Slow divergence integral on a Möbius band

Peer-reviewed author version

HUZAK, Renato (2018) Slow divergence integral on a Möbius band. In: JOURNAL OF DIFFERENTIAL EQUATIONS, 266 (10), p. 6179-6203..

DOI: 10.1016/j.jde.2018.11.002
Handle: http://hdl.handle.net/1942/27416

Slow divergence integral on a Möbius band

Abstract

The slow divergence integral has proved to be an important tool in the study of slow-fast cycles defined on an orientable two-dimensional manifold (e.g. $\mathbb{R}^2$). The goal of our paper is to study 1*–canard cycle* and 2*–canard cycle* bifurcations on a non-orientable two-dimensional manifold (e.g. the *Möbius band*) by using similar techniques. Our focus is on smooth slow-fast models with a Hopf breaking mechanism. The same results can be proved for a jump breaking mechanism and non-generic turning points. The slow-fast bifurcation problems on the Möbius band require the study of the 2*–return map* attached to such 1– and 2–canard cycles. We give a simple sufficient condition, expressed in terms of the slow divergence integral, for the existence of a *period-doubling bifurcation* near the 1–canard cycle. We also prove the finite cyclicity property of "singular" 1– and 2–homoclinic loops ("regular" 1–homoclinic loops of finite codimension have been studied by Guimond).

1 Introduction

In the study of limit cycles appearing in slow-fast vector fields on an orientable 2-manifold, one typically uses the notion *slow divergence integral* attached to the *first iterate of the Poincaré map* (i.e. the 1–return map). See e.g. [DR96, KS01, DMD08, DMDR11]. The purpose of our paper is to initiate the study of limit cycles in slow-fast vector fields on the Möbius band involving the slow divergence integral related to *the second iterate of the Poincaré map* (often called the second return map or the 2*–return map*). To see how the 2–return map comes into play, let's consider a simple planar slow-fast system $X_{\epsilon,b}$ (depending possibly on an extra finite dimensional parameter):

$$\left\{ \begin{array}{rcl} \dot{x} &=& y \\ \dot{y} &=& -xy + \epsilon\big(b - x + O(x^2)\big) + O(\epsilon y^2) \end{array} \right. \tag{1}$$

where $\epsilon \geq 0$ is a singular perturbation parameter and $b \sim 0$ is a breaking parameter. ($X_{\epsilon,b}$ represents a normal form for a slow-fast Hopf point (see e.g. [Dum11, DMDR11]).) The fast subsystem $X_{0,b}$ of (1) consists of the line of singularities $\{y = 0\}$ (often called the *critical curve* or the *slow curve*) and *fast orbits*, given by parabolas $y = -\frac{1}{2}x^2 + c$. See Figure 1(a). All singularities of the critical curve are normally hyperbolic (attracting when $x > 0$ and repelling when $x < 0$), except the origin where we deal with a nilpotent *contact point.* We distinguish between two types of limit periodic sets, at level $\epsilon = 0$, that can produce limit cycles after perturbations: the contact point $(x, y) = (0, 0)$ and *canard cycles*, consisting of a fast orbit and the part of the critical curve between

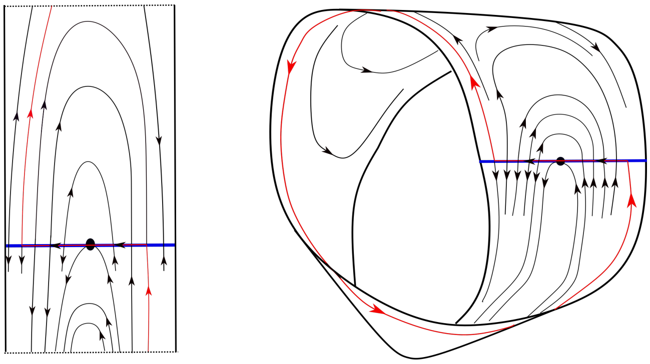

Figure 1: A fast subsystem defined on the Möbius band with indication of the 1–canard cycle turning around the Möbius band. (b) The two ends in (a) are glued together with a halftwist.

the α–limit set and the ω–limit set of the fast orbit. The canard cycles are slow-fast cycles that contain both attracting and repelling parts of the critical curve. To find the cyclicity of a canard cycle, one studies fixed points of the 1–return map, or, equivalently, zeros of a difference map, defined using two transition maps, one related to the attracting part of the critical curve and the other to the repelling part (see e.g. [DMD05, Dum11]). We suppose that the slow dynamics of $X_{\epsilon,b}$ along the critical curve, given by $x' = -1 + O(x)$, has no singularities (i.e., $x' < 0$). This implies that the slow divergence integral of $X_{\epsilon,b}$ along the critical curve is well-defined (see e.g. [Dum11] or Section 2).

Our model (1) can provide much richer dynamics if we consider it on the Möbius band (see Figure 1(b)). (We glue the two ends in Figure 1(a) together with a halftwist). Besides the contact point and the canard cycles we also detect a so-called 1*–canard cycle* consisting of a fast orbit, turning around the Möbius band, and the part of the critical curve between the α-limit set and the ω-limit set of the fast orbit. An *n-limit cycle* is a limit cycle, with $\epsilon > 0$, in a tubular neighborhood of the 1–canard cycle, which intersects a section, transversal to the 1–canard cycle, n times (see Section 2). A simple geometric argument shows that in our model at most one 1–limit cycle can be created and that n–limit cycles, with $n > 2$, are not possible. See Section 2 for more details. If the slow divergence integral, computed along the slow part of the 1-canard cycle, is nonzero, then 2–limit cycles are not possible; a 1-limit cycle can be created if we vary $b \sim 0$ (see Theorem 2.2). *We show, under the condition that the slow divergence integral has a simple zero, that a* 2*-limit cycle can be created by a period-doubling bifurcation as we vary the breaking parameter* $b \sim 0$ (see Theorem 2.3). Using an idea of Khovanskii (see [Kho91, MR12]) we also prove that under the same condition on the slow divergence integral at most one 2–limit cycle can be created in a small ϵ-uniform tubular neighborhood of the 1–canard cycle (see Theorem 2.4). The case of higher multiplicity zeros in the slow divergence integral is a topic of further study.

We call a 1-canard cycle a *singular* 1*–homoclinic loop* if one endpoint of its slow part is a hyperbolic saddle of the slow dynamics. The singular 1–

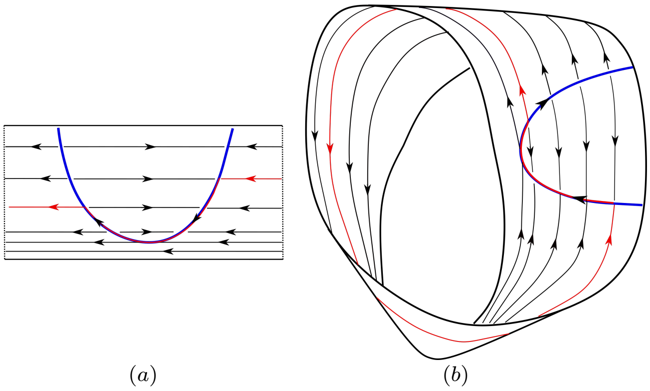

Figure 2: A fast subsystem defined on the Möbius band with indication of the 1–canard cycle turning around the Möbius band.

homoclinic loop is a singular variant of the well-known 1–homoclinic loop on the Möbius band, studied in [Gui99] (see also Section 3). In [Gui99], studying a 2-return map, an explicit bound for the cyclicity of the 1–homoclinic loop for all finite codimensions is given (like in our model, only 1– and 2–limit cycles are possible). Studying the 2-return map, we prove that the cyclicity of the singular 1–homoclinic loop is equal to one: one 1-limit cycle can be created by varying the breaking parameter $b \sim 0$, and the 2–limit cycles are not possible. See Theorem 2.5. Due to the presence of a hyperbolic saddle in the slow dynamics, the slow divergence integral is not well-defined and we have to study the so-called full divergence integral near the hyperbolic saddle (for more details see Section 3.6).

Roughly speaking, a 2*–canard cycle* is a limit periodic set on the Möbius band which contains attracting and repelling parts of the critical curve and turns around the Möbius band twice (for a precise definition see Section 2). When the slow dynamics is regular, we use the slow divergence integral to study 2–limit cycles near the 2–canard cycle (see Theorems 2.6 and 2.7). Like in the case of the 1-canard cycle, we focus on the slow divergence integral with zeros of multiplicity 0 or 1. We can also define a *singular* 2*–homoclinic loop* and prove the finite cyclicity property of such a limit periodic set on the Möbius band (see Section 2 and Theorem 2.8).

In Section 2.1 we define the framework within which we study 1– and 2–canard cycles. For the sake of readability, our focus is on the generic turning point (i.e. the generic Hopf breaking mechanism). The methods we present in our paper in this special framework can be used in a more general framework of non-generic turning points and jump breaking mechanisms (for more details see Section 4). We state our main results in Section 2.2. We prove the results in Section 3.

2 Definitions and statement of results

2.1 Definitions on the smooth Möbius band

Denote by M a smooth Möbius band ("smooth" means C^∞-smooth). Let $(\epsilon, \mu) \sim (0, 0) \in \mathbb{R} \times \mathbb{R}^l$, with $\epsilon \geq 0$, and let $X_{\epsilon,\mu} : M \to TM$ be a smooth (ϵ, μ)-family of vector fields on M (TM is the tangent bundle of M). We suppose $X_{\epsilon,\mu}$ has a slow-fast structure, with a singular perturbation parameter ϵ and with a *generic turning point* (or equivalently, a slow-fast Hopf point) $p \in M$ for $(\epsilon, \mu) = (0, 0)$. More precisely, we suppose that there exists a local chart on M around p in which the vector field $X_{\epsilon,\mu}$ is locally expressed, up to smooth equivalence, as:

$$\left\{ \begin{array}{rcl} \dot{x} & = & y \\ \dot{y} & = & -xy + \epsilon\big(b(\mu) - x + x^2 g(x, \epsilon, \mu)\big) + \epsilon y^2 H(x, y, \epsilon, \mu), \end{array} \right. \tag{2}$$

for some smooth functions g and H, $b(0) = 0$ and for a smooth submersion b at $\mu = 0$. Using the local submersion theorem we can suppose that $\mu = (b, \lambda)$ where we call the parameter $b := b(\mu) \sim 0$ a *breaking parameter* (see e.g. [Dum11]). The generic turning point $p \in M$ is represented by $(x, y) = (0, 0)$ in the local coordinates. We further assume that $X_{0,\mu}$ has a smooth μ-family of one dimensional embedded manifolds m_μ containing singularities of $X_{0,\mu}$ (in the local coordinates, m_μ is given by $\{y = 0\}$), and that $m_0 = m^- \cup \{p\} \cup m^+$, where m^- (resp. m^+) is normally attracting (resp. normally repelling). See Figure 3. In the local coordinates, m^- (resp. m^+) is given by $\{x > 0, y = 0\}$ (resp. $\{x < 0, y = 0\}$). We suppose that the *slow dynamics* is nonzero on $m^- \cup m^+$, pointing towards p on m^- and away from p on m^+.

We assume that the family m_μ of slow curves is located in an open orientable submanifold $\widetilde{M}$ of M such that we can directly use the results [DMD05, DMD08, Dum11, DMDR11] for slow-fast vector fields defined on a two-dimensional smooth orientable manifold. Working with such an orientable submanifold, we can choose a volume form and define the divergence of (the restriction of) the vector field $X_{\epsilon,\mu}$. As we will see in later sections, the divergence integral along orbits of $X_{\epsilon,\mu}$, near the slow curves, is closely related to the *slow divergence integral* along the slow curves. The slow divergence integral is independent of the chosen volume form and the local chart (see e.g. [DMD08]).

Before we give a precise definition of the slow divergence integral, let us define:

Definition 1 (1– and 2–canard cycles)**.** *Let $\Sigma_+, \Sigma_- \subset \widetilde{M} \subset M$ be sections transverse to the fast orbits of $X_{0,\mu}$, with $\mu \sim 0$ (see Figure 3). We parametrize Σ_+ (resp. Σ_-) by a local coordinate u (resp. v). Suppose that all points u of Σ_+ lie in the basin of attraction of m^- and in the basin of repulsion of m^+, when $\epsilon = 0$. We denote by $\alpha(u, \mu)$ (or shortly $\alpha(u)$) the α-limit on m^+ of the fast orbit of $X_{0,\mu}$, characterized by $u \in \Sigma_+$, and we denote by $\omega(u, \mu)$ (or shortly $\omega(u)$) the ω-limit on m^- of the fast orbit of $X_{0,\mu}$ through $u \in \Sigma_+$ and turning around the Möbius band M. We define:*

(a) Let $u_0 \in \Sigma_+$. For $(\epsilon, \mu) = (0, 0)$, we define a limit periodic set L_{u_0} as follows: L_{u_0} consists of the fast orbit of $X_{0,0}$ through u_0 and the piece of the slow curve m_0 between $\omega(u_0)$ and $\alpha(u_0)$, including $\omega(u_0)$, $\alpha(u_0)$ and

the turning point p. (The fast orbit through u_0 turns around M, intersects Σ_- in v_0 and tends to $\omega(u_0)$.) See Figure 3(a). We call L_{u_0} ***1–canard cycle****.*

(b) Let $u_0, u_1 \in \Sigma_+$ and $u_0 < u_1$. For $(\epsilon, \mu) = (0, 0)$, we define a limit periodic set L_{u_0,u_1} as follows: L_{u_0,u_1} consists of the fast orbit of $X_{0,0}$ through u_0, turning around M and intersecting Σ_- in v_0, the piece of the slow curve m_0 between $\omega(u_0)$ and $\alpha(u_1)$, the fast orbit of $X_{0,0}$ through u_1, turning around M and intersecting Σ_- in v_1, and the piece of the slow curve m_0 between $\omega(u_1)$ and $\alpha(u_0)$. See Figure 3(b). We call L_{u_0,u_1} a ***2–canard cycle****.*

Definition 2 (1 and 2–periodic orbits)**.** *Let L_{u_0} and L_{u_0,u_1} be 1– and 2–canard cycles introduced in Definition 1.*

(a) Let $V \subset M$ be a small tubular neighborhood of L_{u_0}. Let $\mathcal{O} \subset V$ be a periodic orbit of $X_{\epsilon,\mu}$, with $\epsilon > 0$. We call $\mathcal{O}$ a 1–periodic orbit if $\mathcal{O}$ intersects the section Σ_+ only once. Isolated 1–periodic orbits are called 1–limit cycles.

