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ON THE CENTERS OF QUANTUM GROUPS OF A_n-TYPE

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Abstract. Let $\mathfrak{g}$ be the finite dimensional simple Lie algebra of type A_n, and let $\overline{U} = U_q(\mathfrak{g}, \Lambda)$ and $U = U_q(\mathfrak{g}, Q)$ be the quantum groups defined over the weight lattice and over the root lattice respectively. In this paper, we find two algebraically independent central elements in $\overline{U}$ for all $n \geq 2$ and give an explicit formula of the Casimir elements for the quantum group $\overline{U} = U_q(\mathfrak{g}, \Lambda)$, which corresponds to the Casimir element of the enveloping algebra $U(\mathfrak{g})$. Moreover, for $n = 2$ we give explicitly generators of the center subalgebras of the quantum groups $\overline{U} = U_q(\mathfrak{g}, \Lambda)$ and $U = U_q(\mathfrak{g}, Q)$.

1. Introduction

1.1. **Background.** Let $\mathfrak{g}$ be the finite dimensional simple Lie algebra of type A_n over the complex number field $\mathbb{C}$. We let $\overline{U} = U_q(\mathfrak{g}, \Lambda)$ and $U = U_q(\mathfrak{g}, Q)$ be the quantum groups defined over the weight lattice and over the root lattice respectively (see [2] and [5]). By the quantum analogue of the Harish-Chandra Theorem, the center of $\overline{U}$ is a polynomial algebra. In [3], a generator set of the center of $\overline{U}$ is given for a generic q (referred to [1]). Unfortunately, these papers do not contain complete proofs.

The situation turns more complicated when one considers the center of U with q being generic. The center subalgebra $Z(U)$ of U is not a polynomial algebra except $n = 1$. In [7], by using the quantized Harish-Chandra Theorem, we proved that the center of U is a finitely generated algebra. In the special case where $n = 2$, the center of U is isomorphic to the algebra generated by x, y, z subject to the relation $xy = z^3$(also see [6]). However, the generators of $Z(U)$ in U are still unknown in general.

Let $\overline{U}_A \subset \overline{U}$ be the Lusztig A-form of $\overline{U}$, where $A = \mathbb{Z}[q, q^{-1}]$. Then $\mathbb{C} \otimes_{\mathbb{Z}} \lim_{q \to 1} \overline{U}_A$ is isomorphic to the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Obviously, the central elements of $\overline{U}_A$ correspond to the central elements of $U(\mathfrak{g})$. Up to a scalar, the Casimir element of $U(\mathfrak{g})$ means the quadratic central element $\sum_i x_i y_i \in U(\mathfrak{g})$, where $\{x_i | 1 \leq i \leq \dim\mathfrak{g}\}$ is a basis of $\mathfrak{g}$ and $\{y_i | 1 \leq i \leq \dim\mathfrak{g}\}$ is the dual dual basis. As far as we know, the quantized Casimir element, the analogue of the Casimir element of $U(\mathfrak{g})$ has not been given.

In this paper, we find two algebraically independent central elements in $\overline{U}$ for $n \geq 2$ and give a quantum analogue of the Casimir element in $\overline{U}$ corresponding to the Casimir element of $U(\mathfrak{g})$. For the type A_2, we give explicitly the generators of the centers $Z(\overline{U})$ and $Z(U)$ respectively.

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1.2. **Main results.** Let $E_i, F_i, K_{\pm\lambda_i}$ be the commonly-used generators of $\overline{U}$ corresponding to the cartan matrix $(a_{i,j} = 2\delta_{i,j} - \delta_{|i-j|,1})$. For $1 \le i \le j \le n$, set

$$\begin{aligned} F_{i,j} &= [\cdots[F_i, F_{i+1}]_q, \cdots, F_j]_q, \\ E_{i,j} &= [\cdots[E_i, E_{i+1}]_{q^{-1}}, \cdots, E_j]_{q^{-1}}, \\ K_{i,j} &= K_{-\lambda_{i-1}+\lambda_i-\lambda_j+\lambda_{j+1}}. \end{aligned}$$

In particular, $F_{i,i} = F_i, E_{i,i} = E_i$ and $K_{i,i} = K_{-\lambda_{i-1}+\lambda_{i+1}}$.

Let σ be the diagram automorphism of $\overline{U}$. Define

$$\begin{aligned} C_1 &= \sum_{i=1}^{n+1} q^{n-2(i-1)} K_{2\lambda_i - 2\lambda_{i-1}} + (q-q^{-1})^2 \sum_{1\le i\le j\le n} (-1)^{j-i} q^{n+1-i-j} F_{i,j} E_{i,j} K_{i,j}, \\ C_n &= \sigma(C_1). \end{aligned}$$

Note that $n \ge 2$, the diagram automorphism σ of $\overline{U}$ is nontrivial and $C_1 \ne C_n$. These two elements also appeared in [3](also see [1]), where they were defined independently.

