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The underdamped Brownian duet and stochastic linear irreversible thermodynamics

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Building on our earlier work [Proesmans *et al.*, Phys. Rev. X 6, 041010 (2016)], we introduce the underdamped Brownian duet as a prototype model of a dissipative system or of a work-to-work engine. Several recent advances from the theory of stochastic thermodynamics are illustrated with explicit analytic calculations and corresponding Langevin simulations. In particular, we discuss the Onsager–Casimir symmetry, the trade-off relations between power, efficiency and dissipation, and stochastic efficiency. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.5001187]

We illustrate Ilya Prigogine's major contributions to thermodynamics, namely, the development of linear irreversible thermodynamics and the concept of an open dissipative system, on a simple exactly solvable model, namely, a periodically driven particle in a harmonic potential. More recently, a spectacular reformulation of thermodynamic has been achieved by focusing on small systems including the effect of fluctuations. We review some key discoveries of this so-called stochastic thermodynamics, including the fluctuation theorem, its relation to the fluctuation-dissipation relation, and the properties of stochastic efficiency, by considering the same model for a Brownian particle driven by a duet of periodic forces.

I. INTRODUCTION

The theory of linear irreversible thermodynamics is typically introduced as a special topic in an advanced class on thermodynamics. It usually focuses on the derivation of the Onsager symmetry and its application to thermo-electric effects, while the Prigogine minimum entropy production theorem is occasionally included. The derivation of the Onsager symmetry itself is often clouded in the somewhat vague Onsager regression hypothesis, stating that fluctuations on average regress in the same way as externally produced perturbations. The discussion of this issue is actually quite subtle. A related jump from statements about the micro world to the macro world concerns the validity of a microscopic derivation for linear response or Green Kubo relations. There are several other concerns: the Onsager symmetry is by no means general. The most obvious generalization is the Onsager–Casimir symmetry, where one needs to distinguish between time-symmetric quantities (such as position) and time-antisymmetric variables (such as velocities and magnetic field). Furthermore, a proper definition of thermodynamic forces and fluxes is required to get a bona fide thermodynamic description including the bilinear law for the corresponding entropy production. Another usual gap in the whole presentation is the lack of the connection with the thermodynamic engine.

The purpose of this paper is to address all of these issues, by introducing a simple exactly solvable model. It consists of a particle in a harmonic potential subject to a time-periodic force. The time-periodicity can be linked to the time-periodic operation of most thermodynamic engines. The full dynamic and thermodynamic description, including the first and the second law, the Onsager coefficient(s) and the thermodynamic efficiency of the related engine, can be derived *via* a simple explicit calculation without any extraneous assumptions. The Onsager coefficients display the Onsager–Casimir symmetry, including the time-reversal of the periodic driving (which needs not be time-symmetric).

The additional purpose is to present the recent spectacular advances in our understanding of the second law by considering its application to small scale systems. Hence we revisit the above scenario for a Brownian particle, i.e., a particle which is small enough to be subject to the thermal fluctuations. Assuming a description in terms of a Langevin equation with the usual additive Gaussian white noise, the above dynamic and thermodynamic discussion can be repeated. This analysis is the generalization to the underdamped case of the Brownian duet considered in Ref. 1. The Langevin description incorporates the property of detailed balance, which reflects the micro-reversibility of the underlying dynamics. We show that the implied fluctuation dissipation response relations are equivalent to the fluctuation theorem, which is the generalization of the second law to small systems. We discuss the implications for the stochastic efficiency of the engine, and show that they are fully described in terms of the Onsager coefficients derived earlier. We illustrate all the properties by Langevin simulations. They are, as expected in the presence of exact analytic results, in full agreement with the theory. We in particular, illustrate several of the surprising findings in this context notably that the reversible efficiency is the least likely in the long time limit for engines operating under time-symmetric driving.

II. UNDERDAMPED PARTICLE IN A HARMONIC POTENTIAL

Consider a particle with mass m, moving in a one-dimensional harmonic potential with spring constant κ, subject to an external time dependent force $F(t) = F_0 g(t)$ and a friction force proportional to the speed with friction coefficient γ. We will refer to the particle as being the system, while its surrounding responsible for the friction force is supposed to be a thermal reservoir at temperature T. The Newton equation of motion for the position of the particle, $Z(t)$, reads

$$m\ddot{Z}(t) = -\gamma\dot{Z}(t) - \kappa Z(t) + F(t). \tag{1}$$

For long enough times, the dependence on the initial position is forgotten and one can concentrate on the following "steady state" time-dependent solution $Z(t)$ of this equation:

$$Z(t) = \frac{2F_0}{\kappa\tau_2}\int_0^\infty dt' e^{-\frac{t'}{2\tau_1}} \frac{\sinh\left(\sqrt{1-4\tau_1/\tau_2}\,\frac{t'}{2\tau_1}\right)}{\sqrt{1-4\tau_1/\tau_2}} g(t-t'). \tag{2}$$

Here, we identified three intrinsic time-scales of the damped harmonic oscillator: the relaxation time in the absence of a spring, $\tau_1 = m/\gamma$, the overdamped relaxation time in the" absence of a mass" $\tau_2 = \gamma/\kappa$, and the oscillation period in the absence of friction $\tau_3 = 2\pi\sqrt{m/\kappa} = 2\pi\sqrt{\tau_1\tau_2}$. Furthermore, in the absence of an external force, the transition from the underdamped to the overdamped regime is described by the critical ratio $\tau_2/\tau_1 = 4$.[2] This also manifests itself in Eq. (2), as both a numerator and denominator inside the integral go to zero in this limit.

We will be particularly interested in the case of a time-periodic forcing $F(t) = F(t + \tau)$. It follows that $Z(t)$ is also periodic with the same period. In the case of a sine modulation $F(t) = F_0 \sin 2\pi t/\tau$, one finds, see also Fig. 1

$$Z(t) = \frac{F_0}{\kappa}\frac{(1-4\pi^2\alpha_1\alpha_2)\sin(2\pi\alpha) - 2\pi\alpha_2\cos(2\pi\alpha)}{(1-4\pi^2\alpha_1\alpha_2)^2 + 4\pi^2\alpha_2^2} \tag{3}$$