(b) Let $V \subset M$ be a small tubular neighborhood of L_{u_0} or L_{u_0,u_1}. Let $\mathcal{O} \subset V$ be a periodic orbit of $X_{\epsilon,\mu}$, with $\epsilon > 0$. We call $\mathcal{O}$ a 2–periodic orbit if $\mathcal{O}$ intersects the section Σ_+ twice. Isolated 2–periodic orbits are called 2–limit cycles.

Definition 3. *Let $X_{\epsilon,\mu}$ be a smooth (ϵ, μ)-family of vector fields on M, defined above, and let L_{u_0} and L_{u_0,u_1} be the limit periodic sets introduced in Definition 1. The cyclicity of L_{u_0} (resp. L_{u_0,u_1}) in the family $X_{\epsilon,\mu}$ is bounded from above by $N \in \mathbb{N}$ if there exists $\epsilon_0 > 0$, $\delta_0 > 0$ and a neighborhood W of 0 in the μ-space such that $X_{\epsilon,\mu}$, with $(\epsilon, \mu) \in [0, \epsilon_0] \times W$, generates at most N limit cycles, lying each within Hausdorff distance δ_0 of L_{u_0} (resp. L_{u_0,u_1}). We call the smallest N with this property the cyclicity of L_{u_0} (resp. L_{u_0,u_1}) in the family $X_{\epsilon,\mu}$, and denote it by $\mathrm{Cycl}(X_{\epsilon,\mu}, L_{u_0})$ (resp. $\mathrm{Cycl}(X_{\epsilon,\mu}, L_{u_0,u_1})$).*

Using simple topological arguments, we see that L_{u_0} can produce at most one 1–periodic orbit (two 1–periodic orbits would have an intersection point, see Remark 4). We also see that L_{u_0} and L_{u_0,u_1} cannot produce n–periodic orbits, with $n > 2$. From this and Lemma 2.1 follows that the cyclicity (see Definition 3) of L_{u_0} (resp. L_{u_0,u_1}) is the number of 2–limit cycles $+1$ (resp. the number of 2–limit cycles).

Lemma 2.1. *Suppose that $X_{\epsilon,\mu}$ has a 2–periodic orbit intersecting Σ_+ in two points: $\bar{u}$ and $\tilde{u}$ ($\bar{u} < \tilde{u}$). Then $X_{\epsilon,\mu}$ also has one 1–periodic orbit intersecting Σ_+ in a point $u' \in]\bar{u}, \tilde{u}[$.*

Lemma 2.1 will be proved in Section 3.2.

The *slow dynamics* of $X_{\epsilon,\mu}$ along the slow curve $m_\mu \subset \widetilde{M}$, away from the turning point, is given by

$$x' = f(x, \mu), \ \mu \sim 0,$$

where f is a smooth function and m_μ is parametrized by a regular parameter x. We suppose that m^- (resp. m^+) is parametrized by $x > 0$ (resp. $x < 0$),

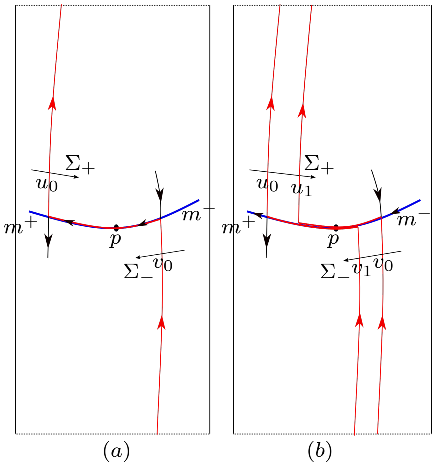

Figure 3: Canard cycles on the Möbius band M turning around M, at level $(\epsilon, \mu) = (0, 0)$. (a) 1–canard cycles intersect Σ_+ only once. (b) 2–canard cycles intersect Σ_+ twice.

and $x = 0$ represents the turning point p. Let us recall that the slow dynamics describes the dynamics of $X_{\epsilon,\mu}$, along the slow curve, when $\epsilon > 0$ and $\epsilon \sim 0$ (for more details see e.g. [DMD08]). Since the slow dynamics is nonzero and points from m^- to m^+ by supposition, we have $f < 0$. Now we can define the slow divergence integral $I_\pm(u, \mu)$ along $m^\pm$:

$$I_+(u, \mu) := \int_{\alpha(u)}^{0} \frac{\operatorname{div} X_{0,\mu} dx}{f(x, \mu)} < 0, \quad I_-(u, \mu) := \int_{\omega(u)}^{0} \frac{\operatorname{div} X_{0,\mu} dx}{f(x, \mu)} < 0, \ \ u \in \Sigma_+, \tag{3}$$

where $\alpha(u) < 0$ and $\omega(u) > 0$. (The slow divergence integral along a slow curve between two points p_1 and p_2 is the integral of the divergence of the vector field $X_{0,\mu}$ along the slow curve from p_1 to p_2 w.r.t. the slow time $\frac{dx}{f(x,\mu)}$.) Note that $\alpha'(u) > 0$, $\omega'(u) < 0$ and $\frac{\partial I_\pm}{\partial u} > 0$ due to the chosen parameterization of Σ_+ (as u increases, the points on Σ_+ are closer to the turning point). For example, the slow dynamics of (2) along $\{y = 0\}$ is given by $x' = -1 + xg(x, 0, \mu)$, and the slow divergence integrals $I_\pm$ of (2) are given by $I_+(u, \mu) = \int_{\alpha(u)}^{0} \frac{-xdx}{-1+xg(x,0,\mu)}$ and $I_-(u, \mu) = \int_{\omega(u)}^{0} \frac{-xdx}{-1+xg(x,0,\mu)}$.

It turns out that the slow divergence integrals in (3) play a crucial role in the study of limit cycles of $X_{\epsilon,\mu}$ bifurcating from L_{u_0} and L_{u_0,u_1} (see Section 2.2).

2.2 Statement of results

2.2.1 Limit cycle bifurcations Hausdorff-close to L_{u_0}

Let $u_0 \in \Sigma_+$ be arbitrary but fixed. For $(u, \mu) \sim (u_0, 0)$, the slow divergence integral along the slow curve from $\omega(u) \in m^-$ to $\alpha(u) \in m^+$ is given by:

$$I(u, \mu) = I_-(u, \mu) - I_+(u, \mu) \tag{4}$$

where $I_\pm$ are defined in (3). When I is nonzero near $(u,\mu)=(u_0,0)$, we have the following result:

Theorem 2.2. *Suppose $X_{\epsilon,\mu}$ satisfies conditions of Section 2.1 and suppose that $I(u,\mu)$ is nonzero near $(u,\mu)=(u_0,0)$. Then $\mathrm{Cycl}(X_{\epsilon,\mu},L_{u_0})=1$ and $X_{\epsilon,\mu}$ has no 2–periodic orbits Hausdorff-close to L_{u_0}. In case $I(u_0,0)<0$ (resp. $I(u_0,0)>0$) any 1–limit cycle bifurcating from L_{u_0} is hyperbolically attracting (resp. hyperbolically repelling).*

Theorem 2.2 will be proved in Section 3.3. Let us note that a similar result (i.e. at most one hyperbolic limit cycle) holds also for a canard cycle of (1), under the condition that the slow divergence integral along the slow part of the canard cycle is nonzero (for more details see e.g. [Dum11]).

If the function $u\to I(u,0)$ has a simple zero at $u=u_0$, then for $\lambda\sim 0$, $\epsilon\sim 0$ and $\epsilon>0$ the b-family $X_{\epsilon,\mu}=X_{\epsilon,b,\lambda}$ undergoes, Hausdorff-close to L_{u_0}, a *period doubling bifurcation*, giving rise to a 2–limit cycle. In this case we do not need the parameter λ.

Theorem 2.3. *Let the family $X_{\epsilon,b,\lambda}$ be as defined in Section 2.1 and let us suppose that the function $u\to I(u,0)$ has a simple zero at $u=u_0$ (i.e. $I(u_0,0)=0$ and $\frac{\partial I}{\partial u}(u_0,0)\neq 0$). Then there are continuous functions $u(\epsilon,\lambda)$ and $b(\epsilon,\lambda)$ defined for $\epsilon\geq 0$, $\epsilon\sim 0$ and $\lambda\sim 0$, smooth for $\epsilon>0$, with $u(0,0)=u_0$ and $b(0,\lambda)=0$, such that for each $\epsilon>0$, $\epsilon\sim 0$ and $\lambda\sim 0$ the b-family $X_{\epsilon,b,\lambda}$ undergoes a period doubling bifurcation at $(u(\epsilon,\lambda),b(\epsilon,\lambda))$.*

More precisely, for each $\epsilon>0$, $\epsilon\sim 0$ and $\lambda\sim 0$ the system $X_{\epsilon,b(\epsilon,\lambda),\lambda}$ has a 1–limit cycle intersecting Σ_+ at $u=u(\epsilon,\lambda)$, with eigenvalue -1. Fixing $(\epsilon,\lambda)\sim(0,0)$, with $\epsilon>0$, there is a smooth curve of 1–limit cycles of $X_{\epsilon,b,\lambda}$ passing through $(u(\epsilon,\lambda),b(\epsilon,\lambda))$, changing the stability at $(u(\epsilon,\lambda),b(\epsilon,\lambda))$, and a smooth curve $\gamma_{\epsilon,\lambda}$ passing through $(u(\epsilon,\lambda),b(\epsilon,\lambda))$ such that $\gamma_{\epsilon,\lambda}\setminus\{(u(\epsilon,\lambda),b(\epsilon,\lambda))\}$ is a union of hyperbolic 2–limit cycles of $X_{\epsilon,b,\lambda}$ and such that $\gamma_{\epsilon,\lambda}$ has a quadratic contact with the line $\{b=b(\epsilon,\lambda)\}$ at $(u(\epsilon,\lambda),b(\epsilon,\lambda))$. In case $\frac{\partial I}{\partial u}(u_0,0)<0$ (resp. $\frac{\partial I}{\partial u}(u_0,0)>0$) the 2-limit cycle is repelling (resp. attracting).

We prove Theorem 2.3 in Section 3.4. As a direct consequence of Theorem 2.3, if $u\to I(u,0)$ has a simple zero at $u=u_0$, then $\mathrm{Cycl}(X_{\epsilon,\mu},L_{u_0})\geq 2$ (there is one 1–limit cycle coexisting with one 2–limit cycle created by the period doubling bifurcation). To prove that, under the same condition on I, $\mathrm{Cycl}(X_{\epsilon,\mu},L_{u_0})\leq 2$, we use a method introduced in [MR12], based on the idea of Khovanskii [Kho91].

Theorem 2.4. *Let the family $X_{\epsilon,\mu}$ be as defined in Section 2.1 and let us suppose that $u\to I(u,0)$ has a simple zero at $u=u_0$. Then $\mathrm{Cycl}(X_{\epsilon,\mu},L_{u_0})=2$.*

Theorem 2.4 will be proved in Section 3.5.

Remark 1. *In fact we will prove the following more general (global) result in Section 3.5. Let $\bar{u},\tilde{u}\in\Sigma_+$, with $\bar{u}<\tilde{u}$, be arbitrary but fixed. Let the family $X_{\epsilon,\mu}$ be as defined in Section 2.1 and let us suppose that the function $u\to I(u,0)$ has precisely one zero in $[\bar{u},\tilde{u}]$ and that $\frac{\partial I}{\partial u}(u,0)\neq 0$ for all $u\in[\bar{u},\tilde{u}]$. Then $\mathrm{Cycl}(X_{\epsilon,\mu},\cup_{u\in[\bar{u},\tilde{u}]}L_u)=2$. Thus, $X_{\epsilon,\mu}$ has at most one 2–limit cycle intersecting the piece of the section Σ_+ parametrized by $u\in[\bar{u},\tilde{u}]$.*

We supposed in Theorems 2.2–2.4 that the slow dynamics is nonzero ($f < 0$). We call the 1–canard cycle L_{u_0} a *singular* 1*–homoclinic loop* if the slow dynamics has a hyperbolic saddle at precisely one corner point for $\mu = 0$: "$f(\omega(u_0), 0) = 0, \frac{\partial f}{\partial x}(\omega(u_0), 0) \neq 0$" or "$f(\alpha(u_0), 0) = 0, \frac{\partial f}{\partial x}(\alpha(u_0), 0) \neq 0$". We prove that such a limit periodic set can produce at most one limit cycle.

Theorem 2.5. *Let the family $X_{\epsilon,\mu}$ be as defined in Section 2.1 and let us suppose that $f(\omega(u_0), 0) = 0, \frac{\partial f}{\partial x}(\omega(u_0), 0) \neq 0$ and $f(x, 0) < 0$ for all $x \in [\alpha(u_0), \omega(u_0)[$. Then* $\text{Cycl}(X_{\epsilon,\mu}, L_{u_0}) = 1$ *and $X_{\epsilon,\mu}$ has no 2–periodic orbits Hausdorff-close to L_{u_0}. When a 1–limit cycle exists, it is hyperbolic and attracting.*

A similar result is true in the case $f(\alpha(u_0), 0) = 0, \frac{\partial f}{\partial x}(\alpha(u_0), 0) \neq 0$ and $f(x, 0) < 0$ for all $x \in]\alpha(u_0), \omega(u_0)]$. A 1–limit cycle bifurcating from L_{u_0} is hyperbolic and repelling.

Theorem 2.5 will be proved in Section 3.6.

2.2.2 Limit cycle bifurcations Hausdorff-close to L_{u_0,u_1}

Let $u_0, u_1 \in \Sigma_+$, with $u_0 < u_1$, be arbitrary but fixed. For $(u, \tilde{u}, \mu) \sim (u_0, u_1, 0)$, we define the so-called *total slow divergence integral* of L_{u_0,u_1}:

$$T(u, \tilde{u}, \mu) = I_-(u, \mu) - I_+(\tilde{u}, \mu) + I_-(\tilde{u}, \mu) - I_+(u, \mu), \tag{5}$$

with $I_\pm$ defined in (3). If T is nonzero near $(u, \tilde{u}, \mu) = (u_0, u_1, 0)$, then L_{u_0,u_1} produces at most one (2–)limit cycle the stability of which depends on the sign of T.

Theorem 2.6. *Suppose $X_{\epsilon,\mu}$ satisfies conditions of Section 2.1 and suppose that T is nonzero near $(u, \tilde{u}, \mu) = (u_0, u_1, 0)$. Then* $\text{Cycl}(X_{\epsilon,\mu}, L_{u_0,u_1}) \leq 1$. *In case $T(u_0, u_1, 0) < 0$ (resp. $T(u_0, u_1, 0) > 0$) any 2–limit cycle bifurcating from L_{u_0,u_1} is hyperbolically attracting (resp. hyperbolically repelling).*

Theorem 2.6 will be proved in Section 3.7. The condition $\{T(u_0, u_1, 0) \neq 0\}$ does not necessarily imply the existence of a limit cycle Hausdorff-close to L_{u_0,u_1}. When the slow divergence integral along $[\alpha(u_1), \omega(u_1)]$ is nonzero (i.e. $I(u_1, 0) \neq 0$), L_{u_0,u_1} cannot produce limit cycles (see Theorem 2.7.1).

