A New Wider Family of Continuous Models: The Extended Cordeiro and de Castro Family

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Abstract

We introduce and study general mathematical properties of a new generator of continuous distributions with three extra parameters called the *extended Cordeiro and de Castro* family. We investigate the asymptotes and shapes. The new density function can be expressed as a linear combination of exponentiated densities based on the same underlying distribution. We derive a power series for the quantile function of this family. We determine explicit expressions for the ordinary and incomplete moments, quantile and generating functions, asymptotic distribution of the extreme values, Shannon and Rényi entropies and order statistics, which hold for any baseline model. We discuss the estimation of the model parameters by maximum likelihood and illustrate the potentiality of the introduced family by means of two applications to real data.

Keywords: Generalized exponential geometric distribution, Generated family, Maximum likelihood, Quantile function, Rényi entropy.

2000 AMS Classification: AMS

1. Introduction

The use of new generators of continuous distributions from classic ones has become very common in recent years. Several attempts have been made to define new families of distributions that extend well-known distributions and at the same time provide great flexibility in modelling data in practice. One example is the *beta-generated family* of distributions pioneered by [9]. A second example is the *gamma-generated family* of distributions defined by [28]. A third example is the *Kumaraswamy family* of distributions proposed by [4]. Recently, [1] proposed a general method to generate new families of distributions.

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Based on the transformer (T-X) generator [1], and using the generalized exponential-geometric (GEG) distribution [25], we propose a new wider family of distributions given by

$$(1.1)\quad F(x) = 1 - \int_0^{-\log[G(x;\boldsymbol{\xi})]} \frac{\alpha\lambda(1-p)\mathrm{e}^{-\lambda t}[1-\mathrm{e}^{-\lambda t}]^{\alpha-1}}{[1-p\,\mathrm{e}^{-\lambda t}]^{\alpha+1}}\mathrm{d}t = 1 - \left[\frac{1-G(x;\boldsymbol{\xi})^{\lambda}}{1-p\,G(x;\boldsymbol{\xi})^{\lambda}}\right]^{\alpha},$$

where $G(x;\boldsymbol{\xi})$ is the underlying cumulative distribution function (cdf) depending on a parameter vector $\boldsymbol{\xi}$ and $\alpha > 0$, $\lambda > 0$ and $p \in (0,1)$ are three additional shape parameters. For each underlying G, the *extended Cordeiro and de Castro-G* ("ECC-G" for short) family of distributions is defined by the cdf (1.1). Equation (1.1) is a wider family of continuous distributions. It includes the *generalized Kumaraswamy class* [4] of distributions, proportional and reversed hazard rate models, Marshal-Olkin family and other sub-families. Some special models are listed in Table 1.

Table 1. Some special models.

λ	α	p	$G(x)$	Reduced distribution
-	-	0	-	Generalized Kumaraswamy distribution [4]
1	1	0	-	$G(x)$
-	1	0	-	Reversed hazard rate model [12]
1	-	0	-	Proportional hazard rate model [12]
-	1	p	-	Marshall-Olkin family of distributions [15]
-	-	0	Generalized Rayleigh	Kumaraswamy generalized Rayleigh distribution [10]
-	-	0	Burr XII distribution	Kumaraswamy Burr XII distribution [20]
-	-	0	Modified Weibull distribution	Kumaraswamy modified Weibull distribution [6]
-	-	0	Pareto distribution	Kumaraswamy Pareto distribution [2]

This paper is organized as follows. In Section 2, we provide a physical interpretation of the ECC-G family. Four special cases of this family are defined in Section 3. Some useful expansions are derived in Section 4. In Section 5, we propose explicit expressions for the moments and generating function using a power series for the quantile function (qf). Further, we present general expressions for the Rényi and Shannon entropies and mean deviations are addressed. Estimation of the model parameters by maximum likelihood is performed in Section 6. Applications to two real data sets illustrate the performance of the new family in Section 7. The paper is concluded in Section 8.

2. The new family

The corresponding probability density function (pdf) to (1.1) is given by

$$(2.1)\qquad f(x;\alpha,\lambda,p,\boldsymbol{\xi}) = \alpha\,\lambda\,(1-p)\,g(x;\boldsymbol{\xi})\,G(x;\boldsymbol{\xi})^{\lambda-1}\,\frac{[1-G(x;\boldsymbol{\xi})^{\lambda}]^{\alpha-1}}{[1-p\,G(x;\boldsymbol{\xi})^{\lambda}]^{\alpha+1}},$$

where $g(x;\boldsymbol{\xi})$ is the parent density. Equation (2.1) will be most tractable when the functions $G(x)$ and $g(x)$ have simple analytic expressions. Hereafter, a random variable X with density function (2.1) is denoted by $X \sim$ ECC-G$(p,\alpha,\lambda,\boldsymbol{\xi})$.