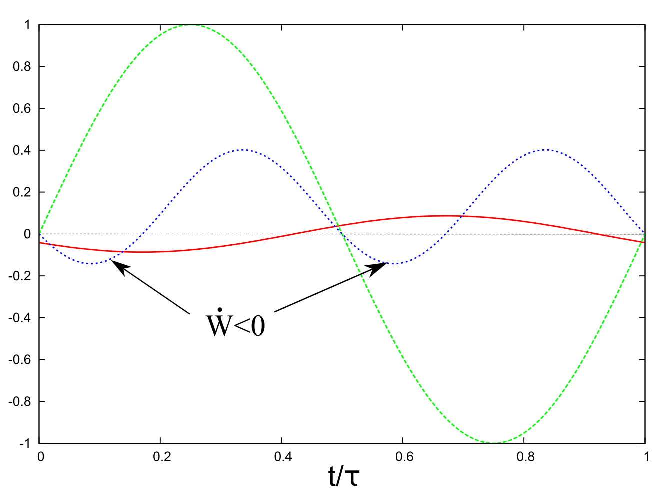

FIG. 1. Scaled position $10Z(t)\kappa/F_0$ (full line), force $F(t)/F_0$ (dashed line), and power $10\dot{W}\tau\kappa/(F_0^2)$ (dotted line) of a bead with $F(t) = F_0\sin(2\pi t/\tau)$, as a function of time for $\tau_1 = \tau_2 = \tau$. Note that while $\bar{\dot{W}} \geq 0$, the system transiently returns part of its energy back to the worksource (cf. the two time-intervals where $\dot{W} \leq 0$).

with

$$\alpha_1 = \tau_1/\tau, \quad \alpha_2 = \tau_2/\tau, \quad \alpha = t/\tau. \tag{4}$$

Having solved the dynamics of the problem, we turn to its thermodynamics. The first law states the conservation of total energy. We assume that the system, consisting of the particle and the spring, has no internal structure or associated internal dynamics. This means that the spring has an energy given by

$$E = mV^2/2 + \kappa Z^2/2 \tag{5}$$

with $V = \dot{Z}$, The time-dependent force is due to an external work reservoir and therefore does not contribute to the internal energy of the system. The power exerted by this external force is given by $\dot{W} = F\dot{Z}$. The notation of power as $\dot{W}$ should not be misinterpreted: the latter is not a full time-derivative, corresponding to the well known fact that there no such thing as a state variable work W. By multiplying the equation of motion Eq. (1) with $\dot{Z}$, one immediately deduces the following balance equation:

$$\dot{E} = \dot{W} + \dot{Q} \tag{6}$$

via which we identify the rate of heat (defined as heat towards the system, i.e., away from the reservoir):

$$\dot{Q} = -\gamma\dot{Z}^2. \tag{7}$$

We recognize the familiar expression of the Joule heating rate $-\dot{Q} = \gamma\dot{Z}^2 \geq 0$, being the heat flux dumped into the reservoir. Again, one should beware of the notation $\dot{Q}$, as this does, in general, not represent the time derivative of a quantity Q [see however comment[3]].

Having identified the heat flux, one can turn to the second law of thermodynamics and the entropy production. We are using here the formulation of thermodynamics for open systems as introduced by Prigogine. The entropy change of a system is the sum of two contributions: the irreversible entropy production $\dot{S}_i$ and the entropy exchange $\dot{S}_e$ with the environment

$$\dot{S} = \dot{S}_i + \dot{S}_e. \tag{8}$$

The entropy flow is given in terms of the heat flux, while the entropy production is nonnegative

$$\dot{S}_e = \frac{\dot{Q}}{T} \quad \dot{S}_i \geq 0. \tag{9}$$

As we are working with a system without an internal structure, we have $\dot{S} = 0$, and therefore

$$\dot{S}_i = -\dot{S}_e = \frac{\gamma\dot{Z}(t)^2}{T}, \tag{10}$$

which is indeed nonnegative.

With the application to thermodynamic engines in mind, we again focus on the case of a time-periodic forcing. $Z(t)$ is then also periodic with the same period, and hence so are all other the thermodynamic quantities, in particular, E and S. The obvious thing to do is to investigate the averages of these quantities over one period. Such an average will be designated by an overbar: $\bar{y} = \int_0^\tau dt\, y(t)/\tau$. Since the energy returns to its original value after each period, one has

$$\bar{\dot{E}} = \bar{\dot{Q}} + \bar{\dot{W}} = 0. \tag{11}$$

Similarly, the system entropy does not change after each period, hence

$$\bar{\dot{S}} = \bar{\dot{S}}_i + \bar{\dot{S}}_e = 0. \tag{12}$$

In combination with the first law, this leads to

$$\bar{\dot{S}}_i = -\frac{\bar{\dot{Q}}}{T} = \frac{\bar{\dot{W}}}{T}. \tag{13}$$

In words, the work performed on the system during each period is dumped, in its entirety, under the form of heat into the reservoir. Prigogine provided another more revealing description of this state of affairs: we are dealing here with a prototype of a dissipative system. The particle is in a time-periodic nonequilibrium state. The nonequilibrium nature of this state entails internal irreversible entropy production. The persistence of this nonequilibirum state is only possible because the system imports a compensating negative entropy flow from the reservoir.

To conclude the analysis, we use the previously derived explicit expression for the heat flow or work flow. One observes that the entropy production (averaged over one period), is quadratic in the amplitude of the driving. One thus reproduced the "usual" expression of the entropy production familiar from usual "steady state" linear irreversible thermodynamics

$$\bar{\dot{S}}_i = \frac{\bar{\dot{W}}}{T} = \mathcal{J}X, \quad \mathcal{J} = LX \quad X = \frac{F_0}{T}. \tag{14}$$

The thermodynamic force X is taken to be the amplitude F_0 of the external driving divided by the reservoir temperature T, in agreement with standard linear irreversible thermodynamics. From $\dot{W} = F\dot{Z}$ with Z given by Eq. (2), one finds the following explicit expression for the Onsager coefficient L

$$L = \frac{2T}{\kappa\tau\tau_2}\int_0^\tau dt \int_0^\infty dt' e^{-\frac{t'}{2\tau_1}} \frac{\sinh\left(\sqrt{1-4\tau_1/\tau_2}\,\frac{t'}{2\tau_1}\right)}{\sqrt{1-4\tau_1/\tau_2}} \times g(t)\dot{g}(t-t'). \tag{15}$$

As is clear from its relation to the non-negative entropy production, this coefficient has to be positive. An explicit proof follows by expressing the periodic forcing in terms of its Fourier series. From

$$F(t) = F_0 \sum_{n=0}^{\infty}\left\{a_{(n,s)}\sin\left(\frac{2\pi nt}{\tau}\right) + a_{(n,c)}\cos\left(\frac{2\pi nt}{\tau}\right)\right\}, \tag{16}$$

one finds that the Onsager coefficient is given by

$$L = \frac{T}{\kappa\tau}\sum_{n=0}^{\infty}\frac{2\pi^2 n^2 \alpha_2\left(a^2_{(n,s)} + a^2_{(n,c)}\right)}{\left(1-4\pi^2 n^2\alpha_1\alpha_2\right)^2 + 4\pi^2 n^2 \alpha_2^2}. \tag{17}$$

The above derivation, while appealing in its simplicity, appears to be unexciting in its implications, aside from the fact that it can serve as a simple prototype model of a dissipative system. From the mathematical point of view, we have merely succeeded in estimating the dissipation per period in terms of a positive coefficient L. To make the connection with an engine, we recall the basic principle of such a construction: it consists of a motor mechanism, corresponding to an entropy producing process, which drives another" entropy consuming" process, i.e., with a negative entropy production.

