} + F(X)y^{k+1}\sum_{i\geq 0}(-k-1)^i g(x)^i y^{ki}.$$

Therefore there exists, according to Theorem 2.1.2, a C^∞ normalizing transformation $Y = z(1 + \psi(X, z))$ reducing (3.2.21) to normalized linear form. We can hence also apply the transformation

$$Y = z(1 + \varphi(x, z)), \text{ where } \varphi(x, z) = \psi(X(x), z),$$

to (3.2.19) to obtain a finitely smooth transition to (3.1.2).
It remains to prove that φ can be expressed as a C^∞ function of $x, s = (1 - x^2)^{1/q}, y$. It suffices to prove that $X(x)$ is C^∞ in x and $(1 - x^2)^{1/q}$. We have

$$X(x) = x\sqrt{\frac{1-s}{x^2}} = x\sqrt{\frac{1-s}{1-s^q}} = x\sqrt{\frac{1}{1+s+\cdots+s^{q-1}}} = x\rho\left((1-x^2)^{1/q}\right),$$

where ρ is C^∞. □

3.3 An invariant and the transition map

Theorem 3.2.1 describes the transformation from (3.1.1) to the integrable system (3.1.2). Since $W := (1 - x^2)^p z^q$ is a constant of motion for (3.1.2), it immediately results to

Corollary 3.3.1. *Consider the C^∞ vector field* (3.1.1) *with $p, q \in \mathbb{N}*$ and gcd(p, q)= 1. There exists a constant of motion $V(x, y)$ of the vector field given by*

$$V(x, y) = (1 - x^2)^p y^q (1 + \psi(x, y))^q,$$

where $\psi(x, y)$ is the function as described in Theorem 3.2.1.

We will use this idea to compute the transition map

$$\Sigma_{\text{in}} \to \Sigma_{\text{out}}$$

with Σ_* as defined in (3.1.3) and using the parametrization there introduced. Suppose the vector field (3.1.1) can be written as

$$\begin{cases} \dot{x} = \frac{q}{2}(1 - x^2), \\ \dot{y} = y\left(px + w^n f(w) + xw^n g(w) + \chi(x)h(y)y^k\right), \end{cases} \tag{3.3.1}$$

where $|f(0)| + |g(0)| \neq 0$ and $h(0) \neq 0$. When we want to determine the dominant term in the transition map, we consider

$$\begin{cases} \dot{x} = \frac{q}{2}(1 - x^2), \\ \dot{y} = y\left(px + w^m f(w) + xw^l g(w) + \chi(x)h(y)y^k\right), \end{cases} \tag{3.3.2}$$

where $f(0)g(0)h(0) \neq 0$ instead of (3.3.1). The exponent n defined in (3.3.1) corresponds to $\min\{m, l\}$, for m, l defined in (3.3.2). In this section we prove the following.

Theorem 3.3.2. *Consider the vector field as given by* (3.3.1)*, where* $|f(0)|+|g(0)| \neq 0$ *and* $h(0) \neq 0$*. The transition map*

$$D : \Sigma_{in} \to \Sigma_{out} : x_0 \mapsto D(x_0),$$

can be written as

$$D(x_0) = -x_0 - (1 + x_0)\delta(x_0),$$

where δ *is a* C^∞ *function in the variables*

$$\left(x_0, (1 - x_0^2)^{np} \log(1 + x_0), (1 - x_0^2)^{np} \log(1 - x_0), (1 - x_0^2)^{1/q}\right). \tag{3.3.3}$$

Moreover when we consider (3.3.2) *with* $f(0)g(0)h(0) \neq 0$*, we have for* x_0 *close to* -1 *(*$x_0 > -1$*):*

$$D(x_0) = -x_0 + \frac{1}{p}(1 - x_0^2)^{mp+1} f(0) \log(1 + x_0)\,(1 + F(x_0))\,, \quad \textit{if} \;\; mq < k, \tag{3.3.4}$$

or

$$D(x_0) = -x_0 - \frac{2}{p} A_k h(0)(1 - x_0^2)^{\frac{pk}{q}+1}(1 + F(x_0)), \quad \textit{if} \;\; k < mq, \tag{3.3.5}$$

where

$$A_k = \int_0^1 \frac{\chi(x)}{(1 - x^2)^{\frac{pk}{q}+1}} dx, \quad \textit{and} \quad \lim_{x_0 \to -1^+} F(x_0) = 0.$$

The proof of this theorem is divided in two parts. First we prove that the transition map can be expressed as a smooth function of the variables (3.3.3) in Section 3.3.1. Then we derive the asymptotics of this map using the first non-linear term of the transformation of Theorem 3.2.1. This is the subject of Section 3.3.2.

The notation used in Theorem 3.3.2 is not natural when one wants to study the asymptotics of the transition map. If we parametrize Σ_{in} by $(-1 + u_0, 1)$ and Σ_{out} by $(1 - u_1, 1)$ (see Figure 3.1), we can reformulate Theorem 3.3.2 as follows.

Theorem 3.3.3. *Consider the vector field as given by* (3.3.1)*, where* $|f(0)|+|g(0)| \neq 0$ *and* $h(0) \neq 0$*. Consider the transverse sections*

$$\Sigma_{in} = \{(-1 + u_0, 1) \mid u_0 > 0, u_0 \approx 0\}, \;\; \textit{and} \; \Sigma_{out} = \{(1 - u_1, 1) \mid u_1 > 0, u_1 \approx 0\}.$$

The transition map

$$\widetilde{D} : \Sigma_{in} \to \Sigma_{out} : u_0 \mapsto u_1 = \widetilde{D}(u_0),$$

can be expressed as a C^∞ *function in the variables*

$$\left(u_0, u_0^{np} \log(u_0), u_0^{1/q}\right).$$

Moreover when we consider (3.3.2) *with* $f(0)g(0)h(0) \neq 0$*, for* $u_0 > 0$ *close to* 0*, we have*

$$\widetilde{D}(u_0) = u_0 + \frac{2^{np+1}}{p} f(0) u_0^{np+1} \log(u_0) \left(1 + \mathrm{o}(1)\right), \quad \text{if } nq < k,$$

or

$$\widetilde{D}(u_0) = u_0 - \frac{2^{\frac{pk}{q}+2}}{p} A_k h(0) u_0^{\frac{pk}{q}+1} \left(1 + \mathrm{o}(1)\right), \quad \text{if } nq > k,$$

where A_k *is given in Theorem 3.3.2.*

In this sense, we provide an alternative proof for the dependence of the transition map as formulated in Theorem 3.3.3. This will be done in Section 3.3.3. The advantage of the notation of Theorem 3.3.2 is that we can exploit the (a)symmetry of terms to obtain the first non-linear term. Therefore we shall not prove the asymptotic expansion as formulated in Theorem 3.3.3 but see this as a direct consequence of Theorem 3.3.2.

Remark 3.3.4. *One can define an asymptotic scale as has been done in [41]. For this we express the variables* (3.3.3) *in terms of the small variable* $u_0 = x_0 + 1$ *(see Theorem 3.3.3) by*

$$u_0^{r_1} \log^{m_1}(u_0) \succ u_0^{r_2} \log^{m_2}(u_0), \tag{3.3.6}$$

if $r_1 > r_2$ *or* $r_1 = r_2$ *and* $m_1 < m_2$ *for* $r_i \in \mathbb{Q}$ *and* $m_i \in \mathbb{N}$*.*

3.3.1 The transition map as a function of the tags

Due to Corollary 3.3.1, we find a C^∞ constant of motion of the system (3.1.1) given by

$$V(x, y) = w\left(1 + \Psi\left(x, y, w^n T_L, w^n T_R, (1 - x^2)^{1/q}\right)\right)^q, \tag{3.3.7}$$

using the first integral of the normally linearized system (3.1.2).

Let us now compute the entry-exit relation. Denote the initial variable on Σ_{in} by x_0 and the corresponding exit variable by x_1. Remark that $x_1 \to 1$ as $x_0 \to -1$, so we write

$$1 - x_1 = (1 + x_0)(1 + \delta(x_0)). \tag{3.3.8}$$

As a matter of fact we will see that δ tends to 0 as x_0 tends to -1, which we will show using the implicit function Theorem with (3.3.8) as ansatz. This form of transition map originates from the fact that it is near-identity due to the symmetry of the eigenvalues of the saddle (see Figure 3.1).

At the cuts Σ_*, the invariant is given by

$$V(x, 1) = (1 - x^2)^p \left(1 + \Psi\left(x, 1, (1 - x^2)^{np} T_L, (1 - x^2)^{np} T_R, (1 - x^2)^{1/q}\right)\right)^q.$$

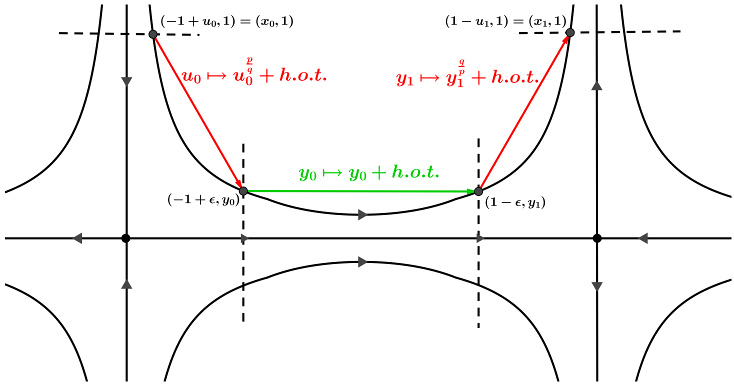


Figure 3.1: Asymptotics of transition near saddle connection

For the sake of notation, denote the LMT-functions as

$$\bar{T}_L = (1-x_0^2)^{np} T_L(x_0), \quad \text{and} \quad \bar{T}_R = (1-x_0^2)^{np} T_R(x_0). \tag{3.3.9}$$

We aim to express δ in terms of $\left(x_0, \bar{T}_L, \bar{T}_R, (1-x_0^2)^{1/q}\right)$ by applying the implicit function Theorem to the equation $V(x_0,1) = V(x_1,1)$ since V is invariant under the flow and we impose that $(x_0,1)$ and $(x_1,1)$ are different points of the same orbit. First we need to express $T_L(x_1), T_R(x_1), (1-x_1^2)$ in terms of x_0 and δ. Observe that

$$T_R(x_1) = -\frac{2}{q}\log(1-x_1) = -T_L(x_0) - \frac{2}{q}\delta + \mathrm{O}\big(\delta^2\big). \tag{3.3.10}$$

Here and in the remainder of this section appearing O-terms are C^∞ in (x_0,δ) near $(-1,0)$. Remark that they can blow up close to $x_0 = 1$, but since we are interested in the behaviour near $x_0 = -1$ this does not pose a problem. The tag $T_L(x)$ is C^∞ at $x = 1$, just as $T_R(x)$ is C^∞ at $x = -1$. We see

$$\begin{aligned} T_L(x_1) &= \frac{2}{q}\log\left(2-(1+x_0)(1+\delta)\right) \\ &= -T_R(x_0) + \frac{2}{q}\log\left(1-\delta\frac{1+x_0}{1-x_0}\right) \\ &= -T_R(x_0) - \frac{2}{q}\delta\frac{1+x_0}{1-x_0} + \mathrm{O}\big(\delta^2\big). \end{aligned} \tag{3.3.11}$$

A simple computation shows

$$(1-x_1^2) = (1-x_0^2)\left(1-2\delta\frac{x_0}{1-x_0} + \mathrm{O}\big(\delta^2\big)\right),$$

hence for a power $r \in \mathbb{Q}^+$,

$$(1-x_1^2)^r = (1-x_0^2)^r\left(1-2r\delta\frac{x_0}{1-x_0} + \mathrm{O}\big(\delta^2\big)\right). \tag{3.3.12}$$

Using the expansions above, more precise (3.3.10) and (3.3.12) for $r = np$, we have

$$\bar{T}_R^{(1)} := (1-x_1^2)^{np} T_R(x_1) = -\bar{T}_L + \left(2np\frac{x_0}{1-x_0}\bar{T}_L - \frac{2}{q}(1-x_0^2)^{np}\right)\delta + \mathrm{O}\big(\delta^2\big), \tag{3.3.13}$$

and by (3.3.11) and (3.3.12),

$$\bar{T}_L^{(1)} := (1-x_1^2)^{np} T_L(x_1) = -\bar{T}_R + \left(2np\frac{x_0}{1-x_0}\bar{T}_R - \frac{2}{q}(1-x_0^2)^{np}\frac{1+x_0}{1-x_0}\right)\delta + \mathrm{O}\big(\delta^2\big). \tag{3.3.14}$$

Denote

$$\Psi_1 := \Psi\left(x_1, 1, \left(1-x_1^2\right)^{np} T_L(x_1), \left(1-x_1^2\right)^{np} T_R(x_1), \left(1-x_1^2\right)^{1/q}\right),$$

where Ψ is introduced in Theorem 3.2.1. This can be expressed as a function of x_0 and δ thanks to (3.3.8), (3.3.12), (3.3.13) and (3.3.14)

$$\Psi_1 = \Psi\left(-x_0 - \delta(1+x_0), 1, \bar{T}_L^{(1)}, \bar{T}_R^{(1)}, (1-x_0^2)^{1/q}\left(1 - \frac{2}{q}\frac{x_0}{1-x_0}\delta + \mathrm{O}\big(\delta^2\big)\right)\right). \tag{3.3.15}$$

Since $V(x_1, 1)$ can be expressed as a function of x_0 and δ using (3.3.12) and (3.3.15), we can search for solutions δ of

$$0 = \Theta\left(\delta, x_0, \bar{T}_L, \bar{T}_R, (1-x_0^2)^{1/q}\right) := \left(\frac{V(x_1,1)}{(1-x_0^2)^p}\right)^{1/q} - \left(\frac{V(x_0,1)}{(1-x_0^2)^p}\right)^{1/q}, \tag{3.3.16}$$

where Θ is C^∞ near $(0,-1,0,0,0)$, such that $V(x_0,1) = V(x_1,1)$ is satisfied. In order to apply the implicit function Theorem to (3.3.16) at $(0,-1,0,0,0)$ and consequently show that we can express δ in terms of $\left(x_0, \bar{T}_L, \bar{T}_R, (1-x_0^2)^{1/q}\right)$, it is sufficient to show that

$$\frac{\partial\Theta}{\partial\delta}(0,-1,0,0,0) \neq 0,$$

since $\Theta(0,-1,0,0,0) = 0$. Notice that

$$\Theta = \left(1 - \frac{2p}{q}\frac{x_0}{1-x_0}\delta + \mathrm{O}\big(\delta^2\big)\right)(1+\Psi_1) - \left(1 + \Psi\left(x_0, 1, \bar{T}_L, \bar{T}_R, (1-x_0^2)^{1/q}\right)\right).$$

We find

$$\frac{\partial\Theta}{\partial\delta} = -\frac{2p}{q}\frac{x_0}{1-x_0}(1+\Psi_1) + \frac{\partial\Psi_1}{\partial\delta} + \mathrm{O}\big(\delta\big), \tag{3.3.17}$$

where

$$\frac{\partial\Psi_1}{\partial\delta} = \mathrm{O}\big(1-x_0^2, \bar{T}_R, \bar{T}_L, \delta\big).$$

Hence we see

$$\frac{\partial\Theta}{\partial\delta}(0,-1,0,0,0) = \frac{p}{q} + \frac{\partial\Psi_1}{\partial\delta}(0,-1,0,0,0) = \frac{p}{q} \neq 0.$$

By the implicit function Theorem, we can thus write

$$x_1 = -x_0 - (1+x_0)\delta(x_0),$$

where we can express

$$\delta(x_0) = \bar{\delta}\left(x_0, \bar{T}_L, \bar{T}_R, (1-x_0^2)^{1/q}\right),$$

for a C^∞ function $\bar{\delta}$ near $(-1,0,0,0)$.

3.3.2 Asymptotics of the transition map

In the previous section we proved that we can express the transition map in terms of

$$\left(x_0, \bar{T}_L, \bar{T}_R, (1-x_0^2)^{1/q}\right). \tag{3.3.18}$$

We now want to compute the asymptotics of the map near $x_0 = -1$. Recall that δ defined in (3.3.8) should be a solution of (3.3.16). Hence if we expand Θ near $\delta = 0$, we ought to solve

$$0 = \Theta\left(0, x_0, \bar{T}_L, \bar{T}_R, (1-x_0^2)^{1/q}\right) + \delta\frac{\partial\Theta}{\partial\delta}\left(0, x_0, \bar{T}_L, \bar{T}_R, (1-x_0^2)^{1/q}\right) + \mathrm{O}\big(\delta^2\big). \tag{3.3.19}$$

From the definition of Θ, it follows immediately that

$$\Theta\mid_{\delta=0} = \Psi\left(-x_0, 1, -\bar{T}_R, -\bar{T}_L, (1-x_0^2)^{1/q}\right) - \Psi\left(x_0, 1, \bar{T}_L, \bar{T}_R, (1-x_0^2)^{1/q}\right), \tag{3.3.20}$$

and using (3.3.17) one can check

$$\frac{\partial\Theta}{\partial\delta}\left(0, x_0, \bar{T}_L, \bar{T}_R, (1-x_0^2)^{1/q}\right) = \frac{2p}{q}\frac{1}{1-x_0} + \dots,$$

where we shall use the notation $\dots$ for a finitely smooth function, smooth on $x_0 \in (-1,1)$, having the same property as the function F in the statement of Theorem 3.3.2. If we want to compute the dominant term of the transition map, i.e. the term of lowest asymptotic order we have to describe (3.3.20). Suppose (3.1.1) can be written as (3.3.2) where $f(0)g(0)h(0) \neq 0$ for some $m, l, k \in \mathbb{N}^*$ with $k \neq mq$. We have to distinguish two cases depending on which of the transformations (Theorem 3.2.7 or Theorem 3.2.9) is dominant, i.e. provides the terms of lowest degree of $(1-x^2)$ in the linearizing transformation of Theorem 3.2.1:

(A) $mq < k$,

(B) $k < mq$.

