IDEAL CLASSES OF THE WEYL ALGEBRA AND NONCOMMUTATIVE PROJECTIVE GEOMETRY (WITH AN APPENDIX BY MICHEL VAN DEN BERGH)

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Abstract. Let $\mathfrak{R}$ be the set of isomorphism classes of right ideals in the Weyl algebra $A = A_1(\mathbb{C})$, and let $\mathfrak{C}$ be the set of isomorphism classes of triples $(V; \mathbb{X}, \mathbb{Y})$, where V is a finite-dimensional (complex) vector space, and $\mathbb{X}$, $\mathbb{Y}$ are endomorphisms of V such that $[\mathbb{X}, \mathbb{Y}] + \mathbb{I}$ has rank 1. Following a suggestion of L. Le Bruyn, we define a map $\theta : \mathfrak{R} \to \mathfrak{C}$ by appropriately extending an ideal of A to a sheaf over a quantum projective plane, and then using standard methods of homological algebra. We prove that θ is inverse to a bijection $\omega : \mathfrak{C} \to \mathfrak{R}$ constructed in [BW] by a completely different method. The main step in the proof is to show that θ is equivariant with respect to natural actions of the group $G = \mathrm{Aut}(A)$ on $\mathfrak{R}$ and $\mathfrak{C}$: for that we have to study also the extensions of an ideal to certain *weighted* quantum projective planes. Along the way, we find an elementary description of θ.

1. Introduction

This is a sequel to our earlier paper [BW]; however, it can, and probably should, be read independently of that work. We first recall the main results of [BW].

Let A be the complex Weyl algebra; that is, A is the associative algebra over $\mathbb{C}$ generated by two elements x and y subject to the relation $[x, y] = 1$. Let $\mathfrak{R}$ be the set of isomorphism classes of finitely generated torsion-free rank one right A-modules. Since A is Noetherian, each right ideal of A is a module of this kind; conversely, because A has a quotient (skew) field, it is easy to see that each such module M is isomorphic to a right ideal of A. For short, we shall often refer to a module M as an "ideal", even when we do not have in mind any particular embedding of M in A. Let G be the automorphism group of A; there is a natural action of G on $\mathfrak{R}$. Finally, for each $n \geq 0$, let $\mathfrak{C}_n$ be the space of equivalence classes (modulo simultaneous conjugation) of pairs $(\mathbb{X}, \mathbb{Y})$ of $n \times n$ matrices over $\mathbb{C}$ such that[1]

$$[\mathbb{X}, \mathbb{Y}] + \mathbb{I} \quad \text{has rank } 1 . \tag{1.1}$$

For brevity, we shall often refer to a point of $\mathfrak{C}_n$ simply as a "pair of matrices". There is a natural action of G on each space $\mathfrak{C}_n$; it is obtained, roughly speaking, by thinking of the pairs $(\mathbb{X}, \mathbb{Y})$ as points dual to the coordinate functions x and y that generate A (see Section 7 below for the precise definition). In [BW] we showed that this action is transitive. Let $\mathfrak{C}$ be the (disjoint) union of the spaces $\mathfrak{C}_n$. The main result of [BW] was the following

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[1] In [BW] we worked with the space of pairs such that $[\mathbb{X}, \mathbb{Y}] - \mathbb{I}$ has rank 1; here we identify these spaces via the map $(\mathbb{X}, \mathbb{Y}) \leftrightarrow (\mathbb{X}^t, \mathbb{Y}^t)$. When $n = 0$, the space $\mathfrak{C}_n$ is supposed to be a point: as in (1.1), we shall sometimes disregard this case.

Theorem 1.1. *There is a bijective map $\omega : \mathfrak{C} \to \mathfrak{R}$ which is equivariant with respect to the action of G.*

Part of the significance of this Theorem becomes clear if we think of the Weyl algebra as a noncommutative deformation of the polynomial ring $\mathbb{C}[x, y]$. In this case the analogue of Theorem 1.1 is elementary, because each isomorphism class of ideals has a unique representative as an ideal of finite codimension; thus in the commutative case, the analogue of our space $\mathfrak{R}$ is the disjoint union of the point Hilbert schemes $\mathrm{Hilb}_n(\mathbb{A}^2)$ of the affine plane. As is well known (and almost tautological), $\mathrm{Hilb}_n(\mathbb{A}^2)$ can be identified with the space of (equivalence classes of) pairs $(\mathbb{X}, \mathbb{Y})$ of *commuting* $n \times n$ matrices possessing a cyclic vector: to an ideal I of finite codimension we assign the pair $(\mathbb{X}, \mathbb{Y})$ of maps on the quotient $\mathbb{C}[x, y]/I$ induced by multiplication by x and y. According to Nakajima (see [N]), our space $\mathfrak{C}_n$ can be obtained from $\mathrm{Hilb}_n(\mathbb{A}^2)$ by a deformation of complex structure. In commutative algebraic geometry it is a basic principle that a (flat) deformation of varieties should give rise to a deformation of any reasonable moduli space of bundles (or coherent sheaves); Theorem 1.1 suggests that this principle should extend also to noncommutative deformations. However, at the present time even the expression "moduli space" seems not to have any precise meaning in the noncommutative case; our space $\mathfrak{R}$, for example, does not to our knowledge possess any *intrinsic* algebraic structure, which is why we referred to it above simply as a set.

The description of the map $\omega : \mathfrak{C} \to \mathfrak{R}$ given in [BW] is not direct, but passes through a third space, the *adelic Grassmannian* $\mathrm{Gr}^{\mathrm{ad}}$ that parametrizes rational solutions of a certain integrable system (the KP hierarchy). Indeed, in [BW] we defined ω to be the composition of a bijection $\beta : \mathfrak{C} \to \mathrm{Gr}^{\mathrm{ad}}$ constructed in [W1] and a bijection $\alpha : \mathrm{Gr}^{\mathrm{ad}} \to \mathfrak{R}$ constructed by Cannings and Holland in [CH]. Theorem 1.1 was then proved by following through what happens to the natural action of G on $\mathfrak{R}$ under the bijections α and β, a tricky process, since the action of G on $\mathrm{Gr}^{\mathrm{ad}}$ is difficult to describe. In any case, although $\mathrm{Gr}^{\mathrm{ad}}$ is an interesting object in its own right, it is hard not to feel that it is *de trop* in the question of classifying ideals of A. Another imperfection in [BW] is that there we gave no explicit description of the inverse map to ω (Cannings and Holland do indeed explain what is the inverse of their bijection α, but no description was known for the inverse of the map β). For these (and other) reasons, we wish to rederive Theorem 1.1 in a way that makes no reference to $\mathrm{Gr}^{\mathrm{ad}}$.

To that end, we take up an idea of L. Le Bruyn [L]. If we think of A as the ring of functions on a "quantum affine plane" $\mathbb{A}^2_q$, then an A-module M (in particular, an ideal) is to be thought of as a vector bundle (or coherent sheaf) over $\mathbb{A}^2_q$. Le Bruyn's idea is to extend M to a sheaf $\mathcal{M}$ over the quantum *projective* plane $\mathbb{P}^2_q$, and then use a noncommutative version of Barth's classification [Bar] of bundles over $\mathbb{P}^2$ to obtain algebraic invariants of M. Here $\mathbb{P}^2_q$ is taken in the sense of M. Artin [A]: following Serre's classic paper [S], Artin suggests to define a (noncommutative) projective variety via its homogeneous coordinate ring $\boldsymbol{A}$, so that a sheaf over such a variety is represented by a graded $\boldsymbol{A}$-module (modulo finite-dimensional modules). We give a quick sketch of this theory in Section 2 below: readers who are not familiar with it may understand the rest of this Introduction by analogy with the commutative case. The homogeneous coordinate ring of $\mathbb{P}^2_q$ is the graded ring of "noncommutative polynomials" $\boldsymbol{A} = \mathbb{C}[X, Y, Z]$, where the generators X, Y

and Z all have degree 1, Z commutes with X and Y, and $[X,Y]=Z^2$. As for the classification of bundles over $\mathbb{P}^2_q$, Le Bruyn suggests to use the version of Beilinson [B], which goes very smoothly in the noncommutative case (cf. [Bo]). The following consequence of that theory is all that will concern us in this paper. Suppose we have a sheaf $\mathcal{F}$ over $\mathbb{P}^2_q$ satisfying the vanishing conditions

$$H^i(\mathbb{P}^2_q,\mathcal{F}(-2)) = H^i(\mathbb{P}^2_q,\mathcal{F}(-1)) = H^i(\mathbb{P}^2_q,\mathcal{F}) = 0 \quad \text{for all } i \neq 1 . \tag{1.2}$$

Then $\mathcal{F}$ is determined by the representation

$$H^1(\mathbb{P}^2_q,\mathcal{F}(-2)) \begin{array}{c}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{array} H^1(\mathbb{P}^2_q,\mathcal{F}(-1)) \begin{array}{c}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{array} H^1(\mathbb{P}^2_q,\mathcal{F}) \tag{1.3}$$

of the indicated quiver with 3 vertices and 6 arrows (and relations reflecting the commutation relations of the algebra $\boldsymbol{A}$). In (1.3) it is understood that each set of 3 arrows is given by multiplication by the generators X, Y and Z of $\boldsymbol{A}$. It turns out that any[2] ideal of A has extensions $\mathcal{M}$ that satisfy (1.2); in particular, we shall see that the (unique) extension whose restriction to the line at infinity in $\mathbb{P}^2_q$ is *trivial* always satisfies (1.2). From now on, $\mathcal{M}$ will denote this extension (it is just at this point that we part company from L. Le Bruyn, who chooses a different extension). Then the three vector spaces $H^1(\mathbb{P}^2_q,\mathcal{M}(-2))$, $H^1(\mathbb{P}^2_q,\mathcal{M}(-1))$ and $H^1(\mathbb{P}^2_q,\mathcal{M})$ have dimensions $(n,n,n-1)$ for some $n \geq 1$. Furthermore, multiplication by Z gives a surjection $H^1(\mathbb{P}^2_q,\mathcal{M}(-1)) \twoheadrightarrow H^1(\mathbb{P}^2_q,\mathcal{M})$ and an isomorphism $H^1(\mathbb{P}^2_q,\mathcal{M}(-2)) \xrightarrow{\sim} H^1(\mathbb{P}^2_q,\mathcal{M}(-1))$. If we now identify $H^1(\mathbb{P}^2_q,\mathcal{M}(-2)) \cong H^1(\mathbb{P}^2_q,\mathcal{M}(-1)) \cong V$ (say) via this isomorphism, then X and Y become endomorphisms of V, and it is easy to see that they satisfy the relation (1.1). In this way, Le Bruyn's (modified) construction gives us a map $\theta : \mathfrak{R} \to \mathfrak{C}$. Our aim is to prove

Theorem 1.2. *The map θ is inverse to the map ω defined in* [BW].

In the present paper we shall not attempt a direct proof of this Theorem; indeed, the machinery involved in the definitions of ω and θ is so different that this appears (at first sight) impossible. Instead, we focus on the equivariance property of ω stated in Theorem 1.1. Because the action of G on each space $\mathfrak{C}_n$ is transitive, the map ω^{-1} is uniquely determined by equivariance once we know its effect on one point in each orbit $\omega(\mathfrak{C}_n)$. Now, there is (at least) one point M in each orbit for which it is possible to check directly that $\omega^{-1}(M) = \theta(M)$; granting that, Theorem 1.2 is equivalent to

Theorem 1.3. *The map θ described above is G-equivariant.*

One might think at first that Theorem 1.3 should follow easily from simple considerations of functoriality: however, the difficulty arises that an automorphism of the affine plane does not (in general) extend to a regular automorphism of the projective plane, but only to a birational automorphism. In algebraic terms, this means that an automorphism of the Weyl algebra does not naturally induce any map on the graded ring $\boldsymbol{A}$, which is the only object we have to work with in the noncommutative case. Our idea for dealing with this problem rests on a theorem

[2]We disregard the case of the free A-module of rank 1, which corresponds to $n = 0$ below.

of Dixmier (see [Di]), which states that the group G is generated by the special automorphisms $\Psi_{r,\lambda}$ and $\Phi_{s,\mu}$ defined by

$$\begin{aligned} \Psi_{r,\lambda}(x) = x \ , \quad \Psi_{r,\lambda}(y) = y + \lambda x^r \ ; \\ \Phi_{s,\mu}(x) = x + \mu y^s \ , \quad \Phi_{s,\mu}(y) = y \ . \end{aligned} \tag{1.4}$$

Clearly, it is enough to prove that θ commutes with the action of these generators. We observe that $\Psi_{r,\lambda}$ and $\Phi_{s,\mu}$ will be homogeneous if we assign to x and y the weights $(1,r)$ and $(s,1)$ (respectively); in geometrical language, this means that these automorphisms extend to (biregular) automorphisms of an appropriate *weighted projective space*. Slightly more generally, we shall work with the weighted projective spaces $\mathbb{P}^2_q(\boldsymbol{w})$ for any weight vector $\boldsymbol{w} = (w_1, w_2)$, where w_1 and w_2 are positive integers (it does not simplify what follows to assume that one of them is equal to 1). By definition, the homogeneous coordinate ring of $\mathbb{P}^2_q(\boldsymbol{w})$ is the graded algebra $\boldsymbol{A}(\boldsymbol{w}) = \mathbb{C}[X,Y,Z]$, where (X,Y,Z) have degrees $(w_1, w_2, 1)$, Z commutes with X and Y, and we have the commutation relation $[X,Y] = Z^{|\boldsymbol{w}|}$ (we set $|\boldsymbol{w}| := w_1 + w_2$). It is not difficult to repeat all of Le Bruyn's considerations for any weight $\boldsymbol{w}$, the only difference being that the quiver that arises is more complicated than the one we saw in (1.3). Thus, each ideal M extends to a sheaf $\mathcal{M}_{\boldsymbol{w}}$ over $\mathbb{P}^2_q(\boldsymbol{w})$, and we obtain a pair of matrices, say $(\mathbb{X}(\boldsymbol{w}), \mathbb{Y}(\boldsymbol{w}))$, acting on the vector space $V(\boldsymbol{w}) := H^1(\mathbb{P}^2_q(\boldsymbol{w}),\, \mathcal{M}_{\boldsymbol{w}}(-1))$. If we take $\boldsymbol{w} = (1,r)$ (or $\boldsymbol{w} = (s,1)$), then the automorphisms $\Psi_{r,\lambda}$ (or $\Phi_{s,\mu}$) extend to automorphisms of the graded ring $\boldsymbol{A}(\boldsymbol{w})$, so we can follow their action on the corresponding pair of matrices $(\mathbb{X}(\boldsymbol{w}), \mathbb{Y}(\boldsymbol{w}))$ by simple functorial considerations. Theorem 1.3 will therefore follow at once from the next theorem, which is perhaps to be considered the main result of this paper.

Theorem 1.4 (Comparison theorem)**.** *The pair of matrices* $(\mathbb{X}(\boldsymbol{w}), \mathbb{Y}(\boldsymbol{w})) \in \mathfrak{C}$ *corresponding to a given ideal of* A *is independent of the choice of* $\boldsymbol{w}$.

The proof of Theorem 1.4 that we shall present here does not compare different weights directly, but instead compares each pair $(\mathbb{X}(\boldsymbol{w}), \mathbb{Y}(\boldsymbol{w}))$ with yet another pair of matrices $(\mathbb{X}, \mathbb{Y})$ which we shall extract from an ideal M in an elementary way (that is, without the use of homological algebra). The construction imitates the elementary treatment in the commutative case, using the representative of finite codimension for an ideal. The Weyl algebra has no (non-trivial) ideals of finite codimension. However, in Section 5 below we shall construct for each M two *fractional* ideals M_x and M_y (both isomorphic to M), together with embeddings r_x and r_y of M_x and M_y as linear subspaces of finite codimension in A. Of course, in that case r_x and r_y cannot be A-module homomorphisms, but r_x will be $\mathbb{C}[y]$-linear and r_y will be $\mathbb{C}[x]$-linear. On the quotient spaces $V_x := A/r_x(M_x)$ and $V_y := A/r_y(M_y)$ we therefore have endomorphisms $\mathbb{Y}$ and $\mathbb{X}$ (respectively) induced by multiplication by y and x. Furthermore, we shall construct a *canonical* isomorphism $\phi : V_x \to V_y$. Identifying V_x and V_y via ϕ, we thus get a pair of matrices $(\mathbb{X}, \mathbb{Y})$ (as usual, defined only up to simultaneous conjugation).

Theorem 1.5. *Let* M *be an ideal of* A, *and for each positive weight vector* $\boldsymbol{w}$ *let* $(\mathbb{X}(\boldsymbol{w}), \mathbb{Y}(\boldsymbol{w}))$ *be the corresponding pair of endomorphisms of the vector space* $V(\boldsymbol{w}) = H^1(\mathbb{P}^2_q(\boldsymbol{w}),\, \mathcal{M}_{\boldsymbol{w}}(-1))$ *described earlier. Then there are isomorphisms* $\alpha_x :$ $V(\boldsymbol{w}) \to V_x$ *and* $\alpha_y : V(\boldsymbol{w}) \to V_y$ *taking* $\mathbb{Y}(\boldsymbol{w})$ *to* $\mathbb{Y}$ *and* $\mathbb{X}(\boldsymbol{w})$ *to* $\mathbb{X}$ *respectively,* *and making the diagram*

$$\begin{array}{ccc} V(\boldsymbol{w}) & \xrightarrow{\alpha_x} & V_x \\ \| & & \downarrow \phi \\ V(\boldsymbol{w}) & \xrightarrow{\alpha_y} & V_y \end{array}$$

commutative.

In other words, the various pairs of matrices $(\mathbb{X}(\boldsymbol{w}), \mathbb{Y}(\boldsymbol{w}))$ and $(\mathbb{X}, \mathbb{Y})$ that we have assigned to an ideal M all coincide. Theorem 1.4 is now clear, since the elementary construction of the pair $(\mathbb{X}, \mathbb{Y})$ does not involve any choice of weights.

The proof of Theorem 1.5 consists of a calculation of $H^1(\mathbb{P}^2_q(\boldsymbol{w}), \mathcal{M}_{\boldsymbol{w}}(-1))$ using the Čech cohomology developed in [Ve], [VW1], [VW2]. A key point is that the quantum "planes" $\mathbb{P}^2_q(\boldsymbol{w})$ have *schematic dimension* (in the sense of [W]) equal to 1, not 2; this means (in geometrical language) that $\mathbb{P}^2_q(\boldsymbol{w})$ can be covered by just two affine open sets, analogous to the (X, Z)-plane and the (Y, Z)-plane in the commutative case. Of course, in that case these two affine open sets fail to catch the point $(0 : 0 : 1)$; but apparently the quantum planes do not have "points" to cause that kind of trouble. It turns out that our pair (M_x, M_y) of special representatives of an ideal is well adapted to calculating the Čech cohomology of this 2-set covering of $\mathbb{P}^2_q(\boldsymbol{w})$. For details we refer to Section 6 below.

After this article appeared as a preprint, M. Van den Bergh succeeded in finding direct homological proofs of our main results: these proofs are presented in the Appendix. In particular, Van den Bergh proves directly that our map θ is bijective, whereas in the main body of the paper we see that only after identifying θ with ω^{-1}: this proof of bijectivity thus still depends on the arguments from the theory of integrable systems used in [BW]. It is interesting that methods from integrable systems and from the theory of derived categories appear here as alternatives to each other (cf. the question raised in the first sentence of [Be]).

The paper is organized as follows. In Section 2 we give a brief introduction to noncommutative projective geometry, and summarize the results we need from the literature on that subject. In Section 3 we introduce the main characters in our story, the Weyl algebra, its various "homogenizations" and the associated projective planes $\mathbb{P}^2_q(\boldsymbol{w})$. Then in Section 4 we show (for arbitrary weights) how to extract from a given ideal of A the pairs of matrices $(\mathbb{X}(\boldsymbol{w}), \mathbb{Y}(\boldsymbol{w})) \in \mathfrak{C}$. The next section explains the elementary construction of the pair of matrices $(\mathbb{X}, \mathbb{Y})$, using the two special representatives of an ideal of A. Then Section 6 gives the calculation of Čech cohomology which identifies $(\mathbb{X}(\boldsymbol{w}), \mathbb{Y}(\boldsymbol{w}))$ with $(\mathbb{X}, \mathbb{Y})$, and hence proves Theorems 1.5 and 1.4. Section 7 then deduces our other main results, filling in some details left vague in the sketch above. Section 8 establishes Beilinson's derived equivalence for the spaces $\mathbb{P}^2_q(\boldsymbol{w})$: although, strictly speaking, this equivalence is not used in the main part of the paper[3], it is scarcely possible to understand the motivation for our construction without at least a cursory reading of this section. Finally, section 9 discusses briefly the relationship of our construction to the original one of Le Bruyn, while Section 10 explains how it fits in with the classification of bundles over $\mathbb{P}^2_q$ given in the recent paper [KKO].

[3] However, it is used in an essential way in the Appendix.