Theorem 2.7. *Let the family $X_{\epsilon,\mu}$ be as defined in Section 2.1. The following statements are true:*

1. *If $I_-(u_1, 0) - I_+(u_1, 0) \neq 0$, then there exists $\epsilon_0 > 0$, $\delta_0 > 0$ and a neighborhood W of 0 in the μ-space such that system $X_{\epsilon,\mu}$, with $(\epsilon, \mu) \in [0, \epsilon_0] \times W$, has no limit cycles lying within Hausdorff distance δ_0 of L_{u_0,u_1}.*
2. *If $I_-(u_1, 0) - I_+(u_1, 0) = 0$ and $I_-(u_0, 0) - I_+(u_0, 0) \neq 0$ (this implies $T(u_0, u_1, 0) \neq 0$), then we have that* $\text{Cycl}(X_{\epsilon,\mu}, L_{u_0,u_1}) \leq 1$. *In case $I_-(u_0, 0) - I_+(u_0, 0) < 0$ (resp. $I_-(u_0, 0) - I_+(u_0, 0) > 0$) any 2–limit cycle bifurcating from L_{u_0,u_1} is hyperbolic and attracting (resp. repelling). Moreover, if $\frac{\partial(I_- - I_+)}{\partial u}(u_1, 0) \neq 0$, then* $\text{Cycl}(X_{\epsilon,\mu}, L_{u_0,u_1}) = 1$.
3. *If $I_-(u_i, 0) - I_+(u_i, 0) = 0$ for $i = 0, 1$ (this implies $T(u_0, u_1, 0) = 0$) and $\frac{\partial(I_- - I_+)}{\partial u}(u_i, 0) \neq 0$ for $i = 0, 1$, then* $\text{Cycl}(X_{\epsilon,\mu}, L_{u_0,u_1}) \leq 2$.

Theorem 2.7 will be proved in Section 3.8.

We suppose in Theorems 2.6–2.7 that the slow dynamics is nonzero. We allow now the slow dynamics to have a hyperbolic saddle at precisely one corner point, $\omega(u_0)$ or $\alpha(u_0)$, for $\mu = 0$ (note that $\alpha(u_0) < \alpha(u_1) < 0 < \omega(u_1) < \omega(u_0)$): "$f(\omega(u_0), 0) = 0$, $\frac{\partial f}{\partial x}(\omega(u_0), 0) \neq 0$" or "$f(\alpha(u_0), 0) = 0$, $\frac{\partial f}{\partial x}(\alpha(u_0), 0) \neq 0$". In this case we call L_{u_0,u_1} a *singular 2–homoclinic loop.*

Theorem 2.8. *Let the family $X_{\epsilon,\mu}$ be as defined in Section 2.1 and let us suppose that $f(\omega(u_0), 0) = 0$, $\frac{\partial f}{\partial x}(\omega(u_0), 0) \neq 0$ and that $f(x, 0) < 0$ for all $x \in [\alpha(u_0), \omega(u_0)[$. Then $\mathrm{Cycl}(X_{\epsilon,\mu}, L_{u_0,u_1}) \leq 1$. Any 2–limit cycle bifurcating from L_{u_0,u_1} is hyperbolic and attracting.*

A similar result is true in the case $f(\alpha(u_0), 0) = 0$, $\frac{\partial f}{\partial x}(\alpha(u_0), 0) \neq 0$ and $f(x, 0) < 0$ for all $x \in]\alpha(u_0), \omega(u_0)]$. Any 2–limit cycle bifurcating from L_{u_0,u_1} is hyperbolic and repelling.

We prove Theorem 2.8 in Section 3.9.

3 Proofs of Theorem 2.2–Theorem 2.8

3.1 Transition maps

In this section we study two transition maps, one along the flow of $X_{\epsilon,\mu}$ from Σ_+ to a section Σ_p, transverse to the slow curve at the turning point p, and the other along the flow of $-X_{\epsilon,\mu}$ from Σ_+ to Σ_p (for a precise definition of Σ_p, see below). This will enable us to study 2–periodic orbits Hausdorff close to L_{u_0} or L_{u_0,u_1}.

We define the section Σ_p as follows. It is well known that the passage near the generic turning point p, from the attracting part m^- to the repelling part m^+, can occur only if $(\epsilon, b) = (\bar{\epsilon}^2, \bar{\epsilon}B)$ where $\bar{\epsilon} \geq 0$, $\bar{\epsilon} \sim 0$ is a new singular perturbation parameter and $B \sim 0$ is the so called *regular breaking parameter*. For more details see [DR96, Dum11]. Then we include the parameter $\bar{\epsilon}$ in the following family blow-up at $(x, y, \bar{\epsilon}) = (0, 0, 0)$ (in the local coordinates (x, y), $p \in M$ is given by $(x, y) = (0, 0)$): $(x, y, \bar{\epsilon}) = (\rho\tilde{x}, \rho^2\tilde{y}, \rho\tilde{\epsilon})$, where $(\tilde{x}, \tilde{y}, \tilde{\epsilon}) \in \mathbb{S}^2$, $\tilde{\epsilon} \geq 0$, $\rho \geq 0$ and $\rho \sim 0$. In the family chart $\{\tilde{\epsilon} = 1\}$, we define $\Sigma_p = \{\tilde{x} = 0\}$. We parametrize Σ_p by $\tilde{y}$, kept in a large compact set. The section Σ_p is thus located on the top of the blow-up locus transversally cutting the heteroclinic orbit on the blow-up locus, at level $B = 0$, connecting m^- and m^+. (For $B = 0$, the dynamics on the blow-up locus is of center type with the center located above the heteroclinic orbit.) This connection becomes broken for $B \neq 0$ in a regular way. See e.g. [DMD05, Dum11] or Theorem 3.1. To prove the results stated in Section 2, it suffices to deal with $X_{\bar{\epsilon}^2,\bar{\epsilon}B,\lambda}$.

We define now the following transition maps for $(\bar{\epsilon}, B, \lambda) \sim (0, 0, 0)$:

1. the forward transition map $\Delta_- : \Sigma_+ \to \Sigma_p$ along the flow of $X_{\bar{\epsilon}^2,\bar{\epsilon}B,\lambda}$;
2. the backward transition map $\Delta_+ : \Sigma_+ \to \Sigma_p$ along the flow of $-X_{\bar{\epsilon}^2,\bar{\epsilon}B,\lambda}$.

The map $\Delta_\pm$ includes a passage near $m^\pm$.

For a fixed $(B, \lambda, \bar{\epsilon}) \sim (0, 0, 0)$, $\bar{\epsilon} > 0$, the system $X_{\bar{\epsilon}^2,\bar{\epsilon}B,\lambda}$ has a 1–periodic orbit passing through the point $u \in \Sigma_+$ if and only if the following holds:

$$\Delta_-(u, B, \lambda, \bar{\epsilon}) = \Delta_+(u, B, \lambda, \bar{\epsilon}). \tag{6}$$

Similarly, fixing $(B, \lambda, \bar{\epsilon}) \sim (0,0,0)$, with $\bar{\epsilon} > 0$, the system $X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda}$ has a 2–periodic orbit passing through the points $u, u' \in \Sigma_+$, with $u \neq u'$, if and only if the following holds:

$$\Delta_-(u, B, \lambda, \bar{\epsilon}) = \Delta_+(u', B, \lambda, \bar{\epsilon}) \text{ and } \Delta_-(u', B, \lambda, \bar{\epsilon}) = \Delta_+(u, B, \lambda, \bar{\epsilon}). \tag{7}$$

Remark 2. *Note that system of equations (7) is symmetric: if (u, u') is a solution of (7), then (u', u) is also solution of (7). These two solutions represent the same 2-periodic orbit. When $u = u'$, (7) reduces to (6).*

Instead of working with (7), it is sometimes more convenient to use the equation for the fixed points $\{P_{B,\lambda,\bar{\epsilon}} \circ P_{B,\lambda,\bar{\epsilon}}(u) = u\}$, where $P_{B,\lambda,\bar{\epsilon}}(u) = \Delta_+^{-1} \circ \Delta_-(u)$ is the 1–return map, or to use the difference equation $\{\Delta_{B,\lambda,\bar{\epsilon}}(u) = 0\}$ where $\Delta_{B,\lambda,\bar{\epsilon}}(u) = P_{B,\lambda,\bar{\epsilon}}(u) - P^{-1}_{B,\lambda,\bar{\epsilon}}(u)$. Note that if $X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda}$, with $\bar{\epsilon} > 0$, has a 2–periodic orbit then $P_{B,\lambda,\bar{\epsilon}}$ (and $P^{-1}_{B,\lambda,\bar{\epsilon}}$) is well-defined in the closed invariant region bounded by the 2–periodic orbit ($X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda}$ has no singularities in that region). This fact will be used in Sections 3.3, 3.6, 3.7 and 3.9.

We say that a function $f(\xi, \bar{\epsilon})$ is *$\bar{\epsilon}$-regularly smooth (resp. $\bar{\epsilon}$-regularly C^k) in ξ* if f is continuous in $(\xi, \bar{\epsilon})$, including $\bar{\epsilon} = 0$, and all partial derivatives of f w.r.t. ξ (resp. all partial derivatives of f w.r.t. ξ up to order k) exist and are continuous in $(\xi, \bar{\epsilon})$, including $\bar{\epsilon} = 0$ (see [Dum11]).

3.1.1 Transition maps with the regular slow dynamics

For a regular slow dynamics, the study of the transition maps relies on [DMD05, Dum11]. The following theorem gives the structure of $\Delta_\pm$.

Theorem 3.1. *There exist $\bar{\epsilon}$-regularly smooth functions $\bar{I}_\pm$ in (u, B, λ) and $\bar{\epsilon}$-regularly smooth functions $f_\pm$ in (B, λ) such that $\bar{I}_\pm(u, B, \lambda, 0) = I_\pm(u, 0, \lambda)$, with $I_\pm$ defined in (3), and such that*

$$\Delta_\pm(u, B, \lambda, \bar{\epsilon}) = f_\pm(B, \lambda, \bar{\epsilon}) \pm \exp\left(\frac{\bar{I}_\pm(u, B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2}\right). \tag{8}$$

Furthermore, $f(0, \lambda, 0) = 0$ and $\frac{\partial f}{\partial B}(0, \lambda, 0) \neq 0$ where $f(B, \lambda, \bar{\epsilon}) := f_-(B, \lambda, \bar{\epsilon}) - f_+(B, \lambda, \bar{\epsilon})$

Proof. Since the slow dynamics is regular along the repelling part m^+ and the passage from Σ_+ to Σ_p in backward time can be studied by using $-X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda}$ restricted to the two-dimensional orientable manifold $\widetilde{M} \subset M$, the expression for Δ_+, given in (8), follows directly from [Dum11] or [DMD05]. The sign in front of the exponential term in Δ_+ is positive due to the chosen parameterizations of Σ_+ and Σ_p.

Let us now consider the forward transition map Δ_-. We split up Δ_- into two parts (see Figure 3):

1. The regular transition map Δ_1 defined by following the orbits of $X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda}$ (in forward time) from Σ_+ to Σ_-, around the Möbius band M. Since $X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda}$ is smooth, Δ_1 is smooth in $(u, B, \lambda, \bar{\epsilon})$, and $\frac{\partial \Delta_1}{\partial u} > 0$ due to the chosen parameterizations of $\Sigma_\pm$.

2. The transition map Δ_2 defined by following the orbits of $X_{\bar\epsilon^2,\bar\epsilon B,\lambda}$ (in forward time) from Σ_- to Σ_p. Like in the case of Δ_+, since the slow dynamics is regular along the attracting part m^-, [Dum11] implies:

$$\Delta_2(v,B,\lambda,\bar\epsilon)=f_-(B,\lambda,\bar\epsilon)-\exp\left(\frac{\tilde I_-(v,B,\lambda,\bar\epsilon)}{\bar\epsilon^2}\right), \tag{9}$$

where f_- and $\tilde I_-$ are $\bar\epsilon$-regularly smooth in (B,λ) and (v,B,λ), respectively, and $\tilde I_-(v,B,\lambda,0)=I_-(v,0,\lambda)$ ($\omega(v)$ in definition of $I_-(v,0,\lambda)$ is the ω-limit of the fast orbit through $v\in\Sigma_-$). We have the negative sign in front of the exponential term in (9) due to the chosen parameterizations of Σ_- and Σ_p.

Using (9) and $\Delta_-=\Delta_2\circ\Delta_1$ we find the expression for Δ_- given in (8).

The properties of f follow from e.g. [Dum11] (B is the breaking parameter). □

Using Theorem 3.1, the equation (6) can be written as:

$$\exp\left(\frac{\bar I_-(u,B,\lambda,\bar\epsilon)}{\bar\epsilon^2}\right)+\exp\left(\frac{\bar I_+(u,B,\lambda,\bar\epsilon)}{\bar\epsilon^2}\right)=f(B,\lambda,\bar\epsilon), \tag{10}$$

and the system (7) can be written as:

$$\begin{cases}\exp\left(\frac{\bar I_-(u,B,\lambda,\bar\epsilon)}{\bar\epsilon^2}\right)+\exp\left(\frac{\bar I_+(u',B,\lambda,\bar\epsilon)}{\bar\epsilon^2}\right)=f(B,\lambda,\bar\epsilon)\\ \exp\left(\frac{\bar I_-(u',B,\lambda,\bar\epsilon)}{\bar\epsilon^2}\right)+\exp\left(\frac{\bar I_+(u,B,\lambda,\bar\epsilon)}{\bar\epsilon^2}\right)=f(B,\lambda,\bar\epsilon).\end{cases} \tag{11}$$

Remark 3. *For the sake of readability, we sometimes write $I_\pm+o(1)$ instead of $\bar I_\pm$ where the term $o(1)$ is $\bar\epsilon$-regularly smooth in (u,B,λ) (or in (u',B,λ)) and tends (u,B,λ)-uniformly (or (u',B,λ)-uniformly) to zero as $\bar\epsilon\to 0$. We write $I_\pm(\cdot)=I_\pm(\cdot,0,\lambda)$.*

Remark 4. *From (10) and Rolle's theorem we have that the system $X_{\bar\epsilon^2,\bar\epsilon B,\lambda}$ can produce at most one 1–limit cycle. Indeed, the derivative of the left-hand side of (10) w.r.t. u, given by*

$$\exp\left(\frac{I_-+o(1)}{\bar\epsilon^2}\right)+\exp\left(\frac{I_++o(1)}{\bar\epsilon^2}\right),$$

is positive, for $\bar\epsilon>0$ and $\bar\epsilon\sim 0$, where the $o(1)$-terms have the properties given in Remark 3. We used the fact that $\frac{\partial I_\pm}{\partial u}>0$ and $\bar\epsilon^2\ln\bar\epsilon=o(1)$.