Case A

Suppose $mq < k$. In this case, the transformation obtained in Theorem 3.2.7 to get rid of the resonant terms is dominant. Obviously in (3.3.20), the first non-zero term which is not symmetric with respect to $x = 0$ provides a dominant contribution. Therefore the transformation of Theorem 3.2.7 can formally be expressed as

$$w_\infty = w - w^{m+1} F_{asym}(x, w) - w^{j+1} F_{sym}(x, w),$$

where $j \geq \min\{m, l\}$ and

$$F_{asym}(-x, w) = -F_{asym}(x, w), \quad \text{and} \quad F_{sym}(-x, w) = F_{sym}(x, w),$$

in terms of the resonant monomials $w_\infty = (1 - x^2)^p z^q$ and $w = (1 - x^2)^p y^q$ due to Lemma 3.2.6. The transformation of Theorem 3.2.7 in terms of z and y can therefore formally be expanded as

$$z = y \left(1 - \frac{1}{q} w^m F_{asym}(x, w) - w^j G(x, w) + \text{h.o.t.} \right),$$

for some G satisfying $G(-x, w) = G(x, w)$ and where the higher order terms contain expressions of degree $m + 1$ or higher in w. The difference (3.3.20) then reduces to

$$\Theta \mid_{\delta=0} = \frac{2}{q} \left(1 - x_0^2\right)^{mp} F_{asym}(x_0, 0)(1 + \ldots).$$

Due to Lemma 3.2.6 and (3.2.13), we can rewrite this as

$$\begin{aligned} \Theta \mid_{\delta=0} &= f(0) \left(1 - x_0^2\right)^{mp} (T_R(x_0) + T_L(x_0))(1 + \ldots) \\ &= f(0) \left(1 - x_0^2\right)^{mp} T_L(x_0)(1 + \ldots). \end{aligned}$$

Combining this with the Taylor expansion given in (3.3.19), we see

$$\delta = -\frac{q}{2p}(1 - x_0) f(0) \left(1 - x_0^2\right)^{mp} T_L(x_0) \left(1 + \ldots\right),$$

Hence

$$D(x_0) = -x_0 + \frac{q}{2p}(1 - x_0^2) f(0) \left(1 - x_0^2\right)^{mp+1} T_L(x_0) \left(1 + \ldots\right),$$

which concludes the proof of Theorem 3.3.2 in the case $k > mq$.

Case B

Suppose $k < mq$. The first higher order term in the transformation of Theorem 3.2.1 can be contributed by the transformation in Theorem 3.2.7 when $lq < k$. However in this case, this contribution is symmetric and will therefore disappear when we consider (3.3.20) similar as we have seen in case A. The main contribution to (3.3.20) is then given by the transformation of Theorem 3.2.9 where we remove the connecting terms. The transformation obtained in Theorem 3.2.1 can formally be written as

$$z = y \left(1 - h(0) y^k \Phi(x) - w^j G(x, w) + \text{h.o.t.} \right),$$

where $G(-x,w)=G(x,w)$ and where $\Phi(x)=\Phi(X\Omega(X))=\tilde{\Phi}(X)$ is a solution of

$$-\frac{1}{2}(1-X^2)\tilde{\Phi}'(X)-pkX\tilde{\Phi}(X)+\frac{\chi(X\Omega(X))}{\Omega(X)}=0,$$

with $(1-x^2)=(1-X^2)^q$ and $\Omega(X)$ is defined in (3.2.20). In the original variable, this translates to solving

$$-\frac{q}{2}(1-x^2)\Phi'(x)-pkx\Phi(x)+\chi(x)=0,$$

hence

$$\Phi(x)=\frac{2}{q}(1-x^2)^{\frac{pk}{q}}\int_0^x\frac{\chi(s)}{(1-s^2)^{\frac{pk}{q}+1}}ds.$$

Remark that

$$\int_0^{x_0}\frac{\chi(s)}{(1-s^2)^{\frac{pk}{q}+1}}ds=-A_k+\ldots,\ \text{and}\ \int_0^{-x_0}\frac{\chi(s)}{(1-s^2)^{\frac{pk}{q}+1}}ds=A_k+\ldots,$$

where A_k is defined in Theorem 3.3.2. Similar as before, we see that the symmetric difference (3.3.20) is given by

$$\Theta\mid_{\delta=0}=-\frac{4}{q}A_kh(0)(1-x_0^2)^{\frac{pk}{q}}(1+\ldots),$$

leading to

$$D(x_0)=-x_0-\frac{2}{p}A_kh(0)(1-x_0^2)^{\frac{pk}{q}+1}(1+\ldots).$$

In Chapter 5 we provide two applications where we illustrate the power of the asymptotic expressions for the transition map. It will allow us to obtain some partial cyclicity results since the non-smooth terms can not be compensated when composed with a regular map.

3.3.3 Alternative expression of the transition map

In this section we prove the first part of Theorem 3.3.3, i.e. we show that the transition map

$$\widetilde{D}:\Sigma_{\text{in}}\to\Sigma_{\text{out}}:u_0\mapsto u_1=\widetilde{D}(u_0),$$

can be expressed as a C^∞-function of the variables

$$\left(u_0,u_0^{np}\log(u_0),u_0^{1/q}\right).$$

Due to Corollary 3.3.1, the transition map $u_1=\widetilde{D}(u_0)$ is given implicitly by

$$V(-1+u_0,1)=V(1-u_1,1).\tag{3.3.21}$$

We are interested in the behaviour of u_0 and u_1 close to 0. Since the function $\psi(x,y)$ occurring in Corollary 3.3.1 can be expressed as a smooth function of

$$\left(x,y,(1-x^2)^{np}\log(1+x),(1-x^2)^{np}\log(1-x),(1-x^2)^{1/q}\right),$$

and by exploiting the fact that $\log(2-u_i)$ is smooth for u_i close to 0 for $i=0,1$, we can write (3.3.21) as

$$2^p u_0^p \left(1+\Psi_L\left(u_0^{1/q}, u_0^{np}\log(u_0)\right)\right) = 2^p u_1^p \left(1+\Psi_R\left(u_1^{1/q}, u_1^{np}\log(u_1)\right)\right), \quad (3.3.22)$$

for some smooth functions Ψ_L, Ψ_R vanishing at the origin. Denote by $v_i = u_i^{1/q}$ for $i=0,1$. After taking the $\frac{1}{pq}$-th power of (3.3.22), this can be written as

$$v_0\left(1+\Psi_L\left(v_0, v_0^{npq}\log(v_0)\right)\right) = v_1\left(1+\Psi_R\left(v_1, v_1^{npq}\log(v_1)\right)\right), \quad (3.3.23)$$

for some new smooth functions Ψ_L, Ψ_R vanishing at the origin. It suffices to prove that v_1 can be expressed as a smooth function of v_0 and $v_0^{npq}\log(v_0)$ satisfying (3.3.23), since this implies that we can express u_1 as a smooth function of $u_0^{1/q}$ and $u_0^{np}\log(u_0)$. Denote the left-hand side of (3.3.23) as

$$z_1 = v_1\left(1+\Psi_R\left(v_1, v_1^{npq}\log(v_1)\right)\right).$$

We want to invert this relation, such that we find an expression

$$v_1 = z_1\left(1+\bar{\Psi}_R\left(z_1, z_1^{npq}\log(z_1)\right)\right),$$

for some smooth function $\bar{\Psi}_R$ and substitute z_1 by the left-hand side of (3.3.23). Therefore denote $V = v_1^{npq}\log(v_1)$ and $Z = z_1^{npq}\log(z_1)$ such that we can consider the following system

$$\begin{cases} z_1 = v_1\left(1+\Psi_R\left(v_1, V\right)\right), \\ Z = v_1^{npq}\left(1+\Psi_R\left(v_1, V\right)\right)^{npq}\log\left[v_1\left(1+\Psi_R\left(v_1, V\right)\right)\right]. \end{cases} \quad (3.3.24)$$

We can simplify the second equation by expanding the logarithm, such that the system (3.3.24) can be rewritten as

$$\begin{cases} z_1 = v_1\left(1+\Psi_R\left(v_1, V\right)\right), \\ Z = V + \tau(v_1, V), \end{cases} \quad (3.3.25)$$

where τ is a smooth function near the origin where it also vanishes. Even more, by considering the explicit expression in (3.3.24), it follows by direct computation that

$$\frac{\partial \tau}{\partial v_1}(0,0) = \frac{\partial \tau}{\partial V}(0,0) = 0.$$

Due to the inverse function Theorem, we can invert the system (3.3.25) since the functions described there are near-identity. Hence there exists a function $\widetilde{\Psi}(z_1, Z)$, smooth near the origin, such that

$$v_1 = z_1\left(1+\widetilde{\Psi}(z_1, z_1^{npq}\log(z_1))\right).$$

If we replace z_1 by the left-hand side of (3.3.23), we can express v_1 as a smooth function of v_0 and $v_0^{npq}\log(v_0)$. By returning to the old variables u_0, u_1, we get that u_1 can be expressed as a smooth function of $u_0, u_0^{np}\log(u_0)$ and $u_0^{1/q}$.

Chapter 4

Saddle connections with symmetrically perturbed eigenvalues

In this chapter, we generalize the techniques of Chapters 2 and 3 to the setting of a smooth family of vector fields similar to (2.1.1) but with a symmetric perturbation in the eigenvalues. We start by giving an adapted version of Theorem 2.1.3. The construction is done in a similar fashion as the Poincaré-Dulac normalization for families of hyperbolic vector fields as described in Section 1.3.2.

Remark that the semi-local normal form of Theorems 2.1.2 and 2.1.3 are robust when we assume that the symmetric eigenvalues of the saddles p and q remain unperturbed. The dependence on some parameter ε is in this case confined to the higher-order terms. This situation typically occurs when the nature of the non-elementary singularity under consideration is fixed. For instance the perturbance of a cusp where the nilpotency is preserved is studied in [41]. However, when one studies an unfolding of these singularities, generically this is broken as is the case for a fake saddle (see [13]). We shall assume that the connection between the saddles remains unbroken and the symmetry of the eigenvalues is preserved to get a finitely smooth normal form (see Theorem 4.1.8).

Then we reduce the system even further by linearizing as we did in Theorem 3.2.1. However we do this by defining an infinite amount of non-smooth variables which we can not yet reduce to a finite set (see Theorem 4.2.5).

4.1 Parameter-dependent semi-local normal forms

In this section we focus on the techniques of Chapter 2 when studying families of vector fields having a fixed saddle connection with symmetric (perturbed) eigenvalues. Due to the normal form procedure in Section 1.3.2, we can at best obtain a finitely smooth normal form. Otherwise the normal form should encompass an infinite amount of resonant terms as was explained in Figure 1.9.

Consider a family of vector fields X_λ where $\lambda \in \Lambda \subset \mathbb{R}^K$ for some $K \in \mathbb{N}$ and some compact set Λ containing the origin. We are interested in families having nearly-resonant saddles with a separatrix connection, i.e. the vector field X_0 satisfies the conditions of Theorem 2.1.3 for some co-prime integers p and q. We assume the following:

(1) The family X_λ has two hyperbolic saddles situated at $s_\pm = (\pm 1, 0)$.

(2) The eigenvalues at both saddles are symmetric, i.e. the linear part at $s_\pm$ of X_λ is given by

$$J(s_\pm) = \pm a_\pm(\lambda) \begin{pmatrix} q & 0 \\ 0 & -p(1+\alpha(\lambda)) \end{pmatrix},$$

where $a_\pm(\lambda) > 0$ and $\alpha(\lambda)$ depends smoothly on λ.

(3) There exists a connection between the saddles $s_\pm$, smoothly depending on λ, which can be straightened to $\{y = 0\}$.

The first assumption is not restrictive. Indeed $s_\pm$ are saddles for the unperturbed vector field X_0. Due to the structural stability of saddles, we know that there exist saddles $\tilde{s}_\pm(\lambda)$ converging to $(\pm 1, 0)$ for λ converging to 0. By means of a coordinate transformation, these saddles can be translated to the points $(\pm 1, 0)$.
Generically when we perturb a vector field of the form (2.1.6), the symmetry in the eigenvalues will be broken. However the symmetry remains unbroken in some applications, for instance when the saddle points are obtained by blowing up a family of vector fields near a fake saddle (Section 5.2). We can assume that the linear part is in diagonal form as a consequence of Propositions 1.3.8 and 1.3.9, since the linear transformation to diagonalize depends smoothly on the parameters. If we rescale time with a positive position-dependent factor, we can assume that $a_\pm = 1$.
Finally, we require that the connection is fixed under perturbation. Although the unstable, respectively stable, manifold of the saddle s_-, respectively s_+, depends smoothly on the parameter λ, it is a non-generic requirement that they coincide. In applications, the connection of symmetric saddles typically occurs on the blow-up locus or the circle at ∞ and therefore it is natural to assume that the connection is fixed. This is also a typical assumption when bounding the cyclicity of a two-saddle cycle, as is done in [23].

4.1.1 Semi-local pre-normal form

A family of vector fields that satisfies conditions (1), (2) and (3) can smoothly be transformed to

$$\begin{cases} \dot{x} = (1-x^2)\left(\frac{q}{2} + (1-x^2)f_1(x,\lambda)\right) + yf_2(x,y,\lambda), \\ \dot{y} = y\left(p(1+\alpha(\lambda))x + (1-x^2)g_1(x,\lambda)\right) + y^2 g_2(x,y,\lambda), \end{cases} \tag{4.1.1}$$

where f_1, f_2, g_1, g_2 are C^∞. We omit the λ-dependence of α from the notation and treat it as a separate variable. Any restriction on α directly translates to a restriction on λ since $\alpha(\lambda)$ is a smooth function. Again we start by considering the scalar vector field on the connection:

$$\dot{x} = (1-x^2)\left(\frac{q}{2} + (1-x^2)f_1(x,\lambda)\right). \tag{4.1.2}$$

Similar as in the unperturbed case (Theorem 2.2.8), we can simplify this equation.

Theorem 4.1.1. *Let $v : (a,b) \times V \to \mathbb{R}$ represent a smooth family of vector fields where $V \subset \mathbb{R}^K$ with exactly 2 singular points $x_1, x_2 \in (a,b)$ for all $\lambda \in V$ such that*

$$v'(x_1) = q > 0 \quad \textit{and} \quad v'(x_2) = -q,$$

where q is independent of λ. Then there exist neighbourhoods O_1 and O_2 of respectively $[x_1, x_2]$ and $[-1,1]$ and a smooth transformation $y : O_1 \times V \to O_2$ such that the equation $\frac{dx}{dt} = v(x,\lambda)$ is transformed into

$$\frac{dy}{dt} = \frac{q}{2}(1-y^2).$$

Proof: The results needed to prove this theorem in the unperturbed system remain valid. Indeed a scalar vector field can be smoothly linearized near a hyperbolic point, where the transformation smoothly depends on the parameter. This can be expanded smoothly to a neighbourhood conjugating the full real line by extension through the flow. The gluing morphism remains the same as in the unperturbed system since the linearized systems are independent of the parameter λ. □

If we apply Theorem 4.1.1, we obtain a transformation simplifying (4.1.2). If we apply this to the full system (4.1.1), we get

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2) + y\tilde{f}_2(x,y,\lambda), \\ \dot{y} = y\left(p(1+\alpha)x + (1-x^2)\tilde{g}_1(x,\lambda)\right) + y^2\tilde{g}_2(x,y,\lambda), \end{cases} \tag{4.1.3}$$

for some smooth $\tilde{f}_2, \tilde{g}_1, \tilde{g}_2$. Next we straighten the stable, respectively unstable, manifold at s_-, respectively s_+ simultaneously. These manifolds depend smoothly on the parameter λ. After a time rescaling of the form $1+\mathrm{O}\big(y\big)$, system (4.1.3) can be written as

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{y} = y\left(p(1+\alpha)x + (1-x^2)G_1(x,\lambda)\right) + y^2 G_2(x,y,\lambda), \end{cases} \tag{4.1.4}$$

where G_1 and G_2 are smooth. Rescaling with

$$y = e^{\int_0^x \frac{2}{q} G_1(s,\lambda)ds}\tilde{y},$$

we obtain the form (where we omit the tilde from the notation):

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{y} = p(1+\alpha)xy + y^2 F(x,y,\lambda), \end{cases} \tag{4.1.5}$$

for some smooth F.

Similar as in the proof of Theorem 1.3.12 we want to put (4.1.5) in a normal form up to some sufficiently high degree N. This can be done by the same transformations as in Lemma 2.3.2, however now dependent on the parameters λ.

Lemma 4.1.2. *Given $n \geq 1$. The effect of the near-identity transformation of the form*

$$(x,y) = (\bar{x}, \bar{y} + h_n(\bar{x},\lambda)\bar{y}^{n+1})$$

on (4.1.5) *is that $y^2F(x,y,\lambda)$ is replaced by*

$$\bar{y}^2F(\bar{x},\bar{y}) + \left(\frac{-q}{2}(1-\bar{x}^2)\frac{\partial h_n}{\partial \bar{x}}(\bar{x},\lambda) - pn(1+\alpha)\,\bar{x}\,h_n(\bar{x},\lambda)\right)\bar{y}^{n+1} + O(\bar{y}^{n+2}). \tag{4.1.6}$$

The typical ODE that we need to solve is of the form

$$-\frac{q}{2}(1-x^2)h'_\lambda(x) - pn(1+\alpha)xh_\lambda(x) + G(x,\lambda) = R(x,\lambda), \tag{4.1.7}$$

where $h_\lambda(x) = h(x,\lambda)$ and $R(x,\lambda)$ is the normal form term that we want to obtain. This simplified term will depend on whether we are close to a resonant level or not. By restricting α to a small neighbourhood of the origin, we can make sure that the terms up to degree N in y do not contain other resonances than the one from the unperturbed system. For instance in Figure 1.9 this is done up to degree 2 in y.
We tackle this problem by considering the qualitative information of the planar system associated to (4.1.7). Therefore we need some results on the existence of local separatrices near nodes. This is constructed by typical normal form theory.