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2. Noncommutative Projective Geometry

As we mentioned in the Introduction, the starting point for noncommutative projective geometry is the following result of Serre (see [S]): the category of coherent sheaves over a (commutative) projective variety X is equivalent to a quotient of the category of finitely generated graded modules over the coordinate ring $\boldsymbol{A}$ of X. This latter category makes sense also for a noncommutative graded ring $\boldsymbol{A}$. In this section we give a brief overview of the theory of noncommutative projective schemes and their cohomology: we introduce the notation, recall definitions and collect some fundamental results needed for understanding the main part of the paper. Our basic reference is [AZ]. More details on graded algebras and modules can be found in [NV], on abelian categories (including Serre quotients) in [G] and [GM], on (Artin-Schelter) regular algebras in [AS], [ATV] and [Ste]. The "schematic" structure on graded algebras and the noncommutative Čech cohomology are introduced and discussed in [Ve], [VW1], [VW2]. For a general overview of the subject we recommend the articles [A], [Sm] and [VW].

2.1. **Graded Algebras and Modules.** We recall that an associative algebra[4] $\boldsymbol{A}$ over a field k is called *graded* (more precisely, $\mathbb{Z}$-graded) if $\boldsymbol{A} = \bigoplus_{i\in\mathbb{Z}} A_i$ as a k-vector space and $A_iA_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{Z}$. We shall assume that $\boldsymbol{A}$ is (both left and right) Noetherian. If $A_i = 0$ for all $i < 0$ and $A_0 = k$ we say that $\boldsymbol{A}$ is *connected*. It is easy to see that any graded connected Noetherian k-algebra is *locally finite*, that is, $\dim_k A_i < \infty$ for all i.

A (right) $\boldsymbol{A}$-module $\boldsymbol{M}$ is *graded* if it has a vector space decomposition $\boldsymbol{M} = \bigoplus_{i\in\mathbb{Z}} M_i$ compatible with the $\mathbb{Z}$-grading on $\boldsymbol{A}$, that is, $M_i\, A_j \subseteq M_{i+j}$ for all i, j. The category of all right graded modules over $\boldsymbol{A}$ will be denoted by $\mathsf{GrMod}(\boldsymbol{A})$: the morphisms in $\mathsf{GrMod}(\boldsymbol{A})$ are *graded* homomorphisms of degree zero.

For each $n \in \mathbb{Z}$ we introduce two functorial operations on graded modules

shift in grading: $\boldsymbol{M} = \bigoplus_{i\in\mathbb{Z}} M_i \;\mapsto\; \boldsymbol{M}(n) := \bigoplus_{i\in\mathbb{Z}} M_{i+n}$;

(left) truncation: $\boldsymbol{M} = \bigoplus_{i\in\mathbb{Z}} M_i \;\mapsto\; \boldsymbol{M}_{\geq n} := \bigoplus_{i\geq n} M_i$.

We say that a graded module $\boldsymbol{M}$ is *left* (respectively, *right*) *bounded* if $\boldsymbol{M}_{\geq n} = \boldsymbol{M}$ (respectively, $\boldsymbol{M}_{\geq n} = 0$) for some $n \in \mathbb{Z}$.

Finitely generated graded modules form a full subcategory in $\mathsf{GrMod}(\boldsymbol{A})$; it is denoted by $\mathsf{grmod}(\boldsymbol{A})$. The shift and truncation functors on $\mathsf{GrMod}(\boldsymbol{A})$ preserve this subcategory. Moreover, if $\boldsymbol{A}$ is left bounded (for example, connected), so are all finitely generated graded modules over $\boldsymbol{A}$. Further, if $\boldsymbol{A}$ is locally finite, then every object in $\mathsf{grmod}(\boldsymbol{A})$ is locally finite as well.

A few words on homological properties of graded modules. First of all, $\mathsf{GrMod}(\boldsymbol{A})$ is an abelian k-linear category with enough projective and injective objects, so for

[4]Throughout the paper we shall denote graded objects (algebras, modules, ...) by (capital) boldface letters ($\boldsymbol{A}$, $\boldsymbol{M}$, ...) distiguishing them from ungraded ones (A, M, ...).

each $n \geq 1$ we may define the functors $\mathrm{Ext}^n_{\boldsymbol{A}}(\boldsymbol{M}, —)$ on $\mathsf{GrMod}(\boldsymbol{A})$ as the right derived of $\mathrm{Hom}_{\boldsymbol{A}}(\boldsymbol{M}, —) \equiv \mathrm{Hom}_{\mathrm{GrMod}}(\boldsymbol{M}, —)$. Next, it is convenient to have some notation for *graded* Ext-groups. Thus, we set

$$\underline{\mathrm{Ext}}^n_{\boldsymbol{A}}(\boldsymbol{M}, \boldsymbol{N}) := \bigoplus_{d \in \mathbb{Z}} \mathrm{Ext}^n_{\boldsymbol{A}}(\boldsymbol{M}, \boldsymbol{N}(d)) \; ;$$

then $\underline{\mathrm{Ext}}^n_{\boldsymbol{A}}(\boldsymbol{M}, —)$ for $n \geq 1$ are the right derived functors of $\underline{\mathrm{Ext}}^0_{\boldsymbol{A}}(\boldsymbol{M}, —) := \underline{\mathrm{Hom}}_{\boldsymbol{A}}(\boldsymbol{M}, —)$. To clarify this definition observe (see [NV], Corollary I.2.12) that

$$\underline{\mathrm{Ext}}^n_{\boldsymbol{A}}(\boldsymbol{M}, \boldsymbol{N}) = \mathrm{Ext}^n_{\mathrm{Mod}(\boldsymbol{A})}(\boldsymbol{M}, \boldsymbol{N}) \quad \text{for all } n \geq 0 \; , \tag{2.1}$$

at least when $\boldsymbol{M}$ is finitely generated. (On the right hand side of (2.1) $\boldsymbol{M}$ and $\boldsymbol{N}$ are regarded as objects in the category $\mathsf{Mod}(\boldsymbol{A})$ of *ungraded* $\boldsymbol{A}$-modules.)

Finally, we mention the following natural isomorphisms (of graded vector spaces):

$$\underline{\mathrm{Ext}}^n_{\boldsymbol{A}}(\boldsymbol{M}, \boldsymbol{N}(d)) \cong \underline{\mathrm{Ext}}^n_{\boldsymbol{A}}(\boldsymbol{M}(-d), \boldsymbol{N}) \cong \underline{\mathrm{Ext}}^n_{\boldsymbol{A}}(\boldsymbol{M}, \boldsymbol{N})(d)$$

valid for all $d \in \mathbb{Z}$ and for all $\boldsymbol{M}, \boldsymbol{N} \in \mathsf{GrMod}(\boldsymbol{A})$.

2.2. **Projective Schemes.** Let $\boldsymbol{A}$ be a Noetherian connected graded k-algebra, and let $\boldsymbol{M}$ be a graded right module over $\boldsymbol{A}$. We say that $m \in \boldsymbol{M}$ is a *τorsion element* of $\boldsymbol{M}$ if $mA_n = 0$ for $n \gg 0$. The τorsion elements form a (graded) submodule in $\boldsymbol{M}$; we denote it by $\boldsymbol{\tau M}$. Equivalently, $\boldsymbol{\tau M}$ is the sum of all finite dimensional (over k) submodules of $\boldsymbol{M}$. In particular, if $\boldsymbol{M}$ is finitely generated, so is $\boldsymbol{\tau M}$ (since we assume $\boldsymbol{A}$ to be Noetherian), and hence $\dim_k \boldsymbol{\tau M} < \infty$ in that case.

A module $\boldsymbol{M}$ is called a *τorsion module* if $\boldsymbol{\tau M} = \boldsymbol{M}$, and *$\tau$orsion-free* if $\boldsymbol{\tau M} = 0$. The full subcategory of $\mathsf{GrMod}(\boldsymbol{A})$ consisting of all τorsion modules will be denoted by $\mathsf{Tors}(\boldsymbol{A})$. Similarly, we shall write $\mathsf{tors}(\boldsymbol{A})$ for the full subcategory of $\mathsf{grmod}(\boldsymbol{A})$ consisting of finitely generated τorsion modules. As we observed above, the latter are precisely the graded modules which have finite dimension as vector spaces over k.

Since both $\mathsf{Tors}(\boldsymbol{A})$ and $\mathsf{tors}(\boldsymbol{A})$ are *dense* subcategories (that is, in any short exact sequence $0 \to \boldsymbol{M}' \to \boldsymbol{M} \to \boldsymbol{M}'' \to 0$ the module $\boldsymbol{M}$ is τorsion if and only if $\boldsymbol{M}'$ and $\boldsymbol{M}''$ are τorsion), we may introduce the quotient categories

$$\mathsf{Tails}(\boldsymbol{A}) := \mathsf{GrMod}(\boldsymbol{A})/\mathsf{Tors}(\boldsymbol{A}) \; , \quad \mathsf{tails}(\boldsymbol{A}) := \mathsf{grmod}(\boldsymbol{A})/\mathsf{tors}(\boldsymbol{A}) \; .$$

These are both abelian categories (see [G], pp. 367–369), the second being a full subcategory of the first; they are equipped with the exact projection functor $\pi : \mathsf{GrMod}(\boldsymbol{A}) \to \mathsf{Tails}(\boldsymbol{A})$ which sends all the τorsion objects in $\mathsf{GrMod}(\boldsymbol{A})$ to zero and is universal (among additive functors) with respect to this property. Throughout the paper we shall denote quotient objects by script letters; for example, if $\boldsymbol{M} \in \mathsf{GrMod}(\boldsymbol{A})$, we write $\mathcal{M} := \pi \boldsymbol{M}$ for the corresponding object in $\mathsf{Tails}(\boldsymbol{A})$. The shift in grading $\boldsymbol{M} \mapsto \boldsymbol{M}(1)$ preserves τorsion modules, hence carries over as an operation on quotient objects. The induced functor $\mathcal{M} \mapsto \mathcal{M}(1)$ on $\mathsf{Tails}(\boldsymbol{A})$ (or on $\mathsf{tails}(\boldsymbol{A})$) is called the *twist functor*.

In general, the description of the morphisms in $\mathsf{Tails}(\boldsymbol{A})$ is somewhat complicated. However, if $\boldsymbol{M}$ is finitely generated, we have simply

$$\mathrm{Hom}_{\,\mathrm{Tails}(\boldsymbol{A})}(\mathcal{M}, \mathcal{N}) \cong \varinjlim \mathrm{Hom}_{\boldsymbol{A}}(\boldsymbol{M}_{\geq n}, \boldsymbol{N}) \; , \tag{2.2}$$

where the system $\{\mathrm{Hom}_{\boldsymbol{A}}(\boldsymbol{M}_{\geq n}, \boldsymbol{N})\}$ is directed by restriction of graded homomorphisms. It is easy to deduce from (2.2) when two objects in $\mathsf{tails}(\boldsymbol{A})$ are isomorphic, namely

$$\mathcal{M} \cong \mathcal{N} \text{ in } \mathsf{tails}(\boldsymbol{A}) \quad \Longleftrightarrow \quad \boldsymbol{M}_{\geq n} \cong \boldsymbol{N}_{\geq n} \text{ in } \mathsf{grmod}(\boldsymbol{A}) \text{ for some } n \ . \tag{2.3}$$

This perhaps explains the use of the word "tails".

If the algebra $\boldsymbol{A}$ is commutative and generated by elements of degree one, then Serre's result tells us that the categories $\mathsf{Tails}(\boldsymbol{A})$ and $\mathsf{tails}(\boldsymbol{A})$ are equivalent to the categories of quasicoherent and coherent sheaves on the projective scheme $X = \mathsf{proj}(\boldsymbol{A})$. For psychological reasons, it is very helpful to use similar language also in the noncommutative case, even though in that case we shall not attempt to give any independent meaning to "$\mathsf{proj}(\boldsymbol{A})$". In what follows we shall refer to the objects of $\mathsf{tails}(\boldsymbol{A})$ (respectively, $\mathsf{Tails}(\boldsymbol{A})$) as coherent (respectively, quasicoherent) sheaves on $X = \mathsf{proj}(\boldsymbol{A})$, even when $\boldsymbol{A}$ is not commutative, and we shall use the notation $\mathcal{O}_X := \pi\boldsymbol{A}$, $\mathsf{coh}(X) := \mathsf{tails}(\boldsymbol{A})$, $\mathsf{Qcoh}(X) := \mathsf{Tails}(\boldsymbol{A})$.

2.3. **Cohomology.** In this section we outline the cohomology theory of coherent sheaves over noncommutative schemes, confining ourselves to results that will be used in the main body of the paper. We keep the assumption that $\boldsymbol{A}$ is a graded connected Noetherian k-algebra.

For each $\mathcal{M} \in \mathsf{Tails}(\boldsymbol{A})$ the functor $\mathrm{Hom}_{\mathsf{Tails}}(\mathcal{M}, \text{—})$ is left exact; since $\mathsf{Tails}(\boldsymbol{A})$ has enough injectives (see [AZ]), the right derived functors $\mathrm{Ext}^n(\mathcal{M}, \text{—})$ are well defined. As in the case of graded modules, we introduce the notation

$$\underline{\mathrm{Ext}}^n(\mathcal{M}, \mathcal{N}) := \bigoplus_{d \in \mathbb{Z}} \mathrm{Ext}^n(\mathcal{M}, \mathcal{N}(d)) \ .$$

Definition 2.1. Let $\mathcal{M} \in \mathsf{Qcoh}(X) \equiv \mathsf{Tails}(\boldsymbol{A})$ be a quasicoherent sheaf over $X = \mathsf{proj}(\boldsymbol{A})$. For each $n \geq 0$ we define the *cohomology groups* of $\mathcal{M}$ by

$$H^n(X, \mathcal{M}) := \mathrm{Ext}^n(\mathcal{O}_X, \mathcal{M}) \ , \tag{2.4}$$

where $\mathcal{O}_X := \pi\boldsymbol{A} \in \mathsf{Qcoh}(X)$.

The *cohomological dimension* of X is then defined by

$$\mathrm{cdim}(X) := \max\{\, n \in \mathbb{N} \ : \ H^n(X, \mathcal{M}) \neq 0 \text{ for some } \mathcal{M} \in \mathsf{Qcoh}(X)\,\} \ .$$

Since $\mathsf{Tails}(\boldsymbol{A})$ is a k-linear category, all the groups (2.4) are vector spaces over k. The graded objects

$$\underline{H}^n(X, \mathcal{M}) := \bigoplus_{d \in \mathbb{Z}} H^n(X, \mathcal{M}(d))$$

are naturally graded right modules over $\boldsymbol{A}$; we refer to them as the *full cohomology modules* of $\mathcal{M}$. By (2.2), if $\mathcal{M} = \pi\boldsymbol{M}$, we have

$$\underline{H}^0(X, \mathcal{M}) \cong \varinjlim \underline{\mathrm{Hom}}_{\boldsymbol{A}}(\boldsymbol{A}_{\geq n}, \boldsymbol{M}) \ .$$

From this it is easy to see that the functor $\omega := \underline{H}^0(X, \text{—})$ is right adjoint to the projection functor $\pi : \mathsf{GrMod}(\boldsymbol{A}) \to \mathsf{Tails}(\boldsymbol{A})$. For any object $\mathcal{M} \in \mathsf{Tails}(\boldsymbol{A})$, the adjunction map $\mathcal{M} \to \pi\omega\mathcal{M}$ is an isomorphism; the other adjunction map $\boldsymbol{M} \to \omega\pi\boldsymbol{M} \cong \underline{H}^0(X, \mathcal{M})$ is the restriction

$$\boldsymbol{M} \cong \underline{\mathrm{Hom}}_{\boldsymbol{A}}(\boldsymbol{A}, \boldsymbol{M}) \to \varinjlim \underline{\mathrm{Hom}}_{\boldsymbol{A}}(\boldsymbol{A}_{\geq n}, \boldsymbol{M}) \cong \underline{H}^0(X, \mathcal{M}) \ . \tag{2.5}$$

Clearly, the kernel of this map is $\boldsymbol{\tau M}$; more generally (see [AZ], Prop. 7.2), there is an exact sequence of graded modules

$$(2.6) \qquad 0 \to \boldsymbol{\tau M} \to \boldsymbol{M} \to \underline{H}^0(X,\mathcal{M}) \to \varinjlim \underline{\mathrm{Ext}}^1_{\boldsymbol{A}}(\boldsymbol{A}/\boldsymbol{A}_{\geq n},\boldsymbol{M}) \to 0 \ .$$

We shall need noncommutative versions of the basic theorems of Serre (finiteness and vanishing of cohomology, and duality). For this we have to impose some additional conditions on our algebra $\boldsymbol{A}$. The least restrictive condition used in the literature is the so-called *χ-condition* (see [AZ]): $\dim_k \underline{\mathrm{Ext}}^n_{\boldsymbol{A}}(\boldsymbol{k_A},\boldsymbol{M}) < \infty$ for all $\boldsymbol{M} \in \mathsf{grmod}(\boldsymbol{A})$ and for all $n \geq 0$. (Here $\boldsymbol{k_A} := \boldsymbol{A}/\boldsymbol{A}_{\geq 1}$ denotes the "trivial" (right) module over $\boldsymbol{A}$.) The algebras occurring in the present paper satisfy a much stronger condition: they are *Artin-Schelter* algebras. We shall concentrate on that case. The definition is as follows (see [AS]).

Definition 2.2. A graded connected algebra $\boldsymbol{A}$ is called *Artin-Schelter* (or *Artin-Schelter regular*) if $\boldsymbol{A}$ has
(i) finite global dimension, say $\mathrm{gl.dim}(\boldsymbol{A}) = d$;
(ii) polynomial growth, that is, $\dim_k A_m \leq \gamma m^p$ for some positive $p \in \mathbb{Z}$, $\gamma \in \mathbb{R}$, and for all $m \geq 0$;
(iii) the (graded) *Gorenstein property*: $\underline{\mathrm{Ext}}^i_{\boldsymbol{A}}(\boldsymbol{k_A},\boldsymbol{A}) = 0$ for all $i \neq d$ and $\underline{\mathrm{Ext}}^d_{\boldsymbol{A}}(\boldsymbol{k_A},\boldsymbol{A}) \cong \boldsymbol{k_A}(l)$ for some integer l (called the *Gorenstein parameter* of $\boldsymbol{A}$).

If $\boldsymbol{A}$ is commutative, then the condition (i) in definition 2.2 already implies that $\boldsymbol{A}$ is isomorphic to a polynomial ring $k[x_0, x_1, \ldots, x_n]$ with some positive grading (see [SZ]). Thus the only commutative Artin-Schelter algebras are polynomial algebras. However, in the noncommutative case there are many interesting examples (see [AS], [ATV], [Ste] and references therein). The projective schemes associated with regular noncommutative algebras are referred to as "quantum projective spaces". The next theorem provides further justification for this terminology.

Theorem 2.3 ([AZ], Theorem 8.1)**.** *Let $\boldsymbol{A}$ be a Noetherian Artin-Schelter algebra of global dimension $d = n+1$, and let $X = \mathsf{proj}(\boldsymbol{A})$. Then $\mathrm{cdim}(X) = n$, and the full cohomology modules of $\mathcal{O}_X := \pi\boldsymbol{A}$ are given by*

$$\underline{H}^i(X,\mathcal{O}_X) \cong \begin{cases} \boldsymbol{A} & \text{if} \quad i = 0 \\ 0 & \text{if} \quad i \neq 0, n \\ \boldsymbol{A}^*(l) & \text{if} \quad i = n \end{cases} ,$$

where l is the Gorenstein parameter of $\boldsymbol{A}$, and $\boldsymbol{A}^*$ denotes the graded dual of $\boldsymbol{A}$ with components $A_i^* := \mathrm{Hom}_k(A_{-i}, k)$.

Now we are in position to state the version of Serre's theorems that we shall use.

Theorem 2.4. *Let $\boldsymbol{A}$ be a Noetherian Artin-Schelter algebra of global dimension $d = n+1$ and Gorenstein parameter l. Let $X = \mathsf{proj}(\boldsymbol{A})$. Then if $\mathcal{M} \in \mathsf{coh}(X)$, we have*
(a) (Finiteness) $\dim_k H^i(X,\mathcal{M}) < \infty$ *for all* $i \geq 0$*;*
(b) (Vanishing) *if* $i \geq 1$*, then* $H^i(X,\mathcal{M}(k)) = 0$ *for all* $k \gg 0$*;*
(c) (Grothendieck-Serre Duality) *there are natural isomorphisms*

$$\mathrm{Ext}^i(\mathcal{M},\mathcal{O}_X(-l)) \cong H^{n-i}(X,\mathcal{M})^* \ ,$$

for $i = 0, 1, 2, \ldots, n$.

The Finiteness and Vanishing theorems have been proved by Artin and Zhang (see [AZ], Theorem 7.4) for any connected Noetherian algebra satisfying the χ-condition. For the Duality theorem we refer to [YZ] (see also [J]): we shall use this Theorem only in Section 10.