The expression in (10) also gives a necessary condition for the existence of 1–limit cycles: $f>0$. The condition "$f>0$" is also necessary condition for the existence of 2–limit cycles (see (11)). Using Theorem 3.1 and the Implicit Function Theorem, applied to (10), we find a unique $\bar\epsilon$-regularly smooth function $B=B_0(u,\lambda,\bar\epsilon)$ in (u,λ) such that $B_0(u,\lambda,0)=0$ and such that $X_{\bar\epsilon^2,\bar\epsilon B_0(u,\lambda,\bar\epsilon),\lambda}$ has 1–periodic orbit passing through the point $u\in\Sigma_+$ (B is the breaking parameter).

3.1.2 Transition maps with a hyperbolic saddle in the slow dynamics

If we have a hyperbolic saddle in the slow dynamics, then we can use results from [DMD08]. Let us suppose that the slow dynamics has a hyperbolic saddle at $\omega(u_0) \in m^-$ at level $\mu = 0$. (A similar study can be done for a hyperbolic saddle at $\alpha(u_0) \in m^+$.) Near $\omega(u_0)$, we have a smooth curve C of hyperbolic saddles of $X_{\bar\epsilon^2,\bar\epsilon B,\lambda} + 0\frac{\partial}{\partial\bar\epsilon}$, and we denote the union of stable manifolds at points of C by M_S. Let ρ be small and positive. We define the set $U_- = \{(u, B, \lambda, \bar\epsilon) : u(B, \lambda, \bar\epsilon) < u < u_0 + \rho, (B, \lambda, \bar\epsilon) \sim (0, 0, 0), \bar\epsilon > 0\}$ where $u(B, \lambda, \bar\epsilon) \sim u_0$ is the smooth intersection of M_S, turning around M, and the section Σ_+ at the level $\bar\epsilon \geq 0$. Note that $u(0, 0, 0) = u_0$. The following theorem gives the structure of the transition map Δ_-.

Theorem 3.2. *For all $k > 0$ there exists $\bar\epsilon_k > 0$ so that Δ_- is C^∞ on $U_- \cap \{\bar\epsilon \leq \bar\epsilon_k\}$ and has a C^k-extension to the closure of $U_- \cap \{\bar\epsilon \leq \bar\epsilon_k\}$. Furthermore,*

$$\frac{\partial\Delta_-}{\partial u}(u, B, \lambda, \bar\epsilon) = -\exp\left(\frac{\mathcal{I}_-(u, B, \lambda, \bar\epsilon)}{\bar\epsilon^2}\right),\ (u, B, \lambda, \bar\epsilon) \in U_- \cap \{\bar\epsilon \leq \bar\epsilon_k\}, \quad (12)$$

where $\mathcal{I}_-$ is $\bar\epsilon$-regularly C^k in (u, B, λ), $\mathcal{I}_-(u, B, \lambda, \bar\epsilon) \to -\infty$ as $(u, B, \lambda, \bar\epsilon) \to (u_0, 0, 0, 0)$ and $\frac{\partial\mathcal{I}_-}{\partial u}(u, B, \lambda, \bar\epsilon) > 0$.

A similar result is true for the transition map Δ_+ in the presence of the hyperbolic saddle $\alpha(u_0)$ (with the $+$ sign in front of the exponential term in (12)).

Proof. Using the notation and the chosen parameterizations from the proof of Theorem 3.1, we have

$$\frac{\partial\Delta_-}{\partial u}(u) = \frac{\partial\Delta_2}{\partial v}(\Delta_1(u))\frac{\partial\Delta_1}{\partial u}(u).$$

It suffices to observe that Δ_1 is smooth, $\frac{\partial\Delta_1}{\partial u} > 0$ and $\frac{\partial\Delta_2}{\partial v}(v)$ has a form similar to (12), with the same smoothness properties (see [DMD08]). □

3.2 Proof of Lemma 2.1

Assume that $X_{\bar\epsilon^2,\bar\epsilon B,\lambda}$, $\bar\epsilon > 0$, has a 2–periodic orbit intersecting Σ_+ in two points $\bar u$ and $\tilde u$, with $\bar u < \tilde u$, i.e. $\Delta_-(\bar u, B, \lambda, \bar\epsilon) = \Delta_+(\tilde u, B, \lambda, \bar\epsilon)$ and $\Delta_-(\tilde u, B, \lambda, \bar\epsilon) = \Delta_+(\bar u, B, \lambda, \bar\epsilon)$. Then we have:

$$\Delta_-(\bar u, B, \lambda, \bar\epsilon) - \Delta_+(\bar u, B, \lambda, \bar\epsilon) > 0 \text{ and } \Delta_-(\tilde u, B, \lambda, \bar\epsilon) - \Delta_+(\tilde u, B, \lambda, \bar\epsilon) < 0,$$

due to the chosen parameterizations of Σ_+ and Σ_p. Now, since the function $u \to \Delta_-(u, B, \lambda, \bar\epsilon) - \Delta_+(u, B, \lambda, \bar\epsilon)$ is continuous for each $\epsilon \sim 0$ and $\epsilon > 0$, there exists $u' \in]\bar u, \tilde u[$ such that $\Delta_-(u', B, \lambda, \bar\epsilon) = \Delta_+(u', B, \lambda, \bar\epsilon)$. Thus, $X_{\bar\epsilon^2,\bar\epsilon B,\lambda}$ has one 1–periodic orbit, passing through $u' \in \Sigma_+$, which coexists with the 2–periodic orbit.

3.3 Proof of Theorem 2.2

The second part of Remark 4 implies $\text{Cycl}(X_{\epsilon,\mu}, L_{u_0}) \geq 1$.

Let I, given in (4), be nonzero near $(u, \mu) = (u_0, 0, 0)$ (i.e. $I_-(u_0, 0, 0) \neq I_+(u_0, 0, 0)$). Let us suppose that for $(B, \lambda, \bar\epsilon) \sim (0, 0, 0)$, $\bar\epsilon > 0$, $X_{\bar\epsilon^2,\bar\epsilon B,\lambda}$ has a 2–periodic orbit intersecting Σ_+ in two points $\bar{u} \sim u_0$ and $\tilde{u} \sim u_0$, with $\bar{u} < \tilde{u}$. Then $\Delta_{B,\lambda,\bar{\epsilon}}(\bar{u}) = \Delta_{B,\lambda,\bar{\epsilon}}(\tilde{u}) = 0$, $P_{B,\lambda,\bar{\epsilon}}(\bar{u}) = \tilde{u}$, $P_{B,\lambda,\bar{\epsilon}}(\tilde{u}) = \bar{u}$ and $P_{B,\lambda,\bar{\epsilon}}([\bar{u},\tilde{u}]) = [\bar{u},\tilde{u}]$, where the maps $\Delta_{B,\lambda,\bar{\epsilon}}$ and $P_{B,\lambda,\bar{\epsilon}}$ are introduced in Remark 2. Using Theorem 3.1 the derivative of $\Delta_{B,\lambda,\bar{\epsilon}}$ can be written as:

$$\begin{aligned}\Delta'_{B,\lambda,\bar{\epsilon}}(u) &= \frac{\Delta'_-(u)}{\Delta'_+(P_{B,\lambda,\bar{\epsilon}}(u))} - \frac{\Delta'_+(u)}{\Delta'_-(P^{-1}_{B,\lambda,\bar{\epsilon}}(u))} \\ &= -\exp\left(\frac{I_-(u) - I_+(P_{B,\lambda,\bar{\epsilon}}(u)) + o(1)}{\bar{\epsilon}^2}\right) \\ &\quad + \exp\left(\frac{I_+(u) - I_-(P^{-1}_{B,\lambda,\bar{\epsilon}}(u)) + o(1)}{\bar{\epsilon}^2}\right),\end{aligned}$$

for all $u \in [\bar{u},\tilde{u}]$. This implies that the equation $\{\Delta'_{B,\lambda,\bar{\epsilon}} = 0\}$ is equivalent, for $\bar{\epsilon} > 0$ and $u \in [\bar{u},\tilde{u}]$, to the following equation:

$$I_-(u) - I_+(P_{B,\lambda,\bar{\epsilon}}(u)) + I_-(P^{-1}_{B,\lambda,\bar{\epsilon}}(u)) - I_+(u) + o(1) = 0, \qquad (13)$$

for a new $o(1)$-term. Since $I_\pm$ are smooth and $u, P_{B,\lambda,\bar{\epsilon}}(u), P^{-1}_{B,\lambda,\bar{\epsilon}}(u) \approx u_0$ for all $u \in [\bar{u},\tilde{u}]$, we have:

$$\begin{aligned}I_-(u) - I_+(P_{B,\lambda,\bar{\epsilon}}(u)) &+ I_-(P^{-1}_{B,\lambda,\bar{\epsilon}}(u)) - I_+(u) \\ &\approx I_-(u_0) - I_+(u_0) + I_-(u_0) - I_+(u_0) \\ &= 2(I_-(u_0) - I_+(u_0)) \neq 0,\end{aligned}$$

for $u \in [\bar{u},\tilde{u}]$. From this and the fact that the $o(1)$-term is (u,B,λ)-uniformly small for $\bar{\epsilon} \sim 0$ follows that (13) has no solutions in $u \in [\bar{u},\tilde{u}]$, or equivalently, $\Delta'_{B,\lambda,\bar{\epsilon}}(u) \neq 0$, for all $u \in [\bar{u},\tilde{u}]$. This is a clear contradiction with the fact that there exists $u' \in]\bar{u},\tilde{u}[$ such that $\Delta'_{B,\lambda,\bar{\epsilon}}(u') = 0$ (Rolle's theorem). Thus, $X_{\bar{\epsilon}^2,\bar{\epsilon}B,\lambda}$ has no 2–periodic orbits Hausdorff close to L_{u_0}, i.e. $\mathrm{Cycl}(X_{\epsilon,\mu}, L_{u_0}) = 1$.

Suppose $X_{\bar{\epsilon}^2,\bar{\epsilon}B,\lambda}$ has a 1–limit cycle passing through $u_1 \in \Sigma_+$, Hausdorff close to L_{u_0} (i.e. $u_1 \sim u_0$), at level $(B,\lambda,\bar{\epsilon})$, with $\bar{\epsilon} > 0$. Then $P_{B,\lambda,\bar{\epsilon}}(u_1) = u_1$ and

$$P'_{B,\lambda,\bar{\epsilon}}(u_1) = -\exp\left(\frac{I_-(u_1) - I_+(u_1) + o(1)}{\bar{\epsilon}^2}\right).$$

Now, if $I_-(u_0) - I_+(u_0) < 0$ (resp. $I_-(u_0) - I_+(u_0) > 0$), then the 1–periodic orbit is hyperbolically attracting (resp. hyperbolically repelling) because $-1 < P'_{B,\lambda,\bar{\epsilon}}(u_1) < 0$ (resp. $P'_{B,\lambda,\bar{\epsilon}}(u_1) < -1$).

3.4 Proof of Theorem 2.3

Let $I(u_0,0,0) = 0$ and $\frac{\partial I}{\partial u}(u_0,0,0) \neq 0$. To prove that a 2–periodic orbit of $X_{\bar{\epsilon}^2,\bar{\epsilon}B,\lambda}$, Hausdorff close to L_{u_0}, can be created by a period doubling bifurcation, we show that the 1–return map $P_{B,\lambda,\bar{\epsilon}}$ fulfils the conditions of the following theorem (Theorem 3.5.1 in [GH83]), for each fixed $(\lambda,\bar{\epsilon}) \sim (0,0)$, $\bar{\epsilon} > 0$:

Theorem 3.3 (period doubling bifurcation)**.** *Let $p_B : \mathbb{R} \to \mathbb{R}$ be a smooth one-parameter family of mappings such that p_{B_0} has a fixed point x_0 with eigenvalue -1. Assume*

(PD1) $\frac{\partial p}{\partial B}\frac{\partial^2 p}{\partial x^2} + 2\frac{\partial^2 p}{\partial x \partial B} \neq 0$ *at* $(x, B) = (x_0, B_0)$*;*

(PD2) $a := \frac{1}{2}\big(\frac{\partial^2 p}{\partial x^2}\big)^2 + \frac{1}{3}\frac{\partial^3 p}{\partial x^3} \neq 0$ *at* $(x, B) = (x_0, B_0)$*.*

Then there is a smooth curve of fixed points of p_B *passing through* (x_0, B_0)*, the stability of which changes at* (x_0, B_0)*. There is also a smooth curve* γ *passing through* (x_0, B_0) *so that* $\gamma \setminus \{(x_0, B_0)\}$ *is a union of hyperbolic period* 2 *orbits. The curve* γ *has a quadratic tangency with the line* $B = B_0$ *at* (x_0, B_0)*. If* a *is positive (resp. negative), the period* 2 *orbits are attracting (resp. repelling).*

The expression in (PD1) is the derivative of $\frac{\partial p}{\partial x}$ w.r.t. B along the curve of the fixed points at (x_0, B_0), multiplied by 2.

