Predator–prey systems with small predator's death rate
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Abstract

The goal of our paper is to study canard relaxation oscillations of predator-prey systems with Holling types of functional response when the death rate of predator is very small and the conversion rate is uniformly positive. This paper is a natural continuation of [LZ13, Li16] where both the death rate and the conversion rate are kept very small. We detect all limit periodic sets that can produce the canard relaxation oscillations after perturbations and study their cyclicity by using singular perturbation theory and the family blow-up.

1 Introduction

The Rosenzweig-MacArthur predator-prey model is typically given by

$$\begin{cases} \dot{x} & = & rx\big(1 - \frac{x}{K}\big) - y\frac{mx}{b+x} \\ \dot{y} & = & y\big(-\delta + c\frac{mx}{b+x}\big), \end{cases} \tag{1}$$

where $x \geq 0$ is the population density of prey, $y \geq 0$ is the population density of predator and all the parameters are strictly positive. The function $P(x) = \frac{mx}{b+x}$ is called the predator response function, $\delta > 0$ represents the death rate of the predator, $c > 0$ is the rate of conversion of prey to predator and the function $x \mapsto rx\big(1 - \frac{x}{K}\big)$ represents the logistic growth model of the prey in the absence of predator. System (1) is a special case of more general predator-prey systems with a Holling type response function P:

$$\begin{cases} \dot{x} & = & rx\big(1 - \frac{x}{K}\big) - yP(x) \\ \dot{y} & = & y\big(-\delta + cP(x)\big), \end{cases} \tag{2}$$

with the same conditions on x, y and the parameters. The function P in (1) is an increasing function and tends to $m > 0$ as $x \to \infty$, and it is often called a Michaelis-Menten function or a response function of Holling type II. We call the function $P(x) = \frac{mx^2}{a+x^2}$, with $m > 0$ and $a > 0$, the response function of Holling type III, and its generalized version is given by $P(x) = \frac{mx^2}{a+bx+x^2}$. The Holling type IV response function or the Monod-Haldane function is given by $P(x) = \frac{mx}{a+bx+x^2}$, and its simplified version is given by $P(x) = \frac{mx}{a+x^2}$. For more details about the predator-prey model (2) with Holling type of functional responses, see e.g. [RX01, LZ13]. We also refer to [FW86, Wol88, ZCW02, LCR08], etc.

When working with predator-prey system (2), we typically assume that the parameters are positive constants kept away from zero. Limit cycles of (2), with

response functions of Holling type II and generalized Holling types III and IV, have been studied in [LZ13, Li16], uniformly in $(\delta, c) \to (0, 0)$. When $\delta = c = 0$, system (2) contains curves of singularities (often called slow curves or critical curves). See e.g. Figure 1. Periodic orbits of (2) that are Hausdorff close to limit periodic sets, at level $\delta = c = 0$, containing pieces of the critical curves and fast orbits are called relaxation oscillations (see e.g. [DR96, DR07]). Thus, [LZ13, Li16] deal with the relaxation oscillations of (2) when $(\delta, c) \sim (0, 0)$, away from the degenerate intersection point $(0, \frac{br}{m})$, using geometric singular perturbation theory [DR96, KS01, DMD05, DMD08].
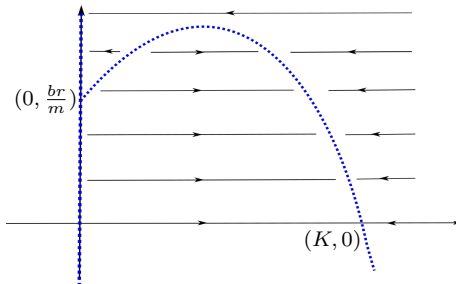

Figure 1: Dynamics of (1), with $\delta = c = 0$.

The goal of our paper is to study another limiting case observed in [LZ13]: $\delta \to 0$ and c is a positive constant. When $\delta = 0$ and $c > 0$, system (2) contains the critical curve $\{x = 0\}$ (note that $P(0) = 0$). Let us focus on system (1) (see Figure 2). All the singularities of the critical curve are normally hyperbolic, except the nilpotent contact point $(x, y) = (0, \frac{br}{m})$. We distinguish between 3 types of limit periodic sets, at level $\delta = 0$, that can generate limit cycles when $\delta \sim 0$ and $\delta > 0$ (see also Section 2): (i) *the nilpotent contact point* $(x, y) = (0, \frac{br}{m})$, (ii) *canard limit cycles* consisting of a fast orbit and the part of the critical curve, with a regular slow dynamics, between the α-limit set and the ω-limit set of that fast orbit and (iii) *the slow-fast 2-saddle limit periodic set* consisting of the hyperbolic saddle $(x, y) = (K, 0)$, the stable manifold of $(x, y) = (K, 0)$, the unstable manifold of $(x, y) = (K, 0)$ and the part of the critical curve between the origin (i.e. the α-limit set of the stable manifold) and the ω-limit set of the unstable manifold. (As we will see in Section 2, the slow dynamics of (1) along the critical curve has a hyperbolic saddle at the origin.)

The goal of our paper is to study the cyclicity of each of these limit periodic sets by using geometric singular perturbation theory and the family blow-up at $(x, y, \delta) = (0, \frac{br}{m}, 0)$. To study the contact point $(x, y) = (0, \frac{br}{m})$, we use the theory of slow-fast Hopf points of codimension 1 developed in [KS01]. To find the cyclicity of the canard limit cycles, we study zeros of the so-called slow-divergence integral of (1), using results from [DMD05]. The paper [DMD08] is a generalization of [DMD05] allowing zeros in the slow dynamics, away from the contact point. Since the slow-fast 2-saddle limit periodic set contains, beside a zero of the slow dynamics at the origin $(x, y) = (0, 0)$, a (hyperbolic) saddle at $(x, y) = (K, 0)$, away from the critical curve, we cannot use the results of [DMD08] directly. *Thus, we have to develop new methods that are suitable for studying the cyclicity of the slow-fast 2-saddle limit periodic set.* Clearly, the

new techniques can be used not only in the framework of (1) or (2), but also in more general planar slow-fast systems.
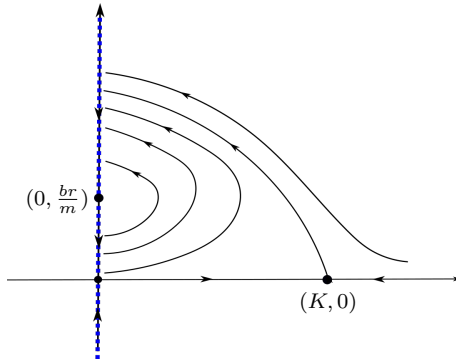

Figure 2: Dynamics of (1) for $\delta = 0$ and $c > 0$.

In the regular case (δ is a positive constant), the system (1) has at most one limit cycle (see [Che81, LC88]). The reason we study limit cycles of (1) uniformly in $\delta \to 0$ is twofold. On one hand, we cannot apply the methods of [Che81, LC88] to the singular case $\delta \to 0$, and we need to use techniques coming from singular perturbation theory, including the family blow-up. On the other hand, our paper provides a good starting point to get familiar with the methods that can be used not only to study the singular Holling type II case (1) but also the more general system (2), with a response function of Holling type III and IV, in the singular case $\delta \to 0$.

In Section 2 we state our main results. Our upper bounds on the number of limit cycles in (1), with $\delta \to 0$, are 1 or 2, depending on the region in the parameter space (see Theorem 2.1–Theorem 2.3). We prove our main results in Section 3.

2 Statement of results

We state our main results in Section 2.3–Section 2.5, clearly distinguishing between three different types of limit periodic sets defined in Section 2.2. Instead of working with the original predator-prey model (1), it is more convenient to work with a polynomial normal form (6) obtained in Section 2.1.