2.4. **Čech Cohomology.** In the classical case when $\boldsymbol{A}$ is commutative and generated by elements of degree 1, we can calculate sheaf cohomology of the projective scheme $X = \mathsf{proj}(\boldsymbol{A})$ using the Čech complex of any affine open covering of X. Restricting a (quasicoherent) sheaf to an affine open set corresponds under Serre's equivalence to (graded) localization of the associated $\boldsymbol{A}$-module. A natural noncommutative generalization of this construction has been suggested recently in [Ve], [VW1], [VW2]. Since in the noncommutative case one can define localization only with respect to Ore sets, the existence of "sufficiently many" (homogeneous) Ore sets is a necessary condition to be imposed on the corresponding graded algebras. The class of such algebras (called *schematic*) is fairly rich and contains many interesting examples (see [VW3]). Translating the definition of a covering into algebraic language, we arrive at the following

Definition 2.5 (see [VW1])**.** A Noetherian graded connected k-algebra $\boldsymbol{A}$ is called *schematic* if there exists a finite number of (two-sided) homogeneous Ore sets $U_1, U_2, \dots, U_s$ in $\boldsymbol{A}$ such that

(*i*) each U_i contains 1, and all $U_i \cap \boldsymbol{A}_{\geq 1}$ are non-empty;

(*ii*) for any collection of elements $(u_1, u_2, \dots, u_s) \in U_1 \times U_2 \times \dots \times U_s$, there is an $m \in \mathbb{N}$ such that

$$\boldsymbol{A}_{\geq m} \subseteq \sum_{i=1}^{s} u_i \boldsymbol{A} \ . \tag{2.7}$$

A collection of Ore sets satisfying the conditions above is called a *covering of* $\boldsymbol{A}$.

If $\boldsymbol{A}$ is schematic, let N denote the least possible number of Ore sets covering $\boldsymbol{A}$. Following [W], we define the *schematic dimension* of $\boldsymbol{A}$ by $\mathrm{sdim}(\boldsymbol{A}) := N - 1$. If $\boldsymbol{A}$ is commutative, $\mathrm{sdim}(\boldsymbol{A})$ coincides with the usual (Krull or cohomological) dimension of the scheme $X = \mathrm{proj}(\boldsymbol{A})$. However, in the noncommutative case the schematic dimension may happen to be smaller than $\mathrm{cdim}(X)$, even for Artin-Schelter algebras (see [W] and Lemma 3.1 below).

The Čech complex of a covering of $\boldsymbol{A}$ is constructed in more or less the usual way, except that an "intersection of open sets" now depends on the order in which the sets intersect. Fix a (finite) covering of $\boldsymbol{A}$, say $\mathfrak{U} = \{U_1, U_2, \dots, U_s\}$. Given $\boldsymbol{M} \in \mathsf{GrMod}(\boldsymbol{A})$ and a $(p+1)$-tuple $(i_0, i_1, \dots, i_p)$ of indices, each i_k being in $\{1, 2, \dots, s\}$, we write

$$\boldsymbol{M}_{i_0 i_1 \dots i_p} := \boldsymbol{M} \underline{\otimes}_{\boldsymbol{A}} \boldsymbol{A}_{U_{i_0}} \underline{\otimes}_{\boldsymbol{A}} \dots \underline{\otimes}_{\boldsymbol{A}} \boldsymbol{A}_{U_{i_p}} \ , \tag{2.8}$$

where $\boldsymbol{A}_{U_{i_k}} := \boldsymbol{A}[U_{i_k}^{-1}]$ is the (graded) localization of $\boldsymbol{A}$ at U_{i_k}. Now, for each $p = 0, 1, 2, \dots$, set

$$\boldsymbol{C}^p(\mathfrak{U}, \mathcal{M}) := \bigoplus_{(i_0, i_1, \dots, i_p)} \boldsymbol{M}_{i_0 i_1 \dots i_p} \in \mathsf{GrMod}(\boldsymbol{A}) \ .$$

Then $\boldsymbol{C}^p(\mathfrak{U}, \mathcal{M})$ form a complex of graded $\boldsymbol{A}$-modules

$$\boldsymbol{C}^{\bullet}(\mathfrak{U}, \mathcal{M}) : \ 0 \to \boldsymbol{C}^0(\mathfrak{U}, \mathcal{M}) \xrightarrow{d^0} \boldsymbol{C}^1(\mathfrak{U}, \mathcal{M}) \xrightarrow{d^1} \dots \tag{2.9}$$

with coboundary maps $d^p : \boldsymbol{C}^p(\mathfrak{U}, \mathcal{M}) \to \boldsymbol{C}^{p+1}(\mathfrak{U}, \mathcal{M})$ defined in the usual way. For example,

$$d^0 : \bigoplus_{i=1}^{s} \boldsymbol{M}_i \to \bigoplus_{i,j=1}^{s} \boldsymbol{M}_{ij}$$

is given by the formula

$$d^0(m_1 \otimes u_1^{-1}, \dots , m_s \otimes u_s^{-1})_{ij} = m_i \otimes u_i^{-1} \otimes 1 - m_j \otimes 1 \otimes u_j^{-1}$$

in $\boldsymbol{M}_{ij} = \boldsymbol{M} \underline{\otimes}_{\boldsymbol{A}} \boldsymbol{A}_{U_i} \underline{\otimes}_{\boldsymbol{A}} \boldsymbol{A}_{U_j}$.

The cohomology of the complex (2.9)

$$\underline{\check{H}}^p(\mathfrak{U}, \mathcal{M}) := h^p[\boldsymbol{C}^\bullet(\mathfrak{U}, \mathcal{M})] \ , \quad p = 0, 1, 2, \dots$$

is called the (full) *Čech cohomology* of the sheaf $\mathcal{M}$ relative to the covering $\mathfrak{U}$. As in the commutative case, we have the following general result.

Theorem 2.6. *For all quasicoherent sheaves $\mathcal{M} = \pi \boldsymbol{M} \in \mathsf{Qcoh}(X)$, and for any covering $\mathfrak{U}$ of $\boldsymbol{A}$, there are natural isomorphisms of graded modules*

$$\underline{H}^p(X, \mathcal{M}) \cong \underline{\check{H}}^p(\mathfrak{U}, \mathcal{M}) \ , \quad p = 0, 1, 2, \dots .$$

For the proof we refer the reader to [VW2]. We shall use Theorem 2.6 only for $p = 0$, in which case it is an elementary exercise. The isomorphism $\underline{H}^0(X, \mathcal{M}) \to \underline{\check{H}}^0(\mathfrak{U}, \mathcal{M})$ is defined as follows. Recall that an element of $\underline{H}^0(X, \mathcal{M})$ is represented by a homomorphism $f : \boldsymbol{A}_{\geq n} \to \boldsymbol{M}$ (for some n). Condition (i) in Definition 2.5 implies that $(\boldsymbol{A}_{\geq n})_{U_i} \cong \boldsymbol{A}_{U_i}$ for any n; so after localization f defines a homomorphism $f_i : \boldsymbol{A}_{U_i} \to \boldsymbol{M}_{U_i}$ for each i. Assigning to f the element $(f_1(1), f_2(1), \dots , f_s(1)) \in \boldsymbol{C}^0(\mathfrak{U}, \mathcal{M})$ defines the desired isomorphism. Combining this with (2.5) we get

Proposition 2.7. *Let $\boldsymbol{M}$ be a graded $\boldsymbol{A}$-module, $\mathcal{M} = \pi \boldsymbol{M}$ the associated sheaf. Then the natural map $\boldsymbol{M} \to \underline{H}^0(X, \mathcal{M}) \to \underline{\check{H}}^0(\mathfrak{U}, \mathcal{M})$ is given by*

$$m \mapsto (m \otimes 1, \dots , m \otimes 1) \ \in \ \boldsymbol{C}^0(\mathfrak{U}, \mathcal{M}) = \boldsymbol{M}_{U_1} \oplus \dots \oplus \boldsymbol{M}_{U_s} \ .$$

We shall denote the d-th graded component of $\underline{\check{H}}^p(\mathfrak{U}, \mathcal{M})$ by $\check{H}^p(\mathfrak{U}, \mathcal{M}(d))$. We then have

$$\underline{\check{H}}^p(\mathfrak{U}, \mathcal{M}) = \bigoplus_{d \in \mathbb{Z}} \check{H}^p(\mathfrak{U}, \mathcal{M}(d)) \ ,$$

in conformity with our usual notation.

3. The Weyl Algebra and its Homogenizations

From now on, we set $k = \mathbb{C}$. Let $A = \mathbb{C}\langle x, y\rangle/([x, y] - 1)$ be the first Weyl algebra over $\mathbb{C}$. Unlike the commutative algebra $\mathbb{C}[x, y]$, the Weyl algebra admits no natural grading; however, it has many natural filtrations.

3.1. **Filtered Rings and Modules.** We recall that a *filtration* on an algebra A is an increasing sequence of linear subspaces

$$\dots \subseteq A_{k-1} \subseteq A_k \subseteq A_{k+1} \ \dots \ , \tag{3.1}$$

indexed by the integers, such that $1 \in A_0$, $\bigcup_{k \in \mathbb{Z}} A_k = A$ and $A_j A_k \subseteq A_{j+k}$ for all $j, k \in \mathbb{Z}$. A filtration is called *positive* if $A_k = 0$ for all $k < 0$. If A is a filtered algebra, then a (right) A-module M is called a *filtered A-module* if there is an increasing sequence of linear subspaces

$$\dots \subseteq M_{k-1} \subseteq M_k \subseteq M_{k+1} \subseteq \dots \ , \tag{3.2}$$

such that $\bigcup_{k\in\mathbb{Z}} M_k = M$ and $M_k A_j \subseteq M_{k+j}$ for all $k, j \in \mathbb{Z}$. We shall assume the filtration (3.2) to be *separated*, meaning that $\bigcap_{k\in\mathbb{Z}} M_k = 0$.

Attached naturally to a filtered algebra are the following two graded algebras:

$$\boldsymbol{A} := \bigoplus_{k\in\mathbb{Z}} A_k \quad , \quad \boldsymbol{GA} := \bigoplus_{k\in\mathbb{Z}} A_k/A_{k-1} \ .$$

The algebra $\boldsymbol{A}$ is called the *Rees algebra* of A with respect to the filtration (3.1). It can be identified with a subring of the ring of Laurent polynomials (in one variable t) with coefficients in A. To be precise, we have

$$\boldsymbol{A} \cong \bigoplus_{k\in\mathbb{Z}} A_k t^k \hookrightarrow A[t, t^{-1}] \ , \tag{3.3}$$

where the grading on $A[t,t^{-1}]$ is defined by $\deg(t) = 1$ and $\deg(a) = 0$ for all elements $a \in A$. The algebra $\boldsymbol{GA}$ is called the *associated graded ring* of A. Under the identification (3.3) there is a natural isomorphism of graded algebras

$$\boldsymbol{GA} \cong \boldsymbol{A}/\langle t \rangle \ , \tag{3.4}$$

where $\langle t \rangle$ denotes the two-sided ideal of $\boldsymbol{A}$ generated by the central element t.

Similarly, if M is a filtered A-module then we have the *Rees module*

$$\boldsymbol{M} := \bigoplus_{k\in\mathbb{Z}} M_k \ \in \ \mathsf{GrMod}(\boldsymbol{A}) \ , \tag{3.5}$$

and the *associated graded module*

$$\boldsymbol{GM} := \bigoplus_{k\in\mathbb{Z}} M_k/M_{k-1} \ \in \ \mathsf{GrMod}(\boldsymbol{GA}) \ .$$

Identifying $\boldsymbol{A}$ with a ring of A-valued Laurent polynomials (see (3.3)), we have $\boldsymbol{M} \cong \bigoplus_{k\in\mathbb{Z}} M_k t^k \hookrightarrow M[t,t^{-1}]$, where $M[t,t^{-1}] := M \otimes_A A[t,t^{-1}]$, and hence, in view of (3.4), the following isomorphisms of graded $\boldsymbol{GA}$-modules

$$\boldsymbol{GM} \cong \boldsymbol{M}/\boldsymbol{M}t \cong \boldsymbol{M} \otimes_{\boldsymbol{A}} \boldsymbol{A}/\langle t \rangle \ \cong \boldsymbol{M} \otimes_{\boldsymbol{A}} \boldsymbol{GA} \ . \tag{3.6}$$

When A is commutative, the above constructions have a simple geometrical meaning: $X := \mathsf{proj}(\boldsymbol{A})$ is a projective scheme containing the affine scheme $\mathsf{Spec}(A)$ as an open subset, and the sheaf $\mathcal{M} := \pi \boldsymbol{M}$ is an extension to X of the sheaf $\tilde{M}$ on $\mathsf{Spec}(A)$ corresponding to M. Thus, from an algebraic point of view, the projective compactification X is determined by the choice of filtration on A, and the extension of $\tilde{M}$ to X is then determined by the choice of filtration on M. Furthermore, $\mathsf{proj}(\boldsymbol{GA})$ is the "hypersurface at infinity" (in our case it will be a line) in X, and $\pi \boldsymbol{GM}$ is the restriction of $\mathcal{M}$ to this hypersurface. We shall use similar language also in the noncommutative case.

3.2. **Weight Filtrations.** We now introduce the class of filtrations on the Weyl algebra A which we shall use in the present paper. Given a pair of positive integers $\boldsymbol{w} := (w_1, w_2)$, we set

$$A_k(\boldsymbol{w}) := \operatorname{span}_{\mathbb{C}}\{\, x^\alpha y^\beta \mid w_1\alpha + w_2\beta \le k \,\} \subset A \tag{3.7}$$

for each $k \in \mathbb{Z}$. Then $\{A_\bullet(\boldsymbol{w})\}$ is a positive locally finite filtration on A with $A_0(\boldsymbol{w}) = \mathbb{C}$. We call (3.7) the *filtration of weight* $\boldsymbol{w}$. In particular, if $\boldsymbol{w} = (1,1)$, this is the standard Bernstein filtration on A.

To describe the Rees algebra associated with (3.7) we use the identification (3.3). Setting $X := x \cdot t^{w_1}$, $Y := y \cdot t^{w_2}$ and $Z := 1 \cdot t$, we observe that $\boldsymbol{A}$ is isomorphic to the graded algebra generated (over $\mathbb{C}$) by 3 elements X, Y and Z (in degrees w_1, w_2 and 1 respectively) subject to the defining relations

$$\begin{aligned} XZ - ZX &= 0\,, \\ YZ - ZY &= 0\,, \\ XY - YX &= Z^{|\boldsymbol{w}|}\,, \end{aligned} \tag{3.8}$$

where $|\boldsymbol{w}| := w_1 + w_2$. We call this algebra the *homogenized Weyl algebra of weight* $\boldsymbol{w}$ and denote it by $\boldsymbol{A}(\boldsymbol{w})$ (or simply by $\boldsymbol{A}$ when there is no danger of confusion).

The following two propositions collect some basic properties of $\boldsymbol{A}(\boldsymbol{w})$.

Proposition 3.1. *For every positive weight vector, the algebras $\boldsymbol{GA}(\boldsymbol{w})$ and $\boldsymbol{A}(\boldsymbol{w})$ are Noetherian Artin-Schelter algebras of global dimensions 2 and 3 respectively. The corresponding Gorenstein parameters are $|\boldsymbol{w}|$ and $|\boldsymbol{w}| + 1$.*

Proof. We have

$$\boldsymbol{GA}(\boldsymbol{w}) \cong \boldsymbol{A}(\boldsymbol{w})/\langle Z\rangle \cong \boldsymbol{S}(\boldsymbol{w})\,, \tag{3.9}$$

where $\boldsymbol{S}(\boldsymbol{w}) := \mathbb{C}[\bar{x}, \bar{y}]$ is the graded *commutative* polynomial ring in two variables of weight $\boldsymbol{w}$ (the first isomorphism in (3.9) is just (3.4), while the second follows immediately from the defining relations (3.8).) Hence $\boldsymbol{GA}(\boldsymbol{w})$ has the properties stated in the lemma. Since $\boldsymbol{GA}(\boldsymbol{w})$ is Noetherian and has global dimension 2, $\boldsymbol{A}(\boldsymbol{w})$ is also Noetherian and has global dimension 3 (see [Lev], Proposition 3.5 and [LO], Theorem II.8.2 respectively). That $\boldsymbol{A}$ is Artin-Schelter follows from [Lev], Theorem 5.10 and Theorem 6.3. According to [ATV], Proposition 2.14, the Gorenstein parameter l is equal to the degree of the inverse of the Poincaré series $P_{\boldsymbol{A}}(s) := \sum_{k\ge 0} \dim_{\mathbb{C}} A_k(\boldsymbol{w})\, s^k$. Since $\boldsymbol{A}$ is isomorphic (as a graded vector space) to the commutative polynomial ring in three variables of weights $(w_1, w_2, 1)$, we have

$$P_{\boldsymbol{A}}(s) = \frac{1}{(1-s^{w_1})(1-s^{w_2})(1-s)}\,,$$

and therefore the Gorenstein parameter is $w_1 + w_2 + 1$. □

Proposition 3.2. *For every positive weight vector, $\boldsymbol{A}(\boldsymbol{w})$ is a schematic algebra of schematic dimension 1.*

Proof. This is proved in [W] in the case $\boldsymbol{w} = (1,1)$; more precisely, it is shown in [W] that $\boldsymbol{A}$ can be covered by the two Ore sets consisting of the powers of X and of Y. The proof in general is similar, but we shall work with a "finer" covering, that is, with larger Ore sets. This covering will be needed in Section 6.

For an element $q(x,y) \in A$ we denote by $\boldsymbol{q}(X,Y,Z) \in \boldsymbol{A}$ its homogenization in $\boldsymbol{A}$, that is, we set $\boldsymbol{q}(X,Y,Z) = Z^d q(X/Z^{w_1}, Y/Z^{w_2})$, where $d = \deg_{\boldsymbol{w}}(q)$. Now define

$$
\begin{aligned}
U_1 &:= \{\, \boldsymbol{q}_1(X,Z) \in \boldsymbol{A} \,|\, q_1(x) \in \mathbb{C}[x] \setminus \{0\} \,\} \,, \\
U_2 &:= \{\, \boldsymbol{q}_2(Y,Z) \in \boldsymbol{A} \,|\, q_2(y) \in \mathbb{C}[y] \setminus \{0\} \,\} \,.
\end{aligned} \tag{3.10}
$$

We claim that $\mathfrak{U} := \{U_1, U_2\}$ is a covering of $\boldsymbol{A}$. Clearly, U_1 and U_2 are multiplicatively closed subsets of $\boldsymbol{A}$; and they satisfy the Ore condition (on both sides) because they consist of (homogeneous) locally ad-nilpotent elements in $\boldsymbol{A}$. On the other hand, for any (nonzero) polynomials $q_1(x)$ and $q_2(y)$, the ideal $q_1(x)A + q_2(y)A$ has finite codimension in A and hence coincides with A. It follows that $1 = q_1(x)a + q_2(y)b$ for some $a, b \in A$, and therefore $Z^r \in \boldsymbol{q}_1\boldsymbol{A} + \boldsymbol{q}_2\boldsymbol{A}$ for some $r > 0$. If either $q_1 \in \mathbb{C}$ or $q_2 \in \mathbb{C}$, the condition (2.7) holds trivially. So we may assume that q_1 and q_2 both have positive degree, say $\deg_{\boldsymbol{w}}(q_1) = w_1 d_1 > 0$ and $\deg_{\boldsymbol{w}}(q_2) = w_2 d_2 > 0$. Then $\boldsymbol{q}_1(X,Z) - X^{d_1} \in Z\boldsymbol{A}$ and $\boldsymbol{q}_2(Y,Z) - Y^{d_2} \in Z\boldsymbol{A}$, hence $X^{kd_1} \in \boldsymbol{q}_1\boldsymbol{A} + Z^k\boldsymbol{A}$ and $Y^{kd_2} \in \boldsymbol{q}_2\boldsymbol{A} + Z^k\boldsymbol{A}$ for any $k \geq 1$. Taking $k = r$, we find that there is an m such that

$$\boldsymbol{A}_{\geq m} \subseteq X^{rd_1}\boldsymbol{A} + Y^{rd_2}\boldsymbol{A} + Z^r\boldsymbol{A} \subseteq \boldsymbol{q}_1\boldsymbol{A} + \boldsymbol{q}_2\boldsymbol{A} \,,$$

which shows that the pair of Ore sets $\{U_1, U_2\}$ covers the algebra $\boldsymbol{A}$, as required. □

Remark. Unlike global or Gel'fand-Kirillov dimension, the schematic dimension distinguishes $\boldsymbol{A}$ from the commutative polynomial algebra $\mathbb{C}[X,Y,Z]$.