Lemma 4.1.3. *Let $X_{(\varepsilon,\gamma)}$ be a family of smooth planar vector fields in a neighbourhood $(\varepsilon,\gamma) \in U \subset \mathbb{R}\times\mathbb{R}^k$ of the origin of the form*

$$\begin{cases} \dot{x} = \lambda x, \\ \dot{y} = (\mu+\varepsilon)y + f(x,y,\gamma), \end{cases} \tag{4.1.8}$$

for $\lambda\mu > 0$ *and some smooth* f *where* $f(x,y,\gamma) = \mathrm{O}\big(\|(x,y)\|^2\big)$*. Then there exists a neighbourhood* $V \subset U$ *of the origin and a smooth conjugation* $(x,y) \mapsto (x,z) = (x, y+\varphi(x,y,\varepsilon,\gamma))$ *defined for* $(\varepsilon,\gamma) \in V$*, where* $\varphi(x,y,\varepsilon,\gamma) = \mathrm{O}\big(\|(x,y)\|^2\big)$*, such that* (4.1.8) *is conjugate to*

- *Case 1:* $\frac{\mu}{\lambda} = N \in \mathbb{N}$*:*

$$\begin{cases} \dot{x} = \lambda x, \\ \dot{y} = (\mu+\varepsilon)y + \beta(\varepsilon,\gamma)x^N, \end{cases} \tag{4.1.9}$$

  for some smooth β*. Moreover, when* $\beta \equiv 0$*, then* (4.1.8) *admits a* C^∞ *integral curve* $y = \psi(x,\varepsilon,\gamma)$ *passing through the origin.*

- *Case 2:* $\frac{\mu}{\lambda} \notin \mathbb{N}$*:*

$$\begin{cases} \dot{x} = \lambda x, \\ \dot{y} = (\mu+\varepsilon)y. \end{cases} \tag{4.1.10}$$

  Moreover, (4.1.8) *admits a* C^∞ *integral curve* $y = \psi(x,\varepsilon,\gamma)$ *passing through the origin.*

Proof: Without loss of generality we can assume that $|\varepsilon| < |\mu|$ such that we do not lose the nodal behaviour of the singularity. The proof of the normalization uses an inductive procedure applying the near-identity transformations

$$y = Y + AX^mY^n,$$

for some $m,n \in \mathbb{N}$, $L = m+n \geq 2$. If we apply this to (4.1.8), a straightforward computation shows

$$\begin{cases} \dot{x} = \lambda x, \\ \dot{Y} = (\mu+\varepsilon)Y + f(x,Y,\gamma) + A\underbrace{\big((1-n)(\mu+\varepsilon) - \lambda m\big)}_{\kappa(\varepsilon)}x^mY^n + \mathrm{O}\big(\|(x,Y)\|^{L+1}\big). \end{cases}$$

It is obvious that when $\kappa(\varepsilon) \neq 0$, for ε in some neighbourhood V_ε of the origin, then we can remove the term of degree (m,n) in f in a way that smoothly depends on (ε,γ). Suppose we are in case 1, i.e. $\mu = \lambda N$ for some $N \in \mathbb{N}$, then it is sufficient to show that

$$(1-n)(N + \frac{\varepsilon}{\lambda}) - m \neq 0. \tag{4.1.11}$$

If we impose the condition $|\varepsilon| < |\lambda|$, then (4.1.11) is satisfied unless $(m,n) = (N,0)$ thus leading to (4.1.9). When the eigenvalues are non-resonant, i.e. when we are in case 2, we know that there is an $M \in \mathbb{N}$ such that

$$M < \frac{\mu}{\lambda} < M+1.$$

By restricting ε to some small neighbourhood V_ε of the origin, we can suppose that

$$M < \frac{\mu}{\lambda} + \frac{\varepsilon}{\lambda} < M + 1, \ \forall \varepsilon \in V_\varepsilon.$$

In order to show that $\kappa(\varepsilon) \neq 0$ for $\varepsilon \in V_\varepsilon$, it is sufficient to show that

$$(1-n)\left(\frac{\mu}{\lambda} + \frac{\varepsilon}{\lambda}\right) - m \neq 0.$$

This is immediate since the part between brackets is not in $\mathbb{N}$ and it can only be zero when $n = 0$. The smooth realizations of the transformation now follows from Borel's Theorem (Theorem 1.2.3) and the Chen Theorem (Theorem 1.3.3) for killing the flat perturbation. The statement about the C^∞ integral curves follows immediately from the existence of these curves in the normalized system. □

We are now able to construct smooth solutions of (4.1.7), for some choice of $R(x, \lambda)$. The technique applied in the following lemmas is similar to the proof of Lemma 2.3.13. First we need to show that the associated two-dimensional system has locally smooth solutions near the saddles using Lemma 4.1.3. Consequently we introduce a parameter-dependent coefficient which should connect these locally smooth solutions. We need to consider two different cases depending on whether we are close to a resonant term of the unperturbed system (Lemma 4.1.5) or to a connecting term (Lemma 4.1.4). Observe that this subdivision corresponds to the resonant, respectively non-resonant case for the node in Lemma 4.1.3.

Lemma 4.1.4. *Let $p, k, q \in \mathbb{N}_0$ such that p and q are relatively prime and suppose that q does not divide k. Then there exists a neighbourhood $V \subset \mathbb{R}^K$ of the origin such that, for every smooth function $F(x, \lambda)$ defined on a open set $U \times V$ where $[-1, 1] \subset U$, there exists a smooth function $C(\lambda) \in \mathbb{R}$ such that the differential equation*

$$-\frac{q}{2}(1-x^2)h'_\lambda(x) - pk(1+\alpha)xh_\lambda(x) + F(x,\lambda) = -C(\lambda)\chi(x), \tag{4.1.12}$$

has a smooth solution $h_\lambda(x)$ in a neighbourhood of $[-1, 1]$.

Proof: Denote by $y = \Phi_F(x, \lambda)$ and $y = \Psi_F(x, \lambda)$, extended to $x \in (-1, 1)$, respectively the smooth invariant manifolds near the nodes $(-1, 0)$ and $(1, 0)$ of the system

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{y} = -pk(1+\alpha)xy + F(x,\lambda). \end{cases} \tag{4.1.13}$$

These depend smoothly on the parameter λ due to Lemma 4.1.3. Indeed, applying the transformation $u = \frac{1+x}{1-x}$, respectively $u = \frac{x-1}{x+1}$, to (4.1.13) turns it in the form (4.1.8), where the ratio of the (unperturbed) eigenvalues is given by $\frac{pk}{q}$. The system is thus locally smoothly linearizable since we are in case 2. Therefore there exists a smooth integral curve in terms of u, which in its turn smoothly depends on x near $x=-1$, respectively $x=1$. Similarly, we can define the smooth invariant manifolds $y=\Phi_\chi(x,\lambda)$ and $y=\Psi_\chi(x,\lambda)$ near the nodes $(-1,0)$ and $(1,0)$ of the system

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{y} = -pk(1+\alpha)xy + \chi(x). \end{cases}$$

The coefficient $C(\lambda)$ is defined as the coefficient such that

$$\Phi_F(x,\lambda) + C(\lambda)\Phi_\chi(x,\lambda) = \Psi_F(x,\lambda) + C(\lambda)\Psi_\chi(x,\lambda), \tag{4.1.14}$$

for $x \in (-1,1)$. By uniqueness of solution, it suffices that this equality is valid for $x=0$. Denote $\Delta_*(x,\lambda) = \Phi_*(x,\lambda) - \Psi_*(-x,\lambda)$ such that $C(\lambda)$ is characterized by

$$\Delta_F(0,\lambda) - C(\lambda)\Delta_\chi(0,\lambda) = 0. \tag{4.1.15}$$

Due to the symmetry of χ, one can check that $\Delta_\chi(x,\lambda) = 2\Phi_\chi(x,\lambda)$. A straightforward computation shows that

$$\Phi_\chi(0,\lambda) = \frac{2}{q}\int_{-1}^{0} \frac{\chi(s)}{(1-s^2)^{\frac{pk}{q}(1+\alpha)}} ds,$$

which is non-zero since we have a positive integrand and converges due to the flatness of χ at -1. Hence thanks to the implicit function Theorem applied to (4.1.15), we know that $C(\lambda)$ depends smoothly on λ. The solution of (4.1.12) in a neighbourhood of $[-1,1]$ is then given by (4.1.14). $\square$

When we are close to resonance, solving (4.1.7) is more challenging. Because of this resonance, expressed as the coefficient β in (4.1.9), logarithmic terms may occur and thus obstruct the existence of locally smooth solutions as in the proof of Lemma 4.1.4. However if we can compensate this resonance such that logarithmic terms do not occur, then we know that every solution through the nodes is smooth. Due to the parameter-dependence, we need to handle both the resonant phenomena as well as the connecting terms when there is no resonance.

Lemma 4.1.5. *Let $p,n,k,q \in \mathbb{N}_0$ such that p and q are relatively prime and $n=kq$. Then there exists a neighbourhood $V \subset \mathbb{R}^K$ of the origin such that, for every smooth function $F(x,\lambda)$ defined on a open set $U \times V$ where $[-1,1] \subset U$, there exist smooth functions $A(\lambda), B(\lambda), C(\lambda)$ with values in $\mathbb{R}$ such that the differential equation*

$$-\frac{q}{2}(1-x^2)h'_\lambda(x) - pn(1+\alpha)xh_\lambda(x) + F(x,\lambda) = (A(\lambda)+xB(\lambda))(1-x^2)^{pk} - C(\lambda)\chi(x), \tag{4.1.16}$$

has a smooth solution $h_\lambda(x)$ in a neighbourhood of $[-1,1]$, where $C(0)=0$.

Proof: Instead of solving (4.1.16), it is sufficient to consider

$$-\frac{1}{2}(1-x^2)h'_\lambda(x)-pk(1+\alpha)xh_\lambda(x)+F(x,\lambda) = (A(\lambda)+xB(\lambda))(1-x^2)^{pk}-C(\lambda)\chi(x), \tag{4.1.17}$$

for any smooth function $F(x,\lambda)$. We restrict λ to a neighbourhood V of the origin such that

$$pk-1 < pk(1+\alpha) < pk+1, \tag{4.1.18}$$

thus $|\alpha| < \frac{1}{pk}$. In this way, the only resonance that can occur is when $\alpha = 0$ as obtained in the proof of Lemma 4.1.3. Decompose $F(x,\lambda)$ as follows

$$F(x,\lambda) = \sum_{i=0}^{pk}(a_i(\lambda)+xb_i(\lambda))(1-x^2)^i + (1-x^2)^{pk+1}G(x,\lambda), \tag{4.1.19}$$

where G is a C^∞ function and a_i, b_i depend smoothly on λ. First we show that there exist smooth integral curves $y = \Phi_F(x,\lambda)$ and $y = \Psi_F(x,\lambda)$ at respectively $x = -1, 1$ of the system

$$\begin{cases} \dot{x} = \frac{1}{2}(1-x^2), \\ \dot{y} = -pk(1+\alpha)xy + F(x,\lambda) - (A(\lambda)+xB(\lambda))(1-x^2)^{pk}, \end{cases} \tag{4.1.20}$$

by defining $A(\lambda)$ and $B(\lambda)$ adequately. By virtue of Lemma 4.1.3, it suffices to prove that the resonant term is identically zero in the local normal forms at $(\pm 1, 0)$ for these well-chosen $A(\lambda)$ and $B(\lambda)$. First we perform smooth coordinate changes of the form

$$y_{i+1} = y_i + (\gamma_i(\lambda)+x\delta_i(\lambda))(1-x^2)^i, \tag{4.1.21}$$

for $i = 0, \ldots, pk-1$, where γ_i and δ_i depend smoothly on λ and $y_0 = y$. By induction we show that in this way, (4.1.20) transforms to

$$\begin{cases} \dot{x} = \frac{1}{2}(1-x^2), \\ \dot{y}_i = -pk(1+\alpha)xy_i + \sum_{j=i}^{pk}\left(a_j^{(i)}(\lambda)+xb_j^{(i)}(\lambda)\right)(1-x^2)^i \\ \qquad - (A(\lambda)+xB(\lambda))(1-x^2)^{pk} + (1-x^2)^{pk+1}G(x,\lambda), \end{cases} \tag{4.1.22}$$

for $i = 0, \ldots, pk$. For $i = 0$, this is true since we have the decomposition (4.1.19). Suppose (4.1.22) is true for some $i \in \{0, \ldots, pk-1\}$. We show that there is a transformation of the form (4.1.21), turning (4.1.22) into a similar vector field but with i replaced by $i+1$. Define

$$\gamma_i(\lambda) = -\frac{b_i^{(i)}(\lambda)}{pk(1+\alpha)-i}, \text{ and } \delta_i(\lambda) = -\frac{a_i^{(i)}(\lambda)}{pk(1+\alpha)-i}. \tag{4.1.23}$$

Observe that these denominators are non-zero for the restriction (4.1.18) since $i \neq pk$. A straightforward computation shows that applying (4.1.21) where the coefficients are given by (4.1.23) to (4.1.22), transforms the second equation to

$$\begin{aligned}\dot{y}_{i+1} = &-pk(1+\alpha)xy_{i+1} + (pk(1+\alpha)-i)\,(\delta_i(\lambda)+x\gamma_i(\lambda))(1-x^2)^i \\ &+\delta_i(\lambda)\left(\frac{1}{2}-pk(1+\alpha(\lambda))+i\right)(1-x^2)^{i+1} \\ &+\sum_{j=i}^{pk}\left(a_j^{(i)}(\lambda)+xb_j^{(i)}(\lambda)\right)(1-x^2)^i \\ &-(A(\lambda)+xB(\lambda))(1-x^2)^{pk}+(1-x^2)^{pk+1}G(x,\lambda),\end{aligned}$$

which can be simplified to the desired form. Hence after these smooth transformations, we have arrived at the system

$$\begin{cases}\dot{x} = \frac{1}{2}(1-x^2), \\ \dot{y}_{pk} = -pk(1+\alpha)xy_{pk} + \left(a_{pk}^{(pk)}(\lambda)+xb_{pk}^{(pk)}(\lambda)\right)(1-x^2)^i \\ \qquad -(A(\lambda)+xB(\lambda))(1-x^2)^{pk}+(1-x^2)^{pk+1}G(x,\lambda).\end{cases}$$

By choosing $A(\lambda) = a_{pk}^{(pk)}(\lambda)$ and $B(\lambda) = b_{pk}^{(pk)}(\lambda)$, we get the system

$$\begin{cases}\dot{x} = \frac{1}{2}(1-x^2), \\ \dot{y}_{pk} = -pk(1+\alpha)xy_{pk} + (1-x^2)^{pk+1}G(x,\lambda).\end{cases} \tag{4.1.24}$$

It is now a straightforward task to show that the associated local normal form of Lemma 4.1.3 has no resonant term. For instance for the local normal form at $(-1,0)$ by applying a coordinate change

$$(x, y_{pk}) = \left(\frac{u-1}{u+1}, (u+1)^{-2pk(1+\alpha)}z\right)$$

to (4.1.24) leads to a vector field of the form

$$\begin{cases}\dot{u} = u, \\ \dot{z} = pk(1+\alpha)z + u^{pk+1}\tilde{F}(u,\lambda),\end{cases}$$

for some locally smooth $\tilde{F}$ and thus has no resonant term since the remaining terms are of higher degree. Due to Lemma 4.1.3, this proves the existence of a smooth integral curve $y = \Phi_F(x,\lambda)$ at $x=-1$ and similarly we can show the existence of a smooth integral curve $y = \Psi_F(x,\lambda)$ at $x = 1$ from (4.1.24).

The existence of similar smooth integral curves $y = \Phi_\chi(x,\lambda)$ and $y = \Psi_\chi(x,\lambda)$ of the system

$$\begin{cases}\dot{x} = \frac{1}{2}(1-x^2), \\ \dot{y} = -pk(1+\alpha)xy + \chi(x),\end{cases} \tag{4.1.25}$$

is immediate from Lemma 4.1.3 due to the flatness of χ at $x = \pm 1$. Similar as in the proof of Lemma 4.1.4, we can prove the existence of a smooth $C(\lambda)$ by use of the implicit function Theorem such that (4.1.17) is smoothly solvable for λ contained in some neighbourhood of the origin also satisfying (4.1.18). □

Due to Lemma 4.1.4 and Lemma 4.1.5, we can apply a similar induction scheme as in the construction of the semi-local normal form (Theorem 2.1.3). However, we can only do this a finite amount of times, since the requirements on λ become more restrictive in each step (see for example (4.1.18)). We can prove the following.

Theorem 4.1.6. *Let $N > 1$ be arbitrary. Consider the smooth vector field* (4.1.5) *where $p, q \in \mathbb{N}$, with $gcd(p,q) = 1$, defined in a neighbourhood $U \times V \subset \mathbb{R}^2 \times \mathbb{R}^k$ of $[-1,1] \times \{0\} \times \{0\}$. There exists a smooth transformation $\Phi : U \times \tilde{V} \to \tilde{U} \times \tilde{V}$, for some neigbourhood $\tilde{U} \subset \mathbb{R}^2$ of $[-1,1] \times \{0\}$ and $\tilde{V} \subset V$ of the origin, such that under this transformation* (4.1.5) *is transformed to a system of the form*

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{y} = y \left(p(1+\alpha)x + \sum\limits_{n \geq 1, nq \leq N} (A_n(\lambda) + xB_n(\lambda))(1-x^2)^{pn} y^{qn} \right. \\ \qquad \left. + \sum\limits_{k \geq 1}^{N} C_k(\lambda)\chi(x) y^k + y^{N+1} F(x,y,\lambda) \right), \end{cases} \tag{4.1.26}$$

where the coefficients A_n, B_n, C_n depend smoothly on $\lambda \in \tilde{V}$.

Proof: We subsequently apply the transformation of Lemma 4.1.2 for $n = 1, \dots, N$. In order to get the term of degree $n+1$ in the desired form, we apply either Lemma 4.1.4 if $\frac{pn}{q} \notin \mathbb{N}$ or Lemma 4.1.5 if $\frac{pn}{q} \in N$ to solve (4.1.7) in each step. □

Remark 4.1.7. *The domain of λ where we can apply Lemma 4.1.4 and 4.1.5 in the proof of Theorem 4.1.6 can not be chosen uniformly. Generically, the domain becomes more and more restrictive as n goes to infinity. Therefore it is not possible to write Theorem 4.1.6 in terms of formal power series as in Theorem 2.3.12.*

As a final step in the normal form procedure, we want to eliminate the term in (4.1.26) containing $F(x,y,\lambda)$. Since this will contain higher order resonances, we expect that this is optimally possible in a finitely smooth way.

4.1.2 Removal of higher order terms

Consider a vector field of the form (4.1.26), which we shall shortly denote as

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{y} = p(1+\alpha)xy + yR(x,y,\lambda) + y^{N+1}F(x,y,\lambda), \end{cases} \tag{4.1.27}$$

for some smooth functions R, representing the resonant and connecting terms, and F, representing the finitely flat remainder, in a neighbourhood of $[-1,1]\times\{0\}\times\{0\}$. The aim of this section is to prove the following.

Theorem 4.1.8. *Let $n \geq 1$ be arbitrary. There exists a $K = K(n)$ and a C^n-transformation $(x, y, \lambda) = (X, \varphi(X, Y, \lambda), \lambda)$ near the origin such that (4.1.5) is C^n-equivalent with the system*

$$\begin{cases} \dot{X} = \frac{q}{2}(1-X^2), \\ \dot{Y} = Y\left(p(1+\alpha)X + \sum\limits_{k\geq 1, kq\leq K(n)} (A_k(\lambda) + XB_k(\lambda))(1-X^2)^{pk}Y^{qk}\right. \\ \qquad\qquad \left. + \sum\limits_{k\geq 1}^{K(n)} C_k(\lambda)\chi(X)Y^k, \right) \end{cases} \tag{4.1.28}$$

where A_k, B_k, C_k are smooth functions of λ. Moreover, $\varphi(x, y, \lambda)$ is of the form

$$\varphi(x, y, \lambda) = y + y^n\psi(x, y, \lambda).$$

Remark 4.1.9. *When $\alpha \equiv 0$, then Theorem 4.1.8 gives a weak version of Theorem 2.1.3 in a finite class of differentiability.*

Proof: First we apply Theorem 4.1.6 to (4.1.5) to obtain a pre-normal form up to degree $N(\tilde{n})$, where we define $\tilde{n} = \tilde{n}(n) > n$ later on. Once we have defined this relation, we define $K(n) = N(\tilde{n}(n))$. For simplicity we will denote the pre-normal form as (4.1.27). Next we prove that there is a conjugation for which (4.1.27) can be $C^{\tilde{n}}$-conjugated to (4.1.28). This proof is divided in three parts as in Section 2.4. First we show that there exist local conjugating morphisms in Lemma 4.1.12. Since this family of transformations is the identity on the x-variable, we see that the conjugation corresponds to the solution of a partial differential equation (4.1.29). Hence the local conjugations can be extended such that their domains of definition contain the interval $(-1, 1)$. Then we glue both transformations together in a finitely smooth way similar as in the proof of Theorem 2.4.1 (see (2.4.4)). This is done by showing that the difference of the two transformations is n-flat when approaching the invariant manifold $x = 1$ in Lemma 4.1.13. We elaborate on this in the rest of this section. □

First we show that the local conjugations exist. This in fact is a direct consequence of Proposition 1.3.11. However by considering the explicit construction of the conjugation in Proposition 1.3.7 and Lemma 1.3.10, we can compute that the transformation remains the identity on the x-variable.