2.1 Normal form of (1)

Let us recall that $r > 0$, $K > 0$, $m > 0$, $b > 0$, $c > 0$ and $\delta \geq 0$ is the singular perturbation parameter in (1). After the rescaling $(\bar{x}, \bar{y}, \bar{b}, \bar{c}, \bar{\delta}) = \left(\frac{x}{K}, \frac{m}{rK}y, \frac{b}{K}, \frac{cm}{r}, \frac{\delta}{r}\right)$, we may assume that (1) has the following form:

$$\begin{cases} \dot{x} &= x\big(1-x\big) - \frac{xy}{b+x} \\ \dot{y} &= y\big(-\delta + c\frac{x}{b+x}\big), \end{cases} \tag{3}$$

where we denote $(\bar{x}, \bar{y}, \bar{b}, \bar{c}, \bar{\delta})$ again by (x, y, b, c, δ). Thus, we have that $x \geq 0$, $y \geq 0$, $b > 0$, $c > 0$ and $\delta \geq 0$ is the singular perturbation parameter ($\delta \sim 0$).

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Using a time rescaling, i.e. multiplication by $b + x > 0$, system (3) becomes a polynomial vector field:

$$\begin{cases} \dot{x} & = & x\big(b + (1-b)x - x^2 - y\big) \\ \dot{y} & = & y\big(-\delta(b+x) + cx\big). \end{cases} \tag{4}$$

When $\delta = 0$, (4) becomes

$$\begin{cases} \dot{x} & = & x\big(b + (1-b)x - x^2 - y\big) \\ \dot{y} & = & cxy, \end{cases} \tag{5}$$

and we call (5) the fast subsystem of (4). The critical curve $\{x = 0\}$ of (5) is normally attracting when $y > b$ and normally repelling when $0 \leq y < b$. When $y = b$, we deal with a nilpotent contact point. See Figure 2. One of the reasons we work with the polynomial system (4), instead of (3), is we want to apply a family blow-up of (4) to the point $(x, y, \delta) = (0, b, 0)$ (see Section 2.2). After translation $\bar{y} = y - b$, we may also assume that the nilpotent contact point is situated at $(x, y) = (0, 0)$. We get

$$\begin{cases} \dot{x} & = & x\big((1-b)x - x^2 - y\big) \\ \dot{y} & = & (y+b)\big(-\delta(b+x) + cx\big). \end{cases} \tag{6}$$

where $x \geq 0$ and $y \geq -b$ (we denote $\bar{y}$ again by y).

2.2 Family blow-up at $(\mathbf{x}, \mathbf{y}, \delta) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$ and the slow dynamics along the critical curve

In this section we show that our slow-fast model (6), along the critical curve $\{x = 0\}$, fits into a general framework of [DMD08]. First, let us focus on the dynamics of (6), with $\delta \sim 0$ and $\delta > 0$, along the critical curve but away from the contact point. Center manifolds of $(6) + 0\frac{\partial}{\partial \delta}$, near the critical curve $\{x = 0\}$, with $y \neq 0$, are given by $x = O(\delta^m)$, for all $m \geq 1$. After substituting these manifolds for x in the y-component of (6), we obtain the dynamics along the critical curve: $\dot{y} = \delta(-b(y+b) + O(\delta))$. We get the *slow dynamics* of (6) after dividing this by δ and letting δ tend to 0:

$$y' = -b(y+b), \quad y \neq 0. \tag{7}$$

Since $b > 0$, the slow dynamics (7) is regular with one (isolated) hyperbolic saddle at $y = -b$, and this singularity does not coincide with the contact point at $y = 0$. Thus, when $y \geq -b$, the slow dynamics points downwards (see also Figure 2).

To study the passage from the attracting part $\{y > 0\}$ to the repelling part $\{y < 0\}$ of the critical curve, we write $\delta = \tilde{\delta}^2$, with $\tilde{\delta} \sim 0$ and $\tilde{\delta} \geq 0$, and blow up the origin in the $(x, y, \tilde{\delta})$-space by using the following blow-up formula:

$$(x, y, \tilde{\delta}) = (\epsilon^2 \bar{x}, \epsilon \bar{y}, \epsilon \bar{\delta}), \ \epsilon \geq 0, \ \epsilon \sim 0, \ \bar{\delta} \geq 0, \ (\bar{x}, \bar{y}, \bar{\delta}) \in \mathbb{S}^2. \tag{8}$$

To study the blown up vector field in a small neighborhood of the blow-up locus $\{\epsilon = 0\}$, we use different charts. In the family directional chart $\{\bar{\delta} = 1\}$, system (6), after division by $\epsilon > 0$, becomes:

$$\begin{cases} \dot{\bar{x}} & = & \bar{x}\big((1-b)\epsilon\bar{x} - \epsilon^3 \bar{x}^2 - \bar{y}\big) \\ \dot{\bar{y}} & = & (\epsilon\bar{y} + b)\big(-(b + \epsilon^2 \bar{x}) + c\bar{x}\big). \end{cases} \tag{9}$$

When $\epsilon = 0$, system (9) has the following form:

$$\begin{cases} \dot{\bar{x}} & = & -\bar{x}\bar{y} \\ \dot{\bar{y}} & = & -b^2 + bc\bar{x}. \end{cases} \tag{10}$$

Clearly, system (10) has a center at $(\bar{x}, \bar{y}) = (\frac{b}{c}, 0)$. In the phase directional chart $\{\bar{y} = 1\}$, system (6) can be written as

$$\begin{cases} \dot{\bar{x}} & = & \bar{x}\big((1-b)\epsilon\bar{x} - \epsilon^3\bar{x}^2 - 1\big) - 2\bar{x}(\epsilon + b)\big(-\bar{\delta}^2(b + \epsilon^2\bar{x}) + c\bar{x}\big) \\ \dot{\epsilon} & = & \epsilon(\epsilon + b)\big(-\bar{\delta}^2(b + \epsilon^2\bar{x}) + c\bar{x}\big) \\ \dot{\bar{\delta}} & = & -\bar{\delta}(\epsilon + b)\big(-\bar{\delta}^2(b + \epsilon^2\bar{x}) + c\bar{x}\big), \end{cases} \tag{11}$$

after division by ϵ. When $\epsilon = \bar{\delta} = 0$, (11) has two singularities: $\bar{x} = -\frac{1}{2bc}$ and $\bar{x} = 0$. At $(\bar{x}, \epsilon, \bar{\delta}) = (-\frac{1}{2bc}, 0, 0)$, (11) has a hyperbolic saddle with eigenvalues $(1, -\frac{1}{2}, \frac{1}{2})$, and at $(\bar{x}, \epsilon, \bar{\delta}) = (0, 0, 0)$, we deal with a semi-hyperbolic singularity (the eigenvalues of the linear part are given by $(-1, 0, 0)$), with a two-dimensional center manifold and with the $\bar{x}$-axis as stable manifold. Each center manifold contains the line of singularities $\{\bar{x} = \bar{\delta} = 0\}$ and has the following form: $\bar{x} = O(\bar{\delta}^m)$, for all $m \geq 1$. *Putting this in the $(\epsilon, \bar{\delta})$-component of (11) we get the following center behavior: $\{\dot{\epsilon} = \epsilon\bar{\delta}^2(-b^2 + O(\epsilon, \bar{\delta})), \dot{\bar{\delta}} = \bar{\delta}^3(b^2 + O(\epsilon, \bar{\delta}))\}$, which has, after division by $\bar{\delta}^2$, an isolated hyperbolic saddle at $(\epsilon, \bar{\delta}) = (0, 0)$.*

If we apply the coordinate change $(\epsilon, \bar{\delta}, t) \to (-\epsilon, -\bar{\delta}, -t)$ to (11), then we obtain the blown-up vector field in the phase directional chart $\{\bar{y} = -1\}$ (see (8)). In the charts $\{\bar{x} = 1\}$ and $\{\bar{x} = -1\}$, we find no extra singularities. See Figure 3.
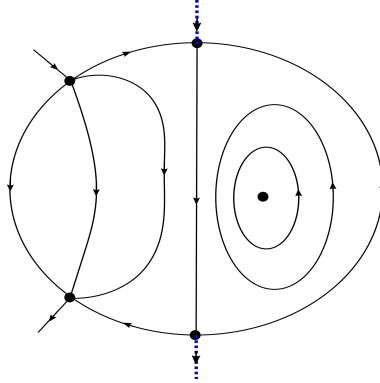

Figure 3: Dynamics on the blow-up locus of (8).