3.3. **Weighted Projective Planes.** Given a weight vector $\boldsymbol{w}$, we write

$$\mathbb{P}^2_q(\boldsymbol{w}) := \mathsf{proj}\, \boldsymbol{A}(\boldsymbol{w}) \quad , \quad \mathbb{P}^1(\boldsymbol{w}) := \mathsf{proj}\, \boldsymbol{GA}(\boldsymbol{w})$$

for the (hypothetical) projective schemes associated to $\boldsymbol{A}$ and $\boldsymbol{GA}$. The identification $\boldsymbol{GA} \cong \boldsymbol{A}/\langle Z \rangle$ provides a natural epimorphism of graded algebras

$$\boldsymbol{i} : \boldsymbol{A} \to \boldsymbol{GA} \,. \tag{3.11}$$

As usual, we have the functors $\boldsymbol{i}_*$ and $\boldsymbol{i}^*$ of restriction and extension of scalars (if $\boldsymbol{M}$ is a (right graded) $\boldsymbol{GA}$-module, then $\boldsymbol{i}_*\boldsymbol{M}$ is the same vector space $\boldsymbol{M}$ with $\boldsymbol{A}$-module structure defined via (3.11), while if $\boldsymbol{M}$ is an $\boldsymbol{A}$-module, $\boldsymbol{i}^*\boldsymbol{M}$ is the $\boldsymbol{GA}$-module $\boldsymbol{M} \underline{\otimes}_{\boldsymbol{A}} \boldsymbol{GA}$). These functors both preserve the classes of finitely generated and of τorsion modules, and hence descend to functors i_* and i^* on the categories of coherent (or quasicoherent) sheaves over $\mathbb{P}^2_q$ and $\mathbb{P}^1$. We shall call $\mathbb{P}^1$ the *line at infinity* in $\mathbb{P}^2_q$ and sometimes denote it by l_∞. If $\mathcal{M}$ is a coherent sheaf over $\mathbb{P}^2_q$, we call $i^*\mathcal{M}$ the *restriction of* $\mathcal{M}$ *to the line at infinity.*

For future use we record the simple

Lemma 3.3. *For any* $\mathcal{M} \in \mathsf{coh}(\mathbb{P}^2_q)$ *there is an exact sequence*

$$\mathcal{M}(-1) \to \mathcal{M} \to i_* i^* \mathcal{M} \to 0 \,,$$

where the first map is induced by multiplication by $Z \in \boldsymbol{A}(\boldsymbol{w})$.

Proof. If $\boldsymbol{M}$ is a graded $\boldsymbol{A}$-module with $\mathcal{M} = \pi\boldsymbol{M}$, it follows at once from (3.6) that the graded quotient $\boldsymbol{M}/\boldsymbol{M}Z$ is canonically isomorphic to $\boldsymbol{i}_*\boldsymbol{i}^*\boldsymbol{M}$. Applying the (exact) functor π to

$$\boldsymbol{M}(-1) \xrightarrow{\cdot Z} \boldsymbol{M} \to \boldsymbol{i}_*\boldsymbol{i}^*\boldsymbol{M} \to 0$$

we get the lemma. □

Remark. Nearly all the results in this paper remain true if we replace $\boldsymbol{A}(\boldsymbol{w})$ by the *commutative* graded algebra $\boldsymbol{A}_0 := \mathbb{C}[X,Y,Z]$ with weights $(w_1, w_2, 1)$. However, except when $w_1 = w_2 = 1$, we do not have a Serre equivalence between the category of coherent sheaves (in the usual sense) over $\mathsf{proj}(\boldsymbol{A}_0)$ and the category $\mathsf{tails}(\boldsymbol{A}_0)$. Our results would refer to the latter category, and so (probably) would not give much information about the usual "weighted projective spaces" studied in (for example) [D], [Do] and [BR]. For a similar reason, our $\mathbb{P}^2_q(\boldsymbol{w})$ are different from the quantum weighted projective planes introduced recently in [Ste1].

4. The Linear Data Associated to an Ideal

Let M be a finitely generated torsion-free rank one right module over the Weyl algebra A. We fix a (positive) filtration of weight $\boldsymbol{w}$ on A. We also fix, temporarily, an embedding of M as an ideal in A; then we have the *induced filtration* $M_k = M \cap A_k$ on M. It is easy to see that, up to an overall shift, the filtration is independent of the choice of embedding: our first task is to normalize this overall shift. Since $M \subseteq A$, the corresponding Rees module $\boldsymbol{M}$ (see (3.5)) is a graded ideal in $\boldsymbol{A} \equiv \boldsymbol{A}(\boldsymbol{w})$. Let $\mathcal{M} = \pi \boldsymbol{M}$ be the associated sheaf over $\mathbb{P}^2_q(\boldsymbol{w})$.

Lemma 4.1. *There is a unique $a \in \mathbb{Z}$ such that the restriction of $\mathcal{M}(a)$ to the line at infinity in $\mathbb{P}^2_q$ is trivial.*

Proof. We have $\boldsymbol{i}^* \boldsymbol{M} = \boldsymbol{GM}$ (see (3.6)). The embedding of $\boldsymbol{M}$ in $\boldsymbol{A}$ induces an embedding of $\boldsymbol{GM}$ in $\boldsymbol{GA}$ as a homogeneous ideal. Now, $\boldsymbol{GA}$ is just a commutative polynomial algebra in two variables; hence if f is the greatest common divisor of the elements of $\boldsymbol{GM}$, then $f^{-1}\boldsymbol{GM}$ is a (homogeneous) ideal of finite codimension in $\boldsymbol{GA}$. Denoting by a the degree of f in $\boldsymbol{GA}$, we therefore have an exact sequence of graded $\boldsymbol{GA}$-modules

$$0 \to \boldsymbol{GM}(a) \to \boldsymbol{GA} \to \boldsymbol{GA}/\boldsymbol{GM}(a) \to 0 \tag{4.1}$$

with finite-dimensional quotient term. The quotient functor π annihilates finite-dimensional modules, so applying π to (4.1), we get the desired isomorphism

$$i^* \mathcal{M}(a) \cong \mathcal{O}_{\mathbb{P}^1} \quad \text{in } \mathsf{coh}()P^1 \, .$$

The uniqueness of a follows from the fact that $\mathcal{O}_{\mathbb{P}^1}(k) \cong \mathcal{O}_{\mathbb{P}^1}$ in $\mathsf{coh}(\mathbb{P}^1)$ only if $k = 0$. Indeed, assuming the contrary, by (2.3) we have $\boldsymbol{GA}(k)_{\geq N} \cong \boldsymbol{GA}_{\geq N}$ for some N, and therefore $\dim_{\mathbb{C}} \boldsymbol{GA}_{n+k} = \dim_{\mathbb{C}} \boldsymbol{GA}_n$ for all $n \geq N$. This implies that the sequence of numbers $(\dim_{\mathbb{C}} \boldsymbol{GA}_n)$ is bounded, which is obviously not the case. □

Lemma 4.2. *Let δ be the minimum filtration degree of elements of M. Then $\delta \geq a$; if M is not cyclic then $\delta > a$.*

Proof. As in the proof of the preceding Lemma, we identify $\boldsymbol{GM}$ with an ideal in the polynomial ring $\boldsymbol{GA}$; then δ is the minimum degree of elements in $\boldsymbol{GM}$, and hence $\delta \geq a$. If $\delta = a$, then $\boldsymbol{GM}$ is cyclic (generated by the greatest common divisor f above), and hence M is also cyclic. □

Proposition 4.3. *The natural map $\boldsymbol{M} \to \underline{H}^0(\mathbb{P}^2_q, \mathcal{M})$ in (2.6) is bijective.*

Proof. It is obvious that $\boldsymbol{\tau M} = 0$, so we have only to prove that the Ext term in (2.6) is zero. Let $\boldsymbol{N} := \boldsymbol{A}/\boldsymbol{M}$. We show first that

$$\varinjlim \underline{\mathrm{Ext}}^1_{\boldsymbol{A}}(\boldsymbol{A}/\boldsymbol{A}_{\geq n}, \boldsymbol{M}) \cong \boldsymbol{\tau N}. \tag{4.2}$$

For brevity, set $\boldsymbol{A}_{<n} := \boldsymbol{A}/\boldsymbol{A}_{\geq n}$. Applying the functor $\underline{\mathrm{Hom}}_{\boldsymbol{A}}(\boldsymbol{A}_{<n}, \text{—})$ to $0 \to \boldsymbol{M} \to \boldsymbol{A} \to \boldsymbol{N} \to 0$, we get the exact sequence

$$\underline{\mathrm{Hom}}_{\boldsymbol{A}}(\boldsymbol{A}_{<n}, \boldsymbol{A}) \to \underline{\mathrm{Hom}}_{\boldsymbol{A}}(\boldsymbol{A}_{<n}, \boldsymbol{N}) \to \underline{\mathrm{Ext}}^1_{\boldsymbol{A}}(\boldsymbol{A}_{<n}, \boldsymbol{M}) \to \underline{\mathrm{Ext}}^1_{\boldsymbol{A}}(\boldsymbol{A}_{<n}, \boldsymbol{A}) \ .$$

The first term in this sequence is obviously zero; and the Gorenstein property (see Definition 2.2) of $\boldsymbol{A}$ implies that the last term is also zero, because $\boldsymbol{A}_{<n}$ has finite length. Thus $\underline{\mathrm{Hom}}_{\boldsymbol{A}}(\boldsymbol{A}_{<n}, \boldsymbol{N}) \cong \underline{\mathrm{Ext}}^1_{\boldsymbol{A}}(\boldsymbol{A}_{<n}, \boldsymbol{M})$ for all n. Passing to the limit as $n \to \infty$, we get (4.2). Hence Proposition 4.3 follows if we show that $\boldsymbol{\tau N} = 0$. Suppose that $a \in A_k$ represents a τorsion element in $\boldsymbol{N}$. This means that for some $n \geq 0$ we have $a\boldsymbol{A}_{\geq n} \subset \boldsymbol{M}$, and hence $aA_n \subset M_{n+k}$. Since $1 \in A_n$, we find that $a \in M_{n+k} \cap A_k = M_k$, and hence a represents zero in $\boldsymbol{N}$. Thus $\boldsymbol{\tau N} = 0$. □

We use Lemma 4.1 to fix the ambiguous shift in the induced filtration. If M is an (embedded) ideal, then the uniqueness of the number a in Lemma 4.1 shows that the filtration $M_k^\circ := M_{k+a}$ on M is independent of the choice of embedding. We shall call this filtration the *normalized* induced filtration on M. From Lemma 4.2, we get

Lemma 4.4. *Let M be an ideal of A with the normalized induced filtration, and let d be the minimum filtration degree of elements of M. Then $d \geq 0$, and if M is not cyclic, then $d > 0$.*

From now on, changing notation, $\mathcal{M}$ will always denote the extension of M to $\mathbb{P}^2_q(\boldsymbol{w})$ determined by the normalized induced filtration (so that $\mathcal{M}|_{l_\infty}$ is trivial). We call $\mathcal{M}$ the *canonical extension* of M. The next Theorem gathers together the information we need about the cohomology of $\mathcal{M}$.

Theorem 4.5. *Let $\mathcal{M}$ be the canonical extension of an ideal of A. Then*

(i) The map $H^1(\mathbb{P}^2_q, \mathcal{M}(k-1)) \to H^1(\mathbb{P}^2_q, \mathcal{M}(k))$ induced by multiplication by Z is injective for $k < 0$ and surjective for $k > -|\boldsymbol{w}|$.

(ii) We have

$$H^0(\mathbb{P}^2_q, \mathcal{M}(k)) = 0 \quad \textit{for } k < 0 \ ,$$
$$H^2(\mathbb{P}^2_q, \mathcal{M}(k)) = 0 \quad \textit{for } k \geq -|\boldsymbol{w}| \ .$$

(iii) Furthermore, if M is not cyclic, we have also $H^0(\mathbb{P}^2_q, \mathcal{M}) = 0$, and

$$\dim_{\mathbb{C}} H^1(\mathbb{P}^2_q, \mathcal{M}) = \dim_{\mathbb{C}} H^1(\mathbb{P}^2_q, \mathcal{M}(-1)) - 1 \ .$$

Proof. The map of sheaves $\mathcal{M}(k-1) \to \mathcal{M}(k)$ induced by multiplication by Z is clearly injective (for any $k \in \mathbb{Z}$). Using Lemma 3.3 and bearing in mind that $i^*\mathcal{M} \cong \mathcal{O}_{\mathbb{P}^1}$ we get the short exact sequence

$$0 \to \mathcal{M}(k-1) \to \mathcal{M}(k) \to i_*\mathcal{O}_{\mathbb{P}^1}(k) \to 0 \ . \tag{4.3}$$

By Theorem 8.3 of [AZ], we have

$$H^i(\mathbb{P}^2_q, \, i_*\mathcal{O}_{\mathbb{P}^1}(k)) \cong H^i(\mathbb{P}^1, \, \mathcal{O}_{\mathbb{P}^1}(k)) \quad \text{for all} \quad i \geq 0 \ .$$

On the other hand, by Theorem 2.3 and Proposition 3.1, we have

$$H^i(\mathbb{P}^1,\, \mathcal{O}_{\mathbb{P}^1}(k)) = \begin{cases} S_k & \text{when} \quad i=0 \\ (S_{-k-|\boldsymbol{w}|})^* & \text{when} \quad i=1 \\ 0 & \text{when} \quad i>1 \end{cases} \tag{4.4}$$

where S_k is the k-th graded component of the commutative polynomial algebra $\boldsymbol{S}(\boldsymbol{w}) = \mathbb{C}[\bar{x}, \bar{y}]$ of weight $\boldsymbol{w}$. Therefore the first and last terms of the exact sequence

$$H^0(\mathbb{P}^2_q, i_*\mathcal{O}_{\mathbb{P}^1}(k)) \to H^1(\mathbb{P}^2_q, \mathcal{M}(k-1)) \to H^1(\mathbb{P}^2_q, \mathcal{M}(k)) \to H^1(\mathbb{P}^2_q, i_*\mathcal{O}_{\mathbb{P}^1}(k))$$

coming from (4.3) are isomorphic to S_k and $(S_{-k-|\boldsymbol{w}|})^*$ respectively. Since the grading on $\boldsymbol{S}(\boldsymbol{w})$ is positive, part (i) of the Theorem follows.

The assertion about H^0 in part (ii) is immediate in view of Proposition 4.3 and Lemma 4.4. To prove the assertion about H^2 we observe (again looking at the long cohomology exact sequence of (4.3)) that the map

$$H^2(\mathbb{P}^2_q, \mathcal{M}(k-1)) \to H^2(\mathbb{P}^2_q, \mathcal{M}(k))$$

is an isomorphism for $k-1 \geq -|\boldsymbol{w}|$. By the Vanishing Theorem 2.4(b), $H^2(\mathbb{P}^2_q, \mathcal{M}(k))$ is zero for $k \gg 0$, hence it is zero for all $k \geq -|\boldsymbol{w}|$.

It remains to prove part (iii) of the Theorem. The fact that $H^0(\mathbb{P}^2_q, \mathcal{M}) = 0$ again follows from Lemma 4.4 and Proposition 4.3. From (4.3) (with $k=0$) we get the exact sequence

$$0 = H^0(\mathbb{P}^2_q, \mathcal{M}) \to \mathbb{C} \to H^1(\mathbb{P}^2_q, \mathcal{M}(-1)) \to H^1(\mathbb{P}^2_q, \mathcal{M}) \to 0\ ,$$

whence the last statement in the Theorem. □

Now, as in the Introduction, let $V(\boldsymbol{w}) := H^1(\mathbb{P}^2_q,\, \mathcal{M}(-1))$. It follows from Theorem 4.5 that multiplication by Z defines *isomorphisms*

$$H^1(\mathbb{P}^2_q,\, \mathcal{M}(-\boldsymbol{w})) \cong H^1(\mathbb{P}^2_q,\, \mathcal{M}(-\boldsymbol{w}+1)) \cong\ \ldots\ \cong H^1(\mathbb{P}^2_q,\, \mathcal{M}(-2)) \cong V(\boldsymbol{w})\ .$$

We identify these spaces, and let $\mathbb{X}$ and $\mathbb{Y}$ be the endomorphisms of $V(\boldsymbol{w})$ induced by (right) multiplication by X and Y. More precisely, if $v \in V(\boldsymbol{w})$, we define

$$\mathbb{X}(v) := v \cdot Z^{-w_1} X \quad , \quad \mathbb{Y}(v) := v \cdot Z^{-w_2} Y\ .$$

Let $n := \dim_{\mathbb{C}} V(\boldsymbol{w})$. By Theorem 4.5, we have $n=0$ if and only if M is cyclic.

Proposition 4.6. *The pair* $(\mathbb{X}, \mathbb{Y})$ *defines a point in the space* $\mathfrak{C}_n$.

Proof. The Proposition is trivial if $n=0$. In general, we calculate:

$$\mathbb{X}\mathbb{Y}(v) \cdot Z = v \cdot Z^{-w_2} Y Z^{-w_1} X Z = v \cdot Z^{-|\boldsymbol{w}|+1} Y X$$

(we used the facts that Z commutes with X and Y, and that $Z^{-|\boldsymbol{w}|+1}$ is still well defined on $V(\boldsymbol{w})$). Similarly, $\mathbb{Y}\mathbb{X}(v) \cdot Z = v \cdot Z^{-|\boldsymbol{w}|+1} XY$. So

$$([\mathbb{X}, \mathbb{Y}] + \mathbb{I})\, v \cdot Z = v \cdot Z^{-|\boldsymbol{w}|+1}(YX - XY + Z^{|\boldsymbol{w}|}) = 0\ .$$

Thus the image of $[\mathbb{X}, \mathbb{Y}] + \mathbb{I}$ is contained in the kernel of $\cdot Z\, : V(\boldsymbol{w}) \to H^1(\mathbb{P}^2_q,\, \mathcal{M})$. By Theorem 4.5, this map is surjective with one-dimensional kernel. Therefore $[\mathbb{X}, \mathbb{Y}] + \mathbb{I}$ has rank 1, as required. □

5. Elementary Constructions

5.1. **Distinguished Representatives.** As usual, let M be a finitely generated rank one torsion-free right A-module. We are going to construct two distinguished realizations of M as fractional ideals of A (that is, submodules of the quotient field Q of A). First, according to [St], Lemma 4.2, we can choose an embedding of M as an ideal which has nonzero intersection with $\mathbb{C}[x] \subset A$. If an element of A (or, later, of the larger algebra $\mathbb{C}(x)[y]$) is written in the form $a = \sum_{i=0}^{n} a_i(x) y^i$ (with $a_n \neq 0$), we call $a_n(x)$ the *leading coefficient* of a. The leading coefficients of all the elements of M form an ideal in $\mathbb{C}[x]$; let $p(x)$ be the (monic) generator of this ideal, and set $M_x := p(x)^{-1} M$. By construction, the fractional ideal M_x has the following properties:

1. $M_x \subset \mathbb{C}(x)[y] \subset Q$ and $M_x \cap \mathbb{C}[x] \neq \{0\}$;
2. all leading coefficients of elements of M_x belong to $\mathbb{C}[x]$;
3. M_x contains an element with constant leading coefficient.

It is easy to see that these properties characterize M_x. More precisely, we have

Lemma 5.1. *Let M_x and M'_x be two fractional ideals of A, both isomorphic to M, and satisfying (1)-(3) above. Let q be an element of Q such that $M'_x = qM_x$. Then q is a constant (and hence $M_x = M'_x$).*

We denote by $\varrho_x : \mathbb{C}(x)[y] \to A$ the map that deletes the "negative part" of the coefficients $a_i(x)$. More precisely, if $a(x)$ is rational function of x, let $a = a_+ + a_-$, where a_+ is a polynomial and a_- vanishes at infinity; then we define

$$\varrho_x \left(\sum a_i(x) y^i \right) := \sum a_i(x)_+ \, y^i \ .$$

We denote by r_x the restriction of ϱ_x to M_x. Then r_x is injective, and $r_x(M_x)$ is a linear subspace of finite codimension in A. Let $V_x := A/r_x(M_x)$. The map ϱ_x commutes with right multiplication by y (though not with right multiplication by x); thus $\cdot y$ induces an endomorphism of V_x. We denote it by $\mathbb{Y}$.

Reversing the roles of x and y in the above construction, we obtain another distinguished representative $M_y \subset \mathbb{C}(y)[x] \subset Q$ for our ideal M, and another finite-dimensional vector space $V_y := A/r_y(M_y)$, together with an endomorphism $\mathbb{X}$ of V_y coming from right multiplication by x. Since M_x and M_y are both isomorphic to M, we have $M_y = \kappa M_x$ for some $\kappa \in Q$; by Lemma 5.1, κ is uniquely determined up to a constant factor. Note that the properties of M_x and M_y imply

$$\kappa \in \mathbb{C}(y)(x) \quad \text{and} \quad \kappa^{-1} \in \mathbb{C}(x)(y) \ . \tag{5.1}$$

Here $\mathbb{C}(x)(y)$ (for example) denotes the space of all elements of Q that have the form $\sum f_i(x)\, g_i(y)$ for some rational functions $f_i(x)$, $g_i(y)$.