Further, we can omit sometimes the dependence on the vector $\boldsymbol{\xi}$ of the parameters and write simply $G(x) = G(x; \boldsymbol{\xi})$.
Furthermore, the basic motivations for the ECC-G family in practice are the following:

i. to make the kurtosis more flexible compared to the baseline model;
ii. to produce a skewness for symmetrical distributions;
iii. to construct heavy-tailed distributions for modeling real data;
iv. to generate distributions with symmetric, left-skewed, right-skewed or reversed-J shape;
v. to define special models with all types of the hazard rate function;
vi. to provide consistently better fits than other generated models under the same underlying distribution.

For $p > 0$, we consider a system formed by α independent components following the Marsha-Olkin cdf (see Table 1) given by

$$H(x) = \frac{(1-p)\, G(x)^{\lambda}}{1 - p\, G(x)^{\lambda}}.$$

Suppose the system fails if any of the α components fails and let X denote the lifetime of the entire system. Then, the cdf of X is

$$F(x) = 1 - [1 - H(x)]^{\alpha} = 1 - \left[\frac{1 - G(x)^{\lambda}}{1 - p\, G(x)^{\lambda}}\right]^{\alpha},$$

which is the proposed generator.

For $p = 0$, a physical interpretation of the ECC-G distribution can be given as follows. Consider a system formed by α independent components and that each component is made up of λ independent sub-components. Suppose that the system fails if any of the α components fails and that each component fails if all of the λ sub-components fail. Let $X_{j1}, \ldots, X_{j\lambda}$ denote the lifetimes of the sub-components within the jth component, $j = 1, \ldots, \alpha$, having a common cdf G. Let X_j denote de lifetime of the jth component, for $j = 1, \ldots, \alpha$, and let X denote the lifetime of the entire system. Then, the cdf of X is

$$\begin{aligned} P(X \le x) &= 1 - P(X_1 > x, \ldots, X_{\alpha} > x) = 1 - P(X_1 > x)^{\alpha} \\ &= 1 - [1 - P(X_1 \le x)]^{\alpha} = 1 - [1 - P(X_{11} \le x, \ldots, X_{1\lambda})]^{\alpha} \\ &= 1 - \left[1 - P(X_{11} \le x)^{\lambda}\right]^{\alpha} = 1 - \left[1 - G(x)^{\lambda}\right]^{\alpha}. \end{aligned}$$

Thus, the family of distributions (2.1) with $p = 0$ is precisely the time to failure of the entire system.

The hazard rate function (hrf) of X becomes

$$h(x; \alpha, \lambda, p, \boldsymbol{\xi}) = \alpha\, \lambda\, (1-p)\, g(x; \boldsymbol{\xi})\, G(x; \boldsymbol{\xi})^{\lambda - 1} \left[\frac{1 - p\, G(x; \boldsymbol{\xi})^{\lambda}}{1 - G(x; \boldsymbol{\xi})^{\lambda}}\right]. \tag{2.2}$$

The ECC-G family can simulated by inverting (1.1). Let $Q_G(u) = G^{-1}(u)$ be the qf of G for $0 < u < 1$. If U has a uniform $U(0, 1)$ distribution, the solution of the nonlinear equation

$$x = F^{-1}(u) = Q_G \left\{ \left[\frac{1-(1-U)^{1/\alpha}}{1-p\,(1-U)^{1/\alpha}} \right]^{1/\lambda} \right\} \tag{2.3}$$

follows the density function (2.1).

3. Special ECC-G distributions

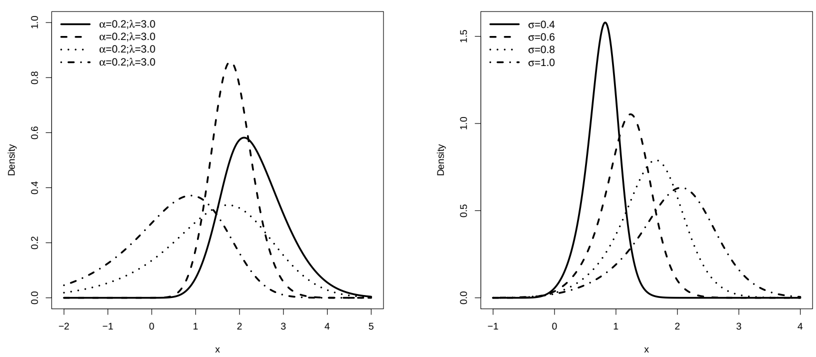
For $p = 0$, we obtain, as an important special case of (2.1), the *Cordeiro and de Castro*'s (CC) class of density functions. This class provides greater flexibility of its tails and can be widely applied in many areas of engineering and biology. Here, we present some special cases of the ECC-G family since it extends several useful distributions in the literature. For all cases listed below, $p \in (0,1)$, $\alpha > 0$ and $\lambda > 0$.