In the following we always assume that $\mathfrak{g}$ is of type $A_n (n \ge 2)$ and q is generic. Our main results are as follows.

Theorem 1.1. *The two elements C_1 and $C_n = \sigma(C_1)$ are central in $\overline{U}$. In particular, they are algebraically independent.*

Theorem 1.2. *Let $\overline{U}_A$ be the A-form of $\overline{U}$ and $\mathfrak{cas} = \frac{1}{4(q-1)^2}(C_1 + C_n - 2n - 2) - \frac{n(n+1)(n+2)}{12}$. Then $\mathfrak{cas} \in \overline{U}_A$ and $\lim_{q\to 1} \mathfrak{cas}$ is the Casimir element of $U(\mathfrak{g})$.*

We call $\mathfrak{cas}$ the quantum Casimir element of $\overline{U}$.

Theorem 1.3. *Let $\mathfrak{g}$ be of type A_2. Then*

(i) *the center $Z(\overline{U})$ of $\overline{U}$ is the polynomial algebra in two variables C_1, C_2;*
(ii) *the center $Z(U)$ of U is the subalgebra generated by three elements C_1^3, C_2^3, C_1C_2.*

2. Basics

2.1. **Lie algebra and its invariant bilinear form.** The complex simple Lie algebra $\mathfrak{g}$ of type A_n is generated by elements $e_i, f_i, h_i (1 \le i \le n)$ subject to the relations:

$$\begin{aligned} &[e_i, f_j] = \delta_{i,j} h_i, [h_i, e_j] = a_{i,j} e_j, [h_i, f_j] = -a_{i,j} f_j, \\ &[e_i, [e_i, e_j]] = 0, [f_i, [f_i, f_j]] = 0, |i-j| = 1, \\ &[e_i, e_j] = 0, [f_i, f_j] = 0, |i-j| > 1, \end{aligned}$$

where $(a_{i,j} = 2\delta_{i,j} - \delta_{|i-j|,1})$ is the Cartan matrix (see [4]).

There exists a unique invariant symmetric bilinear form on $\mathfrak{g}$ determined by

$$(e_i, f_j) = \delta_{i,j},$$

which is a nonzero scalar of the Killing form.

The Cartan subalgebra $\mathfrak{h}$ can be identified by its dual $\mathfrak{h}^*$ via

$$\gamma : h_i \mapsto \alpha_i,$$

satisfying $\alpha(h) = (\gamma^{-1}(\alpha), h)$. Consequently, there exists a unique bilinear form on $\mathfrak{h}^*$ such that $(\lambda, \mu) = \lambda(\gamma^{-1}(\mu)), \forall \lambda, \mu \in \mathfrak{h}^*$.

Let $\{x_i | 1 \leq i \leq \dim\mathfrak{g}\}$ be an arbitrary basis of $\mathfrak{g}$, and let $\{y_i | 1 \leq i \leq \dim\mathfrak{g}\}$ be the dual basis associated to $(,)$. It is well known that

$$\sum_{i=1}^{\dim\mathfrak{g}} x_i y_i$$

is the Casimir elements of $\mathfrak{g}$, independent of the choice of x_i's.

For example, $\mathfrak{g}$ has a Chevalley basis $\{x_\alpha, h_i | \alpha \in \Phi, 1 \leq i \leq n\}$ such that

$$\begin{aligned} &x_{\alpha_i} = e_i, x_{-\alpha_i} = f_i, [x_\alpha, x_{-\alpha}] = \gamma^{-1}(\alpha), \\ &[x_\alpha, x_\beta] = N_{\alpha,\beta} x_{\alpha+\beta}, \text{ if } \alpha + \beta \neq 0, \end{aligned}$$

where $N_{\alpha,\beta} \in \{0, \pm 1\}$ and Φ is the root system of $\mathfrak{g}$. The dual basis is given as follows:

$$\{x_{-\alpha}, \gamma^{-1}(\lambda_i) | \alpha \in \Phi, 1 \leq i \leq n\} = \{x_\alpha, \gamma^{-1}(\lambda_i) | \alpha \in \Phi, 1 \leq i \leq n\}.$$

As usual, let $\Lambda = \sum_{i=1}^n \mathbb{Z}\lambda_i$ and $Q = \sum_{i=1}^n \mathbb{Z}\alpha_i$ respectively denote the weight lattice and the root lattice, where λ_i and α_i stand for the fundamental weight and the simple root associated to index i. For convenience, we let $\lambda_0 = \lambda_{n+1} = 0$. Thus, we have $\alpha_i = -\lambda_{i-1} + 2\lambda_i - \lambda_{i+1}$.

