LIMBURGS UNIVERSITAIR CENTRUM

FACULTEIT WETENSCHAPPEN

LOCAL BIFURCATIONS OF QUADRATIC VECTOR FIELDS

proefschrift
tot het bekomen van de graad van
doctor in de wetenschappen
door

Peter Fiddelaers

Promotor

Prof. dr. F. Dumortier

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To my family

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Contents

Chapter 1

Introduction and Preliminaries.

1.1 Introduction

Consider two-dimensional differential systems of the form

$$\begin{cases} \dfrac{dx}{dt} & = P(x, y, \lambda) \\ \dfrac{dy}{dt} & = Q(x, y, \lambda) \end{cases} \tag{1.1}$$

in which P, Q are polynomials and $\lambda \in I\!R^k$.
As the parameter λ is varied changes may occur in the qualitative structure of the solutions for certain parameter values . These changes are called bifurcations and the parameter values are called bifurcation values . In this work we shall focus upon the bifurcation of equilibria . Since the analysis of such bifurcations is performed by studying the vector field near the equilibrium point these bifurcations are referred to as local .
The problems we study are motivated by a desire to catalog all bifurcations which occur in quadratic systems as a step toward solving Hilbert's sixteenth problem on the maximum number of limit cycles .
In chapters 2 and 3 we treat the following question : " How complicated can local bifurcations be in quadratic systems or in other terms , which phenomena can be described by germs of families of quadratic vector fields on the plane ? ".
We focus on two aspects of this problem . The first one considers the singularities themselves , emphasizing their codimension as a quadratic system and as a general system . The second one concerns the relation between the quadratic unfolding of

the singularity and its (generic) unfolding as a general system.
The treatment of the first problem is described in chapter 2 . We make a distinction between singularities having a finite codimension or having infinite codimension as a general system . The singularities of the first kind are listed completely , while those of the second kind are only checked to be non-isolated , or isolated but hamiltonian , or having an axis of symmetry after a linear coordinate change , or approachable by centers or integrable saddles (these cases are of course not mutually distinct).
Since a purely linear singularity of the first kind is necessarily hyperbolic , we can-restrict to non-hyperbolic singularities of degree exactly two . The semi-hyperbolic ones of the first kind have a codimension at most 3 . The same happens with the singularities having an infinitisimal rotation as 1-jet , as well with the nilpotent singularities of cusp type . For all these singularities we give in chapter 3 a quadratic unfolding representing a versal one (versal among general systems) .
As the bifurcation diagram of the cusp of codimension 3 is not algebraic (and even not analytic) , we obtain a negative answer to a question raised by Coppel in [Cop] asking to characterize the phase portraits of quadratic systems by means of algebraic inequalities on the coefficients.
Nilpotent singularities of saddle and elliptic type occur in codimension 3 and 4. The elliptic points of codimension 4 and type 2 are conjectured to be of infinite codimension. One also encounters nilpotent singularities of saddle-node type of codimension 4. The remaining nilpotent singularities are of the second kind .
In chapter 3 we give a quadratic example of a generic unfolding for all saddles of codimension 3 and 4 , for all elliptic points of codimension 3 and for all saddle-nodes of codimension 4. We show that the elliptic points of codimension 4 and type 1 do not have a quadratic generic unfolding . Essentially the reason is that the nilpotent focus of codimension 3 can not be given a quadratic model.
In chapter 4 we consider the generic unfolding of the nilpotent saddle of codimension 4. In a first section we examine which local bifurcations occur in this unfolding. The fact that there are no Hopf bifurcations of codimension ≥ 3 leads to our conjecture that two is the maximum number of limit cycles which bifurcate out of such a singularity. In a second section we propose a bifurcation diagram.
Many proofs in this work essentially consist of formal calculations - calculations of normal forms - which were performed on an Apollo 4000 using Macsyma , and using programs from Rand and Armbruster [RA].

1.2 Preliminaries

1.2.1 Definitions

Definition

A singularity of a C^k vector field on $\mathbb{R}^n$ is a triple $(\mathbb{R}^n, X, p)$ such that X is a C^k vector field on $\mathbb{R}^n$ with $X(p) = 0$.

Definition

Two vector fields X and Y on $\mathbb{R}^n$ with $X(0) = Y(0) = 0$ are **germ-equivalent** in 0 if they coincide on some neighborhood of 0.

The equivalence classes for this equivalence relation are called **germs** of vector fields in 0.
Let G^n denote the set of germs of C^∞ vector fields on $\mathbb{R}^n$ in 0.

Definition

Let $\tilde{X}, \tilde{Y} \in G^n$. Then $\tilde{X}$ and $\tilde{Y}$ are **(k-jet)-equivalent** if for some (and hence for all) representatives X and Y of resp. $\tilde{X}$ and $\tilde{Y}$, we have $X - Y = O(||x||^{k+1})$, i.e.

$$\exists c, \delta > 0 \quad s.t. \quad ||X(x) - Y(x)|| < c||x||^{k+1} \qquad \forall ||x|| < \delta$$

An equivalence class for this equivalence relation is called a **k-jet** of a germ of a C^∞ vector field in 0 (or a k-jet of a C^∞ vector field in 0).
We denote it by $j_k(\tilde{X})(0)$ or even $j_k(X)(0)$.

If we choose coordinates on $\mathbb{R}^n$, then a k-jet of a vector field X is nothing else than the set of partial derivatives up to order k of the component functions of X in 0.
By this , it is clear that there exists a natural (1-1) correspondance between the set J_k^n of k-jets of vector fields on $\mathbb{R}^n$ in 0 and the space of vector fields Y on $\mathbb{R}^n$ such that $Y(0) = 0$, and such that the component functions of Y are polynomials of degree $\leq$ k.
This correspondance induces on J_k^n an $\mathbb{R}$-vector space-structure , as well as a natural Euclidean topology .

We also consider the following mappings :

$$j_k : \begin{cases} G^n \mapsto J^n_k \\ \tilde{X} \to j_k(\tilde{X})(0) \end{cases} \quad \text{and} \quad \Pi_{lk} : \begin{cases} J^n_l \mapsto J^n_k \\ j_l(\tilde{X})(0) \to j_k(\tilde{X})(0) \end{cases} \quad \text{for} \quad l \geq k$$

Since $\Pi_{lk} \circ \Pi_{ml} = \Pi_{mk}$ for $m \geq l \geq k$ and $\Pi_{ll} = Id$, we can define the inverse limit of the sets J^n_l for the mappings Π_{lk}. We denote it by J^n_∞ and we call its elements ∞-jets.

We also have a map $j_\infty : G^n \mapsto J^n_\infty$, which is the inverse limit of the j_k , and we denote the image of $\tilde{X} \in G^n$ by $j_\infty(\tilde{X})(0)$. This is in fact nothing else than the Taylor expansion of $\tilde{X}$ in 0.

Let us also consider the mappings $\Pi_k : J^n_\infty \mapsto J^n_k$, $j_\infty(\tilde{X})(0) \to j_k(\tilde{X})(0)$.
On G^n (resp. J^n_∞) we take the coarsest topology for which the projections j_k (resp. Π_k) are continous.

Definition
A set $A \subset J^n_k$ is **semi-algebraic** if it is the union of a finite number of sets which can be defined by polynomial equalities and polynomials inequalities. (For the definition of the codimension of a semi-algebraic subset we refer to [D1]).

A **semi-algebraic** subset $A \subset G^n$ (resp. J^n_∞) is a subset which for some k is of the form $A = j_k^{-1}(A_k)$ (resp. $A = \Pi_k^{-1}(A_k)$) , where A_k is some-algebraic subset of J^n_k.

In the space of germs G^2 we consider the action of the group of germs of diffeomorphisms fixing 0 in $I\!R^2$ (C^∞**-conjugacy** defined by $g^*X(x) = (dg_x)^{-1}X(g(x))$) as well as the action of the group of pairs (f,g) consisting of the germ of a strictly positive function and the germ of a diffeomorphism fixing 0 (C^∞**-equivalence**). This last action is defined by $((f,g).X)(x) = f(x)g^*X(x)$, and the group operation by $(f,g).(f',g') = (f.(f' \circ g), g' \circ g)$.
These differentiable actions on the germs induce algebraic actions on each space J^2_k. In a fixed J^2_k the subset of jets conjugate or equivalent to a certain given jet (This means an orbit of one of the given group-actions.) forms a submanifold ; the set of jets conjugate or equivalent to the jets belonging to a given semi-algebraic subset

forms a semi-algebraic subset (Seidenberg-Tarski [Ta1]).

Using these operations we are going to subdivide G^2 into semi-algebraic subsets. Define

$$\begin{aligned} W_1 = & \{X \mid Sp(j_1(X)(0)) \cap i\mathbb{R} = \emptyset\} \\ W_2 = & \{X \mid \Delta(j_1(X)(0)) = 0, Tr(j_1(X)(0)) \neq 0\} \\ = & \{X \mid Sp(j_1(X)(0)) = \{0, \lambda\} \quad \text{with} \quad \lambda \neq 0\} \\ W_3 = & \{X \mid \Delta(j_1(X)(0)) > 0, Tr(j_1(X)(0)) = 0\} \\ = & \{X \mid Sp(j_1(X)(0)) = \{\pm i\lambda\} \quad \text{with} \quad \lambda \neq 0\} \\ W_4 = & \{X \mid \Delta(j_1(X)(0)) = 0, Tr(j_1(X)(0)) = 0, j_1(X)(0) \neq 0\} \\ = & \{X \mid Sp(j_1(X(0)) = \{0\}, j_1(X)(0) \neq 0\} \\ W_5 = & \{X \mid j_1(X)(0) \equiv 0\} \end{aligned}$$

where $\Delta(j_1(X)(0))$ is the determinant , $Tr(j_1(X)(0))$ is the trace and $Sp(j_1(X)(0))$ is the spectrum of the matrix of first partial derivatives of the component functions of X in 0.
The sets W_1, W_2, W_3, W_4, W_5 are of codimension 0,1,1,2,4.

By the Hartman-Grobman theorem (see [PdM]) we know that the $X \in W_1$ are topological stable. They form the **codimension 0** singularities.

If $X \in W_5$ we say that X has **codimension 4**. (Actually it would be better to say that X has codimension ≥ 4.)

We are going to subdivide the sets W_j $(j = 2, 3, 4)$ into sets W_{ji} by looking at the higher order jets. For this construction we work with a set of ∞-jets in Takens normal form (see section 3 of this paragraph): W_j'.
We shall define sets $W_{ji}' \subset W_j'$ such that

$$\begin{aligned} W_{ji} = & \{X \mid \exists \text{ diffeomorphism } g \text{ (or a pair } (f,g)) \text{ with } g : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) \\ & \text{such that } g^*X \in W_{ji}' \quad (\text{or} \quad (f,g).X \in W_{ji}')\} \end{aligned}$$

The actual definitions are:

$$W_2' = \{X \mid j_\infty(X)(0) = (\sum_{i=2}^{\infty} a_i x^i)\frac{\partial}{\partial x} + (\lambda + \sum_{i=1}^{\infty} b_i x^i) y \frac{\partial}{\partial y} \quad \text{with} \quad \lambda \neq 0\}$$

Define $W_{21}' = W_2'$ and $W_{2i}' = \{X \in W_2' \mid a_j = 0 \ \forall j \leq i\}$, $\forall i \geq 2$.
If $X \in W_{2i} \backslash W_{2(i+1)}$ we say that X is a **semi-hyperbolic singularity of codimension i**.
If $X \in W_{2i}$, $\forall i$, we say that X is a **semi-hyperbolic of codimension** ∞.

$$\begin{aligned} W_3' = \{X \mid j_\infty(X)(0) = \ & [\lambda + \sum_{i=1}^{\infty} a_i (x^2+y^2)^i](x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}) \\ & + [\sum_{j=1}^{\infty} b_j (x^2+y^2)^j](x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}) \quad \text{with} \quad \lambda \neq 0\} \end{aligned}$$

Define $W_{31}' = W_3'$ and $W_{3i}' = \{X \in W_3' \mid b_j = 0 \quad \forall j \leq i-1\}$, $\forall i$.
If $X \in W_{3i} \backslash W_{3(i+1)}$ we say that X is a **Hopf singularity of codimension i.**
If $X \in W_{3i}$, $\forall i$, we say that X is of ∞ **codimension.**

$$W_4' = \{X \mid j_\infty(X)(0) = y\frac{\partial}{\partial x} + (\sum_{i=2}^{\infty} a_i x^i + y \sum_{i=1}^{\infty} b_i x^i)\frac{\partial}{\partial y}\}$$

Define $W_{41}' = \{X \in W_4' \mid a_2 \neq 0\}$ and $W_{42}' = \{X \in W_4' \mid a_2 = 0\}$. If $X \in W_4$, then we say that X is (-or better has at the origin a-) singularity of cusp-type. This is a first subdivision of W_4'. In chapter 2 we will refine this subdivision.

Definition
$\tilde{X}, \tilde{Y} \in G^n$ are **topologically** (or C^0-) **equivalent** if for some (and hence for all) representatives X, Y of resp. $\tilde{X}, \tilde{Y}$, there exist neighborhoods U and V of 0 in $\mathbb{R}^n$ and a homeomorphism $h : U \to V$, mapping orbits of X to orbits of Y.

If we denote the flow of a vector field X by $\phi_X : D \subset \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, then the condition in this definition means that :

If $x \in U$ and $\phi_X(x,[o,t]) \subset U$ for $t > 0$, then there is some $t' > 0$ such that $h(\phi_X(x,[0,t])) = \phi_Y(h(x),[0,t'])$.

Linked to this notion is the stronger notion of C^0-conjugacy.

Definition
$\tilde{X}, \tilde{Y} \in G^n$ are called **C^0-conjugate** if there exist a homeomorphism h from a neighborhood V of 0 onto a neighborhood W of 0 such that $h(\phi_X(x,t)) = \phi_Y(h(x),t)$ (as long as we remain in V , resp. W).

Definition
A C^r **k-parameter family of vector fields** on $I\!R^n$, (X_μ) , where $\mu \in I\!R^k$ denotes the parameter , is defined to be a vector field

$$X_\mu(x_1,\ldots,x_n) = \sum_{i=1}^{n} X_i(x_1,\ldots,x_n,\mu)\frac{\partial}{\partial x_i}$$

where the coefficient functions X_i are C^r with respect to $(x_1,\ldots,x_n,\mu) \in I\!R^n \times I\!R^k$.

When X_μ is defined only on a neighborhood of $(0,0) \in I\!R^n \times I\!R^k$ we call it a **local family**. If the neighborhood does not matter we consider **germs of families**. Such a germ (X_μ) will be called a k-parameter **unfolding** of X_o.

Definition
Let (X_μ) be a k-parameter family on $I\!R^n$. The **bifurcation set** Σ of the family (X_μ) is defined to be the complement in $I\!R^k$ (or some neighborhood of $0 \in I\!R^k$) of the values μ , for which there exist neighborhoods U_μ of $0 \in I\!R^n$ and a neighborhood V_μ of μ in $I\!R^k$ such that $\forall \mu' \in V_\mu$ there exsist homeomorphism $h_{\mu'} : U_\mu \to U_\mu$ mapping orbits of X_μ to orbits of $X_{\mu'}$.

Definition
Two k-parameter families (X_μ) and (Y_μ) on the same space of parameters $I\!R^k$ are called **fiber-C^0-equivalent over the identity** or **(fiber-C^0,Id)-equivalent** if

there exist homeomorphisms h_μ such that for each $\mu \in I\!R^k$, h_μ is a C^0-equivalence between the vector fields X_μ and Y_μ.

If h_μ depends continously on μ we say that the families (X_μ) and (Y_μ) are C^0-equivalent over the Id , or (C^0, Id)-equivalent . We skip ' over the Id ' if we also admit a change in the parameter space , i.e. h_μ is a C^0-equivalence between X_μ and $Y_{k(\mu)}$ with k a homeomorphism in the parameter space . When k is a C^r-diffeomorphism $(0 \leq r \leq \infty)$ we speak about a (fiber-C^0,C^r)-equivalence or a (C^0, C^r)-equivalence.

Remarks :

1. In the same way one can define similar notions with fiber-C^0-conjugacy and C^0-conjugacy.

2. For local families around $(0,0)$ one imposes the conditions that $h_o(0) = 0$ and $h_\mu(x)$ must only be defined for (x, μ) belonging to a neighborhood $V \times W$ of $(0,0)$ in $I\!R^n \times I\!R^k$, with $\{(h_\mu(x), \mu) | (x, \mu) \in V \times W\}$ a neighborhood of $(0,0)$. So these relations induce equivalence relations for local families (and also for germs of families).

Definition

If $\phi : (I\!R^l, 0) \mapsto (I\!R^k, 0)$ (or ϕ defined only on a neighborhood of 0) is a C^r-mapping $(0 \leq r \leq \infty)$ and (X_μ) is a family with parameter $\mu \in I\!R^k$ (defined on a neighborhood of 0) we call **family C^r-induced** by ϕ the family $Y_\epsilon := X_{\phi(\epsilon)}$ with parameter $\epsilon \in I\!R^l$.

Definition

An unfolding (X_μ) of X_0 is called a (C^0, C^r)-**versal unfolding** if all unfoldings of X_o are C^0-equivalent over the identity to an unfolding C^r-induced from (X_μ).

In the same way one has the notion of (fiber-C^0,C^r)-versal unfolding.

A versal unfolding of a singularity of a vector field $(X, 0)$, if it exists does not only describes all the singularities near $(X, 0)$, it describes also the possible transitions between these singularities as well as the global phenomena emanating from the singularity.

The notion of versal unfolding leads to a specification of the codimension of a singularity.

Definition (codimension of a singularity)
Let $(\mathbb{R}^n, X, p)$ be a singularity for which there is a versal unfolding known. Then we define the codimension of the singularity $(\mathbb{R}^n, X, p)$ as the minimal number of parameters necessary to describe a versal unfolding.

1.2.2 Center manifolds

Vector fields

For a linear vector fields L on $\mathbb{R}^n$ there exists a decomposition $\mathbb{R}^n = E^s \oplus E^u \oplus E^c$, invariant under e^{tL} , such that the eigenvalues of $L^s = L|E^s$ have a negative real part , those of $L^u = L|E^u$ a positive real part and those of $L^c = L|E^c$ a zero real part.
We call L **semi-hyperbolic** if $dim(E^s \oplus E^u) \neq 0$ and $dim(E^c) \neq 0$.

Theorem [HPS]
Let $(\mathbb{R}^n, X, 0)$ be a C^r singularity of a vector field , $r \in \mathbb{N}^* = \mathbb{N}\backslash\{0\}$,
with $E^s \oplus E^u \oplus E^c$ the decomposition associated to DX_o and with DX_o semi-hyperbolic at 0 . Then there exists C^r-manifolds W^s, W^u, W^c containing 0 , invariant under the flow of X such that
W^s is tangent to E^s at 0 and $j_1(X|W^s)(0) = DX_o|E^s$
W^u is tangent to E^u at 0 and $j_1(X|W^u)(0) = DX_o|E^u$
W^c is tangent to E^c at 0 and $j_1(X|W^c)(0) = DX_o|E^c$

W^s is called the stable manifold , W^u the unstable manifold , W^c the center manifold of X . W^s and W^u are unique , while W^c is not necessarily unique . If X is C^∞ , then W^s and W^u are C^∞ while W^c is not necessarily C^∞.

Theorem ([PS],[PT])

Let X and W^c be like in the previous theorem with $dim(W^c) = m$.
Then there exists a $p, 0 \leq p \leq n-m$, such that the germ of X at 0 is C^0-conjugate to the germ of

$$X' = \sum_{i=1}^{m} \tilde{X}_i(z_1,\ldots,z_m)\frac{\partial}{\partial z_i} + \sum_{i=m+1}^{m+p} z_i\frac{\partial}{\partial z_i} - \sum_{i=m+p+1}^{n} z_i\frac{\partial}{\partial z_i}$$

where $(z_1,\ldots,z_m)$ is a coordinate system on W^c , $(z_1,\ldots,z_n)$ is a coordinate system on $\mathbb{R}^n$ extending $(z_1,\ldots,z_m)$ and $\sum_{i=1}^{m}\tilde{X}_i\frac{\partial}{\partial z_i} = X|W^c$.
Moreover , if

$$Y = \sum_{i=1}^{m} \tilde{Y}_i(z_1,\ldots,z_m)\frac{\partial}{\partial z_i} + \sum_{i=m+1}^{m+p} z_i\frac{\partial}{\partial z_i} - \sum_{i=m+p+1}^{n} z_i\frac{\partial}{\partial z_i}$$

and if $\sum_{i=1}^{m}\tilde{Y}_i\frac{\partial}{\partial z_i}$ is C^0-equivalent (resp. C^0-conjugate) to $\sum_{i=1}^{m}\tilde{X}_i\frac{\partial}{\partial z_i}$, then X is C^0-equivalent (resp. C^0-conjugate) to Y.

Families of vector fields

Let now (X_μ) be a C^∞ family on $\mathbb{R}^n$ with $X_o(0) = 0$, $\mu \in \mathbb{R}^k$ and $E^u \oplus E^s \oplus E^c$ the invariant decomposition associated to $DX_o(0)$. One can consider X_μ as a vector field on $\mathbb{R}^k \times \mathbb{R}^n$ with $\mathbb{R}^k$-component zero .
The spaces $E^u, E^s, \mathbb{R}^k \oplus E^c$ are respectively the unstable , stable and center spaces of the linearization of this field . So $\forall l \geq 1$ there exists a manifold W^c of class C^l in $\mathbb{R}^k \times \mathbb{R}^n$ of dimension $m+k$ and tangent at 0 to $\mathbb{R}^k \oplus E^c$. Let $\tilde{X}_\mu$ be the restriction of X_μ to this manifold , $(\tilde{X}_\mu)$ is a family on $\mathbb{R}^m$ with parameters $\mu \in \mathbb{R}^k$.

Theorem

Let X_μ be a germ of a family at $0 \in \mathbb{R}^n$ with parameters $\mu \in \mathbb{R}^k$ such that the center manifold W^c of X_o at 0 has dimension $m, 0 \leq m \leq n$. Then

(1) $\exists p, 0 \leq p \leq n-m$ such that the germ of the family X_μ is C^0-conjugate over the identity to the germ of

$$X'_\mu = \tilde{X}_\mu(y_1,\ldots,y_m) + \sum_{i=m+1}^{m+p} y_i\frac{\partial}{\partial y_i} - \sum_{i=m+p+1}^{n} y_i\frac{\partial}{\partial y_i}$$

where $(\mu_1, \ldots, \mu_k, y_1, \ldots, y_n)$ *is a coordinate system on* $\mathbb{R}^k \times \mathbb{R}^n$ *, extending a coordinate system on* W^c.

(2) If

$$Y_\mu = \sum_{i=1}^{m} \tilde{Y}_i(y_1, \ldots, y_m, \mu) + \sum_{i=m+1}^{m+p} y_i \frac{\partial}{\partial y_i} - \sum_{i=m+p+1}^{n} y_i \frac{\partial}{\partial y_i}$$

and if $\tilde{Y}_\mu = \sum_{i=1}^{m} \tilde{Y}_i \frac{\partial}{\partial y_i}$ *is (fiber-*C^0*)-equivalent (resp. (fiber-*C^0*)-conjugate) to* $\tilde{X}_\mu$ *, then* Y_μ *is (fiber-*C^0*)-equivalent (resp. (fiber-*C^0*)-conjugate) to* X_μ*.*

Hence , if $\tilde{X}_\mu$ is a versal unfolding of $\tilde{X}_o$, then X_μ is a versal unfolding of X_o.

1.2.3 Normal form theorems

Normal forms for vector fields

Let X be a C^r vector field on $\mathbb{R}^n$,defined on a neighborhood of 0 , with $X(0) = 0$ and $DX(0) = L$, $r \in \mathbb{N}^* \cup \{\infty\}$.
Let H^h denote the vector space of polynomial vector fields on $\mathbb{R}^n$ which are homogeneous of degree h.
Let $[L, -]_h : H^h \mapsto H^h$ be the linear map which assigns to each $Y \in H^h$ the Lieproduct $[L, Y]$.
Recall $[\sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}, \sum_{i=1}^{n} Y_i \frac{\partial}{\partial x_i}] = \sum_{i=1}^{n} (\sum_{j=1}^{n} X_j \frac{\partial Y_i}{\partial x_j} - Y_j \frac{\partial X_i}{\partial x_j}) \frac{\partial}{\partial x_i}$.
We now consider the splitting $H^h = B^h \oplus G^h$, where $B^h = Im([L, -]_h)$ and G^m is some complement.

Theorem[T1]
Let X, L, B^h *and* G^h *be as above . Then there is a* C^∞*-diffeomorphism* $\phi : (\mathbb{R}^n, 0) \mapsto (\mathbb{R}^n, 0)$ *such that* $X' = \phi_*(X)$ *is of the form :*

$$X' = L + g_2 + \ldots + g_r + R_r$$

where $g_i \in G^i$ *,* $\forall i = 2, \ldots, r$ *and* $j_r(R_r)(0) \equiv 0$.

Remark

The proof can be found in [GH] . It is constructive , and it can be used to implement the calculations in specific examples.

Normal form for unfoldings

Concerning normal forms for families (X_μ) on $\mathbb{R}^n$ one can do something similar.
Let $L = DX_o(0)$ and $\overline{J}^n_k$ the space of k-jets of vector fields on $\mathbb{R}^n$ which are not necessarily zero at the origin.
Consider the map $[L, -]_k : \overline{J}^n_k \mapsto \overline{J}^n_k$. $\overline{B}^k$ is again defined to be the image of $[L, -]_k$ and $\overline{G}^k$ is some complementary subspace : $\overline{B}^k \oplus \overline{G}^k = \overline{J}^n_k$.

Theorem[T3]

Let X be a C^∞ p-parameter family of vector fields on $\mathbb{R}^n$ (defined in the neighborhood of 0) , then for $l \in \mathbb{N}^* \cup \{\infty\}$ and for every $k \in \mathbb{N}^* \cup \{\infty\}$ there exists a C^∞-diffeomorphism $\phi : \mathbb{R}^p \times \mathbb{R}^n \mapsto \mathbb{R}^p \times \mathbb{R}^n$ with $\Pi \circ \phi = \Pi$ (for $\Pi : \mathbb{R}^p \times \mathbb{R}^n \mapsto \mathbb{R}^p, (\lambda, x) \to \lambda$) such that the l-jet of $\phi_*(X) = \tilde{X}$ has the form :

$$j_l(\tilde{X})(0) = j_l(X_o)(0) + \sum_{i_1 \ldots i_p, \sum_j i_j = 1}^{\sum_j i_j = l} \lambda_1^{i_1} \ldots \lambda_p^{i_p} Z_{i_1 \ldots i_p} + O(||\lambda||)O(||x||^{k+1})$$

where all $Z_{i_1 \ldots i_p} \in \overline{G}^k$.

Remarks :

1) A priori one can of course already bring $j_l(X_o)(0)$ into the normal form.
2) If for fixed k and l , $X_1, \ldots, X_r$ denotes a basis for $\overline{G}^k$, then the previous theorem together with the first remark , imply that the family X is C^∞-conjugate over the identity to $L + \sum_{i=1}^r f_i(\lambda)X_i + O(||\lambda||)O(||x||^{k+1}) + O((||\lambda|| + ||x||)^{l+1})$, where $f_i(\lambda)$ are polynomials in λ of degree $\leq l$.

Chapter 2

Singularities of finite codimension.

Among the singularities which we encounter in differential equations

$$\begin{cases} \dfrac{dx}{dt} &= P(x,y) \\ \dfrac{dy}{dt} &= Q(x,y) \end{cases} \tag{2.1}$$

with P and Q polynomial of degree ≤ 2, we will make a distinction between the singularities which are of finite codimension as a singularity of a general differential equation on the plane and singularities which are of infinite codimension.
The first will be called singularities of the first kind , while the latter will be said to be of the second kind. Singularities of the second kind will be detected by showing that they are either non-isolated,or that they are Hamiltonian , or that they have an axis of symmetry after a linear coordinate change , or that they are approachable by centers or by integrable saddles .

2.1 Hyperbolic singularities

By this we mean singularities with no eigenvalues on the imaginary axis. These singularities are of codimension 0, as well among quadratic systems as among general systems . In both cases the hyperbolic singularities form an open and dense set. In the set of quadratic systems we work with the coefficient topology, while in the set of germs of singular C^∞ vector fields we work with the usual inverse limit topology of the jet-spaces. These hyperbolic singularities are stable for topological equivalence.