Proposition 4.1.10. *Let $\tilde{n} \geq 1$ be arbitrary. Let F_1, F_2 and w_2 be C^∞-functions where $F_2(x, 0, \lambda) = w_2(x, 0, \lambda) = 0$. There exists a finite $N(\tilde{n}) \in \mathbb{N}$ such that the* *system*

$$\begin{cases} \dot{x} = F_1(x), \\ \dot{y} = F_2(x, y, \lambda) + \tau w_2(x, y, \lambda), \end{cases}$$

is $C^{\tilde{n}}$ *equivalent with the system*

$$\begin{cases} \dot{x} = F_1(x), \\ \dot{y} = F_2(x, y, \lambda), \end{cases}$$

on some neighbourhood V *of* $M = \{y = 0\}$ *as long as* w_2 *is* $N(\tilde{n})$*-flat at all points of* $M = \{y = 0\}$ *and there is a compact* $C \subset M$ *such that* F_1 *is supported in* C *and*

$$F_2(x, y, \lambda) + \tau w_2(x, y, \lambda) < -\mu,$$

for some $\mu > 0$, $\tau \in [0, 1]$ *and* $(x, y, \lambda) \in V$. *Moreover, the* $(n-1)$*-jet of the equivalence* φ *is the identity on* M *and is of the form*

$$\varphi(x, y) = (x, y + y^n \psi(x, y)).$$

Proof: We apply Lemma 1.3.10 to the system. Hence we need to show that we can solve (1.3.12) for some smooth h. Due to the global bound on the hyperbolicity, we can apply the results from [32], where it is shown that (1.3.12) has a $C^{\tilde{n}}$- solution which is $\tilde{n}$-flat on M. The fact that this transformation leaves the first variable unchanged follows from the explicit form of (1.3.8). Indeed $D(F + \tau w)$ is in this case a lower triangular matrix, as will be $G(\gamma(x, y, \lambda, \tau, t))$. Adding the fact that the first component of w is zero, we get that the first component of h is identically zero, hence the first component of φ_τ remains the identity. □

Remark 4.1.11. *When the vector field of Proposition 4.1.10 is considered close to a hyperbolic saddle, then the value* $N(\tilde{n})$ *depends, other than* $\tilde{n}$, *only on the stable and unstable eigenvalue of the saddle (see Theorem 1.3.12). In fact a lower bound for* $N(\tilde{n})$ *is defined in [5] for diffeomorphisms. Therefore the local conjugations of Theorem 4.1.8 at both saddles have the same required flatness* $N(\tilde{n})$ *due to the symmetry of the system with respect to the eigenvalues.*

If we apply Proposition 4.1.10 to the pre-normal form of (4.1.26), we can show the following.

Lemma 4.1.12. *Let* $\tilde{n} > 1$ *be arbitrary and consider* (4.1.5). *Let* $N(\tilde{n})$ *be the corresponding value as obtained in Proposition 4.1.10 which only depends on the eigenvalues at* $(\pm 1, 0)$. *There exists* $C^{\tilde{n}}$*-equivalences* $(x, y, \lambda) = (X, \varphi_\pm(X, Y, \lambda), \lambda)$, *where*

$$\varphi_\pm(x, y, \lambda) = y + y^{\tilde{n}} \psi_\pm(x, y, \lambda),$$

which is $C^{\tilde{n}}$ *in a neighbourhood of* $(\pm 1, 0)$ *such that* (4.1.5) *is conjugated to* (4.1.28).

Observe that the transformations $\varphi_\pm$ satisfy a partial differential equation

$$\begin{aligned}\frac{q}{2}(1-x^2)\varphi_x + y\left(p(1+\alpha)x + R(x,y,\lambda)\right)\varphi_y \\ = \varphi\left(p(1+\alpha)x + R(x,\varphi,\lambda) + \varphi^N F(x,\varphi,\lambda)\right),\end{aligned} \tag{4.1.29}$$

where we used the notation from (4.1.27). Let $\delta > 0$ and $\delta' > 0$ be chosen such that φ_+ is defined on a neighbourhood of $[1-2\delta, 1] \times [-\delta', \delta'] \times \{0\}$. By means of the method of characteristics applied to (4.1.29), we can assume that φ_- is defined for $(x,y) \in \Sigma = \{1-\delta\} \times (-\delta'', \delta'')$ for some $\delta'' > 0$ where we can assume $\delta' \geq \delta''$. In fact this is similarly proven as in Lemma 2.4.2. We consider the difference of these transformations, i.e.

$$\Delta(x,y,\lambda) = \varphi_-(x,y,\lambda) - \varphi_+(x,y,\lambda).$$

Observe that this difference is $\tilde{n}$-flat at $y=0$ on Σ. Following (4.1.29), this difference should satisfy the partial differential equation

$$\frac{q}{2}(1-x^2)\Delta_x + y\left(p(1+\alpha)x + R(x,y,\lambda)\right)\Delta_y = \Delta\left(p(1+\alpha)x + H(x,y,\Delta,\lambda)\right), \tag{4.1.30}$$

where

$$H(x,y,\Delta,\lambda) = \int_0^1 \frac{\partial h}{\partial y}(x, \varphi_+(x,y,\lambda) + z\Delta, \lambda)dz,$$

with

$$h(x,y,\lambda) = y\left(R(x,y,\lambda) + y^N F(x,y,\lambda)\right).$$

We now want to show the following result

Lemma 4.1.13. *Let $n \geq 1$ be arbitrary. If $\tilde{n}$ is large enough, then the exists a $C^{\tilde{n}}$-solution $\Delta(x_E, y_E, \lambda)$, equal $\varphi_- - \varphi_+$ on Σ, of (4.1.30) which is n-flat for $x_E \to 1$.*

Proof: Since Δ should satisfy (4.1.30), we can construct a solution using the method of characteristics. The propagation of the difference is given by

$$\begin{cases}\dot{x} = \frac{q}{2}(1-x^2), \\ \dot{y} = y\left(p(1+\alpha)x + R(x,y,\lambda)\right), \\ \dot{\Delta} = \Delta\left(p(1+\alpha)x + H(x,y,\Delta,\lambda)\right),\end{cases} \tag{4.1.31}$$

with initial values

$$x(0) = 1-\delta,\ y(0) = s,\ \Delta(0) = \Delta_0(s) = \varphi_-(s) - \varphi_+(s) = s^n\tilde{\Delta}_0(s),$$

for some continuous $\tilde{\Delta}_0$. Since we are interested in a neighbourhood of $x=1$, we apply the translation $u = \frac{x-1}{x+1}$. By means of a near-identity transformation

$$(y_1, \Delta_1) = (\psi(u,y,\lambda), \xi(u,y,\Delta,\lambda)), \tag{4.1.32}$$

we can put (4.1.31) in pre-normal form

$$\begin{cases} \dot{u} = -qu, \\ \dot{y}_1 = y_1 \left[p(1+\alpha) + f_\lambda \left(u^p y_1^q \right) + F(x,y,\lambda) \right] \\ \dot{\Delta}_1 = \Delta_1 \left[p(1+\alpha) + g_\lambda \left(u^p y_1^q, u^p \Delta_1^q \right) + G(u,y_1,\Delta_1,\lambda) \right], \end{cases} \tag{4.1.33}$$

similar to the pre-normal form in (1.3.11) where F and G are of degree $\tilde{n}$ or higher. The initial conditions in the new variables are now given by

$$\begin{aligned} & u(0) = u_0 = -\frac{\delta}{2-\delta},\ y_1(0) = \psi(u(0), s, \lambda) = s(1 + \text{h.o.t.}), \\ & \Delta_1(0) = \xi(u(0), s, \Delta_0(s), \lambda) = \Delta_0(s)(1 + \text{h.o.t.}). \end{aligned} \tag{4.1.34}$$

In order to define Δ in (x_E, y_E) for $1-\delta < x_E < 1$ but close to 1, we need to define Δ_1 at the point

$$(u_1, y_{1,E}) = \left(\frac{x_E - 1}{x_E + 1}, \psi\left(\frac{x_E - 1}{x_E + 1}, y_E, \lambda \right) \right).$$

For this we need to determine an initial value s_E on Σ, in terms of the variables in (4.1.31), and a time T_E such that

$$(u(T_E), y_1(T_E)) = (u_1, y_{1,E}).$$

A straightforward computation show that

$$T_E = -\frac{1}{q} \ln \left(\frac{u_1}{u_0} \right), \tag{4.1.35}$$

which indeed is a positive value. In order to determine s_E, we compute the transition of the first two equations of (4.1.33) in reverse time with initial value $(u_1, y_{1,E})$ to (u_0, s_E) on the cut Σ. Due to Theorem 1.3.18, we see that

$$s_{1,E} = \frac{1}{u_0} \left(u_1 y_{1,E} \right)^{\frac{p}{q}(1+\alpha)} \left(A(\lambda) + F(u_1 y_{1,E}, \lambda) \right), \tag{4.1.36}$$

where $F(z,\lambda)$ is of Mourtada type in the variable z and $A(\lambda)$ is a $C^{\tilde{n}}$ positive function and $s_{1,E} = \psi(u_0, s_E, \lambda)$. Due to the fact that ψ in (4.1.32) is near-identity, we know that s_E approaches 0 as $s_{1,E}$ approaches zero and vice-versa. We reparametrize time in (4.1.33) such that the transition time from $(u_0, s_{1,E})$ on Σ to $(u_1, y_{1,E})$ is given by 1, i.e.

$$\begin{cases} \dot{u} = -qT_E u, \\ \dot{y}_1 = y_1 T_E \left[p(1+\alpha) + f_\lambda \left(u^p y_1^q \right) + F(x,y,\lambda) \right] \\ \dot{\Delta}_1 = \Delta_1 T_E \left[p(1+\alpha) + g_\lambda \left(u^p y_1^q, u^p \Delta_1^q \right) + G(u,y_1,\Delta_1,\lambda) \right], \end{cases} \tag{4.1.37}$$

with initial values

$$u(0) = u_0,\ y_1(0) = s_{1,E},\ \Delta_1(0) = \Delta_{1,0} = \xi(u_0, s_E, \Delta_0(s_E), \lambda).$$

A straightforward computation show that

$$u(t) = u_0 e^{-qT_E t} = u_0 \left(\frac{u_1}{u_0}\right)^t.$$

Denote $z_1 = u^p y_1^q$. It is shown in [52] that $z_1(t)$ with initial value $z_1(0) = u_0^p s_{1,E}^q$ is given by

$$z_1(t) = \sum_{i=1}^{\tilde{N}} e^{p\alpha T_E t} Q_i(\Omega(t,\alpha),\lambda) u_0^{pi} s_{1,E}^{qi} + \psi_{\tilde{N}}\left(u_0^p s_{1,E}^q, \lambda\right), \tag{4.1.38}$$

where $\psi_{\tilde{N}}$ is an $\tilde{N}$-flat function for some $\tilde{N} > 0$, Q_i is a polynomial of degree $\leq i-1$ in Ω and where Ω is given by

$$\Omega(t,\alpha) = \begin{cases} \frac{e^{p\alpha T_E t}-1}{p\alpha} & \alpha \neq 0 \\ T_E t & \alpha = 0. \end{cases}$$

If we replace the value (4.1.35) of T_E into (4.1.38), we get

$$z_1(t) = \sum_{i=1}^{\tilde{N}} \left(\frac{u_0}{u_1}\right)^{\frac{p\alpha}{q}t} Q_i(\Omega(t,\alpha),\lambda) u_0^{pi} s_{1,E}^{qi} + \psi_{\tilde{N}}\left(u_0^p s_{1,E}^q, \lambda\right), \tag{4.1.39}$$

where

$$\Omega(t,\alpha) = \begin{cases} \frac{\left(\frac{u_0}{u_1}\right)^{\frac{p\alpha}{q}t}-1}{p\alpha} & \alpha \neq 0 \\ \frac{1}{q}\ln\left(\frac{u_0}{u_1}\right)t & \alpha = 0. \end{cases}$$

We now focus on the third equation of (4.1.37), i.e.

$$\dot{\Delta}_1 = \Delta_1 T_E \left[p(1+\alpha) + g_\lambda\left(z_1(t), u^p(t)\Delta_1^q\right) + G(u, y_1, \Delta_1, \lambda)\right]. \tag{4.1.40}$$

Denote

$$\Delta_1(t) = \Delta_{1,0} e^{p(1+\alpha)T_E t}\Theta(t).$$

This transforms (4.1.40) into

$$\dot{\Theta} = T_E \Theta \left[g_\lambda\left(z_1, u_0^p \Delta_{1,0}^q \left(\frac{u_0}{u_1}\right)^{p\alpha t}\Theta^q\right) + \tilde{G}(t,\Theta,\lambda)\right], \ \Theta(0) = 1. \tag{4.1.41}$$

Observe that since $g_\lambda(0,0) = 0$, we can rewrite (4.1.41) as

$$\dot{\Theta} = \Theta \tilde{G}_\lambda(u_1, u_1\ln(u_1), t),$$

for some $C^{\tilde{n}}$-function G_λ, similar to the proof of Lemma 2.4.3. Therefore, the solution of (4.1.41) is $C^{\tilde{n}}$ dependent on u_1 and $u_1\ln(u_1)$ and can be written as $\Theta(t, u_1, u_1\ln(u_1))$. Since these monomial u_1 and $u_1\ln(u_1)$ are of Mourtada type, it is easy to see that the function

$$u_1^{\tilde{k}}\Theta(1, u_1, u_1\ln(u_1)),$$

has bounded partial derivatives with respect to u_1 up to order $\tilde{k} < \tilde{n}$, for u_1 close to zero.
Since $\Delta_{1,0}$ is $\tilde{n}$-flat in s_E, and thus in $s_{1,E}$, we get from (4.1.36) that there exists an $L > 0$ depending on $\tilde{n}, p, q$ for $\tilde{n}$ sufficiently large, such that

$$\Delta_{1,0} = u_1^L \tilde{\Delta}(u_1, y_E).$$

The desired expression $\Delta_1(u_1, y_{1,E})$ is thus given by

$$u_1^L \tilde{\Delta}(u_1, y_E) \left(\frac{u_0}{u_1}\right)^{\frac{p}{q}(1+\alpha)} \Theta(1). \tag{4.1.42}$$

By choosing L big enough, we can make sure that $\Theta(1)$ has bounded derivatives up to degree n and the function given in (4.1.42) is flat of degree n. □

4.2 Finitely smooth family linearization

In this section, we want to apply a similar procedure as in Section 3.2, but now applied to a vector field of the form

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{y} = y\left(p(1+\alpha)x + \sum_{i=1}^{N}(A_i(\lambda)x + B_i(\lambda))(1-x^2)^{pi}y^{qi} + \chi(x)\sum_{j=1}^{qN} C_j(\lambda)y^j\right), \end{cases} \tag{4.2.1}$$

where the coefficients A_i, B_i and C_i depend smoothly on the parameter λ and χ is defined in (2.1.5).

First we consider (4.2.1) without connecting terms, i.e.

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{y} = y\left(p(1+\alpha)x + \sum_{i=1}^{N}(A_i(\lambda)x + B_i(\lambda))(1-x^2)^{pi}y^{qi}\right). \end{cases} \tag{4.2.2}$$

We want to reduce this to an easily integrable system by means of a formal transformation involving tags. Denote by $z = (1-x^2)^p y^q$ the resonant monomial of (4.2.2), such that

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{z} = Axz + z^2\sum_{i=0}^{N-1}(C_i(\lambda)(1+x) + D_i(\lambda)(1-x))z^i, \end{cases} \tag{4.2.3}$$

where C_i and D_i are linear combinations of A_{i+1} and B_{i+1} for $i = 0, \ldots, N-1$ and $A = pq\alpha$.
Observe that in contrast to (3.2.2), there is an additional term, linear in z, due to

the perturbation of the eigenvalues of the saddles. In order to reduce (4.2.3), we need to introduce a series of tags with a compensator-like behaviour. For instance, the Ecalle-Roussarie compensator defined in (1.3.15) is a smooth function of x_0 for $x_0 > 0$, satisfying

$$\frac{\partial \omega}{\partial x_0} = -\frac{\alpha_0 \omega + 1}{x_0},$$

and appears due to the perturbation of the eigenvalue by $\alpha_0 = q\alpha(\varepsilon)$ in (1.3.13). Here we define the S_L and S_R tags of order one as the smooth functions defined on $(-1, 1)$ with $S_L(0) = S_R(0) = 0$ where their time-derivatives $\dot{S}_L, \dot{S}_R$ satisfy

$$\dot{S}_L = (1-x) - AxS_L,\ \dot{S}_R = (1+x) - AxS_R. \tag{4.2.4}$$

The time-position relation between x and t is defined by the first equation of (4.2.3). Define for every word $* \in \mathcal{W}$ in the alphabet $\{L, R\}$, by induction on the length $\ell(*)$, the tags

$$\begin{aligned} \dot{S}_{*L} &= (1-x)S_* - A(\ell(*)+1)xS_{*L},\ S_{*L}(0) = 0, \\ \dot{S}_{*R} &= (1+x)S_* - A(\ell(*)+1)xS_{*R},\ S_{*R}(0) = 0. \end{aligned} \tag{4.2.5}$$

Observe that these tags correspond to the tags defined in (3.2.3) and (3.2.4) for $\alpha = 0$. It is difficult to express the higher order terms in terms of the first tags as we did in Proposition 3.2.2, due to the compensator-like behaviour. Indeed, a direct computation shows that the solutions of (4.2.5) are given by

$$S_{*L}(x) = \frac{2}{q}(1-x^2)^{p\alpha(\ell(*)+1)} \int_0^x \frac{S_*(s)}{(1+s)(1-s^2)^{p\alpha(\ell(*)+1)+1}} ds,$$

and

$$S_{*R}(x) = \frac{2}{q}(1-x^2)^{p\alpha(\ell(*)+1)} \int_0^x \frac{S_*(s)}{(1-s)(1-s^2)^{p\alpha(\ell(*)+1)+1}} ds,$$

where the difficulty lies in the appearance of powers which are not necessarily integers. We can however show the following adaptation of Theorem 3.2.3, where we denote by $\mathcal{W}_k$ the words of length k in $\mathcal{W}$.