Thus, the passage along the critical curve $\{x = 0\}$, with $y \geq -b$, satisfies the assumptions of [DMD08]. Now we define the canard limit cycle $\Gamma_y^{(b,c)}$, $y \in]-b, 0[$, consisting of a regular orbit of the fast subsystem of (6) and the part of the critical curve between the α-limit set $(0, y)$ and the ω-limit set $(0, F(y))$ of that regular orbit, where the *fast relation function* F depends on (b, c) and $F(y) > 0$. Since the slow dynamics (7) is regular on the segment $[y, F(y)]$ for each $y \in]-b, 0[$, we can use the so-called *slow divergence integral* of (6) along $[y, F(y)]$ to study limit cycles of (6) Hausdorff close to $\Gamma_y^{(b,c)}$ (see [DMD08] or

[DMD05]). The slow divergence integral is defined as:

$$I(y, b, c) := \int_y^{F(y)} \frac{sds}{-b(s+b)}, \ y \in]-b, 0[, \tag{12}$$

where we use the fact that $-y$ is the divergence of (6), with $\delta = 0$, along the critical curve $\{x = 0\}$ and $d\tau = \frac{dy}{-b(y+b)}$ (see (7)). Following [DMD05] or [DMD08], the cyclicity of $\Gamma_{y_0}^{(b_0,c_0)}$, with $y_0 \in]-b_0, 0[$, near $b = b_0 > 0$ and $c = c_0 > 0$, is bounded by the multiplicity of zero of $I(y, b_0, c_0)$ at $y = y_0$, increased by one.

We denote by $\Gamma_{-b}^{(b,c)}$ the slow-fast 2-saddle limit periodic set in (6), at level $\delta = 0$, consisting of the horizontal orbit $y = -b$ connecting the hyperbolic saddle $(x, y) = (0, -b)$ of the slow dynamics with the hyperbolic saddle $(x, y) = (1, -b)$, the unstable manifold of $(x, y) = (1, -b)$, the ω-limit set $(x, y) = (0, F(-b))$ of the unstable manifold and the critical curve $[-b, F(-b)]$. Note that $\Gamma_y^{(b,c)} \to \Gamma_{-b}^{(b,c)}$ in the Hausdorff sense as $y \to -b$. Since the segment $[-b, F(-b)]$ contains the singularity of the slow dynamics, we cannot use (12) to study the cyclicity of $\Gamma_{-b}^{(b,c)}$ (see [DMD08]). See Sections 2.3 and 3.1 for more details.

Remark 1. *Let us explain why the slow-fast 2-saddle limit periodic set $\Gamma_{-b}^{(b,c)}$ is well defined for each $b > 0$ and $c > 0$, i.e. let us show that $F(-b) < \infty$ for all $b > 0$ and $c > 0$. Notice that the unstable manifold of the hyperbolic saddle $(x, y) = (1, -b)$ in (6) with $\delta = 0$ is given by $x = 1 - \frac{y+b}{b+c+1} + O((y+b)^2)$. Since the vector field (6), with $\delta = 0$, points inwards the (positively invariant) set $D := \{0 \le x \le 1, y \ge -b\}$ along the line $\{x = 1, y > -b\}$, the unstable manifold enters the set D and stays in D. For the system (6), with $\delta = 0$, the parabola $y = (1-b)x - x^2$ and the critical curve $x = 0$ represent the x-nullcline (see Figure 4). To show that $F(-b) < \infty$ for $b > 0$, we have to study (6) near $y = +\infty$. After a coordinate change $(x, y) = (X, \frac{1}{Y})$, with $Y \sim 0$ and $Y > 0$, and multiplication by Y, system (6), for $\delta = 0$, becomes*

$$\begin{cases} \dot{X} & = & X\left((1-b)XY - X^2Y - 1\right) \\ \dot{Y} & = & -cXY^2(1+Yb). \end{cases} \tag{13}$$

where $X \ge 0$ and $Y \ge 0$. Singularities of (13) are given by $\{X = 0\}$; all the singularities are semi-hyperbolic and attracting. Notice also that $\{Y = 0\}$ is the stable manifold of the singular point $(X, Y) = (0, 0)$ and that the (local) stable manifold of the singular point $(X, Y) = (0, Y_0)$, with $Y_0 \ge 0$ and $Y_0 \sim 0$, is the graph of the unique solution of the initial value problem $\{\frac{dY}{dX} = \frac{-cY^2(1+Yb)}{(1-b)XY - X^2Y - 1}, \ Y(0) = Y_0\}$. Since the stable manifolds are unique, we have that $F(-b) < \infty$.

2.3 Cyclicity of $\Gamma_{-\mathbf{b}}^{(\mathbf{b},\mathbf{c})}$

As explained in Section 1 and Section 2.2, the limit periodic set $\Gamma_{-b}^{(b,c)}$ contains one "singular" hyperbolic saddle and one "regular" hyperbolic saddle. Using [Mou90], [DMD08] and the fact that the connection between the two saddles is unbroken, we show that the singular hyperbolic saddle is dominant, and we obtain the following cyclicity result:
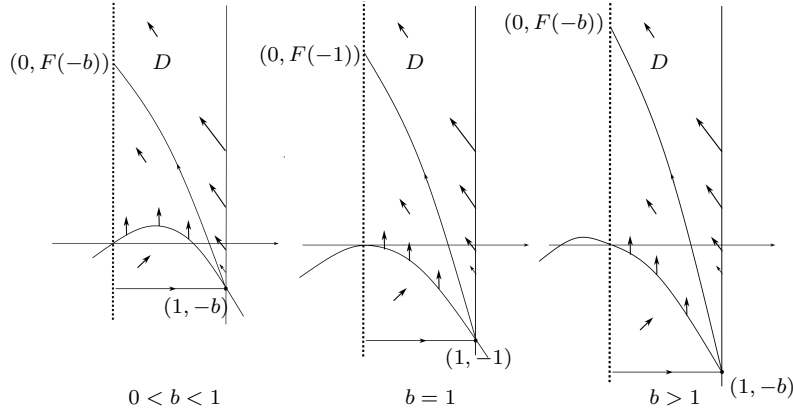
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Figure 4: Nullclines and direction field for (6) with $\delta = 0$.

Theorem 2.1. *Let $b_0 > 0$ and $c_0 > 0$. There exist a small $\delta_0 > 0$, a neighborhood $\mathcal{W}$ of (b_0, c_0) in the (b, c)-space and a (Hausdorff) neighborhood $\mathcal{V}$ of $\Gamma_{-b_0}^{(b_0, c_0)}$ such that system (6) has at most one limit cycle in $\mathcal{V}$, for each value $(\delta, b, c) \in [0, \delta_0] \times \mathcal{W}$. If the limit cycle exists, it is hyperbolic and repelling.*

Theorem 2.1 will be proved in Section 3.1.