2.2 Semi-hyperbolic singularities

By this we mean singularities with 0 as a simple eigenvalue or in other terms $j_1X(0) \sim_L by\frac{\partial}{\partial y}$ with $b \neq 0$. ($\sim_L$ stands for linear conjugation .)
The normal form theorem permits to give following expression to $j_\infty X(0)$:

$$\left(\sum_{i=2}^{\infty} a_i x^i\right) \frac{\partial}{\partial x} + \left(b + \sum_{i=1}^{\infty} b_i x^i\right) y \frac{\partial}{\partial y} \tag{2.2}$$

X is said to be of codimension k if $a_{k+1} \neq 0$ while $a_j = 0$ for $j \leq k$.
We will show that in case of singularities of the first kind only semi-hyperbolic singularities of codimension ≤ 3 occur, and the general as well as the quadratic codimensions are the same.
For this purpose we can suppose that the system (2.1) is in canonical form (given by Jordan normal form theory):

$$\begin{cases} \dfrac{dx}{dt} &= ax^2 + bxy + cy^2 \\ \dfrac{dy}{dt} &= y + dx^2 + exy + fy^2 \end{cases} \tag{2.3}$$

THEOREM 1
The quadratic system (2.3) has :

(1) *a saddle-node of codimension 1 if* $a \neq 0$

(2) *a saddle or a node of codimension 2 if* $a = 0, bd \neq 0$

(3) *a saddle-node of codimension 3 if* $a = 0, b = 0, cd \neq 0$

If $a = 0, b = 0, c = 0$ *or if* $a = 0, d = 0$ *there is a curve of singularities so that the codimension is* ∞ .

Proof: Instead of calculating normal form (2.2) , we use the method of center manifolds because we will need it later on . As the center manifold is tangent to

$E^c = \{(x,y) \in I\!R^2 | y = 0\}$ we can represent it as a (local) graph

$$W^c = \{(x,y) \in I\!R^2 | y = h(x)\};$$

where $h : U \subset I\!R \to I\!R$,C^∞ , is defined on some neighborhood $U \subset I\!R$ of 0, with $h(0) = h'(0) = 0$. The restriction to the center manifold is given by

$$\frac{dx}{dt} = ax^2 + bxh(x) + (h(x))^2 \tag{2.4}$$

Now we calculate the Taylorseries of h(x) at x=0. Substituting $y = h(x)$ in the second component of (2.3), we obtain

$$h'(x)(ax^2 + bxh(x) + c(h(x))^2) - (h(x) + dx^2 + exh(x) + f(h(x))^2) = 0$$

with conditions $h(0) = h'(0) = 0$.
So we find

$$h(x) = -dx^2 + d(e-2a)x^3 - d(df + e^2 - 5ae - 2bd + 6a^2)x^4 + O(x^5)$$

Substituting this Taylor approximation in (2.4) , we obtain for the behaviour on the center manifold

$$\frac{dx}{dt} = ax^2 - bdx^3 + (bd(e-2a) + cd^2)x^4 + O(x^5) \tag{2.5}$$

From this it is readily seen that the theorem holds.
The conclusion about the codimension as a quadratic system follows from the Tarski-Seidenberg theorem (appendix of [A.R.]).

Remark:

Concerning the codimension of a non-isolated singularity we recall a result from [D1].

In J^2_∞ there exists a pro-algebraic subset A of ∞ codimension such that all $X \in G^2$ with $j_\infty(X)(0) \in J^2_\infty \backslash A$ satisfy a Lojasiewicz inequality ; i.e. there are $k, c, \delta > 0$

such that

$$||X(x)|| \geq c||x||^k \qquad \text{for} \quad ||x|| < \delta$$

Hence , if $X \in G^2$ has at the origin a non-isolated singularity , $j_\infty(X)(0)$ must belong to A.

2.3 Singularities with an infinitesimal rotation as 1-jet

By this we mean singularities with a pair of non-zero purely imaginary eigenvalues. The formal normal form for these singularities is :

$$(1+\sum_{i=1}^{\infty} a_i(x^2+y^2)^i)(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}) + (\sum_{i=1}^{\infty} b_i(x^2+y^2)^i)(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}) \tag{2.6}$$

X is said to be of codimension k if $b_k \neq 0$ while $b_j = 0$ for $j \leq k-1$. In case all b_i are zero , which is clearly a situation of infinite codimension , X is formally a center.

Using the Jordan normal form theorem , multiplication with a positive number and a rotation we may suppose that system (1.1) has the form :

$$\begin{cases} \dfrac{dx}{dt} &= -y + ax^2 + bxy + cy^2 \\ \dfrac{dy}{dt} &= x + dx^2 + exy - dy^2 \end{cases} \tag{2.7}$$

As has been shown by Bautin in [Bau], these singularities can be of codimension 1, 2 or 3 and in that case the codimension is the same among quadratic systems, or they are centers and are integrable.

THEOREM 2 (Bautin)

Define

$$\begin{aligned} B_1 &= (a+c)(b-2d) \\ B_2 &= (a+c)d(e+2a)(e-3a-5c) \\ B_3 &= (a+c)d(e+2a)(d^2+2c^2+ac) \end{aligned}$$

The quadratic system (2.7) has at the origin :

(1) *a Hopf-point of codimension 1 (or a fine focus of order 1) if* $B_1 \neq 0$

(2) *a Hopf-point of codimension 2 if* $B_1 = 0, B_2 \neq 0$

(3) *a Hopf-point of codimension 3 if* $B_1 = B_2 = 0, B_3 \neq 0$

(4) *a center if* $B_1 = B_2 = B_3 = 0$

2.4 Nilpotent singularities

A nilpotent singularity is a singularity with a nilpotent linear part; by this we mean a linear part linearly conjugate to $y\frac{\partial}{\partial x}$.
Again using the Jordan normal form theorem , we can suppose that the system is in the canonical form:

$$\begin{cases} \dfrac{dx}{dt} &= y + ax^2 + bxy + cy^2 \\ \dfrac{dy}{dt} &= dx^2 + exy + fy^2 \end{cases} \tag{2.8}$$

Up to a linear conjugacy we may even assume d to be 0 or 1.
First we treat the case d=1

THEOREM 3
When d =1 , system (2.8) has at the origin:

(1) *a cusp singularity of codimension 2 if* $e + 2a \neq 0$

(2) *a cusp singularity of codimension 3 if* $e + 2a = 0, b + 2f \neq 0, c \neq a(f - b - 2a^2)$

Under all further restrictions , the cusp singularity is Hamiltonian or is symmetric with respect to an axis after a linear change of coordinates, and hence of ∞ codimension.

Proof: Using the Takens normal form theorem [T1], we can formally give system (2.8) the following expression:

$$\begin{cases} \dfrac{du}{dt} & = v \\ \dfrac{dv}{dt} & = \sum_{n\geq 2} a_n u^n + v \sum_{n\geq 2} b_n u^{n-1} \end{cases} \tag{2.9}$$

where (u,v) are related to (x,y) by a near-identity transformation (in the terminology of [G.H.]). Using Macsyma one easily finds the family of such transformations which perform these calculations up to terms of order 4 .The family is:

$$\begin{cases} x = & r_6 v^4 + ((2r_3 + r_1^2)c + (r_4 + r_1 r_2)b + r_2^2 a + r_5)uv^3 \\ & +r_4 v^3 + ((6r_1 c + r_2 b + 2r_3 + r_1^2)f \\ & +(2r_2 c - 2r_4 + r_1 r_2)e + (3r_1 b + 2r_2 a)c + r_2 b^2 \\ & +(2r_3 + r_1^2)b + (2r_1 r_2 - 4r_4)a + r_2^2)u^2 v^2/2 \\ & +(2r_1 c + r_2 b + r_3)uv^2 + r_2 v^2 + (6(c + r_1)f^2 \\ & +(r_2 e + 4bc + 9r_1 b + 6r_2 a)f + (2c^2 - 5r_1 c - 4r_2 b \\ & -5r_3 + r_1^2)e + 2ac^2 + (b^2 - 16r_1 a + 10r_2)c \\ & +3r_1 b^2 - 6r_2 ab + (2r_1^2 - 10r_3)a - 4r_4 + 14r_1 r_2)u^3 v/6 \\ & +(2(c + r_1)f - r_2 e + bc + 2r_1 b - 2r_2 a)u^2 v/2 \\ & +(c + r_1)uv + (6f^3 + 11bf^2 + ((-3c - 10r_1)e - 12ac + 6b^2 \\ & -12r_1 a + 6r_2)f + (-3bc - 10r_1 b + 6r_2 a)e + 20c^2 \\ & +4(7r_1 - ab)c + b^3 - 2(8r_1 a + 7r_2)b - 14r_3 + 32r_1^2)u^4/24 \\ & +(2f^2 + 3bf - 2r_1 e - c(e + 4a) + b^2 - 4r_1 a - 2r_2)u^3/6 \\ & +(f + b)u^2/2 + u \end{cases} \tag{2.10}$$

$$
\begin{cases}
y = & r_5 v^4 + ((2r_3 + r_1^2)f + (r_4 + r_1 r_2)e + r_2^2)uv^3 \\
& +r_3 v^3 + (6r_1 f^2 + 3r_2 ef + (3r_1 c + r_2 b - r_3 + r_1^2)e \\
& +2r_2 c - 6r_3 a + 2r_4 + 2r_1 r_2)u^2 v^2/2 \\
& +(2r_1 f + r_2 e)uv^2 + r_1 v^2 \\
& +(6f^3 + ((6c - r_1)e - 24r_1 a + 6r_2)f - 3r_2 e^2 \\
& +(bc + r_1 b - 12r_2 a)e + 2c^2 \\
& +16r_1 c + 6r_2 b - 2r_3 + 14r_1^2)u^3 v/6 \\
& +(2f^2 + ce - 4r_1 a + 2r_2)u^2 v/2 + fuv + v \\
& -(11af^2 - 12(c + r_1)f + (8ac + 4r_1 a + 6r_2)e \\
& -2(3b + 2a^2)c + ab^2 - 12r_1 b - 16r_1 a^2 + 4r_2 a)u^4/12 \\
& +(-af + c + r_1)u^3 - au^2
\end{cases}
\tag{2.11}
$$

For the normal form , we get

$$
\begin{cases}
\dfrac{du}{dt} & = v + p_1(u,v) \\
\dfrac{dv}{dt} & = u^2 + a_3 u^3 + a_4 u^4 + b_1 uv + b_2 u^2 v + b_3 u^3 v + p_2(u,v)
\end{cases}
\tag{2.12}
$$

where

$p_i(u,v) = O(|u,v|^5), i = 1,2$, $b_1 = e + 2a$, b_2 depends on a parameter of the family in such a way that we may assume that $b_2 = 0$ without loss of generality. This choice once made:

$b_3 = -(b + 2f)(af - c - ab - 2a^3)$

a_3, a_4 are polynomials in a, b, c, e, f and the parameters $r_1 \ldots r_6$ of the family of transformations.

From now on we suppose that $b_2 = 0$.

Using the transformation :

$$
\begin{cases}
U = & u \\
V = & v + p_1(u,v)
\end{cases}
$$

system (2.12) is transformed into :

$$\begin{cases} \dfrac{dU}{dt} = & V \\ \dfrac{dV}{dt} = & U^2 + a_3U^3 + a_4U^4 + O(U^5) \\ & +V(b_1U + b_3U^3 + O(U^4)) + V^2O(|U,V|^3) \end{cases} \tag{2.13}$$

Now we consider the corresponding dual form :

$$VdV - (U^2 + a_3U^3 + a_4U^4 + O_1(U))dU - V(b_1U + b_3U^3 + O(U^4))dU - V^2O(|U,V|^3)dU$$

with $O_1(U) = O(U^5)$. Performing the coordinate transformation

$$\begin{cases} r = & U(1 + 3a_3U/4 + 3a_4U^2/5 + O_2(U))^{(1/3)} \\ s = & V \end{cases}$$

with $dO_2(U) = O_1(U)dU$, we get

$$sds - r^2dr - s(b_1r + 3\alpha b_1r^2 + (2(2\beta + \alpha^2)b_1 + b_3)r^3 + O(r^4))dr - s^2O(|r,s|^3)dr$$

where $\alpha = -a_3/4$ and $\beta = (15a_3^2 - 16a_4)/80$
So we see that system (2.8) is C^∞-equivalent to

$$\begin{cases} \dfrac{dr}{dt} = & s \\ \dfrac{ds}{dt} = & r^2 + s(b_1r + 3\alpha b_1r^2 + (2(2\beta + \alpha^2)b_1 + b_3)r^3 \\ & +O(r^4)) + s^2O(|r,s|^3) \end{cases} \tag{2.14}$$

From this and [DRS1] the first two statements follow immediately. To have a cusp of codimension > 3 at the origin, one of the following conditions (C1) or (C2) must hold:

$$(C1): \begin{cases} e = -2a \\ b = -2f \end{cases} \qquad (C2): \begin{cases} e & = -2a \\ c & = a(f - b - 2a^2) \end{cases}$$

Under conditions (C1), system (2.8) is Hamiltonian.
Suppose now that conditions (C2) are fullfilled.
Making use of the linear coordinate change $x = X + aY, y = Y$ system (2.8) is transformed into:

$$\begin{cases} \dfrac{dX}{dt} &= Y + (b + 2a^2)XY \\ \dfrac{dY}{dt} &= X^2 + (f - a^2)Y^2 \end{cases} \tag{2.15}$$

System (2.15) is invariant under the transformation $(X, Y, t) \to (X, -Y, -t)$; and consequently it is symmetric with respect to the X-axis.

Remark :

Concerning the codimension , we refer to the normal form theory as presented in [Bro].
A singularity with $y\dfrac{\partial}{\partial x}$ as 1-jet, can be transformed by a C^∞-diffeomorphism Φ into one with a formal normal form:

$$\begin{cases} \dfrac{du}{dt} &= v + \sum_{n\geq 2} c_n u^n \\ \dfrac{dv}{dt} &= \sum_{n\geq 2} a_n u^n \end{cases} \tag{2.16}$$

If such a singularity belongs to some Lie-algebra , like the Lie-algebra of Hamiltonian vector fields or the Lie-algebra $\{X | \Psi_* X = \pm X\}$ (with $\Psi(x,y) = (x, -y)$), then Φ can be chosen in a way that the normal form belongs to the same Lie-algebra.
In that way, in both cases, all c_n in (2.16) need to be zero. This is the same as saying that all b_n in (2.9) are zero , and it is clearly a situation of infinite codimension.

Next we treat the case d=0.

Here also it is easy to find the family of near-identity transformations which convert system (2.8) into the normal form (2.9) up to terms of order 4 . The family is

$$\begin{cases} x = & r_6 v^4 + ((2r_3 + r_1^2)c + (r_4 + r_1 r_2)b + r_2^2 a + r_5)uv^3 \\ & +r_4 v^3 + ((6r_1 c + r_2 b + 2r_3 + r_1^2)f \\ & +(2r_2 c - 2r_4 + r_1 r_2)e + (3r_1 b + 2r_2 a)c + r_2 b^2 \\ & +(2r_3 + r_1^2)b + (2r_1 r_2 - 4r_4)a)u^2 v^2/2 \\ & +(2r_1 c + r_2 b + r_3)uv^2 + r_2 v^2 + (6(c + r_1)f^2 \\ & +(r_2 e + 4bc + 9r_1 b + 6r_2 a)f \\ & +(2c^2 - 5r_1 c - 4r_2 b - 5r_3 + r_1^2)e \\ & +2ac^2 + (b^2 - 16r_1 a)c + 3r_1 b^2 - 6r_2 ab \\ & +(2r_1^2 - 10r_3)a)u^3 v/6 \\ & +(2(c + r_1)f - r_2 e + bc + 2r_1 b - 2r_2 a)u^2 v/2 + (c + r_1)uv \\ & +(6f^3 + 11bf^2 + ((-3c - 10r_1)e - 12ac + 6b^2 - 12r_1 a)f \\ & +(-3bc - 10r_1 b + 6r_2 a)e - 4abc + b^3 - 16r_1 ab)u^4/24 \\ & +(2f^2 + 3bf - 2r_1 e - c(e + 4a) + b^2 - 4r_1 a)u^3/6 \\ & +(f + b)u^2/2 + u \end{cases} \tag{2.17}$$

$$\begin{cases} y = & r_5 v^4 + ((2r_3 + r_1^2)f + (r_4 + r_1 r_2)e)uv^3 + r_3 v^3 \\ & +(6r_1 f^2 + 3r_2 ef + (3r_1 c + r_2 b - r_3 + r_1^2)e \\ & -6r_3 a)u^2 v^2/2 + (2r_1 f + r_2 e)uv^2 + r_1 v^2 \\ & +(6f^3 + ((6c - r_1)e - 24r_1 a)f - 3r_2 e^2 \\ & +(bc + r_1 b - 12r_2 a)e)u^3 v/6 \\ & +(2f^2 + ce - 4r_1 a)u^2 v/2 + fuv + v \\ & -(11af^2 + 4a(2c + r_1)e - 4a^2 c + ab^2 \\ & -16r_1 a^2)u^4/12 - afu^3 - au^2 \end{cases} \tag{2.18}$$

The normal form reads:

$$\begin{cases} \dfrac{du}{dt} & = v \\ \dfrac{dv}{dt} & = a_3 u^3 + a_4 u^4 + b_2 uv + b_3 u^2 v + b_4 u^3 v \end{cases} + O(|u,v|^5) \tag{2.19}$$

where $a_3 = -ae$, $a_4 = -a(f(e-2a) + eb)/2$,$b_2 = e + 2a$, $b_3 = (f(e-2a) + eb)/2$ and b_4 is a polynomial in a, b, c, e, f and the parameter r_1 of the family of transformations .

Using (2.19), we can prove the next theorem:

THEOREM 4

When d=0, the following statements can be made concerning the singularity at the origin in system (2.8):

(1) *If*

$$ae < 0 \quad b \neq (2a-e)f/e \quad \begin{cases} e \neq -2a & \text{nilpotent saddle of cod 3} \\ e = -2a & \text{nilpotent saddle of cod 4} \end{cases}$$

(2) *If*

$$ae > 0 \quad \begin{cases} b \neq (2a-e)f/e \\ e \neq 3a \end{cases} \quad \begin{cases} e \neq 2a & \text{elliptic singularity of cod 3} \\ e = 2a & \text{elliptic singularity of cod 4} \\ & \text{(which we call of type 1)} \end{cases}$$

(3) *If*

$$\begin{cases} e = 3a \neq 0 \quad 3b + f \neq 0 \\ (f-2b)(3f-b) + 25ac \neq 0 \end{cases} \quad \text{elliptic singularity of cod 4 (which we call of type 2)}$$

(4) *If*

$$\begin{cases} e = 3a \neq 0 \quad 3b + f \neq 0 \\ (f-2b)(3f-b) + 25ac = 0 \end{cases}$$

elliptic singularity which is approachable by centers , and hence of ∞ cod

(5) *If*

$$ae \neq 0 \quad b = (2a-e)f/e$$

the singularity is symmetric with respect to an axis after a linear change of coordinates , and hence of ∞ cod

(6) *If*

$$a \neq 0 \quad e = 0 \qquad \begin{cases} f \neq 0 & \textit{nilpotent saddle-node of cod 4} \\ f = 0 & \textit{the singularity is non-isolated,} \\ & \textit{and hence of } \infty \textit{ cod} \end{cases}$$

(7) *If*

$$a = 0 \qquad \textit{the singularity is non-isolated, and hence of } \infty \textit{ cod}$$

Proof:
We prove (2) and (4), since the proof of the other statements is completely analogous to the proof of theorem 2 .
One can show that system (2.19) is C^∞- equivalent to the system :

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -aex^3 + y((e+2a)x + \dfrac{(e-3a)(f(e-2a)+be)}{5e}x^2 + O(x^3)) + y^2 O(|x,y|^2) \end{cases}$$

After the rescaling $x = \alpha r$, $y = \beta s$, $t = \gamma\tau$, we get :

$$\begin{cases} \dfrac{dr}{d\tau} = \dfrac{\beta\gamma}{\alpha}s \\ \dfrac{ds}{d\tau} = \dfrac{-ae\alpha^3\gamma}{\beta}r^3 + (e+2a)\alpha\gamma rs + \dfrac{(e-3a)(eb+(e-2a)f)\alpha^2\gamma}{5e}r^2 s + sO(|r,s|^3) \end{cases}$$

We take $\alpha = \beta\gamma$.
The coefficient of r^3 is then $-(ae\alpha^3\gamma)/\beta = -ae\beta^2\gamma^4$, and we impose $\beta^2\gamma^4 ae = 1$. It is interesting to remark that the condition in (3) are chosen to have the r^2s-coefficient different from zero . Making the coefficient of r^2s equal to ± 1 has no effect on the coefficient of rs . For this coefficient we have $(e+2a)\alpha\gamma = (e+2a)\beta\gamma^2 = |e+2a|/\sqrt{ae}$, if we choose
sign $\beta =$ sign $(e+2a)$.
Now

$$(e-2a)^2 \geq 0 \Leftrightarrow e^2 + 4ae + 4a^2 \geq 8ae \Leftrightarrow (e+2a)^2 \geq 8ae \Leftrightarrow \frac{|e+2a|}{\sqrt{ae}} \geq 2\sqrt{2}$$

Conclusion:
If $e \neq 2a$ we have an elliptic singularity of codimension 3 , if $e = 2a$ we have an elliptic singularity of codimension 4 (of type 1).
Here ends the proof of statement (2) . Under the conditions of (4) and using linear coordinate changes and rescaling of time system (2.8) can be transformed into :

$$Z_0 = (y + x^2 + xy - 2y^2/25)\frac{\partial}{\partial x} + 3xy\frac{\partial}{\partial y}$$

Let us introduce

$$Z_k : \begin{cases} \dfrac{dx}{dt} &= y + \dfrac{25+4k}{25}x^2 + xy - \dfrac{2}{25}y^2 \\ \dfrac{dy}{dt} &= -kx + \dfrac{k}{5}x^2 + \dfrac{75+2k}{25}xy \end{cases}$$

For $k > 0$, Z_k has a center at the origin .

Remarks:

a. Concerning the nilpotent focus.

When $d = 0$, the x-axis is an invariant line of system (2.8). Hence a degenerate nilpotent focus cannot exist in a quadratic system.

b. Concerning the saddle (resp. elliptic point) of codimension 4.

We call it a saddle (resp. elliptic point) of codimension 4 since the normal form theorem together with the Tarski-Seidenberg decision theorem show that these singularities lay on a semi-algebraic set of codimension 4 . For the elliptic point , the distinction between type 1 and type 2 has to do with the value of certain coefficients in the normal form.
For nilpotent elliptic points of codimension 4 and type 1 is the 4-jet C^∞-equivalent to:

$$y\frac{\partial}{\partial x} + (-x^3 + y(2\sqrt{2}x + \epsilon_2 x^2 + fx^3))\frac{\partial}{\partial y}$$

with $\epsilon_2 = \pm 1$ anf $f \in I\!R$.

For nilpotent elliptic points of codimension 4 and type 2 is the 4-jet C^∞-equivalent to:

$$y\frac{\partial}{\partial x} + (-x^3 + y(bx + fx^3))\frac{\partial}{\partial y}$$

with $b > 2\sqrt{2}$ and $f \in I\!R$.

c. Concerning the codimension of a singularity which is approachable by centers . Let $W \subset G^2$ be the set of those germs of vector fields $X = X_1\frac{\partial}{\partial x_1} + X_2\frac{\partial}{\partial x_2}$ on $I\!R^2$ for which the eigenvalues of $(\frac{\partial X_i}{\partial x_j}(0))_{i,j=1,2}$ are non-zero purely imaginary . These germs of vector fields can be given the formal normal form :

$$(\lambda + \sum_{i=1}^{\infty} a_i(x^2+y^2)^i(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}) + (\sum_{i=1}^{\infty} b_i(x^2+y^2)^i)(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$$

with $\lambda \neq 0$.
We define $\widetilde{V_k}$ to be the set of those germs $X \in W$ whose $(2k-1)$-jet is in normal form with $b_1 = \ldots = b_{k-1} = 0 \quad (k \geq 2)$.
The sets V_k are now defined by :

$$V_k = \{X \in W | \exists \phi : I\!R^2 \mapsto I\!R^2 \quad C^\infty\text{diffeomorphism such that} \quad \phi_*(X) \in \widetilde{V_k}\}$$

Using the Tarski-Seidenberg theorem it follows that V_k is a semi-algebraic subset of G^2 of codimension k. So , if X can be approached by centers , X belongs to $\overline{\bigcap_k V_k} \subset \bigcap_k \overline{V_k}$, a subset of infinite codimension .

d. Concerning the elliptic point of codimension 4 and type 2.

Using Macsyma it is easy to prove that all quadratic systems (2.8) with $e = 3a \neq 0$, $3b + f \neq 0$ and $(f - 2b)(3f - b) + 25ac \neq 0$ are C^∞-equivalent to

$$\left\{\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x^3 + \tfrac{5}{\sqrt{3}}xy \end{aligned}\right. \quad + O(|x,y|^8) \qquad (2.20)$$

So we see that the coefficients $b_3, \ldots, b_7$ of the normal form (2.9) all can be made zero . For this reason we conjecture that these singularities are of ∞ codimension.

2.5 Homogeneous Singularities

In this section we will show that every quadratic homogeneous singularity is approachable by saddles or integrable saddles. We may suppose that , after performing a rotation , we have the following expression

$$\left\{ \begin{array}{rcl} \dfrac{dx}{dt} & = & ax^2 + bxy + cy^2 \\ \dfrac{dy}{dt} & = & exy + fy^2 \end{array} \right. \tag{2.21}$$

THEOREM 5

The following statements can be made concerning the singularity at the origin in system (2.21) :

(1) *If*

$a(af^2 - bef + ce^2) \leq 0$ *the singularity is approachable by centers , and hence of ∞ cod.*

(2) *If*

$a(af^2 - bef + ce^2) > 0$ *the singularity is approachable by integrable saddles.*

Proof:

First we prove (1).

Suppose $a \neq 0$, $e \neq 0$ and $a(af^2 - bef + ce^2) < 0$ Let us introduce

$$X_\epsilon : \left\{ \begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = P^{-1} \begin{pmatrix} 0 & -\epsilon \\ \epsilon & 0 \end{pmatrix} P \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} ax^2 + bxy + cy^2 \\ exy + fy^2 \end{pmatrix} \right.$$

where $P = \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix}$ with $\alpha = \frac{f}{e}$ and $\beta^2 = -\frac{af^2 - bef + ce^2}{ae^2}$ and $\epsilon \in I\!R$.
Using the transformation

$$\begin{cases} u = & x + \alpha y \\ v = & \beta y \end{cases}$$

one easily checks that system X_ϵ has a center at the origin for $\epsilon \neq 0$.
In case $a = 0$, $e = 0$ or $a(af^2 - bef + ce^2) = 0$ we take a small perturbation of the quadratic terms in order to bring them to non-zero values .
Next we prove (2) .
Suppose $a \neq 0$, $e \neq 0$ and $a(af^2 - bef + ce^2) > 0$. In this situation we introduce the following system :

$$Y_\epsilon : \quad \left\{ \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = P^{-1} \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix} P \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} ax^2 + bxy + cy^2 \\ exy + fy^2 \end{pmatrix} \right.$$

where $P = \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix}$ with $\alpha = \frac{f}{e}$ and $\beta^2 = \frac{af^2 - bef + ce^2}{ae^2}$ and $\epsilon \in I\!R$.
One easily checks that system Y_ϵ has at the origin a saddle with first four dual Lyapounov coefficients equal to zero . (The dual Lyapounov coefficient of order 1 is the divergence .) Such a saddle is integrable (see [JR]).
When $a = 0$ or $e = 0$ we take a small perturbation of the quadratic terms to bring them to non-zero values .

Remark:

Suppose that the planar system X has at the origin a hyperbolic saddle. Then X can be given the formal normal form ([Joy]) :

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \sum_{n=0}^{\infty} (x^2 - y^2)^n \begin{pmatrix} a_n & b_n \\ b_n & a_n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

If X is a quadratic system and the first four dual Lyapounov coefficients are zero,

then we know from [JR] that $a_n = 0 \ \forall n$. Hence, such saddles as well as the homogeneous vector fields approachable by them belong to a pro-algebraic subset of infinite codimension. But it is by no means clear that this will have topological consequences.
In her thesis [Khe] , F. Khechichine gives indications that homogeneous singularities satisfying condition (2) have a versal quadratic unfolding. So one can expect that those singularities are of finite codimension in the sense of the definition of codimension of chapter 1.

Chapter 3

Quadratic generic local bifurcations of codimension ≤ 4.

In this chapter we deal with the relation between the quadratic unfolding of the singularity and its generic (or versal) unfolding as a general system . We will show that every quadratic singularity of finite codimension , except for the elliptic points of codimension 4 , has a generic (or versal) unfolding among the quadratic systems.

3.1 Nilpotent Bifurcations

In this section we treat the nilpotent singularities . We show that every nilpotent singularity of the first kind , except for the nilpotent elliptic points of codimension 4 , does have a generic unfolding among the quadratic systems . The reason of the non-existence of a generic unfolding of the elliptic point of codimension 4 and type 1 essentially is that the nilpotent focus of codimension 3 cannot be given a quadratic model . We do not treat the elliptic points of codimension 4 and type 2 because we conjecture them to be of ∞ codimension.