Next, we describe the linear isomorphism $\phi : V_x \to V_y$ mentioned in the Introduction. In the next section we shall see how this isomorphism arises naturally from a calculation of Čech cohomology. Let $\Phi : r_x(M_x) \to r_y(M_y)$ be the isomorphism defined by

$$\Phi(m) := r_y \left(\kappa \cdot r_x^{-1}(m) \right)$$

(the dot denotes multiplication in Q). We shall extend Φ to a linear isomorphism (also denoted by Φ) from A to itself: ϕ will then be the induced map on the quotient spaces. The extension of Φ to A is defined as follows. Note first that for any $a \in A$ there are polynomials $g(y)$ such that $a\, g(y) \in r_x(M_x)$ (for example, we can take g to be the characteristic polynomial of the map $\mathbb{Y}$ above). For each $a \in A$, we choose such a polynomial $g(y)$, and set

$$\Phi(a) := \varrho_y \left(\kappa \cdot r_x^{-1}[\, a\, g(y)\,] \cdot g(y)^{-1} \right) \ . \tag{5.2}$$

Using the $\mathbb{C}[y]$-linearity of the map ϱ_x, it is easy to check that $\Phi(a)$ is independent of the choice of g; in particular, if $a \in r_x(M_x)$ we can choose $g = 1$, so Φ is indeed an extension of the map that we started with. The reader may like to prove at this point that Φ and ϕ are isomorphisms (with inverses defined in a similar way, interchanging x and y). This will follow from the results of the next section, so we omit the proof here.

5.2. **An Example.** For $n \geq 1$, let $M = x^{n+1}A + (xy+n)A$. In this case the spaces V_x and V_y have dimension n, and (with suitable choice of basis) the matrices $\mathbb{X}$ and $\mathbb{Y}$ are

$$\mathbb{X} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \quad , \quad \mathbb{Y} = \begin{pmatrix} 0 & 1-n & 0 & \dots & 0 \\ 0 & 0 & 2-n & \dots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \ .$$

Although very elementary, the calculation is not short enough to reproduce in full here. We just indicate the main steps, leaving some details for the reader. First, we have $M_x = x^{-1}M$; as a basis for V_x we can take the residue classes (modulo $r_x(M_x)$) of the elements $1, x, \dots, x^{n-1}$. Since $x^k y + (n-k)x^{k-1} \in r_x(M_x)$ for $1 \leq k \leq n-1$, and $y \in r_x(M_x)$, it follows that the matrix $\mathbb{Y}$ is as above. The calculation of $\mathbb{X}$ is a little harder; however, using formula (5.2), it is straightforward to check that $\Phi(x^k) = x^k$ for $0 \leq k \leq n-1$. So we can again choose the residue classes of $1, x, \dots, x^{n-1}$ (now modulo $r_y(M_y)$) as a basis for V_y, and the matrix of ϕ is then the identity. Since $x^n \in r_y(M_y)$, it follows that the matrix $\mathbb{X}$ is as above. In this example we have

$$\kappa = (xy)^{-n} y\,(xy+1)\,(xy+2)\,\dots\,(xy+n-1)\,x \quad \text{and} \quad \kappa^{-1} = 1 + n\,x^{-1}y^{-1} \ .$$

5.3. **The Associated Graded Ideals.** Our last goal in this section is to establish an important property of the element κ which we shall need later, namely, that multiplication by κ preserves the $\boldsymbol{w}$-filtration on A for every weight vector $\boldsymbol{w} = (w_1, w_2)$. Slightly more generally than in Section 3.2, we shall allow w_1 and w_2 to be any non-negative integers[5] that are not both zero. We denote by $\mathfrak{v}_{\boldsymbol{w}}$ the valuation on A corresponding to $\boldsymbol{w}$, that is, if $a \neq 0$ then $\mathfrak{v}_{\boldsymbol{w}}(a)$ is the least integer k such that $a \in A_k(\boldsymbol{w})$ (and $\mathfrak{v}_{\boldsymbol{w}}(0) := -\infty$). We extend $\mathfrak{v}_{\boldsymbol{w}}$ to Q by setting

$$\mathfrak{v}_{\boldsymbol{w}}(ab^{-1}) = \mathfrak{v}_{\boldsymbol{w}}(a) - \mathfrak{v}_{\boldsymbol{w}}(b) \ ,$$

and let $Q_k(\boldsymbol{w}) := \{\, q \in Q \mid \mathfrak{v}_{\boldsymbol{w}}(q) \leq k \,\}$. Then $\{Q_\bullet(\boldsymbol{w})\}$ is a separated filtration on Q extending the original filtration on A. Changing notation slightly from Section 3.2, we denote the associated graded algebra by $\boldsymbol{G_w Q}$ and write

$$\sigma_{\boldsymbol{w}} \, : \, Q \to \boldsymbol{G_w Q}$$

[5]That is, we now allow one of w_i to be zero. More generally still, we could work with non-integer (real) "weights" as in [Di], [LM].

for the *symbol map*: if $\mathfrak{v}_{\boldsymbol{w}}(q) = k$, then $\sigma_{\boldsymbol{w}}(q)$ is the class of q in $Q_k(\boldsymbol{w})/Q_{k-1}(\boldsymbol{w})$. As usual, we identify $\boldsymbol{G}_{\boldsymbol{w}}\boldsymbol{A} \equiv \boldsymbol{G}\boldsymbol{A}(\boldsymbol{w})$ with the polynomial algebra $\mathbb{C}[\bar{x}, \bar{y}]$, where $\bar{x} := \sigma_{\boldsymbol{w}}(x)$ and $\bar{y} := \sigma_{\boldsymbol{w}}(y)$; then $\boldsymbol{G}_{\boldsymbol{w}}\boldsymbol{Q}$ is identified with the subalgebra of $\mathbb{C}(\bar{x}, \bar{y})$ spanned by quotients of homogeneous polynomials.

For short, we write σ_y and $\boldsymbol{G}_y$ instead of $\sigma_{(0,1)}$ and $\boldsymbol{G}_{(0,1)}$. Property (2) of $M_x = p(x)^{-1}M$ says that $\boldsymbol{G}_y M_x \subseteq \mathbb{C}[\bar{x}, \bar{y}]$ (even though $M_x \not\subset A$). More precisely, if $a_n(x)$ is the leading coefficient of $a \in M$ then $\sigma_y(a) = a_n(\bar{x})\bar{y}^n$. It follows that $p(\bar{x})$ is the greatest common divisor of the elements of $\boldsymbol{G}_y M$ so that $\boldsymbol{G}_y M_x = p(\bar{x})^{-1}\boldsymbol{G}_y M$ is an ideal of finite codimension[6] in $\mathbb{C}[\bar{x}, \bar{y}]$. More generally, we have

Proposition 5.2. *Let* $\boldsymbol{w}$ *be any weight vector (as specified above). Then*
(*i*) $\boldsymbol{G}_{\boldsymbol{w}} M_x$ *and* $\boldsymbol{G}_{\boldsymbol{w}} M_y$ *are ideals of finite codimension in* $\mathbb{C}[\bar{x}, \bar{y}]$*;*
(*ii*) $\boldsymbol{G}_{\boldsymbol{w}} M_x = \boldsymbol{G}_{\boldsymbol{w}} M_y$ *in* $\mathbb{C}[\bar{x}, \bar{y}]$*;*
(*iii*) *the symbol* $\sigma_{\boldsymbol{w}}(\kappa)$ *is constant.*

Proof. An argument of Letzter and Makar-Limanov (see [LM], Lemma 2.1) shows that $\boldsymbol{G}_{\boldsymbol{w}} M_x$ is contained in $\mathbb{C}[\bar{x}, \bar{y}]$. Proposition 2.4′ of [LM] then shows that it has finite codimension (and also that this codimension is independent of $\boldsymbol{w}$). Interchanging the roles of x and y, we obtain the same result for $\boldsymbol{G}_{\boldsymbol{w}} M_y$.

Since M_x and M_y are both isomorphic to M, the ideals $\boldsymbol{G}_{\boldsymbol{w}} M_x$ and $\boldsymbol{G}_{\boldsymbol{w}} M_y$ are isomorphic. An ideal class of $\mathbb{C}[\bar{x}, \bar{y}]$ has a *unique* representative of finite codimension, so (*ii*) follows from (*i*).

Since $M_y = \kappa M_x$, we have $\boldsymbol{G}_{\boldsymbol{w}} M_y = \sigma_{\boldsymbol{w}}(\kappa)\boldsymbol{G}_{\boldsymbol{w}} M_x$ (the symbol map is multiplicative). So (*iii*) follows from (*ii*). □

Corollary 5.3. *For any positive weight vector* $\boldsymbol{w}$*, the filtration induced on* M_x *(or* M_y *) by the* $\boldsymbol{w}$*-filtration on* Q *coincides with the normalized induced filtration of Section 4.*

Proof. This is a reformulation of Proposition 5.2(*i*) (cf. the proof of Lemma 4.1 above). □

Remark. If we compare the formula $M_y = \kappa M_x$ with Proposition 6.2 in [BW], we see that κ can be identified with the formal integral operator K that plays a basic role in the theory of integrable systems; more precisely, if W is the point of $\mathrm{Gr}^{\mathrm{ad}}$ that corresponds to the ideal M then $\kappa = K_{b(W)}$, where b is the bispectral involution on $\mathrm{Gr}^{\mathrm{ad}}$. We shall not make any use of this remark in the present paper; however, it points the way to a more direct proof of Theorem 1.2.

6. The Comparison Theorem

As usual, let $\mathcal{M}$ be the canonical extension of a noncyclic ideal M of A. Our aim in this section is to calculate the groups $H^1(\mathbb{P}^2_q(\boldsymbol{w}), \mathcal{M}(k))$ (for $-|\boldsymbol{w}| \le k \le -1$) using the Čech complex of the covering $\mathfrak{U}$ introduced in Section 3.2: this will enable us to identify these groups with the spaces V_x and V_y in Section 5. Although it would, of course, be possible to calculate $\underline{\check{H}}^1(\mathfrak{U}, \mathcal{M})$ directly from the complex (2.9), this does not appear to yield the answer in the form we want. Instead, we choose a large integer p (eventually we shall let $p \to \infty$), and denote by $\mathcal{N}_p$ the

[6] equal to the codimension of $r_x(M_x)$ in A.

restriction of $\mathcal{M}$ to the p-th infinitesimal neighbourhood of the line at infinity in $\mathbb{P}^2_q$; that is, $\mathcal{N}_p$ is the quotient term in the exact sequence

$$0 \to \mathcal{M} \xrightarrow{\cdot Z^p} \mathcal{M}(p) \to \mathcal{N}_p \to 0 \ . \tag{6.1}$$

In what follows we shall mostly omit the subscript p, writing $\mathcal{N} \equiv \mathcal{N}_p$. We shall assume that p is chosen so that

$$H^1(\mathbb{P}^2_q,\, \mathcal{M}(p+k)) = 0 \quad \text{for all } k \geq -|\boldsymbol{w}| \tag{6.2}$$

(this is possible by the Vanishing Theorem 2.4(b)). From (6.1), we then get the exact sequence

$$0 \to H^0(\mathbb{P}^2_q,\, \mathcal{M}(p+k)) \to H^0(\mathbb{P}^2_q,\, \mathcal{N}(k)) \to H^1(\mathbb{P}^2_q,\, \mathcal{M}(k)) \to 0 \tag{6.3}$$

for any k in the range $-|\boldsymbol{w}| \leq k \leq -1$. We are going to calculate $H^0(\mathbb{P}^2_q, \mathcal{N}(k))$ via a Čech complex, and then obtain $H^1(\mathbb{P}^2_q,\, \mathcal{M}(k))$ as the quotient term in (6.3). We first record the following fact.

Lemma 6.1. *For any* $k \geq -|\boldsymbol{w}|$, *we have*

$$\dim_{\mathbb{C}} H^0(\mathbb{P}^2_q,\, \mathcal{N}(k)) = \dim_{\mathbb{C}} A_{p+k} - \dim_{\mathbb{C}} A_k \ .$$

Proof. From (6.1) we easily find (using (6.2) and Theorem 4.5) that

$$H^1(\mathbb{P}^2_q,\, \mathcal{N}(k)) = H^2(\mathbb{P}^2_q,\, \mathcal{N}(k)) = 0 \quad \text{for all } k \geq -|\boldsymbol{w}| \ ,$$

so $\dim_{\mathbb{C}} H^0(\mathbb{P}^2_q,\, \mathcal{N}(k)) = \chi(\mathbb{P}^2_q,\, \mathcal{N}(k))$ (where χ denotes the Euler characteristic). From (6.1) again, we then get

$$\dim_{\mathbb{C}} H^0(\mathbb{P}^2_q,\, \mathcal{N}(k)) = \chi(\mathbb{P}^2_q,\, \mathcal{M}(p+k)) - \chi(\mathbb{P}^2_q,\, \mathcal{M}(k)) \ . \tag{6.4}$$

On the other hand, by (4.3) the Euler characteristics of the sheaves $\mathcal{M}(r)$ satisfy

$$\chi(\mathbb{P}^2_q,\, \mathcal{M}(r)) = \chi(\mathbb{P}^2_q,\, \mathcal{M}(r-1)) + \chi(\mathbb{P}^1,\, \mathcal{O}_{\mathbb{P}^1}(r)) \ ;$$

and by (4.4), $\chi(\mathbb{P}^1,\, \mathcal{O}_{\mathbb{P}^1}(r)) = \dim_{\mathbb{C}} S_r = \dim_{\mathbb{C}} A_r - \dim_{\mathbb{C}} A_{r-1}$ for all $r > -|\boldsymbol{w}|$. By Theorem 4.5, we have $\chi(\mathbb{P}^2_q,\, \mathcal{M}(-|\boldsymbol{w}|)) = -n$, so by induction

$$\chi(\mathbb{P}^2_q,\, \mathcal{M}(r)) = \dim_{\mathbb{C}} A_r - n \quad \text{for all } r \geq -|\boldsymbol{w}| \ . \tag{6.5}$$

Combining (6.4) and (6.5) yields the Lemma. □

For a while now we shall work with the special representative M_x for the class of M (see Section 5); to simplify the notation we drop the suffix x and denote M_x simply by M. We have the (normalized) filtration on M induced by the $\boldsymbol{w}$-filtration on Q (see Corollary 5.3). As usual, let $\boldsymbol{M} = \bigoplus_{k\in\mathbb{Z}} M_k$ be the corresponding homogenization of M, and let $\boldsymbol{N}$ denote the quotient term in the exact sequence

$$0 \to \boldsymbol{M} \xrightarrow{\cdot Z^p} \boldsymbol{M}(p) \to \boldsymbol{N} \to 0 \quad , \tag{6.6}$$

so that $\pi\boldsymbol{N} = \mathcal{N}$. Let $\mathfrak{U} = (U_1, U_2)$ be the covering of $\boldsymbol{A}$ defined by (3.10). We identify the localizations $\boldsymbol{A}_{U_1}$ and $\boldsymbol{A}_{U_2}$ with subalgebras of the homogenization $\boldsymbol{Q}$ of the Weyl quotient field Q: specifically, we have

$$\boldsymbol{A}_{U_1} = \bigoplus_{k\in\mathbb{Z}} \mathbb{C}(x)[y]_k \quad \text{and} \quad \boldsymbol{A}_{U_2} = \bigoplus_{k\in\mathbb{Z}} \mathbb{C}(y)[x]_k$$

(here and below, the subscript k refers to the filtration induced from Q). In a similar way, the embedding of M in Q allows us to identify the localizations of

$\boldsymbol{M}$ with subspaces of $\boldsymbol{Q}$ (the tensor products in Definition 2.8 then become multiplication in $\boldsymbol{Q}$). First, the conditions $M \cap \mathbb{C}[x] \neq \{0\}$, $M \subset \mathbb{C}(x)[y]$ imply that $\boldsymbol{M}_{U_1} = \boldsymbol{A}_{U_1}$. To calculate $\boldsymbol{M}_{U_2}$, we use the other distinguished representative $M_y = \kappa M$. For the corresponding homogenizations, we have $\boldsymbol{M}_y = \boldsymbol{\kappa} \boldsymbol{M}$. As above, the localization of $\boldsymbol{M}_y$ with respect to U_2 is just $\boldsymbol{A}_{U_2}$; and by Proposition 5.2(*iii*), multiplication by κ preserves the $\boldsymbol{w}$-filtration on Q, that is $\boldsymbol{\kappa} \in Q_0$. We therefore have

$$\boldsymbol{M}_{U_1} = \bigoplus_{k\in\mathbb{Z}} \mathbb{C}(x)[y]_k \quad \text{and} \quad \boldsymbol{M}_{U_2} = \bigoplus_{k\in\mathbb{Z}} \kappa^{-1}\mathbb{C}(y)[x]_k \ .$$

Finally, because localization is an exact functor, we can calculate $\boldsymbol{N}_{U_1}$ and $\boldsymbol{N}_{U_2}$ by localizing the exact sequence (6.6). The result is

$$\boldsymbol{N}_{U_1} = \bigoplus_{k\in\mathbb{Z}} \mathbb{C}(x)[y]_{p+k}/\mathbb{C}(x)[y]_k \ , \ \boldsymbol{N}_{U_2} = \bigoplus_{k\in\mathbb{Z}} \kappa^{-1}\mathbb{C}(y)[x]_{p+k}/\kappa^{-1}\mathbb{C}(y)[x]_k \ .$$

The repeated localizations $\boldsymbol{N}_{U_iU_j}$ similarly get identified with certain (easily specified) subspaces of $\bigoplus_{k\in\mathbb{Z}} Q_{p+k}/Q_k$. In what follows we shall denote elements of degree k in $\boldsymbol{N}_{U_1}$ and $\boldsymbol{N}_{U_2}$ by $\bar{n}_1$ and $\bar{n}_2$, where it is understood that $n_1 \in \mathbb{C}(x)[y]_{p+k}$, $n_2 \in \kappa^{-1}\mathbb{C}(y)[x]_{p+k}$, and that the bars denote residue classes modulo elements in Q_k.

Proposition 6.2. *With the identifications explained above,* $\check{H}^0(\mathfrak{U}, \mathcal{N}(k))$ *is the subspace of* $(\boldsymbol{N}_{U_1})_k \oplus (\boldsymbol{N}_{U_2})_k$ *consisting of all pairs* $(\bar{n}_1, \bar{n}_2)$ *such that* $n_1 - n_2 \in Q_k$. *Furthermore, the map* $M_{p+k} \cong \check{H}^0(\mathfrak{U}, \mathcal{M}(p+k)) \to \check{H}^0(\mathfrak{U}, \mathcal{N}(k))$ *coming from (6.1) sends* m *to* $(\bar{m}, \bar{m})$.

Proof. The first statement is obvious, because the coboundary map

$$d^0 \ : \ \boldsymbol{N}_{U_1} \oplus \boldsymbol{N}_{U_2} \to \boldsymbol{N}_{U_1U_1} \oplus \boldsymbol{N}_{U_1U_2} \oplus \boldsymbol{N}_{U_2U_1} \oplus \boldsymbol{N}_{U_2U_2}$$

takes $(\bar{n}_1, \bar{n}_2)$ to $(0, \bar{n}_1 - \bar{n}_2, \bar{n}_2 - \bar{n}_1, 0)$. The second statement follows from Proposition 2.7. □

Now, for each $k \in \mathbb{Z}$, define a map

$$\gamma_k \ : \ \check{H}^0(\mathfrak{U}, \mathcal{N}(k)) \to A_{p+k}/A_k \tag{6.7}$$

by setting $\gamma_k(\bar{n}_1, \bar{n}_2) := \overline{\varrho_x(n_1)}$, where $\varrho_x : \mathbb{C}(x)[y] \to A$ is as in Section 5.

Proposition 6.3. *The map* γ_k *is an isomorphism for all* $k \geq -|\boldsymbol{w}|$.

Proof. By Lemma 6.1, the two spaces in (6.7) have the same (finite) dimension if $k \geq -|\boldsymbol{w}|$, so it is enough to prove that γ_k is injective if $k \geq -|\boldsymbol{w}|$. In fact, γ_k is injective for all $k \in \mathbb{Z}$. To see that, let $(\bar{n}_1, \bar{n}_2) \in \check{H}^0(\mathfrak{U}, \mathcal{N}(k))$ and suppose $\gamma_k(\bar{n}_1, \bar{n}_2) = 0$; we have to show that $\bar{n}_1 = \bar{n}_2 = 0$. Equivalently, by Proposition 6.2, we are given $n_1 \in \mathbb{C}(x)[y]_{p+k}$ and $n_2 \in \kappa^{-1}\mathbb{C}(y)[x]_{p+k}$ such that $n_1 - n_2 \in Q_k$ and $\varrho_x(n_1) \in Q_k$; we have to show that n_1 and n_2 are in Q_k. Clearly, it is enough to show that $n_2 \in Q_k$. We extend ϱ_x to a map from $\mathbb{C}(x)(y)$ to $\mathbb{C}[x](y) = \mathbb{C}(y)[x]$ by setting (as in Section 5)

$$\varrho_x\left(\sum f_i(x) g_i(y)\right) := \sum f_i(x)_+ \, g_i(y) \ .$$

It is easy to see that ϱ_x is well defined and respects the $\boldsymbol{w}$-filtration for any $\boldsymbol{w}$. Note that $\kappa^{-1}\mathbb{C}(y)[x] \subset \mathbb{C}(x)(y)$ by (5.1), so $n_2 \in \mathbb{C}(x)(y)$. We have $n_1 - n_2 \in Q_k$, hence $\varrho_x(n_1) - \varrho_x(n_2) \in Q_k$; since we are given $\varrho_x(n_1) \in Q_k$, we get $\varrho_x(n_2) \in Q_k$.