III. UNDERDAMPED HARMONIC DUET

To build a genuine engine, we repeat the above analysis in the presence of two external forces, i.e., we set

$$F(t) = F_1(t) + F_2(t) \quad F_1(t) = F_1 g_1(t) \quad F_2(t) = F_2 g_2(t). \tag{18}$$

One now distinguishes the work done by each of the forces

$$\dot{W} = \dot{W}_1 + \dot{W}_2, \quad \dot{W}_1 = F_1\dot{Z}, \; \dot{W}_2 = F_2\dot{Z}. \tag{19}$$

Considering time-periodic modulations with the same period, we note that Eqs. (8), (11), and (14) remain valid, with the above replacement for the expression of the (total) work. The system can now act like a catalyst in chemistry: it returns to its original state after each period, having mediated the exchange of work between the two work sources. Inserting Eq. (19) into Eq. (14), one finds (again after averaging over one period):

$$\bar{\dot{S}}_i = \frac{\bar{\dot{W}}_1 + \bar{\dot{W}}_2}{T} = \mathbf{X}^\dagger \mathbf{L}\mathbf{X} = \mathcal{J}^\dagger \mathbf{X}. \tag{20}$$

Hence, instead of a single force X, flux J and Onsager coefficient L, one now has two forces $\mathbf{X} = (X_1 = F_1/T,\ X_2 = F_2/T)^\dagger$ with corresponding fluxes $\mathcal{J} = (\mathcal{J}_1, \mathcal{J}_2)$ linked by a 2×2 Onsager matrix $\mathbf{L}$, $\mathcal{J} = \mathbf{L}\mathbf{X}$. These thermodynamic fluxes give direct access to the work fluxes to the associated reservoir. Furthermore, it is now possible to extract work $\bar{\dot{W}}_1 \le 0$, i.e., the worksource 1 is receiving work, provided worksource 2 "pays for it" by delivering positive work $\bar{\dot{W}}_2 \ge -\bar{\dot{W}}_1 \ge 0$. The efficiency of this" work-to-work-converter" is obviously given by

$$\bar{\eta} = -\frac{\bar{\dot{W}}_1}{\bar{\dot{W}}_2} = -\frac{\mathcal{J}_1 X_1}{\mathcal{J}_2 X_2} = -\frac{L_{11}X_1^2 + L_{12}X_1X_2}{L_{21}X_1X_2 + L_{22}X_2^2} \le 1. \tag{21}$$

To get the explicit expression of the output power $-\bar{\dot{W}}_1 = -\mathcal{J}_1 X_1$ and efficiency $\bar{\eta}$ in terms of the applied thermodynamic forces $\mathbf{X}$, we need the expression for the Onsager matrix $\mathbf{L}$. The latter can be basically copied from the expression Eq. (15) following the splitting of the single force into a duet of forces. One finds

$$L_{ij} = \frac{2T}{\kappa\tau\tau_2}\int_0^\tau dt \int_0^\infty dt' e^{-\frac{t'}{2\tau_1}} \frac{\sinh\left(\sqrt{1-4\tau_1/\tau_2}\,\frac{t'}{2\tau_1}\right)}{\sqrt{1-4\tau_1/\tau_2}} \times \dot{g}_i(t) g_j(t-t'). \tag{22}$$

One can again decompose the Onsager coefficients in terms of Fourier components. Setting

$$F_i(t) = F_{i,0}\sum_{n=1}^{\infty}\left\{a_{(i,n,s)}\sin\left(\frac{2\pi n t}{\tau}\right) + a_{(i,n,c)}\cos\left(\frac{2\pi n t}{\tau}\right)\right\}, \tag{23}$$

one finds that the Onsager coefficients L_{ij} are given by the following bilinear expression in terms of the Fourier amplitudes (σ, $\rho = s$, c refer to sine and cosine contributions, respectively)

$$L_{ij} = \sum_{\sigma,\rho=\{s,c\}}\sum_{n',n=1}^{\infty} a_{(i,n',\sigma)} L_{(i,n',\sigma),(j,n,\rho)} a_{(j,n,\rho)} \tag{24}$$

with

$$\begin{aligned} L_{(i,n',\sigma),(j,n,\sigma)} &= \frac{T}{\kappa\tau}\delta_{n,n'}\frac{2n^2\pi^2\alpha_2 a_{(i,n,\sigma)} a_{(j,n,\sigma)}}{(1-4\alpha_1\alpha_2 n^2\pi^2)^2 + 4\pi^2 n^2\alpha_2^2}, \\ L_{(i,n',s),(j,n,c)} &= \frac{T}{\kappa\tau}\delta_{n,n'}\frac{n\pi(1-4\alpha_1\alpha_2 n^2\pi^2)}{(1-4\alpha_1\alpha_2 n^2\pi^2)^2 + 4\pi^2 n^2\alpha_2^2}, \\ L_{(i,n',c),(j,n,s)} &= -L_{(i,n',s),(j,n,c)}. \end{aligned} \tag{25}$$

We note that different frequencies do not couple to one another. This is of course a consequence of the linearity of the underlying dynamics and hence not a deep symmetry principle. Furthermore, the matrix consists of symmetric and anti-symmetric parts. This observation is put in the proper perspective by considering the Onsager matrix $\tilde{L}_{ij}$ for the time-reversed driving, $\tilde{g}_i(t) = g_i(-t)$

$$\begin{aligned} \tilde{L}_{ij} &= \frac{2T}{\tau\kappa}\int_0^\tau dt \int_0^\infty dt' e^{-\frac{t'}{2\tau_1}} \frac{\sinh\left(\frac{\sqrt{1-4\tau_1/\tau_2}}{2\tau_1}t'\right)}{\tau_2\sqrt{1-4\tau_1/\tau_2}} \\ &\quad \times \dot{g}_i(-t) g_j(-t+t') \\ &= \frac{2T}{\tau\kappa}\int_0^\tau dt \int_0^\infty dt' e^{-\frac{t'}{2\tau_1}} \frac{\sinh\left(\frac{\sqrt{1-4\tau_1/\tau_2}}{2\tau_1}t'\right)}{\tau_2\sqrt{1-4\tau_1/\tau_2}} \\ &\quad \times \dot{g}_j(t) g_i(t-t') \\ &= L_{ji}. \end{aligned} \tag{26}$$

In the transition to the second line, we have used a partial integration with respect to t and shifted the time-axis of t (using the fact that $g_i(t)$ is time-periodic). We thus conclude that the Onsager matrix satisfies the Onsager–Casimir symmetry relation, i.e., it is symmetric upon inverting the quantities that are odd under time reversal (Fig. 2). This completes the thermodynamic analysis of the harmonic duet functioning as a work-to-work converter. In Sec. IV, we review some of its consequences for the efficiency of the engine.