2.4 Cyclicity of $\Gamma_{\mathbf{y}}^{(\mathbf{b}, \mathbf{c})}$

To study zeros of the slow divergence integral (12), first we have to find properties of the fast relation function F on $]-b, 0[$ by using a first integral of (6), with $\delta = 0$. When $c = 2b + 2$, we find the first integral by using Darboux theory of integrability, and we get the following cyclicity result:

Theorem 2.2. *Let $b_0 > 0$, $c_0 = 2b_0 + 2$. Suppose that $C \subset]-b_0, 0[$ is compact. The following statements are true:*

1. *If $0 < b_0 < 1$, then there exist $\delta_0 > 0$, a neighborhood $\mathcal{W}$ of (b_0, c_0) in the (b, c)-space such that the set $\cup_{y \in C} \Gamma_y^{(b_0, c_0)}$ can produce at most two limit cycles in (6), for each value $(\delta, b, c) \in [0, \delta_0] \times \mathcal{W}$.*

2. *If $b_0 \geq 1$, then there exist $\delta_0 > 0$, a neighborhood $\mathcal{W}$ of (b_0, c_0) in the (b, c)-space such that the set $\cup_{y \in C} \Gamma_y^{(b_0, c_0)}$ can produce at most one limit cycle in (6), for each value $(\delta, b, c) \in [0, \delta_0] \times \mathcal{W}$. If the limit cycle exists, then it is hyperbolic and repelling.*

Theorem 2.2 will be proved in Section 3.2. The case $c \neq 2b + 2$ is a topic of further studies.

2.5 Cyclicity of the contact point

To study the number of limit cycles close to a contact point, we typically blow up the contact point and detect all possible limit periodic sets on the blow-up locus that can produce limit cycles after perturbation: a center, closed orbits

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surrounding the center and a singular cycle consisting of two semi-hyperbolic singularities on the equator of the blow-up locus and the regular orbits that are heteroclinic to them (see Figure 3). When $b \neq 1$, we can bring our system (6), near the contact point at the origin, to a normal form for a slow-fast Hopf point of codimension 1 studied in e.g. [KS01], and we obtain at most one (hyperbolic) limit cycle.

Theorem 2.3. *Let $b_0 > 0$ and $c_0 > 0$. The following statements are true:*

1. *If $0 < b_0 < 1$, then there exist $\delta_0 > 0$, a neighborhood $\mathcal{W}$ of (b_0, c_0) in the (b, c)-space and a neighborhood $\mathcal{V}$ of $(x, y) = (0, 0)$ such that system (6) has a repelling hyperbolic focus in $\mathcal{V}$ and at most one (hyperbolically attracting) limit cycle in $\mathcal{V}$, for each value $(\delta, b, c) \in]0, \delta_0] \times \mathcal{W}$.*

2. *If $b_0 > 1$, then there exist $\delta_0 > 0$, a neighborhood $\mathcal{W}$ of (b_0, c_0) in the (b, c)-space and a neighborhood $\mathcal{V}$ of $(x, y) = (0, 0)$ such that system (6) has an attracting hyperbolic focus in $\mathcal{V}$ and at most one (hyperbolically repelling) limit cycle in $\mathcal{V}$, for each value $(\delta, b, c) \in]0, \delta_0] \times \mathcal{W}$.*

Theorem 2.3 will be proved in Section 3.3.

To study the degenerate case $b = 1$, we have to deal with higher order Melnikov theory near the center and closed orbits on the blow-up locus, on one hand, and with the so-called birth of canards near the singular cycle (see e.g. [DR09]), on the other. This is the topic of further studies.

3 Proofs of Theorem 2.1–Theorem 2.3

3.1 Proof of Theorem 2.1

Let $b_0 > 0$ and $c_0 > 0$ be arbitrary positive constants, and $(b, c) \sim (b_0, c_0)$. System (6) has a hyperbolic saddle at $(x, y) = (0, -b)$ with a ratio of eigenvalues $-\tilde{\delta}^2$, for each $\tilde{\delta} > 0$ and $\tilde{\delta} \sim 0$ ($\delta = \tilde{\delta}^2$). This hyperbolic saddle originates from the hyperbolic singularity $y = -b$ in the slow dynamics (7). We also deal with a hyperbolic saddle at $(x, y) = (1, -b)$, for all $\tilde{\delta} > 0$ and $\tilde{\delta} \sim 0$, with eigenvalues $\rho_1 = -(1 + b) < 0$ and $\rho_2 = c - \tilde{\delta}^2(b + 1) > 0$. *Note that the connection between the two saddles is unbroken, i.e. the orbit inside $\{y = -b\}$ connects $(x, y) = (0, -b)$ and $(x, y) = (1, -b)$, for all $\tilde{\delta} > 0$, $\tilde{\delta} \sim 0$ and $(b, c) \sim (b_0, c_0)$.* See Figure 5.

We define a difference map near the slow-fast 2-saddle limit periodic set $\Gamma_{-b_0}^{(b_0, c_0)}$:

$$\Delta(x, \tilde{\delta}, b, c) := \Delta_+(x, \tilde{\delta}, b, c) - \Delta_-(x, \tilde{\delta}, b, c), \ (\tilde{\delta}, b, c) \sim (0, b_0, c_0),$$

where Δ_+ (resp. Δ_-) is a transition map of (6) in forward time (resp. in backward time) from S to T (see Figure 5). The horizontal section S is parametrized by x (we give a precise definition of S later in this section) and the section $T \subset \{\bar{y} = 0\}$ is defined in the family directional chart of (8) and parametrized by $\bar{x} \sim 0$, where $(\bar{x}, \bar{y})$ are the coordinates of (9). Thus, T is transverse to the connection $\{\bar{x} = 0\}$, on the blow-up locus, between the attracting part of the critical curve and the repelling part of the critical curve (see Figure 3). It is clear that limit cycles of (6), close to $\Gamma_{-b_0}^{(b_0, c_0)}$ in the Hausdorff sense, are given

as zeros of the difference map Δ. *In the rest of this section, we show that Δ has at most one zero in x (counting multiplicity), for each fixed $\tilde{\delta} > 0$, $\tilde{\delta} \sim 0$ and $(b,c) \sim (b_0, c_0)$.*
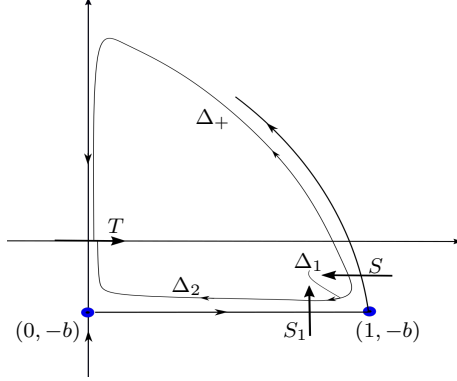

Figure 5: Limit cycles of (6), Hausdorff close to $\Gamma_{-b}^{(b,c)}$, can be studied as zeros of a difference map $\Delta := \Delta_+ - \Delta_-$ where Δ_+ (resp. $\Delta_- = \Delta_2 \circ \Delta_1$) is a transition map defined by following the trajectories of (6) in forward time (resp. in backward time) from the section S to the section T.

First, we study the transition map Δ_-. We split up Δ_- into two parts (see Figure 5):

1. The Dulac map Δ_1 near the hyperbolic saddle $(x,y) = (1,-b)$ defined by following the orbits of (6) in backward time from S to S_1. We study the transition Δ_1 by using results from e.g. [Mou90].

2. The transition map Δ_2 defined by following the trajectories of (6) in backward time from S_1 to T. This transition map includes the passage near the critical curve with a hyperbolic saddle in the slow dynamics and it has been studied in detail in [DMD08] or [HDMD13].

