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Normal linearization and transition map near a saddle connection with symmetric resonances

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Abstract

We consider a heteroclinic connection in a planar system, between two symmetric hyperbolic saddles of which the eigenvalues are resonant. Starting with a C^∞ normal form, defined globally near the connection, we normally linearize the vector field by using finitely smooth tags of logarithmic form. We furthermore define an asymptotic entry–exit relation, and we discuss on two examples how to deal with counting limit cycles near a limit periodic set involving such a connection.

Keywords: Planar vector fields, Saddle connection, Invariant, Linearization, Poincaré map, Cyclicity

2010 MSC: 34C14, 34C20, 37C10, 37C15, 37C29

1. Introduction

There has been extensive research on bounding the number of isolated periodic orbits bifurcating from graphics (the cyclicity) in analytic planar vector fields in the context of Hilbert’s 16th problem following an idea of Roussarie ([1]). Graphics are formed by a finite sequence of heteroclinic connections that together with the connected singular points topologically form a circle. For instance in [2] the authors reduce the problem of finding a uniform bound on the number of limit cycles in quadratic vector fields to the study of 121 graphics. The classical way to do this is by studying the map of first return of such a graphic in order to get an upper bound. However these computations tend to be difficult in general especially in a neighborhood of singularities. Using normal form theory (see chapter 2 of [3]) one can simplify the local calculations (e.g. near a hyperbolic saddle, see [4]). When the graphic contains non-elementary singularities, for example in cuspidal loops (see [5]), one usually uses advanced techniques like a blow-up of the vector field near the singularity.

Here we will present a tool that may be useful in dealing with graphics that contain two hyperbolic saddles. More specifically, we consider in this paper C^∞ vector fields in the plane with two hyperbolic saddles A and B having a heteroclinic connection (see figure

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1). Without loss of generality we can assume that $A = (-1, 0)$ and $B = (1, 0)$. We impose that the linearization of the vector field about A (resp. about B) has a $-p : q$ (resp. $p : -q$) resonant spectrum; p and q positive and relatively prime integers. In this

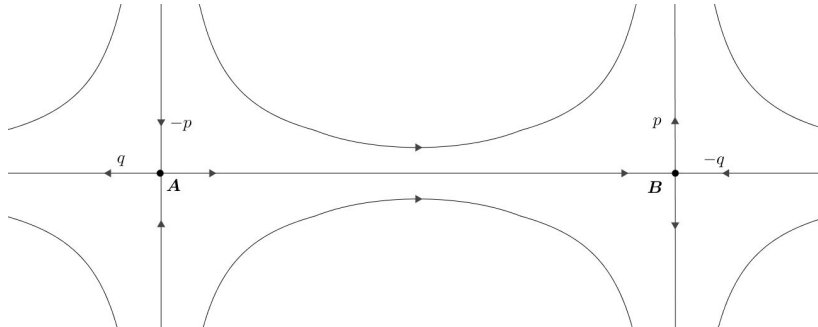


Figure 1: Saddle connection with symmetric $q : -p$ spectrum

paper, we do not consider unfoldings, i.e. here we do not consider families of vector fields in which the parameters either break the saddle connection and/or perturb the ratios of eigenvalues. This setting, where the ratios of eigenvalues is fixed and the saddle connection is unbroken is often encountered when studying polynomial vector fields at ∞ , or when blowing up nilpotent or degenerate singular points; the saddle connection is then found as a segment on the equator of the blow-up circle.

Under the imposed conditions a C^∞ normal form (up to time rescaling) near the connection has been obtained in [6]:

$$\begin{cases} \dot{x} &= \frac{q}{2}(1-x^2) \\ \dot{y} &= y(px + w^n f(w) + xw^n g(w) + \chi(x)h(y)), \end{cases} \quad (1)$$

where $w = (1-x^2)^{p/q}$ and χ is infinitely flat at $x = \pm 1$, $n \geq 1$ and all occurring functions are C^∞ . For readers familiar with local normal form theory, it might be beneficial to realize that the local normal forms about A and B have resonant terms of the form $(1+x)^{p/q}$ and $(1-x)^{p/q}$. The expression $xw^n g(w)$ represents the part of the normal form where B behaves truly reversible w.r.t. A ; it is the symmetric part. The expression $w^n f(w)$ represents the anti-symmetric part. The function $\chi(x)h(y)$ contains the so-called connecting terms (terminology from [6]); it is only present when $q \neq 1$. We will see that these terms may have an effect that is distinguishably different from the effect of the resonant terms on the dynamics near the connection.

The setup is quite specific and oriented towards symmetric resonant saddles, i.e. saddles with reciprocal saddle quantities. Following the discussion in [7] this is the most degenerate case when studying 2-saddle cycles, since besides the local resonances there is a supplementary resonance between the two saddles. Cyclicity results for these kinds of hyperbolic 2-polycycles have already been proven in [8] however when studying specific cases one loses information near the fixed connection. The regular transition inbetween the saddles can be of importance, even more than the non-smooth transition close to the saddles themselves. For instance in [5], it can be seen that the regular transition will add a non-smooth contribution to the transition map which is dominant as we will also prove in section 5.1. More non-degenerate cases consider connections between non-resonant

saddles, i.e. where the ratio of the eigenvalues (here given by p/q) is not rational, or situations where the anti-symmetry is dropped and therefore has no additional resonance between the saddles. Cyclicity results for these cases can be found in [9].

The goal of this paper is to establish a transition map along the connection and to apply it to cyclicity problems, counting the number of limit cycles nearby a given limit periodic set. Following the idea of linearizing individual saddles using logarithmic expressions (see e.g. [10]), we show in section 3 that we can normally linearize (1) in terms of the y -variable in a similar way using the local logarithmic expressions $\log(1-x)$ and $\log(1+x)$. This is obtained by a near-identity coordinate transformation $(x, y) \mapsto (x, Y) = (x, \psi(x, y))$ which is C^∞ in these logarithmic expressions. The resulting normally linearized equation is given by

$$\begin{cases} \dot{x} &= \frac{q}{2}(1-x^2) \\ \dot{Y} &= pxY. \end{cases} \quad (2)$$

A precise statement is given in Theorem 2.1. Clearly, this model can be integrated since $(1-x^2)^p Y^q$ is a first integral of the system. Moreover the map $\Sigma_{\text{in}}^{\text{lin}} \rightarrow \Sigma_{\text{out}}^{\text{lin}}$ is trivially given by $x_0 \mapsto -x_0$, where

$$\begin{aligned} \Sigma_{\text{in}}^{\text{lin}} &=]-1, -1 + \delta[\times \{Y_0\}, \\ \Sigma_{\text{out}}^{\text{lin}} &=]1 - \delta, 1[\times \{Y_0\}, \end{aligned}$$

for any given $Y_0 > 0$ and $\delta \in]0, 1[$. Using the normal form transformation we can then specify an invariant for the original system (1) and obtain qualitative information on the map $\Sigma_{\text{in}} \rightarrow \Sigma_{\text{out}}$, where

$$\begin{aligned} \Sigma_{\text{in}} &=]-1, -1 + \delta[\times \{y_0\}, \\ \Sigma_{\text{out}} &=]1 - \delta, 1[\times \{y_0\}. \end{aligned} \quad (3)$$

A precise statement for the asymptotics of the transition map is given in Theorem 2.3.

2. Statement of the results

Theorem 2.1. *Consider the C^∞ vector field (1) with $p, q \in \mathbb{N}_*$ and $\gcd(p, q) = 1$. There exists a near-identity coordinate change $(x, y) \mapsto (x, Y) = (x, y(1 + \psi(x, y)))$, preserving $y = 0$ and bringing (1) in the form (2). Moreover ψ is of the form*

$$\psi(x, y) = \Psi(x, y, w^n \log(1+x), w^n \log(1-x), (1-x^2)^{1/q}),$$

where Ψ is C^∞ near $[-1, 1] \times \{(0, 0, 0)\} \times [0, 1]$ and $w = (1-x^2)^p y^q$.

As observed before, we know that $W := (1-x^2)^p Y^q$ is a constant of motion for (2), hence from Theorem 2.1 it results to

Corollary 2.2. *Consider the C^∞ vector field (1) with $p, q \in \mathbb{N}_*$ and $\gcd(p, q) = 1$. There exists a constant of motion $V(x, y)$ of the vector field given by*

$$V(x, y) = (1-x^2)^p y^q (1 + \psi(x, y))^q,$$

where $\psi(x, y)$ is the function as described in theorem 2.1.

We will use this idea to compute the transition map

$$\Sigma_{\text{in}} \rightarrow \Sigma_{\text{out}}$$

with Σ_* as defined in (3) and using the parametrization there introduced. Suppose the vector field (1) can be written as

$$\begin{cases} \dot{x} = \frac{q}{2}(1-x^2), \\ \dot{y} = y(px + w^n f(w) + xw^n g(w) + \chi(x)h(y)y^k), \end{cases} \quad (4)$$

where $h(0) \neq 0$ and $f(0) \neq 0$. In section 4 we prove the following.

Theorem 2.3. *Consider the vector field as given by (4), where $h(0) \neq 0$ and $f(0) \neq 0$. The transition map*

$$D : \Sigma_{\text{in}} \rightarrow \Sigma_{\text{out}} : x_0 \mapsto D(x_0),$$

can be written as

$$D(x_0) = -x_0 - (1+x_0)\delta(x_0),$$

where δ is a C^∞ function in the variables

$$\left(x_0, (1-x_0^2)^{np} \log(1+x_0), (1-x_0^2)^{np} \log(1-x_0), (1-x_0^2)^{1/q} \right).$$

Moreover,

$$D(x_0) = -x_0 + \frac{1}{p}f(0)(1-x_0^2)^{np+1} \log(1+x_0) (1+F(x_0)), \quad \text{if } nq < k, \quad (5)$$

and

$$D(x_0) = -x_0 - \frac{2}{p}Ah(0)(1-x_0^2)^{\frac{pk}{q}+1} (1+F(x_0)), \quad \text{if } nq > k, \quad (6)$$

where

$$A = \int_0^1 \frac{\chi(x)}{(1-x^2)^{\frac{pk}{q}+1}} dx, \quad \text{and} \quad \lim_{x_0 \rightarrow -1^+} F(x_0) = 0.$$

Remark 2.4. Theorem 2.3 remains true when $f(0) = 0$ and $g(0) \neq 0$ in (4). In this case the exponent n in (5) should be replaced by the exponent of the first non-zero term in the expansion of $w^n f(w)$. We will clarify this claim in remarks 3.3 and 4.3 without going into further details. The asymptotics for (6) stays the same if $nq > k$. Note that $k = nq$ does not appear in the normal form from [6].