2.2. **Quantum group.** The simply-connected type quantum group $\overline{U} = U_q(\mathfrak{g}, \Lambda)$ is a q-analogue of the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. As an associative algebra over $\mathbb{C}(q)$, $\overline{U}$ is generated by the elements $E_i, F_i (1 \leq i \leq n)$ and $K_\lambda (\lambda \in \Lambda)$ subject to the relations:

$$\begin{aligned} &K_0 = 1, K_\lambda K_\mu = K_{\lambda+\mu}, K_\lambda e_i K_{-\lambda} = q^{(\lambda,\alpha_i)}, K_\lambda f_i K_{-\lambda} = q^{-(\lambda,\alpha_i)} f_i \\ &[E_i, F_j] = \delta_{i,j} \frac{K_{\alpha_i} - K_{-\alpha_i}}{q - q^{-1}}, \\ &[E_i, E_j] = 0, [F_i, F_j] = 0, |i - j| > 1, \end{aligned}$$

and the q-Serre relations:

$$[E_i, [E_i, E_j]_{q^{-1}}]_q = 0, [F_i, [F_i, F_j]_{q^{-1}}]_q = 0, |i - j| = 1,$$

where $[a, b]_v = ab - vba$, for all $a, b \in \overline{U}$ and $v \in \mathbb{C}(q)$.

We arrange the sets $\{F_{i,j} | 1 \leq i \leq j \leq n\}$ and $\{E_{i,j} | 1 \leq i \leq j \leq n\}$ in numerical order so that we have:

$$\begin{aligned} \{F_{i,j} | 1 \leq i \leq j \leq n\} &= \{\mathfrak{F}_i | 1 \leq i \leq n(n+1)/2\}, \\ \{E_{i,j} | 1 \leq i \leq j \leq n\} &= \{\mathfrak{E}_i | 1 \leq i \leq n(n+1)/2\}. \end{aligned}$$

In this way, $\overline{U}$ has a PBW type basis (one is referred to [5], see the Theorem in 8.24 for the PBW type basis of U):

$$\big\{\mathfrak{F}_1^{i_1} \cdots \mathfrak{F}_{n(n+1)/2}^{i_{n(n+1)/2}} K_\lambda \mathfrak{E}_1^{j_1} \cdots \mathfrak{E}_{n(n+1)/2}^{j_{n(n+1)/2}} | i_k, j_k \in \mathbb{N}, \lambda \in \Lambda\big\}.$$

The quantum group $U = U_q(\mathfrak{g}, Q)$ is the subalgebra of $\overline{U}$ generated by elements $E_i, F_i (1 \leq i \leq n)$ and $K_\alpha (\alpha \in Q)$, this is the quantized enveloping algebra in the Jantzen's sense.

The diagram automorphism σ of $\overline{U}$ is defined via

$$\sigma(E_i) = E_{n+1-i}, \sigma(F_i) = F_{n+1-i}, \sigma(K_{\lambda_i}) = K_{\lambda_{n+1-i}}.$$

Note that $\alpha_i = -\lambda_{i-1} + 2\lambda_i - \lambda_{i+1}$. We have $\sigma(K_{\alpha_i}) = K_{\alpha_{n+1-i}}$. The restriction $\sigma|_U$ of σ on U is also an automorphism.

2.3. **Lusztig $\mathbb{Z}[q,q^{-1}]$-form.** Let $A = \mathbb{Z}[q,q^{-1}]$ be the Laurent polynomial ring in variable q. The Lusztig A-form of U is an A-algebra U_A generated by the elements:

$$E_i^{(N)} = E_i^N/[N]_q!,\ F_i^{(N)} = F_i^N/[N]_q!,\ 1 \le i \le n, N \ge 1.$$

Since U_A is an A-algebra and $[E_i, F_j] = \delta_{i,j}\frac{K_{\alpha_i} - K_{-\alpha_i}}{q-q^{-1}}$, the limit of U_A as $q \to 1$ can be well defined in the sense of $K_\alpha = \exp(\hbar\gamma^{-1}(\alpha))$, where $\hbar = \log q$. Then

$$\lim_{q\to 1}\frac{K_{\alpha_i} - K_{-\alpha_i}}{q-q^{-1}} = \lim_{\hbar\to 0}\frac{K_{\alpha_i} - K_{-\alpha_i}}{q-q^{-1}} = h_i.$$