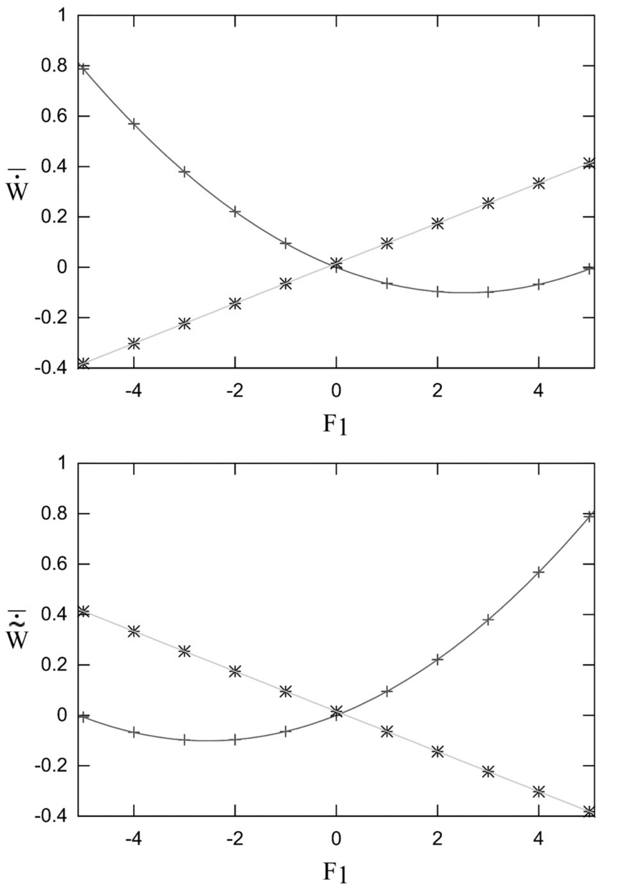

FIG. 2. $\bar{\dot{W}}_1$ and $\bar{\dot{W}}_2$ for the Brownian duet with $F_1(t) = F_1\cos(2\pi t/\tau)$, $F_2(t) = F_2\sin(2\pi t/\tau)$, $F_2 = 1$, $\tau_1 = \tau_2 = \tau$, for time-forward (top) and time-reversed process (bottom). The diagonal terms L_{11} and L_{22} induce a quadratic dependence in F_1 and a constant contribution for $\bar{\dot{W}}_1$ and $\bar{\dot{W}}_2$, respectively, while the off-diagonal terms give linear contributions. A straighforward fitting procedure leads to $L_{11} = L_{22} = \tilde{L}_{11} = \tilde{L}_{22} = 0.0156$, $L_{12} = -L_{21} = \tilde{L}_{21} = -\tilde{L}_{12} = 0.0795$, verifying the Onsager–Casimir symmetry.

IV. EFFICIENCY OF THE HARMONIC DUET

The trade-off between the efficiency, power and dissipation of an engine is an important issue, which has been extensively discussed in the literature.[4–35] Going back to early work by Moritz von Jacobi on maximizing the output power, an interesting scenario consists in optimizing thermodynamic features with respect to the load force. In the present setting of a time-periodic driving, we will assume that the time-dependence of functions $g_1(t)$ and $g_2(t)$ is specified. We select an output load amplitude F_1, such that it maximizes the output power or efficiency, or minimizes dissipation. These three different regimes are identified by the subscript notation MP, ME or mD, respectively. Power $P = -\bar{\dot{W}}_1$, efficiency η and dissipation $\bar{\dot{S}}$ are, *via* their definitions, linked to each other in the linear regime as follows:

$$T\bar{\dot{S}}_i = P\left(\frac{1}{\eta} - 1\right). \tag{27}$$

In Ref. 25, we proved that the values of these quantities in the MP, ME, and mD regimes are further constrained by the following set of relations:[12,24,25]

$$\begin{aligned}\bar{\eta}_{MP} &= \frac{P_{MP}}{2P_{MP}-P_{ME}}\bar{\eta}_{ME}\\ T\bar{\dot{S}}_{i,mD} &= \left(\frac{1}{\bar{\eta}_{MP}}-\frac{1}{\bar{\eta}_{ME}^2}-1\right)P_{MP}+\frac{1}{\bar{\eta}_{ME}^2}P_{ME},\\ P_{mD} &= P_{MP}-\frac{1}{\bar{\eta}_{ME}^2}(P_{MP}-P_{ME}).\end{aligned} \tag{28}$$

In the present case, the Onsager matrix obeys the following symmetry relation:[25]

$$L_{12} = \pm L_{21}. \tag{29}$$

Under this condition, one can derive the additional result that the power at minimum dissipation vanishes, implying the following simplification of Eq. (28)

$$\begin{aligned}&P_{mD}=0,\;\; T\dot{S}_{i,mD}=\left(\frac{1}{\bar{\eta}_{MP}}-2\right)P_{MP},\\ &\frac{P_{ME}}{P_{MP}}=1-\bar{\eta}_{ME}^2,\;\; \bar{\eta}_{MP}=\frac{\bar{\eta}_{ME}}{1+\bar{\eta}_{ME}^2}.\end{aligned} \tag{30}$$

An illustrative verification of all these relations is given in Fig. 3. In this figure, we show the power, efficiency, and dissipation for a harmonic duet with time-symmetric driving, meaning that the symmetry relation, Eq. (29), is satisfied. One can easily verify that Eq. (30) are indeed obeyed.