Remark 2.5. We can reformulate Theorem 2.3 in a more natural way for the asymptotics by parametrizing Σ_1 by $(-1+u_0, 1)$ and Σ_2 by $(1-u_1, 1)$. In this way, we can express the transition map $u_1 = \tilde{D}(u_0)$ as a C^∞ in the variables

$$\left(u_0, u_0^{np} \log(u_0), u_0^{1/q} \right).$$

Moreover, for $u_0 > 0$ close to 0,

$$\tilde{D}(u_0) = u_0 + \frac{2^{np+1}}{p} f(0) u_0^{np+1} \log(u_0) (1 + o(1)), \quad \text{if } nq < k,$$

and

$$\tilde{D}(u_0) = u_0 - \frac{2^{\frac{pk}{q}+2}}{p} Ah(0)u_0^{\frac{pk}{q}+1} (1 + o(1)), \quad \text{if } nq > k,$$

where A is given in theorem 2.3.

In section 5 we discuss on two examples how this result can be used to bound the number of limit cycles near a graphic containing a heteroclinic connection of aforementioned type. A crucial observation thereto is the non-smoothness of the principle term in $D(x_0)$.

The present paper is organized as follows. In section 3 we construct the transformation mentioned in theorem 2.1. First we eliminate the resonant terms by an induction process (section 3.1) before removing the connecting terms (section 3.2). Using the invariant of corollary 2.2, we compute the transition map (section 4) by considering it as a C^∞ function of finitely smooth variables (section 4.1). Then we compute the dominant non-linear term of this map in section 4.2. We illustrate these techniques and results on two applications: a cuspidal loop (section 5.1) and a loop through a fake saddle (section 5.2). We conclude on discussing this method and its applicability in section 6.

3. Proof of Theorem 2.1

3.1. Reduction of the resonant part

Consider (1) and recall that $w = (1 - x^2)^p y^q$, so

$$\dot{w} = qw^{n+1}f(w) + qwx^{n+1}g(w) + qw\chi(x)h(y).$$

For the moment we focus on the resonant part of (1), i.e. we neglect $\chi(x)h(y)$. A simple manipulation of the functions $f + xg$ leads to

$$\begin{cases} \dot{x} &= \frac{q}{2}(1 - x^2) \\ \dot{w} &= (1 - x)w^{n+1}F_L(w) + (1 + x)w^{n+1}F_R(w), \end{cases} \quad (7)$$

where F_L and F_R are a linear combination of the original f and g , more precisely

$$F_L(w) = \frac{q}{2}(f(w) - g(w)), \quad \text{and} \quad F_R(w) = \frac{q}{2}(f(w) + g(w)).$$

Later we will see the effect of our manipulations on the full system. Our intention is to increase the order of w in the equation for \dot{w} step by step using changes of coordinates in w . In the easier setting where one locally works around a single saddle, it is possible to remove the saddle's resonant terms using finitely smooth expressions involving logarithms (see e.g. [11]). Here we will extend this idea and therefore introduce the notion of tags.

3.1.1. Tags

In this paragraph we will introduce a series of tags which are functions of x , defined on $(-1, 1)$, by recursion. First we define T_L and T_R as the tags of order 1 satisfying $T_L(0) = T_R(0) = 0$ where we impose that their time-derivatives, denoted as \dot{T}_L and \dot{T}_R , should satisfy

$$\dot{T}_L = (1 - x), \quad \text{and} \quad \dot{T}_R = (1 + x).$$

The time dependence of x is expressed in the first line of (7). A direct computation shows that

$$T_L(x) = \frac{2}{q} \log(1+x), \quad \text{and} \quad T_R(x) = -\frac{2}{q} \log(1-x). \quad (8)$$

are the unique solutions satisfying the requirements. We recursively define T_* for any word $*$ composed of the alphabet $\{L, R\}$ as solutions of

$$\dot{T}_{*L} = (1-x)T_*, \quad T_{*L}(0) = 0, \quad \dot{T}_{*R} = (1+x)T_*, \quad T_{*R}(0) = 0,$$

more specifically

$$T_{*L}(x) = \int_0^x \frac{2 T_*(s)}{q(1+s)} ds, \quad T_{*R}(x) = \int_0^x \frac{2 T_*(s)}{q(1-s)} ds. \quad (9)$$

Unlike in the case [11], the tags do not easily admit a closed expression: tags of order 2 may already contain dilogarithm expressions and order 3 tags may even be more complicated. We do however show the following proposition:

Proposition 3.1. *Let $k \geq 1$. The tags T_* of order k (i.e. of word length k in $*$) are of the form*

$$T_*(x) = P_*(x, T_L(x)) + Q_*(x, T_R(x)),$$

where $P_*(x, u)$, resp. $Q_*(x, u)$, is polynomial in u with C^∞ coefficients in x of degree $L(*)$, resp. $R(*)$, equal to the amount of the letter L , resp. R , appearing in the word $*$.

Proof. For $k = 1$ this is obviously true. Suppose it is true for $k \geq 1$. This means that for words $*$ of length k we have

$$T_*(x) = \sum_{i=1}^{L(*)} f_*^i(x) T_L^i + \sum_{j=1}^{R(*)} g_*^j(x) T_R^j, \quad (10)$$

where the functions f_*^i and g_*^j are C^∞ . We show that the expression for T_{*R} is similar to (10) but the second summation is expanded to $R(*) + 1$. The case T_{*L} is treated similarly.

By the recursive definition (9), it suffices to show that for every positive integer k and C^∞ functions f and g ,

$$\int_0^x f(s) \frac{\log^k(1-s)}{1-s} ds = \sum_{i=0}^{k+1} F_i(x) \log^i(1-x), \quad (11)$$

and

$$\int_0^x g(s) \frac{\log^k(1+s)}{1-s} ds = G(x) \log(1-x) + \sum_{i=0}^k H_i(x) \log^i(1+x), \quad (12)$$

for some C^∞ functions F_i, G and H_i . Observe that for any C^∞ function f , we have

$$\int_0^x \frac{f(s)}{1-s} ds = -f(1) \log(1-x) + G(x),$$

for some C^∞ function G . Similarly, by partial integration we have

$$\int_0^x f(s) \log(1-s) ds = F(x) \log(1-x) + \int_0^x \frac{F(s)}{1-s} ds = (F(x) - F(1)) \log(1-x) + G(x),$$

for some C^∞ functions F and G . By induction on $n \in \mathbb{N}$ it follows:

$$\int_0^x f(s) \log^n(1-s) ds = \sum_{i=0}^n F_i^n(x) \log^i(1-x), \quad (13)$$

for some C^∞ functions F_i^n ($i = 0, \dots, n$) since by partial integration

$$\int_0^x f(s) \log^n(1-s) ds = (F(x) - F(1)) \log^n(1-x) - \int_0^x G(s) \log^{n-1}(1-s) ds,$$

where G is C^∞ and F is a C^∞ primitive function of f . From these observations (11) immediately follows since

$$\int_0^x f(s) \frac{\log^k(1-s)}{1-s} ds = f(1) \int_0^x \frac{\log^k(1-s)}{1-s} ds + \int_0^x g(x) \log^k(1-s) ds.$$

To deal with (12), we now define C^∞ bump functions $\chi_L(x)$ and $\chi_R(x) = \chi_L(-x)$ such that $\chi_L(x) + \chi_R(x) = 1$, and χ_L is locally 1, resp. 0, near $x = -1$, resp. $x = 1$. The integral in (12) can be separated in

$$\begin{aligned} \int_0^x g(s) \frac{\log^k(1+s)}{1-s} ds &= \int_0^x \left(\frac{g(s)}{1-s} \chi_L(s) \right) \log^k(1+s) ds \\ &\quad + \int_0^x \left(g(s) \log^k(1+s) \chi_R(s) \right) \frac{1}{1-s} ds. \end{aligned}$$

The expressions between brackets in each of the integrals are now C^∞ functions and since a similar result as (13) holds for $\log(1+x)$, (12) follows from all of the above. \square

3.1.2. Formal reduction of the resonant part using tags

We show that we can formally eliminate the resonant terms in (7). Normal linearization amounts to finding a perturbation w_∞ of $w = (1-x^2)^p y^q$ for which $\dot{w}_\infty = 0$, in other words we seek a first integral of the form $w_\infty = w + w^2 \bar{\psi}(x, w)$. The new coordinate $Y(x, y)$ is chosen such that $w_\infty = (1-x^2)^p Y^q$, i.e.

$$Y = y(1 + (1-x^2)^p y^q \bar{\psi}(x, (1-x^2)^p y^q))^{1/q}, \quad (14)$$

will give the required normal linear form, eliminating completely the resonant part. Denote by \mathcal{W} the set of words with alphabet $\{L, R\}$ and define for every $k \in \mathbb{N}_0$ the set \mathcal{W}_k of words with length k .