The derivative of $P_{B,\lambda,\bar{\epsilon}}$ w.r.t. u is given by

$$\frac{\partial P_{B,\lambda,\bar{\epsilon}}}{\partial u}(u) = \frac{\frac{\partial \Delta_-}{\partial u}(u, B, \lambda, \bar{\epsilon})}{\frac{\partial \Delta_+}{\partial u}(P_{B,\lambda,\bar{\epsilon}}(u), B, \lambda, \bar{\epsilon})}, \tag{14}$$

with

$$\frac{\partial \Delta_\pm}{\partial u}(u, B, \lambda, \bar{\epsilon}) = \pm \exp\left(\frac{\hat{I}_\pm(u, B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2}\right)$$

where functions $\hat{I}_\pm$ are $\bar{\epsilon}$-regularly smooth in (u, B, λ) and $\hat{I}_\pm(u, B, \lambda, 0) = I_\pm(u, 0, \lambda)$ (see Theorem 3.1). Since the function $u \to I_-(u, 0, 0) - I_+(u, 0, 0)$ has a simple zero at $u = u_0$, $f(0, 0, 0) = 0$ and $\frac{\partial f}{\partial B}(0, 0, 0) \neq 0$, with f defined in Theorem 3.1, we can apply the Implicit Function Theorem to the following $\bar{\epsilon}$-regularly smooth in (u, B, λ) system

$$\begin{cases} \Delta_-(u, B, \lambda, \bar{\epsilon}) - \Delta_+(u, B, \lambda, \bar{\epsilon}) = 0 \\ \hat{I}_-(u, B, \lambda, \bar{\epsilon}) - \hat{I}_+(u, B, \lambda, \bar{\epsilon}) = 0, \end{cases}$$

and find a solution $(\lambda, \bar{\epsilon}) \to (u(\lambda, \bar{\epsilon}), B(\lambda, \bar{\epsilon}))$, $\bar{\epsilon}$-regularly smooth in λ, with $u(0, 0) = u_0$ and $B(0, 0) = 0$. From this and (14) follows

$$P_{B(\lambda,\bar{\epsilon}),\lambda,\bar{\epsilon}}(u(\lambda, \bar{\epsilon})) = u(\lambda, \bar{\epsilon}) \text{ and } \frac{\partial P_{B(\lambda,\bar{\epsilon}),\lambda,\bar{\epsilon}}}{\partial u}(u(\lambda, \bar{\epsilon})) = -1,$$

for all $(\lambda, \bar{\epsilon}) \sim (0, 0)$ and $\bar{\epsilon} > 0$. Thus, for each $(\lambda, \bar{\epsilon}) \sim (0, 0)$ and $\bar{\epsilon} > 0$, $P_{B(\lambda,\bar{\epsilon}),\lambda,\bar{\epsilon}}$ has a fixed point $u(\lambda, \bar{\epsilon})$ with eigenvalue -1.

Let us write $P(u) = P_{B,\lambda,\bar{\epsilon}}(u)$ and $\Delta_\pm(u) = \Delta_\pm(u, B, \lambda, \bar{\epsilon})$. Using (14) we have:

$$\frac{\partial^2 P}{\partial u^2}(u) = \frac{\frac{\partial^2 \Delta_-}{\partial u^2}(u)}{\frac{\partial \Delta_+}{\partial u}(P(u))} - \frac{\big(\frac{\partial \Delta_-}{\partial u}(u)\big)^2 \frac{\partial^2 \Delta_+}{\partial u^2}(P(u))}{\big(\frac{\partial \Delta_+}{\partial u}(P(u))\big)^3} \tag{15}$$

and

$$\begin{aligned} \frac{\partial^3 P}{\partial u^3}(u) = & \frac{\frac{\partial^3 \Delta_-}{\partial u^3}(u)}{\frac{\partial \Delta_+}{\partial u}(P(u))} - 3\frac{\frac{\partial \Delta_-}{\partial u}(u)\frac{\partial^2 \Delta_-}{\partial u^2}(u)\frac{\partial^2 \Delta_+}{\partial u^2}(P(u))}{\big(\frac{\partial \Delta_+}{\partial u}(P(u))\big)^3} \\ & - \frac{\big(\frac{\partial \Delta_-}{\partial u}(u)\big)^3 \frac{\partial^3 \Delta_+}{\partial u^3}(P(u))}{\big(\frac{\partial \Delta_+}{\partial u}(P(u))\big)^4} + 3\frac{\big(\frac{\partial \Delta_-}{\partial u}(u)\big)^3 \big(\frac{\partial^2 \Delta_+}{\partial u^2}(P(u))\big)^2}{\big(\frac{\partial \Delta_+}{\partial u}(P(u))\big)^5}. \end{aligned} \tag{16}$$

Since $\frac{\partial\Delta_+}{\partial u}(P(u)) = -\frac{\partial\Delta_-}{\partial u}(u)$ and $P(u) = u$ at $(u,B) = (u(\lambda,\bar\epsilon), B(\lambda,\bar\epsilon))$, from (15) and (16) follows that

$$\frac{\partial^2 P}{\partial u^2}(u) = -\frac{\frac{\partial^2(\Delta_- - \Delta_+)}{\partial u^2}(u)}{\frac{\partial\Delta_-}{\partial u}(u)}, \text{ at } (u,B) = (u(\lambda,\bar\epsilon), B(\lambda,\bar\epsilon)), \tag{17}$$

and

$$\frac{\partial^3 P}{\partial u^3}(u) = -\frac{\frac{\partial^3(\Delta_- + \Delta_+)}{\partial u^3}(u)}{\frac{\partial\Delta_-}{\partial u}(u)} + 3\frac{\frac{\partial^2\Delta_-}{\partial u^2}(u)\frac{\partial^2\Delta_+}{\partial u^2}(u) - \left(\frac{\partial^2\Delta_+}{\partial u^2}(u)\right)^2}{\left(\frac{\partial\Delta_-}{\partial u}(u)\right)^2} \tag{18}$$

at $(u,B) = (u(\lambda,\bar\epsilon), B(\lambda,\bar\epsilon))$. Using (17) and (18) we find the quantity a in (PD2):

$$a = \frac{3\left(\frac{\partial^2\Delta_-}{\partial u^2}(u)\right)^2 - 3\left(\frac{\partial^2\Delta_+}{\partial u^2}(u)\right)^2 - 2\frac{\partial\Delta_-}{\partial u}(u)\frac{\partial^3(\Delta_- + \Delta_+)}{\partial u^3}(u)}{6\left(\frac{\partial\Delta_-}{\partial u}(u)\right)^2} \tag{19}$$

where $(u,B) = (u(\lambda,\bar\epsilon), B(\lambda,\bar\epsilon))$.

Using $\Delta_+ \circ P = \Delta_-$ we find

$$\frac{\partial P}{\partial B}(u) = \frac{\frac{\partial\Delta_-}{\partial B}(u) - \frac{\partial\Delta_+}{\partial B}(P(u))}{\frac{\partial\Delta_+}{\partial u}(P(u))}, \tag{20}$$

and then, using (14) and (20), we get

$$\begin{aligned}\frac{\partial^2 P}{\partial u\partial B}(u) =& \frac{\frac{\partial^2\Delta_-}{\partial u\partial B}(u)}{\frac{\partial\Delta_+}{\partial u}(P(u))} - \frac{\frac{\partial\Delta_-}{\partial u}(u)\frac{\partial^2\Delta_+}{\partial u\partial B}(P(u))}{\left(\frac{\partial\Delta_+}{\partial u}(P(u))\right)^2} \\ &- \frac{\frac{\partial\Delta_-}{\partial u}(u)\frac{\partial^2\Delta_+}{\partial u^2}(P(u))\left(\frac{\partial\Delta_-}{\partial B}(u) - \frac{\partial\Delta_+}{\partial B}(P(u))\right)}{\left(\frac{\partial\Delta_+}{\partial u}(P(u))\right)^3}.\end{aligned} \tag{21}$$

Since $\frac{\partial\Delta_+}{\partial u}(P(u)) = -\frac{\partial\Delta_-}{\partial u}(u)$ and $P(u) = u$ at $(u,B) = (u(\lambda,\bar\epsilon), B(\lambda,\bar\epsilon))$, from (20) and (21) follows that

$$\frac{\partial P}{\partial B}(u) = -\frac{\frac{\partial(\Delta_- - \Delta_+)}{\partial B}(u)}{\frac{\partial\Delta_-}{\partial u}(u)}, \ (u,B) = (u(\lambda,\bar\epsilon), B(\lambda,\bar\epsilon)), \tag{22}$$

and

$$\frac{\partial^2 P}{\partial u\partial B}(u) = -\frac{\frac{\partial^2(\Delta_- + \Delta_+)}{\partial u\partial B}(u)}{\frac{\partial\Delta_-}{\partial u}(u)} + \frac{\frac{\partial^2\Delta_+}{\partial u^2}(u)\frac{\partial(\Delta_- - \Delta_+)}{\partial B}(u)}{\left(\frac{\partial\Delta_-}{\partial u}(u)\right)^2} \tag{23}$$

at $(u,B) = (u(\lambda,\bar\epsilon), B(\lambda,\bar\epsilon))$. Combining (17), (22) and (23) we get the quantity (PD1):

$$\frac{\frac{\partial(\Delta_- - \Delta_+)}{\partial B}(u)\frac{\partial^2(\Delta_- + \Delta_+)}{\partial u^2}(u) - 2\frac{\partial\Delta_-}{\partial u}(u)\frac{\partial^2(\Delta_- + \Delta_+)}{\partial u\partial B}(u)}{\left(\frac{\partial\Delta_-}{\partial u}(u)\right)^2}, \tag{24}$$

where $(u,B) = (u(\lambda,\bar\epsilon), B(\lambda,\bar\epsilon))$.

To further simplify the quantities given in (19) and (24) and to show that they are nonzero, we use the following simple but important lemma:

Lemma 3.4. *Let $m \in \mathbb{N}$, $m \geq 1$. Then we have:*

$$\bar{\epsilon}^{2m} \frac{\partial^{m+1}\Delta_\pm}{\partial u^{m+1}}(u) = \pm\left(\left(\frac{\partial I_\pm}{\partial u}(u)\right)^m + o(1)\right) \exp\left(\frac{\hat{I}_\pm(u,B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right),$$

where $\hat{I}_\pm(u,B,\lambda,\bar{\epsilon})$ are defined after (14), $I_\pm(u) = I_\pm(u,0,\lambda)$ and the $o(1)$-term is $\bar{\epsilon}$-regularly smooth in (u,B,λ) and tends uniformly to zero as $\bar{\epsilon} \to 0$.

Since $\hat{I}_-(u,B,\lambda,\bar{\epsilon}) = \hat{I}_+(u,B,\lambda,\bar{\epsilon})$ at $(u,B) = (u(\lambda,\bar{\epsilon}), B(\lambda,\bar{\epsilon}))$, Lemma 3.4 implies

$$\begin{cases} \bar{\epsilon}^{2m} \frac{\partial^{m+1}\Delta_-}{\partial u^{m+1}}(u) = \left(\left(\frac{\partial I_-}{\partial u}(u)\right)^m + o(1)\right) \frac{\partial \Delta_-}{\partial u}(u) \\ \bar{\epsilon}^{2m} \frac{\partial^{m+1}\Delta_+}{\partial u^{m+1}}(u) = -\left(\left(\frac{\partial I_+}{\partial u}(u)\right)^m + o(1)\right) \frac{\partial \Delta_-}{\partial u}(u), \end{cases} \tag{25}$$

at $(u,B) = (u(\lambda,\bar{\epsilon}), B(\lambda,\bar{\epsilon}))$. As a simple consequence of (25), (19) can be written as:

$$a = \frac{\left(\frac{\partial I_-}{\partial u}(u)\right)^2 - \left(\frac{\partial I_+}{\partial u}(u)\right)^2 + o(1)}{6\bar{\epsilon}^4}, \ (u,B) = (u(\lambda,\bar{\epsilon}), B(\lambda,\bar{\epsilon})).$$

Since $\frac{\partial I_\pm}{\partial u} > 0$, the quantity a is positive (resp. negative) if $\frac{\partial I_-}{\partial u}(u_0,0,0) - \frac{\partial I_+}{\partial u}(u_0,0,0) > 0$ (resp. $\frac{\partial I_-}{\partial u}(u_0,0,0) - \frac{\partial I_+}{\partial u}(u_0,0,0) < 0$), for each fixed $(\lambda,\bar{\epsilon}) \sim (0,0)$, $\bar{\epsilon} > 0$.

Similarly, using (25), the quantity (24) becomes:

$$\frac{\frac{\partial(\Delta_- - \Delta_+)}{\partial B}(u)\left(\frac{\partial I_-}{\partial u}(u) - \frac{\partial I_+}{\partial u}(u)\right) + o(1)}{\bar{\epsilon}^2 \frac{\partial \Delta_-}{\partial u}(u)}, \tag{26}$$

where $(u,B) = (u(\lambda,\bar{\epsilon}), B(\lambda,\bar{\epsilon}))$. Since $\frac{\partial(\Delta_- - \Delta_+)}{\partial B}$ is nonzero (see Theorem 3.1), (26) is nonzero, for each fixed $(\lambda,\bar{\epsilon}) \sim (0,0)$, $\bar{\epsilon} > 0$. Thus, putting all the informations together, we have proved that for each fixed $(\lambda,\bar{\epsilon}) \sim (0,0)$, $\bar{\epsilon} > 0$, the B-family $X_{\bar{\epsilon}^2,\bar{\epsilon}B,\lambda}$ undergoes a period doubling bifurcation at $(u,B) = (u(\lambda,\bar{\epsilon}), B(\lambda,\bar{\epsilon}))$. This implies that for each fixed $\epsilon > 0$, $\epsilon \sim 0$ and $\lambda \sim 0$ the b-family $X_{\epsilon,b,\lambda}$ undergoes a period doubling bifurcation at $(u,b) = (u(\lambda,\sqrt{\epsilon}), \sqrt{\epsilon}B(\lambda,\sqrt{\epsilon}))$. Since the functions $(\lambda,\bar{\epsilon}) \to u(\lambda,\bar{\epsilon}), B(\lambda,\bar{\epsilon})$ are $\bar{\epsilon}$-regularly smooth in λ, the functions $(\lambda,\epsilon) \to (u(\lambda,\sqrt{\epsilon}), \sqrt{\epsilon}B(\lambda,\sqrt{\epsilon})$ are ϵ-regularly smooth in λ.

3.5 Proof of Theorem 2.4

Let $\bar{u},\tilde{u} \in \Sigma_+$, with $\bar{u} < \tilde{u}$, be arbitrary but fixed, and let us suppose that the function $u \to I(u,0,0)$ has one zero (counting multiplicity) in $[\bar{u},\tilde{u}]$, and $\frac{\partial I}{\partial u}(u,0,0) \neq 0$ for all $u \in [\bar{u},\tilde{u}]$. *In this section we prove that for each fixed $(B,\lambda,\bar{\epsilon}) \sim (0,0,0)$, with $\bar{\epsilon} > 0$, the system (11) has at most 3 solutions (counted without their multiplicity) in $(u,u') \in [\bar{u},\tilde{u}]\times[\bar{u},\tilde{u}]$.* From this and the symmetry of (11), explained in Remark 2, follows that $X_{\epsilon,\mu}$ has at most one 2–limit cycle, intersecting the piece of the section Σ_+ parametrized by $u \in [\bar{u},\tilde{u}]$. This implies, using Lemma 2.1, that $\mathrm{Cycl}(X_{\epsilon,\mu}, \cup_{u\in[\bar{u},\tilde{u}]}L_u) \leq 2$. On the other hand, Theorem 2.3 implies $\mathrm{Cycl}(X_{\epsilon,\mu}, \cup_{u\in[\bar{u},\tilde{u}]}L_u) \geq 2$. Thus, $\mathrm{Cycl}(X_{\epsilon,\mu}, \cup_{u\in[\bar{u},\tilde{u}]}L_u) = 2$. As a special case we obtain Theorem 2.4 (under the conditions of Theorem 2.4, we choose $\bar{u}, \tilde{u} \sim u_0$ such that $\bar{u} < u_0 < \tilde{u}$).