Moreover, we have the following identification:

$$\mathbb{C}\otimes_{\mathbb{Z}}\lim_{q\to 1} U_A \cong U(\mathfrak{g}).$$

We let $\overline{U}_A$ be the A-algebra generated by the elements:

$$E_i^{(N)} = E_i^N/[N]_q!,\ F_i^{(N)} = F_i^N/[N]_q!,\ K_\lambda,\ \frac{K_\lambda - K_{-\lambda}}{q-q^{-1}},\ 1 \le i \le n, N \ge 1, \lambda \in \Lambda.$$

The limit of $\overline{U}_A$ as $q \to 1$ can be defined in a similar way. In particular, $\lim\limits_{q\to 1} K_\lambda = 1$ and $U(\mathfrak{g})$ is also identified with $\mathbb{C}\otimes_{\mathbb{Z}}\lim\limits_{q\to 1}\overline{U}_A$. In particular, with this identification, $\lim\limits_{q\to 1} F_{i,j}$ and $\lim\limits_{q\to 1} E_{i,j}$ correspond respectively to the root vectors $x_{-\alpha}$ and x_α with roots $\pm\alpha = \pm(\alpha_i + \cdots + \alpha_j)$. It follows that

$$(\lim_{q\to 1} F_{i,j}, \lim_{q\to 1} E_{i,j}) = (-1)^{j-i}.$$

2.4. **Quantized Harish-Chandra isomorphism.** The algebra $\overline{U}$ is Λ-graded with homogeneous spaces

$$\overline{U}_\nu \quad = \quad \{u | K^\mu u K^{-\mu} = q^{(\mu,\nu)}\}.$$

Let $\overline{U}^0$ be the subalgebra generated by $K^\mu (\mu \in \Lambda)$. Identify $\overline{U}$ as the triangular decomposition $\overline{U}^- \otimes \overline{U}^0 \otimes \overline{U}^+$. Then $\overline{U}_0$ has a decomposition

$$\overline{U}_0 = \overline{U}^0 \oplus \bigoplus_{\nu > 0}\overline{U}^-_{-\nu}\overline{U}^0\overline{U}^+_\nu.$$

Let $\pi : \overline{U}_0 \to \overline{U}^0$ be the projection with respect to this decomposition. Then π is an algebra homomorphism.

Let $\Gamma : \overline{U}^0 \to \overline{U}^0$ be an algebra automorphism defined by

$$\Gamma(K_{\lambda_i}) \quad = \quad q^{-(n+1-i)i/2}K_{\lambda_i}.$$

Let W be the Weyl group and $(\overline{U}^0)_{ev}$ be the subalgebra generated by $K_\lambda (\lambda \in 2\Lambda)$. Then $\Gamma \circ \pi$ is the quantized Harish-Chandra isomorphism from the center $Z(\overline{U})$ of $\overline{U}$ to the algebra $(\overline{U}^0_{ev})^W$ of W-invariants in $\overline{U}^0_{ev}$. Moreover, it is also an isomorphism from the center $Z(U)$ of U to $(U^0_{ev})^W := U \cap (\overline{U}^0_{ev})^W$. The algebra $(\overline{U}^0_{ev})^W$ is obviously generated by the elements

$$\sum_{\omega \in W} K_{\omega(2\lambda_i)},\ i = 1, \cdots, n.$$

In particular, when $n=2$, the invariant subalgebra $(\overline{U}^0_{ev})^W$ can be generated by two elements:

$$\begin{aligned} Z_1 &= K_{2\lambda_1} + K_{2\lambda_2-2\lambda_1} + K_{-2\lambda_2}, \\ Z_2 &= K_{-2\lambda_1} + K_{2\lambda_1-2\lambda_2} + K_{2\lambda_2}. \end{aligned}$$

and $(U^0_{ev})^W$ can be generated by three elements (see [6] and [7])

$$\begin{aligned} Z_3 &= K_{6\lambda_1} + K_{6\lambda_2-6\lambda_1} + K_{-6\lambda_2}, \\ Z_4 &= K_{-6\lambda_1} + K_{6\lambda_1-6\lambda_2} + K_{6\lambda_2}, \\ Z_5 &= K_{2\lambda_1+2\lambda_2} + K_{-2\lambda_1+4\lambda_4} + K_{4\lambda_1-2\lambda_2} + K_{2\lambda_1-4\lambda_2} + K_{-4\lambda_1+2\lambda_2} + K_{-2\lambda_1-2\lambda_2}. \end{aligned}$$

2.5. **Some useful lemmas.**

Lemma 2.1. *The following equations hold for* $1 \le i \le n$:

$$[E_i, [E_i, E_{i\pm 1}]_{q^{\pm 1}}]_{q^{\mp 1}} = [F_i, [F_i, F_{i\pm 1}]_{q^{\pm 1}}]_{q^{\mp 1}} = 0.$$