But we claim that if $n \in \kappa^{-1}\mathbb{C}(y)[x]$ then $\mathfrak{v}_{\boldsymbol{w}}(n) = \mathfrak{v}_{\boldsymbol{w}}(\varrho_x(n))$, hence $n_2 \in Q_k$, as required. To prove the last claim, write $n = \kappa^{-1}q$ with $q \in \mathbb{C}(y)[x]$. By Proposition 5.2(iii), we may normalize κ so that $\sigma_{\boldsymbol{w}}(\kappa) = 1$. We then have $n = q + q'$, where $\mathfrak{v}_{\boldsymbol{w}}(q') < \mathfrak{v}_{\boldsymbol{w}}(q)$; hence $\mathfrak{v}_{\boldsymbol{w}}(n) = \mathfrak{v}_{\boldsymbol{w}}(q)$. Since $q \in \mathbb{C}(y)[x] \Rightarrow \varrho_x(q) = q$, we have $\varrho_x(n) = q + \varrho_x(q')$, and $\mathfrak{v}_{\boldsymbol{w}}(\varrho_x(q')) \le \mathfrak{v}_{\boldsymbol{w}}(q') < \mathfrak{v}_{\boldsymbol{w}}(q)$. Hence $\mathfrak{v}_{\boldsymbol{w}}(\varrho_x(n)) = \mathfrak{v}_{\boldsymbol{w}}(q) = \mathfrak{v}_{\boldsymbol{w}}(n)$, as claimed above. □

Combining the maps γ_k for all $k \in \mathbb{Z}$, we now get an (injective) map

$$\boldsymbol{\gamma} : \underline{\check{H}}^0(\mathfrak{U}, \mathcal{N}_p) \to \bigoplus_{k\in\mathbb{Z}} A_{p+k}/A_k \ . \tag{6.8}$$

The $\mathbb{C}[y]$-linearity of the map ϱ_x implies that $\boldsymbol{\gamma}$ is a homomorphism of graded $\mathbb{C}[Y,Z]$-modules, where on the right Y acts as right multiplication by y and Z acts by embedding successive filtration components.

We now focus our attention on the degrees k in the range $-|\boldsymbol{w}| \le k \le -1$. In this case $A_k = 0$, so γ_k is simply an isomorphism $\check{H}^0(\mathfrak{U}, \mathcal{N}(k)) \to A_{p+k}$. Looking back at the exact sequence (6.3) and using the last statement in Proposition 6.2, we see that γ_k induces isomorphisms

$$H^1(\mathbb{P}^2_q, \mathcal{M}(k)) \xrightarrow{\sim} A_{p+k}/r_x(M_{p+k}) \quad \text{for } -|\boldsymbol{w}| \le k \le -1 \tag{6.9}$$

(recall that r_x denotes the restriction of ϱ_x to M). We can now let $p \to \infty$. It is easy to check that the map $\underline{\check{H}}^0(\mathfrak{U}, \mathcal{N}_p) \to \underline{\check{H}}^0(\mathfrak{U}, \mathcal{N}_{p+1})$ induced by multiplication by Z is compatible with embedding of components $A_{p+k}/A_k \hookrightarrow A_{p+k+1}/A_k$ on the right of (6.8). It follows that the isomorphisms (6.9) are compatible with the embeddings $A_{p+k}/r_x(M_{p+k}) \hookrightarrow A_{p+k+1}/r_x(M_{p+k+1})$; hence, letting $p \to \infty$ in (6.9), we get isomorphisms

$$\alpha_k : H^1(\mathbb{P}^2_q, \mathcal{M}(k)) \to V_x \ , \quad -|\boldsymbol{w}| \le k \le -1 \ , \tag{6.10}$$

where (as in Section 5) $V_x := A/r_x(M)$. Further, the $\mathbb{C}[Y,Z]$-linearity of $\boldsymbol{\gamma}$ implies that the isomorphisms α_k take multiplication by Y and Z (when defined) on the left of (6.10) to (right) multiplication by y and to the identity map (respectively) on the right. It follows at once that the isomorphism

$$\alpha_x := \alpha_{-1} : H^1(\mathbb{P}^2_q, \mathcal{M}(-1)) = V(\boldsymbol{w}) \to V_x$$

takes the map $\mathbb{Y}(\boldsymbol{w})$ of Section 4 to the map $\mathbb{Y}$ of Section 5, as claimed in Theorem 1.5.

To obtain the other isomorphism α_y in Theorem 1.5, we have only to repeat all the above, starting from the representative M_y rather than M_x. We sketch a few details to fix the notation for the last calculation below. To avoid confusion, we continue to denote M_x by M, so that $M_y = \kappa M$. Further, we continue to identify $\check{H}^0(\mathfrak{U}, \mathcal{N}(k))$ with the space described in Proposition 6.2, so that in the new argument we work with the realization $\kappa\, \check{H}^0(\mathfrak{U}, \mathcal{N}(k))$. The crucial map

$$\gamma'_k : \kappa\, \check{H}^0(\mathfrak{U}, \mathcal{N}(k)) \to A_{p+k}/A_k$$

is then defined by $\gamma'_k(\kappa\,\bar{n}_1, \kappa\,\bar{n}_2) := \overline{\varrho_y(\kappa\, n_2)}$. Passing to a quotient and letting $p \to \infty$, we get the required isomorphism $\alpha_y : V(\boldsymbol{w}) \to V_y$ exactly as before.

To complete the proof of Theorem 1.5, it remains to show that the isomorphism $\alpha_y\, \alpha_x^{-1} : V_x \to V_y$ coincides with the map ϕ in Section 5. To do that, we return temporarily to the case of finite $p \gg 0$ and (for $-|\boldsymbol{w}| \le k \le -1$) let Φ_k be the map that makes the diagram

$$\begin{array}{ccccc} M_{p+k} & \longrightarrow & \check{H}^0(\mathfrak{U}, \mathcal{N}(k)) & \xrightarrow{\gamma_k} & A_{p+k} \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow \Phi_k \\ \kappa\, M_{p+k} & \longrightarrow & \kappa\, \check{H}^0(\mathfrak{U}, \mathcal{N}(k)) & \xrightarrow{\gamma'_k} & A_{p+k} \end{array}$$

commutative. In this diagram the first two vertical arrows are just multiplications by κ; the horizontal maps $M_{p+k} \to A_{p+k}$ and $\kappa\, M_{p+k} \to A_{p+k}$ are r_x and r_y, respectively. Let $a \in A_{p+k}$, and (as in Section 5) choose a polynomial $g(y)$ so that $a\, g(y) \in r_x(M)$, say $a\, g(y) = r_x(m)$: so here $m \in M_{p+k+N}$, where N is the (weighted) degree of g. Let $\gamma_k^{-1}(a) = (\bar{n}_1\, , \bar{n}_2)$; then $\gamma_k^{-1}(a\, g(y)) = (\overline{n_1\, g(y)}\, , \overline{n_2\, g(y)})$. On the other hand (by the last assertion in Proposition 6.2) $\gamma_k^{-1}(a\, g(y)) = (\bar{m}\, , \bar{m})$, hence $n_2\, g(y) - m \in Q_{k+N}$. Multiplying on the left by $\kappa \in Q_0$ and on the right by $g(y)^{-1} \in Q_{-N}$, we get

$$\kappa\, n_2 - \kappa\, m\, g(y)^{-1} \in Q_k \; . \tag{6.11}$$

Note that both terms in (6.11) belong to $\mathbb{C}(y)[x]$. Now we can calculate:

$$\begin{aligned} \Phi_k(a) &= \gamma'_k(\kappa\, \gamma_k^{-1}(a)) \\ &= \gamma'_k\, (\kappa\, \bar{n}_1\, , \kappa\, \bar{n}_2) \\ &= \varrho_y(\kappa\, n_2) \\ &= \varrho_y\left(\kappa\, m\, g(y)^{-1}\right) \qquad (\text{ by } (6.11)\,) \\ &= \varrho_y\left(\kappa\, r_x^{-1}[\, a\, g(y)\,]\, g(y)^{-1}\right) \; . \end{aligned}$$

Letting $p \to \infty$ we get the isomorphism $\Phi : A \to A$ already defined in Section 5 (cf. formula (5.2)). It follows at once that $\alpha_y\, \alpha_x^{-1} = \phi$, because these are both derived from Φ by passing to the quotients. That completes the proof of Theorem 1.5; as explained in the Introduction, the Comparison Theorem 1.4 follows immediately.

7. Proof of Theorem 1.3 and Theorem 1.2

The natural action of $G = \mathrm{Aut}(A)$ on $\mathfrak{R}$ can be defined in two (equivalent) ways. First, if $M \subseteq A$ is an *embedded* ideal, we can make $\sigma \in G$ act on M pointwise: $\sigma(M) = \{\, \sigma(m) \mid m \in M \,\}$. This leads to a well defined (left) action of G on the space of isomorphism classes $\mathfrak{R}$, and is the definition used in [BW]. Alternatively, we have simply $\sigma(M) \cong \rho_* M$, where $\rho = \sigma^{-1}$ is the automorphism inverse to σ.

Now suppose that σ preserves the $\boldsymbol{w}$-filtration on A for some weight vector $\boldsymbol{w}$. Then ρ extends to a graded automorphism $\boldsymbol{\rho}$ of $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{w})$, and the functor $\boldsymbol{\rho}_* : \mathsf{grmod}(\boldsymbol{A}) \to \mathsf{grmod}(\boldsymbol{A})$ descends to a functor ρ_* on the quotient category $\mathsf{coh}(\mathbb{P}^2_q(\boldsymbol{w}))$. In general, there will be no $\boldsymbol{w}$-filtration preserved by σ: however, that is the case if σ is one of the generators of G in (1.4). Slightly more generally[7], let σ be an automorphism of A of the form

$$\sigma(x) = x \;, \quad \sigma(y) = y + f(x) \;,$$

[7]We need this generality to deal with the case when r or s in (1.4) is 0.

where $f(x) = \sum_0^r a_i x^i$ is a polynomial in x of degree r. Let $\boldsymbol{w} = (1, N)$, where $N \geq r$. Then σ preserves the $\boldsymbol{w}$-filtration, and the extension of $\rho = \sigma^{-1}$ to $\boldsymbol{A}$ is defined on generators by the formulas

$$\boldsymbol{\rho}(X) = X \ , \quad \boldsymbol{\rho}(Y) = Y - \sum_{i=0}^{r} a_i X^i Z^{N-i} \ , \quad \boldsymbol{\rho}(Z) = Z \ .$$

If M is an ideal of A and $\mathcal{M}$ is its canonical extension, then it is easy to see that the canonical extension of $\sigma(M) = \rho_* M$ is $\rho_* \mathcal{M}$. Now, for any $\mathcal{F} \in \mathsf{coh}(\mathbb{P}^2_q(\boldsymbol{w}))$ there are natural isomorphisms of graded $\boldsymbol{A}$-modules

$$\underline{H}^i(\mathbb{P}^2_q,\, \rho_* \mathcal{F}) \cong \boldsymbol{\rho_*} \underline{H}^i(\mathbb{P}^2_q,\, \mathcal{F})$$

(cf. [AZ], p. 283). Thus we may identify $\underline{H}^1(\mathbb{P}^2_q,\, \rho_* \mathcal{M})$ with $\underline{H}^1(\mathbb{P}^2_q,\, \mathcal{M})$ as a graded vector space, but multiplication by $X,\, Y,\, Z$ on $\underline{H}^1(\mathbb{P}^2_q,\, \mathcal{M})$ is then replaced by multiplication by $\boldsymbol{\rho}(X),\, \boldsymbol{\rho}(Y),\, \boldsymbol{\rho}(Z)$. It follows at once that if $(\mathbb{X},\, \mathbb{Y})$ is the pair of matrices associated to M by the construction of Section 4, then the pair associated to $\sigma(M)$ is $(\mathbb{X},\, \mathbb{Y} - f(\mathbb{X}))$. A similar argument (interchanging the roles of x and y) shows that if σ is an automorphism of the form

$$\sigma(x) = x + g(y) \ , \quad \sigma(y) = y \ ,$$

then σ sends $(\mathbb{X},\, \mathbb{Y})$ to $(\mathbb{X} - g(\mathbb{Y}),\, \mathbb{Y})$. These are exactly the formulas that defined the action of G on $\mathfrak{C}$ in [BW], so the proof of Theorem 1.3 is complete.

Remark. This action of G on $\mathfrak{C}$ perhaps deserves comment, since it is not immediately obvious that it is well defined. Indeed, if $\sigma \in G$ we are proposing to define $\sigma(\mathbb{X},\, \mathbb{Y})$ by writing σ as a product of generators $\Psi_{r,\lambda}$ and $\Phi_{s,\mu}$; since the matrices $(\mathbb{X},\, \mathbb{Y})$ do not satisfy the defining relation of the algebra A, it is not *a priori* clear that the result is independent of the choice of the representation for σ. The best way out of this difficulty is to appeal to a theorem of Makar-Limanov [M] which implies that the relations satisfied by $\Psi_{r,\lambda}$ and $\Phi_{s,\mu}$ in G are the same as the relations satisfied by the corresponding automorphisms of the *free* associative algebra $\mathbb{C}\langle x, y\rangle$. In [BW] this problem did not arise, because we knew in advance that the map $\omega : \mathfrak{C} \to \mathfrak{R}$ was bijective, so we had only to transfer to $\mathfrak{C}$ the natural action of G on $\mathfrak{R}$.

. As explained in the Introduction, to prove Theorem 1.2 we have now only to check that the maps θ and ω^{-1} agree for one point in each G-orbit $\omega(\mathfrak{C}_n) \subset \mathfrak{R}$. A suitable point is (the class of) the ideal $I = y^{n+1}A + (yx - n)A$; this is the formal Fourier transform of the ideal M in Section 5.2. As in [BW], we identify A with the ring $\mathbb{C}[z, \partial/\partial z]$ of differential operators with polynomial coefficients by $x \leftrightarrow \partial/\partial z$, $y \leftrightarrow z$. Then $I \cap \mathbb{C}[z] \neq \{0\}$, so we can calculate that the Cannings-Holland map sends I to the point

$$W = z^{-1} \Big\{ f \in \mathbb{C}[z] \mid f^{(n)}(0) = 0 \Big\} \ \in \ \mathrm{Gr}^{\mathrm{ad}} \ .$$

The (reduced stationary) Baker function of this point is

$$\tilde{\psi}_W(x, z) = 1 - n\, x^{-1} z^{-1} \ .$$

If $\mathbb{X}_n$, $\mathbb{Y}_n$ are the two matrices found in the example of Section 5.2, then we have

$$\tilde{\psi}_W(x, z) = \det \big\{ \mathbb{I} - (x\mathbb{I} - \mathbb{Y}_n)^{-1} (z\mathbb{I} + \mathbb{X}_n)^{-1} \big\} \ ,$$

which means that the map ω^{-1} sends I to the pair of matrices $(\mathbb{Y}_n, -\mathbb{X}_n)$. The Fourier transform on $\mathfrak{R}$ corresponds to the map $(\mathbb{X}, \mathbb{Y}) \mapsto (-\mathbb{Y}, \mathbb{X})$ on matrices, hence indeed ω^{-1} sends the ideal M to $(\mathbb{X}_n, \mathbb{Y}_n) = \theta(M)$.

8. The Beilinson Equivalence

For each $i \in \{0, 1, \dots, |\boldsymbol{w}|\}$, we set $\mathcal{E}_i := \mathcal{O}_{\mathbb{P}^2_q(\boldsymbol{w})}(i)$, and $\mathcal{E} := \bigoplus_{i=0}^{|\boldsymbol{w}|} \mathcal{E}_i$. Let

$$B := \mathrm{Hom}(\mathcal{E}, \mathcal{E}) = \bigoplus_{i,j=0}^{|\boldsymbol{w}|} \mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j)$$

be the algebra of endomorphisms of $\mathcal{E}$. We consider the (left exact) functor $\mathrm{Hom}(\mathcal{E}, —)$, which takes (quasi)coherent sheaves over $\mathbb{P}^2_q(\boldsymbol{w})$ to right B-modules. Using the fact that $\mathbb{P}^2_q(\boldsymbol{w})$ has finite cohomological dimension and the Finiteness Theorem 2.4(a), we see that $\mathrm{Hom}(\mathcal{E}, —)$ extends to a functor on bounded derived categories

$$\mathbf{R}\mathrm{Hom}(\mathcal{E}, —) \,:\, D^b(\mathsf{coh}\,\mathbb{P}^2_q) \to D^b(\mathsf{mod}\,B)\ , \tag{8.1}$$

where $\mathsf{mod}(B)$ denotes the category of finite-dimensional right B-modules. The following statement is the analogue for $\mathbb{P}^2_q(\boldsymbol{w})$ of a theorem of Beilinson (see [B]) for the usual projective spaces $\mathbb{P}^n$.

Theorem 8.1. *The functor (8.1) is an equivalence of categories.*

Proof. According to [Bo], Theorem 6.2, it is enough to check that the sequence of sheaves $(\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_{|\boldsymbol{w}|})$ (regarded as 0-complexes in $D^b(\mathsf{coh}\,\mathbb{P}^2_q)$) is a *complete strongly exceptional collection.* Here "complete" means that these objects generate $D^b(\mathsf{coh}\,\mathbb{P}^2_q)$ as a triangulated category, while "strongly exceptional" means that

(a) $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j) = 0$ if $i > j$,

(b) $\mathrm{Ext}^k(\mathcal{E}_i, \mathcal{E}_j) = 0$ for all $i, j \in \{0, 1, \dots, |\boldsymbol{w}|\}$ and $k \neq 0$.

Property (a) is trivial, since

$$\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j) \cong \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2_q}(i), \mathcal{O}_{\mathbb{P}^2_q}(j)) \cong H^0(\mathbb{P}^2_q(\boldsymbol{w}),\, \mathcal{O}_{\mathbb{P}^2_q}(j-i)) \cong A_{j-i}(\boldsymbol{w})\ .$$

Similarly, $\mathrm{Ext}^k(\mathcal{E}_i, \mathcal{E}_j) \cong H^k(\mathbb{P}^2_q(\boldsymbol{w}),\, \mathcal{O}_{\mathbb{P}^2_q}(j-i))$. By Theorem 2.3 and Proposition 3.1 we have $H^1(\mathbb{P}^2_q,\, \mathcal{O}_{\mathbb{P}^2_q}(r)) = 0$ and $H^2(\mathbb{P}^2_q,\, \mathcal{O}_{\mathbb{P}^2_q}(r)) \cong (A_{r-|\boldsymbol{w}|-1})^*$ for all r. If $i, j \in \{0, 1, \dots, |\boldsymbol{w}|\}$ then $j - i \leq |\boldsymbol{w}|$, hence $(j-i) - |\boldsymbol{w}| - 1 < 0$. Property ($b$) above follows.

It remains to show that the collection $(\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_{|\boldsymbol{w}|})$ is complete. Denote by E the smallest strictly full triangulated subcategory of $D^b(\mathsf{coh}\,\mathbb{P}^2_q)$ containing the objects $\mathcal{O}_{\mathbb{P}^2_q}, \mathcal{O}_{\mathbb{P}^2_q}(1), \dots, \mathcal{O}_{\mathbb{P}^2_q}(|\boldsymbol{w}|)$. We must show that $E = D^b(\mathsf{coh}\,\mathbb{P}^2_q)$. Since any derived category is generated by its abelian core, it suffices to prove that the 0-complexes $(\cdots \to 0 \to \mathcal{M} \to 0 \to \cdots)$ are in E for all $\mathcal{M} \in \mathsf{coh}(\mathbb{P}^2_q)$. We know that $\boldsymbol{A}$ has finite global dimension, so every $\boldsymbol{M} \in \mathsf{grmod}(\boldsymbol{A})$ has a finite projective resolution. Moreover, every graded projective $\boldsymbol{A}$-module is a finite direct sum of shifts of $\boldsymbol{A}$ (see [CE], Theorem 6.1). Therefore every $\mathcal{M} \in \mathsf{coh}(\mathbb{P}^2_q)$ has a finite resolution by finite direct sums of sheaves $\mathcal{O}_{\mathbb{P}^2_q}(m)$. Such a resolution gives a complex isomorphic to $\mathcal{M}$ in the derived category, and hence (see [GM], Chapter III, § 5, Exercise 4(b)), we need only to show that $\mathcal{O}_{\mathbb{P}^2_q}(m)$ belongs to E for any $m \in \mathbb{Z}$.