V. STOCHASTIC DYNAMICS

The analysis of Secs. III and IV can be extended to the study of a periodically driven Brownian particle in a

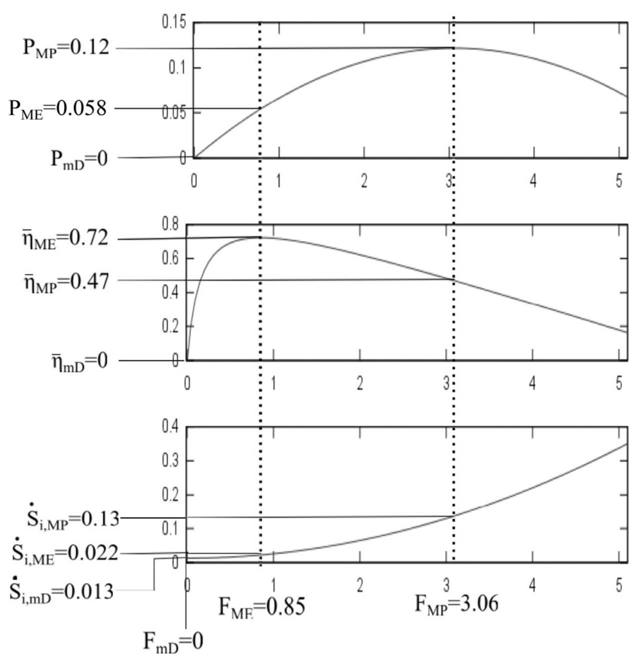

FIG. 3. (a) Power, (b) efficiency, and (c) dissipation as a function of F_1, for a Brownian duet, with $F_1(t)=F_1\cos(2\pi t/\tau)$, $F_2(t)=F_2\sin(2\pi t/\tau)$, $\tau_1=\tau_2=\tau$ and scaling $\tau=\kappa=F_2=1$. One verifies that $P_{ME}/P_{MP}=0.478=1-\bar{\eta}_{ME}^2, \bar{\eta}_{ME}/(1+\bar{\eta}_{ME}^2)=0.47=\bar{\eta}_{MP}$, $P_{mD}=0$, and $(1/\bar{\eta}_{MP}-2)P_{MP}=0.013=T\dot{S}_{i,mD}$.

harmonic potential, by adding a noise term to the equation of motion, Eq. (1)

$$m\ddot{z}(t) = -\gamma\dot{z}(t)-\kappa z(t)+F(t)+\sqrt{2\gamma k_B T}R(t). \tag{31}$$

$R(t)$ is the delta correlated noise

$$\langle R(t)\rangle = 0,\quad \langle R(t)R(t')\rangle = \delta(t-t'). \tag{32}$$

The amplitude of the noise is chosen such that it obeys the fluctuation dissipation relation. We are using a lower case notation to stress that the position $z(t)$ of the particle is now a stochastic, fluctuating quantity. However, as the above equation is linear, one immediately verifies that $Z(t)=\langle z(t)\rangle$ still obeys Eq. (1). Similarly one finds that the stochastic power, again using the lower case convention to identify the corresponding stochastic quantities

$$\dot{w}_i = F_i(t)\dot{z}(t), \tag{33}$$

reduces upon averaging to the previously introduced power $\langle\dot{w}_i\rangle=\dot{W}_i$. For this reason, the linear thermodynamics of Sec. III describes the ensemble properties of the above stochastic model. In particular, ensemble averaged response and efficiency are quantified in terms of the aforementioned Onsager–Casimir coefficients.

In the remainder of this paper we however show how other stochastic properties of the model are also linked to the Onsager coefficients. The first connection is not really novel, as it is an expression of the famous fluctuation dissipation theorem (although here derived in the context of a time-periodic system). We consider the fluctuations in the power output

$$C_{ij}(t) = \langle w_i w_j\rangle - \langle w_i\rangle\langle w_j\rangle, \tag{34}$$

where the notation w_i stands for the change over a time interval $[0, t]$

$$w_i = \int_0^t dt'\dot{w}_i(t'). \tag{35}$$

We omit the explicit dependence on t for notational simplicity, whenever it is clear from the context. In the sppendix, we derive the following expression for $C_{ij}(t)$ in the limit $t=n\tau$ with the number n of cycles large. The $\sim$ sign denotes, here and in the sequel, an equality to dominant order in t

$$\begin{aligned}C_{ij}(n\tau) \sim \frac{2k_BT^3nX_iX_j}{\kappa\tau_2}\int_0^\tau dt'\int_0^\infty dt''\big(\dot{g}_i(t')g_j(t'-t'')\\ +\dot{g}_j(t')g_i(t'-t'')\big)\frac{e^{-\frac{t''}{2\tau_1}}\sinh\left(\sqrt{1-4\tau_1/\tau_2}\,\frac{t'}{2\tau_1}\right)}{\sqrt{1-4\tau_1/\tau_2}}.\end{aligned} \tag{36}$$

Comparison with Eq. (22) leads to the fluctuation-dissipation relation

$$C_{ij}(n\tau) \sim k_BT^2n\tau X_iX_j(L_{ij}+L_{ji}). \tag{37}$$

VI. FLUCTUATION THEOREM

While we have reproduced the above fluctuation dissipation relation by an explicit calculation, its validity can be derived directly from the generalization of the second law, describing small scale nonequilibrium systems. To formulate this so-called fluctuation theorem, one needs to define the stochastic analogues of the entropy, energy, heat, and work. These quantities will be denoted by the lower case notation of their macroscopic counterparts. We refer to Refs. 36 and 37 for an introduction to this stochastic thermodynamics,[38,39] and briefly review the main result that is relevant here. The stochastic system entropy s obeys a Prigogine balance equation

$$\dot{s} = \dot{s}_i + \dot{s}_e \quad \dot{s}_e = \dot{q}/T, \tag{38}$$

$\dot{q}$ is the stochastic heat flux into the system. Contrary to its macroscopic average, the stochastic entropy production $\dot{s}_i$ need not be positive. In fact, the second law is replaced by a symmetry relation for the probability distribution of this quantity: the fluctuation theorem states that the probability to have a positive stochastic entropy production rate is exponentially more likely then to have the corresponding negative entropy production rate in the process with time-inverted driving[40–44]

$$\frac{p(\Delta s_i)}{\tilde{p}(-\Delta s_i)} \sim e^{\frac{\Delta s_i}{k_B}} \tag{39}$$

with $\Delta s_i = \int_0^t dt' \dot{s}_i(t')$. We note in passing that, by multiplying with $\tilde{p}_t(-\Delta s_i)$ and integrating over Δs_i, one finds the so-called integral version of the fluctuation theorem, which in turn implies by Jensen's inequality the usual second law property for the ensemble average

$$\langle e^{\frac{\Delta s_i}{k_B}} \rangle = 1 \to \langle \Delta s_i \rangle \geq 0. \tag{40}$$

For the application to the present situation, one needs a "stronger" fluctuation theorem expressed in terms of the individual fluxes[45,46]

$$\frac{p_t(w_1, w_2)}{\tilde{p}_t(-w_1, -w_2)} \sim e^{\frac{w_1+w_2}{k_B T}}, \tag{41}$$

where we have used the fact that the stochastic entropy production in the long time limit is equal to the work input divided by the temperature, i.e., $\Delta s_i = \Delta s - \Delta s_e \sim -\Delta s_e = -q/T = (w_1 + w_2)/T$. Due to the linearity of Eq. (31), the work distribution is Gaussian[47]

$$p_t(w_1, w_2) = \frac{1}{2\pi\sqrt{\det C}} e^{-\frac{1}{2}\sum_{i,j}\left(w_i - \langle w_i\rangle_t\right) C_{ij}^{-1}\left(w_j - \langle w_j\rangle_t\right)} \tag{42}$$

and an analogous result for the time-inverted dynamics. One finds in the long time limit that

$$\frac{2(w_1+w_2)t^2}{k_B T} \sim -\sum_{i,j}\left(w_i - \langle w_i\rangle_t\right) C_{ij}^{-1}\left(w_j - \langle w_j\rangle_t\right) + \sum_{i,j}\left(w_i + \langle \tilde{w}_i\rangle_t\right) \tilde{C}_{ij}^{-1}\left(w_j + \langle \tilde{w}_j\rangle_t\right). \tag{43}$$