Since the proofs in the different cases follow a same procedure we only give the proofs in the case of the saddle and elliptic points of codimension 3 and in the case of the saddle of codimension 4 . In an appendix we give the coordinate transformations which one can use in the proof of the other cases.

3.1.1 Nilpotent Cusps

Let X_0 be a germ of a vector field at $0 \in \mathbb{R}^2$ with $X_0(0) = 0$ and with nilpotent 1-jet , then by [T1] the 2-jet of X_0 is C^∞-conjugate to

$$y\frac{\partial}{\partial x} + (\alpha x^2 + \beta xy)\frac{\partial}{\partial y} \tag{3.1}$$

Topologically , such an X_0 with $\alpha \neq 0$ and β whatsoever looks like a cusp.

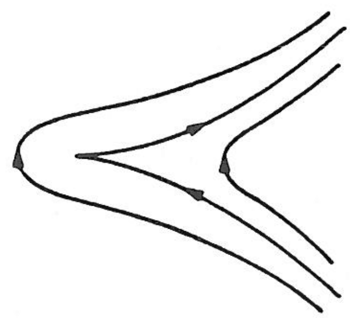


fig. 3.1

In generic 2-parameter families one may suppose that $\alpha.\beta \neq 0$. In that case the 2-jet is C^∞-equivalent to

$$y\frac{\partial}{\partial x} + (x^2 \pm xy)\frac{\partial}{\partial y} \tag{3.2}$$

The germs of vector fields whose 2-jet is C^∞-equivalent to (3.2+) (resp. (3.2-)) constitute a submanifold Σ^2_{C+} (resp. Σ^2_{C-}) of codimension 4 in the space of all germs of vector fields at $0 \in \mathbb{R}^2$ (and of codimension 2 in the space of all germs of singular vector fields in $0 \in \mathbb{R}^2$).

The genericity of a 2-parameter unfolding X_λ of X_0 can be defined by the fact that the mapping $((x,y),\lambda) \to j^2X_\lambda(x,y)$ cuts $\Sigma^2_{C\pm}$ transversally in $(0,0)$.

In [Bog] Bogdanov has shown that 2 families cutting transversally Σ^2_{C+} (resp. Σ^2_{C-})

are (fiber-C^0 , C^0)-equivalent . Such a transversal family is called a generic Bogdanov-Takens bifurcation (or generic bifurcation of the cusp of codimension 2).
The family

$$X_\lambda = y\frac{\partial}{\partial x} + (x^2 + \mu + y(\nu \pm x))\frac{\partial}{\partial y} \tag{3.3}$$

where $\lambda = (\mu, \nu)$ is the parameter , is called the standard Bogdanov-Takens bifurcation or quadratic Bogdanov-Takens bifurcation . The bifurcation diagram and the phase portraits of the family (3.3-) are represented in figure 3.2.

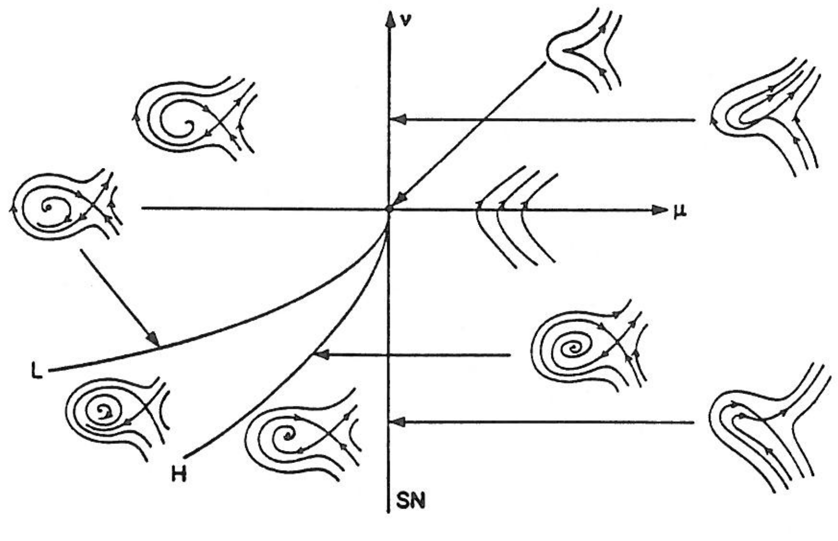


fig. 3.2

One can also show that 2-parameter families X_λ cutting $\Sigma^2_{C\pm}$ transversally can be brought - up to C^∞-equivalence - in the following simplified form , called a normal form :

$$y\frac{\partial}{\partial x} + (x^2 + \mu(\lambda) + y(\nu(\lambda) \pm x + x^2 h(x,\lambda) + y^2 Q(x,y,\lambda))\frac{\partial}{\partial y} \tag{3.4}$$

where μ, ν, h and Q are C^∞ and Q is N-flat for an a priori given N .
The transversality of the map $((x,y),\lambda) \mapsto j^2X_\lambda(x,y)$ with respect to $\Sigma^2_{C\pm}$ expresses

itself as

$$\frac{D(\mu,\nu)}{D(\lambda_1,\lambda_2)}(0) \neq 0 \tag{3.5}$$

Using this normal form theory we can show that every quadratic cusp singularity of codimension 2 has a generic unfolding among the quadratic systems.

THEOREM 1

The 2-parameter family

$$C_{(\lambda_1,\lambda_2)}: \quad \begin{cases} \dfrac{dx}{dt} = y + ax^2 + bxy + cy^2 \\ \dfrac{dy}{dt} = \lambda_1 + \lambda_2 y + x^2 + exy + fy^2 \end{cases} \tag{3.6}$$

with $e + 2a \neq 0$, *is a generic bifurcation of a cusp codimension 2.*

In generic 3-parameter families one also finds unfoldings of singularities with nilpotent 1-jet which are however more degenerate than the cusp of codimension 2. More precise , one also encounters the cases where the 2-jet of X_0 is C^∞-conjugate to

$$y\frac{\partial}{\partial x} + (\alpha x^2 + \beta xy)\frac{\partial}{\partial y} \tag{3.7}$$

with ($\alpha \neq 0$ and $\beta = 0$) or ($\alpha = 0$ and $\beta \neq 0$) . We proceed with the case ($\alpha \neq 0$ and $\beta = 0$) .

To start with we will recall some results from [DRS1] .

A singularity with nilpotent 1-jet and whose 2-jet is C^∞-equivalent to $y\frac{\partial}{\partial x} + \alpha x^2\frac{\partial}{\partial y}$ with $\alpha \neq 0$ is called a cusp singularity of codimension ≥ 3 . The set of germs of such vector fields constitute a semi-algebraic subset of codimension 3 in the space of all germs of singular vector fields in $0 \in I\!R^2$. Denote this set by Σ^3_C . ($\Sigma^2_C = \Sigma^2_{C+} \cup \Sigma^2_{C-} \cup \Sigma^3_C$) .

One can prove (see [DRS1]) that each $X_0 \in \Sigma^3_C$ has a 4-jet C^∞-equivalent to

$$y\frac{\partial}{\partial x} + (x^2 + \gamma x^3 y)\frac{\partial}{\partial y} \tag{3.8}$$

One defines Σ^4_C by the condition $\gamma = 0$; Σ^4_C is a semi-algebraic subset of codimension 1 in Σ^3_C and $\Sigma^3_C = \Sigma^3_{C+} \cup \Sigma^3_{C-} \cup \Sigma^4_C$, where $\Sigma^3_{C\pm}$ is the submanifold of codimension 3 consisting of germs of singular vector fields whose 4-jet is C^∞-equivalent to $y\frac{\partial}{\partial x} + (x^2 \pm x^3 y)\frac{\partial}{\partial y}$.

The genericity condition of 3-parameter families X_λ with $X_0 \in \Sigma^3_{C\pm}$ consists in the transversality of the mapping $((x,y),\lambda) \to j^4 X_\lambda(x,y)$ with respect to $\Sigma^3_{C\pm}$ in $(0,0)$.

An example of such a family is given by

$$\widetilde{X}^{\pm}_\lambda = y\frac{\partial}{\partial x} + (x^2 + \mu + y(\nu_0 + \nu_1 x \pm x^3))\frac{\partial}{\partial y} \tag{3.9}$$

with $\lambda = (\mu, \nu_0, \nu_1)$. Its bifurcation set (in (μ,ν_0,ν_1)-space) consists of several surfaces . The $\{\mu = 0\}$-plane outside the origin is a bifurcation surface of saddle-node type . The other surfaces of bifurcation are situated in the half space $\{\mu < 0\}$. They can best be visualized by drawing their trace on the half-sphere $S = \{(\mu,\nu_0,\nu_1) | \mu < 0, \mu^2 + \nu_0^2 + \nu_1^2 = \epsilon^2\}$ for $\epsilon > 0$ sufficiently small . The bifurcation set is a cone based on its trace with S. (see fig. 3.3)

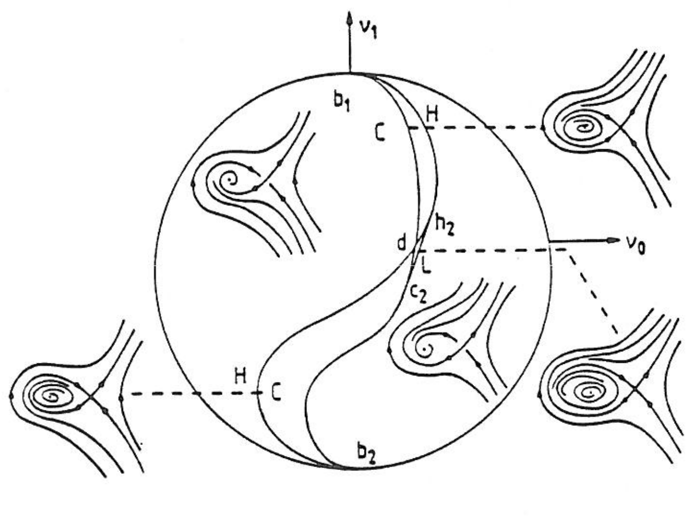


fig. 3.3

The main result of [DRS1] can be stated as follows :
A local 3-parameter family in $(0,0) \in I\!R^2 \times I\!R^3$ transversally cutting $\Sigma^3_{C\pm}$ at $(0,0)$ is (fiber-C^0,C^0)-equivalent to $\widetilde{X}^{\pm}_{\lambda}$.

Another result of [DRS1] is the following :
3-parameter families X_λ cutting $\Sigma^3_{C\pm}$ transversally can be brought - up to C^∞-equivalence - in the form :

$$
\begin{aligned}
y\frac{\partial}{\partial x} + (x^2 + \mu(\lambda) + \; & yK(y,\lambda)(\nu_0(\lambda) + \nu_1(\lambda)x \\
& + \alpha(\lambda)x^2 \pm x^3 + x^4 h(x,\lambda)) + y^2 Q(x,y,\lambda))\frac{\partial}{\partial y}
\end{aligned}
\qquad (3.10)
$$

where $\mu, \nu_0, \nu_1, \alpha, K$ and Q are C^∞-functions with $K(0,\lambda) \equiv 1, \alpha(0) = 0$ and Q is of order N in (x,y,λ) , where N is arbitrarily high , but given a priori .

The genericity condition - the transversality of the map $((x,y),\lambda) \to j^4 X_\lambda(x,y)$ with respect to $\Sigma^3_{C\pm}$ at $(0,0)$ - translates itself as

$$
\frac{D(\mu,\nu_0,\nu_1)}{D(\lambda_1,\lambda_2,\lambda_3)}(0) \neq 0 \qquad (3.11)
$$

Using this result we can prove that every quadratic cusp singularity of codimension 3 has a generic unfolding among the quadratic systems.

THEOREM 2

The 3-parameter family

$$
C_{(\lambda_1,\lambda_2,\lambda_3)} : \quad \left\{
\begin{aligned}
\frac{dx}{dt} &= y + ax^2 + bxy + cy^2 \\
\frac{dy}{dt} &= \lambda_1 + \lambda_2 y + x^2 + (\lambda_3 - 2a)xy + fy^2
\end{aligned}
\right. \qquad (3.12)
$$

with $b+2f \neq 0$ *and* $c \neq a(f-b-2a^2)$ *is a generic unfolding of a cusp of codimension 3 .*

Remarks :

We may observe that $C_{(\lambda_1,\lambda_2,\lambda_3)}$ is an example of a family of quadratic systems showing for certain parameter values two limit cycles , which disappear (or appear) in a generic double cycle (L) , in a Hopf bifurcation of codimension 2 (h2) , in a homoclinic loop of codimension 2 (c2) or in the simultaneous occurence of a Hopf bifurcation of codimension 1 (h) and a homoclinic loop of codimension 1 (c) . This phenomenon was also observed in [Rou].

At the end of his article " A Survey of Quadratic Systems " ([Cop]) Coppel raised the following question : "Is it possible to characterize the phase portraits of quadratic systems by means of algebraic inequalities on the coefficients ?" . As the contact in $c2$ between the line of double cycles L and the line of homoclinic loop bifurcations C is a flat contact (see [DRS1]), we know that the germ of this bifurcation set at the origin is not C^∞-diffeomorphic to the germ of an algebraic variety , even not to the germ of an analytic variety . This provides in a negative answer to Coppel's question.

3.1.2 Nilpotent saddles and elliptic points of codimension 3

To start with we treat the remaining singularities of codimension 3 , namely in case ($\alpha = 0$ and $\beta \neq 0$).

One can show that the germs of vector fields at $0 \in \mathbb{R}^2$ whose 1-jet is nilpotent and whose 2-jet is C^∞-conjugate to $y\frac{\partial}{\partial x}+\beta xy\frac{\partial}{\partial y}$ with $\beta \neq 0$, have a 4-jet C^∞-conjugate to

$$y\frac{\partial}{\partial x} + (\epsilon x^3 + dx^4 + bxy + ax^2y + cx^3y)\frac{\partial}{\partial y} \tag{3.13}$$

with $b > 0, \epsilon = 0, \pm 1$ and $a, b, c, d \in \mathbb{R}$.

It was shown in [D1] that the topological type of such a germ is determined by its 3-jet , if $\epsilon \neq 0$ and $b \neq 2\sqrt{2}$ in case $\epsilon = -1$.

Adding the extra condition

$$5\epsilon a - 3bd \neq 0 \tag{3.14}$$

one can show that the 4-jet is C^∞-equivalent to

$$y\frac{\partial}{\partial x} + (\epsilon_1 x^3 + bxy + \epsilon_2 x^2 y + f x^3 y)\frac{\partial}{\partial y} \tag{3.15}$$

with $\epsilon_{1,2} = \pm 1, b > 0, f \in \mathbb{R}$.(For more information we refer to [DRS2].)

The topological type falls into one of the following classes:

(1) The saddle case : $\epsilon_1 = 1$, any b and ϵ_2 (a degenerate saddle).
We denote by $\Sigma^3_{S\pm}(\epsilon_2 = \pm 1)$ the subsets of germs with such a 4-jet . They all have the same topological type .

(2) The focus case : $\epsilon_1 = -1$ and $0 < b < 2\sqrt{2}$ (a degenerate focus) .
We denote by $\Sigma^3_{F\pm}(\epsilon_2 \pm 1)$ the corresponding subsets of germs .

(3) The elliptic case : $\epsilon_1 = -1$ and $b > 2\sqrt{2}$ (an elliptic point).
Notation : $\Sigma^3_{E\pm}(\epsilon_2 = \pm 1)$

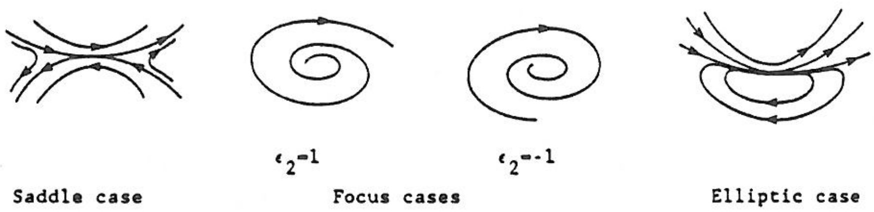


fig. 3.4

The article [DRS2] is devoted to the following :
Let X_λ and Y_λ be two local 3-parameter families with X_0 and Y_0 belonging to the same set $\Sigma^3_{S\pm}, \Sigma^3_{F\pm}$ or $\Sigma^3_{E\pm}$. Suppose that both families are generic in the sense that the mapping $((x,y),\lambda) \in \mathbb{R}^2 \times \mathbb{R}^3 \to j^4 X_\lambda(x,y)$ is transverse to the sets $\Sigma^3_{S\pm}, \Sigma^3_{F\pm}$ or $\Sigma^3_{E\pm}$ at $(0,0)$. Then they are (fiber-C^0,C^0)-equivalent .

An example of such a generic family in each case is given by

$$\widetilde{X}_\lambda = y\frac{\partial}{\partial x} + (\epsilon_1 x^3 + \mu_2 x + \mu_1 + y(\nu + bx + \epsilon_2 x^2))\frac{\partial}{\partial y} \tag{3.16}$$

where $\lambda = (\mu_1, \mu_2, \nu)$, and $b > 0, b \neq 2\sqrt{2}, \epsilon_{1,2} = \pm 1$.

As in the cusp case one can prove that 3-parameter families cutting $\Sigma^3_{S\pm}, \Sigma^3_{F\pm}$ or $\Sigma^3_{E\pm}$ transversally can be brought - up to C^∞-equivalence - into a normal form . This normal form is :

$$\begin{aligned} X_\lambda &= y\frac{\partial}{\partial x} + (\epsilon_1 x^3 + \mu_2(\lambda)x + \mu_1(\lambda) \\ &\quad + y(\nu(\lambda) + b(\lambda)x + \epsilon_2 x^2 + x^3 h(x,\lambda)) + y^2 Q(x,y,\lambda))\frac{\partial}{\partial y} \end{aligned} \tag{3.17}$$

where $\epsilon_{1,2} = \pm 1, \mu_1(\lambda), \mu_2(\lambda), \nu(\lambda), h(x,\lambda)$ (with $\mu_1(0) = \mu_2 0) = \nu(0) = 0, b(0) > 0)$ are C^∞-functions and Q is a C^∞-function of order N in (x, y, λ) , where N is arbitrarily high , but given a priori .

The transversality condition of the mapping $((x,y),\lambda) \to j^4 X_\lambda(x,y)$ with respect to Σ^3_{SEF} amounts to

$$\frac{D(\mu_1, \mu_2, \nu)}{D(\lambda_1, \lambda_2, \lambda_3)}(0) \neq 0 \tag{3.18}$$

Using this normal form theory we can show that every quadratic nilpotent saddle or elliptic point of codimension 3 has a quadratic generic unfolding.

THEOREM 3

(1) *The 3-parameter family*

$$S_{(\lambda_1,\lambda_2,\lambda_3)} : \quad \begin{cases} \dfrac{dx}{dt} = y + ax^2 + bxy + cy^2 \\ \dfrac{dy}{dt} = \lambda_1 + \lambda_2 x + \lambda_3 y + exy + fy^2 \end{cases} \tag{3.19}$$

with $ae < 0, b \neq \frac{(2a-e)f}{e}$ *and* $e \neq -2a$ *is a generic unfolding of a nilpotent saddle of codimension 3 .*

(2) *The 3-parameter family*

$$E^1_{(\lambda_1,\lambda_2,\lambda_3)} : \quad \begin{cases} \dfrac{dx}{dt} = y + ax^2 + bxy + cy^2 \\ \dfrac{dy}{dt} = \lambda_1 + \lambda_2 x + \lambda_3 y + exy + fy^2 \end{cases} \tag{3.20}$$

with $ae > 0, b \neq \frac{(2a-e)f}{e}$ *and* $e \neq a, 2a$ *and* $3a$ *is a generic unfolding of a nilpotent elliptic point of codimension 3 .*

(3) *The 3-parameter family*

$$E^2_{(\lambda_1,\lambda_2,\lambda_3)} : \quad \begin{cases} \dfrac{dx}{dt} = y + ax^2 + bxy + cy^2 \\ \dfrac{dy}{dt} = \lambda_1 + \lambda_2 x + \lambda_3 x^2 + axy + fy^2 \end{cases} \tag{3.21}$$

with $b \neq f$ *is a generic unfolding of a nilpotent elliptic point of codimension 3.*

Proof :

Before starting the proof we give the reduction of a family X_λ - cutting Σ^3_{SEF} transversally - into normal form :

$$\begin{aligned} X_\lambda \;&=\; y\frac{\partial}{\partial x} + (\epsilon_1 x^3 + \mu_2(\lambda)x + \mu_1(\lambda) \\ &\quad + y(\nu(\lambda) + b(\lambda)x + \epsilon_2 x^2 + x^3 h(x,\lambda)) + y^2 Q(x,y,\lambda))\frac{\partial}{\partial y} \end{aligned} \tag{3.22}$$

where $\epsilon_{1,2} = \pm 1, \mu_1(\lambda), \mu_2(\lambda), \nu(\lambda), h(x,\lambda)$ (with $\mu_1(0) = \mu_2(0) = \nu(0) = 0, b(0) > 0$) are C^∞-functions and Q is a C^∞-function of order N in (x, y, λ) , where N is arbitrarily high , but given a priori .

Although this reduction is a known technique we recall it here because it is of fundamental importance for the proof of the theorem . We give this conversion to the normal form in successive steps , prescribing each time for which it becomes necessary , the addition of the supplementary required hypothesis . For further information we refer to [DRS2] . Let us start with a k-parameter family X_λ with the unique hypothesis :

(Hyp 1) $j^1X_0(0)$ is linearly conjugate to $y\frac{\partial}{\partial x}$.

So , up to linearly conjugacy , we may suppose : $j^1X_0(0) = y\frac{\partial}{\partial x}$. As we know from [T3] , the family X_λ can be put - by C^∞-equivalence - in the following normal form:

$$y\frac{\partial}{\partial x} + (F(x,\lambda) + yG(x,\lambda))\frac{\partial}{\partial y} + Q_1(x,y,\lambda)\frac{\partial}{\partial x} + Q_2(x,y,\lambda)\frac{\partial}{\partial y} \tag{3.23}$$

where Q_1 and Q_2 are of order $O((||m|| + ||\lambda||)^N)$ for a certain N that one can chose arbitrarily big , $m = (x,y)$, F and G are C^∞-functions in (x,λ) and we may suppose that they are polynomial of degree N in x .
The λ-dependent coordinate change :

$$\begin{cases} X = x \\ Y = y + Q_1(x,y,\lambda) \end{cases}$$

transforms the family (3.23) into

$$Y\frac{\partial}{\partial X} + (F(X,\lambda) + YG(X,\lambda) + Q_2'(X,Y,\lambda))\frac{\partial}{\partial Y}$$

where $Q_2' = O((||M|| + ||\lambda||)^{N-1})$, $M = (X,Y)$.

Changing $N-1$ into N , (X,Y) into (x,y) and omitting the prime , we find back the expression (3.23) with $Q_1 \equiv 0$. We proceed by developping Q_2 in powers of y :

$$Q_2(x,y,\lambda) = \tilde{F}(x,\lambda) + y\tilde{G}(x,\lambda) + y^2\tilde{Q}(x,y,\lambda)$$

So , with an evident change of notation , we obtain that X_λ is C^∞-equivalent to :

$$y\frac{\partial}{\partial x} + (F(x,\lambda) + yG(x,\lambda) + y^2Q(x,y,\lambda))\frac{\partial}{\partial y} \tag{3.24}$$

where Q is of order N , $F(0,0) = \frac{\partial F}{\partial x}(0,0) = G(0,0) = 0$.
Next we introduce a second hypothesis :

(Hyp 2) $\frac{\partial^2 F}{\partial x^2}(0,0) = 0, \frac{\partial^3 F}{\partial x^3}(0,0) \neq 0$ and $\frac{\partial G}{\partial x}(0,0) \neq 0$.

Now we reduce $F(x,\lambda)$ to $\epsilon_1 x^3 + \mu_2(\lambda)x + \mu_1(\lambda) \qquad (\epsilon_1 = \pm 1)$.

Hypothesis 2 implies that $F(x,0)dx$ is the differential of a function of order 4 at $x = 0$. Such a function admits as universal unfolding :

$$\epsilon_1 \frac{x^4}{4} + \mu_2 \frac{x^2}{2} + \mu_1$$

where the term $\epsilon_1 = \pm 1$ has the sign of $\frac{\partial^3 F}{\partial x^3}(0,0)$.
Hence , there exists a differentiable mapping $\mu(\lambda) = (\mu_1(\lambda), \mu_2(\lambda))$ and a family of diffeomorphisms depending on the parameter λ :

$$U_\lambda(x) = u(\lambda)x + O(x^2) + O(||\lambda||)$$

such that

$$U_\lambda^*(F(x,\lambda)dx) = (\epsilon_1 x^3 + \mu_2(\lambda)x + \mu_1(\lambda))dx$$

with $\mu_1(0) = \mu_2(0) = 0$.
Performing the C^∞-equivalence $\tilde{U}_\lambda : (x,y) \to (U_\lambda(x), y)$ on the dual family $\omega_\lambda = ydy - (F(x,\lambda) + yG(x,\lambda) + y^2Q(x,y,\lambda))dx$, we get :

$$\omega_\lambda \sim ydy - [\epsilon_1 x^3 + \mu_2(\lambda)x + \mu_1(\lambda) + y\tilde{G}(x,\lambda) + y^2\tilde{Q}(x,y,\lambda)]dx$$

where $\sim$ stands for C^∞-equivalence .

As $\tilde{G}dx = U_\lambda^*(Gdx)$, $\tilde{Q}dx = U_\lambda^*(Qdx)$, $U(x,0) = u(0)x + O(x^2)$ and $U(x,\lambda) = O(|x| + ||\lambda||)$, the functions $\tilde{G}, \tilde{Q}$ have the same properties as above : $\frac{\tilde{G}}{\partial x}(0,0) \neq 0$ and $\tilde{Q} = O((||m|| + ||\lambda||)^N)$.

So , with an obvious change of notation we have :

$$X_\lambda \sim y\frac{\partial}{\partial x} + (\epsilon_1 x^3 + \mu_2(\lambda)x + \mu_1(\lambda) + yG(x,\lambda) + y^2 Q(x,y,\lambda))\frac{\partial}{\partial y} \qquad (3.25)$$

with $G(0,0) = 0, \frac{\partial G}{\partial x}(0,0) \neq 0$ and Q is of order N.

(Hyp 3) $\frac{\partial^2 G}{\partial x^2}(0,0) \neq 0$ and $\frac{\partial G}{\partial x}(0,0) \neq 2\sqrt{2}$ in case $\epsilon_1 = -1$

The next step in the conversion to normal form is the reduction to $G(x,\lambda) = \nu(\lambda) + b(\lambda)x + \epsilon_2 x^2 + O(x^3)$ with $b(0) = b > 0$ and $\epsilon_2 = \pm 1$.

Let $G(x,\lambda) = \nu(\lambda) + b(\lambda)x + c(\lambda)x^2 + O(x^3)$.
We have that $\nu(0) = 0, b(0) \neq 0$ and $c(0) \neq 0$.