Instead of studying the number of solutions of (13) when the slow divergence integral I has a simple zero, it is more convenient to study the number of solutions of system (11) by using a method introduced in [MR12]. The paper [MR12] is devoted to the study of the number of limit cycles bifurcating from canard cycles with two breaking parameters, and a system containing exponential functions, similar to (11), has been studied in [MR12].

The main difficulty lies in the fact that the limit $\bar{\epsilon} = 0$ of the system (11) is degenerate. Our goal is, therefore, to replace (11) with a new system, nonsingular for $\bar{\epsilon} = 0$, as explained in [MR12]. Let us first recall results from [MR12], which we will use in our proof.

- **Regular pair of foliations**. Let us suppose that $\Psi(u, u')$ and $\Phi(u, u')$ are two smooth functions defined on a rectangle $R = [\bar{U}_1, \tilde{U}_1] \times [\bar{U}_2, \tilde{U}_2]$ and let us suppose that $\frac{\partial \Psi}{\partial u}$, $\frac{\partial \Psi}{\partial u'}$, $\frac{\partial \Phi}{\partial u}$ and $\frac{\partial \Phi}{\partial u'}$ are nonzero for all $(u, u') \in R$. We further assume that the equation $\{\det(\Psi, \Phi)(u, u') = 0\}$ for contact points is equivalent on R to an equation $\{E(u, u') = 0\}$, where E is a smooth function on R, and where $\frac{\partial E}{\partial u}$ and $\frac{\partial E}{\partial u'}$ are nonzero for all $(u, u') \in R$. (Equivalent means $\det(\Psi, \Phi) = F.E$, where the factor F is a smooth nowhere zero function on R.) Now we can define a *regular pair of foliations* $(\widetilde{\Psi}, \widetilde{\Phi})$ on R as follows: the curves $\{\Psi(u, u') = \alpha\}$ (resp. $\{\Phi(u, u') = \beta\}$) are the leaves of foliation $\tilde{\Psi}$ (resp. $\tilde{\Phi}$). Each leaf and the curve $\{E(u, u') = 0\}$ are simple connected curves. For more details see [MR12].

  Let γ_1 and γ_2 be two smooth simple curves in $\mathbb{R}^2$ and let $q \in \gamma_1 \cap \gamma_2$. We say that γ_1 and γ_2 have *intersection multiplicity* $k \geq 1$ *at* q if and only if γ_1 and γ_2 are graphs of smooth functions $x \to y = f_1(x)$ and $x \to y = f_2(x)$ in a neighborhood of q (in local coordinates (x, y), q is given by $(x, y) = (0, 0)$) such that the function $x \to f_1(x) - f_2(x)$ has a zero of multiplicity $k \geq 1$ at $x = 0$. If $q \notin \gamma_1 \cap \gamma_2$, then we say that the intersection multiplicity at q is zero. See e.g. Definition 2 in [MR12].

  The following proposition relates the number of intersection points (counting multiplicity) of two leaves $\{\Psi(u, u') = \alpha\}$ and $\{\Phi(u, u') = \beta\}$ in R with the number of intersection points (counting multiplicity) of the curve $\{E(u, u') = 0\}$ and one of these two leaves in R (see Proposition 23 in [MR12]).

  Proposition 3.5. *Let $(\widetilde{\Psi}, \widetilde{\Phi})$ be a regular pair of foliations on R as defined above and let $\alpha, \beta \in \mathbb{R}$ be arbitrary but fixed. Let $\mathcal{N}(\alpha, \beta)$ be the number of intersection points of $\{\Psi(u, u') = \alpha\}$ with $\{\Phi(u, u') = \beta\}$ in R, counting multiplicity, and let $\mathcal{N}(\beta)$ be the number of intersection points of $\{E(u, u') = 0\}$ with $\{\Phi(u, u') = \beta\}$ in R, counting multiplicity. If $\mathcal{N}(\beta)$ is finite, then*

  $$\mathcal{N}(\alpha, \beta) \leq \mathcal{N}(\beta) + 1. \tag{27}$$

  The same result is true if we use the number of intersection points $\mathcal{N}(\alpha)$ of $\{E(u, u') = 0\}$ with $\{\Psi(u, u') = \alpha\}$.

To find at most 3 solutions of (11) in $[\bar{u}, \tilde{u}] \times [\bar{u}, \tilde{u}]$, for each $(B, \lambda, \bar{\epsilon}) \sim (0, 0, 0)$, with $\bar{\epsilon} > 0$, we use Proposition 3.5 twice. The system (11) is a special case of the more general system

$$\begin{cases} \exp\left(\frac{\bar{I}_-(u,B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right) + \exp\left(\frac{\bar{I}_+(u',B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right) = \alpha \\ \exp\left(\frac{\bar{I}_-(u',B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right) + \exp\left(\frac{\bar{I}_+(u,B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right) = \beta \end{cases} \tag{28}$$

where $\alpha, \beta \in \mathbb{R}$, and it suffices to prove that (28) has at most 3 solutions in $[\bar{u}, \tilde{u}] \times [\bar{u}, \tilde{u}]$, for each fixed $(B, \lambda, \bar{\epsilon}) \sim (0, 0, 0)$, with $\bar{\epsilon} > 0$, and $\alpha, \beta \in \mathbb{R}$.

We denote by $\Psi_{B,\lambda,\bar{\epsilon}}(u, u'), \Phi_{B,\lambda,\bar{\epsilon}}(u, u')$ the functions on the left-hand side of (28) and we prove that they define a regular pair of foliations on $[\bar{u}, \tilde{u}] \times [\bar{u}, \tilde{u}]$, for each fixed $(B, \lambda, \bar{\epsilon}) \sim (0, 0, 0)$, with $\bar{\epsilon} > 0$. Since $\frac{\partial I_\pm}{\partial u} > 0$, the first order partial derivatives of $\Psi_{B,\lambda,\bar{\epsilon}}$ and $\Phi_{B,\lambda,\bar{\epsilon}}$ w.r.t. u and u' are not zero for all $(u, u') \in [\bar{u}, \tilde{u}] \times [\bar{u}, \tilde{u}]$. Further we have:

$$\begin{aligned} \det(\Psi_{B,\lambda,\bar{\epsilon}}, \Phi_{B,\lambda,\bar{\epsilon}}) =& \frac{\partial \Psi_{B,\lambda,\bar{\epsilon}}}{\partial u} \frac{\partial \Phi_{B,\lambda,\bar{\epsilon}}}{\partial u'} - \frac{\partial \Psi_{B,\lambda,\bar{\epsilon}}}{\partial u'} \frac{\partial \Phi_{B,\lambda,\bar{\epsilon}}}{\partial u} \\ =& \exp\left(\frac{I_-(u) + o(1)}{\bar{\epsilon}^2}\right) \exp\left(\frac{I_-(u') + o(1)}{\bar{\epsilon}^2}\right) \\ &- \exp\left(\frac{I_+(u') + o(1)}{\bar{\epsilon}^2}\right) \exp\left(\frac{I_+(u) + o(1)}{\bar{\epsilon}^2}\right) \\ =& \exp\left(\frac{I_-(u) + I_-(u') + o(1)}{\bar{\epsilon}^2}\right) - \exp\left(\frac{I_+(u) + I_+(u') + o(1)}{\bar{\epsilon}^2}\right). \end{aligned}$$

This implies that the equation $\{\det(\Psi_{B,\lambda,\bar{\epsilon}}, \Phi_{B,\lambda,\bar{\epsilon}})(u, u') = 0\}$ of the contact points between the two foliations $\widetilde{\Psi}_{B,\lambda,\bar{\epsilon}}$ and $\widetilde{\Phi}_{B,\lambda,\bar{\epsilon}}$ is equivalent for $\bar{\epsilon} > 0$ to $\{E_{B,\lambda,\bar{\epsilon}}(u, u') = 0\}$ with

$$E_{B,\lambda,\bar{\epsilon}}(u, u') = I_-(u) - I_+(u') + I_-(u') - I_+(u) + o(1),$$

where the $o(1)$-term is $\bar{\epsilon}$-regularly smooth in (u, u', B, λ). Since

$$\frac{\partial (I_- - I_+)}{\partial u}(u, 0, 0) \neq 0$$

for all $u \in [\bar{u}, \tilde{u}]$, the first order partial derivatives of $E_{B,\lambda,\bar{\epsilon}}$ w.r.t. u and u' are not zero for all $(u, u') \in [\bar{u}, \tilde{u}] \times [\bar{u}, \tilde{u}]$. Thus, $(\widetilde{\Psi}_{B,\lambda,\bar{\epsilon}}, \widetilde{\Phi}_{B,\lambda,\bar{\epsilon}})$ is a regular pair of foliations.

We define the following system:

$$\begin{cases} E_{B,\lambda,\bar{\epsilon}}(u, u') = I_-(u) - I_+(u') + I_-(u') - I_+(u) + o(1) = 0 \\ \Phi_{B,\lambda,\bar{\epsilon}}(u, u') = \exp\left(\frac{\bar{I}_-(u',B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right) + \exp\left(\frac{\bar{I}_+(u,B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right) = \beta. \end{cases} \tag{29}$$

Following Proposition 3.5, if we denote by $\mathcal{N}_{B,\lambda,\bar{\epsilon}}(\alpha, \beta)$ (resp. $\mathcal{N}_{B,\lambda,\bar{\epsilon}}(\beta)$) the number of solutions of (28) (resp. (29)), counting multiplicity, in $[\bar{u}, \tilde{u}] \times [\bar{u}, \tilde{u}]$, then

$$\mathcal{N}_{B,\lambda,\bar{\epsilon}}(\alpha, \beta) \leq 1 + \mathcal{N}_{B,\lambda,\bar{\epsilon}}(\beta).$$

We have

$$\begin{aligned}\det(E_{B,\lambda,\bar\epsilon},\Phi_{B,\lambda,\bar\epsilon}) =& \frac{\partial E_{B,\lambda,\bar\epsilon}}{\partial u}\frac{\partial \Phi_{B,\lambda,\bar\epsilon}}{\partial u'} - \frac{\partial E_{B,\lambda,\bar\epsilon}}{\partial u'}\frac{\partial \Phi_{B,\lambda,\bar\epsilon}}{\partial u}\\ =& \left(\frac{\partial I_-}{\partial u}(u) - \frac{\partial I_+}{\partial u}(u) + o(1)\right)\exp\left(\frac{I_-(u')+o(1)}{\bar\epsilon^2}\right)\\ &-\left(\frac{\partial I_-}{\partial u}(u') - \frac{\partial I_+}{\partial u}(u') + o(1)\right)\exp\left(\frac{I_+(u)+o(1)}{\bar\epsilon^2}\right).\end{aligned}$$

Clearly, the equation $\{\det(E_{B,\lambda,\bar\epsilon},\Phi_{B,\lambda,\bar\epsilon})(u,u')=0\}$ is equivalent for $\bar\epsilon>0$ to $\{\bar E_{B,\lambda,\bar\epsilon}(u,u')=0\}$ where

$$\bar E_{B,\lambda,\bar\epsilon}(u,u') = I_-(u') - I_+(u) + o(1),$$

where the $o(1)$-term is $\bar\epsilon$-regularly smooth in (u,u',B,λ). (We used the fact that the derivative $\frac{\partial(I_- - I_+)}{\partial u}$ has a fixed sign on the segment $[\bar u,\tilde u]$.) Since $E_{B,\lambda,\bar\epsilon}$ and $\Phi_{B,\lambda,\bar\epsilon}$ define a regular pair of foliations on $[\bar u,\tilde u]\times[\bar u,\tilde u]$ ($\frac{\partial I_\pm}{\partial u}>0$), Proposition 3.5 implies that

$$\mathcal{N}_{B,\lambda,\bar\epsilon}(\beta) \le 1 + \mathcal{N}_{B,\lambda,\bar\epsilon},$$

where $\mathcal{N}_{B,\lambda,\bar\epsilon}$ is the number of solutions (counting multiplicity) of the system $\{I_-(u) - I_+(u') + I_-(u') - I_+(u) + o(1) = 0, I_-(u') - I_+(u) + o(1) = 0\}$ in $[\bar u,\tilde u]\times[\bar u,\tilde u]$, or equivalently the system

$$\begin{cases} I_-(u) - I_+(u') + o(1) = 0\\ I_-(u') - I_+(u) + o(1) = 0.\end{cases} \tag{30}$$

Thus, we have proved that

$$\mathcal{N}_{B,\lambda,\bar\epsilon}(\alpha,\beta) \le 2 + \mathcal{N}_{B,\lambda,\bar\epsilon},$$

for each $(B,\lambda,\bar\epsilon)\sim(0,0,0)$, with $\bar\epsilon>0$, and $\alpha,\beta\in\mathbb{R}$. Now, it suffices to show that $\mathcal{N}_{B,\lambda,\bar\epsilon}=1$. Denote by $u_0\in[\bar u,\tilde u]$ the simple zero of $I(u,0,0)$. Since $\frac{\partial I_\pm}{\partial u}(u_0,0,0)>0$ and $\frac{\partial I}{\partial u}(u_0,0,0)\neq 0$, the Implicit Function Theorem implies that the system (30) has one solution $(u,u')=(U(B,\lambda,\bar\epsilon),U'(B,\lambda,\bar\epsilon))$ (counting multiplicity) in a small $(B,\lambda,\bar\epsilon)$-uniform neighborhood of (u_0,u_0), where functions $U(B,\lambda,\bar\epsilon),U'(B,\lambda,\bar\epsilon)$ are continuous and $U(0,0,0)=U'(0,0,0)=u_0$. There are no other solutions of (30):

1. $I_-(u,0,0) - I_+(u,0,0)\neq 0$, for all $u\in[\bar u,\tilde u]$ and $u\neq u_0$ (besides u_0 there are no extra zeros of $I(u,0,0)$ in $[\bar u,\tilde u]$);
2. when $u<u'$ (resp. $u'<u$), we have $I_-(u,0,0)-I_+(u',0,0) < I_-(u',0,0)-I_+(u,0,0)$ (resp. $I_-(u',0,0)-I_+(u,0,0) < I_-(u,0,0)-I_+(u',0,0)$) because the functions $u\to I_\pm(u)$ are strictly increasing.