Noting that this should hold for any values of the w_i, one has

$$\mathbf{C}(t) \sim \tilde{\mathbf{C}}(t), \tag{44}$$

$$\mathbf{C}^{-1}(t)\left(\langle \mathbf{w}\rangle_t + \langle \tilde{\mathbf{w}}\rangle_t\right) \sim \frac{1}{k_B T}\mathbf{1}, \tag{45}$$

$$\langle \mathbf{w}\rangle_t^\dagger \mathbf{C}^{-1}(t)\langle \mathbf{w}\rangle_t \sim \langle \tilde{\mathbf{w}}\rangle_t \mathbf{C}^{-1}(t)\langle \tilde{\mathbf{w}}\rangle_t, \tag{46}$$

where $\mathbf{1} = (1, 1)$ and where we used the fact that C is by definition symmetric. Plugging Eq. (45) into Eq. (46) gives

$$\mathbf{1}^\dagger \mathbf{C}(t)\mathbf{1} \sim 2k_B T \mathbf{1}\langle \mathbf{w}\rangle_t = 2k_B T^2 n\tau \mathbf{X}^\dagger \mathbf{L}\mathbf{X} \tag{47}$$

with $\mathbf{X} = (X_1, X_2)^\dagger$ and where we consider $t = n\tau$ with n large in the last equality sign. As this equation should hold for any choice of $\mathbf{X}$, one reproduces Eq. (37), i.e., the fluctuation theorem reproduces the fluctuation-dissipation theorem.

VII. STOCHASTIC EFFICIENCY

A recent discovery in the field of stochastic thermodynamics has to do with the properties of the stochastic efficiency.[48–65] The latter is defined as

$$\eta = -\frac{w_1}{w_2}. \tag{48}$$

One expects that this stochastic quantity will converge to the thermodynamic efficiency $\bar{\eta}$ in the limit of long times t, as both work fluxes will converge to their average value in the large time limit, $w_i \sim \dot{\bar{W}}_i t$ for $i = 1, 2$, implying that the results from Sec. IV remain valid, when one focuses on averaged values. The approach of this limit however holds some surprises, which can be nicely illustrated in the present model. The probability distribution for the efficiency is given by

$$p_t(\eta) = \int dw_1 \int dw_2\, p(w_1, w_2)\delta\left(\eta + \frac{w_1}{w_2}\right). \tag{49}$$

As the probability distribution associated with the work is purely Gaussian, cf. Eq. (42), the efficiency distribution can be calculated exactly for all times

$$p_t(\eta) = \frac{e^{c(\eta)}\left[2 + \frac{|a(\eta)|}{\sqrt{b(\eta)}} e^{\frac{a(\eta)^2}{2b(\eta)}}\sqrt{2\pi}\,\mathrm{erf}\left(\frac{|a(\eta)|}{\sqrt{2b(\eta)}}\right)\right]}{2b(\eta)\pi\sqrt{\det \mathbf{C}(t)}} \tag{50}$$

with

$$\begin{aligned} a(\eta) &= \frac{C_{22}(t)\eta\langle w_1\rangle_t - C_{11}(t)\langle w_2\rangle_t + C_{12}(t)\left(\langle w_1\rangle_t - \eta\langle w_2\rangle_t\right)}{\det \mathbf{C}(t)}, \\ b(\eta) &= \frac{C_{11}(t) + 2C_{12}(t)\eta + C_{22}(t)\eta^2}{\det \mathbf{C}(t)}, \\ c(\eta) &= -\frac{C_{22}(t)\langle w_1\rangle_t^2 - 2C_{12}(t)\langle w_1\rangle_t\langle w_2\rangle_t - C_{11}(t)\langle w_2\rangle_t^2}{2\det \mathbf{C}(t)}. \end{aligned} \tag{51}$$

One can straightforwardly check that

$$p_t(\eta) \sim \eta^{-2} \tag{52}$$

for $\eta \to \pm\infty$. This implies that the moments, and in particular, the average and the cumulant generating function, do not exist. While this may seem to be counter-intuitive, one has to realize that the efficiency is not an "additive" quantity, but rather the ratio of "additive" quantities, and therefore has some unusual properties. Furthermore, the macroscopic efficiency is well defined and given by $\bar{\eta} = -\lim_{t\to\infty}\langle w_1\rangle/\langle w_2\rangle$. The properties of the stochastic efficiency are particularly interesting as one approaches this asymptotic limit. The probability distribution for the efficiency converges to a delta function centered at the macroscopic efficiency $\bar{\eta}$, with all other efficiencies exponentially unlikely. More explicitly, this asymptotic behavior is described by the so-called large-deviation-function $J(\eta)$[66]

$$J(\eta) = -\lim_{\eta\to\infty}\frac{1}{t}\ln p_t(\eta). \tag{53}$$

By applying this limit to Eq. (50) and combining it with the fluctuation-dissipation result, Eq. (37), one finds that $J(\eta)$ can be expressed as follows in terms of the Onsager matrix:

$$J(\eta) = \frac{1}{4k_B}\frac{\left(\begin{bmatrix} X_1 & \eta X_2 \end{bmatrix}\boldsymbol{L}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right)^2}{\begin{bmatrix} X_1 & \eta X_2 \end{bmatrix}\boldsymbol{L}\begin{bmatrix} X_1 \\ \eta X_2 \end{bmatrix}}. \tag{54}$$

One verifies the following remarkable properties. $J(1)$ is invariant under a transposition of the Onsager matrix implying $J(1) = \tilde{J}(1)$. Furthermore, $J(\eta)$ has a unique maximum at $\eta = 1$ if the Onsager matrix is symmetric, $L_{12} = L_{21}$, which is the case when the driving is time-symmetric. In particular, the probability distribution will intersect at reversible efficiency in the case of time-asymmetric driving, while for time-symmetric protocols a minimum emerges at reversible efficiency in the efficiency distribution. These properties are verified *via* analytic calculations and simulations in Figs. 4 and 5. For these simulations we used a standard Euler–Maruyama integrator.