Theorem 4.2.1. *There exists a formal transformation*

$$z_\infty = z - \sum_{k=1}^{\infty} z^{k+1} \sum_{* \in \mathcal{W}_k} F_*(z, \lambda) S_*,$$

where F_* are polynomial functions and the tags S_* are defined by (4.2.4) *and* (4.2.5), *such that* (4.2.3) *transforms to*

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{z}_\infty = Axz_\infty. \end{cases}$$

Proof: Denote by $z_0 = z$ and $\mathcal{W}_0 = \{1\}$. We show that there exist coefficients $C_{*,i}(\lambda), D_{*,i}(\lambda)$ for every $* \in \mathcal{W}$ and $i = 0, \dots, N-1$, such that

$$\dot{z}_k = Axz_k + z^{k+2} \sum_{*\in\mathcal{W}_k} \sum_{i=0}^{(k+1)(N-1)} \left(C_{*,i}(\lambda)(1+x) + D_{*,i}(\lambda)(1-x)\right) z^i S_*, \tag{4.2.6}$$

where we inductively define

$$z_{k+1} = z_k - z^{k+2} \sum_{*\in\mathcal{W}_k} \sum_{i=0}^{(k+1)(N-1)} \left(C_{*,i}(\lambda)S_{*R} + D_{*,i}(\lambda)S_{*L}\right) z^i. \tag{4.2.7}$$

It is obvious that (4.2.6) corresponds to (4.2.3) for $k = 0$. Therefore suppose (4.2.6) is true for some $k \geq 0$ and apply (4.2.7). By a straightforward computation we get

$$\begin{aligned}\dot{z}_{k+1} = {} & Axz_{k+1} + Axz^{k+2} \sum_{*\in\mathcal{W}_k} \sum_{i=0}^{(k+1)(N-1)} i\left(C_{*,i}(\lambda)S_{*R} + D_{*,i}(\lambda)S_{*L}\right) z^i \\ & - z^{k+3} \left(\sum_{*\in\mathcal{W}_k} \sum_{i=0}^{(k+1)(N-1)} (k+2+i)\left(C_{*,i}(\lambda)S_{*R} + D_{*,i}(\lambda)S_{*L}\right) z^i \right) \\ & \cdot \left(\sum_{j=0}^{N-1} (C_j(\lambda)(1+x) + D_j(\lambda)(1-x)) z^j \right).\end{aligned}$$

By replacing x by $\frac{1}{2}\left((1+x)-(1-x)\right)$ in the second term, and by observing that the term for $i = 0$ is zero, we see that this expression is again of the form (4.2.6). $\square$

A natural question to ask is the asymptotic behaviour of the formal transformation of Theorem 4.2.1 with respect to $x = \pm 1$. For this we denote

$$\bar{S}_*(x) = (1-x^2)^{\ell(*)} S_*(x),$$

for any $* \in \mathcal{W}$ where $\ell(*)$ denotes the length of the word. From (4.2.4), a straightforward computation shows that $\bar{S}_L(x)$ and $\bar{S}_R(x)$ satisfy

$$\begin{aligned}\frac{q}{2}(1-x^2)\frac{d\bar{S}_L}{dx} &= (1-x)(1-x^2) - q(p\alpha+1)x\bar{S}_L, \\ \frac{q}{2}(1-x^2)\frac{d\bar{S}_R}{dx} &= (1+x)(1-x^2) - q(p\alpha+1)x\bar{S}_R.\end{aligned} \tag{4.2.8}$$

Similarly, from (4.2.5), we get

$$\begin{aligned}\frac{q}{2}(1-x^2)\frac{d\bar{S}_{*L}}{dx} &= (1-x)(1-x^2)\bar{S}_* - q(\ell(*)+1)(p\alpha+1)x\bar{S}_{*L}, \\ \frac{q}{2}(1-x^2)\frac{d\bar{S}_{*R}}{dx} &= (1+x)(1-x^2)\bar{S}_* - q(\ell(*)+1)(p\alpha+1)x\bar{S}_{*R}.\end{aligned} \tag{4.2.9}$$

These adapted tags are of Mourtada type (see Definition 1.3.17) at $x = \pm 1$ as we prove in the following theorem.

Theorem 4.2.2. *Consider the function $\bar{S}_*(x)$ for some $* \in \mathcal{W}$ defined by* (4.2.8) *and* (4.2.9). *For any $k \geq 0$, we have*

$$\lim_{x\to -1^+}(1-x^2)^k \frac{d^k \bar{S}_*}{dx^k}(x) = \lim_{x\to 1^-}(1-x^2)^k \frac{d^k \bar{S}_*}{dx^k}(x) = 0. \tag{4.2.10}$$

Proof: Define recursively for any $* \in \mathcal{W}$ the functions

$$\bar{S}_*^i(x) = (1-x^2)\frac{d\bar{S}_*^{i-1}}{dx}(x),$$

for $i \geq 1$, where $\bar{S}_*^0(x) = \bar{S}_*(x)$. It suffices to prove that

$$\lim_{x\to -1^+} \bar{S}_*^k(x) = \lim_{x\to 1^-} \bar{S}_*^k(x) = 0, \tag{4.2.11}$$

for any $k \geq 0$. Indeed by an induction argument, we see that

$$\bar{S}_*^k(x) = (1-x^2)^k \frac{d^k \bar{S}_*}{dx^k}(x) + G_*^k\left(x, \bar{S}_*^0, \ldots, \bar{S}_*^{k-1}\right),$$

for some smooth function G_*^k where

$$G_*^k(x, 0, \ldots, 0) = 0.$$

Therefore showing (4.2.11) for any $k \geq 0$ immediately proves (4.2.10) for any $k \geq 0$. We will prove that (4.2.11) is true for any $k \geq 0$ by induction on the length $\ell(*)$. First we show that the statement is true for $\bar{S}_L$. The proof for $\bar{S}_R$ is completely similar. Due to (4.2.8), we know that the graph $y = \bar{S}_L(x)$ is tangent to the vector field

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{y} = -q(p\alpha+1)xy + (1-x^2)(1-x). \end{cases} \tag{4.2.12}$$

If we impose that

$$p\alpha + 1 > \frac{1}{2}, \tag{4.2.13}$$

we know that (4.2.12) has two singularities $(\pm 1, 0)$ which are nodes. Similar as in the proof of Lemma 2.3.13 we can show that the solution passing through the origin also passes through the nodes since it can not diverge to infinity. This means that

$$\lim_{x\to -1^+} \bar{S}_L(x) = \lim_{x\to 1^-} \bar{S}_L(x) = 0. \tag{4.2.14}$$

Now from (4.2.8), it immediately follows that

$$\bar{S}_L^1(x) = \frac{2}{q}(1-x^2)(1-x) - 2(p\alpha+1)x\bar{S}_L(x).$$

By induction on k, it is an easy exercise to show that

$$\bar{S}_L^k(x) = (1-x^2)H_L^k(x) + \sum_{i=0}^{k-1} F_L^{k,i}(x)\bar{S}_L^i(x), \tag{4.2.15}$$

for some smooth functions $H_L^k, F_L^{k,i}(x)$ for $i = 0, \dots, k-1$. Hence by induction on k, starting from (4.2.14), using (4.2.15), we can show that

$$\lim_{x \to -1^+} \bar{S}_L^k(x) = \lim_{x \to 1^-} \bar{S}_L^k(x) = 0,$$

for any $k \geq 0$.
Suppose (4.2.11) is true for any word $* \in \mathcal{W}_m$, $m \geq 1$. In order to finish the induction procedure, it suffices to prove

$$\lim_{x \to -1^+} \bar{S}_{*L}^k(x) = \lim_{x \to 1^-} \bar{S}_{*L}^k(x) = 0, \tag{4.2.16}$$

and similar for $\bar{S}_{*R}^k(x)$ for any $k \geq 0$. Since the proofs are similar, we will only prove the result for $\bar{S}_{*L}^k(x)$. Due to (4.2.9), we know the graph $y = \bar{S}_{*L}(x)$ is tangent to the vector field

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{y} = -q(m+1)(p\alpha+1)xy + (1-x^2)(1-x)\bar{S}_*(x). \end{cases}$$

Since we assumed that (4.2.11) for $k = 0$ and we still impose (4.2.13), we have that this describes a vector field with two nodal singularities $(\pm 1, 0)$. Similar as in the case $\bar{S}_L$, we see that the solution passing through the origin passes through the nodes and hence

$$\lim_{x \to -1^+} \bar{S}_{*L}(x) = \lim_{x \to 1^-} \bar{S}_{*L}(x) = 0.$$

As before, this together with (4.2.11) induces (4.2.16), since for any $k \geq 1$ we can write

$$\bar{S}_{*L}^k(x) = \sum_{i=0}^{k-1} \left(H_{*L}^{k,i}(x)\bar{S}_*^i(x) + F_{*L}^{k,i}(x)\bar{S}_{*L}^i(x) \right),$$

for some smooth functions $H_{*L}^{k,i}, F_{*L}^{k,i}$ for $i = 0, \dots, k-1$. □

The formal transformation obtained in Theorem 4.2.1 induces a transformation in the original variable y as it did in (3.2.9) which we can apply to the full system (4.2.1). Hence we have the following result.

Theorem 4.2.3. *Consider the vector field* (4.2.1) *with* $p, q \in \mathbb{N}$ *and* $gcd(p,q) = 1$. *There exists a near-identity coordinate change which is finitely smooth in* x *and formal in* y

$$(x, y) \mapsto (x, y_\infty) = (x, y(1 + \varphi(x, y, \lambda))),$$

formally bringing (4.2.1) *to the vector field*

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{y}_\infty = y_\infty \left(p(1+\alpha)x + \sum_{j=1}^{\infty} F_j(x,\lambda) y_\infty^j \right), \end{cases} \tag{4.2.17}$$

where the functions F_j are smooth for x in a neighbourhood of $[-1,1]$ and are infinitely flat at $x = \pm 1$. Formally, in terms of $z = (1-x^2)^p y^q$, the transformation φ corresponds to the transformation obtained in Theorem 4.2.1.

The remaining connecting terms can be removed in a similar fashion as in Theorem 3.2.9.

Theorem 4.2.4. *Consider* (4.2.17) *with $p, q \in \mathbb{N}$ and $gcd(p,q) = 1$. There exists a near-identity coordinate change which is finitely smooth in x and formal in y*

$$(x, y_\infty) \mapsto (x, Y) = (x, y_\infty(1 + \psi(x, y_\infty, \lambda))),$$

formally bringing (4.2.17) *to the vector field*

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{Y} = p(1+\alpha)xY. \end{cases} \tag{4.2.18}$$

Moreover, the transformation ψ can formally be decomposed as

$$\psi(x, y, \lambda) = \sum_{i \geq 1} G_i(x, s, \lambda) y^i,$$

where $s = (1-x^2)^{\frac{(1+\alpha)}{q}}$ and G_i is C^∞.

Proof: Let $Y_0 = y_\infty$. We prove that there exists a sequence of near-identity transformations $Y_{j+1} = Y_j(1 + \psi_j(x, Y_j, \lambda))$ which can be smoothly decomposed as a function of x, s, λ such that for every $i \geq 0$, we have

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{Y_j} = Y_j \left(p(1+\alpha)x + \sum\limits_{i=j+1}^{\infty} F_{j,i}(x,\lambda) Y_j^i \right), \end{cases} \tag{4.2.19}$$

where the functions $F_{j,i}$ are smooth for x in a neighbourhood of $[-1,1]$ and are infinitely flat at $x = \pm 1$. For $j = 0$, this immediately corresponds to (4.2.17), so suppose it is true for some $j \geq 0$. Then we apply the transformation

$$Y_j = Y_{j+1} + H_j(x, \lambda) Y_{j+1}^{j+1},$$

where $H_j(x, \lambda)$ is a solution of

$$-\frac{q}{2}(1-x^2)\frac{\partial H_j(x,\lambda)}{\partial x} - pj(1+\alpha)x H_j(x,\lambda) + F_{j,j+1}(x,\lambda) = 0. \tag{4.2.20}$$

By virtue of Lemma 4.1.2, this transforms (4.2.19) to

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{Y}_{j+1} = Y_{j+1} \left(p(1+\alpha)x + \sum\limits_{i=j+2}^{\infty} F_{j,i}(x,\lambda) Y_{j+1}^i + \mathrm{O}\big(Y_{j+1}^{j+2}\big) \right). \end{cases} \tag{4.2.21}$$

Similar as in the proof of Theorem (3.2.9), we can show that these higher-order terms are all infinitely flat at $x = \pm 1$. Therefore (4.2.21) is of the form (4.2.19) for j replaced by $j+1$. Hence it suffices to prove that there exists a solution $H_j(x,\lambda)$ of (4.2.20) that depends smoothly on (x,s,λ). This is obvious since we can take

$$H_j(x,\lambda) = \frac{2}{q} s^{pk} \int_0^x \frac{F_{j,i}(u,\lambda)}{(1-u^2)^{\frac{pk(1+\alpha)}{q}+1}} du,$$

where due to the flatness of F the last integral is a smooth function of x. $\square$

If we combine the formal transformations of Theorem 4.2.3 and Theorem 4.2.4 and realize these transformations using Borel's Theorem 1.2.5 combined with the removal of flat terms as we did in the proof of Theorem 4.1.8 we get the following result.

Theorem 4.2.5. *Let $n \geq 1$ be arbitrary and consider* (4.2.1) *with $p, q \in \mathbb{N}$ and gcd$(p,q) = 1$. There exists a finitely smooth near-identity coordinate change*

$$(x,y) \mapsto (x,Y) = (x, y(1 + \Phi(x,y,\lambda))),$$

bringing (4.2.1) *in the form* (4.2.18). *Moreover, we can express Φ as a C^n function in the variables*

$$\left(x, y, (\bar{S}_*)_{*\in\mathcal{W}}, (1-x^2)^{\frac{1+\alpha}{q}}\right),$$

where the tags $\bar{S}_*$ are defined in (4.2.8) *and* (4.2.9).

Chapter 5

Application to homoclinic connections

In this chapter we illustrate the results of Chapter 2 and Chapter 3. We study graphics having only one singularity which is non-elementary. We confine ourselves to the case where the singularity is either a cusp (Section 5.1) or a fake saddle (Section 5.2). When we consider perturbations, we will always assume that the singularity remains unperturbed. This chapter is also part of [12] which is accepted for publication in Journal of Differential Equations.

5.1 The cusp

We consider a vector field unfolding a cusp-singularity as in [41]. We assume that the vector fields are already written in Loray's normal form (see [36]). In this paper, the author distinguishes two cases:

$$\alpha_n : \begin{cases} \dot{x} = 2y + 2xh^n \left(f(h) + xg(h)\right), \\ \dot{y} = 3x^2 + 3yh^n \left(f(h) + xg(h)\right), \end{cases} \tag{5.1.1}$$

where $f(0) \neq 0$ and

$$\beta_n : \begin{cases} \dot{x} = 2y + 2xh^n \left(hf(h) + xg(h)\right), \\ \dot{y} = 3x^2 + 3yh^n \left(hf(h) + xg(h)\right), \end{cases} \tag{5.1.2}$$

where $g(0) \neq 0$ with $h := x^3 - y^2$ and f and g are C^∞ in each case. We perform a quasi-homogeneous blow-up $(x, y) = (r^2 \cos\theta, r^3 \sin\theta)$ leading to two hyperbolic saddles on the blow-up locus with reciprocal saddle quantities (see Figure 5.1). One can check that the two cusp separatrices of α_n (respectively β_n) approach the origin in the directions $\theta = \pm\theta_0$ for some $\theta_0 \in \left]0, \frac{\pi}{2}\right[$.

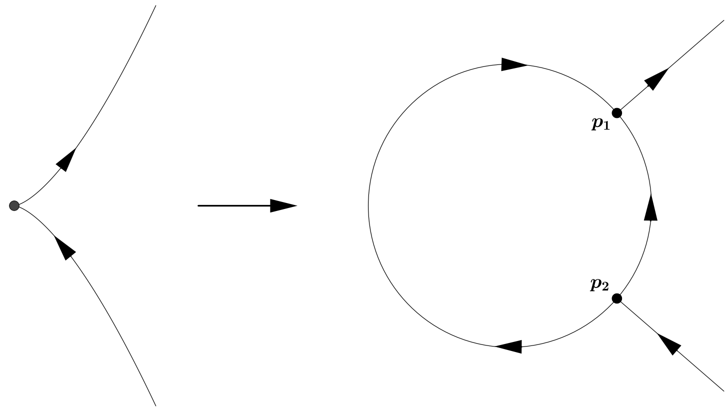


Figure 5.1: Quasi-homogeneous blow-up of a cusp

In order to get the transition map near the right part of the blow-up locus, we consider the directional chart (see Figure 5.2)

$$(x, y) = (Y^2, Y^3 X). \tag{5.1.3}$$

The variable Y serves as the radial variable whereas X acts as (projectivized) angular variable.

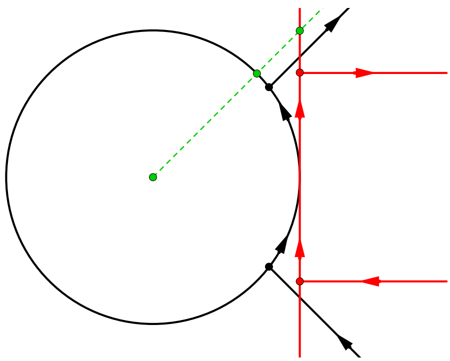


Figure 5.2: Projective blow-up (5.1.3)

To study the part to the left of the singularity one would first think of considering a similar directional chart, which would give us information on the directions $\theta \in \left]\frac{\pi}{2}, \frac{3\pi}{2}\right[$. However, as this region does not include θ_0 it is better to replace this directional chart by a chart using a rational parametrization of the parabola that approximates a circle near $\theta = \pi$ (see Figure 5.3):

$$(x, y) = (Y^2(X^2 - 1), 2Y^3 X). \tag{5.1.4}$$

Again, the variable Y serves as the radial variable whereas X acts as angular variable. In fact it reveals convenient for the computations to do some scaling; we will hence use instead:

$$(x, y) = (aY^2(4X^2 - 1), 4Y^3X), \quad \text{where} \quad a := \frac{2}{3}2^{1/3}. \tag{5.1.5}$$

As will be explained in (5.1.9), this choice of a is necessary to have a nice factorization of the $\dot{X}$ equation.