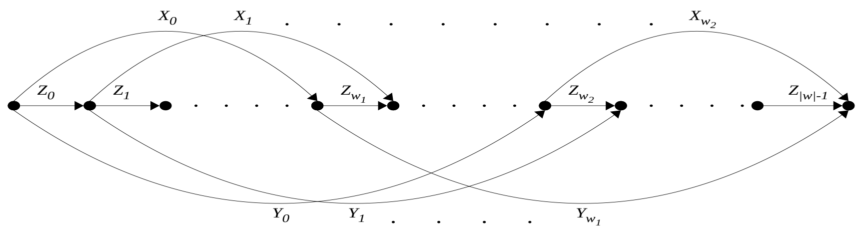


FIGURE 1.

According to [Ste] (combine Proposition 2.5(ii) and Corollary 2.6(ii)), the trivial $\boldsymbol{A}$-module $\boldsymbol{A}/\boldsymbol{A}_{\geq 1} \cong \mathbb{C}$ has a graded resolution of the form

$$0 \to \boldsymbol{A}(-|\boldsymbol{w}|-1) \to \boldsymbol{A}(-w_2-1) \oplus \boldsymbol{A}(-w_1-1) \oplus \boldsymbol{A}(-|\boldsymbol{w}|) \to \\ \to \boldsymbol{A}(-w_1) \oplus \boldsymbol{A}(-w_2) \oplus \boldsymbol{A}(-1) \to \boldsymbol{A} \to \mathbb{C} \to 0\ . \tag{8.2}$$

By shifting degrees in (8.2) and passing to the quotient category we get (for any integer m) a Koszul-type exact sequence in $\mathsf{coh}(\mathbb{P}^2_q)$:

$$\begin{aligned} 0 \quad \to \quad & \mathcal{O}(m) \to \mathcal{O}(m+w_1) \oplus \mathcal{O}(m+w_2) \oplus \mathcal{O}(m+1) \to \\ \to \quad & \mathcal{O}(m+w_2+1) \oplus \mathcal{O}(m+w_1+1) \oplus \mathcal{O}(m+|\boldsymbol{w}|) \to \mathcal{O}(m+|\boldsymbol{w}|+1) \to 0\ . \end{aligned}$$

Letting $m=0$ above, we observe that $\mathcal{O}(|\boldsymbol{w}|+1)$ is quasi-isomorphic to a complex each term of which is in E; therefore, $\mathcal{O}(|\boldsymbol{w}|+1) \in E$. Similarly, for $m=-1$, it follows that $\mathcal{O}(-1) \in E$. Arguing in this way by induction (going both in negative and positive directions), we conclude that $\mathcal{O}(m) \in E$ for all $m \in \mathbb{Z}$. This finishes the proof of the theorem. □

We can now explain the significance of the vanishing conditions (1.2) (and their generalization in Theorem 4.5). Given a coherent sheaf $\mathcal{M}$ over $\mathbb{P}^2_q$, we may regard it as a 0-complex in $D^b(\mathsf{coh}\,\mathbb{P}^2_q)$. By definition, the functor $\mathbf{R}\mathrm{Hom}(\mathcal{E}, \text{—})$ then maps $\mathcal{M}$ to a complex of B-modules whose cohomology in degree $-k$ is

$$\mathrm{Ext}^k(\mathcal{E},\, \mathcal{M}) \;\cong\; \bigoplus_{i=0}^{|\boldsymbol{w}|} \mathrm{Ext}^k(\mathcal{O}_{\mathbb{P}^2_q}(i),\, \mathcal{M}) \;\cong\; \bigoplus_{i=0}^{|\boldsymbol{w}|} H^k(\mathbb{P}^2_q,\, \mathcal{M}(-i))\ .$$

In our case Theorem 4.5 tells us that this cohomology vanishes for all $k \neq 1$. A complex with cohomology only in one degree is isomorphic (in the derived category) to its cohomology; thus, essentially, the Beilinson equivalence assigns to our sheaf $\mathcal{M}$ the single B-module $\bigoplus_{i=0}^{|\boldsymbol{w}|} H^1(\mathbb{P}^2_q,\, \mathcal{M}(-i))$.

To make contact with the language of quivers used in the Introduction, we have only to note that the algebra B is isomorphic to the path algebra of the quiver[8] shown in Fig. 1, with the relations

[8]In the case $\boldsymbol{w} = (1,1)$, this is just the quiver indicated in (1.3).

$$
(8.3)\qquad
\begin{aligned}
Z_{i+w_1} X_i - X_{i+1} Z_i &= 0\,, & i &= 0,1,\ldots,w_2-1\,;\\
Z_{j+w_2} Y_j - Y_{j+1} Z_j &= 0\,, & j &= 0,1,\ldots,w_1-1\,;\\
X_{w_2} Y_0 - Y_{w_1} X_0 &= Z_{|\boldsymbol{w}|-1} Z_{|\boldsymbol{w}|-2} \ldots Z_1 Z_0\,.
\end{aligned}
$$

That means that a (left) B-module can be identified with a representation of this quiver with relations. Indeed, let e_i denote the identity map in $\operatorname{Hom}(\mathcal{E}_i, \mathcal{E}_i) \subset B$: these are mutually orthogonal idempotents in B, and $e_0 + e_1 + \ldots + e_{|\boldsymbol{w}|} = 1_B$. Hence any B-module V decomposes: $V = \bigoplus_i V_i$, where $V_i := e_i V$. Further, each element of $\operatorname{Hom}(\mathcal{E}_i, \mathcal{E}_j) \subset B$ maps V_i to V_j. As we have seen above, $\operatorname{Hom}(\mathcal{E}_i, \mathcal{E}_j)$ is naturally isomorphic to $A_{j-i}(\boldsymbol{w})$; hence the generators X, Y, and Z of $\boldsymbol{A}$ determine maps X_α, Y_α and Z_α as indicated in Fig. 1 (the vertices in this diagram represent the spaces V_i). The relations (8.3) follow from the defining relations (3.8) of the algebra $\boldsymbol{A}(\boldsymbol{w})$. In this way, each B-module determines a representation of the above quiver with relations. The construction of a B-module from a quiver representation is equally straightforward.

9. Le Bruyn's Moduli Spaces

As we mentioned in the Introduction, our construction differs from that in Le Bruyn's paper [L] only in the different choice of extension of an A-module M to a sheaf over $\mathbb{P}^2_q$. In this section we clarify the relationship between the two constructions. We confine ourselves to the (basic) case $\boldsymbol{w} = (1,1)$.

As usual, let M be an ideal of A, with the normalized induced filtration, and let d be the minimum filtration degree of elements of M. We recall (see Lemma 4.4) that $d \geq 1$ (unless M is cyclic: this case has to be excluded from some of the statements below). It follows from Theorem 4.5 that the sheaf $\mathcal{F} = \mathcal{M}(d-1)$ satisfies the vanishing conditions (1.2), so that $\mathcal{M}(d-1)$ (and hence M) is determined by the quiver representation

$$
(9.1)\qquad H^1(\mathbb{P}^2_q, \mathcal{M}(d-3)) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; H^1(\mathbb{P}^2_q, \mathcal{M}(d-2)) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; H^1(\mathbb{P}^2_q, \mathcal{M}(d-1))\,.
$$

Le Bruyn uses a slightly more subtle fact (see [Ba], Corollary 7.2): because the sheaf $\mathcal{F} = \mathcal{M}(d-2)$ also satisfies (1.2), M is determined by the left hand part

$$
H^1(\mathbb{P}^2_q, \mathcal{M}(d-3)) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; H^1(\mathbb{P}^2_q, \mathcal{M}(d-2))
$$

of (9.1). For each pair of non-negative integers (r,s), let $\tilde{\mathfrak{M}}(r,s)$ be the space of isomorphism classes of quintuples $(V, W; X_1, X_2, X_3)$, where V and W are vector spaces of dimensions r and s (respectively) and X_i are linear maps from V to W. Thus each ideal M of A determines (and is determined by) a point of $\tilde{\mathfrak{M}}(r,s)$, where $r = \dim_{\mathbb{C}} H^1(\mathbb{P}^2_q, \mathcal{M}(d-3))$ and $s = \dim_{\mathbb{C}} H^1(\mathbb{P}^2_q, \mathcal{M}(d-2))$. Denoting by $\mathfrak{M}(r,s)$ the subset of points of $\tilde{\mathfrak{M}}(r,s)$ that arise in this way, we obtain the main result of [L]: the space of ideals $\mathfrak{R}$ decomposes as the disjoint union of the "moduli spaces" $\mathfrak{M}(r,s)$ $(r,s \geq 0)$.

The relationship of this to our decomposition of $\mathfrak{R}$ becomes clear if we calculate the dimensions r and s in terms of d and our invariant n. Because of (1.2), $-r$ and $-s$ are equal to the Euler characteristics $\chi(\mathbb{P}^2_q,\, \mathcal{M}(d-3))$ and $\chi(\mathbb{P}^2_q,\, \mathcal{M}(d-2))$

respectively. From (6.5) we get

$$\chi(\mathbb{P}^2_q, \mathcal{M}(k)) = \frac{1}{2}(k+1)(k+2) - n \quad \text{for } k \geq -2$$

(in fact, this formula is true for all $k \in \mathbb{Z}$). In particular, we have

$$r = n - \frac{1}{2}(d-2)(d-1) \quad , \quad s = n - \frac{1}{2}d(d-1) \ . \tag{9.2}$$

From this we notice immediately that the space $\mathfrak{M}(r, s)$ is empty if $r < s$, while in general each point of $\mathfrak{M}(r, s)$ determines an ideal M for which our parameters are given by

$$n = \frac{1}{2}\left[(r-s)^2 + r + s\right] \quad , \quad d = r - s + 1 \ . \tag{9.3}$$

The map θ then gives us a point of $\mathfrak{C}_n$. We thus have

Proposition 9.1. *Let $\mathfrak{C}_n(d)$ be the subspace of $\mathfrak{C}_n$ corresponding to ideals of minimum filtration degree d. Then the construction explained above defines a bijection $\mathfrak{M}(r, s) \to \mathfrak{C}_n(d)$, where the numbers (r, s) and (n, d) are related by (9.2) and (9.3).*

The commutative analogue of this decomposition of $\mathfrak{C}_n$ (into the subspaces $\mathfrak{C}_n(d)$) forms part of the much studied[9] *Brill-Noether theory*: in that case $\mathfrak{C}_n$ is replaced by the point Hilbert scheme $\mathrm{Hilb}_n(\mathbb{A}^2)$ and $\mathfrak{C}_n(d)$ by the subvariety $\mathrm{Hilb}_n(d)$ (say) of n-tuples of points of $\mathbb{A}^2$ that lie on a curve of degree d, but not on one of degree $d-1$. The detailed structure of this stratification of $\mathrm{Hilb}_n(\mathbb{A}^2)$ seems quite complicated (see, for example, [BH], [R]). However, using the fact that there is a curve of degree d through any $d(d+3)/2$ points in the plane, it is easy to see that (for $n > 0$) $\mathrm{Hilb}_n(d)$ is non-empty if and only if we have $1 \leq d \leq D$, where D is the least integer such that $n \leq D(D+3)/2$. Furthermore, the dimension of $\mathrm{Hilb}_n(d)$ is then given by

$$\dim_{\mathbb{C}} \mathrm{Hilb}_n(D) = 2n \ , \quad \dim_{\mathbb{C}} \mathrm{Hilb}_n(d) = n + \frac{1}{2}d(d+3) \quad \text{if} \quad d < D \ .$$

We expect that the situation is the same in the noncommutative case. It might be interesting to study this decomposition of $\mathfrak{C}_n$ in more detail to see to what extent it is simply a deformation of what we have in the commutative case.

10. Ideals and Bundles

In the case $\boldsymbol{w} = (1, 1)$, the authors of [KKO] establish a bijection between $\mathfrak{C}$ and the space $\mathfrak{L}$ of all line bundles (suitably defined) over $\mathbb{P}^2_q$ that are trivial on the line at infinity. This bijection is constructed using monads, following the original approach of Barth to classifying bundles over projective spaces (see [Bar], [N]). Here we shall check that the result of [KKO] is essentially equivalent to the bijectivity of our map $\theta : \mathfrak{R} \to \mathfrak{C}$. Most of what follows is valid for any positive weight $\boldsymbol{w}$. The key step is the following lemma, which may be of independent interest.

Lemma 10.1. *Let M be a finitely generated rank one torsion-free A-module, and let $\{M_\bullet\}$ be any filtration of M by finite-dimensional subspaces M_k. Suppose the associated graded module $\boldsymbol{GM}$ is* essentially torsion-free *(meaning that $\boldsymbol{GM}_{\geq N}$ is torsion-free for some N). Then for any embedding of M in A there is an integer k_0 such that $M_k = M \cap A_{k-k_0}$ for all $k \geq N$. In other words, the given filtration on M essentially coincides with an induced filtration.*

[9] We thank A. Iarrobino for information on this subject.

Proof. Let $\mathfrak{u}$ denote the valuation on M corresponding to the given filtration, that is, if $m \in M$ then $\mathfrak{u}(m)$ is the smallest integer k such that $m \in M_k$. Let $\mathfrak{v}$ be the valuation on A corresponding to the ring filtration we are using. Then

$$\mathfrak{u}(ma) \leq \mathfrak{u}(m) + \mathfrak{v}(a) \quad \text{for all } m \in M \text{ and } a \in A \ . \tag{10.1}$$

The assumption on $\boldsymbol{GM}$ is equivalent to

$$\mathfrak{u}(ma) = \mathfrak{u}(m) + \mathfrak{v}(a) \quad \text{if} \quad \mathfrak{u}(m) \geq N \ . \tag{10.2}$$

Fix an embedding of M in A. Then $M \otimes_A Q = Q$, hence every element q of the Weyl quotient field Q can be written in the form $q = ma^{-1}$ with $m \in M$ and $a \in A$. Moreover, we can choose m so that $\mathfrak{u}(m) \geq N$. Indeed, for any $b \in A$ we have $ma^{-1} = mb(ab)^{-1}$, and we cannot have $\mathfrak{u}(mb) < N$ for all b because M_N is finite-dimensional.

Now define a function $F : Q \to \mathbb{Z}$ as follows: if $q = ma^{-1}$ as above with $\mathfrak{u}(m) \geq N$, let $F(q) := \mathfrak{u}(m) - \mathfrak{v}(a)$. To show that F is well defined, suppose that $ma^{-1} = nb^{-1}$ are two such expressions for q. Since A is an Ore domain, we have $b^{-1}a = pr^{-1}$ for some $p, r \in A$. So $mr = np$ and $ar = bp$. Using (10.2) and the similar fact for the filtration on A, we get

$$\mathfrak{u}(m) + \mathfrak{v}(r) = \mathfrak{u}(n) + \mathfrak{v}(p) \quad \text{and} \quad \mathfrak{v}(a) + \mathfrak{v}(r) = \mathfrak{v}(b) + \mathfrak{v}(p) \ .$$

Hence $\mathfrak{u}(m) - \mathfrak{v}(a) = \mathfrak{u}(n) - \mathfrak{v}(b)$, as desired.

Now, if $\mathfrak{u}(m) \geq N$ then $F(1) = F(mm^{-1}) = \mathfrak{u}(m) - \mathfrak{v}(m)$. Thus, setting $k_0 := F(1)$, we have

$$\mathfrak{v}(m) = \mathfrak{u}(m) - k_0 \quad \text{if} \quad \mathfrak{u}(m) \geq N \ . \tag{10.3}$$

Also, for *any* element $m \in M$, if we choose $a \in A$ so that $\mathfrak{u}(ma) \geq N$, we have (using (10.1)) $\mathfrak{v}(ma) = \mathfrak{v}(m) + \mathfrak{v}(a) = \mathfrak{u}(ma) - k_0 \leq \mathfrak{u}(m) + \mathfrak{v}(a) - k_0$, so

$$\mathfrak{v}(m) \leq \mathfrak{u}(m) - k_0 \quad \text{for all} \quad m \in M \ . \tag{10.4}$$

Let $\{M'_\bullet\}$ denote the induced filtration on M. By (10.4), we have

$$m \in M_k \ \Leftrightarrow \ \mathfrak{u}(m) \leq k \ \Rightarrow \ \mathfrak{v}(m) \leq k - k_0 \ \Leftrightarrow \ m \in M'_{k-k_0} \ ,$$

that is,

$$M_k \subseteq M'_{k-k_0} \quad \text{for all } k \ . \tag{10.5}$$

Similarly, by (10.3), if $k \geq N$, we have

$$m \notin M_k \ \Leftrightarrow \ \mathfrak{u}(m) > k \ \Rightarrow \ \mathfrak{u}(m) \geq N \ \Rightarrow$$
$$\Rightarrow \ \mathfrak{v}(m) = \mathfrak{u}(m) - k_0 \ \Rightarrow \ \mathfrak{v}(m) > k - k_0 \ \Leftrightarrow \ m \notin M'_{k-k_0} \ ;$$

equivalently, $m \in M'_{k-k_0} \ \Rightarrow \ m \in M_k$, and hence

$$M'_{k-k_0} \subseteq M_k \quad \text{for all } k \geq N \ . \tag{10.6}$$

The lemma now follows at once from (10.5) and (10.6). □

Lemma 10.2. *Let M be an ideal of A, and let $\mathcal{M}$ be its canonical extension to $\mathbb{P}^2_q$. Then $\mathcal{M}$ is a bundle in the sense of* [KKO] *(Definition 5.4).*

Proof. According to [KKO] (see Section 5.3), we have to show that

$$\mathrm{Ext}^i(\mathcal{M}, \mathcal{O}(k)) = 0 \quad \text{for } k \gg 0 \text{ and } i > 0 .$$

By Serre Duality (see Theorem 2.4(c)), it is equivalent to show that $H^0(\mathbb{P}^2_q\,, \mathcal{M}(k))$ and $H^1(\mathbb{P}^2_q\,, \mathcal{M}(k))$ vanish for $k \ll 0$. The vanishing of H^0 is part of Theorem 4.5(ii). The statement about H^1 follows from Theorem 4.5 (i) by a useful argument which would not be available in the commutative case. Because the spaces $H^1(\mathbb{P}^2_q\,, \mathcal{M}(k))$ are finite-dimensional, Theorem 4.5 (i) implies that the maps $H^1(\mathbb{P}^2_q\,, \mathcal{M}(k-1)) \to H^1(\mathbb{P}^2_q\,, \mathcal{M}(k))$ induced by multiplication by Z are isomorphisms for $k \ll 0$. If we use these isomorphisms to identify the spaces $H^1(\mathbb{P}^2_q\,, \mathcal{M}(k))$ for $k \ll 0$, the action of the generators X and Y gives us a finite-dimensional representation of the Weyl algebra, which is impossible unless the representation space is zero. □

Proposition 10.3. *Let* $\mu : \mathfrak{R} \to \mathfrak{L}$ *be the map that sends an ideal class to its canonical extension. Then* μ *is bijective.*

Proof. The inverse map $\nu : \mathfrak{L} \to \mathfrak{R}$ is constructed as follows. Let $\mathcal{M}$ be a line bundle over $\mathbb{P}^2_q$, trivial over l_∞, and let $\boldsymbol{M} = \oplus M_k$ be a graded $\boldsymbol{A}$-module with $\pi\boldsymbol{M} = \mathcal{M}$. By (the proof of) Lemma 6.1 of [KKO], $\mathcal{M}$ can be embedded in a direct sum of sheaves $\mathcal{O}(k)$; hence the $\boldsymbol{A}$-module $\boldsymbol{M}$ is essentially torsion-free. We set $M := \varinjlim M_k$, where the direct limit is taken over the maps $\cdot Z : M_{k-1} \to M_k$. Then M is a rank one torsion-free A-module, filtered by (the images of) the components M_k. Forgetting this filtration, we obtain a map $\nu : \mathfrak{L} \to \mathfrak{R}$. Since $\mathcal{M}$ is trivial over l_∞, we have $\boldsymbol{GM}_{\geq k} \cong \boldsymbol{GA}_{\geq k}$ for $k \gg 0$, and therefore the filtration on M coincides (in sufficiently high degrees) with the normalized induced filtration (see Lemma 10.1). It follows easily that $\nu = \mu^{-1}$. □

We omit the proof that the bijection $\theta\nu : \mathfrak{L} \to \mathfrak{C}$ coincides[10] with the map constructed in [KKO]. Although we do not know a reference for this fact, it is very unsurprising, since Beilinson's equivalence is in essence a generalization of the monad construction (cf. [B]).

Appendix A. Appendix by Michel Van den Bergh

In this appendix we give alternative proofs of Theorems 1.3 and 1.4. Our proof of Theorem 1.4 does not rely on Čech cohomology. Furthermore our proof of Theorem 1.3 does not rely on the properties of weighted projective spaces with $w \neq (1,1)$. So it is in fact independent of Theorem 1.4!

After the authors of this paper had proposed me to write this appendix they succeeded in simplifying some of my original arguments and they have gracefully allowed me to consult some of their private notes which contained similar ideas. These contributions have allowed me to streamline the presentation below.

The main idea behind the new proofs is that while the map (a priori dependent on w) which associates linear data to ideals seems hard to understand, the inverse of that map is given by a simple formula (see (A.5) below) whose properties are transparent.