Consider the linear coordinate change , depending on the parameter λ :
$U_\lambda : (x,y) \to (\alpha(\lambda)x, \beta(\lambda)y)$. Applying it to X_λ we obtain :

$$\begin{aligned}(U_\lambda)^*(X_\lambda) &= \frac{\beta}{\alpha}y\frac{\partial}{\partial x} + \frac{1}{\beta}(\epsilon_1\alpha^3 x^3 + \mu_2\alpha x + \mu_1 \\ &\quad +\beta y(\nu + \alpha b x + \alpha^2 c x^2 + O(x^3)) + \beta^2 y^2 Q(\alpha x, \beta y, \lambda))\frac{\partial}{\partial y}\end{aligned}$$

Taking $|\beta| = \alpha^2$, sign(α)=sign(β)=sign($b(0)$) and $|\alpha(\lambda)| = \frac{1}{|c(\lambda)|}$, we see that U_λ is a C^∞-equivalence which transforms the family X_λ into a new one with the desired properties . The transversality condition of $j^4 X_\lambda(m)$ with respect to $\Sigma^3_{S\pm}$ amounts to :

$$\frac{D(\mu_1,\mu_2,\nu)}{D(\lambda_1,\lambda_2,\nu)}(0) \neq 0 \tag{3.26}$$

So , to prove the genericity of the family

$$S_{(\lambda_1,\lambda_2,\lambda_3)} : \begin{cases} \dfrac{dx}{dt} = y + ax^2 + bxy + cy^2 \\ \dfrac{dy}{dt} = \lambda_1 + \lambda_2 x + \lambda_3 y + exy + fy^2 \end{cases}$$

with $ae < 0, b \neq \dfrac{(2a-e)f}{e}$ and $e \neq -2a$ we have to show that bringing the family S_λ into the normal form (3.22) the mapping $I\!R^3 \mapsto I\!R^3, \lambda \to (\mu_1(\lambda), \mu_2(\lambda), \nu(\lambda))$ satisfies the condition

$$\frac{D(\mu_1,\mu_2,\nu)}{D(\lambda_1,\lambda_2,\nu)}(0) \neq 0$$

To compute this determinant we treat every parameter seperately .
Consider the family

$$\begin{cases} \dfrac{dx}{dt} = y + ax^2 + bxy + cy^2 \\ \dfrac{dy}{dt} = \lambda_3 y + exy + fy^2 \end{cases} \tag{3.27}$$

Using the coordinate transformation

$$\begin{cases} x = t_1(U,V) - \dfrac{c\lambda_3}{2}V^2 - \dfrac{c(b+f)\lambda_3}{3}U^3 \\ y = t_2(U,V) \end{cases}$$

with

$$\begin{cases} t_1(U,V) = & U + \dfrac{(b+f)}{2}U^2 + cUV + \dfrac{c(b+2f)}{2}U^2V \\ & + \dfrac{(b^2-4ac+3bf+2f^2-ec)}{6}U^3 \\ t_2(U,V) = & V - aU^2 + fUV - afU^3 + \dfrac{2f^2+ec}{2}U^2V \end{cases} \tag{3.28}$$

and a change of coordinates of the form $u = U$, $v = V + O(|U, V|^4)$ one checks that system (3.27) is C^∞-conjugate to

$$\begin{cases} \dfrac{du}{dt} = & v \\ \dfrac{dv}{dt} = & -a\lambda_3 u^2 - eau^3 + O(u^4) + v(\lambda_3 + (e+2a)u \\ & (f(e-2a) + be + O(\lambda_3))u^2 + O(u^3)) + v^2 O(|u,v|^2) \end{cases}$$

Now we consider the corresponding family of dual forms :

$$vdv - (-a\lambda_3 u^2 - eau^3 + O(u^4))du - v(\lambda_3 + (e+2a)u + O(u^2))du - v^2 O(|u,v|^2)du \quad (3.29)$$

Denote $f(u, \lambda_3) = -a\lambda_3 u^3/3 - eau^4/4 + O(u^5)$.
Using an adapted form of the preparation theorem ([T2]) we find a coordinate change of the form

$$u = \Phi(U, \lambda_3) = \phi_1(\lambda_3)U + \phi_2(\lambda_3)U^2 + O(U^3)$$

with $\phi_1(0) = (-ae)^{-1/4}$ such that

$$f(\Phi(U, \lambda_3), \lambda_3) = U^4/4 + \eta(\lambda_3)U^3/3$$

with

$$\eta(\lambda_3) = -\frac{a\lambda_3}{(-ae)^{3/4}} + O(\lambda_3^2)$$

Performing this coordinate transformation to the family (3.28) we get

$$vdv - (U^3 + \eta(\lambda_3)U^2)dU - v(\lambda_3\phi_1(\lambda_3) + (2\lambda_3\phi_2(\lambda_3)$$

$$+(e+2a)\phi_1(\lambda_3)^2)U + O(U^2))dU - v^2 O(|U,v|^2))dU$$

Further we use the translation

$$\begin{cases} U = r - \eta(\lambda_3)/3 \\ v = s \end{cases}$$

and we obtain

$$sds - (r^3 - (\eta(\lambda_3))^2 r/3 + 2(\eta\lambda_3))^3/27)dr - s(\lambda_3\phi_1(\lambda_3) - (2\lambda_3\phi_2(\lambda_3) + e + 2a)(\phi_1(\lambda_3)^2)\eta(\lambda_3)/3 + O((\lambda_3)^2) + O(r))dr - s^2 O(|r,s|^2)dr$$

From this we can conclude that $\dfrac{\partial\mu_1}{\partial\lambda_3}(0) = 0$, $\dfrac{\partial\mu_2}{\partial\lambda_3}(0) = 0$ and $\dfrac{\partial\nu}{\partial\lambda_3}(0)$ is proportional to $\dfrac{1}{(-ae)^{1/4}}(1 - \dfrac{e+2a}{3a})$.

Next we consider the family

$$\begin{cases} \dfrac{dx}{dt} = y + ax^2 + bxy + cy^2 \\ \dfrac{dy}{dt} = \lambda_2 x + exy + fy^2 \end{cases} \tag{3.30}$$

Using the coordinate transformation

$$\begin{cases} x = t_1(U,V) + \lambda_2 c^2 U^3 \\ y = t_2(U,V) + \lambda_2 c U^2 + \dfrac{cf\lambda_2}{2} U^3 \end{cases}$$

and a coordinate change of the form $u = U$,$v = V + O(|U,V|^4)$, one checks that system (3.30) is C^∞-conjugate to

$$\begin{cases} \dfrac{du}{dt} = & v \\ \dfrac{dv}{dt} = & \lambda_2 u + \dfrac{b-f}{2}\lambda_2 u^2 - (ae + O(\lambda_2))u^3 + O(u^4) \\ & +v((e + 2a + O(\lambda_2))u + (f(e-2a) + eb + O(\lambda_2))u^2 + O(u^3)) \\ & +v^2 O(|u,v|^2) \end{cases}$$

Now we consider the corresponding family of dual forms

$$vdv - (\lambda_2 u + \frac{(b-f)}{2}\lambda_2 u^2 + (-ae + O(\lambda_2))u^3 + O(u^4))du - vO(|u,v|)du \quad (3.31)$$

Denote $f(u,\lambda_2) = \lambda_2 \frac{u^2}{2} + \frac{(b-f)}{2}\lambda_2 \frac{u^3}{3} + (-ae + O(\lambda_2))\frac{u^4}{4} + O(u^5)$.
Using the preparation theorem we find a coordinate change of the form

$$u = \Phi(U,\lambda_2) = \phi_1(\lambda_2)U + O(U^2)$$

with $\phi_1(0) = (-ae)^{1/4}$ such that

$$f(\Phi(U,\lambda_2),\lambda_2) = \frac{U^4}{4} + \xi_2(\lambda_2)\frac{U^3}{3} + \xi_1(\lambda_2)\frac{U^2}{2}$$

with

$$\xi_1(\lambda_2) = \frac{1}{(-ae)^{2/4}}\lambda_2 + O(\lambda_2^2) \qquad \xi_2\lambda_2) = O(\lambda_2)$$

Performing this coordinate transformation to the family (3.31) we get

$$vdv - (U^3 + \xi_2(\lambda_2)U^2 + \xi_1(\lambda_2)U)dU - vO(|\lambda_2, U, v|)dU$$

Further we use the translation

$$\begin{cases} U = r - \xi_2(\lambda_2)/3 \\ v = s \end{cases}$$

and we obtain

$$sds - (r^3 + (\xi_1(\lambda_2) + O(\lambda_2^2))r + O(\lambda_2^2))dr - sO(|\lambda_2, r, s|)dr$$

From this we may conclude that $\frac{\partial \mu_1}{\partial \lambda_2}(0) = 0$ and $\frac{\partial \mu_2}{\partial \lambda_2}(0)$ is proportional to $(-ae)^{1/2}$.

To conclude we consider the family

$$\left\{\begin{array}{l}\dfrac{dx}{dt}=y+ax^2+bxy+cy^2\\ \dfrac{dy}{dt}=\lambda_1+exy+fy^2\end{array}\right. \tag{3.32}$$

Using the coordinate transformation

$$\left\{\begin{array}{l}x=r\\ \\ y=c\lambda_1 r+s\end{array}\right.$$

followed by the transformation

$$\left\{\begin{array}{rl}r= & t_1(U,V)+\dfrac{3c^2\lambda_1}{2}U^2+\dfrac{c^2(2f+3b)\lambda_1}{6}U^3\\ s= & t_2(U,V)-\dfrac{c(f+2b)\lambda_1}{2}U^2-2c^2\lambda_1 UV\\ & -\dfrac{c(8f^2+9bf-4ce+2ac+b^2)\lambda_1}{6}U^3\end{array}\right.$$

and a coordinate change of the form $u=U$, $v=V+O(\lambda_1^2)O(|U,V|^2)+O(|U,V|^4)$ one checks that system (3.32) is C^∞-conjugate to

$$\left\{\begin{array}{rl}\dfrac{du}{dt}= & v\\ \dfrac{dv}{dt}= & \lambda_1-\lambda_1 fu+(ce\lambda_1+O(\lambda_1^2))u^2-(ae+O(\lambda_1))u^3+O(u^4)\\ & v(O(\lambda_1)+ge+2a+O(\lambda_1))u+(f(e-2a)+be+O(\lambda_1))u^2+O(u^3))\\ & +v^2(O(\lambda_1)O(|u,v|)+O(|u,v|^2)\end{array}\right.$$

Now we consider the corresponding family of dual forms

$$vdv-(\lambda_1-\lambda_1 fu+(ce\lambda_1+O(\lambda_1^2))u^2$$

$$-(ae+O(\lambda_1))u^3+O(u^4))du-vO(|\lambda_1,u,v|)du \tag{3.33}$$

Denote $f(u,\lambda_1) = \lambda_1 u - \frac{\lambda_1 f}{2}u^2 + \frac{ce\lambda_1}{3}u^3 - \frac{ae + O(\lambda_1)}{4}O(u^4) + O(u^5)$.
Using the preparation theorem we find a coordinate change of the form

$$u = \Phi(U,\lambda_1) = \phi(\lambda_1)U + O(U^2)$$

with $\phi_1(0) = (-ae)^{-1/4}$ such that

$$f(\Phi(U,\lambda_1),\lambda_1) = \zeta_o(\lambda_1)U + \zeta_1(\lambda_1)\frac{U^2}{2} + \zeta_2(\lambda_1)\frac{U^3}{3} + \frac{U^4}{4}$$

with

$$\zeta_0(\lambda_1) = \frac{\lambda_1}{(-ae)^{1/4}} + O(\lambda_1^2) \qquad \zeta_1(\lambda_1) = O(\lambda_1) = \zeta_2(\lambda_1)$$

Performing this coordinate transformation to the family (2.33) we get

$$vdv - (U^3 + \zeta_2(\lambda_1)U^2 + \zeta_1(\lambda_1)U + \zeta_0(\lambda_1))dU - vO(|\lambda, U, v|)dU$$

Further we use the translation

$$\begin{cases} U = r - \zeta_2(\lambda_1)/3 \\ v = s \end{cases}$$

and we obtain

$$sds - (r^3 + O(\lambda_1)r + \zeta_0(\lambda_1) + O(\lambda_1^2))dr - sO(|\lambda_1, r, s|)dr$$

and we may conclude that $\frac{\partial\mu_1}{\partial\lambda_1}(0)$ is proportional to $(-ae)^{-1/4}$.
This ends the genericity of the family $S_{(\lambda_1,\lambda_2,\lambda_3)}$.

The proof of the genericity of the family $E^1_{(\lambda_1,\lambda_2,\lambda_3)}$ with $e \neq a$ goes in the same way . In case $e = a$ the family $E^1_{(\lambda_1,\lambda_2,\lambda_3)}$ is not generic . One easily checks that $\frac{\partial\mu_1}{\partial\lambda_3}(0) = \frac{\partial\mu_2}{\partial\lambda_3}(0) = \frac{\partial\nu}{\partial\lambda_3}(0) = 0$. Therefore we consider the family $E^2_{(\lambda_1,\lambda_2,\lambda_3)}$.

Using the transformation

$$\begin{cases} x = t_1(U,V) \\ y = t_2(U,V) + c\lambda_3 U^3 \end{cases}$$

with $t_1(U,V)$, $t_2(U,V)$ of (3.28) (with $e = a$) and a change of coordinates of the form $u = U$, $v = V + O(|U,V|^4)$ one easily checks that the family $E^2_{(0,0,\lambda_3)}$ is C^∞-conjugate to

$$\begin{cases} \dfrac{du}{dt} = & v \\ \dfrac{dv}{dt} = & \lambda_3 u^2 - (a^2 - b\lambda_3)u^3 + O(u^4) \\ & +v(3au + \dfrac{(f-b)a + 2c\lambda_3}{2}u^2 + O(u^3)) \\ & +v^2 O(|u,v|^2) \end{cases}$$

Now we consider the corresponding family of dual forms

$$vdv - (\lambda_3 u^2 - (a^2 - b\lambda_3)u^3 + O(u^4))du$$

$$-v(3au\frac{(f-b)a + 2c\lambda_3}{2}u^2 + O(u^3))du - v^2 O(|u,v|^2)du \tag{3.34}$$

Denote $f(u,\lambda_3) = -\dfrac{a^2 - b\lambda_3}{4}u^4 + \dfrac{\lambda_3}{3}u^3 + O(u^5)$.
Using the preparation theorem we find the existence of a coordinate change of the form

$$u = \Phi(U,\lambda_3) = \phi_1(\lambda_3)U + O(U^2)$$

with $\phi_1(0) = (a^2)^{1/4}$ such that

$$f(\Phi(U,\lambda_3),\lambda_3) = -\frac{U^4}{4} + \eta(\lambda_3)\frac{U^3}{3}$$

with $\eta(\lambda_3) = (a^2)^{-3/4}\lambda_3 + O(\lambda_3^2)$.

Performing this transformation to the family (3.34) we get

$$vdv - (-U^3 + \eta(\lambda_3)U^2)dU - v(3a\phi_1^2(\lambda_3)U + O(U^2))dU - v^2O(|U,v|^2)dU$$

Further we use the translation

$$\begin{cases} U = r + \eta(\lambda_3)/3 \\ v = s \end{cases}$$

and we obtain

$$sds - (-r^3 + \frac{2\eta(\lambda_3)^2}{3}r + \frac{2\eta(\lambda_3)^3}{27})dr - s(a\phi_1^2(\lambda_3)\eta(\lambda_3) + O(\lambda_3^2) + O(r))dr - s^2O(|r,s|^2)dr$$

So we see that $\frac{\partial\mu_1}{\partial\lambda_3}(0) = \frac{\partial\mu_2}{\partial\lambda_3}(0) = 0$ and $\frac{\partial\nu}{\partial\lambda_3}(0) \neq 0$. For the parameters λ_1 and λ_2 the proofs are similar as in the saddle case , and so we may conclude that the family $E^2_{(\lambda_1,\lambda_2,\lambda_3)}$ is generic .

3.1.3 Nilpotent saddles , elliptic points and saddle-nodes of codimension 4

In quadratic systems there are also singularities with nilpotent 1-jet which are however more degenerate than the preceding ones , as we have seen in chapter 1. More precise , one encounters singularities which belong to one of the following submanifolds (of codimension 4 in the space of all germs of singular vector fields at $0 \in \mathbb{R}^2$):

(1)

$$\Sigma^4_{S\pm} = \{X | j^4X(0) \sim y\frac{\partial}{\partial x} + (x^3 + y(\pm x^2 + fx^3))\frac{\partial}{\partial y}, f \in \mathbb{R}\}.$$

Such singularities are called nilpotent saddles of codimension 4 .

(2)

$$\Sigma^4_{E_1^\pm} = \{X | j^4X(0) \sim y\frac{\partial}{\partial x} + (-x^3 + y(2\sqrt{2}x \pm x^2 + fx^3))\frac{\partial}{\partial y}, f \in \mathbb{R}\}.$$

Such singularities are called nilpotent elliptic points of codimension 4 and type 1 .

(3)

$$\Sigma^4_{SN\pm} = \{X | j^4X(0) \sim y\frac{\partial}{\partial x} + (x^4 + y(bx \pm x^2 + fx^3))\frac{\partial}{\partial y}, b > 0, f \in I\!R\}.$$

Such singularities are called nilpotent saddle-nodes of codimension 4 .

Similarly as before we call a 4-parameter family X_λ with $X_0 \in \Sigma^4_{S\pm}, \Sigma^4_{E_1^\pm}$ or $\Sigma^4_{SN\pm}$ generic if the mapping $((x,y),\lambda) \longrightarrow j^4X_\lambda(x,y)$ is transversal with the corresponding submanifold. (As said above we do not look at the elliptic points of codimension 4 and type 2.)

One can show that 4-parameter families X_λ cutting $\Sigma^4_{S\pm}, \Sigma^4_{E_1^\pm}$ or $\Sigma^4_{SN\pm}$ transversally can be brought - up to C^∞-equivalence - into the following normal forms :

1) If $X_0 \in \Sigma^4_{S\pm}$, this normal form reads

$$y\frac{\partial}{\partial x} + (x^3 + \mu_2(\lambda)x + \mu_1(\lambda) + y(\nu(\lambda) + b(\lambda)x$$

$$+\epsilon_2 x^2 + x^3 h(x,\lambda)) + y^2 Q(x,y,\lambda))\frac{\partial}{\partial y} \tag{3.35}$$

where $\epsilon_2 = \pm 1, \mu_1(\lambda), \mu_2(\lambda), \nu(\lambda), b(\lambda)$ (with $\mu_1(0) = \mu_2(0) = \nu(0) = b(0) = 0$) are C^∞-functions and Q is a C^∞-function of order N in (x,y,λ).
The genericity of the family amounts to

$$\frac{D(\mu_1,\mu_2,\nu,b)}{D(\lambda_1,\lambda_2,\lambda_3,\lambda_4)} \neq 0 \tag{3.36}$$

(2) If $X_0 \in \Sigma^4_{E_1^\pm}$, this normal form reads

$$y\frac{\partial}{\partial x} + (-x^3 + \mu_2(\lambda)x + \mu_1(\lambda) + y(\nu(\lambda) + b(\lambda)x$$

$$+\epsilon_2 x^2 + x^3 h(x,\lambda)) + y^2 Q(x,y,\lambda))\frac{\partial}{\partial y} \tag{3.37}$$

where $\epsilon_2 = \pm 1, \mu_1(\lambda), \mu_2(\lambda), \nu(\lambda), b(\lambda)$ (with $\mu_1(0) = \mu_2(0) = \nu(0) = 0$, $b(0) = 2\sqrt{2}$) are C^∞-functions and Q is a C^∞-function of order N in (x, y, λ). The genericity of the family amounts to

$$\frac{D(\mu_1, \mu_2, \nu, b)}{D(\lambda_1, \lambda_2, \lambda_3, \lambda_4)} \neq 0 \tag{3.38}$$

(3) If $X_0 \in \Sigma^4_{SN\pm}$, this normal form reads

$$y\frac{\partial}{\partial x} + (x^4 + \mu_3(\lambda)x^2 + \mu_2(\lambda)x + \mu_1(\lambda) + y(\nu(\lambda) + b(\lambda)x$$

$$+\epsilon_2 x^2 + x^3 h(x, \lambda)) + y^2 Q(x, y, \lambda)\frac{\partial}{\partial y} \tag{3.39}$$

where $\epsilon_2 = \pm 1, \mu_1(\lambda), \mu_2(\lambda), \nu(\lambda), b(\lambda)$ (with $\mu_1(0) = \mu_2(0) = \mu_3(0) = \nu(0) = 0$, $b(0) > 0$) are C^∞-functions and Q is a C^∞-function of order N in (x, y, λ). The genericity of the family amounts to

$$\frac{D(\mu_1, \mu_2, \mu_3, \nu)}{D(\lambda_1, \lambda_2, \lambda_3, \lambda_4)} \neq 0 \tag{3.40}$$

Using these normal forms one can prove that every quadratic nilpotent saddle of codimension 4 and nilpotent saddle-node of codimension 4 has a generic unfolding among the quadratic systems .

THEOREM 5

(1) *The 4-parameter family*

$$S_{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)} : \begin{cases} \dfrac{dx}{dt} = y + ax^2 + bxy + cy^2 \\ \dfrac{dy}{dt} = \lambda_1 + \lambda_2 x + \lambda_3 y + (\lambda_4 - 2a)xy + fy^2 \end{cases} \tag{3.41}$$

with $a \neq 0$ and $b + 2f \neq 0$ is a generic unfolding of a nilpotent saddle of codimension 4 .

(2) *The 4-parameter family*

$$SN_{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)} : \begin{cases} \dfrac{dx}{dt} = y + ax^2 + bxy + cy^2 \\ \dfrac{dy}{dt} = \lambda_1 + \lambda_2 x + \lambda_3 y + \lambda_4 x^2 + fy^2 \end{cases} \tag{3.42}$$

with $a \neq 0$ and $f \neq 0$ is a generic unfolding of a nilpotent saddle-node of codimension 4.

Proof :

We only prove case (1) . For case (2) we refer to the appendix of this section.
To prove the genericity of the family $S_{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}$ we have to show that bringing this family into the normal form (3.35) condition (3.36) is satisfied. Again we treat the parameters separately.
Using the transformation

$$\begin{cases} x = t_1(U,V) - c\lambda_4 U^3/6 \\ y = t_2(U,V) + c\lambda_4 U^2 V/2 \end{cases}$$

with $t_1(U,V)$, $t_2(U,V)$ of (3.28) (with e=-2a) and a change of coordinates of the form $u = U$, $v = V + O(|U,V|^4)$ we get that the family $S_{(0,0,0,\lambda_4)}$ is C^∞-conjugated to

$$\begin{cases} \dfrac{du}{dt} = v \\ \dfrac{dv}{dt} = (2a^2 - a\lambda_4)u^3 + O(u^4) + v(\lambda_4 u \\ \qquad - \dfrac{2(b+f)a + \lambda_4(b+f)}{2}u^2 + O(u^3)) + v^2 O(|u,v|^3) \end{cases}$$

From this it is readily seen that $\dfrac{\partial\mu_1}{\partial\lambda_4}(0) = \dfrac{\partial\mu_2}{\partial\lambda_4}(0) = \dfrac{\partial\nu}{\partial\lambda_4}(0) = 0$ and $\dfrac{\partial b}{\partial\lambda_4}(0) \neq 0$.
For the parameters λ_1 , λ_2 and λ_3 the proofs are exactly the same as in the case of the nilpotent saddle of codimension 3.

3.1.4 On the versal unfolding of the elliptic point of codimension 4 and type 1

In this section we will consider all elliptic points of codimension 4 and type 1 as obtained in theorem 4 of chapter 1 and we will show that such a singularity can never have a quadratic versal unfolding. This fact does not really need to be further argumented since we already know that a quadratic nilpotent focus does not occur, while near the elliptic point of codimension 4 and type 1 (as a general system) one obtains nilpotent foci merely by making the coefficient before $xy\frac{\partial}{\partial y}$ in the normal form (3.37) less than $2\sqrt{2}$.
Nevertheless we will make the necessary normal form calculations in order to show that this coefficient before $xy\frac{\partial}{\partial y}$ always undergoes a generic fold near such an elliptic point of codimension 4 and type 1.

As seen in theorem 4 of chapter 1 the expression of these quadratic vector fields is :

$$\begin{cases} \dfrac{dx}{dt} = y + ax^2 + bxy + cy^2 \\ \dfrac{dy}{dt} = 2axy + fy^2 \end{cases} \tag{3.43}$$

with $a \neq 0, b \neq 0$. We may suppose $a > 0$.
Let us consider quadratic perturbations having at the origin a singularity with 1-jet $y\frac{\partial}{\partial x}$ and which is not of cusp type.

$$\begin{cases} \dfrac{dx}{dt} = y + (a + a_{20})x^2 + (b + a_{11})xy + (c + a_{02})y^2 \\ \dfrac{dy}{dt} = (2a + b_{11})xy + (f + b_{02})y^2 \end{cases} \tag{3.44}$$

We perform the coordinate change

$$\begin{cases} u = x \\ v = y + (a + a_{20})x^2 + (b + a_{11})xy + (c + a_{02})y^2 \end{cases}$$

and get :

$$\begin{cases} \dfrac{du}{dt} = v \\ \dfrac{dv}{dt} = -c_{30}u^3 + O(u^4) + v(c_{11}u + O(u^2)) + v^2O(1) \end{cases} \tag{3.45}$$

with

$$\begin{cases} c_{30} = 2a^2 + a(b_{11} + 2a_{20}) + a_{20}b_{11} \\ c_{11} = 4a + b_{11} + 2a_{20} \end{cases} \tag{3.46}$$

Using Macsyma it is easy to show that system (3.45) is C^∞-conjugate to :

$$\begin{cases} \dfrac{du}{dt} = v \\ \dfrac{dv}{dt} = -c_{30}u^3 + O(u^4) + v(c_{11}u + O(u^2)) + v^2O(|u, v|^2) \end{cases}$$

From now on we work with the family of dual forms:

$$vdv + (c_{30}u^3 + O(u^4))du - v(c_{11}u + O(u^2))du - v^2O(|u, v|^2)du \tag{3.47}$$

There exists a transformation of the form

$$\begin{cases} r = c_{30}^{1/4}u + O(u^2) \\ s = v \end{cases}$$

which changes the family (3.47) into

$$sds + r^3dr - s((c_{11}/\sqrt{c_{30}})r + O(r^2))dr - s^2O(|r, s|^2)dr$$

Direct calculation shows that the coefficient of $-rsdr, c_{11}/\sqrt{c_{30}}$, is always greater or equal than $2\sqrt{2}$.Moreover , the Taylor expansion of this coefficient is given by:

$$2\sqrt{2} + \frac{\sqrt{2}(b_{11} - 2a_{20})^2}{16a^2} + O(|b_{11}, a_{20}|^3)$$

So we see that the coefficient undergoes a fold.

We end by proving the genericity of this fold. For this purpose we have to show that bringing the family

$$X_\lambda : \begin{cases} \dfrac{dx}{dt} = y + ax^2 + bxy + cy^2 \\ \dfrac{dy}{dt} = \lambda_1 + \lambda_2 x + \lambda_3 y + 2axy + fy^2 \end{cases}$$

with $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ into the normal form

$$y\frac{\partial}{\partial x} - [-x^3 + \mu_2(\lambda)x + \mu_1(\lambda) + y(\nu(\lambda) + b(\lambda)x \pm x^2 + x^3 h(x,\lambda)) + y^2 Q(x,y,\lambda)]\frac{\partial}{\partial y}$$

the map $\mathbb{R}^3 \to \mathbb{R}^3$, $\lambda \mapsto (\mu_1(\lambda), \mu_2(\lambda), \nu(\lambda))$ satisfies the condition

$$\frac{D(\mu_1, \mu_2, \nu)}{D(\lambda_1, \lambda_2, \lambda_3)}(0) \neq 0$$

The proof is completely similar to the proof of theorem 3 .

Appendix

Transformations used in the proof of the genericity of the family $C_{(\lambda_1,\lambda_2)}$

For the parameter λ_2 we used the transformation:

$$\begin{cases} x = t_1(U,V) - c\lambda_2 U^2/2 \\ y = t_2(U,V) \end{cases}$$

with

$$\begin{cases} t_1(U,V) = U + cUV + (b+f)U^2/2 \\ t_2(U,V) = V - aU^2 + fUV \end{cases} \tag{3.48}$$

For the parameter λ_1 we used the translation

$$\begin{cases} x = u \\ y = c\lambda_1 u + v \end{cases}$$

followed by the transformation

$$\begin{cases} u = t_1(U,V) \\ v = t_2(U,V) \end{cases}$$

with $t_1(U,V)$ and $t_2(U,V)$ of (3.48).

Transformations used in the proof of the genericity of the family $C_{(\lambda_1,\lambda_2,\lambda_3)}$.