Theorem 3.2. *There exists a formal transformation*

$$w_\infty = w - \sum_{k=1}^{\infty} w^{kn+1} \sum_{* \in \mathcal{W}_k} F_*(w) T_*,$$

where F_* are C^∞ functions and the tags T_* are defined by (8) and (9), such that (7) transforms to

$$\begin{cases} \dot{x} = \frac{g}{2}(1-x^2), \\ \dot{w}_\infty = 0. \end{cases}$$

Proof. Let $w_0 = w$. We claim that by appropriately choosing

$$w_{k+1} = w_k - w^{(k+1)n+1} \sum_{* \in \mathcal{W}_{k+1}} F_*(w) T_*(w), \quad k \geq 0, \quad (15)$$

for some C^∞ functions F_* , we can ensure that

$$\dot{w}_k = w^{(k+1)n+1} \left(\sum_{* \in \mathcal{W}_k} F_{*L}(w)(1-x)T_* + F_{*R}(w)(1+x)T_* \right), \quad (16)$$

for some C^∞ functions F_{*L} and F_{*R} , $* \in \mathcal{W}_k$. We show how these are defined in the induction step below. Since the order in w of the words of length k increases with k , it will imply that \dot{w}_k becomes flatter with growing k . The limit w_∞ of this transformation is of the desired form and due to the growing flatness will satisfy $\dot{w}_\infty = 0$. The claim is obviously true for $k = 0$. Let us now proceed under the induction hypothesis that the claim is correct up to order k , i.e. (16) holds. Define

$$w_{k+1} = w_k - w^{(k+1)n+1} \left(\sum_{* \in \mathcal{W}_k} F_{*L}(w) T_{*L} + F_{*R}(w) T_{*R} \right).$$

A simple computation shows that

$$\begin{aligned} \dot{w}_{k+1} &= \sum_{* \in \mathcal{W}_k} \underbrace{\left(-w^{n+1} F_L(w) \frac{d(w^{(k+1)n+1} F_{*L}(w))}{dw} \right)}_{w^{(k+2)n+1} F_{*LL}(w)} (1-x) T_{*L} \\ &+ \sum_{* \in \mathcal{W}_k} \underbrace{\left(-w^{n+1} F_R(w) \frac{d(w^{(k+1)n+1} F_{*L}(w))}{dw} \right)}_{w^{(k+2)n+1} F_{*LR}(w)} (1+x) T_{*L} \\ &+ \sum_{* \in \mathcal{W}_k} \underbrace{\left(-w^{n+1} F_L(w) \frac{d(w^{(k+1)n+1} F_{*R}(w))}{dw} \right)}_{w^{(k+2)n+1} F_{*RL}(w)} (1-x) T_{*R} \\ &+ \sum_{* \in \mathcal{W}_k} \underbrace{\left(-w^{n+1} F_R(w) \frac{d(w^{(k+1)n+1} F_{*R}(w))}{dw} \right)}_{w^{(k+2)n+1} F_{*RR}(w)} (1+x) T_{*R}, \end{aligned}$$

which is equivalent to (16) for $k + 1$. □

Remark 3.3. In the setting of remark 2.4, one may choose first to delete only a finite part of the symmetric resonant terms, corresponding to the function g . Clearly, the transformation formula (14) will in that case be even in x and such an even near-identity transformation will have an effect on the asymmetric part of the resonant terms, i.e. the function g will be altered. However, the principal term of f remains exactly the same.

3.1.3. Finitely smooth reduction of the resonant part

From (15) and the result of Proposition 3.1, we find that the transformation in each step can be expressed as a C^∞ function in $(x, y, w^n \log(1+x), w^n \log(1-x))$. Using Borel's theorem, there exists a function $\psi(x, y, w^n \log(1+x), w^n \log(1-x))$, formally equal to w_∞ , so that $\dot{\psi}$ is formally identically zero. Using the techniques in [6], it is possible to adapt ψ by a flat function to make $\dot{\psi}$ truly zero as a function. We will not repeat this construction here.

Remark 3.4. The functions $w^n \log(1-x)$ and $w^n \log(1+x)$ are of Logarithmic Mourta type (LMT) near resp. 1 and -1 (see [12] or [11]) and C^∞ in $] -1, 1[$. The loss of smoothness is thus located at the points $x = \pm 1$.

3.2. Removing the connecting terms

Returning to the full system (1), we have under the new set of coordinates

$$\dot{w} = qw\chi(x)\tilde{h}(x, y),$$

for some finitely smooth \tilde{h} . However the non-smoothness only occurs at $x = \pm 1$. Since χ is infinitely flat at these two points, the product $\chi(x)\tilde{h}(x, y)$ persists as a C^∞ function, infinitely flat at $x = \pm 1$. In terms of the original (x, y) coordinates, this gives

$$\begin{cases} \dot{x} &= \frac{q}{2}(1-x^2) \\ \dot{y} &= y(px + \chi(x)\tilde{h}(x, y)). \end{cases} \quad (17)$$

Theorem 3.5. Consider the vector field (17). There exists a near-identity coordinate change

$$(x, y) \mapsto (x, Y) = (x, Y(1 + \varphi(x, Y))),$$

bringing (17) in the form (2). Moreover φ is of the form

$$\varphi(x, y) = \Phi\left(x, y, (1-x^2)^{1/q}\right),$$

where Φ is C^∞ near $[-1, 1] \times \{0\} \times [0, 1]$.

Proof. Write $1-x^2 = (1-X^2)^q$ and therefore

$$x = X\Omega(X), \quad \text{where } \Omega(X) = \sqrt{\frac{1 - (1-X^2)^q}{X^2}}. \quad (18)$$

Observe that $\Omega(X)$ is a C^∞ strictly positive function for $X \in (-\sqrt{2}, \sqrt{2})$. This change of coordinates maps $[-1, 1]$ to itself, although in a finitely smooth way at the boundary. After division by $\Omega(X)$, the effect of (18) on system (17) is:

$$\begin{cases} \dot{X} &= \frac{1}{2}(1-X^2) \\ \dot{y} &= y\left(pX + \frac{1}{\Omega(X)}\chi(X\Omega(X))\tilde{h}(X\Omega(X), y)\right). \end{cases}$$

Since transformation (18) fixes $x = \pm 1$, the second term remains flat and therefore \dot{y} can be written as

$$\dot{y} = y(pX + H(X, y)) \quad (19)$$

for some C^∞ function H that is flat at $X = \pm 1$. Since this system has a saddle connection with ratios of eigenvalues $-p : 1$ and $p : -1$, its normal form has no connecting terms (i.e. $h = 0$ in the form (1)); it even has no resonant terms due to the flatness of H . Therefore there exists, according to [6], a C^∞ normalizing transformation $Y = y(1 + \psi(X, y))$ reducing it to normalized form. We can hence also apply the transformation

$$Y = y(1 + \varphi(x, y)), \quad \text{where } \varphi(x, y) = \psi(X(x), y),$$

to (17) in (x, y) -coordinates to obtain a finitely smooth transition to (2).

It remains to prove that φ can be expressed as a C^∞ function of $x, s = (1 - x^2)^{1/q}, y$. Note that

$$X(x) = \text{sign}(x)\sqrt{1 - (1 - x^2)^{1/q}},$$

so it suffices to prove that $X(x)$ is C^∞ in x and $(1 - x^2)^{1/q}$. We have

$$X(x) = x\sqrt{\frac{1-s}{x^2}} = x\sqrt{\frac{1-s}{1-s^q}} = x\sqrt{\frac{1}{1+s+\dots+s^{q-1}}} = x\rho\left((1-x^2)^{1/q}\right),$$

where ρ is C^∞ . □

Applying subsequently the C^∞ version of theorem 3.2 and theorem 3.5 to (1) finishes the proof of theorem 2.1.

4. Proof of Theorem 2.3

4.1. The transition map as a function of the tags

As discussed in section 2 (corollary 2.2), we find a C^∞ constant of motion of the system (1) given by

$$V(x, y) = w \left(1 + \Psi \left(x, y, w^n T_L, w^n T_R, (1 - x^2)^{1/q} \right) \right)^q, \quad (20)$$

using the first integral of the normally linearized system.

Let us now compute the entry-exit relation. Denote the initial variable on Σ_{in} by x_0 and the corresponding exit variable by x_1 . Remark that $x_1 \rightarrow 1$ as $x_0 \rightarrow -1$, so we write

$$1 - x_1 = (1 + x_0)(1 + \delta(x_0)). \quad (21)$$

As a matter of fact we will see that δ tends to 0 as x_0 tends to -1 , which we will show using the implicit function theorem with (21) as ansatz. This form of transition map originates from the fact that it is near-identity due to the symmetry of the eigenvalues of the saddle (see figure 2).

At the cuts, the invariant is given by

$$V(x, 1) = (1 - x^2)^p \left(1 + \Psi \left(x, 1, (1 - x^2)^{np} T_L, (1 - x^2)^{np} T_R, (1 - x^2)^{1/q} \right) \right)^q.$$

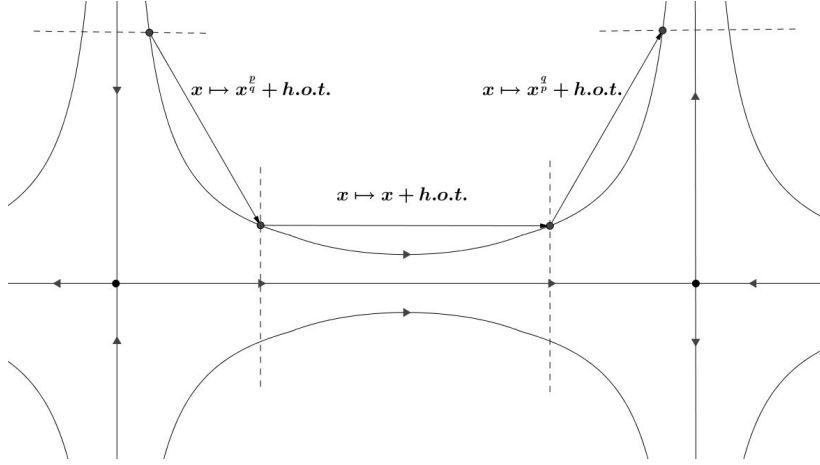


Figure 2: Asymptotics of transition near saddle connection

For the sake of notation, denote the LMT-functions as

$$\bar{T}_L = (1 - x_0^2)^{np} T_L(x_0), \quad \text{and} \quad \bar{T}_R = (1 - x_0^2)^{np} T_R(x_0). \quad (22)$$

We aim to express δ in terms of $(x_0, \bar{T}_L, \bar{T}_R, (1 - x_0^2)^{1/q})$ by applying the implicit function theorem to the equation $V(x_0, 1) = V(x_1, 1)$ since V is invariant under the flow and we impose that $(x_0, 1)$ and $(x_1, 1)$ are different points of the same orbit.