The Dulac map Δ_1, near the hyperbolic saddle $(1,-b_0)$, has the following form in the C^k-normal form coordinates (see e.g. [Mou90, DEMR96]):

$$\Delta_1(x, \tilde{\delta}, b, c) = x^\rho (1 + \Psi(x, \tilde{\delta}, b, c)), \tag{14}$$

where $x \sim 0$, $x > 0$, $\rho = -\frac{\rho_2}{\rho_1} = \frac{c}{1+b} - \tilde{\delta}^2 > 0$, Ψ is a C^∞-function for $x > 0$ and $(\tilde{\delta}, b, c) \sim (0, b_0, c_0)$ and

$$\forall n \in \mathbb{N}_0 : \lim_{x \to 0} x^n \frac{\partial^n \Psi}{\partial x^n} = 0, \text{ uniformly in } (\tilde{\delta}, b, c) \sim (0, b_0, c_0). \tag{15}$$

In the normal form coordinates, which we denote again by (x,y), the hyperbolic saddle corresponds to the origin $(x,y) = (0,0)$, $S \subset \{y = 1\}$ is parametrized by $x \sim 0$ and $S_1 \subset \{x = 1\}$ is parametrized by $y \sim 0$. Using (14) and (15), we obtain:

$$\frac{\partial \Delta_1}{\partial x}(x, \tilde{\delta}, b, c) = x^{\rho-1}(\rho + \Phi(x, \tilde{\delta}, b, c)), \tag{16}$$

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where Φ has the property (15). On the other hand, the derivative of the transition map Δ_2 has the following form (see [DMD08] or Section 3.8.5 of [HDMD13]):

$$\frac{\partial \Delta_2}{\partial y}(y, \tilde{\delta}, b, c) = \frac{1}{\tilde{\delta}^4} \exp \frac{1}{\tilde{\delta}^2} \left(\int_{\alpha(y, \tilde{\delta}, b, c)}^1 \frac{-1 + \tilde{\delta}^2}{s} ds + O(1) \right)$$

$$= \frac{1}{\tilde{\delta}^4} \exp \frac{1}{\tilde{\delta}^2} \left((1 - \tilde{\delta}^2) \ln \alpha(y, \tilde{\delta}, b, c) + O(1) \right), \qquad (17)$$

where the section S_1 is parametrized by the above normal form coordinate $y \sim 0$, α is a C^k-function in $(y, \tilde{\delta}, b, c) \sim (0, 0, b_0, c_0)$, with $\frac{\partial \alpha}{\partial y} > 0$, and the $O(1)$-term in (17) is $\tilde{\delta}$-regularly C^k in (y, b, c), i.e. $O(1)$ and all its derivatives up to order k w.r.t. (y, b, c) are continuous including at $\tilde{\delta} = 0$. The integral in (17) is the divergence integral of (6) near the hyperbolic saddle $(0, -b_0)$, multiplied by $\tilde{\delta}^2$ and calculated in C^k-normal form coordinates $\{\dot{v}_1 = \tilde{\delta}^2 v_1, \dot{v}_2 = -v_2\}$ from $\{v_2 = 1\}$, parametrized by $v_1 \sim 0$, to $\{v_1 = 1\}$, parametrized by $v_2 \sim 0$. We supposed that the orbit starting at a point on the section S_1, parametrized by $y \sim 0$, intersects the section $\{v_2 = 1\}$ in a point with $v_1 = \alpha(y, \tilde{\delta}, b, c)$. Since the connection between $(0, -b)$ and $(1, -b)$ is unbroken, we can write $\alpha = y\tilde{\alpha}$, where $\tilde{\alpha} > 0$.

Using (16) and (17), the derivative of Δ_- w.r.t. x can be written as:

$$\frac{\partial \Delta_-}{\partial x}(x) = \frac{\partial \Delta_2}{\partial y}(\Delta_1(x)) \frac{\partial \Delta_1}{\partial x}(x)$$

$$= \frac{1}{\tilde{\delta}^4} \exp \frac{1}{\tilde{\delta}^2} \left((1 - \tilde{\delta}^2) \ln \alpha(\Delta_1(x)) + \tilde{\delta}^2(\rho - 1) \ln x + \right.$$

$$\left. + O(1) + \tilde{\delta}^2 \ln(\rho + \Phi(x)) \right) \qquad (18)$$

where, for the sake of readability, we write x instead of $(x, \tilde{\delta}, b, c)$ and y instead of $(y, \tilde{\delta}, b, c)$. Note that the term $O(1) + \tilde{\delta}^2 \ln(\rho + \Phi(x))$ in (18) is bounded. Since $\alpha = y\tilde{\alpha}$ and $\tilde{\alpha} > 0$, (14) implies that:

$$\alpha(\Delta_1(x)) = x^\rho (1 + \Psi(x)) \tilde{\alpha}(\Delta_1(x)),$$

where $(1 + \Psi(x))\tilde{\alpha}(\Delta_1(x))$ is strictly positive and bounded. Thus, (18) has the following form:

$$\frac{\partial \Delta_-}{\partial x}(x) = \frac{1}{\tilde{\delta}^4} \exp \frac{1}{\tilde{\delta}^2} \left((\rho - \tilde{\delta}^2) \ln x + O_-(1) \right) \qquad (19)$$

for a bounded function $O_-(1)$.

Since the slow dynamics (7) is regular for $y \geq 0$, we use e.g. [DMD05] to find the derivative of Δ_+ w.r.t. x:

$$\frac{\partial \Delta_+}{\partial x}(x) = \frac{1}{\tilde{\delta}^4} \exp \frac{1}{\tilde{\delta}^2} O_+(1) \qquad (20)$$

where $O_+(1)$ is $\tilde{\delta}$-regularly C^k in (x, b, c), thus bounded. Let's introduce the analytic function $E(\beta_1, \beta_2) = \frac{\exp \beta_1 - \exp \beta_2}{\beta_1 - \beta_2} > 0$, $\beta_1 \neq \beta_2$, and $E(\beta_1, \beta_1) =$

$\exp \beta_1$. Combining (19) and (20) we get the expression for the derivative of the difference map:

$$\frac{\partial \Delta}{\partial x}(x) = \frac{\partial(\Delta_+ - \Delta_-)}{\partial x}(x)$$
$$= \frac{1}{\tilde{\delta}^6} E(\beta_1, \beta_2)\Big(-(\rho - \tilde{\delta}^2)\ln x + O(1)\Big) \tag{21}$$

where $O(1)$ is a bounded function, $\beta_1 = \frac{1}{\tilde{\delta}^2}O_+(1)$ and $\beta_2 = \frac{1}{\tilde{\delta}^2}\big((\rho - \tilde{\delta}^2)\ln x + O_-(1)\big)$. It thus suffices to study zeros of the function

$$-(\rho - \tilde{\delta}^2)\ln x + O(1) \tag{22}$$

introduced in (21). Since $-(\rho - \tilde{\delta}^2)\ln x \to +\infty$ as $x \to 0$, (22) is strictly positive for $x > 0$, $x \sim 0$ and $(\tilde{\delta}, b, c) \sim (0, b_0, c_0)$. Rolle's theorem implies that the difference map Δ has at most one zero (counting multiplicity) w.r.t. $x \sim 0$, $x > 0$, i.e. $\Gamma_{-b_0}^{(b_0, c_0)}$ can produce at most one limit cycle. It the limit cycle exists, it is hyperbolic and repelling because (22) is strictly positive. This completes the proof of Theorem 2.1.

3.2 Proof of Theorem 2.2

Let b_0 be an arbitrary positive constant and $c_0 = 2b_0 + 2$. We suppose that $C \subset] - b_0, 0[$ is an arbitrary compact set. We study zeros of the slow divergence integral $I(y, b_0, c_0)$, given in (12), w.r.t. $y \in C$, depending on the constant b_0.