3.6 Proof of Theorem 2.5

Assume that $f(\omega(u_0),0)=0$, $\frac{\partial f}{\partial x}(\omega(u_0),0)\neq 0$ and $f(x,0)<0$ for all $x\in[\alpha(u_0),\omega(u_0)[$. First, let us prove that $\mathrm{Cycl}(X_{\epsilon,\mu},L_{u_0})\le 1$, i.e. there are no 2-periodic orbits Hausdorff close to L_{u_0}. Suppose, on the contrary, that for $(B,\lambda,\bar\epsilon)\sim(0,0,0)$, $\bar\epsilon>0$, $X_{\bar\epsilon^2,\bar\epsilon B,\lambda}$ has a 2–periodic orbit intersecting Σ_+ in two points $\bar{u} \sim u_0$ and $\tilde{u} \sim u_0$, with $u(B,\lambda,\bar{\epsilon}) < \bar{u} < \tilde{u}$, where $u(B,\lambda,\bar{\epsilon})$ is defined in Section 3.1.2. Then Rolle's theorem implies the existence of $u' \in]\bar{u},\tilde{u}[$ such that $\Delta'_{B,\lambda,\bar{\epsilon}}(u') = 0$ where $\Delta_{B,\lambda,\bar{\epsilon}}$ is defined in Remark 2 (see also Section 3.3). On the other hand, we have for $u \in [\bar{u},\tilde{u}]$:

$$\begin{aligned}\Delta'_{B,\lambda,\bar{\epsilon}}(u) =& \frac{\frac{\partial \Delta_-}{\partial u}(u)}{\frac{\partial \Delta_+}{\partial u}(P_{B,\lambda,\bar{\epsilon}}(u))} - \frac{\frac{\partial \Delta_+}{\partial u}(u)}{\frac{\partial \Delta_-}{\partial u}(P^{-1}_{B,\lambda,\bar{\epsilon}}(u))} \\ =& - \exp\left(\frac{\mathcal{I}_-(u,B,\lambda,\bar{\epsilon}) - I_+(P_{B,\lambda,\bar{\epsilon}}(u)) + o(1)}{\bar{\epsilon}^2}\right) \\ &+ \exp\left(\frac{I_+(u) - \mathcal{I}_-(P^{-1}_{B,\lambda,\bar{\epsilon}}(u),B,\lambda,\bar{\epsilon}) + o(1)}{\bar{\epsilon}^2}\right),\end{aligned}$$

where $\frac{\partial \Delta_-}{\partial u}$ is given in Theorem 3.2 and we get $\frac{\partial \Delta_+}{\partial u}$ from (8) (note that the slow dynamics is regular along the repelling part m^+ of the critical curve). Since $u, P_{B,\lambda,\bar{\epsilon}}(u), P^{-1}_{B,\lambda,\bar{\epsilon}}(u) \sim u_0$ for $u \in [\bar{u},\tilde{u}]$, I_+ and $o(1)$ are bounded functions and $\mathcal{I}_-(u,B,\lambda,\bar{\epsilon}) \to -\infty$ as $(u,B,\lambda,\bar{\epsilon}) \to (u_0,0,0,0)$, the exponents of the above exponential functions have opposite signs. Thus, $\Delta'_{B,\lambda,\bar{\epsilon}}$ is nonzero for all $u \in [\bar{u},\tilde{u}]$. This is a contradiction with $\Delta'_{B,\lambda,\bar{\epsilon}}(u') = 0$.

Since $\mathcal{I}_-(u,B,\lambda,\bar{\epsilon}) - I_+(u)$ is negative, any 1-limit cycle bifurcating from L_{u_0} is hyperbolically attracting (see Section 3.3). We have $\text{Cycl}(X_{\epsilon,\mu}, L_{u_0}) \geq 1$ due to the presence of the breaking parameter B (for more details see Section 3.8.5 of [HDMD13]).

We use a similar proof in the case $f(\alpha(u_0),0) = 0$, $\frac{\partial f}{\partial x}(\alpha(u_0),0) \neq 0$ and $f(x,0) < 0$ for all $x \in]\alpha(u_0),\omega(u_0)]$.

3.7 Proof of Theorem 2.6

Let $u_0, u_1 \in \Sigma_+$, with $u_0 < u_1$, be arbitrary but fixed, and let us suppose that $T(u_0,u_1,0) \neq 0$, where T is the total slow divergence integral defined in (5). Suppose, on the contrary, that for $(B,\lambda,\bar{\epsilon}) \sim (0,0,0)$, $\bar{\epsilon} > 0$, $X_{\bar{\epsilon}^2,\bar{\epsilon}B,\lambda}$ has two 2–periodic orbits, one intersecting Σ_+ in two points $\bar{u} \sim u_0$ and $\tilde{u} \sim u_1$, and the other in $\bar{\bar{u}} \sim u_0$ and $\tilde{\tilde{u}} \sim u_1$. Then we have $\bar{u} < \bar{\bar{u}} < \tilde{\tilde{u}} < \tilde{u}$ or $\bar{\bar{u}} < \bar{u} < \tilde{u} < \tilde{\tilde{u}}$. Suppose without loss of generality that $\bar{u} < \bar{\bar{u}} < \tilde{\tilde{u}} < \tilde{u}$. Then $\Delta_{B,\lambda,\bar{\epsilon}}(\bar{u}) = \Delta_{B,\lambda,\bar{\epsilon}}(\bar{\bar{u}}) = 0$, $P_{B,\lambda,\bar{\epsilon}}(\bar{u}) = \tilde{u}$, $P_{B,\lambda,\bar{\epsilon}}(\bar{\bar{u}}) = \tilde{\tilde{u}}$, $P_{B,\lambda,\bar{\epsilon}}([\bar{u},\bar{\bar{u}}]) = [\tilde{\tilde{u}},\tilde{u}]$ and $P^{-1}_{B,\lambda,\bar{\epsilon}}([\bar{u},\bar{\bar{u}}]) = [\tilde{\tilde{u}},\tilde{u}]$. Using Rolle's theorem we find $u' \in]\bar{u},\bar{\bar{u}}[$ such that $\Delta'_{B,\lambda,\bar{\epsilon}}(u') = 0$. On the other hand we get (see Section 3.3):

$$\begin{aligned}\Delta'_{B,\lambda,\bar{\epsilon}}(u) =& - \exp\left(\frac{I_-(u) - I_+(P_{B,\lambda,\bar{\epsilon}}(u)) + o(1)}{\bar{\epsilon}^2}\right) \\ &+ \exp\left(\frac{I_+(u) - I_-(P^{-1}_{B,\lambda,\bar{\epsilon}}(u)) + o(1)}{\bar{\epsilon}^2}\right),\end{aligned}$$

where $u \in [\bar{u},\bar{\bar{u}}]$. The equation $\{\Delta'_{B,\lambda,\bar{\epsilon}}(u) = 0\}$ is equivalent for $\bar{\epsilon} > 0$ and $u \in [\bar{u},\bar{\bar{u}}]$ to an equation given in (13). Since $T(u_0,u_1,0) \neq 0$, $u \sim u_0$, $P_{B,\lambda,\bar{\epsilon}}(u), P^{-1}_{B,\lambda,\bar{\epsilon}}(u) \sim u_1$ for all $u \in [\bar{u},\bar{\bar{u}}]$, (13) has no solutions w.r.t. $u \in [\bar{u},\bar{\bar{u}}]$. This is a contradiction with $\Delta'_{B,\lambda,\bar{\epsilon}}(u') = 0$. Thus, $\text{Cycl}(X_{\epsilon,\mu}, L_{u_0,u_1}) \leq 1$.

Using the above expression for $\Delta'_{B,\lambda,\bar{\epsilon}}$ it can be easily seen that any 2–limit cycle bifurcating from L_{u_0,u_1} is hyperbolic and attracting (resp. repelling) when $T(u_0,u_1,0) < 0$ (resp. $T(u_0,u_1,0) > 0$).

3.8 Proof of Theorem 2.7

Let $u_0, u_1 \in \Sigma_+$, with $u_0 < u_1$, be arbitrary but fixed.

Proof of Theorem 2.7.1. Suppose that $(u, u') \sim (u_0, u_1)$ is a solution of the system (11) for $(B, \lambda, \bar{\epsilon}) \sim (0, 0, 0)$ and $\bar{\epsilon} > 0$. Then (11) implies

$$\exp\left(\frac{\bar{I}_+(u', B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2}\right) - \exp\left(\frac{\bar{I}_+(u, B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2}\right) = \exp\left(\frac{\bar{I}_-(u', B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2}\right) - \exp\left(\frac{\bar{I}_-(u, B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2}\right). \quad (31)$$

The expression (31) can be written as:

$$\exp\left(\frac{I_+(u') + o(1)}{\bar{\epsilon}^2}\right)\left(1 - \exp\left(\frac{I_+(u) - I_+(u') + o(1)}{\bar{\epsilon}^2}\right)\right) = \exp\left(\frac{I_-(u') + o(1)}{\bar{\epsilon}^2}\right)\left(1 - \exp\left(\frac{I_-(u) - I_-(u') + o(1)}{\bar{\epsilon}^2}\right)\right), \quad (32)$$

where we write $\bar{I}_\pm(., B, \lambda, \bar{\epsilon}) = I_\pm(.) + o(1)$ (see Remark 3). Since $u < u'$ and the functions $u \to I_\pm(u)$ are strictly increasing, we have $I_\pm(u) - I_\pm(u') < 0$ and we can write (32) as

$$\exp\left(\frac{I_+(u') + o(1)}{\bar{\epsilon}^2}\right)\Big(1 + o(1)\Big) = \exp\left(\frac{I_-(u') + o(1)}{\bar{\epsilon}^2}\right)\Big(1 + o(1)\Big),$$

or equivalently, as

$$\exp\left(\frac{I_+(u') + o(1)}{\bar{\epsilon}^2}\right) = \exp\left(\frac{I_-(u') + o(1)}{\bar{\epsilon}^2}\right) \quad (33)$$

for new $o(1)$-functions. The equation (33) is equivalent for $\bar{\epsilon} > 0$ to

$$I_-(u') - I_+(u') + o(1) = 0. \quad (34)$$

(All the $o(1)$-terms in the above expressions are $\bar{\epsilon}$-regularly smooth in (u, u', B, λ) and tend uniformly to zero as $\bar{\epsilon} \to 0$.) Thus, each solution $(u, u') \sim (u_0, u_1)$ of (11) satisfies (34). From this, the assumption $I_-(u_1, 0) - I_+(u_1, 0) \neq 0$ and $u' \sim u_1$ follows that (11) has no solutions $(u, u') \sim (u_0, u_1)$ when $(B, \lambda, \bar{\epsilon}) \sim (0, 0, 0)$ and $\bar{\epsilon} > 0$. Thus, there are no 2-periodic orbits Hausdorff close to L_{u_0,u_1}.

Proof of Theorem 2.7.2. Assume that $I_-(u_1, 0) - I_+(u_1, 0) = 0$ and $I_-(u_0, 0) - I_+(u_0, 0) \neq 0$. Then $T(u_0, u_1, 0) = I_-(u_0, 0) - I_+(u_0, 0) \neq 0$ and from Theorem 2.6 it follows the first part of Theorem 2.7.2.

Moreover, suppose that $\frac{\partial(I_- - I_+)}{\partial u}(u_1, 0) \neq 0$. Our goal is to show that $\mathrm{Cycl}(X_{\epsilon,\mu}, L_{u_0,u_1}) = 1$. Using the Implicit Function Theorem, we find a function $B = B_0(\lambda, \bar{\epsilon})$, $\bar{\epsilon}$-regularly smooth in λ, such that $B_0(\lambda, 0) = 0$ and

$$f(B_0(\lambda, \bar{\epsilon}), \lambda, \bar{\epsilon}) = 0,$$

where f is introduced in Theorem 3.1. This implies that

$$f(B, \lambda, \bar{\epsilon}) = l(B, \lambda, \bar{\epsilon}).(B - B_0(\lambda, \bar{\epsilon}))$$

where $l(0,0,0) \neq 0$ (without loss of generality we can take $l(0,0,0) > 0$). If we define an adapted breaking parameter $\bar{B} = B - B_0(\lambda, \bar{\epsilon}) \sim 0$ (see e.g. [Dum11]), then the system (11) can be written as:

$$\begin{cases} \exp\left(\frac{\bar{I}_-(u,\bar{B},\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right) + \exp\left(\frac{\bar{I}_+(u',\bar{B},\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right) = \bar{B} \\ \exp\left(\frac{\bar{I}_-(u',\bar{B},\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right) + \exp\left(\frac{\bar{I}_+(u,\bar{B},\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right) = \bar{B}, \end{cases} \tag{35}$$

where the new functions $\bar{I}_\pm$ are $\bar{\epsilon}$-regularly smooth in $(u, \bar{B}, \lambda)$ or in $(u', \bar{B}, \lambda)$ and $\bar{I}_\pm(\cdot, \bar{B}, \lambda, 0) = I_\pm(\cdot)$. Since the partial derivative of the functions on the left-hand side of (35) w.r.t. $\bar{B}$ is flat in $\bar{\epsilon}$, using the Implicit Function Theorem in each equation of (35) we can change (35) to

$$\begin{cases} \exp\left(\frac{\bar{I}_-(u,u',\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right) + \exp\left(\frac{\bar{I}_+(u',u,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right) = \bar{B} \\ \exp\left(\frac{\bar{I}_-(u',u,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right) + \exp\left(\frac{\bar{I}_+(u,u',\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right) = \bar{B}, \end{cases} \tag{36}$$

where the functions $\bar{I}_\pm$ are $\bar{\epsilon}$-regularly smooth in (u, u', λ) and $\bar{I}_\pm(u, u', \lambda, 0) = I_\pm(u)$ and $\bar{I}_\pm(u', u, \lambda, 0) = I_\pm(u')$. The advantage of working with system (36) is that the breaking parameter $\bar{B}$ does not appear in the left-hand terms of (36). We can write (36) as

$$\Theta_{\lambda,\bar{\epsilon}}(u, u') = (\Theta^1_{\lambda,\bar{\epsilon}}(u, u'), \Theta^1_{\lambda,\bar{\epsilon}}(u', u)) = (\bar{B}, \bar{B}),\ (u, u') \sim (u_0, u_1).$$

Like in the proof of Theorem 2.7.1, we can show that the equation $\{\Theta^1_{\lambda,\bar{\epsilon}}(u, u') = \Theta^1_{\lambda,\bar{\epsilon}}(u', u)\}$, with $(u, u') \sim (u_0, u_1)$, is equivalent for $\bar{\epsilon} > 0$ to

$$I_-(u') - I_+(u') + o(1) = 0,\ (u, u') \sim (u_0, u_1), \tag{37}$$

where the term $o(1)$ is $\bar{\epsilon}$-regularly smooth in (u, u', λ) and tends uniformly to zero as $\bar{\epsilon} \to 0$. Since $I_-(u_1, 0) - I_+(u_1, 0) = 0$ and $\frac{\partial(I_- - I_+)}{\partial u}(u_1, 0) \neq 0$, the Implicit Function Theorem implies existence of a function $u' = \mathcal{U}(u, \lambda, \bar{\epsilon})$, $\bar{\epsilon}$-regularly smooth in (u, λ), such that $\mathcal{U}(u_0, 0, 0) = u_1$ and such that $u' = \mathcal{U}(u, \lambda, \bar{\epsilon})$ is the solution of (37). Thus, we have

$$\Theta^1_{\lambda,\bar{\epsilon}}(u, \mathcal{U}(u, \lambda, \bar{\epsilon})) = \Theta^1_{\lambda,\bar{\epsilon}}(\mathcal{U}(u, \lambda, \bar{\epsilon}), u),\ \text{for all } (u, \lambda, \bar{\epsilon}) \sim (u_0, 0, 0),\ \bar{\epsilon} > 0.$$

This implies that $X_{\bar{\epsilon}^2,\bar{\epsilon}B,\lambda}$ has a 2-periodic orbit intersecting Σ_+ in the points $(u, \mathcal{U}(u, \lambda, \bar{\epsilon})) \sim (u_0, u_1)$, for each fixed $(u, \lambda, \bar{\epsilon}) \sim (u_0, 0, 0)$, $\bar{\epsilon} > 0$, and for $\bar{B} = \Theta^1_{\lambda,\bar{\epsilon}}(u, \mathcal{U}(u, \lambda, \bar{\epsilon})) \sim 0$ $(B = \bar{B} + B_0(\lambda, \bar{\epsilon}))$. Thus, $\text{Cycl}(X_{\epsilon,\mu}, L_{u_0,u_1}) = 1$.