First we need to express $T_L(x_1), T_R(x_1), (1 - x_1^2)$ in terms of x_0 and δ . Observe that

$$T_R(x_1) = -\frac{2}{q} \log(1 - x_1) = -T_L(x_0) - \frac{2}{q} \delta + O(\delta^2). \quad (23)$$

Here and in the remainder of this section appearing O -terms are C^∞ in (x_0, δ) near $(-1, 0)$. Remark that they can blow up close to $x = 1$, but since we are interested in the behaviour near $x = -1$ this does not pose a problem. The tag $T_L(x)$ is C^∞ at $x = 1$, just as $T_R(x)$ is C^∞ at $x = -1$. We see

$$T_L(x_1) = -T_R(x_0) - \frac{2}{q} \delta \frac{1 + x_0}{1 - x_0} + O(\delta^2). \quad (24)$$

A simple computation shows

$$(1 - x_1^2) = (1 - x_0^2) \left(1 - 2\delta \frac{x_0}{1 - x_0} + O(\delta^2) \right),$$

hence for a power $r \in \mathbb{Q}^+$,

$$(1 - x_1^2)^r = (1 - x_0^2)^r \left(1 - 2r\delta \frac{x_0}{1 - x_0} + O(\delta^2) \right). \quad (25)$$

Using the expansions above, more precise (23) and (25) for $r = np$, we have

$$\bar{T}_R^{(1)} := (1 - x_1^2)^{np} T_R(x_1) = -\bar{T}_L + \left(2np \frac{x_0}{1 - x_0} \bar{T}_L - \frac{2}{q} (1 - x_0^2)^{np} \right) \delta + O(\delta^2), \quad (26)$$

and by (24) and (25)

$$\bar{T}_L^{(1)} := (1-x_1^2)^{np}T_L(x_1) = -\bar{T}_R + \left(2np\frac{x_0}{1-x_0}\bar{T}_R - \frac{2}{q}(1-x_0^2)^{np}\frac{1+x_0}{1-x_0}\right)\delta + O(\delta^2). \quad (27)$$

Denote

$$\Psi_1 = \Psi \left(x_1, (1-x_1^2)^{np}T_L(x_1), (1-x_1^2)^{np}T_R(x_1), (1-x_1^2)^{1/q} \right), \quad (28)$$

where Ψ is introduced in Theorem 2.1. This can be expressed as a function of x_0 and δ thanks to (21), (27), (26) and (25)

$$\Psi_1 = \Psi \left(-x_0 - \delta(1+x_0), 1, \bar{T}_L^{(1)}, \bar{T}_R^{(1)}, (1-x_0^2)^{1/q} \left(1 - \frac{2}{q}\frac{x_0}{1-x_0}\delta + O(\delta^2) \right) \right), \quad (29)$$

where Ψ is introduced in Theorem 2.1. Since $V(x_1, 1)$ can be expressed as a function of x_0 using (25) and (29), we can search for solutions δ of

$$0 = \Theta \left(\delta, x_0, \bar{T}_L, \bar{T}_R, (1-x_0^2)^{1/q} \right) := \left(\frac{V(x_1, 1)}{(1-x_0^2)^p} \right)^{1/q} - \left(\frac{V(x_0, 1)}{(1-x_0^2)^p} \right)^{1/q}, \quad (30)$$

where Θ is C^∞ near $(0, -1, 0, 0, 0)$, such that $V(x_0, 1) = V(x_1, 1)$ is satisfied. In order to apply the implicit function theorem to (30) at $(0, -1, 0, 0, 0)$ and consequently show that we can express δ in terms of $(x_0, \bar{T}_L, \bar{T}_R, (1-x_0^2)^{1/q})$, it is sufficient to show that

$$\frac{\partial \Theta}{\partial \delta}(0, -1, 0, 0, 0) \neq 0,$$

since $\Theta(0, -1, 0, 0, 0) = 0$. Notice that

$$\Theta = \left(1 - \frac{2p}{q}\frac{x_0}{1-x_0}\delta + O(\delta^2) \right) (1 + \Psi_1) - \left(1 + \Psi \left(x_0, 1, \bar{T}_L, \bar{T}_R, (1-x_0^2)^{1/q} \right) \right).$$

We find

$$\frac{\partial \Theta}{\partial \delta} = -\frac{2p}{q}\frac{x_0}{1-x_0}(1 + \Psi_1) + \frac{\partial \Psi_1}{\partial \delta} + O(\delta), \quad (31)$$

where

$$\frac{\partial \Psi_1}{\partial \delta} = O(1-x_0^2, \bar{T}_R, \bar{T}_L, \delta).$$

Hence we see

$$\frac{\partial \Theta}{\partial \delta}(0, -1, 0, 0, 0) = \frac{p}{q} + \frac{\partial \Psi_1}{\partial \delta}(0, -1, 0, 0, 0) = \frac{p}{q} \neq 0.$$

By the implicit function theorem, we can thus write

$$x_1 = -x_0 - (1+x_0)\delta(x_0),$$

where we can express

$$\delta(x_0) = \bar{\delta} \left(x_0, \bar{T}_L, \bar{T}_R, (1-x_0^2)^{1/q} \right),$$

for a C^∞ function $\bar{\delta}$ at $(-1, 0, 0, 0)$.

Remark 4.1. We can give an alternative proof of the first part of theorem 2.3 by using the notation of remark 2.5. Here we give a sketch of the proof. Due to corollary 2.2, the transition map $u_1 = \tilde{D}(u_0)$ is given implicitly by

$$V(-1 + u_0, 1) = V(1 - u_1, 1). \quad (32)$$

Denote by $v_i = u_i^{1/q}$ for $i = 0, 1$. After taking the $\frac{1}{npq}$ -th power of (32), this can be written as

$$v_0 (1 + \Psi_L(v_0, v_0^{npq} \log(v_0))) = v_1 (1 + \Psi_R(v_1, v_1^{npq} \log(v_1))), \quad (33)$$

for some smooth functions Ψ_L, Ψ_R . Denote

$$z_1 = v_1 (1 + \Psi_R(v_1, v_1^{npq} \log(v_1))).$$

It now suffices to invert this relation, such that we find an expression

$$v_1 = z_1 (1 + \bar{\Psi}_R(z_1, z_1^{npq} \log(z_1))),$$

for some smooth function $\bar{\Psi}_R$ and substitute z_1 by the left-hand side of (33). This is done by denoting $V = v_1^{npq} \log(v_1)$ and $Z = z_1^{npq} \log(z_1)$ and applying the implicit function theorem to the system

$$\begin{cases} z_1 = v_1 (1 + \Psi_R(v_1, V)), \\ Z = v_1^{npq} (1 + \Psi_R(v_1, V))^{npq} \log[v_1 (1 + \Psi_R(v_1, V))]. \end{cases}$$

After returning to the original variables u_0, u_1 it's easy to see that we can express $\tilde{D}(u_0)$ as a smooth function of the variables $(u_0, u_0^{np} \log(u_0), u_0^{1/q})$.

4.2. Asymptotics of the transition map

In the previous section we proved that we can express the transition map in terms of

$$(x_0, \bar{T}_L, \bar{T}_R, (1 - x_0^2)^{\frac{1}{q}}) = (x_0, (1 - x_0^2)^{np} T_L(x_0), (1 - x_0^2)^{np} T_R(x_0), (1 - x_0^2)^{\frac{1}{q}}). \quad (34)$$

We now want to compute the asymptotics of the map near $x_0 = -1$.

Remark 4.2. One can define an asymptotic scale as has been done in [5]. For this we express the variables (34) in terms of the small variable $u_0 = x_0 + 1$ (see remark 2.5) by

$$u_0^{r_1} \log^{m_1}(u_0) \succ u_0^{r_2} \log^{m_2}(u_0),$$

if $r_1 > r_2$ or $r_1 = r_2$ and $m_1 < m_2$ for $r_i \in \mathbb{Q}$ and $m_i \in \mathbb{N}$.

Recall that δ defined in (21) should be a solution of (30). Hence if we expand Θ near $\delta = 0$, we ought to solve

$$0 = \Theta(0, x_0, \bar{T}_L, \bar{T}_R, (1 - x_0^2)^{1/q}) + \delta \frac{\partial \Theta}{\partial \delta} (0, x_0, \bar{T}_L, \bar{T}_R, (1 - x_0^2)^{1/q}) + O(\delta^2). \quad (35)$$

From the definition of Θ , it follows immediately that

$$\Theta|_{\delta=0} = \Psi\left(-x_0, 1, -\bar{T}_R, -\bar{T}_L, (1-x_0^2)^{1/q}\right) - \Psi\left(x_0, 1, \bar{T}_L, \bar{T}_R, (1-x_0^2)^{1/q}\right), \quad (36)$$

and using (31) one can check

$$\frac{\partial\Theta}{\partial\delta}\left(0, x_0, \bar{T}_L, \bar{T}_R, (1-x_0^2)^{1/q}\right) = \frac{2p}{q} \frac{1}{1-x_0} + o(1).$$

Here $o(1)$ denotes a finitely smooth function of x_0 which converges to 0 for x_0 going to -1 . If we want to compute the dominant term of the transition map, i.e. the term of lowest asymptotic order we have to describe (36). Suppose (1) can be written as (4) where $f(0) \neq 0$ and $h(0) \neq 0$ for some n, k with $k \neq nq$. We have to distinguish two cases depending on which of the transformations (section 3.1 or section 3.2) is dominant, i.e. provides the terms of lowest degree of $(1-x^2)$ in the linearising transformation of theorem 2.1:

- (A) $k > nq$,
- (B) $k < nq$.

4.2.1. Case A

Suppose $k > nq$. In this case, the transformation discussed in section 3.1 provides the lowest order terms in the linearising transformation. From theorem 3.2, it formally is given by

$$w_\infty = w - w^{n+1} (F_L(0)T_L + F_R(0)T_R) + \text{h.o.t.},$$

where the higher order terms contain expressions of higher degree in w . The transformation of theorem 2.1 can be written as

$$Y = y \left(1 - \frac{1}{q} w^n (F_L(0)T_L + F_R(0)T_R) + \text{h.o.t.} \right),$$

Asymptotically at $x = -1$, we have

$$\Psi|_{y=1} = -\frac{1}{q} (F_L(0)T_L(x) + F_R(0)T_R(x)) (1 + o(1)).$$

Therefore (36) reduces to

$$\Theta|_{\delta=0} = \frac{1}{q} (F_R(0) + F_L(0)) \bar{T}_L (1 + o(1)).$$

Combining the above with the Taylor expansion given in (35), we get

$$\delta = -\frac{1}{2p} (1-x_0) (F_R(0) + F_L(0)) \bar{T}_L (1 + o(1)),$$

Hence

$$\begin{aligned} D(x_0) &= -x_0 + \frac{1}{2p} (1-x_0^2) (F_R(0) + F_L(0)) \bar{T}_L (1 + o(1)) \\ &= -x_0 + \frac{q}{2p} (1-x_0^2) f(0) \bar{T}_L (1 + o(1)), \end{aligned}$$

which concludes the proof of theorem 2.3 in the case $k > nq$.

Remark 4.3. In the setting of remark 2.4, it's easy to see that the transition map doesn't change if the lowest order term of g is lower than the lowest order term of f . Indeed, thanks to remark 3.3 we can get these orders at the same height using a symmetric transformation. However, this has no effect on the lowest order of δ since in (36) the symmetric part cancels out.