For the parameters λ_2 and λ_3 we used the transformation:

$$\begin{cases} x = t_1(U,V) \\ y = t_2(U,V) \end{cases} \tag{3.49}$$

with

$$
\begin{cases}
t_1(U,V) = & U + (b+f)U^2/2 + (2c - a(b+2f))UV/3 \\
& (b^2 - 2ac + 3bf + 2f^2)U^3/6 + (bc - 2ab^2 + (4c - 6ab)f - 4af^2)U^2V/6 \\
& -c(4af + 2c + 2ab)UV^2/3 + (54f^3 + (99b + 80a^2)f^2 \\
& (-118ac + 54b^2 + 80a^2b)f + 128c^2 - 14abc + 9b^3 + 20a^2b^2)U^4/216 \\
& (-12af^3 + 12(c - 2ab)f^2 + (3(b + 4a^2)c - 15ab^2)f \\
& 6a^2bc - 3ab^3)U^3V/18 + (48a^2f^3 + (96a^2b - 384ac)f^2 \\
& (-17c^2 - 30abc + 5a^2b^2)f - 8bc^2 - 7ab^2c + a^2b^3)U^2V^2/18 \\
& (4acf(af + c + ab) + c^3 + abc(2c + ab))UV^3/9
\end{cases}
$$

$$
\begin{cases}
t_2(U,V) = & V + AU^2 - fUV - (a(b+2f) + c)V^2/3 \\
& (2c - 5af - ab)U^3/3 + (4af^2 + 2a^2b + 3f^2 - ac)U^2V/3 \\
& -2f(ab + c + af)UV^2/3 - (57af^2 + 12(2c - 3ab - 4a^3)f \\
& 6(b + 6a^2)c + 15ab^2 + 24a^3b)U^4/36 \\
& (27f^3 + 94a^2f^2 - (41ac + 67a^2b)f - 8c^2 - 16abc + 10a^2b^2)U^3V/27 \\
& -(18af^3 + (9c + 9ab + 4a^3)f^2 + 2a^2f(2ab - 7c)f - 8ac^2 \\
& -7a^2bc + a^3b^2)U^2V^2/9 + (4af^2(af + c + ab) \\
& (c^2 + 2abc + a^2b^2)f)UV^3/9
\end{cases}
$$

For the parameter λ_1 we used the translation

$$
\begin{cases}
x = u \\
y = -(2af - 2c + ab)\lambda_1 u/3 + v
\end{cases}
$$

followed by the transformation

$$\begin{cases} u = t_1(U,V) \\ v = t_2(U,V) \end{cases}$$

with $t_1(U,V)$ and $t_2(U,V)$ of (3.49)

Transformations used in the proof of the genericity of the family $SN_{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}$.
For the parameter λ_4 we used the transformation

$$\begin{cases} x = t_1(U,V) \\ y = t_2(U,V) \end{cases}$$

with

$$\begin{cases} t_1(U,V) = & U + (f+b)U^2/2 + cUV + (2f^2+3bf-4ac+b^2)U^3/6 \\ & c(b+2f)U^2V/2 + (b^3-4abc+6b^2f-12acf+11bf^2+6f^3)U^4/24 \\ & (b^2c+2a^2c+4bcf+6cf^2)U^3V/6 \\ t_2(U,V) = & V - aU^2 + fUV - afU^3 + f^2U^2V \\ & +a(4ac-b^2-11f^2)U^4/12 + f^3U^3V \end{cases} \tag{3.50}$$

For the parameter λ_3 we used the transformation

$$\begin{cases} x = t_1(U,V) - c\lambda_3U^2/2 - c(b+f)\lambda_3U^3/3 \\ y = t_2(U,V) \end{cases}$$

with $t_1(U,V)$ and $t_2(U,V)$ of (3.50).
For the parameter λ_2 we used the transformation

$$\begin{cases} x = t_1(U,V) \\ y = t_2(U,V) + c\lambda_2U^2 \end{cases}$$

with $t_1(U,V)$ and $t_2(U,V)$ of (3.50).

For the parameter λ_1 we used the translation

$$\begin{cases} x = u \\ y = c\lambda_1 u + v \end{cases}$$

followed by the transformation

$$\begin{cases} u = t_1(U, V) \\ v = t_2(U, V) + c\lambda_2 U^2 \end{cases}$$

with $t_1(U, V)$ and $t_2(U, V)$ of (3.50).

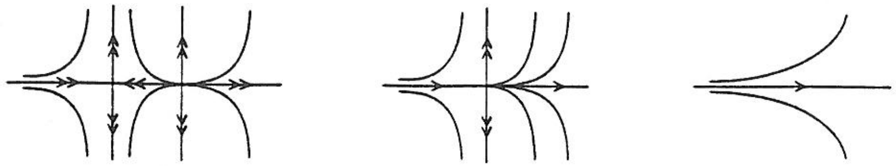
3.2 Semi-hyperbolic bifurcations

In this section we treat the semi-hyperbolic bifurcation . By this we mean an unfolding of a semi-hyperbolic singularity X of finite codimension . We will show that every such quadratic singularity has a quadratic versal unfolding . The standard model to which all codimension k semi-hyperbolic bifurcations can be reduced to by C^0-equivalence is :

$$X_k^{\pm,\pm} = (\nu_0 + \nu_1 x + \ldots + \nu_{k-1} x^{k-1} \pm x^{k+1})\frac{\partial}{\partial x} \pm y \frac{\partial}{\partial y} \tag{3.51}$$

where the $(\nu_0, \ldots, \nu_{k-1})$ are independent parameters .
As is well known the bifurcation diagrams of these bifurcations only deal with singularities and are given by catastrophy theory . See figures 3.5-3.7.
Semi-hyperbolic bifurcation of codimension 1

$$(x^2 + \lambda)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$



$$(x^2 + \lambda)\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

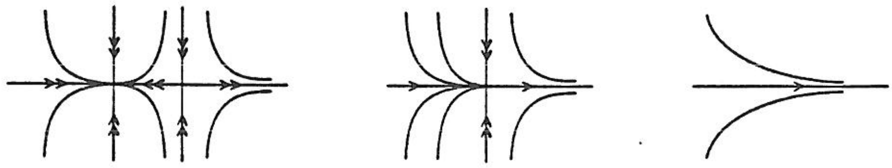


fig. 3.5

Semi-hyperbolic bifurcation of codimension 2

$$(\pm x^3 + \lambda x + \mu)\frac{\partial}{\partial x} \pm y\frac{\partial}{\partial y}$$

there are four cases , of which we represent the (+,-)-one :

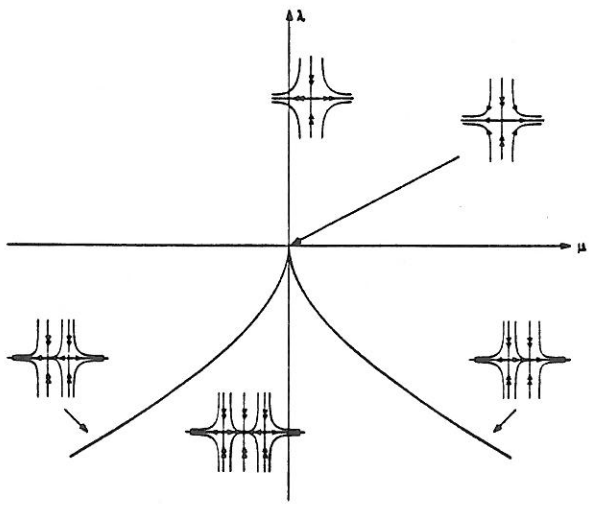


fig. 3.6

Semi-hyperbolic bifurcation of codimension 3

$$(x^4 + \lambda x^2 + \mu x + \nu)\frac{\partial}{\partial x} \pm y\frac{\partial}{\partial y}$$

The bifurcation diagram looks as follows :

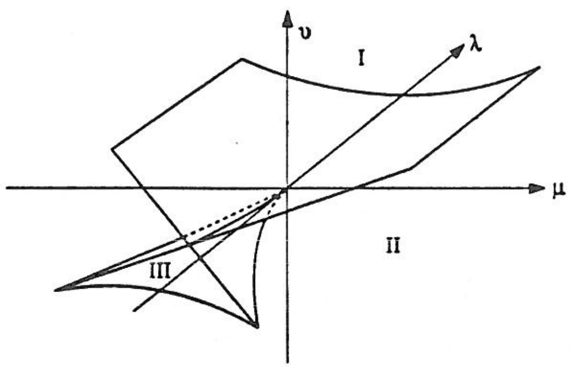


fig. 3.7

Vector fields in regions I,II,III have respectively 0,2 and 4 singularities.

We proceed by recalling that a C^∞-family X_λ with X_0 semi-hyperbolic of codimension k can be brought -up to C^∞-conjugacy - into the normal form :

$$[\sum_{i=0}^{k+1} a_i(\lambda)x^i + Q_1(x,y,\lambda)]\frac{\partial}{\partial x} \pm [(\sum_{i=0}^{k} b_i(\lambda)x^i)y + Q_2(x,y,\lambda)]\frac{\partial}{\partial y} \tag{3.52}$$

with $a_i(\lambda), b_i(\lambda), Q_j(x,y,\lambda)$ of class C^∞ ; $a_i(0) = 0, \forall i \in \{0,\ldots,k\}$, $a_{k+1} \neq 0$, $b_0(0) > 0$ and $Q_j(x,y,\lambda) = O(||x,y||^{k+2}) + O(||\lambda||^2)$.

Proposition

Let X_λ be a k-parameter unfolding so that the mapping $A : \mathbb{R}^k \mapsto \mathbb{R}^k$, $\lambda \to (a_0(\lambda),\ldots,a_{k-1}(\lambda))$ has maximal rank at the origin .
Then X_λ is versal , i.e X_λ is (C^0,C^0)-equivalent to $X_k^{\pm,\pm}$.

Proof :

Multiplying (3.52) with the function $g(x,\lambda) = [\sum_{i=0}^{k} b_i(\lambda)x^i]^{-1}$, it is easily seen that X_λ is C^∞-equivalent to :

$$\widetilde{X_\lambda}(x,y) = [\sum_{i=0}^{k+1} \alpha_i(\lambda)x^i + \tilde{Q}_1(x,y,\lambda)]\frac{\partial}{\partial x}] \pm [y + \tilde{Q}_2(x,y,\lambda)]\frac{\partial}{\partial y}$$

where $\tilde{Q}_i(x,y,\lambda) = O(||x,y||^{k+2}) + O(||\lambda||^2)$.

Using the expansion $g(x,\lambda) = \sum_{j=0}^{k+1} g_j(\lambda)x^j + O(x^{k+2})$, we find that
$\alpha_i(\lambda) = \sum_{j=0}^{i} g_j(\lambda)a_{i-j}(\lambda)$.
It is readily seen that $\alpha_i(0) = 0$, $\forall i \in \{0,\ldots,k\}$, $\alpha_{k+1}(0) \neq 0$ and the mapping $\lambda \to (\alpha_0(\lambda),\ldots,\alpha_{k-1}(\lambda))$ has maximal rank at the 0.

To continue we make a reduction to the center manifold . For any $r \in \mathbb{N}$ we know the existence of a C^{r+1}-center manifold W^c . Since this manifold is tangent at the origin $(x,y,\lambda) = (0,0,0)$ to the (x,λ)-space , we can approximate it as a (local) graph

$$W^c = \{(x,y,\lambda) \in \mathbb{R}^2 \times \mathbb{R}^k | y = h(x,\lambda)\}$$

where h is defined on some neighborhood $U \subset I\!R \times I\!R^k$ of $(0,0) \in I\!R \times I\!R^k$ with $h(0,0) = 0$ and $Dh_{(0,0)} \equiv O$.
We may approximate h as a Taylor series

$$h(x,\lambda) = \sum_{j\geq 0} h_j(\lambda)x^j.$$

Following a same reasoning for the vector field $\widetilde{X}_o$ as in the proof of theorem 1 of chapter 2 , we find that $h_j(0) = 0 \quad \forall j \in \{0,\ldots,k+1\}$.

Next we consider the reduction of $\tilde{X}_\lambda(x,y)$ to the center manifold W^c :

$$\begin{aligned} Y_\lambda(x) &= f(x,\lambda)\frac{\partial}{\partial x} \\ &= [\beta_0(\lambda) + \beta_1(\lambda)x + \ldots + \beta_k(\lambda)x^k + \beta_{k+1}(\lambda)x^{k+1} + O(x^{k+2})]\frac{\partial}{\partial x} \end{aligned}$$

Using the fact that $h(x,\lambda) = O(||\lambda||^2) + O(||\lambda||)O(x) + O(x^{k+1})$, we find that $\beta_0(0) = \ldots = \beta_k(0) = 0$, $\beta_{k+1}(0) \neq 0$ and that the mapping $\lambda \rightarrow (\beta_0(\lambda),\ldots,\beta_{k-1}(\lambda))$ has maximal rank at the origin .

Using the division theorem of Lassalle [Las] and a same reasoning as in theorem 6.3 of [Brö] we find that

$$f(x,\lambda) = \pm Q(x,\lambda)[x^{k+1} + \mu_k(\lambda)x^k + \ldots + \mu_1(\lambda)x + \mu_0(\lambda)]$$

with $Q(x,\lambda), \mu_i(\lambda)$ of class C^l , $l = [\frac{r-(k+1)}{k+1}]$. Q, μ_i are defined on a neighborhood of the origin , with $Q(0,0) > 0$ and $\mu_i(0) = 0, \quad \forall i$.

Since $\lambda \rightarrow (\beta_0(\lambda),\ldots,\beta_{k-1}(\lambda))$ has maximal rank at the origin , one easily checks that the map $\lambda \rightarrow (\mu_0(\lambda),\ldots,\mu_{k-1}(\lambda))$ has maximal rank at the origin .

So $f(x,\lambda)\dfrac{\partial}{\partial x}$ is C^0-equivalent to

$$\pm(x^{k+1} + \mu_k(\lambda)x^k + \mu_{k-1}x^{k-1} + \ldots + \mu_1(\lambda)x + \mu_0(\lambda))\frac{\partial}{\partial x}$$

To conclude we use the translation $x = u - \frac{\mu_k(\lambda)}{k+1}$ to transform this expression into (we write x instead of u) :

$$\pm(x^{k+1} + \delta_{k-1}(\lambda)x^{k-1} + \ldots + \delta_0(\lambda))\frac{\partial}{\partial x}$$

where $\delta_i(\lambda) = \mu_i(\lambda) + O(\lambda^2)$.

So , using a theorem of Palis-Takens (see section 1.2.2) , we may conclude that X_λ is (C^0, C^0)-equivalent to the family $X_k^{\pm,\pm}$.

THEOREM 6

(1) *The 1-parameter family*

$$SH_{(\lambda_0)} : \begin{cases} \dfrac{dx}{dt} = \lambda_0 + ax^2 + bxy + cy^2 \\ \dfrac{dy}{dt} = y + dx^2 + exy + fy^2 \end{cases} \qquad \textit{with } a \neq 0 \qquad (3.53)$$

is a versal semi-hyperbolic bifurcation of codimension 1 .

(2) *The 2-parameter family*

$$SH_{(\lambda_0,\lambda_1)} : \begin{cases} \dfrac{dx}{dt} = \lambda_0 + \lambda_1 x + bxy + cy^2 \\ \dfrac{dy}{dt} = y + dx^2 + exy + fy^2 \end{cases} \qquad \textit{with } bd \neq 0 \qquad (3.54)$$

is a versal semi-hyperbolic bifurcation of codimension 2 .

(3) *The 3-parameter family*

$$SH_{(\lambda_0,\lambda_1,\lambda_2)} : \begin{cases} \dfrac{dx}{dt} = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + cy^2 \\ \dfrac{dy}{dt} = y + dx^2 + exy + fy^2 \end{cases} \qquad \textit{with } cd \neq 0 \qquad (3.55)$$

is a versal semi-hyperbolic bifurcation of codimension 3 .

Proof :

Instead of doing the normal form calculations , we prove this theorem using the center manifold reduction . We only give the proof of (3) , the proofs of (1) and (2) go in the same way . We know that the center manifold can be represented as a local graph

$$y = h(x, \lambda) = h_0(\lambda) + h_1(\lambda)x + h_2(\lambda)x^2 + O(x^3)$$

One easily checks that $h_0(\lambda) = O(||\lambda||^2), h_1(\lambda) = O(||\lambda||)$ and $h_2(0) = -d$ (see chapter 2) . For the behaviour on the center manifold we find :

$$\begin{aligned}
\dot{x} &= \lambda_0 + \lambda_1 x + \lambda_2 x^2 + ch(x, \lambda)^2 \\
&= (\lambda_0 + ch_0^2(\lambda)) + (\lambda_1 + 2ch_0(\lambda)h_1(\lambda))x + (\lambda_2 + 2ch_0(\lambda)h_2(\lambda))x^2 + O(x^3) \\
&= f(x, \lambda)
\end{aligned}$$

with $f(x, 0) = cd^2x^4 + O(x^5)$ (see chapter 2).
From the previous theorem it is clear that this ends the proof .

3.3 Hopf-Takens bifurcations

In this section we study the C^∞-unfoldings of singularities of finite codimension whose 1-jet has a pair of non-zero purely imaginary eigenvalues . We will show that every such quadratic singularity has a versal unfolding among the quadratic vector fields. The standard model to which all codimension k Hopf-Takens bifurcation can be reduced to by (weak-C^∞,C^∞)-equivalence is :

$$X_k^\pm = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} - (\pm(x^2+y^2)^k + \mu_{k-1}(x^2+y^2)^{k-1} + \dots$$

$$+\mu_1(x^2+y^2) + \mu_0)(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}) + O(|x,y|^{2k+1}) \qquad (3.56)$$

where the $(\mu_0, \dots, \mu_{k-1})$ are independent parameters . We call the families $X_k^\pm$ the 'standard generalized Hopf bifurcations' or the 'standard Hopf-Takens bifurcations'.

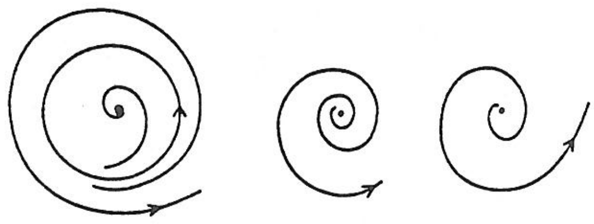
Definition

Two k-parameter families X_μ and Y_μ are called **(weak-C^∞, C^∞)-equivalent** if there exist a neighborhood U of $0 \in \mathbb{R}^2$, C^∞ local diffeomorphisms h_μ defined on U and a C^∞-diffeomorphism g defined on a neighborhood V of $0 \in \mathbb{R}^k$ such that for each $\mu \in V$, h_μ sends singularities of X_μ to singularities of $Y_{g(\mu)}$ preserving the type (sink or source) and sends closed orbits of X_μ to closed orbits of $Y_{g(\mu)}$ also preserving their repelling or attracting nature .

As is well known the bifurcation diagrams of these bifurcations only deal with the number of limit cycles around a singular point . See figures 3.8-3.10.

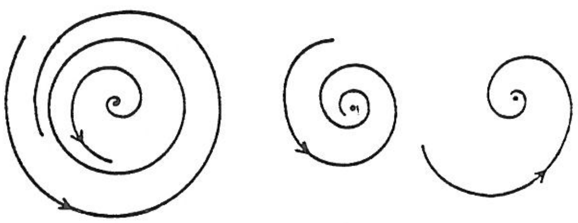
Codimension 1 : The Andronov-Hopf bifurcation

$$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} + ((x^2 + y^2) + \lambda)(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$$



$\lambda < 0$ $\qquad$ $\lambda = 0$ $\qquad$ $\lambda > 0$

$$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} - ((x^2 + y^2) + \lambda)(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$$



$\lambda < 0$ $\qquad$ $\lambda = 0$ $\qquad$ $\lambda > 0$

fig. 3.8

Codimension 2 : The Hopf-Takens bifurcation of codimension 2

$$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \pm ((x^2+y^2)^2 + \mu_1(x^2+y^2) + \mu_0)(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$$

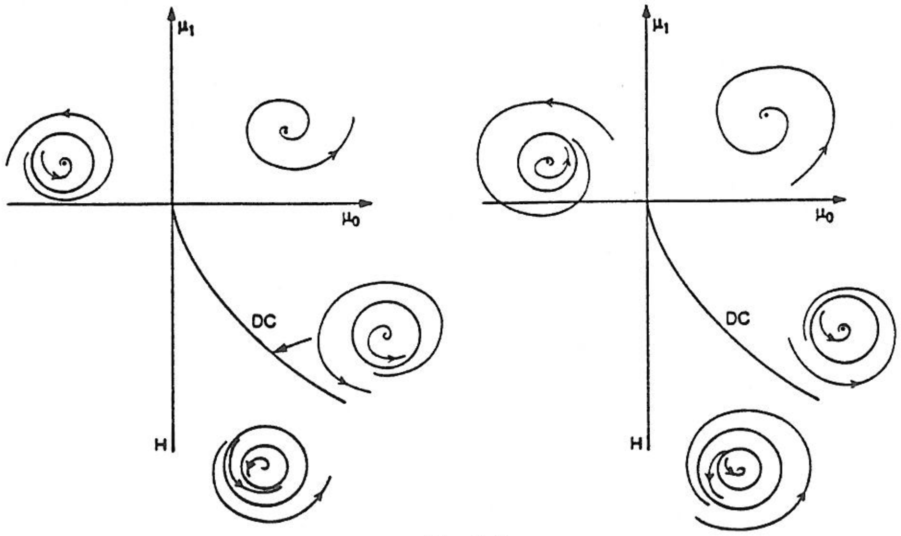


fig 3.9

Codimension 3 : The Hopf-Takens bifurcation of codimension 3

$$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \pm ((x^2+y^2)^3 + \mu_2(x^2+y^2)^2 + \mu_1(x^2+y^2) + \mu_0)(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$$

The bifurcation diagram looks as follows :

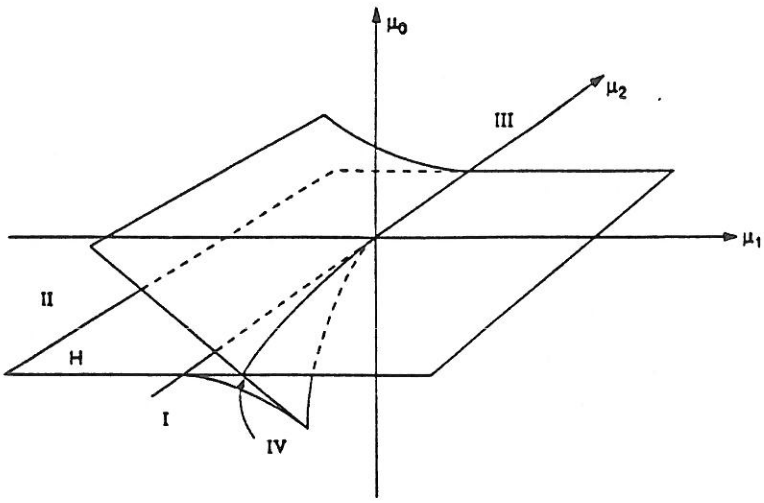


fig 3.10

In the regions I,II,III,IV the vector fields have respectively 1,2,0 or 3 limit cycles.

We proceed by recalling that a C^∞-family X_λ with X_o a Hopf-Takens bifurcation of codimension k can be brought - up to C^∞-conjugacy - into the normal form :

$$[\sum_{i=0}^{k} b_i(\lambda)(x^2+y^2)^i](x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}) +$$

$$[\sum_{j=0}^{k} a_j(\lambda)(x^2+y^2)^j](x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}) + \tilde{X}(x,y,\lambda) \tag{3.57}$$

with $a_0(0) = \ldots = a_{k-1} = 0$, $a_k(0) \neq 0$, $b_0(0) > 0$ and $\tilde{X} = O(||x,y||^{2k+2})$.

Proposition

Let X_λ be a k-parameter unfolding so that the mapping $A : \mathbb{R}^k \mapsto \mathbb{R}^k$,
$\lambda \rightarrow (a_0(\lambda), \ldots, a_{k-1}(\lambda))$ has maximal rank at the origin .
Then X_λ is versal , i.e X_λ is (weak-C^∞ , C^∞)-equivalent to $X_k^\pm$.

Proof :

In order to check the transversality of the family X_λ we repeat the proof of [T2]. Each member in the family (3.57) can be blown-up and we obtain a C^∞ k-parameter family Y of vector fields on $S^1 \times \mathbb{R}$ of the form ;

$$\begin{aligned} Y &= b(\theta,r,\lambda)\frac{\partial}{\partial\theta} + a(\theta,r,\lambda)\frac{\partial}{\partial r} \\ &= (b_o(\lambda) + b_1(\lambda)r^2 + \ldots + b_k(\lambda)r^{2k} + r^{2k+1}\xi(\theta,r,\lambda))\frac{\partial}{\partial\theta} \\ &\quad +(a_o(\lambda) + a_1(\lambda)r^2 + \ldots + a_k(\lambda)r^{2k} + r^{2k+1}\eta(\theta,r,\lambda))r\frac{\partial}{\partial r} \end{aligned} \tag{3.58}$$

The topological properties of the family (3.56) in the neighborhood of $\{r = 0\}$ do not change if we multiply by $[b(\theta,r,\lambda)]^{-1}$, since this function is non-zero and positive on some neighborhood of $\{r = 0\} \times \{0\}$ in $S^1 \times \mathbb{R} \times \mathbb{R}^k$. We obtain :

$$Y_2 = \frac{\partial}{\partial\theta} + (c_o(\lambda) + c_1(\lambda)r^2 + \ldots + c_k(\lambda)r^{2k} + r^{2k+1}\zeta(\theta,r,\lambda))r\frac{\partial}{\partial r} \tag{3.59}$$

with $c_o(0) = \ldots = c_{k-1}(0) = 0$, $c_k(0) \neq 0$ and the mapping $\lambda \rightarrow (c_o(\lambda), \ldots, c_{k-1}(\lambda))$ has maximal rank at the origin .

Up to some C^∞-coordinate change Φ , preserving the parameter , we can put this family in such a form that the closed orbits (near the origin) of $\Phi_*(Y_2) = Y_3$ are of the form $\{r = r_o\}$ with $r_o \in \mathbb{R}$. If $\phi_{Y_2,t}(\theta, r, \lambda)$ denotes the flow of Y_2 , and $\phi^r_{Y_2,t}(\theta, r, \lambda)$ the r-component , we define Φ by :

$$\Phi(\theta, r, \lambda) = (\theta, \frac{1}{2\pi}\int_0^{2\pi} \phi^r_{Y_2,u}(\theta, r, \lambda)du, \lambda)$$

Since $D\Phi(0,0,0) = Id$, Φ is a local C^∞-diffeomorphism at $(0,0,0)$, and we get :

$$Y_3 = \frac{\partial}{\partial\theta} + (d_o(\lambda) + d_1(\lambda)r^2 + \ldots + d_k(\lambda)r^{2k} + r^{2k+1}\zeta(\theta, r, \lambda))r\frac{\partial}{\partial r}$$

with $d_o(0) = \ldots = d_{k-1}(0) = 0$, $d_k(0) \neq 0$ and the mapping $\lambda \rightarrow (d_o(\lambda), \ldots, d_{k-1}(\lambda))$ has maximal rank at the origin .

The Poincaré -mapping of Y_3 is of the form :

$$P(\lambda, r) = (1 + \alpha_o(\lambda))r + \alpha_1(\lambda)r^3 + \ldots + \alpha_{k-1}(\lambda)r^{2k+1} + r^{2k+2}R(\lambda, r)$$

with $\alpha_o(0) = \ldots = \alpha_{k-1}(0) = 0$, $\alpha_k(0) \neq 0$ and $\lambda \rightarrow (\alpha_o(\lambda), \ldots, \alpha_{k-1}(\lambda))$ has maximal rank at the origin .
Let us now consider

$$D(\lambda, r) = r(P(\lambda, r) - r - P(\lambda, -r) - r)$$

We see that :

(1) $D(\lambda, 0) = 0$

(2) $D(\lambda, r) = D(\lambda, -r)$

(3) $D(\lambda, r_o) = 0$ iff r_o is the radius of a closed orbit of Y_3

(4) $D(\lambda, r_o) > (<)0$ iff $P(\lambda, r_o) < (>)0$

We call a function with these properties a 'displacement function' of the unfolding .

$$D(\lambda, r) = 2\alpha_o(\lambda)r^2 + 2\alpha_1(\lambda)r^4 + \ldots + 2\alpha_k(\lambda)r^{2k+2} + r^{2k+3}\tilde{R}(\lambda, r)$$

A displacement function of the standard generalized Hopf bifurcation $X^k_{\pm}$ can be given by:

$$D^k_{\pm}(\mu, r) = \pm r^{2k+2} + \mu_{k-1}r^{2k} + \ldots + \mu_0 r^2$$

By the Malgrange preparation theorem (for even functions) we know that in some neighborhood of $(r = 0, \lambda = 0)$ there exists $U(\lambda, r)$, $f_o(\lambda), \ldots f_k, (\lambda)$, all C^∞ and $U(0,0) > 0$, such that

$$D(\lambda, r) = U(\lambda, r)(\pm r^{2k+2} + f_k(\lambda)r^{2k} + \ldots + f_1(\lambda)r^2 + f_0(\lambda))$$

Since $D(\lambda, 0) = 0 \ \forall\lambda$, we necessarily have $f_o(\lambda) \equiv 0$.
Putting $\mu_0(\lambda) = f_1(\lambda), \ldots, \mu_{k-1}(\lambda) = f_k(\lambda)$:

$$\begin{aligned} D(\lambda, r) &= U(\lambda, r)(\pm r^{2k+2} + \mu_{k-1}(\lambda)r^{2k} + \ldots + \mu_0(\lambda)r^2) \\ &= U(\lambda, r)D^k_{\pm}(\mu_0(\lambda), \ldots, \mu_{k-1}(\lambda), r) \end{aligned}$$

It is readily seen that the mapping $\lambda \longrightarrow (\mu_0(\lambda), \ldots, \mu_{k-1}(\lambda))$ has maximal rank at the origin .
From this we may conclude that X_λ is a versal unfolding .

Using this proposition (and Macsyma) we easily proof that every quadratic Hopf singularity of finite codimension has a quadratic versal unfolding.