Proof. They are the q-Serre relations. □

Lemma 2.2. *The following hold for* $1 \le i \le n$:

$$[E_i, [E_{i-1}, [E_i, E_{i+1}]_{q^{\pm 1}}]_{q^{\pm 1}}] = [F_i, [F_{i-1}, [F_i, F_{i+1}]_{q^{\pm 1}}]_{q^{\pm 1}}] = 0.$$

Proof. We only check $[E_i, [E_{i-1}, [E_i, E_{i+1}]_q]_q] = 0$, the proof for other cases is similar. In fact,

$$\begin{aligned} & [E_i, [E_{i-1}, [E_i, E_{i+1}]_q]_q] \\ = \;& E_iE_{i-1}E_iE_{i+1} - qE_iE_{i-1}E_{i+1}E_i - qE_iE_iE_{i+1}E_{i-1} + q^2E_iE_{i+1}E_iE_{i-1} \\ & -E_{i-1}E_iE_{i+1}E_i + qE_{i-1}E_{i+1}E_iE_i + qE_iE_{i+1}E_{i-1}E_i - q^2E_{i+1}E_iE_{i-1}E_i \\ = \;& E_iE_{i-1}E_iE_{i+1} - qE_iE_iE_{i+1}E_{i-1} + q^2E_iE_{i+1}E_iE_{i-1} \\ & -E_{i-1}E_iE_{i+1}E_i + qE_{i-1}E_{i+1}E_iE_i - q^2E_{i+1}E_iE_{i-1}E_i \\ = \;& \frac{1}{q+q^{-1}}(E_iE_iE_{i-1}E_{i+1} + E_{i-1}E_iE_iE_{i+1}) - qE_iE_iE_{i+1}E_{i-1} \\ & + \frac{q^2}{q+q^{-1}}(E_iE_iE_{i+1}E_{i-1} + E_{i+1}E_iE_iE_{i-1}) \\ & - \frac{1}{q+q^{-1}}(E_{i-1}E_iE_iE_{i+1} + E_{i-1}E_{i+1}E_iE_i) + qE_{i-1}E_{i+1}E_iE_i \\ & - \frac{q^2}{q+q^{-1}}(E_{i+1}E_iE_iE_{i-1} + E_{i+1}E_{i-1}E_iE_i) \\ = \;& \Big(\frac{1}{q+q^{-1}} - q + \frac{q^2}{q+q^{-1}}\Big)(E_iE_iE_{i-1}E_{i+1} + E_{i-1}E_{i+1}E_iE_i) = 0. \end{aligned}$$

□

Lemma 2.3. *The following equations hold for* $i \ne j$*:*

$$\begin{aligned} [E_i, E_{i,j}]_q = 0, &\qquad [E_{i-1}, E_{i,j}]_{q^{-1}} = E_{i-1,j}, \\ [E_j, E_{i,j}]_{q^{-1}} = 0, &\qquad [E_{j+1}, E_{i,j}]_q = -qE_{i,j+1}. \end{aligned}$$

Moreover, if $k \ne i-1, i, j, j+1$*, then*

$$[E_k, E_{i,j}] = 0.$$

Proof. Follow from Lemma 2.2, the q-Serre relations and the definition of $E_{i,j}$. □

Lemma 2.4. *If $i \neq j$, then for any k, we have*

$$[E_k, F_{i,j}] = \delta_{i,k} F_{i+1,j} K_{-\alpha_i} - q\delta_{j,k} E_{i,j-1} K_{\alpha_j}.$$

Proof. If $k \neq i, j$, it is clear that $[E_k, F_{i,j}] = 0$. If $k = i$, then

$$\begin{aligned}
[E_k, F_{i,j}] &= [\cdots[[E_i, F_i], F_{i+1}]_q, \cdots, F_j]_q \\
&= \frac{1}{q-q^{-1}}[\cdots[K_{\alpha_i} - K_{-\alpha_i}, F_{i+1}]_q, \cdots, F_j]_q \\
&= [\cdots[F_{i+1}, F_{i+2}]_q, \cdots, F_j]_q K_{-\alpha_i} = F_{i+1,j} K_{-\alpha_i}.
\end{aligned}$$

If $k = j$, then

$$\begin{aligned}
[E_k, F_{i,j}] &= [\cdots[F_i, F_{i+1}]_q, \cdots, [E_j, F_j]]_q \\
&= \frac{1}{q-q^{-1}}[[\cdots[F_i, F_{i+1}]_q, \cdots, F_{j-1}]_q, K_{\alpha_j} - K_{-\alpha_j}]_q \\
&= -q[\cdots[F_{i+1}, F_{i+2}]_q, \cdots, F_{j-1}]_q K_{\alpha_j} = -qF_{i,j-1} K_{\alpha_j}.
\end{aligned}$$

□

Lemma 2.5. *If $k \neq i-1, i, j, j+1$, then*

$$[E_k, F_{i,j} E_{i,j} K_{i,j}] = 0.$$

Proof. If $k \neq i-1, i, j, j+1$, then $[E_k, K_{i,j}] = 0$. The rest follows from Lemmas 2.3 and Lemma 2.4. □

Lemma 2.6. *The group-like elements $K_{\lambda_i}(1 \leq i \leq n)$are algebraically independent.*

Proof. We only prove for $n = 2$. The proof for general n is similar.