3.1. The ECC-Normal (ECCN) distribution. The ECCN distribution is defined from (2.1) by taking $G(x)$ and $g(x)$ to be the cdf and pdf of the normal $N(\mu, \sigma^2)$ distribution. Its density function is given by

$$f(x) = \frac{\alpha\,\lambda\,(1-p)}{\sigma}\,\phi\left(\frac{x-\mu}{\sigma}\right)\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\lambda-1} \frac{[1-\Phi(\frac{x-\mu}{\sigma})^{\lambda}]^{\alpha-1}}{[1-p\,\Phi(\frac{x-\mu}{\sigma})^{\lambda}]^{\alpha+1}}, \tag{3.1}$$

where $x \in \mathbb{R}$, $\mu \in \mathbb{R}$ is a location parameter, $\sigma > 0$ is a scale parameter, and $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and cdf of the standard normal distribution, respectively. A random variable with density function (3.1) is denoted by $X \sim \mathrm{ECCN}(p, \alpha, \lambda, \mu, \sigma^2)$. For $\mu = 0$, $\sigma = 1$ and $p \to 0$, we obtain the standard Kumaraswamy-normal (KwN) distribution. Furthermore, the KwN distribution with $\lambda = 1$ and $\alpha = 1$ reduces to the normal distribution.

Plots of the ECCN density function for some values of α, λ, μ and p and different values of σ are displayed in Figure 1. Based on these plots, we note that the parameter σ has the same dispersion property such as in the normal density.



(a) $p = 0.5, \mu = 1$ and $\sigma = 0.5$ (b) $p = 0.9, \mu = 0, \alpha = 1$ and $\lambda = 4$

Figure 1. Plots of the ECCN density function for some parameter values.

3.2. The ECC-Weibull (ECCW) distribution. Taking $G(x)$ as the Weibull cdf with scale parameter $\beta > 0$ and shape parameter $c > 0$, say $G(x) = 1 - \mathrm{e}^{-(\beta x)^c}$, it follows from equation (2.1) the ECCW density function (for $x > 0$)

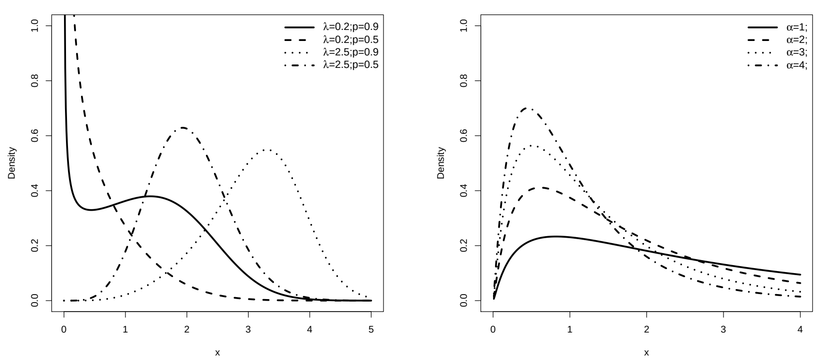
$$f(x) = \alpha\,\lambda\,(1-p)\,c\,\beta^c\,x^{c-1}\,\frac{\left[1-\mathrm{e}^{-(\beta x)^c}\right]^{\lambda-1}}{\mathrm{e}^{(\beta x)^c}}\,\frac{\left[1-\left(1-\mathrm{e}^{-(\beta x)^c}\right)^{\lambda}\right]^{\alpha-1}}{\left[1-p\,(1-\mathrm{e}^{-(\beta x)^c})^{\lambda}\right]^{\alpha+1}}. \tag{3.2}$$

For $p = 0$ and $\alpha = \lambda = 1$, the ECCW distribution reduces to the classical Weibull distribution. A random variable with density function (3.2) is denoted by $X \sim \mathrm{ECCW}(p, \alpha, \lambda, \beta, c)$. For $c = 1$, the ECCW model becomes the Kumaraswamy-exponential-geometric (KwEG) distribution. The Kumaraswamy-Weibull (KwW) distribution follows as a special case when $p \to 0$.

The hrf corresponding to (3.2) is given by

$$h(x) = \alpha\,\lambda\,(1-p)\,c\,\beta^c\,x^{c-1}\,\mathrm{e}^{-(\beta x)^c}\,\left[1-\mathrm{e}^{-(\beta x)^c}\right]^{\lambda-1}\left[\frac{1-p\left(1-\mathrm{e}^{-(\beta x)^c}\right)^{\lambda}}{1-\left(1-\mathrm{e}^{-(\beta x)^c}\right)^{\lambda}}\right]. \tag{3.3}$$

Figures 2 and 3 display plots of the ECCW density and hrf for selected parameter values, respectively.

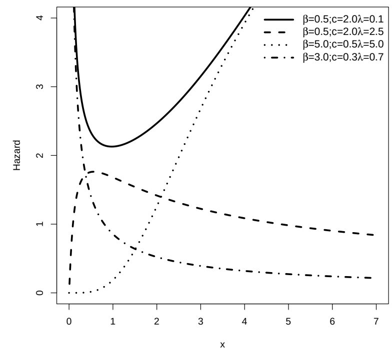


(a) $\alpha = 2.0, c = 2.0$ and $\beta = 0.5$ (b) $p = 0.3, \lambda = 5.0, c = 0.5$ and $\beta = 2.0$

Figure 2. Plots of the ECCW density function for some parameter values.