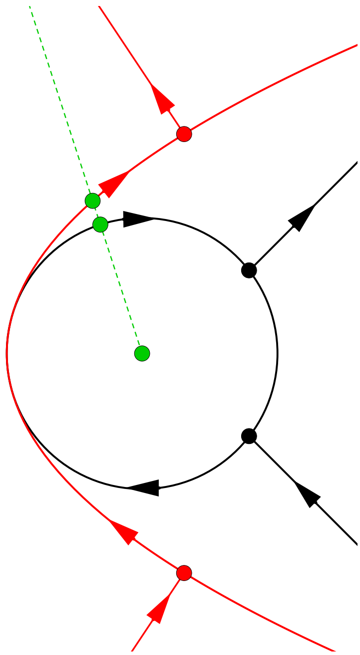


Figure 5.3: Parabolic blow-up (5.1.4)

5.1.1 Projective blow-up

In this section we consider the vector fields in Loray normal form in the blow-up chart (5.1.3). For α_n we find, after division by the non-negative factor Y:

$$\alpha_n : \begin{cases} \dot{X} = 3(1 - X^2), \\ \dot{Y} = YX + H^n \left(f(H) + Y^2 g(H)\right), \end{cases} \tag{5.1.6}$$

where $f(0) \neq 0$ and $H = (1 - X^2)Y^6$. Similarly, β_n becomes

$$\beta_n : \begin{cases} \dot{X} = 3(1 - X^2), \\ \dot{Y} = YX + H^n \left((1 - X^2)Y^6 f(H) + Y^2 g(H)\right), \end{cases} \tag{5.1.7}$$

where $g(0) \neq 0$. In both cases, the saddle connection lies on $Y = 0$ between $X = \pm 1$.

5.1.2 Parabolic blow-up

In this section we consider the vector fields in the blow-up chart given by (5.1.5). A straightforward computation gives

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \frac{1}{4aY^3(8X^2+1)} \begin{pmatrix} 6XY & -a(4X^2-1) \\ -2Y^2 & 4aXY \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \tag{5.1.8}$$

Let us first compute $\dot{X}$

$$\dot{X} = \frac{3Y}{4a(8X^2+1)}(16X^2 - a^3(4X^2-1)^3).$$

For the special value $a^3 = 16/27$, this simplifies to

$$\dot{X} = \frac{4Y(8X^2+1)}{9a}(1-X^2). \tag{5.1.9}$$

After division of (5.1.8) by the non-negative factor $\frac{a^2}{4}(1+8X^2)Y$ and normalizing by

$$Y = (8X^2+1)^{-1/3}\bar{Y},$$

we obtain

$$\alpha_n : \begin{cases} \dot{X} = 3(1-X^2), \\ \dot{\bar{Y}} = X\bar{Y} + \frac{27a}{4(8X^2+1)^{2/3}}\bar{H}^n(f(\bar{H}) + a\frac{4X^2-1}{(8X^2+1)^{2/3}}\bar{Y}^2 g(\bar{H})), \end{cases} \tag{5.1.10}$$

and

$$\beta_n : \begin{cases} \dot{X} = 3(1-X^2), \\ \dot{\bar{Y}} = X\bar{Y} + \frac{27a}{4(8X^2+1)^{2/3}}\bar{H}^n(\bar{H}f(\bar{H}) + a\frac{4X^2-1}{(8X^2+1)^{2/3}}\bar{Y}^2 g(\bar{H})), \end{cases} \tag{5.1.11}$$

where $\bar{H} = -\frac{16}{27}(1-X^2)\bar{Y}^6$.

5.1.3 The normalizing transformation

All the previous vector fields can be written in the form

$$\begin{cases} \dot{x} = 3(1-x^2), \\ \dot{y} = y\left(x + F(x)y^k + \mathrm{O}(y^l)\right), \end{cases} \tag{5.1.12}$$

where $k = 6n-1, l = 6n+1$, respectively $k = 6n+1, l = 6n+5$, for α_n, respectively β_n and where F is not identically zero. We show in this section that the first non-zero term gives rise to a non-zero connecting term in the normal form (2.1.6) which is of lower order than the resonant terms, i.e. we are in the case $k < mq$ of Theorem 3.3.2. As an application of Theorem 2.1.3, we have the following.

Lemma 5.1.1. *Let $k = 6n - 1$ for some $n \geq 1$ or $k = 6n + 1$ for some $n \geq 0$. There exists a smooth coordinate transformation $(x, y) \mapsto (x, z) = (x, \varphi(x, y))$, such that system* (5.1.12)*, with F not identically* 0*, is orbitally equivalent to*

$$\begin{cases} \dot{x} = 3(1 - x^2), \\ \dot{z} = z\left[x + \gamma\chi(x)z^k + \chi(x)z^{k+1}f_1(z)\right. \\ \qquad\qquad \left. + (1 - x^2)^l z^{6l}\left(f_2\left((1 - x^2)z^6\right) + xf_3\left((1 - x^2)z^6\right)\right)\right], \end{cases} \tag{5.1.13}$$

where $\gamma \neq 0$, $k < 6l$ and f_1, f_2, f_3 are C^∞. Moreover, we have

$$y = z + G(x)z^{k+1} + \mathrm{O}(z^{l+1}),$$

where G is a C^∞ solution of

$$-3(1 - x^2)G'(x) - kxG(x) + F(x) = \gamma\chi(x). \tag{5.1.14}$$

Following the method of proving Lemma 2.3.13, the coefficient γ in (5.1.13) has the property that it is the unique coefficient for which (5.1.14) has a smooth solution G in a neighbourhood of $[-1, 1]$ and depends on the function F. For each of the cases above, we want to compute the coefficient γ. For this, we need the following lemma.

Lemma 5.1.2. *Let $p, k, q \in \mathbb{N}_0$ such that $gcd(p, q) = 1$ and $\lambda = \frac{pk}{q} \notin \mathbb{N}$. Let $N = \lfloor \lambda \rfloor$. There exists an $\gamma \in \mathbb{R}$ such that the differential equation*

$$\frac{-q}{2}\frac{dh(x)}{dx}(1 - x^2) - pkxh(x) + (1 - x^2)^{N+1} = \gamma\chi(x), \tag{5.1.15}$$

has a C^∞ solution in a neighbourhood of $[-1, 1]$. Moreover,

$$\gamma \int_0^1 \frac{\chi(u)}{(1 - u^2)^{\lambda+1}} du = \frac{1}{2}\frac{\sqrt{\pi}\,\Gamma(1 - \alpha)}{\Gamma\left(\frac{3}{2} - \alpha\right)},$$

where $\alpha = \lambda - N$.

Proof: It only remains to explicitly compute γ, since the existence of such a γ that (5.1.15) has a smooth solution is guaranteed by virtue of Lemma 2.3.13. The coefficient γ corresponds to the unique value where the curves $y = \varphi_\gamma(x)$ and $y = \psi_\gamma(x)$, tangent to the vector field

$$\begin{cases} \dot{x} = \frac{q}{2}(1 - x^2), \\ \dot{y} = -pkxy + (1 - x^2)^{N+1} - \gamma\chi(x), \end{cases} \tag{5.1.16}$$

and given by the locally smooth solutions near the nodes $(-1, 0)$ and $(1, 0)$, coincide on the interval $(-1, 1)$. In order to compute γ, we need to provide explicit expressions for the C^∞-graphs $y = \varphi_\gamma(x)$ and $y = \psi_\gamma(x)$ (see Figure 5.4).

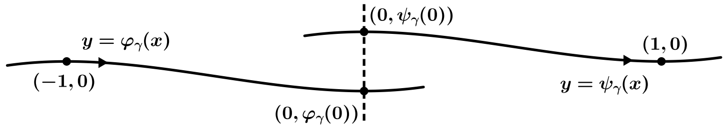


Figure 5.4: Connecting the smooth graphs

One can easily verify that

$$\varphi_\gamma(x) = (1-x^2)^\lambda \frac{2}{q}\int_{-1}^{x}(1-u^2)^{-\alpha}du - (1-x^2)^\lambda\gamma\frac{2}{q}\int_{-1}^{x}\frac{\chi(u)}{(1-u^2)^{\lambda+1}}du, \quad (5.1.17)$$

where $\alpha = \lambda - N \in (0,1)$, is a solution of (5.1.15). It remains to prove that this solution is locally C^∞ near $x = -1$. We claim that

$$\frac{2}{q}(1-x^2)^\lambda \int_{-1}^{x}(1-u^2)^{-\alpha}du = \frac{2^{1-\alpha}}{q(1-\alpha)}(1+x)^{N+1}(1-x)^\lambda \text{hypergeom}\left([\alpha, 1-\alpha], [2-\alpha], \frac{x+1}{2}\right),$$

where the hypergeometric function is defined in Section 1.2.3. Since this is C^∞ near $x = -1$, we know that (5.1.17) describes the local unstable manifold. We prove the claim using Euler's formula for hypergeometric functions (1.2.9):

$$\begin{aligned}
\int_{-1}^{x}(1-u^2)^{-\alpha}du &\overset{z=u+1}{=} \int_{0}^{x+1} z^{-\alpha}(2-z)^{-\alpha}dz \\
&\overset{z=t(1+x)}{=} 2^{-\alpha}(1+x)^{1-\alpha}\int_0^1 t^{-\alpha}\left(1-t\left(\frac{1+x}{2}\right)\right)^{-\alpha}dt \\
&= \frac{2^{-\alpha}}{1-\alpha}(1+x)^{1-\alpha}\frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)\Gamma(1)}\int_0^1 t^{-\alpha}\left(1-t\left(\frac{1+x}{2}\right)\right)^{-\alpha}dt \\
&= \frac{2^{-\alpha}}{1-\alpha}(1+x)^{1-\alpha}\text{hypergeom}\left([\alpha, 1-\alpha], [2-\alpha], \frac{x+1}{2}\right).
\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
\psi_\gamma(x) = &-\frac{2^{1-\alpha}}{q(1-\alpha)}(1-x)^{N+1}(1+x)^\lambda \text{hypergeom}\left([\alpha, 1-\alpha], [2-\alpha], \frac{1-x}{2}\right) \\
&- (1-x^2)^\lambda\frac{2\gamma}{q}\int_1^x \frac{\chi(u)}{(1-u^2)^{\lambda+1}}du,
\end{aligned}$$

describes the local stable manifold near $x = 1$. We have $\varphi_\gamma(0) = \psi_\gamma(0)$ if and only if

$$\frac{2^{-\alpha}}{1-\alpha}\text{hypergeom}\left([\alpha, 1-\alpha], [2-\alpha], \frac{1}{2}\right) = \gamma\int_0^1 \frac{\chi(u)}{(1-u^2)^{\lambda+1}}du.$$

The result now follows from the fact

$$\text{hypergeom}\left([\alpha, 1-\alpha], [2-\alpha], \frac{1}{2}\right) = \frac{2^{-1+\alpha}\sqrt{\pi}\Gamma(2-\alpha)}{\Gamma\left(\frac{3}{2}-\alpha\right)}.$$

□

Thanks to Lemma 5.1.1 and Lemma 5.1.2 the first coefficient in the normal form of (5.1.6) can be deduced.

Corollary 5.1.3. *There exists a C^∞ function $h(x)$ defined in a neighbourhood of $[-1,1]$ that satisfies*

$$-3\frac{dh(x)}{dx}(1-x^2)-(6n-1)xh(x)+(1-x^2)^n=\gamma\chi(x),$$

requiring

$$\gamma\int_0^1 \frac{\chi(u)}{(1-u^2)^{n+5/6}}du=\frac{1}{2}\frac{\sqrt{\pi}\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{2}{3}\right)}.$$

The other blow-up vector fields (5.1.7), (5.1.10) and (5.1.11) need to be treated in a similar way. Lemma 5.1.1 applied to these vector fields provides us a differential equation (5.1.14) which should be smoothly solvable. By choosing γ wisely, as a consequence of Lemma 2.3.13, this is possible in a similar way as in the proof of Lemma 5.1.2. First we consider (5.1.7).

Corollary 5.1.4. *There exists a C^∞ function $h(x)$ defined in a neighbourhood of $[-1,1]$ that satisfies*

$$-3\frac{dh(x)}{dx}(1-x^2)-(6n+1)xh(x)+(1-x^2)^n=\gamma\chi(x),$$

requiring

$$\gamma\int_0^1 \frac{\chi(u)}{(1-u^2)^{n+7/6}}du=-\frac{\sqrt{\pi}\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{4}{3}\right)}.$$

Proof: It is a straightforward computation to see that

$$-3\frac{df(x)}{dx}(1-x^2)-(6n+1)xf(x)+(1-x^2)^n+2(1-x^2)^{n+1}=0,$$

has a smooth solution

$$f(x)=3x(1-x^2)^n.$$

Hence to solve the ODE as stated in the corollary, it suffices to solve

$$-3\frac{dg(x)}{dx}(1-x^2)-(6n+1)xg(x)-2(1-x^2)^{n+1}=\gamma\chi(x).$$

The solution h is then simply given by $f+g$. In order to solve for a smooth g, we can apply Lemma 5.1.2. □

If we treat (5.1.10), we have

Corollary 5.1.5. *There exists a C^∞ function $h(x)$ defined in a neighbourhood of $[-1,1]$ that satisfies*

$$-3\frac{dh(x)}{dx}(1-x^2)-(6n-1)xh(x)+(1-x^2)^n(8x^2+1)^{-2/3}=\gamma\chi(x)$$

requiring

$$\gamma\int_0^1 \frac{\chi(t)}{(1-t^2)^{n+5/6}}dt=\frac{1}{3}\frac{\pi^{3/2}}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)}.$$

Proof: Similar as in the proof of Lemma 5.1.2, we need to connect both local smooth solutions. The local smooth solution near $x=-1$ is given by

$$\Phi_\gamma(x)=\frac{1}{3}(1-x^2)^{n-1/6}\int_{-1}^x \frac{(8t^2+1)^{-2/3}}{(1-t^2)^{5/6}}-\gamma\frac{\chi(t)}{(1-t^2)^{n+5/6}}dt.$$

We see that this is smooth near $x=-1$ as follows

$$\begin{aligned}\int_{-1}^x \frac{(8t^2+1)^{-2/3}}{(1-t^2)^{5/6}}dt &\overset{z=t+1}{=} \int_0^{x+1}\frac{(8(z-1)^2+1)^{-2/3}}{z^{5/6}(2-z)^{5/6}}dz\\ &\overset{z=u(x+1)}{=} (x+1)^{1/6}\int_0^1 \frac{(8(u(x+1)-1)^2+1)^{-2/3}}{u^{5/6}(2-u(x+1))^{5/6}}du\end{aligned}$$

This integral, which smoothly depends on x, is finite on a neighbourhood of $x=-1$. Similarly the right smooth manifold is given by

$$\Psi_\gamma(x)=\frac{1}{3}(1-x^2)^{n-1/6}\int_{1}^x \frac{(8t^2+1)^{-2/3}}{(1-t^2)^{5/6}}-\gamma\frac{\chi(t)}{(1-t^2)^{n+5/6}}dt.$$

Hence we want

$$\gamma\int_0^1 \frac{\chi(t)}{(1-t^2)^{n+5/6}}dt=\int_0^1 \frac{(8t^2+1)^{-2/3}}{(1-t^2)^{5/6}}dt.$$

The expression as stated in the corollary can be found using MAPLE. □

Finally for (5.1.11), we have

Corollary 5.1.6. *There exists a C^∞ function $h(x)$ defined in a neighbourhood of $[-1,1]$ that satisfies*

$$-3\frac{dh(x)}{dx}(1-x^2)-(6n+1)xh(x)+\frac{4x^2-1}{(8x^2+1)^{4/3}}(1-x^2)^n=\gamma\chi(x),$$

requiring

$$\gamma\int_0^1 \frac{\chi(t)}{(1-t^2)^{n+7/6}}dt=-\frac{\sqrt{3}}{2}\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)}{\sqrt{\pi}}.$$

Proof: Denote

$$F(x)=\frac{4x^2-1}{(8x^2+1)^{4/3}}.$$

We need to do a similar trick as in the proof of Corollary 5.1.4, to elevate the degree of $(1-x^2)$. This is done in the following way. Consider

$$-3\frac{dh(x)}{dx}(1-x^2)-(6n+1)xh(x)+F(x)(1-x^2)^n(1+2(1-x^2))$$
$$-2(1-x^2)^{n+1}F(x)=\gamma\chi(x).$$

By a straightforward computation using partial integration and an argument similar as the one in the proof of Corollary 5.1.5, we see that the left smooth invariant manifold is given by

$$\Phi_\gamma(x)=xF(x)(1-x^2)^n-\frac{1}{3}(1-x^2)^{n+1/6}\int_{-1}^{x}\frac{3uF'(u)+2F(u)}{(1-u^2)^{1/6}}+\gamma\frac{\chi(u)}{(1-u^2)^{7/6}}du,$$

and the right smooth invariant manifold is given by

$$\Psi_\gamma(x)=xF(x)(1-x^2)^n-\frac{1}{3}(1-x^2)^{n+1/6}\int_{1}^{x}\frac{3uF'(u)+2F(u)}{(1-u^2)^{1/6}}+\gamma\frac{\chi(u)}{(1-u^2)^{7/6}}du.$$

As before, $\Phi_\gamma(0)=\Psi_\gamma(0)$ amounts to

$$\gamma\int_0^1\frac{\chi(u)}{(1-u^2)^{7/6}}du=-\int_0^1\frac{3uF'(u)+2F(u)}{(1-u^2)^{1/6}}du$$
$$=-\frac{\sqrt{3}}{2}\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)}{\sqrt{\pi}}.$$

The expression as stated in the corollary can be found using MAPLE. □

5.1.4 The transition maps and cyclicity

Theorem 3.3.2 states that the transition maps for the vector fields α_n are of the form

$$D(x_0)=-x_0+2\gamma A_{6n-1}(1-x_0^2)^{n+5/6}+\text{h.o.t.},$$

where γ denotes the first non-zero coefficient in the normal form and

$$A_{6n-1}=\int_0^1\frac{\chi(t)}{(1-t^2)^{n+5/6}}dt$$

The notation 'h.o.t.' denotes the higher order terms with respect to the variable $1+x_0$ as explained in remark 3.3.4. Hence for (5.1.6) by Corollary 5.1.3, we have

$$D(x_0)=-x_0+f(0)\frac{\sqrt{\pi}\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{2}{3}\right)}(1-x_0^2)^{n+5/6}+\text{h.o.t.},\tag{5.1.18}$$

and for (5.1.10), by Corollary 5.1.5, we have

$$D(x_0)=-x_0+\frac{27a}{2}\left(-\frac{16}{27}\right)^n f(0)\frac{\pi^{3/2}}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)}(1-x_0^2)^{n+5/6}+\text{h.o.t.}.\tag{5.1.19}$$

Similarly for the vector fields β_n we have

$$D(x_0) = -x_0 + 2\gamma A_{6n+1}(1-x_0^2)^{n+7/6} + \text{h.o.t.},$$

where γ denotes the first non-zero coefficient and

$$A_{6n+1} = \int_0^1 \frac{\chi(t)}{(1-t^2)^{n+7/6}}\, dt.$$

Hence the transition map for (5.1.7) is given by (see Corollary 5.1.4)

$$D(x_0) = -x_0 - 2g(0)\frac{\sqrt{\pi}\,\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{4}{3}\right)}(1-x_0^2)^{n+7/6} + \text{h.o.t.},$$

and for (5.1.11) by (see Corollary 5.1.6)

$$D(x_0) = -x_0 - \frac{\sqrt{3}}{a}\left(-\frac{16}{27}\right)^n g(0)\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)}{\sqrt{\pi}}(1-x_0^2)^{n+7/6} + \text{h.o.t.}.$$

We combine the results in each of the blow-up maps (5.1.3) and (5.1.5) to get an upper bound on the cyclicity of a cuspidal loop where the cusp is locally smoothly conjugated to (5.1.1) or (5.1.2). Write

$$\Sigma_{\text{in}} = \{(-1+x, 1) \mid |x| << 1\}, \text{ and } \Sigma_{\text{out}} = \{(1-y, 1) \mid |y| << 1\},$$

and use x, respectively y to parametrize Σ_{in}, respectively Σ_{out}. In this way, the transition map $D : \Sigma_{\text{in}} \to \Sigma_{\text{out}}$ can be considered as a one-dimensional function $y = D(x)$. The two cases $x > 0$ and $x < 0$ corresponding to either the blow-up chart (5.1.3) or (5.1.5) correspond to two different types of limit cycles, namely the interior and exterior ones. In this way we can bound the cyclicity of the inner or outer limit cycles separately and also bound the true (two-sided) cyclicity as we explain shortly here. Without going in too much detail, the transition map near the blow-up locus of (5.1.1) is given by

$$D(x) = \begin{cases} x + x^{n+5/6}(\kappa f(0) + \mathrm{o}(1)), & \text{if } x \geq 0, \\ x + |x|^{n+5/6}(\eta f(0) + \mathrm{o}(1)), & \text{if } x < 0, \end{cases} \tag{5.1.20}$$

for some non-zero κ, η related to the coefficients in (5.1.18), respectively (5.1.19). Observe that this map is only C^n. When there is a cuspidal loop, one can consider the inverted regular transition $R : \Sigma_{\text{in}} \to \Sigma_{\text{out}}$ near the loop. The most degenerate case is when

$$R(x) = x + \text{h.o.t.}.$$

Since this is C^∞ at $x = 0$, its asymptotic expansion does not contain non-smooth terms which can compensate for the non-smooth terms in the map D. Hence if we look at the difference map $\Delta(x) = D(x) - R(x)$, we know that

$$\Delta^{(n)}(x) = \begin{cases} x^{5/6}(\bar{\kappa} f(0) + \mathrm{o}(1)), & \text{if } x \geq 0, \\ |x|^{5/6}(\bar{\eta} f(0) + \mathrm{o}(1)), & \text{if } x < 0. \end{cases} \tag{5.1.21}$$

This has only one zero in a neighbourhood U of $x = 0$. Due to Rolle's Theorem, we know that Δ can not have more than $n + 1$ zeroes in U. This allows us to give a partial cyclicity result since zeroes of the difference map correspond to limit cycles of the system. We can put an upper bound on the cyclicity of a specific family of vector fields perturbating from the cuspidal loop where the cusp is conjugated to (5.1.1) or (5.1.2) as follows.