We assume that

$$\zeta := \sum_{i,j} c_{i,j} K_{\lambda_1}^i K_{\lambda_2}^j = 0,$$

for finitely many nonzero $c_{i,j} \in \mathbb{C}(q)$.

Let V be a weight module with a weight vector v corresponding to the weight $\lambda = k\alpha_1 + l\alpha_2$. Then

$$\zeta \cdot v = \Big(\sum_{i,j} c_{i,j} q^{ik+jl}\Big) v = 0,$$

and hence

$$\sum_{i,j} c_{i,j} q^{ik+jl} = 0.$$

Let $i_0 = \max\{i | c_{i,j} \neq 0\}$, $j_0 = \max\{j | c_{i_0,j} \neq 0\}$ and $k' = j_0 + 1, l' = 1$. Then the integers $ik' + jl'$ such taht $c_{i,j} \neq 0$ are mutually different. Let $\{\eta_1, \cdots, \eta_N\}$ be an arrangement of such integers. So the matrix $(a_{r,s} = q^{(s-1)\eta_r})$ is a vandermonde matrix, which is invertible when q is generic.

Consider $k = rk', l = rl'$ for $r = 1, 2, \cdots$. Then $\sum_{i,j} c_{i,j} q^{ik+jl} = 0$ implies that all $c_{i,j}$ are zeros. Thus the lemma holds. □

3. Proof for main results

3.1. **Proof of Theorem 1.1.** By definition, we have

$$\begin{aligned}
[E_1, C_1] &= [E_1, q^n K_{2\lambda_1} + q^{n-2} K_{2\lambda_2-2\lambda_1} + (q-q^{-1})^2 \sum_{1\le i\le j\le 2} (-1)^{j-i} q^{n+1-i-j} F_{i,j}E_{i,j}K_{i,j}] \\
&= q^{n-1}K_{\lambda_2}[E_1, qK_{\alpha_1} + q^{-1}K_{-\alpha_1} + (q-q^{-1})^2 F_1E_1] \\
&\quad +(q-q^{-1})^2 \sum_{j\ge 2} [E_1, q^{n-1-j}F_{2,j}E_{2,j}K_{2,j} - q^{n-j}F_{1,j}E_{1,j}K_{1,j}] \\
&= (q-q^{-1})^2 \sum_{j\ge 2} q^{n-1-j}(F_{2,j}E_{1,j}K_{2,j} - q[E_1, F_{1,j}]E_{1,j}K_{1,j}) \\
&= (q-q^{-1})^2 \sum_{j\ge 2} q^{n-1-j}(F_{2,j}E_{1,j}K_{2,j} - qF_{2,j}K_{-\alpha_1}E_{1,j}K_{1,j}) = 0.
\end{aligned}$$

The proof for $[E_n, C_1] = 0$ is similar.

For $1 < i < n$, we compute

$$\begin{aligned}
[E_i, C_1] &= q^{n+1-2i}K_{\lambda_{i-1}+\lambda_{i+1}}[E_i, qK_{\alpha_i} + q^{-1}K_{-\alpha_i} + (q-q^{-1})^2 F_iE_i] \\
&\quad +(q-q^{-1})^2 \sum_{j\ge i+1} q^{n-i-j}(-1)^{j-i-1}[E_i, F_{i+1,j}E_{i+1,j}K_{i+1,j} - qF_{i,j}E_{i,j}K_{i,j}] \\
&\quad +(q-q^{-1})^2 \sum_{j\le i-1} q^{n+1-i-j}(-1)^{j-i+1}[E_i, qF_{j,i-1}E_{j,i-1}K_{j,i-1} - F_{j,i}E_{j,i}K_{j,i}] \\
&= (q-q^{-1})^2 \sum_{j\ge i+1} q^{n-i-j}(-1)^{j-i-1}(F_{i+1,j}E_{i,j}K_{i+1,j} - qF_{i+1,j}K_{-\alpha_i}E_{i,j}K_{i,j}) \\
&\quad +(q-q^{-1})^2 \sum_{j\le i-1} q^{n+1-i-j}(-1)^{j-i+1}(-q^2F_{j,i-1}E_{j,i}K_{j,i-1} + qF_{j,i-1}K_{\alpha_i}E_{j,i}K_{j,i}) = 0.
\end{aligned}$$

So far we have proved $[E_i, C_1] = 0$ for all i. In a similar way, we obtain $[F_i, C_1] = 0$ for all i. Note that $C_1 \in \overline{U}_0$. So C_1 is a central element. By definition, C_n is also a central element.