The above long-time properties are in fact generic, as is clear by deriving them directly from the fluctuation theorem. From

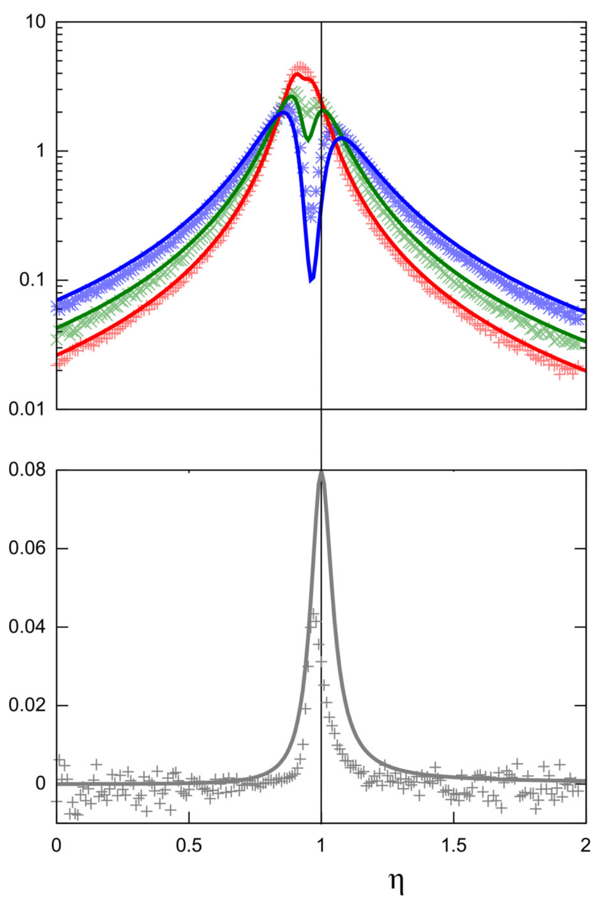

FIG. 4. Stochastic efficiency of a Brownian duet with time-symmetric driving: $g_1(t) = 10\cos(2\pi t/\tau)$ and $g_2(t) = 10\cos(2\pi t/\tau) + \cos(4\pi t/\tau)$ with thermodynamic forces $X_1 = -X_2 = 10$, which leads to $\bar{\eta} = 0$. Upper panel: probability distribution of the efficiency after 32 (red), 64 (green), and 128 (blue) cycles with analytical results and simulation data. Lower panel: large deviation function of the efficiency, with analytical results and extrapolation from simulation data (using the extrapolation procedure described in Refs. 55 and 57).

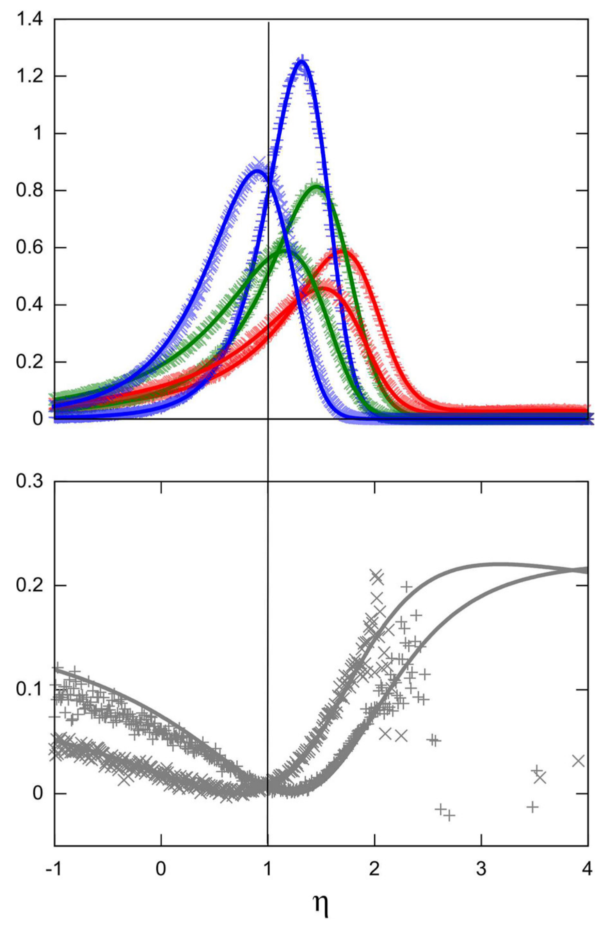

FIG. 5. Stochastic efficiency of a Brownian duet with time-asymmetric driving: $g_1(t) = 10\cos(2\pi t/\tau)$ and $g_2(t) = 10\cos(2\pi(t/\tau - 0.4))$ with thermodynamic forces $X_1 = 2$, $X_2 = 1$. Upper panel: probability distribution of the efficiency after 16 (red), 32 (green), and 64 (blue) cycles with analytical results and simulation data. Lower panel: large deviation function of the efficiency, with analytical results and extrapolation from simulation data. Note that the deviations between analytic and simulation results for larger values of η are due to insufficient statistics.

$$\frac{p_t(w_1,w_2)}{\tilde{p}_t(-w_1,-w_2)}\sim e^{\frac{w_1+w_2}{k_BT}}, \tag{55}$$

one finds that the large deviation function $I(w_1, w_2)$ of the joint work

$$I(w_1,w_2)=-\lim_{t\to\infty}\frac{1}{t}\ln p_t(w_1t,w_2t), \tag{56}$$

obeys the symmetry property

$$I(w_1,w_2)-\tilde{I}(-w_1,-w_2)=\frac{w_1+w_2}{k_BT} \tag{57}$$

with an analogous relation for the time-inverted dynamics. This large deviation function for the work fluxes is related to the large deviation function for the efficiency *via* the so called contraction principle

$$J(\eta)=\min_{-w_1/w_2=\eta}I(w_1,w_2)=\min_{\lambda}I(-\eta\lambda,\lambda). \tag{58}$$

Note that for reversible efficiency, $\eta=1$, one has $w_1+w_2=0$, and therefore, using Eq. (56)

$$J(1)=\min_{\lambda}I(-\lambda,\lambda)=\min_{\lambda}\tilde{I}(\lambda,-\lambda)=\tilde{J}(1), \tag{59}$$

i.e., the large deviation functions for the efficiency of the time-forward and time-reversed process intersect at $\eta = 1$. For the time-symmetric case, we note that the minimisation in Eq. (58) includes $\lambda = 0$, and therefore

$$J(\eta)\le I(0,0). \tag{60}$$

On the other hand, Eq. (56) implies

$$I(\lambda,-\lambda)+I(-\lambda,\lambda)=0, \tag{61}$$

and as $I(w_1, w_2)$ is generally convex, this implies

$$J(1)=\min_{\lambda}I(\lambda,-\lambda)=I(0,0). \tag{62}$$

Combining with Eq. (60) implies that $J(1)$ is the maximum of $J(\eta)$, and reversible efficiency becomes the least likely efficiency.