When $(\delta, b, c) = (0, b_0, c_0)$, using Darboux theory of integrability, we find the first integral of (6), divided by x (i.e. the first integral of the system $\{\dot{x} = (1 - b_0)x - x^2 - y, \dot{y} = c_0(y + b_0)\}$):

$$H(x, y) = (2b_0 + 2)\arctan \frac{x + b_0}{\sqrt{y + b_0}} + 2\sqrt{y + b_0}, \ y > -b_0.$$

Since $H(0, y) = H(0, F(y))$ for all $y \in]-b_0, 0[$, with F introduced in (12), it can be easily seen that F is analytic on $]-b_0, 0]$ and that the derivative of F has the following form:

$$F'(y) = \frac{h(y)}{h(F(y))}, \ y \in]-b_0, 0[, \tag{23}$$

where $h(y) = \frac{y}{\sqrt{y+b_0}(b_0^2 + b_0 + y)}$. Using (12) and (23), the derivative of the slow divergence integral w.r.t. y can be written as:

$$\frac{\partial I}{\partial y}(y, b_0, c_0) = -\frac{1}{b_0}\Big(\frac{F(y)F'(y)}{F(y) + b_0} - \frac{y}{y + b_0}\Big)$$
$$= \frac{h(y)}{b_0\sqrt{F(y) + b_0}\sqrt{y + b_0}}\Big(\sqrt{F(y) + b_0}(b_0^2 + b_0 + y) -$$
$$- \sqrt{y + b_0}(b_0^2 + b_0 + F(y))\Big). \tag{24}$$

From (24) it follows that $\{\frac{\partial I}{\partial y} = 0\}$ is equivalent to $\{L(y) = 0\}$, on $]-b_0, 0[$, where

$$L(y) = \sqrt{F(y) + b_0}(b_0^2 + b_0 + y) - \sqrt{y + b_0}(b_0^2 + b_0 + F(y)). \tag{25}$$

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On the other hand, (23) implies that $\{z = F(y)\}$ is an orbit of the system $\{\dot{y} = h(z), \dot{z} = h(y)\}$, where $y, z > -b_0$. If we multiply this system by $\sqrt{y + b_0}(b_0^2 + b_0 + y)\sqrt{z + b_0}(b_0^2 + b_0 + z) > 0$, we get:

$$\begin{cases} \dot{y} &= z\sqrt{y + b_0}(b_0^2 + b_0 + y) \\ \dot{z} &= y\sqrt{z + b_0}(b_0^2 + b_0 + z) \end{cases} \tag{26}$$

where $y, z > -b_0$. System (26) has a hyperbolic saddle at the origin $(y, z) = (0, 0)$ (see Figure 6).
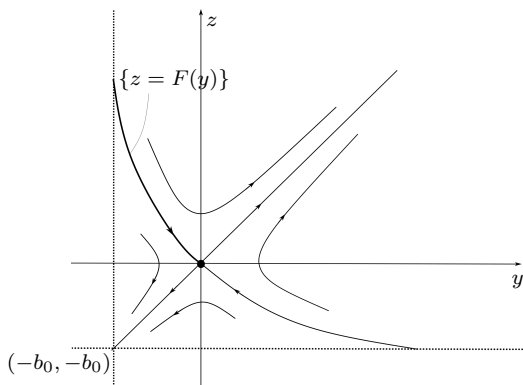

Figure 6: Dynamics of system (26).

In the rest of this section, we show that there exists at most one intersection (counting multiplicity) between $\{z = F(y)\}$ and $\{\mathcal{L}(y, z) = 0\}$ in the region $\{-b_0 < y < 0, z > 0\}$ where the function $\mathcal{L}$ is given by

$$\mathcal{L}(y, z) = \sqrt{z + b_0}(b_0^2 + b_0 + y) - \sqrt{y + b_0}(b_0^2 + b_0 + z). \tag{27}$$

(We replace $F(y)$ by z in (25).) This will imply that $\frac{\partial I}{\partial y}$ has at most one zero (counting multiplicity) on $]-b_0, 0[$. It can be easily seen that each solution of $\{\mathcal{L}(y, z) = 0\}$ satisfies $y = z$ or

$$z = b_0 \frac{b_0(b_0^2 - 1) - y}{y + b_0}. \tag{28}$$

When $0 < b_0 < 1$ (resp. $b_0 \geq 1$), we prove that there is one transversal intersection (resp. there are no intersections) between the curve $\{z = F(y)\}$ and the curve given by (28), in $\{-b_0 < y < 0, z > 0\}$ (see Figure 7). To prove it, first we calculate the number of contact points between the curve $\{\mathcal{L}(y, z) = 0\}$ and the orbits of system (26). The contact points are given by:

$$\nabla \mathcal{L}(y, z) \cdot \left(z\sqrt{y + b_0}(b_0^2 + b_0 + y), y\sqrt{z + b_0}(b_0^2 + b_0 + z)\right) = 0 \tag{29}$$

where $\mathcal{L}(y, z) = 0$. Using (27), we have that

$$\nabla \mathcal{L}(y, z) = \left(\sqrt{z + b_0} - \frac{1}{2}\frac{b_0^2 + b_0 + z}{\sqrt{y + b_0}}, -\sqrt{y + b_0} + \frac{1}{2}\frac{b_0^2 + b_0 + y}{\sqrt{z + b_0}}\right),$$
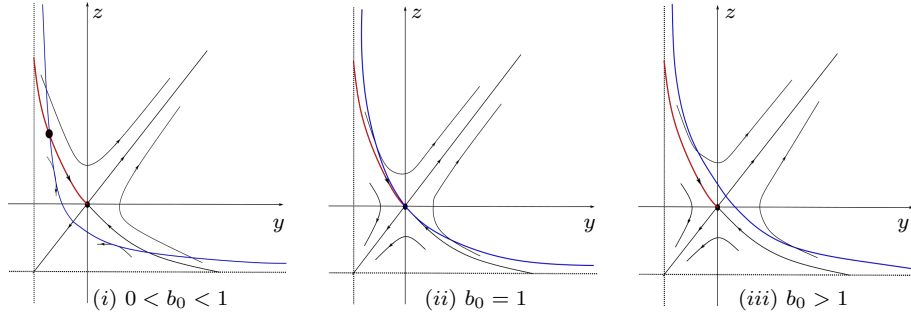
Figure 7: When $0 < b_0 < 1$, there exist one transversal intersection between $\{z = F(y)\}$ and $\{\mathcal{L}(y, z) = 0\}$ in the region $\{-b_0 < y < 0, z > 0\}$. When $b_0 \geq 1$, there are no intersections in $\{-b_0 < y < 0, z > 0\}$.

and (29) becomes

$$(y - z)\left(\frac{1}{2}(b_0^2 + b_0 + y)(b_0^2 + b_0 + z) - (b_0^2 + b_0)\sqrt{y + b_0}\sqrt{z + b_0}\right) = 0.$$

We focus on

$$\frac{1}{2}(b_0^2 + b_0 + y)(b_0^2 + b_0 + z) - (b_0^2 + b_0)\sqrt{y + b_0}\sqrt{z + b_0} = 0. \qquad (30)$$

When $\mathcal{L}(y, z) = 0$, then (30) is equivalent to

$$y^2 + b_0^2(b_0^2 - 1) = 0.$$

We distinguish between two cases:

1. When $0 < b_0 < 1$, we have the following contact points between the curve $\{\mathcal{L}(y, z) = 0\}$ and the orbits of system (26): $P_1 = (-b_0\sqrt{1 - b_0^2}, b_0\sqrt{1 - b_0^2})$, $P_2 = (b_0\sqrt{1 - b_0^2}, -b_0\sqrt{1 - b_0^2})$ and the line $y = z$. Since z in (28) tends to $+\infty$ as $y \to -b_0$, there exists at least one intersection point between $\{z = F(y)\}$ and $\{\mathcal{L}(y, z) = 0\}$ in the region $\{-b_0 < y < 0, z > 0\}$ (see Figure 7(i)). If we suppose that there are at least 2 intersection points (counting multiplicity) between $\{z = F(y)\}$ and $\{\mathcal{L}(y, z) = 0\}$ in $\{-b_0 < y < 0, z > 0\}$, then there are at least 2 contact points in $\{-b_0 < y < 0, z > 0\}$. This is in clear contradiction with the fact that there is only one contact point (P_1) in $\{-b_0 < y < 0, z > 0\}$. Thus, there exists one transversal intersection point between $\{z = F(y)\}$ and $\{\mathcal{L}(y, z) = 0\}$ in $\{-b_0 < y < 0, z > 0\}$. This implies that $\frac{\partial I}{\partial y}$ has one simple zero on $]-b_0, 0[$. Using Rolle's theorem and the fact that $I(0, b_0, c_0) = 0$, we have that $I(y, b_0, c_0)$ has at most one simple zero on $]-b_0, 0[$. Thus, at most two limit cycles can be created.