THEOREM 7

(1) *The 1-parameter family*

$$H_{\mu_0} : \begin{cases} \dfrac{dx}{dt} = \mu_0 x - y + ax^2 + bxy + cy^2 \\ \dfrac{dy}{dt} = x + \mu_0 y + dx^2 + exy - dy^2 \end{cases} \tag{3.60}$$

with $(a+c)(b-2d) \neq 0$ *is a versal Hopf-Takens bifurcation of codimension 1.*

(2) *The 2-parameter family*

$$H_{(\mu_0,\mu_1)}: \quad \begin{cases} \dfrac{dx}{dt} = \mu_0 x - y + ax^2 + (2d+\mu_1)xy + cy^2 \\ \dfrac{dy}{dt} = x + \mu_0 y + dx^2 + exy - dy^2 \end{cases} \tag{3.61}$$

with $(a+c)d(e+2a)(e-3a-5c) \neq 0$ *is a versal Hopf-Takens bifurcation of codimension 2 .*

(3) *The 3-parameter family*

$$H_{(\mu_0,\mu_1,\mu_2)}: \quad \begin{cases} \dfrac{dx}{dt} = \mu_0 x - y + ax^2 + (2d+\mu_1)xy + cy^2 \\ \dfrac{dy}{dt} = x + \mu_0 y + dx^2 + (3a+5c+\mu_2)xy - dy^2 \end{cases} \tag{3.62}$$

with $(a+c)d(d^2+2c^2+ac) \neq 0$ *is a versal Hopf-Takens bifurcation of codimension 3 .*

Proof :

The genericity of the family H_{μ_0} is readily seen.

To prove the genericity of the family $H_{(\mu_0,\mu_1)}$ we have to show that bringing the family into the normal form (3.57) the mapping $(\mu_0,\mu_1) \to (a_0(\mu_0,\mu_1), a_1(\mu_0,\mu_1))$ has maximal rank at the origin. Therefore we treat every parameter seperately. Using the coordinate transformation

$$\begin{cases} x = u - (d+\mu_1)u^2/3 + (a-2c+e)uv/3 + (d+\mu_1)v^2/3 \\ \quad -(e^2+ce+7ae+6c^2+2ac+4a^2+2d\mu_1+\mu_1^2)u^3/12 \\ \quad -(4d(e+2a)+\mu_1(4e-5c+7a))u^2v/8 \\ \quad +(e^2-17ce-5ae-6c^2-22ac+4a^2-16\mu_1 d-7\mu_1^2)uv^2/24 \end{cases}$$

$$\begin{cases} y = & v + (a+2c-e)u^2/3 + (2d-\mu_1)uv/3 + (2a+c+e)v^2/3 \\ & (2d(8c+6a-e) + \mu_1(2e-c+5a))u^3/12 \\ & (6ac - 4a^2 - 13ce - 11ac - 9e^2 - 12d\mu_1 + 3\mu_1^2)u^2v/24 \\ & (8d(e+2c+4a) + \mu_1(3a+3c-8e))uv^2/24 \end{cases}$$

one easily checks that the family $H_{(0,\mu_1)}$ is C^∞-equivalent to

$$\begin{cases} \dfrac{du}{dt} = -v + (a+c)\mu_1 u(u^2+v^2)/8 \\ \dfrac{dv}{dt} = u + (a+c)\mu_1 v(u^2+v^2)/8 \end{cases} + O(|u,v|^4)$$

From this we may conclude that the family $H_{(\mu_0,\mu_1)}$ is generic.
The genericity of the family $H_{(\mu_0,\mu_1,\mu_2)}$ can be proved in an analogous way . Since the expressions are rather long we prefer not to include them here.

3.4 Appendix

In this appendix we explain our method by which we construct our generic quadratic unfoldings. We elucidate the problems we met during the calculations and the way we solved of them.
The procedure of [DF] by which we construct generic quadratic unfoldings consists of the next steps : (we treat the case of a nilpotent saddle of codimension 3)
We start with a quadratic vector field $S_0 \in \Sigma^3_{S+}$ and we determine the near-identity transformation Φ such that the 3-jet of $\Phi_*(S_0)$ is transformed into the normal form:

$$y\frac{\partial}{\partial x} + (x^3 + bxy \pm x^2y)\frac{\partial}{\partial y}$$

with $b > 0$.
So we find that

$$S_0 : \begin{cases} \dfrac{dx}{dt} = & y + x^2 - 2xy \\ \dfrac{dy}{dt} = & -xy \end{cases}$$

is transformed into

$$R_0 : \begin{cases} \dfrac{du}{dt} = & v \\ \dfrac{dv}{dt} = & u^3 + uv + u^2 v \end{cases} + O(|u,v|^4)$$

by the near-identity transformation

$$\Phi : \begin{cases} x = & u - \dfrac{u^2}{2} + \dfrac{2u^3}{3} \\ y = & v - u^2 \end{cases}$$

Next we apply the transformation to the vector field $S_0 + S_2$ where

$$S_2(x,y) = P_2(x,y)\frac{\partial}{\partial x} + Q_2(x,y)\frac{\partial}{\partial y}$$

with

$$P_2(x,y) = \sum_{\substack{i+j=0 \\ i,j \geq 0}}^{i+j=2} a_{ij} x^i y^j \qquad \text{and} \qquad Q_2(x,y) = \sum_{\substack{i+j=0 \\ i,j \geq 0}}^{i+j=2} b_{ij} x^i y^j$$

By choosing $P_2(x,y)$ and $Q_2(x,y)$ such that the 2-jet of $\Phi_*(S_0 + S_2)$ already is in normal form (??) , we find that the quadratic family

$$S_{(a,b,c)} : \begin{cases} \dfrac{dx}{dt} = & y + x^2 - 2xy \\ \dfrac{dy}{dt} = & a + bx + cy + (b+c)x^2 - xy \end{cases}$$

is transformed into

$$\tilde{S}_{(a,b,c)} : \begin{cases} \dfrac{du}{dt} = & v \\ \dfrac{dv}{dt} = & a + bu + cv + uv + (3 - 4b - 6c)u^3/3 + u^2 v \end{cases} + O(|u,v|^4)$$

by the transformation Φ.Then we have to prove the genericity of the family $\tilde{S}_{(a,b,c)}$. The greatest disadvantages of this working method are the following :

The use of several parameters at the same time complicates the calculations and it also reduces the surveyability of the expressions considerably.
The required computertime is very long , sometimes more than 24 hours (CPU-time) on an Apollo 4000.

The above considerations forced us to change our working method. To construct quadratic generic unfoldings we now proceed as follows: We start with a vector field $S_0 \in \Sigma^3_{S+}$, f.i.

$$S_0 : \begin{cases} \dfrac{dx}{dt} = & y + x^2 - 2xy \\ \dfrac{dy}{dt} = & -xy \end{cases}$$

and we add terms wich correspond to the important terms in the normal form (3.17): $(\mu_1(\lambda), \mu_2(\lambda), \nu(\lambda))$. So we get the quadratic family :

$$S'_{(a,b,c)} : \begin{cases} \dfrac{dx}{dt} = & y + x^2 - 2xy \\ \dfrac{dy}{dt} = & a + bx + cy - xy \end{cases}$$

In order to prove the genericity of the family $S'_{(a,b,c)}$ we deal with every parameter seperately . The above method works in general , except for one particular case , namely the elliptic point of codimension 3 with $a = e$ (see theorem 3 of chapter 3).

Chapter 4

Generic 4-parameter family on the plane : unfolding of the nilpotent saddle of codimension 4.

In this chapter we consider a generic 4-parameter family X_λ written in the normal form :

$$y\frac{\partial}{\partial x} + (x^3 + \mu_2 x + \mu_1 + y(\nu + bx + \epsilon x^2 + x^3 h(x,\lambda)) + y^2 Q(x,y,\lambda))\frac{\partial}{\partial y}$$

where $\epsilon = \pm 1$, $\lambda = (\mu_1, \mu_2, \nu, b)$,$h(x,\lambda)$ is C^∞-function , $Q(x,y,\lambda)$ is a C^∞-function of order N in (x,y,λ) , where N is arbitrarily high .
Remark that one can change $\epsilon = -1$ into $\epsilon = +1$ by means of the coordinate change $(x, y, \mu_1, \mu_2, \nu, b, t) \to (-x, y, -\mu_1, \mu_2, -\nu, b, -t)$. This change is not a C^∞-equivalence since we admit a reversal of time . So from now on we work with $\epsilon = +1$.
In a first section we investigate which local bifurcations occur in the family X_λ . We find that there are Hopf singularities of codimension 1 and 2 , semi-hyperbolic bifurcations of codimension 1 and 2 , Bogdanov-Takens bifurcations , cusp bifurcations of codimension 3 and nilpotent saddle point bifurcations of codimension 3. The fact that there are no foci of codimension 3 leads to our conjecture that 2 is the maximum number of limit cycles that bifurcate from a nilpotent saddle of codimension 4. It is clear that the b-axis is a line of nilpotent saddle bifurcations of codimension 3. In a second section we propose a bifurcation diagram for the transition between a nilpotent saddle of codimension 3 with $b > 0$ and a nilpotent saddle of codimension

3 with $b < 0$.

4.1 Local bifurcations

The critical points of X_λ are given by $y = 0$ and $x^3 + \mu_2 x + \mu_1 = 0$.
Let SN be the zero set of the discriminant of this last equation :
$SN = \{(\mu_1, \mu_2, \nu, b) | 27\mu_1^2 + 4\mu_2^3 = 0\}$. We now verify that the critical points are non-degenerate outside SN.
Let $m_o = (x_o, 0)$ be any critical point . Taking $x = x_o + X, y = Y$ we calculate the 2-jet of X_λ at m_o :

$$\begin{aligned} j^2 X_\lambda(m_o) = \; & Y\frac{\partial}{\partial X} + (-Det(x_o,\lambda)X + Tr(x_o,\lambda)Y + 3x_o X^2 \\ & +(b + 2x_o + 3x_o^2 h(x_o,\lambda) + x_o^3 \frac{\partial h}{\partial x}(x_o,\lambda))XY + Q(x_o,0,\lambda)Y^2)\frac{\partial}{\partial Y} \end{aligned}$$

where

$$\begin{cases} -Det(x_o,\lambda) = 3x_o^2 + \mu_2 \\ Tr(x_o,\lambda) = \nu + bx_o + x_o^2 + x_o^3 h(x_o,\lambda) \end{cases}$$

In particular we see that :

$$j^1 X_\lambda(m_o) = \begin{pmatrix} 0 & 1 \\ -Det(x_o,\lambda) & Tr(x_o,\lambda) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The determinant , $Det(x_o,\lambda)$, of the 1-jet is non zero when $\lambda \notin SN$ and the saddle or focus/node nature is given by the sign of $Det(x_o,\lambda)$. So the problem reduces to the study of the roots of the cubic equation : $x^3 + \mu_2 x + \mu_1 = 0$. There exists 3 non-degenerate points in the region $R_- = \{27\mu_1^2 + 4\mu_2^3 < 0\}$ and 1 non-degenerate point in the region $R_+ = \{27\mu_1^2 + 4\mu_2^3 > 0\}$. The nature of these points can be described as follows :
a focus or node is located between 2 saddles for $\lambda \in R_-$; there exists a hyperbolic saddle for $\lambda \in R_+$.

4.1.1 The Hopf singularities

The set of Hopf singularities of any codimension is contained in the set obtained by elimination of x_o from the 2 equations :

$$\begin{cases} Tr(x_o,\lambda) = \nu + bx_o + x_o^2 + x_o^3 h(x_o,\lambda) = 0 \\ x_o^3 + \mu_2 x_o + \mu_1 = 0 \end{cases}$$

This is the set of values of λ where X_λ has some critical point $(x_o, 0)$ with vanishing trace.

In order to calculate the first Lyapounov coefficient at the point $(x_o, 0)$, we perform the translation :

$$\begin{cases} x' = x - x_o \\ y' = y \end{cases}$$

Omitting the primes we obtain :

$$y\frac{\partial}{\partial x} + (x^3 + 3x_o x^2 + (3x_o^2 + \mu_2)x + y(\nu + b(x + x_o) + x^2 + 2x_o x + x_o^2$$
$$+ x_o^3 h(x_o,\lambda) + ((x + x_o)^3 h(x + x_o,\lambda) - x_o^3 h(x_o,\lambda))) + y^2 Q(x + x_o, y, \lambda))\frac{\partial}{\partial y}$$

which , because $\nu + bx_o + x_o^2 + x_o^3 h(x_o,\lambda) = 0$, gives :

$$\begin{aligned} y\frac{\partial}{\partial x} + & \quad (x^3 + 3x_o x^2 + (3x_o^2 + \mu_2)x + y((b + 2x_o)x + x^2 \\ + & \quad ((x + x_o)^3 h(x + x_o,\lambda) - x_o^3 h(x_o,\lambda))) + y^2 Q(x + x_o, y, \lambda))\frac{\partial}{\partial y} \end{aligned}$$

As the singularity must be a focus $(Det(x_o,\lambda) > 0)$ we have :

$$3x_o^2 + \mu_2 < 0$$

The coordinate change:

$$\begin{cases} x = x' \\ y = (-(3x_o^2 + \mu_2))^{1/2} y' \end{cases}$$

gives , omitting the primes and writing $\Delta = Det(x_o, \lambda) = -(3x_o^2 + \mu_2)$

$$\sqrt{\Delta} y \frac{\partial}{\partial x} \quad +[\frac{1}{\sqrt{\Delta}}(x^3 + 3x_o x^2 - \Delta x) + y((b + 2x_o)x + x^2$$
$$+((x + x_o)^3 h(x + x_o, \lambda) - x_o^3 h(x_o, \lambda))) + \sqrt{\Delta} y^2 Q(x + x_o, \sqrt{\Delta} y, \lambda)] \frac{\partial}{\partial y}$$

Multiplying this expression by $\frac{1}{\sqrt{\Delta}}$ we find :

$$y \frac{\partial}{\partial x} \quad +[-x + \frac{3x_o}{\Delta} x^2 + \frac{x^3}{\Delta} + y(\frac{(b + 2x_o)}{\sqrt{\Delta}} x + \frac{x^2}{\sqrt{\Delta}}$$
$$+\frac{1}{\sqrt{\Delta}}((x + x_o)^3 h(x + x_o, \lambda) - x_o^3 h(x_o, \lambda)) + y^2 Q(x + x_o, \sqrt{\Delta} y, \lambda)] \frac{\partial}{\partial y}$$

The Lyapounov coefficient of order 1 is given by (see [ALGM , DRS2]):

$$-\frac{1}{\sqrt{\Delta}}(1 + \overline{A} + 3\Delta \frac{\partial Q}{\partial y}(x_o, 0, \lambda)) + \frac{b + 2x_o + \overline{B}}{\sqrt{\Delta}}(\frac{-3x_o - \Delta Q(x_o, 0, \lambda)}{\Delta}) \qquad (4.1)$$

where

$$\overline{A} = \text{coefficient of } x^2 \text{ in } (x + x_o)^3 h(x + x_o, \lambda) - x_o^3 h(x_o, \lambda)$$
$$\overline{B} = \text{coefficient of } x \text{ in } (x + x_o)^3 h(x + x_o, \lambda) - x_o^3 h(x_o, \lambda)$$

Clearly $\overline{A}$, $\overline{B}$ are of the form $\overline{A} = x_o A, \overline{B} = x_o^2 B$. The expression (4.1) has the same sign as :

$$(-3x_o - \Delta Q(x_o, 0, \lambda))(b + 2x_o + x_o^2 B) - \Delta(1 + x_o A + 3\Delta \frac{\partial Q}{\partial y}(x_o, 0, \lambda))$$

Using the new parameter $c = b + 2x_o + x_o^2 B$ this expression becomes:

$$-3x_o c - \Delta(1 + x_o A + cQ(x_o, 0, \lambda) + 3\Delta \frac{\partial Q}{\partial y}(x_o, 0, \lambda)) \qquad (4.2)$$

The Hopf singularities of codimension ≥ 2 can only appear when this last expression is zero . Therefore let us now suppose that this holds.

Next we calculate the second Lyapounov coefficient . Using Macsyma we find that this coefficient has the same sign as an expression of the form:

$$x_o^2 c^2 f(x_o, c) \tag{4.3}$$

with $f(0,0) < 0$.
We prefer not to include the expression nor the calculation because they are both very long . Since we are only looking for small values of x_o and c we may conclude from (4.3) that we must have $x_o = 0$ or $c = 0$ to have Hopf singularities of codimension > 2 . But $x_o = 0$ or $c = 0$, together with the condition that expression (4.2) is zero, imply that $\Delta = 0$. *This shows that there are no Hopf singularities of codimension > 2 in the unfolding of the nilpotent saddle of codimension 4.*

Remarks :

1. In his work [Bau] , Bautin showed that for quadratic systems at most three small-amplitude limit cycles can bifurcate out of a critical point of focus or center type. From [Bog] , [DRS1], [DRS2] we know that for nilpotent singularities of codimension ≤ 3 this number is at most 2 . The maximum number of limit cycles which can bifurcate out of a single critical point of a quadratic system is still a matter of investigation . The fact that there are no Hopf singularities of codimension > 2 in the generic unfolding of the nilpotent saddle of codimension 4 is the first reason to conjecture that two is the maximum number of limit cycles that bifurcate out of such a singularity.

2. In attempt to prove the genericity of the Hopf singularities of codimension 2 we performed the necessary normal form calculations. But unfortunately, the obtained expressions were to complicated to use further on.

3. Following a same procedure as above one can show that a nilpotent elliptic point of codimension 4 and type 1 and a nilpotent saddle-node of codimension 4 can not be approached by Hopf singularities of codimension ≥ 3.

4. On the other hand every nilpotent elliptic point of codimension ≥ 4 and type 2 can be approached by Hopf singularities of codimension 3. This can be seen as follows:

From chapter 1 we know that these singularities have the following form :

$$\begin{cases} \dfrac{dx}{dt} = & y + ax^2 + bxy + cy^2 \\ \dfrac{dy}{dt} = & 3axy + fy^2 \end{cases} \tag{4.4}$$

with $a \neq 0, 3b + f \neq 0$ and $\alpha := (f - 2b)(3f - b) + 25ac \neq 0$.
Using the coordinate transformation :

$$\begin{cases} u = & 3ax + fy \\ v = & y \end{cases}$$

system (4.4) is transformed into

$$\begin{cases} \dfrac{du}{dt} = & 3av + \dfrac{1}{3}u^2 + \dfrac{3b+f}{3}uv + \dfrac{f^2 + 9ac - 3bf}{3}v^2 \\ \dfrac{dv}{dt} = & uv \end{cases}$$

The linear coordinate change $u = \dfrac{3}{\gamma}x', v = \dfrac{1}{a\gamma^2}y'$ and the change of time $t = \gamma t'$ with $\gamma = \dfrac{3a}{3b+f}$ give , after omitting the primes :

$$Z_o : \begin{cases} \dfrac{dx}{dt} = & y + x^2 + xy + \delta y^2 \\ \dfrac{dy}{dt} = & 3xy \end{cases}$$

with $\delta = \dfrac{f^2 - 3bf + 9ac}{(3b+f)^2}$. (The condition $\alpha \neq 0$ implies that $\delta \neq -\dfrac{2}{25}$.)
Let us introduce the family

$$Z_k : \begin{cases} \dfrac{dx}{dt} = & y + (1 + \dfrac{4k}{25})x^2 + xy + \delta y^2 \\ \dfrac{dy}{dt} = & -kx + \dfrac{k}{5}x^2 + (3 + (\dfrac{125\delta + 12}{25})k)xy \end{cases}$$

For $k > 0$, Z_k has at the origin a Hopf singularity of codimension 3.

4.1.2 Bifurcations along the set SN

The vector field X_λ has a degenerate singular point for $\lambda \in SN = \{27\mu_1^2 + 4\mu_2^3 = 0\}$. Let $(x_o, 0)$ be this point . We have :

$$\left\{ \begin{array}{l} x_o^3 + \mu_2 x_o + \mu_1 = 0 \\ \\ 3x_o^2 + \mu_2 = 0 \end{array} \right. \tag{4.5}$$

The nilpotent bifurcations

The trace at the point $(x_o, 0)$ is given by

$$Tr(x_o, \lambda) = \nu + bx_o + x_o^2 + x_o^3 h(x_o, \lambda) \tag{4.6}$$

The point$(x_o, 0)$ is nilpotent if $Tr(x_o, \lambda) = 0$.
Define $NB = SN \cap \{Tr(x_o, \lambda) = 0\}$.
Let $\lambda^o \in NB \backslash \{0\}$, $\lambda^o = (\mu_1^o, \mu_2^o, \nu^o, b^o)$.
Let $\mu_1 = \mu_1^o + M_1$, $\mu_2 = \mu_2^o + M_2$, $\nu = \nu^o + N$, $b = b^o + B$, $x = x_o + X$, $y = Y$ and $\Lambda = (M_1, M_2, N, B)$.
We develop the family X_λ in the coordinates X, Y and the parameters Λ ; x_o enters in the formula as an arbitrarily small extra parameter .
Taking into account that $x_o^3 + \mu_2 x_o + \mu_1 = 0$, $3x_o^2 + \mu_2 = 0$ and $Tr(x_o, \lambda^o) = 0$ we have

$$\begin{aligned} X_{\lambda^o + \Lambda} = Y\frac{\partial}{\partial X} \quad & +[(M_2 x_o + M_1) + M_2 X + 3x_o X^2 + X^3 \\ & +Y(N + Bx_o + (b^o + 2x_o + B)X + X^2 \\ & +((X + x_o)^3 h(X + x_o, \lambda^o + \Lambda) - x_o^3 h(x_o, \lambda^o))) \\ & +Y^2 Q(X + x_o, Y, \lambda^o + \Lambda)]\frac{\partial}{\partial Y} \end{aligned}$$

To determine the type of the singularity $(x_o, 0)$ we consider the 4 jet of X_{λ^o}

at $(x_o, 0)$:

$$j^4 X_{\lambda^o}((x_o,0)) = Y\frac{\partial}{\partial X} + (X^3 + b_{20}X^2 + Y(b_{11}X + b_{21}X^2 + b_{31}X^3) + Y^2(b_{02} + b_{12}X + b_{03}Y + b_{22}X^2 + b_{13}XY + b_{04}Y^2))\frac{\partial}{\partial Y} \tag{4.7}$$

where

$$\begin{cases} b_{20} = 3x_o & b_{11} = b^o + 2x_o + \dfrac{\partial}{\partial x}(x^3 h(x,\lambda^o))_{x=x_o} \\ b_{21} = 1 + \dfrac{1}{2}\dfrac{\partial^2}{\partial x^2}(x^3 h(x,\lambda^o))_{x=x_o} & b_{31} = \dfrac{1}{6}\dfrac{\partial^3}{\partial x^3}(x^3 h(x,\lambda^o))_{x=x_o} \\ b_{02} = Q(x_o,0,\lambda^o) & b_{03} = \dfrac{\partial Q}{\partial y}(x_o,0,\lambda^o) \\ b_{12} = \dfrac{\partial Q}{\partial x}(x_o,0,\lambda^o) & b_{13} = \dfrac{\partial^2 Q}{\partial x \partial y}(x_o,0,\lambda_o) \\ b_{22} = \dfrac{1}{2}\dfrac{\partial^2 Q}{\partial x^2}(x_o,0,\lambda_o) & b_{04} = \dfrac{1}{2}\dfrac{\partial^2 Q}{\partial y^2}(x_o,0,\lambda^o) \end{cases}$$

First we treat the case $x_o \neq 0$.
If $b_{11} \neq 0$: $(x_o, 0)$ is a cusp singularity of codimension 2.
Suppose $b_{11} = 0$. Using a linear coordinate transformation , working with dual forms and using Macsyma one shows that (4.7) is C^∞-equivalent to :

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = x^2 + y(c_{31}x^3 + O(x^4)) + y^2 O(|x,y|^3) \end{cases} \tag{4.8}$$

with $c_{31} = (b_{20})^{-4}(-b_{21} + b_{21}b_{20}b_{02} + b_{31}b_{20} - 3b_{20}^2 b_{03}) = (3x_o)^{-4}(-1 + O(x_o))$.
Since we are only looking for x_o small , we can conclude that the singularity $(x_o, 0)$ is a cusp singularity of codimension 3 .

Next we treat the case $x_o = 0$.
The condition $x_o = 0$, together with (4.5) and (4.6) , imply that $\mu_1^o = \mu_2^o = \nu^o = 0$;

therefore $b^o \neq 0$.
Using Macsyma it is easily seen that $j^4 X_{(0,0,0,b^o)}(0,0)$ is C^∞-equivalent to :

$$\left\{ \begin{array}{rcl} \dfrac{dx}{dt} &=& y \\ \dfrac{dy}{dt} &=& x^3 + c_{40}x^4 + O(x^5) + y(c_{11}x + c_{21}x^2 + O(x^3)) + y^2 O(|x,y|^3) \end{array} \right. \qquad (4.9)$$

with $c_{11} = b^o$, $c_{21} = \dfrac{b_{11}b_{02} + 2b_{21}}{2}$, $c_{40} = \dfrac{b_{02}}{2}$. Condition (3.14) becomes

$$5c_{21} - 3c_{11}c_{40} = 5 + O(b^o)$$

Since we are looking for b^o small , we can conclude that $(0,0)$ is a nilpotent saddle of codimension 3 .

We proved the following theorem:

THEOREM

In each nilpotent case the family $X_{\lambda^o + \Lambda}$ is a generic unfolding of the vector field X_{λ^o}.

The proof of this theorem follows a same procedure as in the previous chapter.

Remark:

About the nature of the limit cycles that bifurcate out of the cusps of cod 3 and the nilpotent saddles of cod 3.
Using the method of dual forms one shows that system (4.9) is C^∞-equivalent to:

$$\left\{ \begin{array}{rcl} \dfrac{dx}{dt} &=& y \\ \dfrac{dy}{dt} &=& x^3 + y(c_{11}x + \dfrac{5c_{21} - 3c_{11}c_{40}}{5}x^2 + O(x^3)) + y^2 O(|x,y|^3) \end{array} \right. \qquad (4.10)$$

So, from (4.8) and (4.9) (see [DRS1] and [DRS2]) we see that the cusps and the saddles are of the same kind. In the region with two limit cycles in the bifurcation diagram we have the following arrangement: the inner one is attracting and the outer one is repelling.

The semi-hyperbolic bifurcations

We suppose now that $\lambda^o \in SN\backslash NB$, so that $Tr(x_o, \lambda^o) \neq 0$.
Let $\lambda^o = (\mu_1^o, \mu_2^o, \nu^o, b^o)$, $\mu_1 = \mu_1^o + M_1$, $\mu_2 = \mu_2^o + M_2$, $\nu = \nu_o + N$, $b = b^o + B$, $x = x_o + X$, $y = Y$ and $\Lambda = (M_1, M_2, N, B)$.
Then

$$X_{\lambda^o+\Lambda} = Y\frac{\partial}{\partial X} + (a(\Lambda) + b(\Lambda)X + c(\Lambda)Y(1 + O(||(X,Y)||) + 3x_oX^2 + X^3)\frac{\partial}{\partial Y}$$

where $a(\Lambda) = M_1 + x_oM_2$, $b(\Lambda) = M_2$, $c(\Lambda) = Tr(x_o, \lambda^o + \Lambda)$ (with $c(0) = Tr(x_o, \lambda^o) \neq 0$).

Lemma

Let $m_o = (x_o, 0)$, then

$$j^2X_{\lambda^o}(m_o) \sim c(0)Y\frac{\partial}{\partial Y} - \frac{3x_o}{c(0)}X^2\frac{\partial}{\partial X} \qquad \text{for } \lambda^o \in SN\backslash NB \text{ and } \lambda^o \notin \{(\mu_1, \mu_2, \nu, b) | \mu_1 = \mu_2 = 0\}$$

$$j^3X_{\lambda^o}(m_o) \sim c(0)Y\frac{\partial}{\partial Y} - \frac{1}{c(0)}X^3\frac{\partial}{\partial X} \qquad \text{for } \lambda^o \in SN\backslash NB \text{ and } \lambda^o \in \{(\mu_1, \mu_2, \nu, b) | \mu_1 = \mu_2 = 0\}\backslash\{0\}$$

Proof

Obviously , for $\lambda^o \in SN\backslash NB$: $j^1X_{\lambda^o}(m_o) = Y\frac{\partial}{\partial X} + c(0)Y\frac{\partial}{\partial Y}$.
The central axis is 0X . Then , each central manifold W has an expression :

$$W : Y = \Psi(X) = KX^2 + O(X^3)$$

The restriction of X_{λ_o} to W has the following orbit equation :

$$\frac{dX}{dt} = \Psi(X)$$

To find the coefficient K , we write that W is invariant by X_{λ^o} , i.e. at the point $(X, \Psi(X))$ the tangent vector to W has the same direction as $X_{\lambda^o}(X, \Psi(X))$:

$$\frac{\dot{Y}}{\dot{X}}(X, \Psi(X)) = \frac{d\Psi}{dX}(X) \tag{4.11}$$

This equation gives :

$$\frac{c(0)KX^2 + 3x_oX^2 + O(X^3)}{KX^2 + O(X^3)} = 2KX + O(X^2)$$

This implies that $c(0)K + 3x_o = 0$ and the first result follows .
If now $\lambda^o \in SN\backslash NB$ and $\lambda^o \in \{(\mu_1, \mu_2, \nu, b)|\mu_1 = \mu_2 = 0\}\backslash\{0\}$ the center manifolds W are of the form :

$$Y = \Psi(X) = KX^3 + O(X^4)$$

Again , applying (4.11) we obtain :

$$\frac{c(0)KX^3 + X^3 + O(X^4)}{KX^3 + O(X^4)} = 3KX^2 + O(X^3)$$

and the result follows .