VIII. CONCLUSIONS

In this work, we have discussed the underdamped periodically driven (Brownian) duet in the terminology of Ilya Prigogine. Entropy production, Onsager coefficients, and Onsager–Casimir symmetry can be easily derived. The analysis provides a pedagogical illustration of a periodically driven dissipative system, and with a duet of forces of a work-to-work convertor. With the addition of thermal noise, the model can be analyzed in full analytic detail in the context of stochastic thermodynamics. In particular, the connection to the fluctuation-dissipation relation, to the fluctuation theorem, and to universal properties of stochastic efficiency can be displayed.

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APPENDIX: WORK CORRELATION FUNCTION C_{IJ}

The solution of the Langevin equation, Eq. (31), for a particle starting at position $z(0)=z_0$ with initial velocity, $v(0)=v_0$ sampled from the periodic steady state distribution Eq. (A3), is given by

$$\begin{aligned}z(t)-\langle z(t)\rangle=&\frac{e^{-\frac{\gamma t}{2m}}\left(\left(\sinh\left(\sqrt{1-4\tau_1/\tau_2}\frac{t}{2\tau_1}\right)+\sqrt{1-4\tau_1/\tau_2}\cosh\left(\sqrt{1-4\tau_1/\tau_2}\frac{t}{2\tau_1}\right)\right)z_0+2\tau_1\sinh\left(\sqrt{1-4\tau_1/\tau_2}\frac{t}{2\tau_1}\right)v_0\right)}{\sqrt{1-4\tau_1/\tau_2}}\\&+\frac{2F_0}{\beta\kappa\tau_1\tau_2}\int_0^\infty dt'e^{-\frac{t'}{2\tau_1}}\frac{\sinh\left(\sqrt{1-4\tau_1/\tau_2}\frac{t}{2\tau_1}\right)}{\sqrt{1-4\tau_1/\tau_2}}R(t-t').\end{aligned} \tag{A1}$$

To evaluate the probability distribution of position and velocity in the periodic steady state, it is convenient to start from the Kramers equation

$$\frac{\partial}{\partial t}p(z,v;t)=-v\frac{\partial}{\partial z}p(z,v;t)+\frac{\partial}{\partial v}(\gamma v p(z,v;t))+\left(\frac{z}{\tau_1\tau_2}+\frac{F_1g_1(t)+F_2g_2(t)}{m}\right)\frac{\partial}{\partial v}p(z,v;t)+\frac{1}{\tau_1 m\beta}\frac{\partial^2}{\partial v^2}p(z,v;t). \tag{A2}$$

One verifies by substitution that the solution reads

$$p(z,v;t)=\frac{\beta\sqrt{\kappa m}}{2\pi}e^{-\frac{\beta}{2}\left(m(v-\langle v\rangle_t)^2+\frac{\kappa}{2}(z-\langle z\rangle_t)^2\right)}, \tag{A3}$$

where $\langle z\rangle_t$ is the solution of Eq. (1) and $\langle v\rangle_t=d\langle z\rangle_t/dt$. Multiplying Eq. (A1) with z_0 and averaging with the distribution given in Eq. (A3) leads to

$$\langle z(0)z(t)\rangle - \langle z(0)\rangle\langle z(t)\rangle = \frac{2e^{-\frac{\gamma t}{2m}}\left(\left(\sinh\left(\sqrt{1-4\tau_1/\tau_2}\frac{t}{2\tau_1}\right)+\sqrt{1-4\tau_1/\tau_2}\cosh\left(\sqrt{1-4\tau_1/\tau_2}\frac{t}{2\tau_1}\right)\right)\right)}{\beta\kappa}$$
$$= \frac{2}{\beta\kappa\tau_2}\int_0^t dt' \frac{e^{-\frac{t'}{2\tau_1}}\sinh\left(\sqrt{1-4\tau_1/\tau_2}\frac{t'}{2\tau_1}\right)}{\sqrt{1-4\tau_1/\tau_2}}. \tag{A4}$$

Note that the right hand side is invariant under a shift of the time axis, and therefore stationary.

Turning to the work distribution, one writes

$$\begin{aligned}
C_{ij}(t) &= \left\langle \int_0^t dt'\dot{w}_i(t') \int_0^t dt''\dot{w}_j(t'')\right\rangle - \left\langle \int_0^t dt'\dot{w}_i(t')\right\rangle \\
&\quad \times \left\langle \int_0^t dt''\dot{w}_j(t'')\right\rangle \\
&= T^2 X_i X_j \int_0^t dt' \int_0^t dt'' \dot{g}_i(t')\dot{g}_j(t'') \\
&\quad \times \left(\langle z(t')z(t'')\rangle - \langle z(t)\rangle\langle z(t')\rangle\right) \\
&= T^2 X_i X_j \int_0^t dt' \int_0^{t'} dt''(\dot{g}_i(t')\dot{g}_j(t'-t'') \\
&\quad + \dot{g}_j(t')\dot{g}_i(t'-t''))\left(\langle z(t'')z(0)\rangle - \langle z(t'')\rangle\langle z(0)\rangle\right).
\end{aligned} \tag{A5}$$

This result further simplifies after partial integration and for $t = n\tau$

$$\begin{aligned}
C_{ij}(n\tau) &= T^2 X_i X_j \int_0^{n\tau} dt' \int_0^{t'} dt''(\dot{g}_i(t')g_j(t'-t'') \\
&\quad + \dot{g}_j(t')g_i(t'-t''))\frac{d}{dt''}\left(\langle z(t'')z(0)\rangle - \langle z(t'')\rangle\langle z(0)\rangle\right) \\
&\quad - T^2 X_i X_j \int_0^{n\tau} dt' \int_0^{t'} dt''(\dot{g}_i(t')g_j(0) \\
&\quad + \dot{g}_j(t')g_i(0))\left(\langle z(t')z(0)\rangle - \langle z(t'')\rangle\langle z(0)\rangle\right) \\
&= \frac{2T^2 X_i X_j}{\beta\kappa\tau_2}\int_0^{n\tau} dt' \int_0^{t'} dt''(\dot{g}_i(t')(g_j(t'-t'') \\
&\quad - g_j(0)) + \dot{g}_j(t')\left(g_i(t'-t'') - g_i(0)\right)) \\
&\quad \times \frac{e^{-\frac{t'}{2\tau_1}}\sinh\left(\sqrt{1-4\tau_1/\tau_2}\frac{t'}{2\tau_1}\right)}{\sqrt{1-4\tau_1/\tau_2}}.
\end{aligned} \tag{A6}$$

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