3.3. The ECC-gamma (ECCG) distribution. Consider the gamma distribution with shape parameter $a > 0$ and scale parameter $b > 0$, where the pdf and cdf (for $x > 0$) are given by

$$g(x) = \frac{b^a}{\Gamma(a)}\,x^{a-1}\,\mathrm{e}^{-bx} \quad \text{and} \quad G(x) = \frac{\gamma(a, bx)}{\Gamma(a)},$$



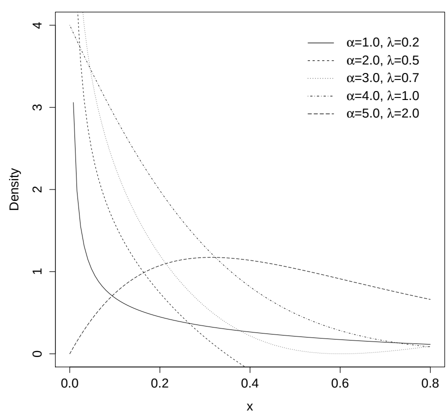
(a) $p = 0.1$ and $\alpha = 2.0$

Figure 3. Plots of the ECCW hrf for some parameter values.

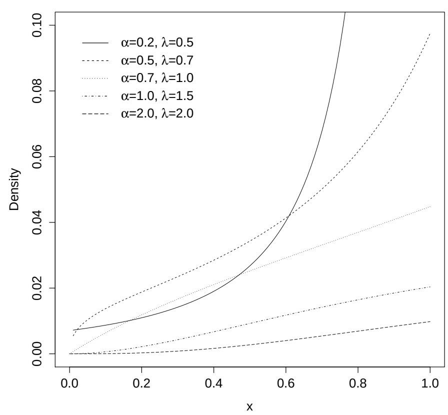
where $\gamma(a, bx)$ is the incomplete gamma function. Inserting these expressions in (2.1) gives the ECCG density function

$$f(x) = \frac{\alpha\,\lambda\,(1-p)\,b^a}{\Gamma(a)^\lambda}\,x^{a-1}\,\mathrm{e}^{-bx}\,\gamma(a,bx)^{\lambda-1}\,\frac{\left[1-\left(\dfrac{\gamma(a,bx)}{\Gamma(a)}\right)^\lambda\right]^{\alpha-1}}{\left[1-p\left(\dfrac{\gamma(a,bx)}{\Gamma(a)}\right)^\lambda\right]^{\alpha+1}}.$$

The Kumaraswamy-gamma (KwG) distribution follows from this model when $p \to 0$. Plots of the ECCG density and its hrf for selected parameter values are displayed in Figures 4 and 5, respectively.

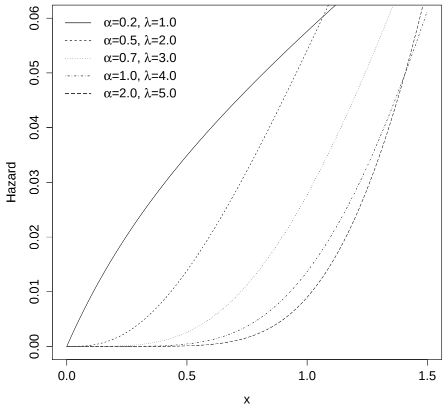


(a) $a = 1, b = 2$ and $p = 0.5$

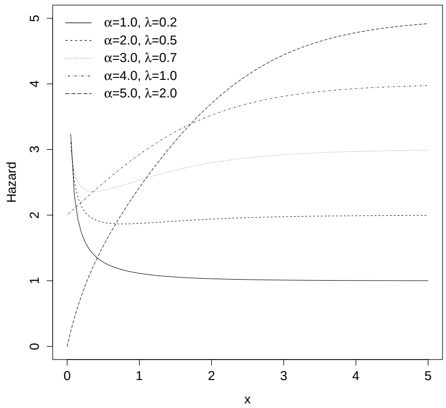


(b) $a = 2, b = 1$ and $p = 0.9$

Figure 4. Plots of the ECCG density function for some parameter values.



(a) $a = 2, b = 1$ and $p = 0.9$



(b) $a = 2, b = 1$ and $p = 0.5$

Figure 5. Plots of the ECCG hrf for some parameter values.