2. When $b_0 \geq 1$, the contact points are given by $y = z$. Suppose there is at least one intersection point between $\{z = F(y)\}$ and $\{\mathcal{L}(y, z) = 0\}$ in $\{-b_0 < y < 0, z > 0\}$. Then there exists at least one contact point between the curve $\{\mathcal{L}(y, z) = 0\}$ and the orbits of system (26) in $\{-b_0 < y < 0, z > 0\}$, which is a contradiction and proves that there are no

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intersection points. See Figure 7(ii)–(iii). This implies that $\frac{\partial I}{\partial y}(y, b_0, c_0)$ has no zeros on $] - b_0, 0[$. Using Rolle's theorem and $I(0, b_0, c_0) = 0$, we have that $I(y, b_0, c_0)$ is strictly positive on $] - b_0, 0[$. Thus, at most one (hyperbolic and repelling) limit cycle can be created.

3.3 Proof of Theorem 2.3

Let $b_0 > 0$, $b_0 \neq 1$ and $c_0 > 0$. We bring (6), near the origin $(x, y) = (0, 0)$, with $\delta = \tilde{\delta}^2$, to a normal form for a slow-fast Hopf point of codimension 1 studied in e.g. [KS01]. (We suppose that $(\tilde{\delta}, b, c) \sim (0, b_0, c_0)$.) The change of coordinates $\bar{x} = -\tilde{\delta}^2 b + (c - \tilde{\delta}^2)x$, with $(x, y) \sim (0, 0)$, changes (6) to

$$\begin{cases} \dot{\bar{x}} &= (\alpha_0 - y)\bar{x} + \tilde{\delta}^2(\tilde{\delta}\beta_0 - by) + \gamma_0 \bar{x}^2 - \frac{1}{(c-\tilde{\delta}^2)^2}\bar{x}^3 \\ \dot{y} &= (y + b)\bar{x}, \end{cases} \tag{31}$$

where

$$\alpha_0 = 2\tilde{\delta}^2 b \frac{1-b}{c - \tilde{\delta}^2} - \frac{3\tilde{\delta}^4 b^2}{(c - \tilde{\delta}^2)^2}, \quad \beta_0 = \tilde{\delta}b^2\frac{1-b}{c - \tilde{\delta}^2} - \frac{\tilde{\delta}^3 b^3}{(c - \tilde{\delta}^2)^2}, \quad \gamma_0 = \frac{1-b}{c - \tilde{\delta}^2} - \frac{3\tilde{\delta}^2 b}{(c - \tilde{\delta}^2)^2}.$$

After the coordinate change $X = (y + b)\bar{x}$, for $(\bar{x}, y) \sim (0, 0)$, (31) becomes

$$\begin{cases} \dot{X} &= (\alpha_0 - y)X + \tilde{\delta}^2\big(\tilde{\delta}b\beta_0 - (b^2 - \tilde{\delta}\beta_0)y - by^2\big) + \frac{1+\gamma_0}{y+b}X^2 \\ &\quad - \frac{1}{(c-\tilde{\delta}^2)^2(y+b)^2}X^3 \\ \dot{y} &= X, \end{cases} \tag{32}$$

where α_0, β_0 and γ_0 are given in (31). The translation $Y = y - \alpha_0$ brings (32) to

$$\begin{cases} \dot{X} &= -YX + \tilde{\delta}^2\Big(\big(\tilde{\delta}b\beta_0 - \alpha_0(b^2 - \tilde{\delta}\beta_0) - b\alpha_0^2\big) - (b^2 - \tilde{\delta}\beta_0 + 2b\alpha_0)Y \\ &\quad - bY^2\Big) + \frac{1+\gamma_0}{Y+\alpha_0+b}X^2 - \frac{1}{(c-\tilde{\delta}^2)^2(Y+\alpha_0+b)^2}X^3 \\ \dot{Y} &= X. \end{cases} \tag{33}$$

After rescaling $(X, Y) = \big((b^2 - \tilde{\delta}\beta_0 + 2b\alpha_0)\tilde{X}, \sqrt{b^2 - \tilde{\delta}\beta_0 + 2b\alpha_0}\tilde{Y}\big)$ and dividing by $\sqrt{b^2 - \tilde{\delta}\beta_0 + 2b\alpha_0}$, (33) changes into

$$\begin{cases} \dot{\tilde{X}} &= -\tilde{Y}\tilde{X} + \tilde{\delta}^2\Big(\tilde{\beta}_0 - \tilde{Y} - \big(1 + O(\tilde{\delta}^2)\big)\tilde{Y}^2\Big) + \big(1 + \frac{1-b}{c} + O(\tilde{\delta}^2, \tilde{X}, \tilde{Y})\big)\tilde{X}^2 \\ \dot{\tilde{Y}} &= \tilde{X}, \end{cases}$$

where

$$\tilde{\beta}_0 = \frac{\tilde{\delta}b\beta_0 - \alpha_0(b^2 - \tilde{\delta}\beta_0) - b\alpha_0^2}{(b^2 - \tilde{\delta}\beta_0 + 2b\alpha_0)^{\frac{3}{2}}} = \tilde{\delta}^2\big(\frac{b-1}{c} + O(\tilde{\delta}^2)\big) =: \tilde{\delta}\bar{\beta}_0. \tag{34}$$

If we write $(\tilde{x}, \tilde{y}) = (\tilde{Y}, \tilde{X})$, we get:

$$\begin{cases} \dot{\tilde{x}} &= \tilde{y} \\ \dot{\tilde{y}} &= -\tilde{x}\tilde{y} + \tilde{\delta}^2\big(\tilde{\delta}\bar{\beta}_0 - \tilde{x} - (1 + O(\tilde{\delta}^2))\tilde{x}^2\big) + \big(1 + \frac{1-b}{c} + O(\tilde{\delta}^2, \tilde{x}, \tilde{y})\big)\tilde{y}^2, \end{cases} \tag{35}$$

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where $\bar{\beta}_0 = \tilde{\delta}\left(\frac{b-1}{c} + O(\tilde{\delta}^2)\right) \sim 0$ is given in (34). When $0 < b_0 < 1$ (resp. $b_0 > 1$), it can be easily seen that (35) has a repelling hyperbolic focus (resp. an attracting hyperbolic focus) near the origin $(\tilde{x}, \tilde{y}) = (0, 0)$, for each $(\tilde{\delta}, b, c) \sim (0, b_0, c_0)$ and $\tilde{\delta} > 0$.

We write the $O(\tilde{y}^2)$-term in (35) as $O(\tilde{y}^2) = \tilde{y}^2 G(\tilde{x}, \tilde{y}, \tilde{\delta}, b, c)$, where G is analytic and $G(0, 0, 0, b_0, c_0) = 1 + \frac{1-b_0}{c_0}$. We denote by $\mathcal{X}_{(\tilde{\delta}, b, c)}$ the vector field $\{\dot{\tilde{x}} = 1,\ \dot{\tilde{y}} = -\tilde{x} + \tilde{y}G(\tilde{x}, \tilde{y}, \tilde{\delta}, b, c)\}$. Let $\tilde{y} = h(\tilde{x}; \hat{y}, \tilde{\delta}, b, c)$, with h analytic and $h(0; \hat{y}, \tilde{\delta}, b, c) = \hat{y}$, be the solutions of

$$\frac{d\tilde{y}}{d\tilde{x}} = -\tilde{x} + \tilde{y}G(\tilde{x}, \tilde{y}, \tilde{\delta}, b, c). \tag{36}$$