Suppose we have a family X_λ of smooth vector fields defined in a neighbourhood V of the parameter λ_0 such that X_{λ_0} contains a cuspidal loop Γ. Suppose that for every $\lambda \in V$ the vector field X_λ has a cusp singularity conjugated to (5.1.1) where $f(0)$ can depend on λ but remains non-zero. The results above remain true, however the coefficients become parameter-dependent. Moreover the coefficients κ and η in (5.1.20) and (5.1.21) can also be parameter-dependent. Even though, the statement that (5.1.21) has at most one zero in a neighbourhood U of $x = 0$ remains valid. Hence we can find a neighbourhood W of Γ such that X_λ has at most $n + 1$ limit cycles contained in W.

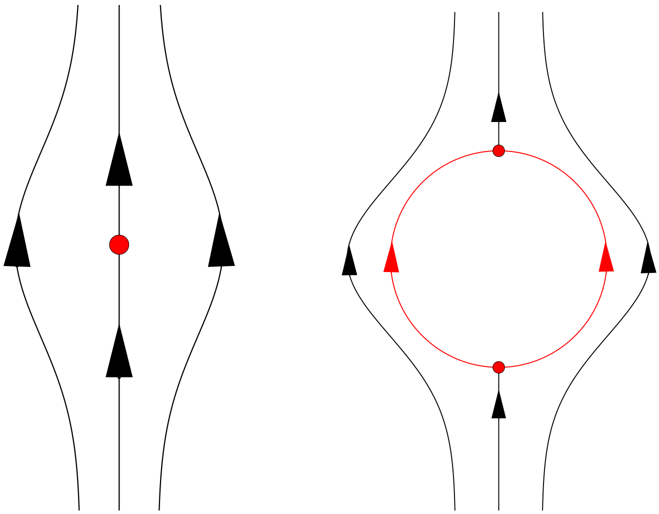
In a similar way we can show that for a family of vector fields X_λ conjugated to (5.1.2) with a cuspidal loop Γ for X_{λ_0} can have at most $n+2$ limit cycles in a neighbourhood W of Γ.

5.2 The fake saddle

Following [13], we consider a degenerate planar singularity of the form

$$\begin{cases} \dot{x} = Ax^2 + bxy + O(\|(x, y)\|^3), \\ \dot{y} = x^2 + y^2 + O(\|(x, y)\|^3), \end{cases} \tag{5.2.1}$$

with $A^2 < 4(1 - b)$ and $b \in (0, 1)$ (see Figure 5.5a). These are the conditions under which the origin has exactly two hyperbolic sectors, an incoming separatrix and an outgoing separatrix. Both separatrices are of center type, and are similar to the two branches of a one-dimensional saddle-node singularity. We show in Section 5.2.1 that after a homogeneous blow-up a saddle connection along the equator connects two hyperbolic saddles with ratios of eigenvalues $b - 1 : 1$ and $1 - b : -1$. As we are interested in this thesis in the study of the resonant case, we confine ourselves to the cases $1 - b$ equal to 1 or $\frac{1}{2}$. Other resonant cases ($1 - b \in \mathbb{Q}$) demand more involved calculations and shall therefore not be handled in this text. Even the case $1 - b = \frac{1}{k}$ with $k \in \mathbb{N}$ requires a non-trivial computation since one needs to compute a residue of some function $\frac{g(x)}{(1-x^2)^{k+1}}$ at $x = 1$ and $x = -1$ in general. Before examining the transition map along the fake saddle by seeing it essentially as a transition through two symmetric saddles, we first put the system in an elementary form in Section 5.2.1 and Section 5.2.2. In Section 5.2.3 we deal with the case $b = 0$, in Section 5.2.4 we deal with $b = \frac{1}{2}$. Since it is merely our intention to demonstrate the applicability of the results in this thesis, we confine ourselves to these two cases.



(a) Phase portrait (b) Blow-up

Figure 5.5: Fake saddle

5.2.1 Persistence of SN-fiber

We blow-up the singularity by writing $(x, y) = (rX, r\sigma)$ (with $\sigma = \pm 1$) (see Figure 5.5b) to find:

$$\begin{cases} \dot{r} = \sigma r(X+1) + \mathrm{O}\big(r^2\big), \\ \dot{X} = \sigma X(\sigma A X - X^2 + b - 1). \end{cases}$$

The origin $(r, X) = (0, 0)$ is a saddle. Then we have two C^∞-separatrices $X = \psi(r)$ and $r = 0$ each of them defined in a neighbourhood of the origin. For $\sigma = +1$, the two saddle points which appear in the polar blow-up are glued in a single saddle point, whose invariant manifold blows down to a C^∞ invariant graph

$$x = y\psi(y),$$

where ψ is defined and smooth in a neighbourhood of 0. This manifold corresponds to the SN-fiber and by a C^∞ change of coordinates we can straighten it to $x = 0$.

5.2.2 Preliminary normal form

Up to a smooth change of coordinates, we can assume that $x = 0$ is an invariant manifold passing through the fake saddle. The behaviour of the vector field on this line is of the form

$$\dot{y} = y^2 + \text{h.o.t.},$$

which can be put in a normal form by a C^∞ transformation, eliminating all higher order terms in the above equation except maybe a resonant cubic term (see [54]). Thanks to this latter transformation, the system takes the form

$$\begin{cases} \dot{x} = Ax^2 + bxy + x\,\mathrm{O}\big(\|(x,y)\|^2\big), \\ \dot{y} = x^2 + y^2 + \sigma y^3 + x\,\mathrm{O}\big(\|(x,y)\|^2\big), \end{cases} \tag{5.2.2}$$

for $\sigma = 0, 1$. We reduce the terms of homogeneous degree 3 and higher as follows:

Lemma 5.2.1. *Consider the vector field* (5.2.2). *There exists a formal conjugacy such that this vector field is conjugated to*

- *Case 1:* $b = 0$

$$\begin{cases} \dot{x} = Ax^2 + x^3 f(x) + Bx^m y, \\ \dot{y} = x^2 + y^2 + \sigma y^3 + x^3 g(x) + xh(x)y^3, \end{cases} \tag{5.2.3}$$

for some $m > 1$, $B \neq 0$, *or*

$$\begin{cases} \dot{x} = Ax^2 + x^3 f(x), \\ \dot{y} = x^2 + y^2 + \sigma y^3 + x^3 g(x) + xh(x)y^3, \end{cases} \tag{5.2.4}$$

- *Case 2:* $b = \frac{1}{N}$, *with* $N \in \mathbb{N}_0$, $N \geq 2$

$$\begin{cases} \dot{x} = Ax^2 + x^3 f(x) + bxy, \\ \dot{y} = x^2 + y^2 + \sigma y^3 + x^3 g(x) + \alpha x^N y^2 + \beta x^{2N} y, \end{cases} \tag{5.2.5}$$

- *Case 3:* $b = \frac{2}{M}$, *with* $M \in \mathbb{N}_0$ *odd,* $M \geq 3$

$$\begin{cases} \dot{x} = Ax^2 + bxy + x^3 f(x), \\ \dot{y} = x^2 + y^2 + \sigma y^3 + x^3 g(x) + \beta x^M y, \end{cases} \tag{5.2.6}$$

- *Case 4:* $b \neq 0$ *and* $b \neq \frac{2}{K}$, *with* $K \in \mathbb{N}_0$

$$\begin{cases} \dot{x} = Ax^2 + bxy + x^3 f(x), \\ \dot{y} = x^2 + y^2 + \sigma y^3 + x^3 g(x). \end{cases} \tag{5.2.7}$$

The functions f, g *and* h *are* C^∞ *in each of the cases.*

Proof: We consider (5.2.2). Suppose the vector field is already in the expected form up to homogeneous degree $N \geq 2$. In order to reduce the term of homogeneous degree $N+1$, we apply a transformation of the form

$$(x, y) = \left(X + CX^kY^l, Y + DX^kY^l\right),$$

where $k \geq 1$, $l \geq 1$ and $N = k + l$ (similar as in the work of Takens [54]) and work by induction on k. To see the effect in the lowest homogeneous degree, it suffices to consider the effect on the quadratic terms of the vector field

$$\begin{cases} \dot{x} = Ax^2 + bxy, \\ \dot{y} = x^2 + y^2. \end{cases}$$

This transforms to

$$\begin{cases} \dot{X} = AX^2 + bXY + C(b - l - bk)X^kY^{l+1} \\ \qquad\qquad\qquad + C(2A - Ak + bD)X^{k+1}Y^l - lCX^{k+2}Y^{l-1}, \\ \dot{Y} = X^2 + Y^2 + D(2 - bk - l)X^kY^{l+1} + (2C - AkD)X^{k+1}Y^l - lDx^{k+2}Y^{l-1}, \end{cases}$$

where we omit the terms of higher order. Remark that $b - l - bk \neq 0$, such that we can eliminate the term X^kY^{l+1} in the first equation of (5.2.2). We can choose D to get rid of the term $X^{k+1}Y^l$ in the second equation, if and only if

$$b \neq \frac{2 - l}{k}.$$

Since $b \in [0, 1)$, this can only happen when $l = 2, b = 0$ and when $l = 1$ and $b = \frac{1}{N}$ for some $N \geq 2$. Next we consider the transformation when $l = 0$ and $k \geq 2$, to see

$$\begin{cases} \dot{X} = AX^2 + bXY + C(b - bk)X^kY + (CA(2 - k) + bD)X^{k+1}, \\ \dot{Y} = X^2 + Y^2 + D(2 - bk)X^kY + (2C - DkA)X^{k+1}. \end{cases}$$

Observe that $b - bk$ can only be zero if $b = 0$ and $2 - bk$ is zero when $b = \frac{1}{N}$ as before or when $b = \frac{2}{M}$ with $M \geq 3$ and odd. With this method it is impossible to eliminate the terms which are independent of y. By induction, we get the forms as described in the lemma except for the case $b = 0$. Here we have after induction

$$\begin{cases} \dot{x} = Ax^2 + x^3f(x) + B(x)y, \\ \dot{y} = x^2 + y^2 + \sigma y^3 + x^3g(x) + xh(x)y^3, \end{cases}$$

for some formal function $B(x)$. In the case that B vanishes, i.e. $B(x) \equiv 0$, we immediately get (5.2.4). Otherwise, we can suppose that there is a $B \neq 0$ and $m > 1$ such that $B(x) = Bx^m(B + xq(x))$. If we apply a transformation $x = \Phi(X)$, where

$$\Phi(X) = X + \mathrm{O}\left(X^2\right),$$

is a smooth solution of

$$BX^m\Phi'(X) = B(\Phi(X)) = \Phi(X)^m(B + \Phi(X)q(\Phi(X)),$$

we get

$$\begin{cases} \dot{X} = AX^2 + X^3\tilde{f}(X) + \frac{B(\Phi(X))}{\Phi'(X)}y = AX^2 + X^3\tilde{f}(X) + BX^m y, \\ \dot{y} = X^2 + y^2 + \sigma y^3 + X^3\tilde{g}(X) + X\tilde{h}(X)y^3. \end{cases}$$

□

In order to compute the transition map near the saddle-node fiber, it suffices to work up to equivalence. In this way, we can simplify even further. The proof is a simple adjustment of the induction argument in the previous lemma where we rescale in each induction step.

Lemma 5.2.2. *Consider the vector field* (5.2.1). *There exists a formal equivalence such that this vector field is equivalent to*

- *Case 1:* $b = 0$
$$\begin{cases} \dot{x} = Ax^2 + x^3 f(x) + Bx^m y, \\ \dot{y} = x^2 + y^2 + x^3 g(x), \end{cases} \tag{5.2.8}$$
for some $m > 1$, $B \neq 0$, *or*
$$\begin{cases} \dot{x} = Ax^2 + x^3 f(x), \\ \dot{y} = x^2 + y^2 + x^3 g(x), \end{cases} \tag{5.2.9}$$

- *Case 2:* $b = \frac{2}{M}$, *with* $M \in \mathbb{N}_0$, $M \geq 3$
$$\begin{cases} \dot{x} = Ax^2 + bxy + x^3 f(x), \\ \dot{y} = x^2 + y^2 + x^3 g(x) + \beta x^M y, \end{cases} \tag{5.2.10}$$

- *Case 3:* $b \neq 0$ *and* $b \neq \frac{2}{K}$, *with* $K \in \mathbb{N}_0$
$$\begin{cases} \dot{x} = Ax^2 + bxy + x^3 f(x), \\ \dot{y} = x^2 + y^2 + x^3 g(x). \end{cases} \tag{5.2.11}$$

The functions f, g *and* h *are formal series in each of the cases.*

Proof: We multiply the acquired vector fields form Lemma 5.2.1 with $1 - \sigma y$. In degree 3, we get an extra term $x^2 y$. This is part of the local normal form, or can be eliminated using the procedure of Lemma 5.2.1 without adding terms of degree three dependent on y. On the higher order terms we then apply the same procedure.

When $b = 0$, we apply after each step of degree $N + 3$ with $N \geq 1$ a reparametrization by multiplying the vector field with $1 - \gamma x^N y$, where γ is the coefficient of $xh(x)$ corresponding to x^N. In this way we construct new terms of degree $N + 3$ of degree at most 1 in y, which we can again eliminate using the normalization procedure.

When $b = \frac{1}{N}$, we multiply the vector field with $1 - \alpha x^N$ after normalization up to degree $N + 2$. All changes are in higher degrees except for a term in degree $N + 2$ with degree 1 in y, which we can eliminate again by normalization. □

Using Borel's Theorem (Theorem 1.2.3), we can realize these transformations as C^∞ functions, however some flat terms arise. We will omit these from the notation, since after a blow-up procedure they contribute to a flat term in the radial variable which can be eliminated according to the results in Section 2.4.

5.2.3 Generic transition map with 1:1-resonance

By Lemma 5.2.2 we consider for $b = 0$ a vector field of the form

$$\begin{cases} \dot{x} = Ax^2 + x^3 f(x) + Bx^m y, \\ \dot{y} = x^2 + y^2 + x^3 g(x), \end{cases} \tag{5.2.12}$$

for some $m > 1$ and $B \neq 0$. Denote $f_0 = f(0)$ and $g_0 = g(0)$ and assume that $m = 2$ in order to simplify computations. We know that the ratio of the eigenvalues of both saddles is -1.
Similar as in Section 5.1, we apply a parabolic blow-up of the form

$$(x, y) = (Y(X^2 - 1), XY). \tag{5.2.13}$$

After multiplication with $\frac{X^2+1}{2Y}$, we get

$$\begin{cases} \dot{X} = \frac{1}{2}(1 - X^2)\left[1 + (1 - X^2)F(X) + F_1(X)Y + \mathrm{O}\big(Y^2\big)\right], \\ \dot{Y} = XY + (1 - X^2)G_1(X)Y + G_2(X)Y^2 + \mathrm{O}\big(Y^3\big), \end{cases} \tag{5.2.14}$$

where

$$\begin{aligned} F(X) &= AX - X^2, \\ F_1(X) &= (1 - X^2)(-g_0 X^4 + f_0 X^3 + (B + 2g_0)X^2 - f_0 X - g_0), \\ G_1(X) &= -X^3 + \frac{1}{2}AX^2 - \frac{1}{2}A, \\ G_2(X) &= -\frac{1}{2}(1 - X^2)^2(-2g_0 X^3 + f_0 X^2 + (B + 2g_0)X - f_0). \end{aligned}$$

In order to compute the dominant term in the transition map, we need to put (5.2.14) in semi-local normal form and identify the first non-zero resonant or connecting term. This is done as follows.