Lemma

The family $X_{\lambda^o+\Lambda}$ is a generic semi-hyperbolic bifurcation of codimension 1 for $\lambda^o \in SN\backslash(NB \cup \{(\mu_1, \mu_2, \nu, b)|\mu_1 = \mu_2 = 0\})$; and a generic semi-hyperbolic bifurcation of codimension 2 for $\lambda^o \in (SN\backslash NB) \cap (\{(\mu_1, \mu_2, \nu, b)|\mu_1 = \mu_2 = 0\}\backslash\{0\})$.

Proof

Suppose $W^\Lambda : Y = \Phi(X, \Lambda)$ is an equation for a central manifold for the family . The restriction of $X_{\lambda^o+\Lambda}$ to W^Λ has the orbit equation :

$$\dot{X} = \Phi(X, \Lambda)$$

where W^Λ is parametrised by X.
Consider to begin with $\lambda^o \in SN\backslash NB$ .We look at

$$\Phi(X,\Lambda) = A(\Lambda) + B(\Lambda)X + K(\Lambda)X^2 + O(X^3)$$

with $A(0) = B(0) = 0$ and $K(0) = -\dfrac{3x_o}{c(0)}$ as calculated above .
We obtain the first order terms of A from equation (4.11) applied to $X_{\lambda^o+\Lambda}$. We have

$$a(\Lambda) + c(\Lambda)A(\Lambda) = O(\Lambda^2)$$

which implies :

$$A(\Lambda) = -\frac{a(\Lambda)}{c(\Lambda)} + O(\Lambda^2)$$

Obviously $da(0) \neq 0$ and so $dA(0) \neq 0$.
If $\lambda^o \in (SN\backslash NB) \cap (\{(\mu_1, \mu_2, v, b) | \mu_1 = \mu_2 = 0\}\backslash\{0\})$, we look at

$$\Phi(X,\Lambda) = A(\Lambda) + B(\Lambda)X + C(\Lambda)X^2 + K(\Lambda)X^3 + O(X^4)$$

Again , formula (4.11) applied to the family $X_{\lambda^o+\Lambda}$ gives :

$$\begin{cases} a(\Lambda) + c(\Lambda)A(\Lambda) = O(\Lambda^2) \\ b(\Lambda) + c(\Lambda)B(\Lambda) = O(\Lambda^2) \end{cases}$$

The independence of $dA(0), dB(0)$ follows from the independence of $da(0), db(0)$.

4.1.3 Proposal of a bifurcation diagram

To give some evidence for our proposal of a bifurcation diagram we use the method of rescaling. We will use the following rescalings:

$$\begin{cases} x = \tau\tilde{x} \\ y = \tau^2\tilde{y} \end{cases} \qquad \begin{cases} \mu_1 = \tau^3\tilde{\mu}_1 \\ \mu_2 = \tau^2\tilde{\mu}_2 \\ \nu = \tau\tilde{\nu} \\ b = \tau\tilde{b} \end{cases}$$

and

$$\begin{cases} x = \tau\overline{x} \\ y = \tau^2\overline{y} \end{cases} \qquad \begin{cases} \mu_1 = \tau^3\overline{\mu}_1 \\ \mu_2 = \tau^2\overline{\mu}_2 \\ \nu = \tau^2\overline{\nu} \\ b = \tau\overline{b} \end{cases}$$

After a rescaling we have a family $X_{\overline{\lambda},\tau}$ where τ is a small parameter and $\overline{\lambda} \in K$, some compact subset of $I\!R^4$. Depending on the nature of the family $X_{\overline{\lambda},0}$ around some value $\overline{\lambda}_o$, the study splits into two cases :

The generic case :
Let $\overline{\lambda}_o$ be a generic bifurcation value for the family $X_{\overline{\lambda},0}$. Then , the bifurcation set of $X_{\overline{\lambda},0}$ is given by transversality conditions and , using an implicit function argument we obtain for $X_{\overline{\lambda},\tau}$ and small τ , a bifurcation set with the same codimension . If σ is the local bifurcation set of $X_{\overline{\lambda},0}$ at $\overline{\lambda}_o$, then the local bifurcation set for $X_{\overline{\lambda},\tau}$ is diffeomorphic to $\sigma \times [0,\epsilon]$ for $\tau \in [0,\epsilon]$, ϵ small enough .

The perturbed Hamiltonian (P.H.) case :
Here , up to multiplication with a positive C^∞-function , we have that $X_{\overline{\lambda},0}$ is a Hamiltonian vector field . Let $\omega_{\overline{\lambda},\tau}$ be the dual form of $X_{\overline{\lambda},\tau}$. Then $\omega_{\overline{\lambda},0} =$ dH for

some C^∞-function H , and we can expand

$$\omega_{\overline{\lambda},\tau} = dH - \tau\omega_D(\overline{\lambda}) + o(\tau)$$

To simplify the discussion suppose that $\forall\tau \quad \omega_{\overline{\lambda},\tau}$ has the same singular points as dH . Let $\sigma=$ [a,b] be a segment in the phase plane which at each point of]a,b] is transversal to a closed component of a level curve of H . Suppose that at a it is also transversal to a closed cycle or it is a non degenerate center of H . We parametrize σ by means of the value of H : $\sigma = [\alpha,\beta] = [H(a), H(b)]$.

Perturbation Lemma*[ALGM]*

Let $\omega_{\overline{\lambda},\tau}, H, \sigma$ be as above . Let K be a compact subset in the parameter space of $\overline{\lambda}$. Then there exists a $T(K) > 0$ such that for all $(\overline{\lambda},\tau) \in K \times [0, T(K)]$:

1) The vectorfield $X_{\overline{\lambda},\tau}$ is transversal to $]\alpha,\beta]$

2) The Poincaré map $P_{\overline{\lambda},\tau}(h)$ of $X_{\lambda,\tau}$, or its inverse $P^{-1}_{\overline{\lambda},\tau}(h)$ is defined on $[\alpha,\beta]$.

3) For $h \in [\alpha,\beta]$, the coordinate defined by the value of H , it holds that

$$P_{\overline{\lambda},\tau}(h) = h + \tau\int_{\gamma_h}\omega_D(\overline{\lambda}) + o(\tau) \tag{4.12}$$

where γ_h is the compact component of $\{H = h\}$ passing through the point $h \in [\alpha,\beta]$, clockwise oriented for the integration.

Let $I(h,\overline{\lambda}) = \int_{\gamma_h}\omega_D(\overline{\lambda})$ be the Abelian integral giving the first order term of formula (4.12).

The fixed points of $P_{\overline{\lambda},\tau}(h)$ are the zeroes of the function

$$G(h,\lambda,\tau) = \frac{P_{\overline{\lambda},\tau}(h) - h}{\tau} = I(h,\overline{\lambda}) + O(\tau) \tag{4.13}$$

Here $O(\tau)$ is a C^∞ function in $(h,\overline{\lambda},\tau)$ of order τ .

Using formula (4.13) it is easy to find conditions for existence and genericity of Hopf bifurcations (at $h = \alpha$) and of limit cycle bifurcations (at $h \neq \alpha$) ; at least for small values of $\tau \neq 0$.

Suppose now that the endpoint β belongs to Γ , where Γ is some hyperbolic singular cycle of H . That is a connected compact piece of $\{H = \beta\}$ made of hyperbolic saddle points and regular arcs . Then the formulas (4.12) and (4.13) are not sufficient to study bifurcations of $X_{\overline{\lambda},\tau}$ near Γ , because the mapping is not longer differentiable at $H = \beta$ for $\tau = 0$. A direct study of $P_{\overline{\lambda},\tau}$ is needed .

Here we establish the list of all bifurcations we encounter in the sequel . Most of them are well known [A , ALGM , S , Sc , GH , CH , DRS1 , DRS2] .

A Codimension 1 bifurcations

A.1 Andronov-Hopf bifurcation (H)

A.2 Semi-hyperbolic bifurcation of codimension 1 (SN)

A.3 Saddle loop bifurcation (L)

We suppose that X_{λ_o} has a hyperbolic saddle $s(\lambda_o)$ with a homoclinic connection Γ . Let σ be a segment transverse to Γ . Let $s(\lambda)$ be the unique singularity of X_λ near $s(\lambda_o)$ and $-u(\lambda)$, $v(\lambda)$ its eigenvalues $(u(\lambda), v(\lambda) > 0)$. Let $W^s(\lambda)$, $W^u(\lambda)$ be the stable and unstable separatrices of X_λ near Γ $(\Gamma = W^s(\lambda_o) = W^u(\lambda_o))$.
Let $\{a(\lambda)\} = W^s(\lambda) \cap \sigma$, $\{b(\lambda)\} = W^u(\lambda) \cap \sigma$ and $\mu(\lambda) = a(\lambda) - b(\lambda)$ $(\mu(\lambda_o) = 0)$.
See figure (4.1) .

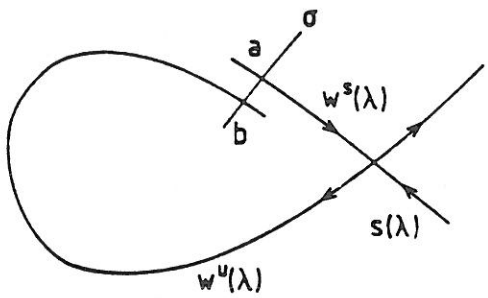


fig. 4.1

Generic case

The bifurcation set is given by $\mu(\lambda) = 0$, $\dfrac{u(\lambda_o)}{v(\lambda_o)} \neq 1.$

P.H.-case :

We suppose that the $\{H = \beta\}$ contains a loop Γ with a hyperbolic saddle s . Then $X_{\overline{\lambda},\tau}$ also has a saddle at s . Let $-u(\overline{\lambda},\tau)$, $v(\overline{\lambda},\tau)$ be the eigenvalues of this saddle .
We have $\dfrac{u(\overline{\lambda},\tau)}{v(\overline{\lambda},\tau)} = 1 - \tau\alpha(\overline{\lambda}) + o(\tau)$.
Let $I(\beta,\overline{\lambda}) = \int_\Gamma \omega_D(\overline{\lambda})$. Then , the bifurcation set is given by $I(\beta,\overline{\lambda}) = 0$, with $\alpha(\overline{\lambda}_o) \neq 0$.

A.4 Saddle connection (SC)

Here , X_{λ_o} has 2 hyperbolic saddles $s_1(\lambda_o), s_2(\lambda_o)$, connected with an unstable manifold manifold $W^u(\lambda_o)$ of $s_1(\lambda_o)$ and a stable one of $s_2(\lambda_o)$. Let σ be a transversal to Γ, let $s_1(\lambda), s_2(\lambda)$ be the unique singularities of X_λ near $s_1(\lambda_o), s_2(\lambda_o)$, $W^s(\lambda)$, $W^u(\lambda)$ the invariant manifolds of $s_1(\lambda), s_2(\lambda)$ near Γ.
Let $\{a(\lambda)\} = \sigma \cap W^s(\lambda), \{b(\lambda)\} = \sigma \cap W^u(\lambda)$ and $\mu(\lambda) = a(\lambda) - b(\lambda)$ with $\{a(\lambda_o)\} = \{b(\lambda_o)\} = \sigma \cap \Gamma$.

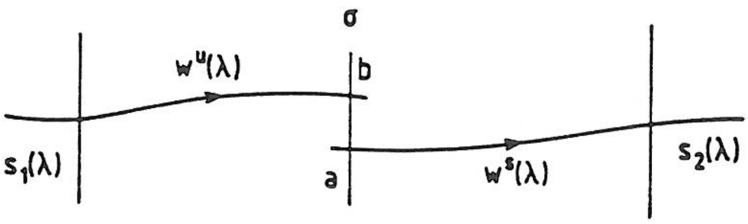


fig. 4.2

Generic case :

The bifurcation set is given by $\mu(\lambda) = 0$.

P.H.-case :

We suppose that the function H has a connection Γ between two saddle points s_1, s_2 contained in the level $\{H = \beta\}$. Let $I(\overline{\lambda}) = \int_\Gamma \omega_D(\overline{\lambda})$.
Then , the equation of the bifurcation set is $I(\overline{\lambda}) = 0$.

A.5 Double Cycle (DC)

We suppose the X_{λ_o} has a semi-stable limit cycle Γ . Let σ be transversal to Γ , and $P(h, \lambda)$ the return map on σ defined for (h, λ) near (h_o, λ_o) , where $\{h_o\} = \sigma \cap \Gamma$.

Generic case :

The equations for the bifurcation set are $P(h, \lambda) = 0$, $\frac{\partial P}{\partial h}(h, \lambda) = 0$ and $\frac{\partial^2 P}{\partial h^2}(h_o, \lambda_o) \neq 0$

P.H.-case :

Let $I(h, \overline{\lambda})$ be the Abelian integral associated to the family . We suppose that for some $h_o \neq \alpha, I(h_o, \overline{\lambda}_o) = \frac{\partial I}{\partial h}(h_o, \overline{\lambda}_o) = 0$ and $\frac{\partial^2 I}{\partial h^2}(h_o, \overline{\lambda}_o) \neq 0$. Then the equation for the bifurcation set is $I(h, \lambda) = \frac{\partial I}{\partial h}(h, \lambda) = 0$.

B Codimension 2 bifurcations

B.1 Degenerate Hopf-Takens bifurcation (DH)

B.2 Semi-hyperbolic bifurcation of codimension 2 (C)

B.3 Bogdanov-Takens bifurcation (TB)

B.4 The degenerate loop (DL)

The vector field X_{λ_o} has a loop Γ through a saddle point $s(\lambda_o)$ where the divergence is zero . Let σ be a transversal to Γ and $s(\lambda)$ the unique singular point of X_λ near $s(\lambda_o)$. Then

$$j^1 X_\lambda(s(\lambda)) \sim x\frac{\partial}{\partial x} - (1-\alpha_o(\lambda))y\frac{\partial}{\partial y}$$

where $\alpha_o(\lambda_o) = 0$.

It is shown in [R] that the return map P_λ on σ has the following expansion (u is a parameter on σ , positive on the side where the return map is defined) :

$$P_\lambda(u) = u + \beta_o(\lambda) + \alpha_o(\lambda)(u\,\omega(u,\lambda) + o(u,\omega)) + \beta_1(\lambda)u + o(u)$$

where $\omega(u,\lambda) = \dfrac{u^{-\alpha_o(\lambda)} - 1}{\alpha_o(\lambda)}$

Generic case :

The equations of the bifurcation set are given by : $\alpha_o(\lambda) = \beta_o(\lambda) = 0$ and $\beta_1(\lambda_o) \neq 0$.

P.H.-case :

Let Γ be a loop for the Hamiltonian H . We suppose that $\Gamma \subset \{H = 0\}$ and that $H > 0$ inside the loop or outside the loop depending on whether the other separatrices are outside the loop or inside . For $h > 0$ near 0 , the Abelian integral I has the following expansion :

$$I(h,\overline{\lambda}) = \overline{\beta}_o(\overline{\lambda}) + \overline{\alpha}_o(\overline{\lambda})h\,log(h) + \overline{\beta}_1(\overline{\lambda})h + o(h)$$

The equations for the bifurcation set are $\overline{\beta}_o(\overline{\lambda}) = \overline{\alpha}_o(\overline{\lambda}) = 0$ and $\overline{\beta}_1(\overline{\lambda}_o) \neq 0$. This bifurcation has been studied in [DRS1] .

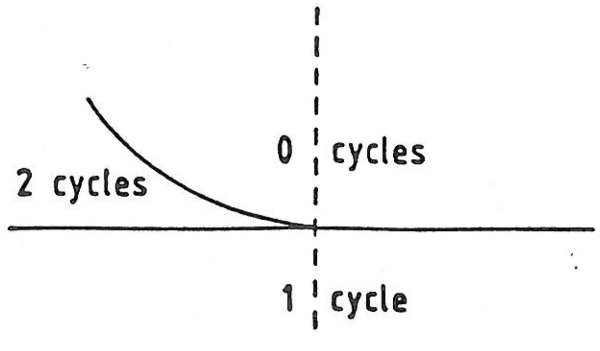


fig. 4.3

B.5 Saddle-node connection (SNC)

The stable (resp. unstable) manifold of a saddle-node $sn(\lambda_o)$ coincides with an unstable (resp. stable) separatrix of a saddle point $s(\lambda_o)$. The bifurcation diagram is given by :

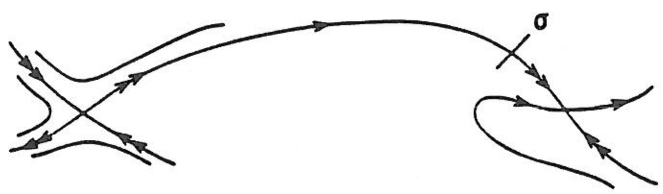


fig. 4.4

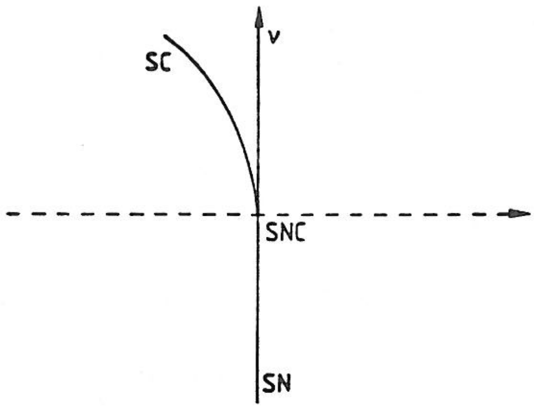


fig. 4.5

See [Sc] for more details .

B.6 Two-saddles cycle (TSC)

We suppose that X_{λ_o} has 2 saddle points $s_1(\lambda_o), s_2(\lambda_o)$ which are connected by two saddle connections Γ_i, Γ_s to make a singular cycle Γ containing 2 saddles .

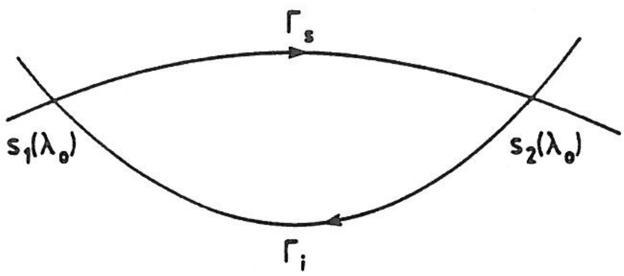


fig. 4.6

Let $\lambda_1(\lambda_o), -\xi_1(\lambda_o)$ be the eigenvalues at $s_1(\lambda_o)$ and $\lambda_2(\lambda_o), -\xi_2(\lambda_o)$ be the eigenvalues at $s_2(\lambda_o)$ ($\lambda_1, \lambda_2, \xi_1, \xi_2 > 0$). The two hyperbolicity ratios $r_1(\lambda_o) = (\xi_1/\lambda_1)(\lambda_o)$ and $r_2(\lambda_o) = (\xi_2/\lambda_2)(\lambda_o)$ are different from 1.

The generic case :

In this case $r = \frac{\xi_1}{\lambda_1} \cdot \frac{\xi_2}{\lambda_2} \neq 1$. The singular cycle is attracting if $r > 1$ and expanding if $r < 1$. Up to orientation we can suppose that we are in the attracting case. Next, up to the order between s_1, s_2 , there are two subcases : the strong attracting case ($r_1 > 1$ and $r_2 > 1$) and the weak attracting case ($r_1 > 1$ and $r_2 < 1$) . These

bifurcations are studied in [DRS2] . The bifurcation diagrams are the following :
The strong attracting case :

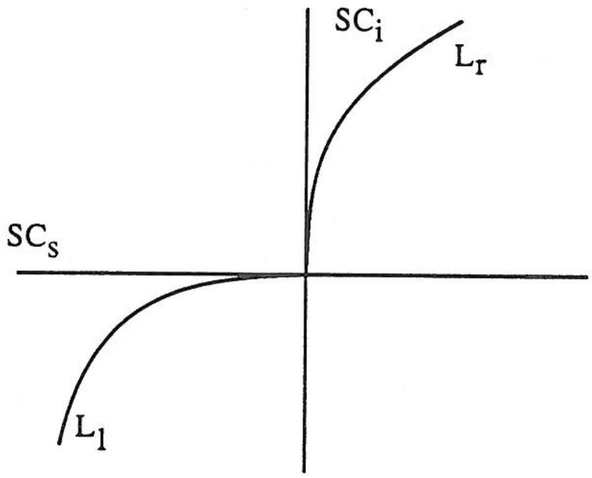


fig. 4.7

The weak attracting case :

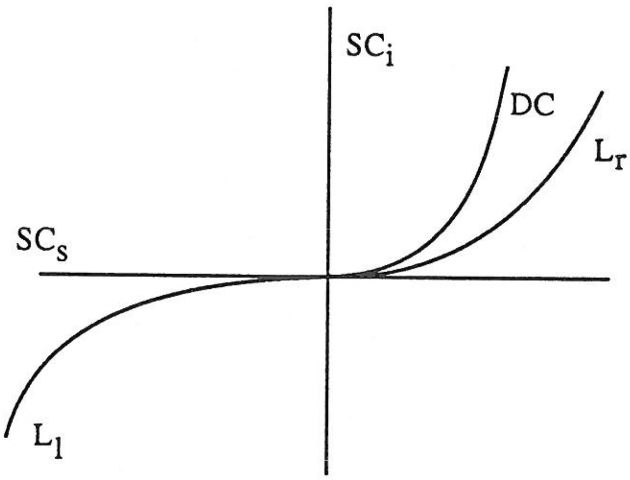


fig. 4.8

The degenerate case :
In this case $r_1 > 1$ and $r_2 < 1$, but the return mapping along Γ is equal to the identity. Therefore $r = r_1 r_2 = 1$. For more information we refer to [DRS2].

C Bifurcations of codimension 3

C.1 Cusp bifurcation of codimension 3 (NC)

C.2 Nilpotent saddle bifurcation of codimension 3 (NS)

C.3 Degenerate two-saddles cycle (DTSC)

Here , X_{λ_o} has 2 saddle points $s_1(\lambda_o), s_2(\lambda_o)$ which are connected by two saddle connection Γ_s, Γ_i to make a singular cycle Γ containing 2 saddles and one of the saddles is divergence free , or in other words has 1 as hyperbolicity rate.
Suppose $r_1(\lambda_o) = 1$ and $r_2(\lambda_o) > 1$. The bifurcation diagram is given by:

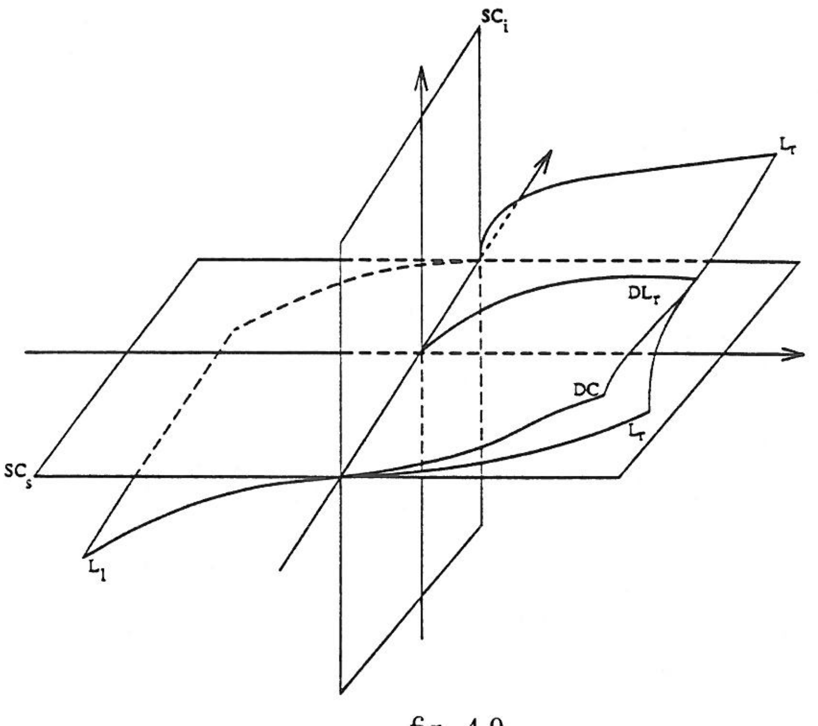


fig. 4.9

The bifurcation diagram is a smooth transition between the diagram of the strong attracting case and that of the weak attracting one. For more information we refer to [Mou].

Rescalings

Consider the family $X_\lambda = y\dfrac{\partial}{\partial x} + (x^3 + \mu_2 x + \mu_1 + y(\nu + bx + x^2 + x^3 h(x,\lambda)) + y^2 Q(x,y,\lambda))\dfrac{\partial}{\partial y}$.

For b constant and non-zero the family $X_{(\mu_1,\mu_2,\nu,b)}$ is a generic unfolding of a saddle of codimension 3 . From [DRS2] we know that the bifurcation diagram is a cone (in the (μ_1,μ_2,ν)-space) with vertex at the origin . More precisely , the parts of the bifurcation set are surfaces (for the codimension 1 strata) and lines (for the codimension 2 ones) which are transversal to the spheres $(\mu_1^2 + \mu_2^2 + \nu^2 = \epsilon^2)$, for ϵ small enough . The intersection of the bifurcation diagram with such a sphere is illustrated in figure (4.10) for $b > 0$. To make a planar picture a point on the sphere S has been deleted . This point has been chosen outside the bifurcation set on the hemisphere $S \cap \{\mu_2 > 0\}$. In the central part of the picture the vertical coordinate is ν ; the horizontal one is μ_1 , oriented to the right .

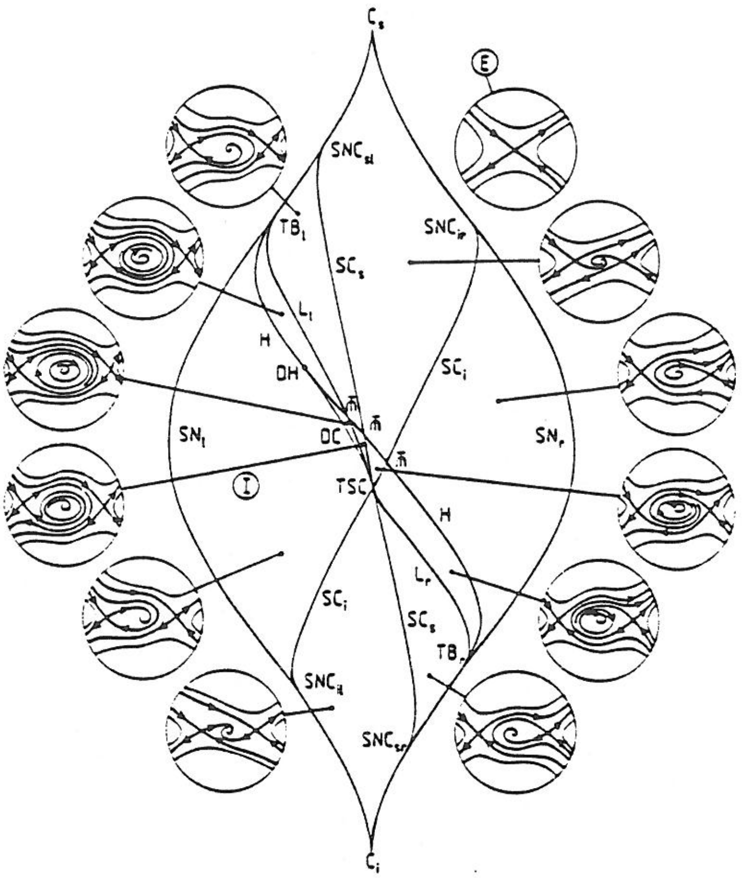


fig. 4.10

Using the linear transformation $(x, y, \mu_1, \mu_2, \nu, b) \to (-x, -y, -\mu_1, \mu_2, \nu, b)$ one finds the bifurcation diagram of a saddle of codimension 3 with $b < 0$. The intersection of the bifurcation sets with a plane $\mu_2 = -\epsilon$, with $\epsilon > 0$ small , is given in figure (4.11).