Proof of Theorem 2.7.3. Assume that $I_-(u_i, 0) - I_+(u_i, 0) = 0$ and $\frac{\partial(I_- - I_+)}{\partial u}(u_i, 0) \neq 0$ for $i = 0, 1$. To show that $\text{Cycl}(X_{\epsilon,\mu}, L_{u_0,u_1}) \leq 2$, we use the idea of Khovanskii explained in Section 3.5. First, we choose a small rectangle $R = [u_0^1, u_0^2] \times [u_1^1, u_1^2]$ such that $u_i^1 < u_i < u_i^2$ and $\frac{\partial(I_- - I_+)}{\partial u}(u, 0) \neq 0$ for all $u \in [u_i^1, u_i^2]$, $i = 0, 1$. Now we follow the same steps as in Section 3.5, working in $(u, u') \in R$. The above condition on the derivative guarantees that in each step we deal with a regular pair of foliations. We show that (28) has at most two solutions in R, for each fixed $(B, \lambda, \bar{\epsilon}) \sim (0, 0, 0)$, with $\bar{\epsilon} > 0$, and $(\alpha, \beta) \in \mathbb{R}$.

More precisely, suppose first that $\frac{\partial(I_- - I_+)}{\partial u}(u_0, 0)$ and $\frac{\partial(I_- - I_+)}{\partial u}(u_1, 0)$ have the same sign. Following Section 3.5, it suffices to prove that the system (30) has no solutions in R, under the same conditions on the parameters. Since $I_-(u_0, 0) - I_+(u_1, 0) < I_-(u_1, 0) - I_+(u_0, 0)$ ($u_0 < u_1$ and $u \to I_\pm(u)$ are strictly increasing), (30) has no solutions in R, up to shrinking R if necessary, uniformly in $(B, \lambda, \bar\epsilon) \sim (0, 0, 0)$. Thus, $\text{Cycl}(X_{\epsilon,\mu}, L_{u_0,u_1}) \leq 2$.

Suppose now that $\frac{\partial(I_- - I_+)}{\partial u}(u_0, 0)$ and $\frac{\partial(I_- - I_+)}{\partial u}(u_1, 0)$ have the opposite sign. Then the equation $\{\det(E_{B,\lambda,\bar\epsilon}, \Phi_{B,\lambda,\bar\epsilon})(u, u') = 0\}$ in Section 3.5 can be written in R as

$$\exp\left(\frac{I_-(u') + o(1)}{\bar\epsilon^2}\right) + \exp\left(\frac{I_+(u) + o(1)}{\bar\epsilon^2}\right) = 0,$$

where the $o(1)$-terms are $\bar\epsilon$-regularly smooth in (u, u', B, λ) and tend uniformly to zero as $\bar\epsilon \to 0$. Since the left-hand side of the above equation is strictly positive for all $(u, u') \in R$, $(B, \lambda, \bar\epsilon) \sim (0, 0, 0)$ and $\bar\epsilon > 0$, the equation is equivalent on R to

$$\bar E_{B,\lambda,\bar\epsilon}(u, u') := (u - u_0) + (u' - u_1) + 1 = 0,$$

where $\bar E_{B,\lambda,\bar\epsilon}$ is strictly positive on R, up to shrinking R if necessary. (The first order partial derivatives of $\bar E_{B,\lambda,\bar\epsilon}$ w.r.t. u and u' are not zero for all $(u, u') \in R$.) This implies that the equation $\{\bar E_{B,\lambda,\bar\epsilon}(u, u') = 0\}$ has no solutions in R, i.e. there are no intersection points of $\{\bar E_{B,\lambda,\bar\epsilon}(u, u') = 0\}$ with $\{E_{B,\lambda,\bar\epsilon}(u, u') = 0\}$. Thus, $\text{Cycl}(X_{\epsilon,\mu}, L_{u_0,u_1}) \leq 2$.

3.9 Proof of Theorem 2.8

Assume that $f(\omega(u_0), 0) = 0$, $\frac{\partial f}{\partial x}(\omega(u_0), 0) \neq 0$ and that $f(x, 0) < 0$ for all $x \in [\alpha(u_0), \omega(u_0)[$. Like in Section 3.7, we suppose that for $(B, \lambda, \bar\epsilon) \sim (0, 0, 0)$, $\bar\epsilon > 0$, $X_{\bar\epsilon^2, \bar\epsilon B, \lambda}$ has two 2–periodic orbits: $(\bar u, \tilde u) \sim (u_0, u_1)$ and $(\bar{\bar u}, \tilde{\tilde u}) \sim (u_0, u_1)$. We have $u(B, \lambda, \bar\epsilon) < \bar u < \bar{\bar u} < \tilde{\tilde u} < \tilde u$ or $u(B, \lambda, \bar\epsilon) < \bar{\bar u} < \bar u < \tilde u < \tilde{\tilde u}$ where the function $u(B, \lambda, \bar\epsilon) \sim u_0$ is defined in Section 3.1.2. Assume that $\bar u < \bar{\bar u} < \tilde{\tilde u} < \tilde u$. Then $\Delta_{B,\lambda,\bar\epsilon}$ and $P_{B,\lambda,\bar\epsilon}$ have the properties on $[\bar u, \bar{\bar u}]$ given in Section 3.7. On the other hand, we have for all $u \in [\bar u, \bar{\bar u}]$:

$$\begin{aligned}\Delta'_{B,\lambda,\bar\epsilon}(u) &= \frac{\frac{\partial \Delta_-}{\partial u}(u)}{\frac{\partial \Delta_+}{\partial u}(P_{B,\lambda,\bar\epsilon}(u))} - \frac{\frac{\partial \Delta_+}{\partial u}(u)}{\frac{\partial \Delta_-}{\partial u}(P^{-1}_{B,\lambda,\bar\epsilon}(u))} \\ &= -\exp\left(\frac{\mathcal{I}_-(u, B, \lambda, \bar\epsilon) - I_+(P_{B,\lambda,\bar\epsilon}(u)) + o(1)}{\bar\epsilon^2}\right) \\ &\quad + \exp\left(\frac{I_+(u) - I_-(P^{-1}_{B,\lambda,\bar\epsilon}(u)) + o(1)}{\bar\epsilon^2}\right),\end{aligned}$$

where $\frac{\partial \Delta_-}{\partial u}(u)$ is given in Theorem 3.2 and functions $\frac{\partial \Delta_+}{\partial u}(u)$, $\frac{\partial \Delta_+}{\partial u}(P_{B,\lambda,\bar\epsilon}(u))$ and $\frac{\partial \Delta_-}{\partial u}(P^{-1}_{B,\lambda,\bar\epsilon}(u))$ are obtained from (8) (the slow dynamics is regular along the segment $[\alpha(u_0), \omega(u_1)]$). Since $\mathcal{I}_-(u, B, \lambda, \bar\epsilon) \to -\infty$ as $(u, B, \lambda, \bar\epsilon) \to (u_0, 0, 0, 0)$ (see Theorem 3.2) and $I_\pm$ are bounded, we have

$$\mathcal{I}_-(u, B, \lambda, \bar\epsilon) - I_+(P_{B,\lambda,\bar\epsilon}(u)) < I_+(u) - I_-(P^{-1}_{B,\lambda,\bar\epsilon}(u))$$

for $u \in [\bar u, \bar{\bar u}]$. Thus, $\Delta'_{B,\lambda,\bar\epsilon}(u)$ is nonzero for all $u \in [\bar u, \bar{\bar u}]$ ($\bar\epsilon > 0$). This is a contradiction with Rolle's theorem. This implies that $\text{Cycl}(X_{\epsilon,\mu}, L_{u_0,u_1}) \leq 1$.

Using the expression for $\Delta'_{B,\lambda,\bar{\epsilon}}$ and the above inequality it can be easily seen that any 2–limit cycle is hyperbolic and attracting.

We use a similar proof in the case $f(\alpha(u_0), 0) = 0, \frac{\partial f}{\partial x}(\alpha(u_0), 0) \neq 0$ and $f(x, 0) < 0$ for all $x \in]\alpha(u_0), \omega(u_0)]$.

4 Discussion

Theorems 2.2–2.4 and Theorems 2.6-2.7 can also be proved in a more general framework when we have a non-generic turning point in smooth vector fields, defined on a two-dimensional smooth orientable submanifold $\widetilde{M}$ of M, which satisfy Assumptions T0-T6 of [DMD05]. A typical example of non-generic turning points is $\{\dot{x} = y, \dot{y} = -x^{2n-1}y + \epsilon(b - x^{2n-1} + O(x^{2n}))\}$, where $n \geq 1$, $\epsilon \geq 0$ is a singular perturbation parameter and $b \sim 0$ is a breaking parameter. When $n = 1$, we deal with a generic turning point (see (2)). The slow dynamics along the slow curve $\{y = 0\}$ is given by $x' = -1 + O(x)$. As in Section 3.1, we can define a new (regular) breaking parameter $B \sim 0$ using the rescaling $(\epsilon, b) = (\bar{\epsilon}^{2n}, \bar{\epsilon}^{2n-1}B)$, and then blow up the origin in the $(x, y, \bar{\epsilon})$ space: $(x, y, \bar{\epsilon}) = (\rho\tilde{x}, \rho^{2n}\tilde{y}, \rho\tilde{\epsilon})$, where $(\tilde{x}, \tilde{y}, \tilde{\epsilon}) \in \mathbb{S}^2$, $\tilde{\epsilon} \geq 0$, $\rho \geq 0$ and $\rho \sim 0$. When $B = 0$, one finds a heteroclinic orbit on the blow-up locus, connecting the attracting part and the repelling part of the slow curve. This connection breaks for $B \neq 0$ in a regular way (see [DMD05]). When the slow dynamics is regular along the slow curve, i.e. nowhere zero (this is included in Assumptions T0–T6), then the slow divergence integral along the slow curve is well defined and we can prove a result for the transition maps similar to Theorem 3.1. We point out that Theorem 3.1 plays a crucial role in our paper and it follows from Theorem 4 in [DMD05] which has been proved in the above general framework.

A result similar to Theorem 3.1 can also be proved if we replace the Hopf breaking mechanism at $p \in M$ with a jump breaking mechanism defined in e.g. [Dum11]. See [Dum11] for more details.

If we deal with a non-generic turning point or a jump mechanism and if we allow a hyperbolic saddle in the slow dynamics, away from the turning point, then we can use a well known framework presented in [DMD08] and we can also prove Theorem 2.5 and Theorem 2.8 in this more general framework. Let us recall that [DMD08] is a natural continuation of [DMD05] where the slow dynamics has isolated singularities, away from the contact point. Theorem 3.2, used in the proof of Theorem 2.5 and Theorem 2.8, follows directly from [DMD08] and a similar result can be proved if we deal with the non-generic turning point. See [DMD08] for more details.

In Theorem 2.5 and Theorem 2.8, we suppose that the slow dynamics has a hyperbolic saddle at precisely one corner point. The case when the slow dynamics has hyperbolic saddles at both corner points, or more generally a finite number of singularities, away from the contact point, is more difficult and it is a topic of further study. The techniques developed in [DMD08] can be used.

References

[DMD05] P. De Maesschalck and F. Dumortier. Time analysis and entry-exit relation near planar turning points. *J. Differential Equations*, 215(2):225–267, 2005.

[DMD08] P. De Maesschalck and F. Dumortier. Canard cycles in the presence of slow dynamics with singularities. *Proc. Roy. Soc. Edinburgh Sect. A*, 138(2):265–299, 2008.

[DMDR11] P. De Maesschalck, F. Dumortier, and R. Roussarie. Cyclicity of common slow-fast cycles. *Indag. Math. (N.S.)*, 22(3-4):165–206, 2011.

[DR96] F. Dumortier and R. Roussarie. Canard cycles and center manifolds. *Mem. Amer. Math. Soc.*, 121(577):x+100, 1996. With an appendix by Cheng Zhi Li.

[Dum11] F. Dumortier. Slow divergence integral and balanced canard solutions. *Qual. Theory Dyn. Syst.*, 10(1):65–85, 2011.

[GH83] J. Guckenheimer and P. Holmes. *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, volume 42 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[Gui99] L.-S. Guimond. Homoclinic loop bifurcations on a Möbius band. *Nonlinearity*, 12(1):59–78, 1999.

[HDMD13] R. Huzak, P. De Maesschalck, and F. Dumortier. Limit cycles in slow-fast codimension 3 saddle and elliptic bifurcations. *J. Differential Equations*, 255(11):4012–4051, 2013.

[Kho91] A. G. Khovanskiĭ. *Fewnomials*, volume 88 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1991. Translated from the Russian by Smilka Zdravkovska.

[KS01] M. Krupa and P. Szmolyan. Relaxation oscillation and canard explosion. *J. Differential Equations*, 174(2):312–368, 2001.

[MR12] L. Mamouhdi and R. Roussarie. Canard cycles of finite codimension with two breaking parameters. *Qual. Theory Dyn. Syst.*, 11(1):167–198, 2012.