3.4. The ECC-beta (ECCB) distribution. Consider the beta distribution with positive shape parameters a and b and pdf and cdf (for $0 < x < 1$) given by

$$g(x) = \frac{1}{B(a,b)}\, x^{a-1}\,(1-x)^{b-1} \qquad \text{and} \qquad G(x) = \frac{I_x(a,b)}{B(a,b)},$$

where $I_x(a,b) = \int_0^x w^{a-1}(1-w)^{b-1}\mathrm{d}w$ is the incomplete beta function and $B(a,b) = \int_0^1 w^{a-1}(1-w)^{b-1}\mathrm{d}w = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function. Inserting these expressions in (2.1) gives the ECCB density function (for $0 < x < 1$)

$$f(x) = \frac{\alpha\,\lambda\,(1-p)}{B(a,b)^{\lambda}}\; x^{a-1}\,(1-x)^{b-1}\, I_x(a,b)^{\lambda-1}\, \frac{\left[1-\left(\frac{I_x(a,b)}{B(a,b)}\right)^{\lambda}\right]^{\alpha-1}}{\left[1-p\,\left(\frac{I_x(a,b)}{B(a,b)}\right)^{\lambda}\right]^{\alpha+1}}.$$

The Kumaraswamy beta (KwB) arises as a special case when $p \to 0$. The beta distribution corresponds to the limiting case: $p \to 0$ and $\alpha = \lambda = 1$. Figure 6 displays plots of the ECCB density function for some parameter values.

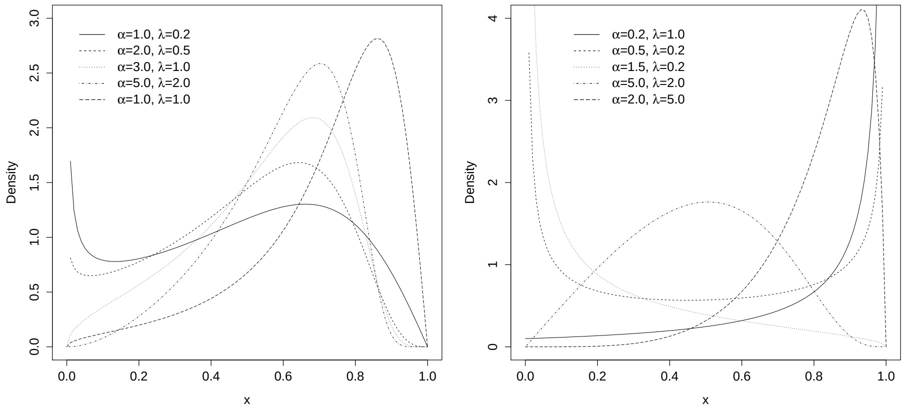
4. Useful expansions

We can demonstrate that the cdf (1.1) of X has the expansion

$$F(x) = 1 - \sum_{j,k=0}^{\infty} w_{j,k}\, H_{(j+k)\lambda}(x), \tag{4.1}$$

where

$$w_{j,k} = (-1)^{j+k}\, p^j \binom{-\alpha}{j}\binom{\alpha}{k}$$



(a) $a = 1.5, b = 2$ and $p = 0.9$ (b) $a = b = 1$ and $p = 0.5$

Figure 6. Plots of the ECCB density function for some parameter values.

and $H_a(x) = G(x)^a$ denotes the exponentiated-G ("exp-G" for short) cumulative distribution. Some structural properties of the exp-G distributions are studied by [17], [13], [18] and among others.

By differentiating (4.1), we obtain

$$f(x;\alpha,\lambda,p,\boldsymbol{\xi}) = \sum_{j,k=0}^{\infty} \omega_{j,k}\, h_{\lambda(j+k+1)}(x), \tag{4.2}$$

where

$$\omega_{j,k} = \frac{\alpha\,\lambda\,(1-p)\,p^j}{\lambda\,(j+k+1)}(-1)^{j+k}\binom{\alpha-1}{j}\binom{-\alpha-1}{k}$$

and $h_{\lambda(j+k+1)}(x;\boldsymbol{\xi}) = \lambda(j+k+1)\,g(x;\boldsymbol{\xi})\,G(x;\boldsymbol{\xi})^{\lambda(j+k+1)-1}$ denotes the exp-G density function with power parameter $\lambda(j+k+1)$. Hereafter, a random variable having this density function is denoted by $Y_{j,k} \sim$ exp-G$(\lambda(j+k+1))$. Equation (4.2) reveals that the ECC-G density function is just a linear combination of exp-G density functions. Thus, some mathematical properties of the new model can be derived from those properties of the exp-G distribution. For example, the ordinary and incomplete moments and moment generating function (mgf) of X can be obtained from those quantities of the exp-G distribution.

The formulae derived throughout the paper can be easily handled in most symbolic computation software platforms such as Maple, Mathematica and Matlab. These platforms have currently the ability to deal with analytic expressions of formidable size and complexity. Established explicit expressions to calculate statistical measures can be more efficient than computing them directly by numerical integration. The infinity limit in these sums can be substituted by a large positive integer such as 20 or 30 for most practical purposes.

5. General properties

5.1. Asymptotes and shapes.

5.1. Proposition. *The asymptotics of equations (1.1), (2.1) and (2.2) as $x \to -\infty$ are given by*

$$
\begin{aligned}
&F(x) \sim \alpha\, G(x)^{\lambda} && \text{as} \quad x \to -\infty,\\
&f(x) \sim \alpha\, \lambda\, g(x)\, G(x)^{\lambda-1} && \text{as} \quad x \to -\infty,\\
&h(x) \sim \alpha\, \lambda\, g(x)\, G(x)^{\lambda-1} && \text{as} \quad x \to -\infty.
\end{aligned}
$$

These equations can provide the effects of the parameters on the tails of the distribution.