We start with the proof of Theorem 1.4. First we introduce some notational conventions. If Q is a quiver with relations then we will identify Q with a $\mathbb{C}$-linear

[10] Actually, there is a difference of sign.

additive category whose objects are the finite direct sums of vertices of Q. This has the effect that we write paths from right to left. Under this formalism the path algebra $\mathbb{C}Q$ of Q is given by the endomorphism ring of the sum of the vertices. Note that $\mathbb{C}(Q^{\mathrm{opp}}) = (\mathbb{C}Q)^{\mathrm{opp}}$. A morphism of quivers is a functor between the associated categories. Such a morphism induces a ring homomorphism between the associated path algebras.

If Q is a quiver with relations then $\mathrm{Mod}(Q)$ is the category of $\mathbb{C}$-linear contravariant functors on Q with values in $\mathbb{C}$-vector spaces. By Yoneda's lemma we obtain a full faithful functor $Q \to \mathrm{Mod}(Q)$ whose image consists of finitely generated projectives. Invoking Morita theory or directly one sees that $\mathrm{Mod}(Q)$ is equivalent to $\mathrm{Mod}(kQ)$, the category of right kQ-modules.

We now use the notations from Section 8. We denote the quiver (with relations) given in Figure 1 by Δ. If we view Δ as a $\mathbb{C}$-linear additive category then it is equivalent to the full subcategory of $\mathrm{coh}\,\mathbb{P}^2_q$ whose objects are finite direct sums of the $\mathcal{O}_{\mathbb{P}^2_q}(i)_{i=0,\dots|w|}$ in such a way that the i'th vertex from the left (counting from 0) corresponds to $\mathcal{O}_{\mathbb{P}^2_q}(i)$. It follows that $\mathbb{C}\Delta = \mathrm{End}(\oplus_{i=0}^{|w|}\mathcal{O}_{\mathbb{P}^2_q}(i)) = \mathrm{End}(\mathcal{E}) = B$.

As noted in Section 8, the functor $\mathbf{R}\mathrm{Hom}_{\mathbb{P}^2_w}(\mathcal{E}, -)$ defines an equivalence between the triangulated categories $D^b_f(\mathrm{coh}(\mathbb{P}^2_q))$ and $D^b(\mathrm{mod}(B))$. The inverse functor is given by $-\overset{\mathbf{L}}{\otimes}_B \mathcal{E}$. It is clear that this equivalence restricts to an equivalence between the following two subcategories

$$\mathcal{X}_1 = \{\mathcal{M} \in \mathrm{coh}(\mathbb{P}^2_q) \mid \mathrm{Ext}^i_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{M}) = 0 \text{ for } i \neq 1\}$$

and

$$\mathcal{Y}_1 = \{M \in \mathrm{mod}(B) \mid \mathrm{Tor}^B_i(M, \mathcal{E}) = 0 \text{ for } i \neq 1\}$$

The inverse equivalences between these categories are given by $\mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{E}, -)$ and $\mathrm{Tor}^B_1(-, \mathcal{E})$.

As before we denote by $\mathbb{P}^1$ the line at infinity in $\mathbb{P}^2_q$. Note that $\mathbb{P}^1$ is a weighted projective line in the sense of [GL]. The inclusion $\mathbb{P}^1 \to \mathbb{P}^2_q$ is denoted by i.

Let us denote by $\mathcal{R}$ the full subcategory of $\mathrm{coh}(\mathbb{P}^2_q)$ whose objects have the property that $\mathcal{M} \not\cong \mathcal{O}_{\mathbb{P}^2}$ and $i^*(\mathcal{M}) \cong \mathcal{O}_{\mathbb{P}^1}$. Using the results in Section 4 one shows that $\mathcal{R} \subset \mathcal{X}_1$ and furthermore that the image of $\mathcal{R}$ under $\mathrm{Ext}^1(\mathcal{E}, -)$ lies in the following category

$$\begin{aligned}\mathcal{C}_1 = \{M \in \mathrm{Mod}(\Delta) \mid\ & M(Z_i) \text{ is an isomorphism for } i = 1, \dots, |w| - 1, \\ & M(Z_0) \text{ is surjective with one dimensional kernel and all } M(i) \\ & \text{are finite dimensional}\}\end{aligned}$$

We define $\mathcal{C}_2$ as the category consisting of triples $(W, \mathbb{X}, \mathbb{Y})$ where W is a finite dimensional vector space and $\mathbb{X}, \mathbb{Y}$ are endomorphisms of W satisfying $\mathrm{rk}([\mathbb{Y}, \mathbb{X}] - \mathrm{Id}) = 1$. Let $M \in \mathcal{C}_1$. Then up to a canonical isomorphism we may assume that $M(1) = \cdots = M(|w|)$ and $M(Z_1) = \cdots = M(Z_{|w|-1}) = \mathrm{Id}$.

Put $W = M(|w|)$, $\mathbb{X} = M(X_1)$, $\mathbb{Y} = M(Y_1)$ and $\mathbb{Z} = M(Z_0)$. Taking into account that M is a contravariant functor we find

$$\begin{gathered} M(X_1) = M(X_2) = \cdots = M(X_{w_2}) = \mathbb{X} \\ M(Y_1) = M(Y_2) = \cdots = M(Y_{w_1}) = \mathbb{Y} \\ M(X_0) = \mathbb{Z}\mathbb{X} \\ M(Y_0) = \mathbb{Z}\mathbb{Y} \end{gathered}$$

It follows that $\mathbb{Z}(\mathbb{Y}\mathbb{X} - \mathbb{X}\mathbb{Y} - \mathrm{Id}) = 0$ and hence $(W, \mathbb{X}, \mathbb{Y}) \in \mathcal{C}_2$. It is clear that this procedure is reversible and defines an equivalence $\mathcal{C}_2 \cong \mathcal{C}_1$.

Let R be the category of non-trivial rank one projective right A-modules (with maps given by isomorphisms). If $N \in R$ then according to Section 4 there exists, up to isomorphism, a unique extension $\mathcal{M}$ of N to $\mathbb{P}^2_q$ which lies in $\mathcal{R}$.

Summarizing we now have a composition of functors:

$$R \xrightarrow{\cong} \mathcal{R} \hookrightarrow \mathcal{C}_1 \cong \mathcal{C}_2 \tag{A.1}$$

(that the first functor is an equivalence follows for example from the easily proved fact that the objects in $\mathcal{C}_2$ are simple objects when considered as representations of the two loop quiver).

Lemma A.1. (A.1) *is an equivalence.*

Proof. Note that we are not allowed deduce this result from Theorem 1.1 since the proof of that theorem depends on Theorem 1.3!

Below we will construct a (left) inverse to (A.1) which is independent of w. This means that in principle we have to prove the lemma only for one particular w. If $w = (1, 1)$ then the lemma can be deduced from the results in [BGK], [KKO], [L] although the point of view in these papers is slightly different.

We will give a proof which works equally well for all w. Perhaps the method has some independent interest.

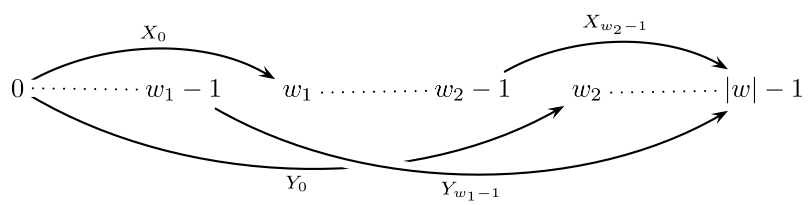
Let $M \in \mathcal{C}_1$. We need to prove two things:

1. $M \in \mathcal{Y}_1$, i.e. $M \overset{\mathbf{L}}{\otimes}_B \mathcal{E}$ has its only non-vanishing cohomology in degree -1. This has the effect that M is in the image of some object in $\mathcal{X}_1$.
2. M is actually in the image of $\mathcal{R}$, i.e. $i^*(H^{-1}(M \overset{\mathbf{L}}{\otimes}_B \mathcal{E})) \cong \mathcal{O}_{\mathbb{P}^1}$.

Now $\mathbb{P}^2_q$ has the pleasant property that if $0 \neq \mathcal{M} \in \mathrm{coh}(\mathbb{P}^2)$ then $i^*(\mathcal{M}) \neq 0$. From this we easily deduce that 1.,2. above are actually equivalent to the following single statement:

3. $Li^*(M \overset{\mathbf{L}}{\otimes}_B \mathcal{E}) \cong \mathcal{O}_{\mathbb{P}^1}[1]$.

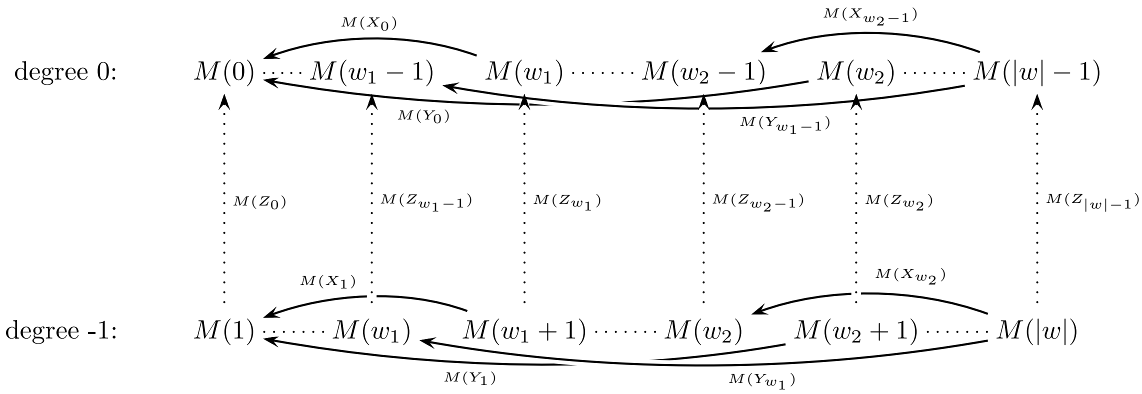
Let $\mathcal{E}_\infty = \oplus_{i=0}^{|w|-1} \mathcal{O}_{\mathbb{P}^1}(i)$ and $B_\infty = \mathrm{End}(\mathcal{E}_\infty)$. Then B_∞ is the path algebra of the quiver Δ_∞



Observing that $\mathbf{R}\mathrm{Hom}(\mathcal{E}_\infty, -)$ defines an equivalence between $D^b(\mathrm{coh}(\mathbb{P}^1))$ and $D^b_f(B_\infty)$ we want to understand the composition

$$D^b_f(B) \xrightarrow{-\overset{\mathbf{L}}{\otimes}\mathcal{E}} D^b(\mathrm{coh}(\mathbb{P}^2_q)) \xrightarrow{Li^*} D^b(\mathrm{coh}(\mathbb{P}^1)) \xrightarrow{\mathbf{R}\mathrm{Hom}(\mathcal{E}_\infty,-)} D^b_f(B_\infty) \tag{A.2}$$

Checking on projectives, and then on complexes of projectives we find that on an object M in $\mathrm{Mod}(\Delta)$ the composition (A.2) is given by a length two complex concentrated in degrees $-1, 0$ of the following form

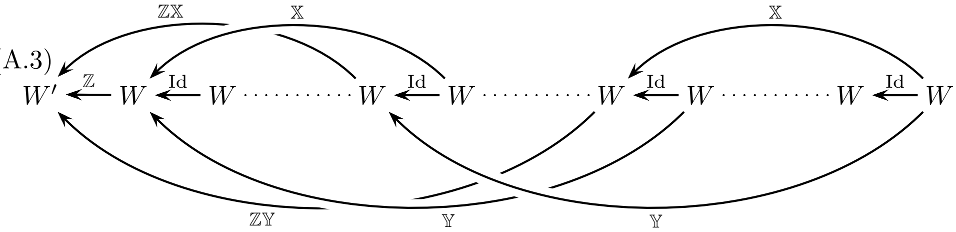


It is now clear that if $M \in \mathcal{C}_1$ then the image of M under the composition (A.2) is equal to $S[1]$ where S is the simple object in $\mathrm{Mod}(\Delta_\infty)$ defined by $\dim S(i) = \delta_{i0}$. Since S corresponds to $\mathcal{O}_{\mathbb{P}^1_w}$ we are done. □

Now we continue with the proof of Theorem 1.4. Theorem 1.4 asserts that the functor (A.1) is independent of w. We prove this by showing that the inverse of (A.1) is independent of w.

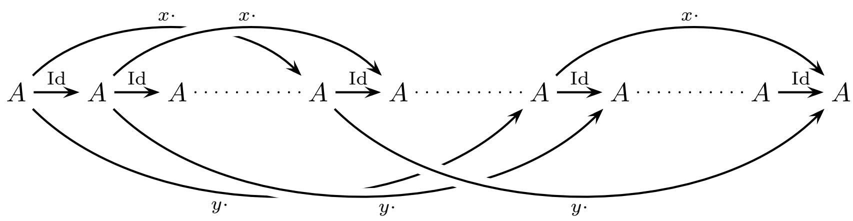
Let $W = (W, \mathbb{X}, \mathbb{Y}) \in \mathcal{C}_2$. Then the associated object M of $\mathrm{Mod}(\Delta)$ looks like

(A.3)



where $W' = W/\mathrm{im}([\mathbb{Y}, \mathbb{X}] - \mathrm{Id})$ and $\mathbb{Z} : W \to W'$ is the quotient map.

Let E be the right A-module which is the restriction of $\mathcal{E}$. Since $\mathcal{E}$ has a left B structure it follows that E is a $B - A$-bimodule. As a $\mathrm{Mod}(\Delta^{\mathrm{opp}}) - A$ object it is given by



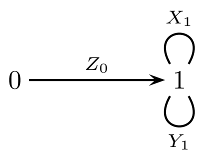
It is clear from the above discussion that the inverse to (A.1) is given by

$$(W, \mathbb{X}, \mathbb{Y}) \mapsto \mathrm{Tor}_1^B(M, E) \tag{A.4}$$

Let $\mathrm{proj}(B)$ be the category of finitely generated projective right B-modules and let Σ be the collection of maps $Z_1, \ldots, Z_{|w|-1}$. Let B_Σ be the universal localization of B at Σ.

Let us recall how B_Σ is constructed [Sch]. We adjoin the inverses of the maps in Σ to $\mathrm{proj}(B)$. Denote the resulting category by $\Sigma^{-1}\mathrm{proj}(B)$. Then $B_\Sigma = \mathrm{End}_{\Sigma^{-1}\mathrm{proj}(B)}(B)$. Now in our case $\mathrm{proj}(B) \cong \Delta$ and under this equivalence B corresponds to the sum of the vertices. It is also clear that $\Sigma^{-1}\mathrm{proj}(B)$ is equivalent to $\Sigma^{-1}\Delta$ which is obtained from Δ by adjoining inverses to the arrows in Σ. Then B_Σ is the endomorphism ring of the sum of the vertices in $\Sigma^{-1}\Delta$, i.e. the path algebra. Thus we obtain $B_\Sigma = \mathbb{C}(\Sigma^{-1}\Delta)$.

Now let Δ^0 be the following quiver.

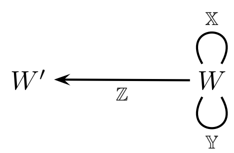


with relation $(X_1Y_1 - Y_1X_1 - \mathrm{Id})Z_0 = 0$. The obvious functor $\Sigma^{-1}\Delta \to \Delta^0$ which sends the arrows in Σ to the identity on the vertex 1 is an equivalence of categories. So we find $\mathrm{Mod}(\Sigma^{-1}\Delta) \cong \mathrm{Mod}(\Delta^0)$. Below we put $B^0 = \mathbb{C}(\Delta^0)$.

Now we return to (A.4). It is clear from the quiver description (A.3) that M may be viewed (necessarily in unique way) as an object in $\mathrm{Mod}(\Sigma^{-1}\Delta)$. Hence M is a right B_Σ-module. In a similar way it follows that E is a $B_\Sigma - A$-bimodule. Then according to [Sch, Thm 4.8(c)] we have $\mathrm{Tor}_1^B(M, E) = \mathrm{Tor}_1^{B_\Sigma}(M, E)$.

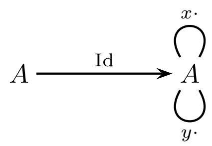
Under the equivalence $\mathrm{Mod}(\Delta^0) \cong \mathrm{Mod}(\Sigma^{-1}\Delta)$ the right B_Σ-module M corresponds to M^0 which is given by

M^0 :



Now $\mathrm{Tor}_1^{B_\Sigma}(-, E)$ is the first left derived functor of the functor $- \otimes_{B_\Sigma} E : \mathrm{Mod}(\Sigma^{-1}\Delta) \to \mathrm{Mod}(A)$. Checking on projectives we see that if we compose this functor with the equivalence $\mathrm{Mod}(\Delta^0) \cong \mathrm{Mod}(\Sigma^{-1}\Delta)$ then it is given by tensoring with the $B^0 - A$-module E^0 which is defined as follows:

E^0 :



Thus we have shown that the inverse to (A.1) is given by the functor

$$(W, \mathbb{X}, \mathbb{Y}) \mapsto \mathrm{Tor}_1^{B^0}(M^0, E^0) \tag{A.5}$$

It is clear that this inverse does not depend on w. This finishes the proof of Theorem 1.4.

We will now prove Theorem 1.3 by showing that (A.5) is compatible with the $\mathrm{Aut}(A)$-actions. Let us recall how these actions are defined. $\mathrm{Aut}(A)$ is generated by the automorphisms $\Psi_{n,\lambda}$ and $\Phi_{m,\mu}$ defined in the introduction. As explained in Section 7 there is an $\mathrm{Aut}(A)$ action on $\mathcal{C}_2$ which on the generators $\Psi_{n,\lambda}$ and $\Phi_{m,\mu}$ is given by $\Psi_{n,\lambda}(\mathbb{X}, \mathbb{Y}) = (\mathbb{X}, \mathbb{Y} - \lambda\mathbb{X}^n)$ and $\Phi_{m,\mu}(\mathbb{X}, \mathbb{Y}) = (\mathbb{X} - \mu\mathbb{Y}^m, \mathbb{Y})$.

We also define an action of $\mathrm{Aut}(A)$ on Δ^0 (viewed as an additive category) by

$$\Psi_{n,\lambda}(X^1) = X^1 \qquad \Phi_{m,\mu}(X^1) = X^1 + \mu(X^1)^m$$
$$\Psi_{n,\lambda}(Y^1) = Y^1 + \lambda(X^1)^n \qquad \Phi_{m,\mu}(Y^1) = Y^1$$
$$\Psi_{n,\lambda}(Z^0) = Z^0 \qquad \Phi_{m,\mu}(Z^0) = Z^0$$

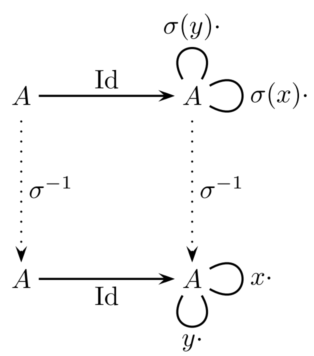
This is well defined because of the Remark in Section 7. Thus we obtain an action of $\mathrm{Aut}(A)$ on B^0 in the obvious way. We obtain a corresponding action on $\mathcal{C}_1$ by putting $\sigma(M^0) = M^0_{\sigma^{-1}}$ where $\sigma \in \mathrm{Aut}(A)$ and $M^0_{\sigma^{-1}}$ is the right B^0-module which is equal to M^0 as a set but whose right B^0-action is twisted by σ^{-1}. The action of $\mathrm{Aut}(A)$ on R is defined similarly.

By checking on the generators $\Psi_{n,\lambda}$ and $\Phi_{m,\mu}$ it is easy to see that the actions of $\mathrm{Aut}(A)$ on $\mathcal{C}_1$ and $\mathcal{C}_2$ are compatible. Hence to prove our claim it is sufficient to prove that the functor $M^0 \mapsto \mathrm{Tor}_1^{B^0}(M^0, E)$ is compatible with the $\mathrm{Aut}(A)$-actions. To prove this we need that

$$_\sigma E \cong E_{\sigma^{-1}} \tag{A.6}$$

as $B^0 - A$-bimodules, since if this is the case then $\mathrm{Tor}_1^{B^0}(M^0_{\sigma^-}, E) \cong \mathrm{Tor}_1^{B^0}(M^0, {}_\sigma E) \cong \mathrm{Tor}_1^{B^0}(M^0, E_{\sigma^{-1}}) \cong \mathrm{Tor}_1^{B^0}(M^0, E)_{\sigma^{-1}}$.

To prove (A.6) we note that $_\sigma E$ is the $\mathrm{Mod}(\Delta^{\mathrm{opp}}) - A$ object given by the top quiver in the diagram below.



It is clear that the map given by the dotted arrows is left B^0-linear and that it twists the right A-action by σ^{-1}. This finishes the proof of Theorem 1.3.

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