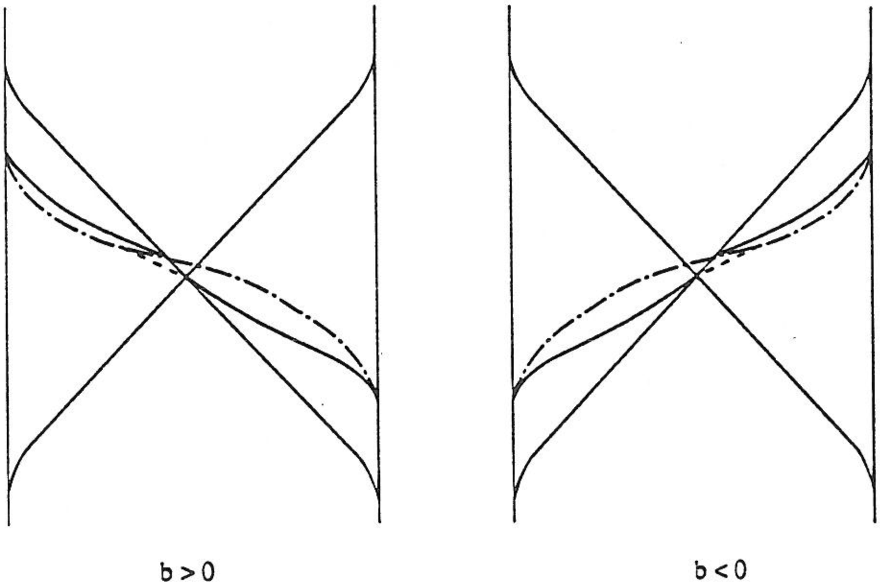


fig. 4.11

In order to clarify the transition between the situation $b > 0$ and the situation $b < 0$ we introduce the principal rescaling:

$$x = \tau\overline{x} \ , \ y = \tau^2\overline{y} \ , \ \mu_1 = \tau^3\overline{\mu}_1 \ , \ \mu_2 = \tau^2\overline{\mu}_2 \ , \ \nu = \tau\overline{\nu} \ , \ b = \tau\overline{b} \ .$$

Using this rescaling family X_λ is transformed into (omitting the bars above x, y)

$$\tau\left\{y\frac{\partial}{\partial x} + (x^3 + \overline{\mu}_2 x + \overline{\mu}_1 + \overline{\nu}y + \tau y(\overline{b}x + x^2) + O(\tau^2)y)\frac{\partial}{\partial y}\right\} \tag{4.14}$$

For each $\tau > 0$, X_λ is C^∞-equivalent to

$$y\frac{\partial}{\partial x} + (x^3 + \overline{\mu}_2 x + \overline{\mu}_1 + \overline{\nu}y + \tau y(\overline{b}x + x^2 + O(\tau)))\frac{\partial}{\partial y} \tag{4.15}$$

For $\tau = 0$ this family becomes:

$$y\frac{\partial}{\partial x} + (x^3 + \overline{\mu}_2 x + \overline{\mu}_1 + \overline{\nu} y)\frac{\partial}{\partial y} \tag{4.16}$$

This is a family of cubic Liénard equations with constant damping.
In the principal chart $\{\overline{\mu}_2 = -1\}$ the phase portraits are:

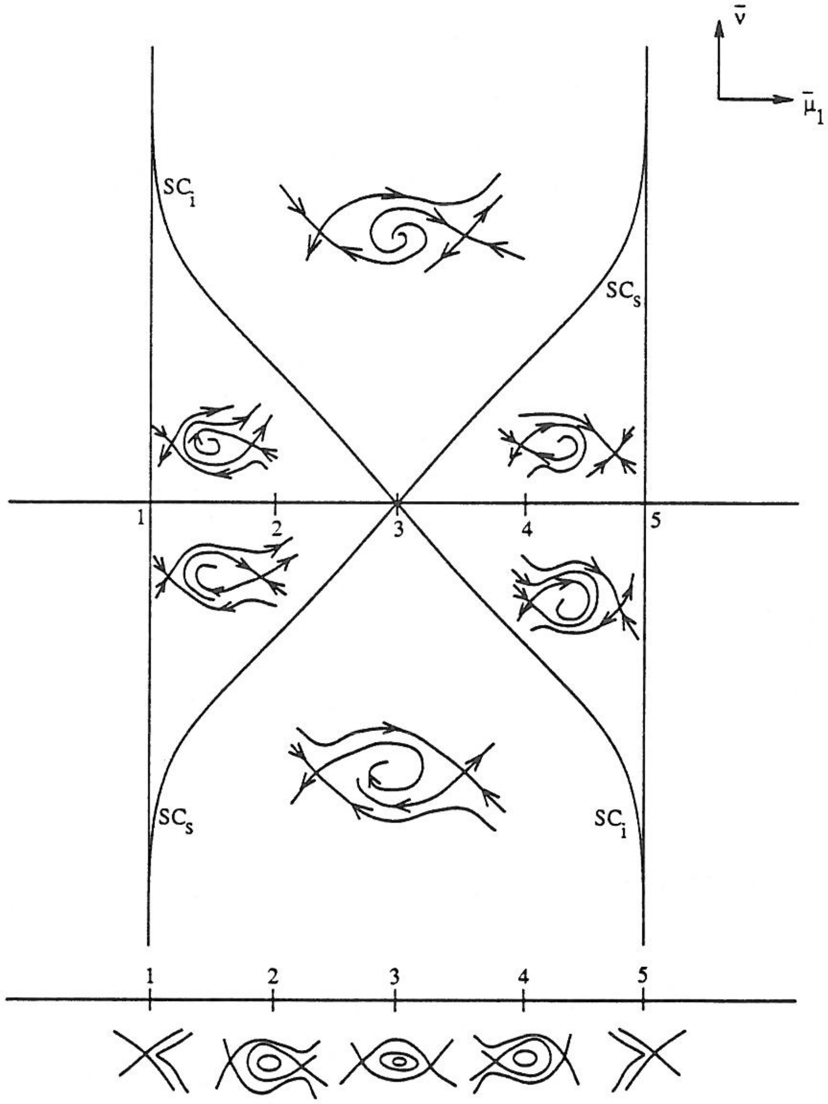


fig. 4.12

Similarly as in [DRS2] and [DR] one shows the genericity of the lines of superior and inferior saddle connections using the rotational property with respect to the parameter $\overline{\nu}$ and the semi-rotational property with respect to the parameter $\overline{\mu}_1$. The divergence of the vector fields (4.16) is equal to $\overline{\nu}$. So there can be no limit cycles for $\overline{\nu} \neq 0$. For $\overline{\nu} = 0$ we have a 1-parameter family of Hamiltonian vector fields.

For $\overline{\nu} \neq 0$, the bifurcation lines remain for τ small. They are stable since they are defined by a transversality condition. For $\overline{\nu} = 0$ the situation degenerates. Therefore we consider the following blow up in parameter space:

$$\begin{cases} \overline{\nu} = u\tilde{\nu} \\ \overline{\mu}_1 = \tilde{\mu}_1 \\ \overline{\mu}_2 = \tilde{\mu}_2 \qquad\qquad \text{with} \quad \tilde{\nu}^2 + r^2 = 1 \\ \overline{b} = \tilde{b} \\ \tau = ur \end{cases}$$

Using this blow up family (4.15) becomes:

$$y\frac{\partial}{\partial x} + (x^3 + \tilde{\mu}_2 x + \tilde{\mu}_1 + uy(\tilde{\nu} + r\tilde{b}x + rx^2) + O(u^2r^2)y)\frac{\partial}{\partial y} \tag{4.17}$$

For $x, y, \tilde{b}, \tilde{\mu}_1, \tilde{\mu}_2$ in a compact there exists a $M > 0$ such that $|\tilde{b}x + x^2 + O(u)| < M$, for u such that $0 \leq u \leq u_o$ (for a certain u_o small enough),

So we have $\tilde{\nu} - rM \leq \tilde{\nu} + r\tilde{b}x + rx^2 + rO(u) \leq \tilde{\nu} + rM$.

If $\tilde{\nu} > 0$ and $0 \leq r \leq \frac{\tilde{\nu}}{2M}$, we have $\frac{\tilde{\nu}}{2} \leq \tilde{\nu} + r\tilde{b}x + rx^2 + rO(u) \leq \frac{3\tilde{\nu}}{2}$.

If $\tilde{\nu} < 0$ and $0 \leq r \leq -\frac{\tilde{\nu}}{2M}$, we have $\frac{3\tilde{\nu}}{2} \leq \tilde{\nu} + r\tilde{b}x + rx^2 + rO(u) \leq \frac{\tilde{\nu}}{2}$.

So, it remains to look at $r = 1$ and $|\tilde{\nu}| \leq M_o$, with $M_o > 0$ fixed and sufficiently big. This is the same as using the following rescaling ("second" rescaling):

$$x = u\tilde{x}, y = u^2\tilde{y}, \mu_1 = u^3\tilde{\mu}_1, \mu_2 = u^2\tilde{\mu}_2, \nu = u^2\tilde{\nu}, b = u\tilde{b}.$$

So, for $r = 1$ family (4.17) becomes:

$$Y_{(\tilde{\lambda},u)} = y\frac{\partial}{\partial x} + (x^3 + \tilde{\mu}_2 x + \tilde{\mu}_1 + uy(\tilde{\nu} + \tilde{b}x + x^2) + O(u^2)y)\frac{\partial}{\partial y} \tag{4.18}$$

In order to study a neighborhood in the original parameters we take $(\tilde{\mu}_1, \tilde{\mu}_2, \tilde{b}) \in S^2 = \{(\tilde{\mu}_1, \tilde{\mu}_2, \tilde{b}) | \tilde{\mu}_1^2 + \tilde{\mu}_2^2 + \tilde{b}^2 = 1\}$, $|\tilde{\nu}| \leq M_o$ ($M_o > 0$,fixed and sufficiently big) and u small enough.

First we treat the case $\tilde{b} = +1$, with $(\tilde{\mu}_1, \tilde{\mu}_2)$ small.

$$Y_{(\tilde{\mu}_1,\tilde{\mu}_2,\tilde{\nu},1,u)} = y\frac{\partial}{\partial x} + (x^3 + \tilde{\mu}_2 x + \tilde{\mu}_1 + uy(\tilde{\nu} + x + x^2) + O(u^2)y)\frac{\partial}{\partial y}$$

Let $(x_o, 0)$ be a nilpotent singularity of $Y_{(\tilde{\mu}_1,\tilde{\mu}_2,\tilde{\nu},1,u)}$ (for u small).
This implies that:

$$\begin{cases} \tilde{\mu}_1 = 2x_o^3 \\ \tilde{\mu}_2 = -3x_o^2 \\ \tilde{\nu} = -x_o - x_o^2 \end{cases}$$

It is easy to check that this singularity is of Bogdanov-Takens type for $x_o \neq 0$ and a nilpotent saddle of codimension 3 for $x_o = 0$.
However what we need is the study of the family $Y_{(\tilde{\mu}_1,\tilde{\mu}_2,\tilde{\nu},\pm1,u)}$ with $(\tilde{\mu}_1, \tilde{\mu}_2, u) \in K \times [0, \epsilon]$, K a small neighborhood and $\epsilon > 0$ small. This is a first problem we do not deal with. Its elaboration may rely on the study performed in [DRS2], but taking care of the fact that u tends to zero.

The case $\tilde{b} = -1$ is similar.

For $\tilde{\mu}_1 = \pm 1$, $\tilde{\mu}_2$ small and $\tilde{b}$ in an arbitrarily large compact we have a structurally stable situation, namely a unique hyperbolic saddle.

Also for $\tilde{\mu}_2 = 1$ we end up with the same structurally stable situation.

At last we consider the case $\tilde{\mu}_2 = -1$ (with $\tilde{\mu}_1, \tilde{b}$ in a arbitrarily large compact).
The equation of the planes (in $(\tilde{\mu}_1, \tilde{b}, \tilde{\nu})$-space) of the degenerate singularities is $\tilde{\mu}_1 = \pm\frac{2}{3\sqrt{3}}$ and for this , the corresponding degenerate singular point is $(x_o, 0) = (\pm\frac{1}{\sqrt{3}}, 0)$.
The limits of the lines of the Bogdanov-Takens singularities , for $u \to 0$, are :

$$\tilde{\mu}_1 = \pm\frac{2}{3\sqrt{3}} \qquad \text{and} \qquad \tilde{\nu} \pm \frac{\tilde{b}}{\sqrt{3}} + \frac{1}{3} = 0$$

After making the translation $x = x_o + X, y = Y$ and using Macsyma to perform the normal form calculations we find on these lines the following points of nilpotent cusps of codimension 3:

$$\tilde{\mu}_1 = \pm\frac{2}{3\sqrt{3}} \quad , \quad \tilde{b} = \mp\frac{2}{\sqrt{3}} \quad , \quad \tilde{\nu} = \frac{1}{3}$$

The equation of the Hopf singularities is given by

$$\begin{cases} x_o^3 - x_o + \tilde{\mu}_1 = 0 \\ Tr(x_o, 0) = u(\tilde{b}x_o + \tilde{\nu} + x_o^2) + O(u^2) = 0 \end{cases}$$

with $x_o \in]-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}[$, since $\Delta := Det(x_o, 0) = 1 - 3x_o^2$ must be strictly greater than 0.
Hence the limit of the Hopf singularities , for $u \to 0$, is :

$$\begin{cases} \tilde{\mu}_1 - x_o + x_o^3 = 0 \\ \tilde{\nu} + \tilde{b}x_o + x_o^2 = 0 \end{cases}$$

with $x_o \in [-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]$.
Taking coordinates (X, Y) around the Hopf point $(x_o, 0)$ $(x = X + x_o, y = Y)$ and

taking into account that $Tr(x_o, 0) = 0$, we find :

$$Y\frac{\partial}{\partial X} + (X^3 + 3x_oX^2 - (1-3x_o^2)X \quad +uY((\tilde{b} + 2x_o + O(u))X + (1+O(u))X^2$$
$$+O(u)O(X^3)) + O(u^2)O(Y^2))\frac{\partial}{\partial Y}$$

Using the coordinate change

$$\begin{cases} X = & u \\ Y = & \sqrt{\Delta}v \end{cases}$$

followed by a multiplication with $\frac{1}{\sqrt{\Delta}}$, we get :

$$v\frac{\partial}{\partial u} + (\frac{1}{\Delta}u^3 + \frac{3x_o}{\Delta}u^2 - u \quad +v(\frac{(\tilde{b} + 2x_o)u + O(u^2)}{\sqrt{\Delta}}u + \frac{u + O(u^2)}{\sqrt{\Delta}}u^2$$
$$+O(u^2)O(u^3)) + O(u^2)O(v^2))\frac{\partial}{\partial v}$$

As the 1-jet is $v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}$, the formula in chapter 4.2.1 of [DRS2] gives that the first Lyapounov coefficient has the same sign as an expression of the form :

$$u(1 - 3x_o\tilde{b} - 9x_o^2) + O(u))$$

So the limit of the Hopf singularities of codimension 2 , for $u \to 0$, is given by

$$\begin{cases} \tilde{\mu}_1 - x_o + x_o^3 = 0 \\ \tilde{\nu} + x_o\tilde{b} + x_o^2 = 0 \\ 1 - 3x_o\tilde{b} - 9x_o^2 = 0 \end{cases}$$

and $x_o \in [-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]\backslash\{0\}$.

Remark that :

(1)

$$\text{when } x_o = \frac{1}{\sqrt{3}} : \qquad \begin{cases} \tilde{\mu}_1 = \dfrac{2}{3\sqrt{3}} \\ \tilde{b} = -\dfrac{2}{\sqrt{3}} \\ \tilde{\nu} = \dfrac{1}{3} \end{cases}$$

(2)

$$\text{when } x_o \downarrow 0 : \qquad \begin{cases} \tilde{\mu}_1 \to 0 \\ \tilde{b} \to +\infty \\ \tilde{\nu} \to -\dfrac{1}{3} \end{cases}$$

(3)

$$\text{when } x_o = -\frac{1}{\sqrt{3}} : \qquad \begin{cases} \tilde{\mu}_1 = -\dfrac{2}{3\sqrt{3}} \\ \tilde{b} = \dfrac{2}{\sqrt{3}} \\ \tilde{\nu} = \dfrac{1}{3} \end{cases}$$

(4)

$$\text{when } x_o \uparrow 0 : \qquad \begin{cases} \tilde{\mu}_1 \to 0 \\ \tilde{b} \to -\infty \\ \tilde{\nu} \to -\dfrac{1}{3} \end{cases}$$

So, near the singularities there are at most 2 limit cycles. The rest of the study (4.18) with $\tilde{\mu}_2 = -1$, especially of the limit cycles and the saddle loops, is a second problem we do not treat.

We can however say more concerning inferior and superior saddle connections. In order to examine the point of two-saddle connection ($\overline{\mu}_1 = 0 = \overline{\nu}$) in the family (4.16) we consider the following blow up in parameter space:

$$\begin{cases} \overline{\nu} = s\nu' \\ \overline{\mu}_1 = s\mu_1' \\ \overline{\mu}_2 = \mu_2' & \text{with} \quad r^2 + (\mu_1')^2 + (\nu')^2 = 1 \\ \overline{b} = b' \\ \tau = sr \end{cases}$$

By this family (4.15) becomes:(omitting primes above x, y)

$$y\frac{\partial}{\partial x} + (x^3 + \mu_2' x + s(\mu_1' + y(\nu' + rb'x + rx^2) + O(s^2r^2)y)\frac{\partial}{\partial y} \tag{4.19}$$

Similarly as we did in the study of family (4.17) one shows that we must look at $r = 1$ and (μ_1', ν') in a large enough compact. This is the same situation as using the rescaling ("third" rescaling):

$$x = sx' \ , \ y = s^2y' \ , \ \mu_1 = s^4\mu_1' \ , \ \mu_2 = s^2\mu_2' \ , \ \nu = s^2\nu' \ , \ b = sb' \ .$$

So, for $r = 1$ family (4.19) becomes:

$$y\frac{\partial}{\partial x} + (x^3 + \mu_2' x + s(\mu_1' + y(\nu' + b'x + x^2)) + O(s^2)y)\frac{\partial}{\partial y}$$

Write

$$X^H = y\frac{\partial}{\partial x} + (x^3 + \mu_2' x)\frac{\partial}{\partial y}$$

X^H is Hamiltonian . The case $\mu_2' = +1$ is of no interest for the bifurcation analysis because X^H is structurally stable . So we consider only $\mu_2' = -1$.

Superior saddle connections (SC_s).

As it is recalled above the equation of the superior saddle connections , at the limit $s \to 0$, is given by :

$$S(\lambda') = \int_{\gamma_s} \omega_D = 0$$

The equation of γ_s is : $y = \frac{1}{\sqrt{2}}(1 - x^2)$, $x \in [-1, 1]$.

So we find :

$$\begin{aligned} S(\lambda') = \int_{\gamma_s} \omega_D &= \int_{\gamma_s} (\mu_1' + y(\nu' + b'x + x^2))dx \\ &= \int_{-1}^{1} (\mu_1' + \frac{1}{\sqrt{2}}(1 - x^2)(\nu' + b'x + x^2))dx \\ &= 2\mu_1' + \frac{2\sqrt{2}}{3}\nu' + \frac{2\sqrt{2}}{15} = 0 \end{aligned}$$

Inferior saddle connections (SC_i).

The equation of γ_i is : $y = \frac{1}{\sqrt{2}}(x^2 - 1)$, $x \in [-1, 1]$, and so we find the following equation , for $s \to 0$, for the inferior saddle connections.

$$\begin{aligned} I(\lambda') &= \int_{\gamma_i} \omega_D \\ &= 2\mu_1' - \frac{2\sqrt{2}}{3}\nu' - \frac{2\sqrt{2}}{15} = 0 \end{aligned}$$

Two saddle connections (TSC)

The intersection of the plane SC_s of superior saddle connections with the plane SC_i of inferior saddle connections gives a line of two saddle connections in the (μ_1', ν', b')-space. The equations of this line are :

$$\begin{cases} \mu_1' = 0 \\ \nu' = -\frac{1}{5} \end{cases}$$

These calculations lead us to the proposal of the following transition between diagrams of figure (4.11).

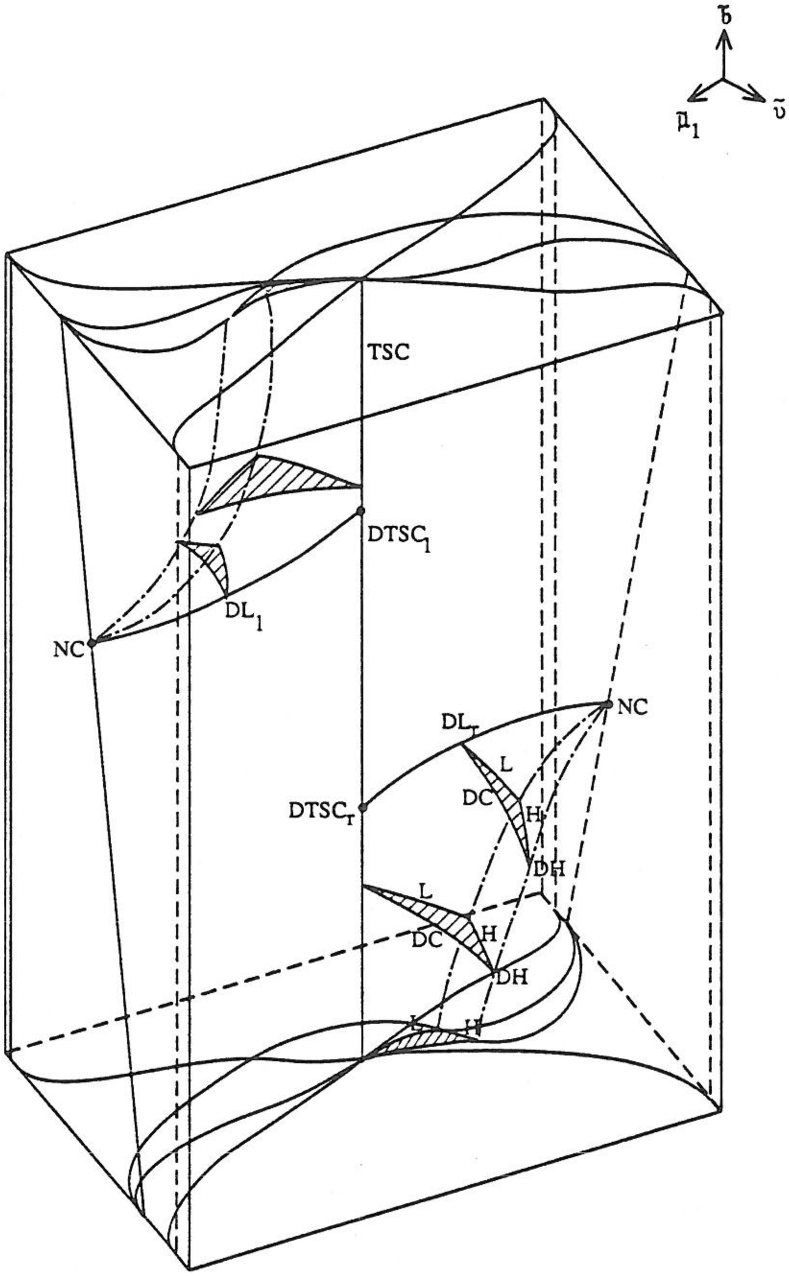


fig. 4.13

Proposal along the line TSC:

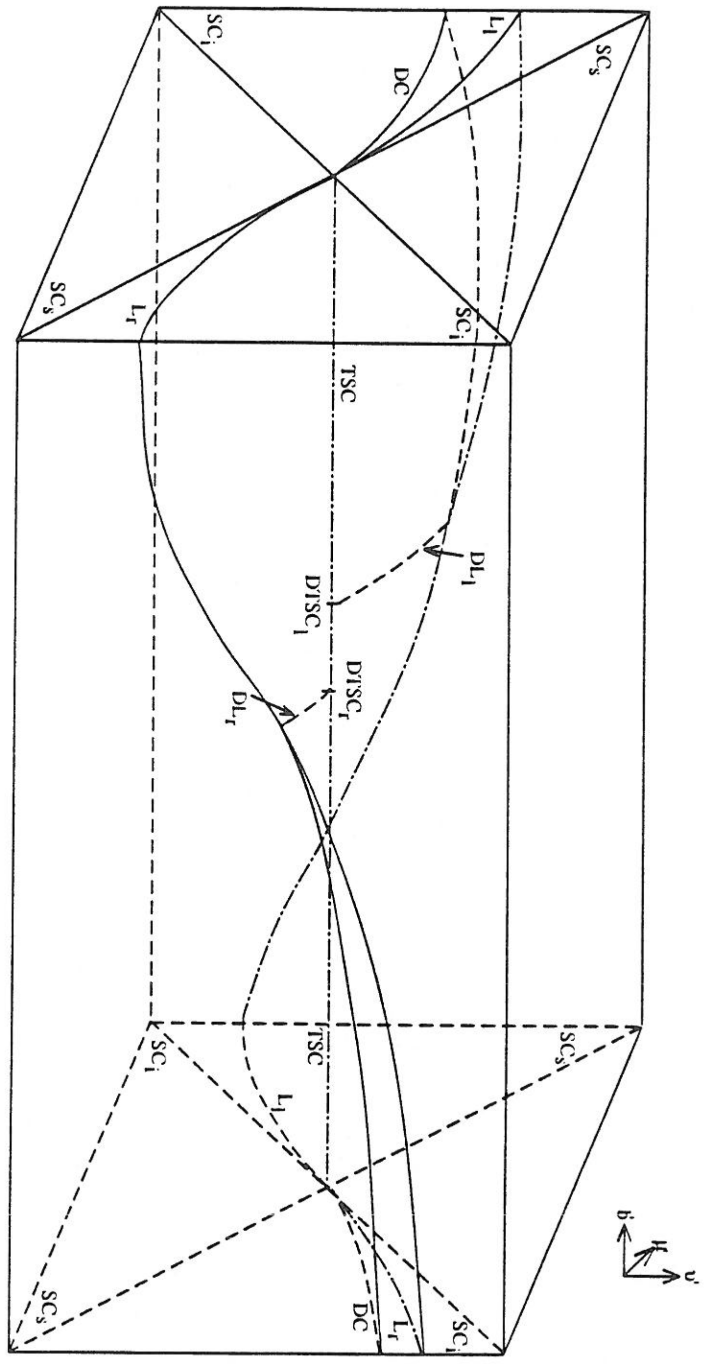


fig. 4.14

This proposal is motivated by the following considerations : Consider the family

$$y\frac{\partial}{\partial x} + (x^3 - x + s(\mu_1' + y(\nu' + b'x + x^2)) + O(s^2)y)\frac{\partial}{\partial y}$$

The singular points $z_1(s)$, $z_2(s)$ are given by

$$\begin{cases} y = 0 \\ x^3 - x + s\mu_1' = 0 \end{cases}$$

Near (-1,0) we introduce $u = x + 1$.

$$(u-1)^3 - (u-1) + s\mu_1' = u^3 - 3u^2 + 2u + s\mu_1' = 0$$

Around $u = 0$, this gives $u = -\frac{1}{2}s\mu_1' + O(s^2)$.
Hence $x = -1 - \frac{1}{2}s\mu_1' + O(s^2)$ for the point $z_1(s)$ and the trace at this point is $Tr(z_1(s)) = s(\nu' - b' + 1) + O(s^2)$.
Near (1,0) we introduce $u = x - 1$.

$$(u+1)^3 - (u+1) + s\mu_1' = u^3 + 3u^2 + 2u + s\mu_1' = 0$$

Around $u = 0$, this gives $u = -\frac{1}{2}s\mu_1' + O(s^2)$.
Hence $x = 1 - \frac{1}{2}s\mu_1' + O(s^2)$ for the point $z_2(s)$ and the trace at this point is $Tr(z_2(s)) = s(\nu' + b' + 1) + O(s^2)$.
In the neighborhood of the line $\mu_1' = 0, \nu' = -1/5$, $Tr(z_1(s)) \approx s(\frac{4}{5} - b') + O(s^2)$ and $Tr(z_2(s)) \approx s(\frac{4}{5} + b') + O(s^2)$. Therefore, for s small , if b' runs from values less than $-\frac{4}{5}$ to values greater than $\frac{4}{5}$, $Tr(z_1(s))$ (resp. $Tr(z_2(s))$ decreases (resp. increases) from positive (resp. negative) values to negative (resp. positive) values. So the hyperbolicity ratio $r(z_1(s))$ (resp. $r(z_2(s))$) increases (resp. decreases) from values smaller (resp. greater) than 1 to values greater (resp. smaller) than 1. We have the following situation:
for $b' < -\frac{4}{5}$: $r(z_1(s)) < 1$ and $r(z_2(s)) > 1$; for $b' > \frac{4}{5}$: $r(z_1(s)) > 1$ and $r(z_2(s)) < 1$; and for intermediate values of b': $r(z_1(s)) < 1$ and $r(z_2(s)) < 1$.
So, for each s small enough but greater than 0, we find back the situation of the generic cases (described in [DRS2] and [Mou]). For $s = 0$ the situation degenerates.

The remaining elaboration can be considered a third problem, which is however a subproblem of the second one. The third rescaling can indeed be seen as a blowing up of the second one, and will be presumably be unavoidable in the study of the second problem.

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