5.2. Proposition. *The asymptotics of equations (1.1), (2.1) and (2.2) when $x \to \infty$ are given by*

$$
\begin{aligned}
&1 - F(x) \sim (\frac{\lambda}{1-p})^{\alpha}\, \bar{G}(x)^{\alpha} && \text{as} \quad x \to \infty\\
&f(x) \sim \alpha(\frac{\lambda}{1-p})^{\alpha}\, g(x)\, \bar{G}(x)^{\alpha-1} && \text{as} \quad x \to \infty,\\
&h(x) \sim \frac{\alpha\, g(x)}{\bar{G}(x)} && \text{as} \quad x \to \infty.
\end{aligned}
$$

The shapes of the density and hazard rate functions can be described analytically. The critical points of the ECC-G density function are the roots of the equation:

$$
(\lambda - 1)\frac{g(x)}{G(x)} + \frac{g'(x)}{g(x)} = \lambda\, g(x)\, G(x)^{\lambda-1}\left[\frac{\alpha - 1}{1 - G(x)^{\lambda}} + \frac{p\,(\alpha+1)}{1 - p\, G(x)^{\lambda}}\right]. \tag{5.1}
$$

There may be more than one root to (5.1). Let $\lambda(x) = \partial^2 \log[f(x)]/\partial x^2$. We have

$$
\begin{aligned}
\lambda(x) &= (\lambda-1)\frac{g'(x)G(x) - g(x)^2}{G(x)^2} + \frac{g''(x)g(x) - g'(x)^2}{g(x)^2}\\
&- \lambda g'(x)G(x)^{\lambda-1}\left[\frac{\alpha-1}{1-G(x)^{\lambda}} + \frac{p\,(\alpha+1)}{1-p\,G(x)^{\lambda}}\right]\\
&- \lambda(\lambda-1)g(x)^2 G(x)^{\lambda-2}\left[\frac{\alpha-1}{1-G(x)^{\lambda}} + \frac{p\,(\alpha+1)}{1-p\,G(x)^{\lambda}}\right]\\
&- \lambda^2 g(x)^2 G(x)^{2\lambda-2}\left[\frac{\alpha-1}{(1-G(x)^{\lambda})^2} + \frac{p\,(\alpha+1)}{(1-p\,G(x)^{\lambda})^2}\right].
\end{aligned}
$$

If $x = x_0$ is a root of (5.1) then it corresponds to a local maximum if $\lambda(x) > 0$ for all $x < x_0$ and $\lambda(x) < 0$ for all $x > x_0$. It corresponds to a local minimum if $\lambda(x) < 0$ for all $x < x_0$ and $\lambda(x) > 0$ for all $x > x_0$. It refers to a point of inflexion if either $\lambda(x) > 0$ for all $x \neq x_0$ or $\lambda(x) < 0$ for all $x \neq x_0$.

The critical point of $h(x)$ are obtained from the equation

$$
\frac{g'(x)}{g(x)} + (\lambda-1)\frac{g(x)}{G(x)} - p\,\lambda\, g(x)\frac{G(x)^{\lambda-1}}{1 - pG(x)^{\lambda-1}} = \lambda\, g(x)\frac{G(x)^{\lambda-1}}{1 - G(x)^{\lambda-1}}. \tag{5.2}
$$

There may be more than one root to (5.2). Let $\tau(x) = d^2 \log[h(x)]/dx^2$. We have

$$\tau(x) = \frac{g''(x)g(x) - [g'(x)]^2}{g(x)^2} + (\lambda - 1)\,\frac{g'(x)G(x) - [g(x)]^2}{G(x)^2}$$
$$-p\,\lambda\,g'(x)\frac{G(x)^{\lambda-1}}{1-p\,G(x)^{\lambda}} - p\,\lambda\,(\lambda-1)\,g(x)^2\frac{G(x)^{\lambda-2}}{1-p\,G(x)^{\lambda}} - p^2\lambda^2 g(x)^2\frac{G(x)^{2\lambda-2}}{[1-p\,G(x)^{\lambda}]^2}$$
$$-\lambda\,g'(x)\frac{G(x)^{\lambda-1}}{1-G(x)^{\lambda}} - \lambda\,(\lambda-1)\,g(x)^2\frac{G(x)^{\lambda-2}}{1-G(x)^{\lambda}} - \lambda^2 g(x)^2\frac{G(x)^{2(\lambda-1)}}{[1-G(x)^{\lambda}]^2} = 0.$$

If $x = x_0$ is a root of (5.2) then it refers to a local maximum if $\tau(x) > 0$ for all $x < x_0$ and $\tau(x) < 0$ for all $x > x_0$. It corresponds to a local minimum if $\tau(x) < 0$ for all $x < x_0$ and $\tau(x) > 0$ for all $x > x_0$. It gives an inflexion point if either $\tau(x) > 0$ for all $x \neq x_0$

5.2. Quantile power series. Power series methods are at the heart of many aspects of applied mathematics and statistics. The qfs are in widespread use in continuous distributions and often find representations in terms of power series. The qf for a distribution has many uses in both the theory and statistical applications. It may be used to generate values of a random variable having $F(x)$ as its distribution function. This fact serves as the basis of a method for simulating a sample from an arbitrary distribution with the aid of a random number generator.