Lemma 5.2.3. *There exists a smooth transformation*

$$(X, Y) \mapsto (x, y) = (X, \varphi(X, Y)),$$

such that the system (5.2.14) *is orbitally equivalent to*

$$\begin{cases} \dot{x} = \frac{1}{2}(1 - x^2), \\ \dot{y} = y\left[x + (\alpha x + \beta)(1 - x^2)y + (1 - x^2)^2 y^2 \left(x\bar{f}\left((1 - x^2)y\right) + \bar{g}\left((1 - x^2)y\right)\right)\right], \end{cases}$$

for some smooth functions $\bar{f}$ and $\bar{g}$. Moreover, we have

$$\beta = B\left(1 - e^{\frac{-2A\pi}{\sqrt{4 - A^2}}}\right).$$

Proof: The existence of a smooth equivalence as stated in the lemma is immediate from Theorem 2.1.2. It remains to compute the coefficient β. This is done by repeating the first steps in the normalization procedure. Denote

$$G(X) = 1 + (1 - X^2)F(X) = \left(1 - X^2 + \frac{A}{2}X\right)^2 + \left(1 - \frac{A^2}{4}\right)X^2,$$

which is strictly positive for $A^2 < 4 = 4(1 - b)$. We divide the vector field by the factor between square brackets in (5.2.14) and apply the transformation

$$Y = \Psi(X)Z,$$

where

$$\Psi(X) = \exp\left(\int_{-\infty}^{\frac{X}{1 - X^2}} \frac{-A}{u^2 + Au + 1} du\right).$$

After a straightforward computation, one can deduce the system

$$\begin{cases} \dot{X} = \frac{1}{2}(1 - X^2), \\ \dot{Z} = XZ + H_1(X)\Psi(X)Z^2 + \mathrm{O}(Z^3), \end{cases}$$

where

$$H(X) = \frac{G_1(X) - XF(X)}{G(X)},$$

and

$$H_1(X) = \frac{G_2(X) - XF_1(X) - (1 - X^2)F_1(X)H(X)}{G(X)}.$$

By direct computation, we can see that we can decompose

$$H_1(X)\Psi(X) = (1 - X^2)(\alpha X + \beta) + (1 - X^2)^2 H_2(X),$$

for some constant α and C^∞ function H_2 and with

$$\beta = B\left(1 - e^{\frac{-2A\pi}{\sqrt{4-A^2}}}\right).$$

By a transformation of the form $(X, Z) = (X, Z_1) = (X, Z + h(X)Z^2)$ we can eliminate the term $(1 - X^2)^2 H_2(x)Z^2$. The rest of the normalization procedure of Theorem 2.1.2 is of the form

$$(X, Z_1) \mapsto (x, y) = \left(X, Z_1 + Z_1^3\psi(X, Z_1)\right),$$

for some smooth function Ψ and thus leaves the coefficient β unchanged. □

Combining Lemma 5.2.3 and Theorem 3.3.2, the transition map of (5.2.12) in the blow-up chart (5.2.13) is asymptotically given by

$$D(x_0) = -x_0 + B\left(1 - e^{\frac{-2A\pi}{\sqrt{4-A^2}}}\right)(1 - x_0^2)^2 \log(1 + x_0) + \text{h.o.t.}. \tag{5.2.15}$$

Observe that blow-up chart (5.2.14) only allows us to describe the dynamics for $x < 0$ of (5.2.12). However if we apply the reflection $x \to -x$ to (5.2.12), then (5.2.13) describes exactly the dynamics for $x > 0$. After the reflection the sign of A and B changes in (5.2.12). If we repeat the discussion above, we compute in this case that the transition map in the blow-up chart is asymptotically given by

$$D(x_0) = -x_0 - B\left(1 - e^{\frac{2A\pi}{\sqrt{4-A^2}}}\right)(1 - x_0^2)^2 \log(1 + x_0) + \text{h.o.t.}. \tag{5.2.16}$$

Combining these results and by considering the difference map Δ as in Section 5.1.4, we see that $\Delta'(x)$ locally has one zero if A and B are non-zero.

Again this can be generalized to a parameter-dependent situation. Suppose X_λ is a family of vector fields such that for all λ in an open set U there is a singularity of the form (5.2.2) with $b = 0$. Suppose X_{λ_0} has a fake saddle loop Γ and is locally equivalent to (5.2.12) with $AB \neq 0$. Then there exists a neighbourhood W of Γ such that X_λ does not contain more than two limit cycles for $\lambda \in U$.

5.2.4 Generic transition map with 1:2-resonance

When $b = \frac{1}{2}$, the saddle quantity is given by $\frac{1}{2}$, respectively 2 at the saddles after blow-up. Using Lemma 5.2.2, the vector field can locally be transformed to

$$\begin{cases} \dot{x} = Ax^2 + x^3 f(x) + \frac{1}{2}xy, \\ \dot{y} = x^2 + y^2 + x^3 g(x) + \beta x^4 y, \end{cases} \tag{5.2.17}$$

for some $\beta \in \mathbb{R}$ and some C^∞ functions f, g. Denote $f_0 = f(0)$ and $g_0 = g(0)$. After blow-up (5.2.13) and multiplying with $\frac{(X^2+1)}{Y}$, we get a vector field of the form

$$\begin{cases} \dot{X} = \frac{1}{2}(1-X^2)\left[1+(1-X^2)F(X)+F_1(X)Y+\mathrm{O}\big(Y^2\big)\right], \\ \dot{Y} = 2XY+(1-X^2)G_1(X)Y+G_2(X)Y^2+\mathrm{O}\big(Y^3\big), \end{cases} \tag{5.2.18}$$

where

$$\begin{aligned} F(X) &= 2AX - 2X^2 + 1, \\ F_1(X) &= 2(1-X^2)^2(g_0X^2 - f_0X - g_0), \\ G_1(X) &= -2X^3 + AX^2 + \frac{1}{2}X - A, \\ G_2(X) &= (1-X^2)^3(-2g_0X + f_0). \end{aligned}$$

Similar as in the case $b = 0$, we put (5.2.18) in semi-local normal form and identify the first non-zero term.

Lemma 5.2.4. *There exists a smooth transformation*

$$(X, Y) \mapsto (x, y) = (X, \varphi(X, Y)),$$

such that the system (5.2.18) *is orbitally equivalent to*

$$\begin{cases} \dot{x} = \frac{1}{2}(1-x^2), \\ \dot{y} = y\left[2x + (\alpha x + \beta)\, z + z^2\left(x\bar{f}\,(z) + \bar{g}\,(z)\right)\right], \end{cases}$$

for some smooth functions $\bar{f}$ and $\bar{g}$, where $z = (1-x^2)^2 y$. Moreover, we have

$$\beta = 2f_0\left(1 + e^{\frac{-3\pi A}{\sqrt{2-A^2}}}\right).$$

Proof: Again, we only need to compute β since the rest of the statement follows immediately from Theorem 2.1.2. We divide the vector field (5.2.18) by the factor in square brackets where we remark that

$$G(X) = 1 + (1-X^2)F(X) = 2\left(1 - X^2 + \frac{A}{2}X\right)^2 + \left(1 - \frac{A^2}{2}\right)X^2$$

is a strictly positive function since $A^2 < 2 = 4(1-b)$. Consequently we apply the transformation

$$Y = \Psi(X)Z, \text{ with } \Psi(X) = e^{\int_{-1}^{X} 2H(s)ds},$$

where

$$H(X) = \frac{G_1(X) - 2XF(X)}{G(X)}.$$

A straightforward computation shows that the vector field can now be written as

$$\begin{cases} \dot{X} = \frac{1}{2}(1 - X^2), \\ \dot{Z} = 2XZ + \big((1 - X^2)^2(\alpha X + \beta) + (1 - X^2)^3 H_2(X)\big)\, Z^2 + \mathrm{O}\big(Z^3\big), \end{cases}$$

for some constant α and C^∞ function H_2 and with

$$\beta = 2f_0 \left(1 + e^{\frac{-3\pi A}{\sqrt{2-A^2}}}\right).$$

Following the normal form procedure of Theorem 2.1.2, we can remove the term $(1 - X^2)^3 H_2(X) Z^2$ by a transformation of the form

$$(X, Y) = \left(\bar{X}, \bar{Y} + \bar{H}(\bar{X})\bar{Y}^2\right),$$

where $\bar{H}(\bar{X})$ is a smooth solution of

$$-\frac{1}{2}(1 - \bar{X}^2)\bar{H}'(\bar{X}) - 2\bar{X}\bar{H}(\bar{X}) + (1 - \bar{X}^2)^3 H_2(\bar{X}) = 0,$$

without changing the coefficients α and β. The higher order terms (with respect to $\bar{Y}$) will then be put in normal form without changing the terms of degree 2. $\square$

By Lemma 5.2.4 and Theorem 3.3.2, we get that the transition map in the blow-up chart (5.2.13) for $x < 0$ is asymptotically given by

$$D(x_0) = -x_0 - f_0 \left(1 + e^{\frac{-3\pi A}{\sqrt{2-A^2}}}\right) (1 - x_0^2)^3 \log(1 + x_0) + \text{h.o.t.},$$

and a similar map for $x > 0$ where A is replaced by $-A$. In a similar way as at the end of Section 5.1.4, we can see that there exists a neighbourhood of 0 where the second derivative of the displacement map Δ has at most one zero when $f_0 \neq 0$.

Consider a family of vector fields X_λ, $\lambda \in U$, with a singularity of the form (5.2.2) with $b = \frac{1}{2}$. Suppose X_{λ_0} has a fake saddle loop Γ and is locally equivalent to (5.2.17) with $f_0 \neq 0$. Then there exists a neighbourhood W of Γ such that X_λ does not contain more than three limit cycles in W for $\lambda \in U$.

Overview and open questions

The primary focus in this thesis was to examine the dynamics near non-elementary singularities as we did in Chapter 5. After a blow-up procedure (see Figure 5.1 and Figure 5.5), these singularities have a similar behaviour near the blow-up locus. More precise, the invariant manifolds of two saddles on the blow-up locus locally divide the phase space in two hyperbolic sectors. Moreover the ratio of the eigenvalues at the saddles are reciprocal. Therefore, in some well-chosen charts, we can consider vector fields of the form

$$\begin{cases} \dot{x} = (1-x^2)\left(\frac{q}{2} + \mathrm{O}\big(1-x^2\big)\right) + \mathrm{O}\big(y\big), \\ \dot{y} = \big(px + \mathrm{O}\big(1-x^2\big)\big)\, y + \mathrm{O}\big(y^2\big). \end{cases}$$

In order to simplify the system near the saddle connection on the blow-up locus, it is natural to construct a formal transformation per induction on the degree of y as we did in Section 2.3, since this denotes the radial component denoting the distance to the blow-up locus and thus the singularity. In contrast, the typical Poincaré-Dulac normalization relies on an induction scheme on the homogeneous degree of both (x, y) (see Section 1.3.1), since the distance to the singularity relies on both variables. The algebraic condition induced by the adjoint action of the linear part is then replaced by an ordinary differential equation (see (2.3.4) and (2.3.5)). In order to solve this in a smooth way, it is qualitatively equivalent to construct a smooth connection of the form $y = \varphi(x)$ through two nodes. At some degrees (resonant degrees), every invariant manifold is smooth up to some logarithmic term due to resonance (see Lemma 2.3.5). Otherwise there exists only one smooth invariant manifold at each of the nodes, which we can connect by adding some well-chosen function (see Lemma 2.3.13). The corresponding connecting terms are non-unique as we discussed in Section 2.3.3. An interesting question is to determine the intrinsic property of a vector field connected to these resonant terms. As we have seen in the applications, the coefficient of the dominant term in the transition map does not rely on the choice of connecting function χ. Since we expect rational exponents, demanding C^∞ normal forms may be too restrictive.

Eventually, after realisation by Borel's Theorem (Theorem 1.2.5) and removal of the flat terms (Section 2.4), we end up with a simplified vector field which is valid near the connection. These semi-local normal forms can be constructed for any value of saddle quantity (Theorems 2.1.2, 2.1.3 and 2.1.4) as long as we assume that they are fixed. We do allow perturbations in higher-order terms.

We reduce the system even further to an easily integrable system by allowing transformations which are only finitely smooth (see Theorem 3.2.1). The non-smooth terms admit nice expressions in the form of logarithmic tags and fractional exponents which both are of Mourtada type near the axes $x = \pm 1$. By exploiting the first integral of the linearized system (see Corollary 3.3.1), we can deduce the smooth dependence of the transition map along the connection in terms of these Mourtada type functions and identify the first non-zero higher order term (see Theorem 3.3.2). From the proof of this theorem (see (3.3.20)), it becomes clear that the first non-zero symmetric term in the semi-local normal (2.1.3), (2.1.6) or (2.1.7) plays a dominant role. This can be logarithmic when the transition trough the saddles themselves is dominant (see (3.3.4)) or given by a term with a fractional exponent when the transition in between the saddles (composed with the Dulac map of one of the saddles) is dominant (see (3.3.5)). This is directly connected to whether the first non-zero symmetric term in the normal form is resonant or connecting. In Chapter 5, we demonstrate both behaviours and we show how we can use this results to derive a partial cyclicity results. If we want to deduce cyclicity results for arbitrary perturbations preserving the nature of the singularity as in [41], we should consider the full transition map up to some degree and apply a division-derivation algorithm. This will be part of future work.

If we also allow a perturbation, although only symmetric, in the eigenvalues of the saddles, it becomes clear that similar results as above remain true (see Chapter 4). However we can only expect the semi-local normal form in this case (see Theorem 4.1.8) to be finitely smooth similar as for a family of vector fields having a saddle (see Section 1.3.2). As in Theorem 3.2.1, we can reduce this system even further by inductively defining terms of Mourtada type. However due to the perturbation in the eigenvalues, the tags are given by an adapted compensator. It is not yet clear how they compare to the typical Ecalle-Roussarie compensators at the saddles and we can not decompose them as in Proposition 3.2.2 since the non-smoothness of the tags is not confined to only one point.

Finally we illustrate in Chapter 5 how to apply the results of this thesis. The technique used here is to manually compute the normal form up to some degree and use Theorem 3.3.2 to see the effect on the transition map. However, since Corollary 3.3.1 provides an expression for an invariant, we can assume a shape of the invariant with undetermined coefficients as ansatz and establish equations for all coefficients appearing therein. Similarly we can exploit the fact that the transition map can be expanded in terms of a finite number of variables. In this way one can provide conditions for a predetermined family of vector field (for instance cubic vector fields having some properties) for which the family has at most $1, 2, \ldots$ limit cycles and try to give explicit parameter values for which this bound is reached.

In the examples, the saddle connection appears after blowing up a singular point which is initially nilpotent or degenerate. The saddle connection on the blow-up locus can be predicted by examining the dominant quasi-homogeneous part of the vector field. Under the condition that the saddles are the only two singularities on the blow-up locus, one might wonder what the relation is between the semi-local normal form after blow-up like we obtain in Chapter 2 and the normal forms obtained from information of the quasi-homogeneous part (before blow-up) like in [35]. This is an open question.

In [41], the transition map is computed in a very elegant and short way: instead of computing the transition map along the real axis, the authors use a complex path and use the monodromy of the two individual saddles to prescribe the dominant term of the transition map. In comparison to [41], our technique is technically more involved but at the same time it is more straightforward and applicable to the general setting. More precisely, when considering a versal unfolding of a cusp as in [21], one needs to consider a three-dimensional family blow-up. We believe that the semi-local normal forms obtained in this thesis can simplify the vector field near the blow-up sphere. This will be subject of further research.

The case of the fake saddle is more involved. The saddle quantities are dependent on b nonetheless that they stay reciprocal. It is worth the effort to compute the full expansion of the transition map in terms of the compensator-like Mourtada type functions appearing in Theorem 4.2.5. The cyclicity results of homoclinic connections of a fake saddle can in this case also include perturbations in the parameter b.

Generically, when one perturbs a saddle connection without breaking the connection itself, one loses the symmetry of the eigenvalues. Therefore, it might be of interest to also compute normal forms for asymmetric pairs of eigenvalues, starting from the scalar vector field (2.2.4) on the connection, where $A \neq 0$.

Nederlandstalige samenvatting

Het doel van deze thesis is om technieken te voorzien die ons in staat stellen om de dynamica van gladde vlakke vectorvelden te bestuderen in de buurt van niet-elementaire singulariteiten. Doorgaans wordt dit gedaan door het opblazen van de singulariteit waar deze vervangen wordt door een cirkel die de richtingen beschrijft waarin men de singulariteit kan benaderen, genaamd de opblaaslocus. In veel gevallen verkrijgen we zo twee hyperbolische zadels op de opblaaslocus met wederkerige verhoudingen van eigenwaarden. Om de transitie in de buurt van de singulariteit te bepalen, moeten we dus de transitie door een zadel-connectie berekenen die de bovengenoemde symmetrie heeft.

De vectorvelden die verschijnen in de opgeblazen faseruimte zijn van de vorm

$$\begin{cases} \dot{x} = (1-x^2)\left(\frac{q}{2} + \mathrm{O}\big(1-x^2\big)\right) + \mathrm{O}\big(y\big), \\ \dot{y} = \big(px + \mathrm{O}\big(1-x^2\big)\big)\, y + \mathrm{O}\big(y^2\big), \end{cases}$$

waar y de radiale richting weergeeft en x de rol speelt van de angulaire variabele. De getallen p en q zijn rechtstreeks verwant met het spectrum van de hyperbolische zadels in de punten $(\pm 1, 0)$. We veronderstellen dat de separatrix-connectie onverbroken blijft.

We starten met het verstrekken van een gladde semi-lokale normaalvorm waaraan het originele vectorveld equivalent is op een gladde manier. Deze equivalentie is geldig in een omgeving van de connectie. We onderscheiden verschillende gevallen afhankelijk van de lokale resonanties van de zadels. Dit wordt op een gelijkaardige manier gedaan als de Poincaré-Dulac normalizatie. Eerst construeren we een formele conjugatie nabij de connectie. In deze inductieve procedure verkrijgen we oftewel resonante termen die ook voorkomen in de lokale normaalvormen oftewel connectiviteitstermen. Deze laatste termen zijn noodzakelijk om de lokale normaalvormen aan elkaar te plakken op een gladde manier. Vervolgens realizeren we deze transformatie als een gladde coördinaatsverandering door gekende lokale resultaten te veralgemenen.

In een volgende stap normaal-lineariseren we de verkregen normaalvormen. Hiermee bedoelen we dat we die termen elimineren die niet linear zijn in functie van de variabele die de loodrechte richting op de connectie voorstelt, i.e. de y-variabele. Hierdoor verliezen we wel de gladheid van de transformatie. Desalniettemin kunnen we deze transformatie uitdrukken als een gladde functie van enkele eindig gladde tags met een logaritmische vorm en termen met fractionele machten. Doordat het normaal-gelineariseerde systeem een eenvoudige eerste integraal heeft, kunnen we hierdoor een invariant definiëren van het originele vectorveld in normaalvorm. Deze invariant leidt op zijn beurt naar een methode om de transitie-afbeelding te berekenen door de zadelconnectie.

Uiteindelijk passen we deze resultaten toe op enkele voorbeelden. Eerst moeten de vectorvelden wel aangepast worden zodat ze geschikt zijn om opgeblazen en genormalizeerd te worden. We gebruiken de typische kaarten van een (quasi-)homogene opblazing als ook een parabolische opblazing. Deze constructie is noodzakelijk wanneer de singulariteiten niet te bevatten zijn in één kaart. Ze wordt verkregen door een stereografische projectie van de opblaascirkel waar we de variabelen niet herschalen. Door voorgaande resultaten toe te passen, verkrijgen we de transitie-afbeeldingen nabij de onderzochte singulariteiten en verwerven zo gedeeltelijke cycliciteitsresultaten voor grafieken met enkel één singulariteit die de vorm onder beschouwing heeft.

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