Now we consider $\Gamma \circ \pi(C_i)$. We have

$$\Gamma \circ \pi(C_1) = \sum_{i=1}^{n+1} K_{2\lambda_i - 2\lambda_{i-1}}, \qquad \Gamma \circ \pi(C_n) = \sum_{i=1}^{n+1} K_{-2\lambda_i+2\lambda_{i-1}}.$$

Thus, for all $i, j \in \mathbb{Z}_+$, we have

$$(\Gamma \circ \pi(C_1))^i (\Gamma \circ \pi(C_n))^j = K_{2i\lambda_1+2j\lambda_n} + \text{other terms involving } \lambda_k.$$

By Lemma 2.6, $K_{\lambda_i}, 1 \le i \le n$, are algebraically independent for $n \ge 2$. So $\Gamma \circ \pi(C_1)$ and $\Gamma \circ \pi(C_n)$ are algebraically independent. It follows that C_1 and C_n are algebraically independent. □

3.2. **Proof of Theorem 1.2.** By definition, we have

$$\begin{aligned}
\mathfrak{cas} &= \frac{(q^{-1}+1)^2}{4} \sum_{i=1}^{n+1} q^{-n+2(i-1)} \Big(\frac{q^{n-2(i-1)}K_{\lambda_i - \lambda_{i-1}} - K_{-\lambda_i+\lambda_{i-1}}}{q-q^{-1}} \Big)^2 - \frac{n(n+1)(n+2)}{12} \\
&\quad + \sum_{1\le i\le j\le n} (-1)^{j-i} q^{n+1-i-j} F_{i,j}E_{i,j}K_{i,j} + \sum_{1\le i\le j\le n} (-1)^{j-i} q^{n+1-i-j} \sigma(F_{i,j}E_{i,j}K_{i,j}).
\end{aligned}$$

Since

$$\begin{aligned}&(q^{-1}+1)\Big(\frac{q^{n-2(i-1)}K_{\lambda_i-\lambda_{i-1}}-K_{-\lambda_i+\lambda_{i-1}}}{q-q^{-1}}\Big)\\ =\ &\frac{q^{n-2(i-1)}-1}{q-1}K_{\lambda_i-\lambda_{i-1}}+(q^{-1}+1)\Big(\frac{q^{n-2(i-1)}K_{\lambda_i-\lambda_{i-1}}-K_{-\lambda_i+\lambda_{i-1}}}{q-1}\Big),\end{aligned}$$

It is obvious that $\mathfrak{cas}$ belongs to the $\mathbb{Z}[q,q^{-1}]$-subalgebra of $\overline{U}_A$ generated by the elements

$$\frac{K_{\lambda_i-\lambda_{i-1}}-K_{-\lambda_i+\lambda_{i-1}}}{q-q^{-1}}, F_k, E_k, K_{k,j}, 1\le i\le n+1, 1\le k\le j\le n.$$

Thus, $\mathfrak{cas}\in\overline{U}_A$. Identifying $\lim\limits_{q\to1}\mathbb{C}\otimes_{\mathbb{Z}}\overline{U}_A$ with $U(\mathfrak{g})$, we see that $\lim\limits_{q\to1}\mathfrak{cas}$ is a central element. Moreover, we have

$$\begin{aligned}&\lim_{q\to1}\frac{q^{n-2(i-1)}-1}{q-1}K_{\lambda_i-\lambda_{i-1}}+(q^{-1}+1)\Big(\frac{q^{n-2(i-1)}K_{\lambda_i-\lambda_{i-1}}-K_{-\lambda_i+\lambda_{i-1}}}{q-1}\Big)\\ =\ &n-2(i-1)+2\gamma^{-1}(\lambda_i-\lambda_{i-1}),\end{aligned}$$

and

$$\begin{aligned}&\lim_{q\to1}\sum_{1\le i\le j\le n}(-1)^{j-i}q^{n+1-i-j}F_{i,j}E_{i,j}K_{i,j}+\lim_{q\to1}\sum_{1\le i\le j\le n}(-1)^{j-i}q^{n+1-i-j}\sigma(F_{i,j}E_{i,j}K_{i,j})\\ =\ &2\sum_{\alpha>0}x_{-\alpha}x_\alpha=-2\gamma^{-1}(\rho)+\sum_{\alpha\in\Phi}x_{-\alpha}x_\alpha,\end{aligned}$$

where the x_α are root vectors such that $(x_\alpha,x_\beta)=\delta_{\alpha+\beta,0}$, and ρ is the half sum of all positive roots.