4.2.2. Case B

Suppose $k < nq$. Here the transformation of theorem 3.5 provides the lowest order terms in the transformation of theorem 2.1. The linearising transformation is asymptotically given by

$$Y = y(1 - h(0)y^k\Phi(x) + \text{h.o.t.}),$$

where $\Phi(x) = \Phi(X\Omega(X)) = \tilde{\Phi}(X)$ is a solution of

$$-\frac{1}{2}(1 - X^2)\tilde{\Phi}'(X) - pkX\tilde{\Phi}(X) + \frac{\chi(X\Omega(X))}{\Omega(X)} = 0,$$

with $(1 - x^2) = (1 - X^2)^q$ and $\Omega(X)$ is defined in (18). In the original variable, this translates to solving

$$-\frac{q}{2}(1 - x^2)\Phi'(x) - pkx\Phi(x) + \chi(x) = 0,$$

hence

$$\Phi(x) = \frac{2}{q}(1 - x^2)^{\frac{pk}{q}} \int_0^x \frac{\chi(s)}{(1 - s^2)^{\frac{pk}{q}+1}} ds.$$

The transformation Ψ at the cuts is thus given by

$$\Psi|_{y=1} = -\frac{2}{q}h(0)(1 - x^2)^{\frac{pk}{q}} \int_0^x \frac{\chi(s)}{(1 - s^2)^{\frac{pk}{q}+1}} ds (1 + o(1)).$$

Remark that

$$\int_0^{x_0} \frac{\chi(s)}{(1 - s^2)^{\frac{pk}{q}+1}} ds = -A + o(1), \text{ and } \int_0^{-x_0} \frac{\chi(s)}{(1 - s^2)^{\frac{pk}{q}+1}} ds = A + o(1),$$

where A is defined in theorem 2.3. Similar as before, we see that the symmetric difference (36) is given by

$$\Theta|_{\delta=0} = -\frac{4}{q}Ah(0)(1 - x_0^2)^{\frac{pk}{q}} (1 + o(1)),$$

leading to

$$D(x_0) = -x_0 - \frac{2}{p}Ah(0)(1 - x_0^2)^{\frac{pk}{q}+1}(1 + o(1)).$$

5. Applications

In this section we illustrate how we can use the results in some applications. They typically originate from the blow-up of a non-elementary singularity.

5.1. The cusp-preserving unfolding

We consider a vector field unfolding a cusp-singularity as in [5]. We assume that the vector fields are already written into Loray's normal form (see [13]). In this paper, the author essentially distinguishes two cases:

$$\alpha_n : \begin{cases} \dot{x} &= 2y + 2xh^n (f(h) + xg(h)), \\ \dot{y} &= 3x^2 + 3yh^n (f(h) + xg(h)), \end{cases} \quad (37)$$

where $f(0) \neq 0$ and

$$\beta_n : \begin{cases} \dot{x} &= 2y + 2xh^n (hf(h) + xg(h)), \\ \dot{y} &= 3x^2 + 3yh^n (hf(h) + xg(h)) \end{cases} \quad (38)$$

where $g(0) \neq 0$ with $h := x^3 - y^2$ and f and g are C^∞ in each case. We perform a quasi-homogeneous blow-up $(x, y) = (r^2 \cos \theta, r^3 \sin \theta)$ leading to two hyperbolic saddles on the blow-up locus with reciprocal saddle quantities. One can check that the two cusp separatrices of α_n (resp. β_n) approach the origin in the directions $\theta = \pm\theta_0$ for some $\theta_0 \in]0, \frac{\pi}{2}[$. To get the transition map near the right part of the blow-up locus, we consider the directional chart

$$(x, y) = (Y^2, Y^3 X). \quad (39)$$

The variable Y serves as the radial variable whereas X acts as (projectivized) angular variable.

To study the part to the left of the singularity one would first think of considering a similar directional chart, which would give us information on the directions $\theta \in]\frac{\pi}{2}, \frac{3\pi}{2}[$. However, as this region does not include θ_0 it is better to replace this directional chart by a chart using a rational parametrization of the parabola that approximates a circle near $\theta = \pi$:

$$(x, y) = (Y^2(X^2 - 1), 2Y^3 X).$$

Again, the variable Y serves as the radial variable whereas X acts as angular variable. In fact it reveals convenient for the computations to do some scaling; we will hence use instead:

$$(x, y) = (aY^2(4X^2 - 1), 4Y^3 X), \quad \text{where } a := \frac{2}{3}2^{1/3}. \quad (40)$$

As will be explained in (44), this choice of a is necessary to have a nice factorization of the \dot{X} -equation.

5.1.1. Blow-up chart (39)

For α_n we find, after division by the non-negative factor Y :

$$\alpha_n : \begin{cases} \dot{X} &= 3(1 - X^2) \\ \dot{Y} &= YX + H^n (f(H) + Y^2g(H)), \end{cases} \quad (41)$$

where $f(0) \neq 0$ and $H = (1 - X^2)Y^6$. Similarly, β_n becomes

$$\beta_n : \begin{cases} \dot{X} &= 3(1 - X^2) \\ \dot{Y} &= YX + H^n ((1 - X^2)Y^6 f(H) + Y^2g(H)), \end{cases} \quad (42)$$

where $g(0) \neq 0$. In both cases, the saddle connection lies on $Y = 0$ between $X = \pm 1$.

5.1.2. Blow-up chart (40)

A straightforward computation gives

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \frac{1}{4aY^3(8X^2+1)} \begin{pmatrix} 6XY & -a(4X^2-1) \\ -2Y^2 & 4aXY \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \quad (43)$$

Let us first compute \dot{X}

$$\dot{X} = \frac{3Y}{4a(8X^2+1)}(16X^2 - a^3(4X^2-1)^3).$$

For the special value $a^3 = 16/27$, this simplifies to

$$\dot{X} = \frac{4Y(8X^2+1)}{9a}(1-X^2). \quad (44)$$

After division of (43) by the non-negative factor $\frac{a^2}{4}(1+8X^2)Y$ and normalizing by

$$Y = (8X^2+1)^{-1/3}\bar{Y},$$

we obtain

$$\alpha_n : \begin{cases} \dot{X} = 3(1-X^2), \\ \dot{\bar{Y}} = X\bar{Y} + \frac{27a}{4(8X^2+1)^{2/3}}\bar{H}^n(f(\bar{H}) + a\frac{4X^2-1}{(8X^2+1)^{2/3}}\bar{Y}^2g(\bar{H})), \end{cases} \quad (45)$$

and

$$\beta_n : \begin{cases} \dot{X} = 3(1-X^2), \\ \dot{\bar{Y}} = X\bar{Y} + \frac{27a}{4(8X^2+1)^{2/3}}\bar{H}^n(\bar{H}f(\bar{H}) + a\frac{4X^2-1}{(8X^2+1)^{2/3}}\bar{Y}^2g(\bar{H})), \end{cases} \quad (46)$$

where $\bar{H} = -\frac{16}{27}(1-X^2)\bar{Y}^6$.

5.1.3. The normalizing transformation

All the previous vector fields can be written in the form

$$\begin{cases} \dot{x} = 3(1-x^2), \\ \dot{y} = y(x + F(x)y^k + O(y^l)), \end{cases} \quad (47)$$

where $k = 6n - 1, l = 6n + 1$, resp. $k = 6n + 1, l = 6n + 5$, for α_n , resp. β_n and where F is not identically 0. We show in this section that the first non-zero term gives rise to a non-zero connecting term in the normal form (4) which is of lower order than the resonant terms, i.e. we are in the setting of section 4.2.2. From [6], we can show the following

Lemma 5.1. *Let $k = 6n - 1$ for some $n \geq 1$ or $k = 6n + 1$ for some $n \geq 0$. There exists a smooth coordinate transformation $(x, y) \mapsto (x, z) = (x, \varphi(x, y))$, such that system (47), with F not identically 0, is orbitally equivalent to*

$$\begin{cases} \dot{x} = 3(1 - x^2), \\ \dot{z} = z \left[x + \gamma\chi(x)z^k + \chi(x)z^{k+1}f_1(z) \right. \\ \quad \left. + (1 - x^2)^l z^{6l} (f_2((1 - x^2)z^6) + xf_3((1 - x^2)z^6)) \right], \end{cases} \quad (48)$$

where $\gamma \neq 0$, $k < 6l$ and f_1, f_2, f_3 are C^∞ . Moreover, we have

$$y = z + G(x)z^{k+1} + O(z^{l+1}),$$

where G is a C^∞ solution of

$$-3(1 - x^2)G'(x) - kxG(x) + F(x) = \gamma\chi(x). \quad (49)$$

Following the discussion in [6], the coefficient γ in (48) has the property that it is the unique coefficient for which (49) has a smooth solution G in a neighbourhood of $[-1, 1]$ and depends on the function F .

For each of the cases above, we want to compute the coefficient γ . For this, we need the following lemma.

Lemma 5.2. *Let $p, k, q \in \mathbb{N}_0$ such that $\gcd(p, q) = 1$ and $\lambda = \frac{pk}{q} \notin \mathbb{N}$. Let $N = \lfloor \lambda \rfloor$. There exists a $\gamma \in \mathbb{R}$ such that the differential equation*

$$\frac{-q}{2} \frac{dh(x)}{dx} (1 - x^2) - pkxh(x) + (1 - x^2)^{N+1} = \gamma\chi(x), \quad (50)$$

has a C^∞ solution in a neighbourhood of $[-1, 1]$. Moreover,

$$\gamma \int_0^1 \frac{\chi(u)}{(1 - u^2)^{\lambda+1}} du = \frac{1}{2} \frac{\sqrt{\pi} \Gamma(1 - \alpha)}{\Gamma(\frac{3}{2} - \alpha)},$$

where $\alpha = \lambda - N$.