By expanding (2.3), we derive explicit expressions for the moments and generating function of the ECC family using a power series for the qf $x = Q(u) = F^{-1}(u)$ of X, which is easily obtained using a linear recurrent equation for its coefficients. If the G qf, say $Q_G(u)$, does not have a closed-form expression, it can usually be expressed in terms of a power series

$$Q_G(u) = \sum_{i=0}^{\infty} a_i\,u^i, \tag{5.3}$$

where the coefficients a_i's are suitably chosen real numbers which depend on the parameters of the G distribution. For several important distributions, such as the normal, Student t, gamma and beta distributions, $Q_G(u)$ does not have explicit expressions but it can be expanded as in equation (5.3). As a simple example, for the normal $N(0,1)$ distribution, $a_i = 0$ for $i = 0, 2, 4, \ldots$ and $a_i = b_{(i-1)/2}$ for $i = 1, 3, 5, \ldots$, where the quantities $b_{(i-1)/2}$ can be determinated recursively from

$$b_{k+1} = \frac{1}{2(2k+3)} \sum_{r=0}^{k} \frac{(2r+1)\,(2k-2r+1)\,b_r\,b_{k-r}}{(r+1)\,(2r+1)}.$$

We have $a_1 = 1$, $a_3 = 1/6$, $a_5 = 7/120$ and $a_7 = 127/7560, \ldots$

Henceforth, we use a result by [11] (see Section 0.314) for a power series raised to a positive integer n (for $n \geq 1$)

$$Q_G(u)^n = \left(\sum_{i=0}^{\infty} a_i\,u^i\right)^n = \sum_{i=0}^{\infty} c_{n,i}\,u^i, \tag{5.4}$$

where the coefficient $c_{n,i}$ (for $i = 1, 2, \ldots$) follows from the recurrence equation (with $c_{n,0} = a_0^n$)

$$(5.5) \qquad c_{n,i} = (i\, a_0)^{-1} \sum_{m=1}^{i} [m(n+1) - i]\, a_m\, c_{n,i-m}.$$

Clearly, the quantity $c_{n,i}$ can be determined from $c_{n,0}, \ldots, c_{n,i-1}$ and then from the quantities $a_0, \ldots, a_i$. The coefficient $c_{n,i}$ can be given explicitly in terms of the coefficients a_i's, although it is not necessary for programming numerically our expansions in any algebraic or numerical software. For the normal $N(0,1)$ distribution, the coefficients $c_{n,i}$ can be obtained from (5.4) using the a_i's given above.

Next, we derive an expansion for the argument of $Q_G(\cdot)$ in (2.3)

$$A = \frac{[1 - (1-u)^{1/\alpha}]^{1/\lambda}}{[1 - p\,(1-u)^{1/\alpha}]^{1/\lambda}}.$$

By using the generalized binomial expansion three times since $u \in (0,1)$, we can write

$$A = \sum_{r,s,t=0}^{\infty} (-1)^{r+s+t}\, p^r \binom{-\lambda^{-1}}{r} \binom{\lambda^{-1}}{s} \binom{(r+s)\alpha^{-1}}{t} u^t.$$

Then, the qf of X can be expressed from (2.3) as

$$(5.6) \qquad Q(u) = Q_G\left(\sum_{t=0}^{\infty} \delta_t\, u^t \right),$$

where

$$\delta_t = \sum_{r,s=0}^{\infty} (-1)^{r+s+t}\, p^r \binom{\lambda^{-1}}{r} \binom{\lambda^{-1}}{s} \binom{(r+s)\alpha^{-1}}{t}.$$

For any underlying G distribution, we combine (5.3) and (5.6) to obtain

$$Q(u) = Q_G\left(\sum_{t=0}^{\infty} \delta_t\, u^t \right) = \sum_{i=0}^{\infty} a_i \left(\sum_{t=0}^{\infty} \delta_t\, u^t \right)^i,$$

and then using (5.4) and (5.5), we have

$$(5.7) \qquad Q(u) = \sum_{t=0}^{\infty} e_t\, u^t,$$

where $e_t = \sum_{i=0}^{\infty} a_i\, d_{i,t}$, $d_{i,0} = \delta_0^i$ and (for $t > 1$)

$$d_{i,t} = (t\, \delta_0)^{-1} \sum_{m=1}^{t} [m(i+1) - t]\, \delta_m\, d_{i,t-m}.$$

Equation (5.7) is the main result of this section. It allows to obtain various mathematical quantities for the ECC-G family as investigated in the next sections.