Using (36), we have $h(0; 0, \tilde{\delta}, b, c) = \frac{\partial h}{\partial \tilde{x}}(0; 0, \tilde{\delta}, b, c) = 0$, $\frac{\partial^2 h}{\partial \tilde{x}^2}(0; 0, \tilde{\delta}, b, c) = -1$, $\frac{\partial^3 h}{\partial \tilde{x}^3}(0; 0, \tilde{\delta}, b, c) = -G(0, 0, \tilde{\delta}, b, c)$, $\frac{\partial h}{\partial \hat{y}}(0; 0, \tilde{\delta}, b, c) = 1$ and $\frac{\partial^2 h}{\partial \tilde{x} \partial \hat{y}}(0; 0, \tilde{\delta}, b, c) = G(0, 0, \tilde{\delta}, b, c)$. The coordinate change $\tilde{y} = h(\tilde{x}; \hat{y}, \tilde{\delta}, b, c)$ brings $\mathcal{X}_{(\tilde{\delta}, b, c)}$ to the box flow $1\frac{\partial}{\partial \tilde{x}}$, and consequently the vector field (35) becomes

$$\begin{cases} \dot{\tilde{x}} &= h(\tilde{x}; \hat{y}, \tilde{\delta}, b, c) \cdot \frac{\partial h}{\partial \hat{y}}(\tilde{x}; \hat{y}, \tilde{\delta}, b, c) \\ \dot{\hat{y}} &= \tilde{\delta}^2\left(\tilde{\delta}\bar{\beta}_0 - \tilde{x} - (1 + O(\tilde{\delta}^2))\tilde{x}^2\right), \end{cases} \tag{37}$$

after multiplication by $\frac{\partial h}{\partial \hat{y}}(\tilde{x}; \hat{y}, \tilde{\delta}, b, c) > 0$. Using the properties of h, (37) can be written as

$$\begin{cases} \dot{\tilde{x}} &= \hat{y}h_1(\tilde{x}, \hat{y}, \tilde{\delta}, b, c) + \tilde{x}^2 h_2(\tilde{x}, \hat{y}, \tilde{\delta}, b, c) \\ \dot{\hat{y}} &= \tilde{\delta}^2\left(\tilde{\delta}\bar{\beta}_0 - \tilde{x} - (1 + O(\tilde{\delta}^2))\tilde{x}^2\right), \end{cases}$$

where $h_1 = 1 + O(\tilde{x}, \hat{y})$, $h_2 = -\frac{1}{2} + O(\tilde{x}, \hat{y})$, $\frac{\partial h_1}{\partial \tilde{x}}(0, 0, \tilde{\delta}, b, c) = 2G(0, 0, \tilde{\delta}, b, c)$ and $\frac{\partial h_2}{\partial \tilde{x}}(0, 0, \tilde{\delta}, b, c) = -\frac{2}{3}G(0, 0, \tilde{\delta}, b, c)$. After rescaling $(\tilde{x}, \hat{y}) = (2x, 2y)$ and multiplication by -1, we finally get the desired normal form for the so-called non-degenerate canard point (see (3.3) of [KS01]):

$$\begin{cases} \dot{x} &= -yH_1(x, y, \tilde{\delta}, b, c) + x^2 H_2(x, y, \tilde{\delta}, b, c) \\ \dot{y} &= \tilde{\delta}^2\left(xH_3(x, y, \tilde{\delta}, b, c) - \lambda\right), \end{cases} \tag{38}$$

with $\lambda = \frac{\tilde{\delta}\bar{\beta}_0}{2}$, $H_k = 1 + O(x, y)$, $k = 1, 2, 3$, $\frac{\partial H_1}{\partial x}(0, 0, 0, b_0, c_0) = 4G(0, 0, 0, b_0, c_0)$, $\frac{\partial H_2}{\partial x}(0, 0, 0, b_0, c_0) = \frac{8}{3}G(0, 0, 0, b_0, c_0)$ and $\frac{\partial H_3}{\partial x}(0, 0, 0, b_0, c_0) = 2$. Following [KS01], the sign of

$$A = -\frac{\partial H_1}{\partial x}(0, 0, 0, b_0, c_0) + 3\frac{\partial H_2}{\partial x}(0, 0, 0, b_0, c_0) - 2\frac{\partial H_3}{\partial x}(0, 0, 0, b_0, c_0) = 4\frac{1-b_0}{c_0}$$

determines the dynamics of (38) near the origin $(x, y) = (0, 0)$. When $0 < b_0 < 1$ (i.e. $A > 0$), then (38) has at most one (hyperbolically repelling) limit cycle in a fixed neighborhood of $(x, y) = (0, 0)$, for each $(\tilde{\delta}, b, c) \sim (0, b_0, c_0)$. When the limit cycle exists, it is generated by a subcritical Hopf bifurcation at $\lambda = 0$, taking $\lambda \sim 0$ as independent (breaking) parameter (see Theorem 4.2 of [KS01]). After changing time, we get Theorem 2.3.1. When $b_0 > 1$ (i.e. $A < 0$), then (38) has at most one (hyperbolically attracting) limit cycle in a

fixed neighborhood of $(x, y) = (0, 0)$, for each $(\tilde{\delta}, b, c) \sim (0, b_0, c_0)$. If the limit cycle exists, it is created by a supercritical Hopf bifurcation at $\lambda = 0$, taking $\lambda \sim 0$ as independent parameter (see Theorem 4.1 of [KS01]). After changing time, this implies Theorem 2.3.2.

Remark 2. *Note that the obtained result (at most one small limit cycle when $b_0 \neq 1$) is compatible with results of [FTV13, Huz16] applied to the normal form (35). In the Liénard plane ($\tilde{y} \to \tilde{y} - \frac{1}{2}\tilde{x}^2$), (35) becomes*

$$\begin{cases} \dot{\tilde{x}} & = & \tilde{y} - \frac{1}{2}\tilde{x}^2 \\ \dot{\tilde{y}} & = & \tilde{\delta}^2\left(\tilde{\delta}\bar{\beta}_0 - \tilde{x} - \left(1 + O(\tilde{\delta}^2)\right)\tilde{x}^2\right) + \left(1 + \frac{1-b}{c} \right. \\ & & \left. + O(\tilde{\delta}^2, \tilde{x}, \tilde{y} - \frac{1}{2}\tilde{x}^2)\right)(\tilde{y} - \frac{1}{2}\tilde{x}^2)^2. \end{cases} \tag{39}$$

After rescaling $(\tilde{x}, \tilde{y}) = (\tilde{\delta}x, \tilde{\delta}^2 y)$, with (x, y) kept in a large compact set, and after division by $\tilde{\delta} > 0$, (39) becomes $\{\dot{x} = y - \frac{1}{2}x^2, \dot{y} = \bar{\beta}_0 - x - (1 + O(\tilde{\delta}^2))\tilde{\delta}x^2 + (1 + \frac{1-b}{c} + O(\tilde{\delta}))\tilde{\delta}(y - \frac{1}{2}x^2)^2\}$. When $\tilde{\delta} = 0$, this system has the center at the origin $(x, y) = (0, 0)$, an integrating factor $-e^{-y}$ and the function $\mathcal{H}(x, y) = e^{-y}(y - \frac{1}{2}x^2 + 1)$ as first integral. The center is represented by $L_1 := \{\mathcal{H}(x, y) = 1\}$ and we denote the closed level curves of $\mathcal{H}$ by L_h, where $h \in]0, 1[$. We write $J_0(h) = \int_{L_h} e^{-y}dx$ and $J_1(h) = -\int_{L_h} e^{-y}x^2dx + (1 + \frac{1-b}{c})\int_{L_h} e^{-y}(y - \frac{1}{2}x^2)^2dx$, for $h \in]0, 1]$, where L_h is oriented counter-clockwise. Since $\int_{L_h} e^{-y}(y - \frac{1}{2}x^2)^2dx = \int_{L_h} e^{-y}x^2dx$ (see [Huz16]), we have that $J_1(h) = \frac{1-b}{c}\int_{L_h} e^{-y}x^2dx$, and based on [FTV13] and $b_0 \neq 1$, it can be proved (see[Huz16]) that the system $\{\frac{d}{dh}J_0, \frac{d}{dh}J_1\}$ of analytic functions is a strict Chebyshev system on $[h_0, 1]$ of degree 1, for all small $h_0 > 0$ and $(b, c) \sim (b_0, c_0)$, i.e. at most one limit cycle can be created.

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