Proof. Smooth solutions of (50) correspond to smooth graphs $y = h(x)$ tangent to the vector field

$$\begin{cases} \dot{x} = \frac{q}{2}(1 - x^2), \\ \dot{y} = -pkxy + (1 - x^2)^{N+1} - \gamma\chi(x). \end{cases} \quad (51)$$

This vector field admits two nodes at $(-1, 0)$ and $(1, 0)$ with respective eigenvalues (q, pk) and $(-q, -pk)$. Since $\frac{pk}{q} \notin \mathbb{N}$, these nodes are non-resonant except for the case $p = 1$ and $q = nk$ for some $n \in \mathbb{N}$. In this case, the Poincaré-Dulac normal form at -1 and 1 admits possibly one resonant term of the form $y^{nk} \frac{\partial}{\partial x}$. However, since the first equation of (51) is independent of y , this resonant term will not appear and therefore (51) is locally smoothly linearizable (as is also the case if the ratio of the eigenvalues does not belong to $\mathbb{N} \cup \mathbb{N}^{-1}$). The curve $y = 0$ is a smooth separatrix of the linearized system and induces the local existence of a smooth separatrix of (51) near $(-1, 0)$ and $(1, 0)$. Locally we can denote these curves as $y = \varphi_\gamma(x)$ and $y = \psi_\gamma(x)$, which are C^∞ at respectively -1 and 1 . By continuation, we can expand their domains of definition such that they

contain $(-1, 1)$, which is only possible since $\dot{x} > 0$ in $(-1, 1)$, whereas the solution can not escape to infinity. This can either be seen by applying the Gronwall inequality to $\frac{dy}{dx} = \frac{y}{x}$ on the interval $[-1 + \varepsilon, 1 - \varepsilon]$, for some $0 < \varepsilon \ll 1$ or by considering the line at infinity by applying the transformation $y = z^{-1}$. In the latter case, we get a saddle connection as in figure 1 on the line at infinity formed by the line at infinity itself and the axes $x = \pm 1$ and therefore no other solutions can diverge to infinity.

If we can find a $\gamma \in \mathbb{R}$ such that $\varphi_\gamma(x) = \psi_\gamma(x)$ for $x \in (-1, 1)$, this provides us with a smooth graph defined in a neighbourhood of $[-1, 1]$ which corresponds to a smooth solution of (50). Due to uniqueness of solutions it suffices to prove that there exists a $\gamma \in \mathbb{R}$ such that $\varphi_\gamma(0) = \psi_\gamma(0)$ and thus the smooth graphs coincide on $(-1, 1)$ (see figure 3). Therefore we compute the C^∞ -graphs $y = \varphi_\gamma(x)$ and $y = \psi_\gamma(x)$ explicitly.

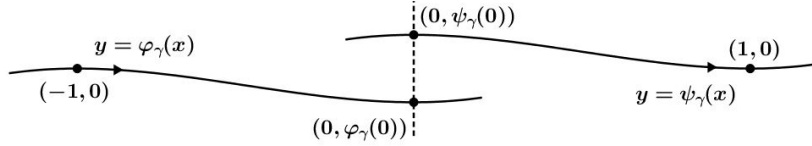


Figure 3: Connecting the smooth graphs

One can easily verify that

$$\varphi_\gamma(x) = (1-x^2)^\lambda \frac{2}{q} \int_{-1}^x (1-u^2)^{-\alpha} du - (1-x^2)^\lambda \gamma \frac{2}{q} \int_{-1}^x \frac{\chi(u)}{(1-u^2)^{\lambda+1}} du, \quad (52)$$

where $\alpha = \lambda - N \in (0, 1)$, is a solution of (50). It remains to prove that this solution is locally C^∞ near $x = -1$. Using Euler's formula for hypergeometric functions, see [14] one can check that

$$\begin{aligned} \frac{2}{q} (1-x^2)^\lambda \int_{-1}^x (1-u^2)^{-\alpha} du = \\ \frac{2^{1-\alpha}}{q(1-\alpha)} (1+x)^{N+1} (1-x)^\lambda \text{hypergeom} \left([\alpha, 1-\alpha], [2-\alpha], \frac{x+1}{2} \right), \end{aligned}$$

where $\text{hypergeom}([\cdot, \cdot], [\cdot], \cdot)$ stands for the Gauss hypergeometric function. Since this is C^∞ near $x = -1$, we know that (52) describes the local unstable manifold. Similarly, we can show that

$$\begin{aligned} \psi_\gamma(x) = -\frac{2^{1-\alpha}}{q(1-\alpha)} (1-x)^{N+1} (1+x)^\lambda \text{hypergeom} \left([\alpha, 1-\alpha], [2-\alpha], \frac{1-x}{2} \right) \\ - (1-x^2)^\lambda \frac{2\gamma}{q} \int_1^x \frac{\chi(u)}{(1-u^2)^{\lambda+1}} du, \end{aligned}$$

describes the locally smooth graph at $x = 1$. We have $\varphi_\gamma(0) = \psi_\gamma(0)$ if

$$\frac{2^{-\alpha}}{1-\alpha} \text{hypergeom} \left([\alpha, 1-\alpha], [2-\alpha], \frac{1}{2} \right) = \gamma \int_0^1 \frac{\chi(u)}{(1-u^2)^{\lambda+1}} du.$$

The result now follows from the fact

$$\text{hypergeom} \left([\alpha, 1-\alpha], [2-\alpha], \frac{1}{2} \right) = \frac{2^{-1+\alpha} \sqrt{\pi} \Gamma(2-\alpha)}{\Gamma(\frac{3}{2}-\alpha)}.$$

□

Thanks to lemmas 5.1 and 5.2 the first coefficient in the normal form of (41) can be deduced. The other blow-up vector fields (42), (45) and (46) need to be treated in a similar way. Lemma 5.1 applied to these vector fields provides us a differential equation (49) which should be smoothly solvable. By choosing γ wisely, this is possible in a similar way as in the proof of lemma 5.2. We only present the results here, since this is not the main objective of this article. First we consider (42).

Lemma 5.3. *There exists a C^∞ function h defined in a neighbourhood of $[-1, 1]$ that satisfies*

$$-3\frac{dh(x)}{dx}(1-x^2) - (6n+1)xh(x) + (1-x^2)^n = \gamma\chi(x),$$

requiring

$$\gamma \int_0^1 \frac{\chi(u)}{(1-u^2)^{n+7/6}} du = -\frac{\sqrt{\pi}\Gamma(\frac{5}{6})}{\Gamma(\frac{4}{3})}.$$

If we treat (45), we have

Lemma 5.4. *There exists a C^∞ function h defined in a neighbourhood of $[-1, 1]$ that satisfies*

$$-3\frac{dh(x)}{dx}(1-x^2) - (6n-1)xh(x) + (1-x^2)^n(8x^2+1)^{-2/3} = \gamma\chi(x)$$

requiring

$$\gamma \int_0^1 \frac{\chi(t)}{(1-t^2)^{n+5/6}} dt = \frac{1}{3} \frac{\pi^{3/2}}{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}.$$

Finally for (46), we have

Lemma 5.5. *There exists a C^∞ function h defined in a neighbourhood of $[-1, 1]$ that satisfies*

$$-3\frac{dh(x)}{dx}(1-x^2) - (6n+1)xh(x) + \frac{4x^2-1}{(8x^2+1)^{4/3}}(1-x^2)^n = \gamma\chi(x),$$

requiring

$$\gamma \int_0^1 \frac{\chi(t)}{(1-t^2)^{n+7/6}} dt = -\frac{\sqrt{3}\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}{2\sqrt{\pi}}.$$

5.1.4. The transition maps and cyclicity

Theorem 2.3 states that the transition maps for the vector fields α_n are of the form

$$D(x_0) = -x_0 + 2\gamma A(1-x_0^2)^{n+5/6} + \text{h.o.t.},$$

where γ denotes the first non-zero coefficient in the normal form and

$$A = \int_0^1 \frac{\chi(t)}{(1-t^2)^{n+5/6}} dt.$$

The notation ‘h.o.t.’ denotes the higher order terms with respect to the variable $1 + x_0$ as explained in remark 4.2. Hence for (41) by lemma 5.2, we have

$$D(x_0) = -x_0 + f(0) \frac{\sqrt{\pi} \Gamma(\frac{1}{6})}{\Gamma(\frac{2}{3})} (1 - x_0^2)^{n+5/6} + \text{h.o.t.}, \quad (53)$$

and for (45), using lemma 5.4, we have

$$D(x_0) = -x_0 + \frac{27a}{2} \left(-\frac{16}{27}\right)^n f(0) \frac{\pi^{3/2}}{\Gamma(\frac{2}{3}) \Gamma(\frac{5}{6})} (1 - x_0^2)^{n+5/6} + \text{h.o.t.} \quad (54)$$

Similarly for the vector fields β_n we have

$$D(x_0) = -x_0 + 2\gamma \tilde{A} (1 - x_0^2)^{n+7/6} + \text{h.o.t.},$$

where γ denotes the first non-zero coefficient and

$$\tilde{A} = \int_0^1 \frac{\chi(t)}{(1-t^2)^{n+7/6}} dt.$$

Hence the transition map for (42) is given by (see lemma 5.3)

$$D(x_0) = -x_0 - 2g(0) \frac{\sqrt{\pi} \Gamma(\frac{5}{6})}{\Gamma(\frac{4}{3})} (1 - x_0^2)^{n+7/6} + \text{h.o.t.},$$

and for (46) by (see lemma 5.5)

$$D(x_0) = -x_0 - \frac{\sqrt{3}}{a} \left(-\frac{16}{27}\right)^n g(0) \frac{\Gamma(\frac{2}{3}) \Gamma(\frac{5}{6})}{\sqrt{\pi}} (1 - x_0^2)^{n+7/6} + \text{h.o.t.}$$

We combine the results in each of the blow-up maps (39) and (40) to get an upper bound on the cyclicity of a cuspidal loop conjugated to (37) and (38) at the origin. Write

$$\Sigma_{\text{in}} = \{(-1 + x, 1) \mid |x| \ll 1\}, \text{ and } \Sigma_{\text{out}} = \{(1 - y, 1) \mid |y| \ll 1\},$$

and use x , resp. y to parametrize Σ_{in} , resp. Σ_{out} . In this way, the transition map $D : \Sigma_{\text{in}} \rightarrow \Sigma_{\text{out}}$ can be considered as a one-dimensional function $y = D(x)$. The two cases $x > 0$ and $x < 0$ corresponding to either the blow-up chart (39) or (39) correspond to two different types of limit cycles, namely the interior and exterior ones. In this way we can bound the cyclicity of the inner or outer limit cycles separately and also bound the true (two-sided) cyclicity as we explain shortly here. Without going in too much detail, the transition map near the blow-up locus of (37) is given by

$$D(x) = \begin{cases} x + x^{n+5/6}(\kappa f(0) + o(1)), & \text{if } x \geq 0, \\ x + |x|^{n+5/6}(\eta f(0) + o(1)), & \text{if } x < 0, \end{cases} \quad (55)$$

for some non-zero κ, η related to the coefficients in (53), resp. (54). Observe that this map is only C^n . When there is a cuspidal loop, one can consider the inverted regular transition $R : \Sigma_{\text{in}} \rightarrow \Sigma_{\text{out}}$ near the loop. The most degenerate case is when

$$R(x) = x + \text{h.o.t.}$$

Since this is C^∞ at $x = 0$, its asymptotic expansion does not contain non-smooth terms which can compensate for the non-smooth terms in the map D . Hence if we look at the difference map $\Delta(x) = D(x) - R(x)$, we know that

$$\Delta^{(n)}(x) = \begin{cases} x^{5/6}(\bar{\kappa}f(0) + o(1)), & \text{if } x \geq 0, \\ |x|^{5/6}(\bar{\eta}f(0) + o(1)), & \text{if } x < 0. \end{cases} \quad (56)$$

This has only one zero in a neighbourhood U of $x = 0$. Due to Rolle's theorem, we know that Δ can not have more than $n + 1$ zeroes in U . This allows us to give a partial cyclicity result since zeroes of the difference map correspond to limit cycles of the system. We can put an upper bound on the cyclicity of a specific family of vector fields perturbing from the cuspidal loop where the cusp is conjugated to (37) or (38) as follows.