It follows that

$$\lim_{q\to1}\mathfrak{cas}\ =\ 2\gamma^{-1}(\rho)+\sum_{\alpha\in\Phi}x_{-\alpha}x_\alpha+\sum_{i=1}^{n+1}\Big(\gamma^{-1}(\lambda_i-\lambda_{i-1})+\frac{n}{2}-(i-1)\Big)^2-\frac{n(n+1)(n+2)}{12}$$

is a quadratic central element. Now the identity

$$\sum_{i=1}^{n+1}\Big(\frac{n}{2}-(i-1)\Big)^2=\frac{n(n+1)(n+2)}{12},$$

and the fact that $U(\mathfrak{h})$ contains no central elements except scalars, imply that $\lim_{q\to1}\mathfrak{cas}$ belongs to $\sum_{\alpha\in\Phi}x_{-\alpha}x_\alpha+U(\mathfrak{h})$.

This forces

$$\lim_{q\to1}\mathfrak{cas}=\sum_{\alpha\in\Phi}x_\alpha x_{-\alpha}+\sum_{i=1}^{n}h_i\gamma^{-1}(\lambda_i)=\sum_{i=1}^{\dim\mathfrak{g}}x_iy_i.$$

□

3.3. Proof of Theorem 1.3.

Proof. Note that the algebra $(\overline{U}^0_{ev})^W$ can be generated by two elements:

$$Z_1=K_{2\lambda_1}+K_{2\lambda_2-2\lambda_1}+K_{-2\lambda_2},\quad Z_2=K_{-2\lambda_1}+K_{2\lambda_1-2\lambda_2}+K_{2\lambda_2}.$$

Since

$$\begin{aligned}\Gamma\circ\pi(C_1)&=K_{2\lambda_1}+K_{2\lambda_2-2\lambda_1}+K_{-2\lambda_2},\\ \Gamma\circ\pi(C_2)&=K_{-2\lambda_1}+K_{2\lambda_1-2\lambda_2}+K_{2\lambda_2},\end{aligned}$$

it follows from the Harish-Chandra isomorphism that the center $Z(\overline{U})$ can be generated by C_1 and C_2.

Note that $3\lambda_1 = 2\alpha_1 - \alpha_2, 3\lambda_2 = 2\alpha_2 - \alpha_1$ and $\lambda_1 + \lambda_2 = \alpha_1 + \alpha_2$. Thus,

$$\begin{aligned} C_1 &= K_{2\lambda_1} + K_{2\lambda_2 - 2\lambda_1} + K_{-2\lambda_2} + (q-q^{-1})^2(qF_1E_1K_{\lambda_2} + q^{-1}F_2E_2K_{-\lambda_1} - F_{1,2}E_{1,2}K_{\lambda_1-\lambda_2}) \\ &= K_{\lambda_2}\Big(K_{\alpha_1} + K_{-\alpha_1} + K_{\alpha_1-2\alpha_2} + (q-q^{-1})^2(qF_1E_1 + q^{-1}F_2E_2K_{-\alpha_1-\alpha_2} - F_{1,2}E_{1,2}K_{-\alpha_2})\Big). \end{aligned}$$

It follows that $C_1 \in K_{\lambda_2}U$ and $C_2 = \sigma(C_1) \in K_{\lambda_1}U$. Therefore, we obtain that $C_1^3, C_2^3, C_1C_2 \in U$. Hence $C_1^3, C_2^3, C_1C_2 \in Z(U)$.

The following calculations:

$$\begin{aligned} \Gamma\circ\pi(C_1^3) &= (\Gamma\circ\pi(C_1))^3 = Z_1^3 = Z_3 + 3Z_5 + 6, \\ \Gamma\circ\pi(C_2^3) &= (\Gamma\circ\pi(C_2))^3 = Z_2^3 = Z_4 + 3Z_5 + 6, \\ \Gamma\circ\pi(C_1C_2) &= \Gamma\circ\pi(C_1)\Gamma\circ\pi(C_2) = Z_1Z_2 = Z_5 + 3, \end{aligned}$$

and the fact that $(U_{ev}^0)^W$ can be generated by Z_3, Z_4, Z_5, together with the quantum Harish-CHandra isomorphism, imply that $Z(U)$ can be generated by C_1^3, C_2^3 and C_1C_2. □

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