Suppose we have a family X_λ of smooth vector fields defined in a neighbourhood V of the parameter λ_0 such that X_{λ_0} contains a cuspidal loop Γ . Suppose that for every $\lambda \in V$ the vector field X_λ has a cusp singularity conjugated to (37) where $f(0)$ can depend on λ but remains non-zero. The results above remain true, however the coefficients become parameter-dependent. Moreover the coefficients κ and η in (55) and (56) can also be parameter-dependent. Even though, the statement that (56) has at most one zero in a neighbourhood U of $x = 0$ remains valid. Hence we can find a neighbourhood W of Γ such that X_λ has at most $n + 1$ limit cycles contained in W .

In a similar way we can show that for a family of vector fields X_λ conjugated to (38) with a cuspidal loop Γ for X_{λ_0} can have at most $n + 2$ limit cycles in a neighbourhood W of Γ .

5.2. The fake saddle

Following [15], we consider a degenerate planar singularity of the form

$$\begin{cases} \dot{x} &= Ax^2 + bxy + O(\|(x, y)\|^3) \\ \dot{y} &= x^2 + y^2 + O(\|(x, y)\|^3) \end{cases} \quad (57)$$

with $A^2 < 4(1 - b)$ and $b \in]0, 1[$. These are the conditions under which the origin has exactly two hyperbolic sectors, an incoming separatrix and an outgoing separatrix. Both separatrices are of center type, and are similar to the two branches of a one-dimensional saddle-node singularity. The reader may verify that after a homogeneous blow-up (as we will do in section 5.2.1), a saddle connection along the equator connects two hyperbolic saddles with ratios of eigenvalues $b - 1 : 1$ and $1 - b : -1$. As we are interested in this paper in the study of the resonant case, we confine ourselves to the cases $1 - b$ equal to 1 or $\frac{1}{2}$. Other resonant cases ($1 - b \in \mathbb{Q}$) demand more involved calculations and shall therefore not be handled in this text. Even the case $1 - b = \frac{1}{k}$ with $k \in \mathbb{N}$ requires a non-trivial computation since one needs to compute a residue of some function $\frac{g(x)}{(1-x^2)^{k+1}}$ at $x = 1$ and $x = -1$ in general. Before examining the transition map along the fake saddle by seeing it essentially as a transition through two symmetric saddles, we first put the system in a elementary form in 5.2.1 and 5.2.2. In 5.2.3 we deal with the case $b = 0$, in 5.2.4 we deal with $b = \frac{1}{2}$; the general case is beyond the scope of this paper as it is merely our intention to demonstrate the applicability of our main result.

5.2.1. Persistence of SN-fiber

We blow-up the singularity by writing $(x, y) = (rX, r\sigma)$ (with $\sigma = \pm 1$) to find:

$$\begin{cases} \dot{r} &= \sigma r(X + 1) + O(r^2) \\ \dot{X} &= \sigma X(\sigma AX - X^2 + b - 1). \end{cases}$$

The origin $(r, X) = (0, 0)$ is a saddle. Then we have two C^∞ -separatrices $X = \psi(r)$ and $r = 0$ each of them defined in a neighbourhood of the origin. For $\sigma = +1$, the two saddle points which appear in the polar blow-up are glued in a single saddle point, whose invariant manifold blows down to a C^∞ invariant graph

$$x = y\psi(y),$$

where ψ is defined and smooth in a neighborhood of 0. This manifold corresponds to the SN-fiber and by a C^∞ change of coordinates we can straighten it to $x = 0$.

5.2.2. Preliminary normal form

Up to a smooth change of coordinate, we can assume that $x = 0$ is an invariant manifold passing through the fake saddle. The behaviour of the vector field on this line is of the form

$$\dot{y} = y^2 + \text{h.o.t.},$$

which can be put in a normal form by a C^∞ transformation, eliminating all higher order terms in the above equation except maybe a resonant cubic term (see [16]). Thanks to this latter transformation, the system takes the form

$$\begin{cases} \dot{x} &= Ax^2 + bxy + xO(\|(x, y)\|^2), \\ \dot{y} &= x^2 + y^2 + \sigma y^3 + xO(\|(x, y)\|^2), \end{cases} \quad (58)$$

for $\sigma = 0, 1$. We reduce the terms of homogeneous degree 3 and higher as follows:

Lemma 5.6. *Consider the vector field (58). There exists a formal conjugacy such that this vector field is conjugated to*

- Case 1: $b = 0$

$$\begin{cases} \dot{x} = Ax^2 + x^3 f(x) + Bx^m y, \\ \dot{y} = x^2 + y^2 + \sigma y^3 + x^3 g(x) + xh(x)y^3, \end{cases} \quad (59)$$

for some $m > 1$, $B \neq 0$, or

$$\begin{cases} \dot{x} = Ax^2 + x^3 f(x), \\ \dot{y} = x^2 + y^2 + \sigma y^3 + x^3 g(x) + xh(x)y^3, \end{cases} \quad (60)$$

- Case 2: $b = \frac{1}{N}$, with $N \in \mathbb{N}_0$, $N \geq 2$

$$\begin{cases} \dot{x} = Ax^2 + x^3 f(x) + bxy, \\ \dot{y} = x^2 + y^2 + \sigma y^3 + x^3 g(x) + \alpha x^N y^2 + \beta x^{2N} y, \end{cases} \quad (61)$$

- *Case 3:* $b = \frac{2}{M}$, with $M \in \mathbb{N}_0$ odd, $M \geq 3$

$$\begin{cases} \dot{x} = Ax^2 + bxy + x^3 f(x), \\ \dot{y} = x^2 + y^2 + \sigma y^3 + x^3 g(x) + \beta x^M y, \end{cases} \quad (62)$$

- *Case 4:* $b \neq 0$ and $b \neq \frac{2}{K}$, with $K \in \mathbb{N}_0$

$$\begin{cases} \dot{x} = Ax^2 + bxy + x^3 f(x), \\ \dot{y} = x^2 + y^2 + \sigma y^3 + x^3 g(x). \end{cases} \quad (63)$$

The functions f, g and h are C^∞ in each of the cases.

Proof. The proof of this lemma is classical in normal form theory. We introduce a change of variables in (58)

$$(x, y) = (X + CX^k Y^l, Y + DX^k Y^l),$$

where $k \geq 1$, $l \geq 1$, and collect the coefficients of the term $X^{k+1}Y^l$. Subsequently we choose C and D such that those new coefficients vanish and continue by induction on the homogeneous degree. Since the lowest degree terms in (58) are of homogeneous degree 2, we need to introduce terms of order 3 to eliminate a term of order 4 similar as in the work of Takens (see [16]). \square

In order to compute the transition map near the saddle-node fiber, it suffices to work up to equivalence. In this way, we can simplify even further.

Lemma 5.7. *There exists a formal equivalence putting (57) into the vector fields as given in lemma 5.6 but with $\sigma = 0$. When $b = 0$, we can even transform to (59) with $h = 0$. When $b = \frac{1}{N}$ with $N \geq 2$, we can get $\alpha = 0$ in (61).*

The proof is a simple adjustment of the induction argument in the previous lemma where we rescale in each induction step.

Using Borel's theorem, we can realize these transformations as C^∞ functions, however some flat terms arise. We will omit these from the notation, since after a blow-up procedure they contribute to a flat term which can be eliminated according to [6].

5.2.3. Generic transition map when $b = 0$

By Lemma 5.7 we consider for $b = 0$ a vector field of the form

$$\begin{cases} \dot{x} = Ax^2 + x^3 f(x) + Bx^m y, \\ \dot{y} = x^2 + y^2 + x^3 g(x), \end{cases} \quad (64)$$

for some $m > 1$ and $B \neq 0$. Denote $f_0 = f(0)$ and $g_0 = g(0)$ and assume that $m = 2$. We know that the ratio of the eigenvalues of both saddles is -1 .

Similar as in section 5.1, we apply a parabolic blow-up of the form

$$(x, y) = (Y(X^2 - 1), XY). \quad (65)$$

After multiplication with $\frac{X^2+1}{2Y}$, we get

$$\begin{cases} \dot{X} = \frac{1}{2}(1-X^2) [1 + (1-X^2)F(X) + F_1(X)Y + O(Y^2)], \\ \dot{Y} = XY + (1-X^2)G_1(X)Y + G_2(X)Y^2 + O(Y^3), \end{cases} \quad (66)$$

where

$$\begin{aligned} F(X) &= AX - X^2, \\ F_1(X) &= (1-X^2)(-g_0X^4 + f_0X^3 + (B+2g_0)X^2 - f_0X - g_0), \\ G_1(X) &= -X^3 + \frac{1}{2}AX^2 - \frac{1}{2}A, \\ G_2(X) &= -\frac{1}{2}(1-X^2)^2(-2g_0X^3 + f_0X^2 + (B+2g_0)X - f_0). \end{aligned}$$

In order to compute the dominant term in the transition map, we need to put (66) in semi-local normal form and identify the first non-zero resonant or connecting term. This is done as follows.

Lemma 5.8. *There exists a smooth transformation $(X, Y) \mapsto (x, y) = (X, \varphi(X, Y))$, such that the system (66) is orbitally equivalent to*

$$\begin{cases} \dot{x} = \frac{1}{2}(1-x^2), \\ \dot{y} = y [x + (\alpha x + \beta)(1-x^2)y + (1-x^2)^2y^2 (x\bar{f}((1-x^2)y) + \bar{g}((1-x^2)y))], \end{cases}$$

for some smooth functions \bar{f} and \bar{g} . Moreover, we have

$$\beta = B \left(1 - e^{\frac{-2A\pi}{\sqrt{4-A^2}}} \right).$$

Proof. The existence of a smooth equivalence as stated in the lemma is immediate from the results of [6]. It remains to compute the coefficient β . This is done by repeating the first steps in the normalization procedure of [6]. Denote

$$G(X) = 1 + (1-X^2)F(X) = \left(1 - X^2 + \frac{A}{2}X \right)^2 + \left(1 - \frac{A^2}{4} \right) X^2,$$

which is strictly positive for $A^2 < 4 = 4(1-b)$. We divide the vector field by the factor between square brackets in (66) and apply the transformation

$$Y = \Psi(X)Z,$$

where

$$\Psi(X) = \exp \left(\int_{-\infty}^{\frac{X}{1-X^2}} \frac{-A}{u^2 + Au + 1} du \right).$$

After a straightforward computation, one can deduce the system

$$\begin{cases} \dot{X} = \frac{1}{2}(1-X^2), \\ \dot{Z} = XZ + H_1(X)\Psi(X)Z^2 + O(Z^3), \end{cases}$$

where

$$H(X) = \frac{G_1(X) - XF(X)}{G(X)},$$

and

$$H_1(X) = \frac{G_2(X) - XF_1(X) - (1 - X^2)F_1(X)H(X)}{G(X)}.$$

We can decompose

$$H_1(X)\Psi(X) = (1 - X^2)(\alpha X + \beta) + (1 - X^2)^2 H_2(X),$$

for some constant α and C^∞ function H_2 and with

$$\beta = B \left(1 - e^{\frac{-2A\pi}{\sqrt{4-A^2}}} \right).$$

By a transformation of the form $(X, Z) = (X, Z_1) = (X, Z + h(X)Z^2)$ we can eliminate the term $(1 - X^2)^2 H_2(x)Z^2$. The rest of the normalization procedure of [6] is of the form

$$(X, Z_1) \mapsto (x, y) = (X, Z_1 + Z_1^3 \psi(X, Z_1)),$$

for some smooth function Ψ and thus leaves the coefficient β unchanged. \square

Combining lemma 5.8 and theorem 2.3, the transition map of (64) in the blow-up chart (65) is asymptotically given by

$$D(x_0) = -x_0 + B \left(1 - e^{\frac{-2A\pi}{\sqrt{4-A^2}}} \right) (1 - x_0^2)^2 \log(1 + x_0) + \text{h.o.t.} \quad (67)$$

Observe that blow-up chart (66) only allows us to describe the dynamics for $x < 0$ of (64). However if we apply the reflection $x \rightarrow -x$ to (64), then (65) describes exactly the dynamics for $x > 0$. After the reflection the sign of A and B changes in (64). If we repeat the discussion above, we compute in this case that the transition map in the blow-up chart is asymptotically given by

$$D(x_0) = -x_0 - B \left(1 - e^{\frac{2A\pi}{\sqrt{4-A^2}}} \right) (1 - x_0^2)^2 \log(1 + x_0) + \text{h.o.t.} \quad (68)$$

Combining these results and by considering the difference map Δ as in section 5.1.4, we see that $\Delta'(x)$ locally has one zero if A and B are non-zero.

Again this can be generalized to a parameter-dependent situation. Suppose X_λ is a family of vector fields such that for all λ in an open set U there is a singularity of the form (58) with $b = 0$. Suppose X_{λ_0} has a fake saddle loop Γ and is locally equivalent to (64) with $AB \neq 0$. Then there exists a neighbourhood W of Γ such that X_λ does not contain more than two limit cycles for $\lambda \in U$.

5.2.4. *Generic transition map when $b = \frac{1}{2}$*

When $b = \frac{1}{2}$, the saddle quantity is given by $\frac{1}{2}$, resp. 2 at the saddles after blow-up. Using Lemma 5.7, the vector field can locally be transformed to

$$\begin{cases} \dot{x} = Ax^2 + x^3f(x) + \frac{1}{2}xy, \\ \dot{y} = x^2 + y^2 + x^3g(x) + \beta x^4y, \end{cases} \quad (69)$$

for some $\beta \in \mathbb{R}$ and some C^∞ functions f, g . Denote $f_0 = f(0)$ and $g_0 = g(0)$. After blow-up (65) and multiplying with $\frac{(X^2+1)}{Y}$, we get a vector field of the form

$$\begin{cases} \dot{X} = \frac{1}{2}(1 - X^2) [1 + (1 - X^2)F(X) + F_1(X)Y + O(Y^2)], \\ \dot{Y} = 2XY + (1 - X^2)G_1(X)Y + G_2(X)Y^2 + O(Y^3), \end{cases} \quad (70)$$

where

$$\begin{aligned} F(X) &= 2AX - 2X^2 + 1, \\ F_1(X) &= 2(1 - X^2)^2(g_0X^2 - f_0X - g_0), \\ G_1(X) &= -2X^3 + AX^2 + \frac{1}{2}X - A, \\ G_2(X) &= (1 - X^2)^3(-2g_0X + f_0). \end{aligned}$$

Similar as in the case $b = 0$, we put (70) in semi-local normal form and identify the first non-zero term.

Lemma 5.9. *There exists a smooth transformation $(X, Y) \mapsto (x, y) = (X, \varphi(X, Y))$, such that the system (70) is orbitally equivalent to*

$$\begin{cases} \dot{x} = \frac{1}{2}(1 - x^2), \\ \dot{y} = y [2x + (\alpha x + \beta)(1 - x^2)^2y + (1 - x^2)^4y^2 (x\bar{f}((1 - x^2)^2y) + \bar{g}((1 - x^2)^2y))], \end{cases}$$

for some smooth functions \bar{f} and \bar{g} . Moreover, we have

$$\beta = 2f_0 \left(1 + e^{\frac{-3\pi A}{\sqrt{2-A^2}}} \right).$$

Proof. Again, we only need to compute β since the rest of the statement follows immediately from [6]. We divide the vector field (70) by the factor in square brackets where we remark that

$$G(X) = 1 + (1 - X^2)F(X) = 2 \left(1 - X^2 + \frac{A}{2}X \right)^2 + \left(1 - \frac{A^2}{2} \right) X^2$$

is a strictly positive function since $A^2 < 2 = 4(1 - b)$. Consequently we apply the transformation

$$Y = \Psi(X)Z, \quad \text{with } \Psi(X) = e^{\int_{-1}^X 2H(s)ds},$$

where

$$H(X) = \frac{G_1(X) - 2XF(X)}{G(X)}.$$

A straightforward computation shows that the vector field can now be written as

$$\begin{cases} \dot{X} = \frac{1}{2}(1 - X^2), \\ \dot{Z} = 2XZ + ((1 - X^2)^2(\alpha X + \beta) + (1 - X^2)^3 H_2(X)) Z^2 + O(Z^3), \end{cases}$$

for some constant α and C^∞ function H_2 and with

$$\beta = 2f_0 \left(1 + e^{\frac{-3\pi A}{\sqrt{2-A^2}}} \right).$$

Following the normal form procedure of [6], we can remove the term $(1 - X^2)^3 H_2(X) Z^2$ by a transformation of the form

$$(X, Y) = (\bar{X}, \bar{Y} + \bar{H}(\bar{X})\bar{Y}^2),$$

where $\bar{H}(\bar{X})$ is a smooth solution of

$$-\frac{1}{2}(1 - \bar{X}^2)\bar{H}'(\bar{X}) - 2\bar{X}\bar{H}(\bar{X}) + (1 - \bar{X}^2)^3 H_2(\bar{X}) = 0,$$

without changing the coefficients α and β . The higher order terms (with respect to \bar{Y}) will then be put in normal form without changing the terms of degree 2. \square

By lemma 5.9 and theorem 2.3, we get that the transition map in the blow-up chart (65) for $x < 0$ is asymptotically given by

$$D(x_0) = -x_0 - f_0 \left(1 + e^{\frac{-3\pi A}{\sqrt{2-A^2}}} \right) (1 - x_0^2)^3 \log(1 + x_0) + \text{h.o.t.},$$

and a similar map for $x > 0$ where A is replaced by $-A$. In a similar way as at the end of section 5.1.4, we can see that there exists a neighbourhood of 0 where the second derivative of the displacement map Δ has at most one zero when $f_0 \neq 0$.

Consider a family of vector fields X_λ , $\lambda \in U$, with a singularity of the form (58) with $b = \frac{1}{2}$. Suppose X_{λ_0} has a fake saddle loop Γ and is locally equivalent to (69) with $f_0 \neq 0$. Then there exists a neighbourhood W of Γ such that X_λ does not contain more than three limit cycles in W for $\lambda \in U$.

6. Discussion

As illustrated in section 5, this paper provides a method to compute the first non-zero higher order term of the Dulac map close to a non-elementary singularity. The technique used in section 5 is to directly compute the normal form up to some degree and use the main result to see the effect on the transition map. Instead of this method one may also assume an expression for the invariant V as an ansatz and establish equations for all coefficients appearing therein; our main theorem is then merely used to prescribe the shape of the invariant V .

Depending upon the situation, either a logarithmic term is dominant in the transition

map (caused by a resonant term in the normal form) or a term with fractional power is dominant in the transition map (caused by a connecting term in the normal form). Section 5 contains examples of both situations.

In section 5, the saddle connection appears after blowing up a singular point which is initially nilpotent or degenerate. The saddle connection on the blow-up locus can be predicted by examining the dominant quasi-homogeneous part of the vector field. Under the condition that the saddles are the only two singularities on the blow-up locus, one might wonder what the relation is between the semi-local normal form after blow-up like we obtain here (and in [6]) and the normal forms obtained from information of the quasi-homogeneous part (before blow-up) like in [17]. This is an open question.

In [5], the transition map is computed in a very elegant and short way: instead of computing the transition map along the real axis, the authors use a complex path and use the monodromy of the two individual saddles to prescribe the dominant term of the transition map. In comparison to [5], our technique is technically more involved but at the same time it is more straightforward and applicable to the general setting. The results in [5] are used to deal with cyclicity of the cuspidal loop, where the authors consider unfoldings of the cusp preserving the singularity. In section 5, we demonstrated how cyclicity results can be deduced from the transition map outside the generality of the context of unfoldings as it was merely our intention to illustrate the computation of the map. We claim that (singularity-preserving) unfoldings can be treated similarly.

The case of the fake saddle is more involved. The saddle quantities are dependent on b nonetheless that they stay reciprocal. In this case, our claim above about the possibility of treating unfoldings is not valid: we believe we need to introduce Ecalle-Roussarie compensators, depending on the parameter b . This will be subject of further research.

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