5.3. Generating function. Here, we provide two general formulae for the moment generating fucntion (mgf) $M(t) = E(\mathrm{e}^{t\,X})$ of X. A first formula for $M(t)$ follows from (4.2) as

$$M(t) = \sum_{j,k=0}^{\infty} \omega_{j,k}\, M_{j,k}(t), \tag{5.8}$$

where $M_{j,k}(t)$ is the mgf of $Y_{j,k}$. Hence, $M(t)$ can be immediately determined from the generating function of the exp-G distribution. We now provide three applications of equation (5.8). For example, the generating functions of the ECC-exponential (with parameter β) (for $t < 1/\beta$), ECC-Pareto (ECCP) (with parameter $\nu > 0$ real non integer) and ECC-standard logistic (ECCSL) (for $t < 1$) distributions are determined from equation (5.8) as

$$M(t) = \sum_{j,k=0}^{\infty} [\lambda(j+k+1)]\, B(\lambda(j+k+1), 1-\beta t)\, \omega_{j,k},$$

$$M(t) = \mathrm{e}^{-t} \sum_{j,k,m=0}^{\infty} [\lambda(j+k+1)]\ B\left(\lambda(j+k+1), 1-m\nu^{-1}\right)\ \omega_{j,k}\, \frac{t^m}{m!},$$

and

$$M(t) = \sum_{j,k=0}^{\infty} [\lambda(j+k+1)]\, B(t+\lambda(j+k+1), 1-t)\, \omega_{j,k},$$

respectively.

Next, we provide a fourth application of (5.8) by taking again as the underlying the Weibull distribution with scale parameter β and shape parameter c (see Section 3.2). The generating function of the exp-Weibull distribution with power parameter $\lambda(j+k+1)$ is given by

$$M_{j,k}(t) = \sum_{r=0}^{\infty} v_{j,k}^{(r)}\, I_r(t), \tag{5.9}$$

where

$$v_{j,k}^{(r)} = \beta\, c^{\beta}\, [\lambda(j+k+1)] \sum_{i=0}^{\infty} (-1)^{i+r} \binom{[\lambda(j+k+1)](i+1)-1}{r},$$

$\delta_r = \beta\,(r+1)^{1/c}$ and

$$I_r(t) = \int_0^{\infty} x^{c-1}\, \exp\{t\,x - (\delta_r\, x)^c\} dx.$$

[21] derived two different formulae for $I_r(t)$ which hold for: (i) $c > 1$ or (ii) for $c = p/q$, where $p \geq 1$ and $q \geq 1$ are co-prime integers. The first representation for $I_r(t)$ is given in terms of the Wright generalized hypergeometric function [27]

defined by

$$
{}_p\Psi_q\left[\begin{array}{c}(\alpha_1,A_1),\cdots,(\alpha_p,A_p)\\(\beta_1,B_1),\cdots,(\beta_q,B_q)\end{array};x\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(\alpha_j+A_j\,n)}{\prod_{j=1}^{q}\Gamma(\beta_j+B_j\,n)}\,\frac{x^n}{n!}.
$$

We can write

$$
\begin{aligned}
I_r(t) &= \sum_{m=0}^{\infty}\frac{t^m}{m!}\int_0^{\infty}x^{m+\beta-1}\exp\{-(\delta_r\,x)^c\}dx=\frac{1}{\beta\,\delta_r^{\beta}}\sum_{m=0}^{\infty}\frac{t^m}{\delta_r^m\,m!}\,\Gamma(mc^{-1}+1)\\
(5.10)\quad &= \frac{1}{\beta\,\delta_r^{\beta}}\,{}_1\Psi_0\left[\begin{array}{c}(1,\beta^{-1})\\-\end{array};\frac{t}{\delta_r}\right].
\end{aligned}
$$

The function $I_r(t)$ exists if $1+\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j>0$.

Based on equations (5.8), (5.9) and (5.10), we obtain (for $\lambda>1$)

$$
M(t)=c^{-1}\sum_{j,k,r=0}^{\infty}\frac{\omega_{j,k}\,v_{j,k}^{(r)}}{\delta_r}\,{}_1\Psi_0\left[\begin{array}{c}(1,c^{-1})\\-\end{array};\frac{t}{\delta_r}\right]. \qquad (5.11)
$$

A second representation for $I_r(t)$ is obtained from the Meijer G-function defined by

$$
G_{p,q}^{m,n}\left(x\,\middle|\,\begin{array}{c}a_1,\ldots,a_p\\b_1,\ldots,b_q\end{array}\right)=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j+t)\prod_{j=1}^{n}\Gamma(1-a_j-t)}{\prod_{j=n+1}^{p}\Gamma(a_j+t)\prod_{j=m+1}^{p}\Gamma(1-b_j-t)}\,x^{-t}dt,
$$

where $\mathrm{i}=\sqrt{-1}$ is the complex unit and L denotes an integration path; see Section 9.3 in [11] for a description of this path. The Meijer G-function contains many integrals with elementary and special functions [22]. From the result $\exp\{-g(x)\}=G_{0,1}^{1,0}\left(g(x)\mid\begin{array}{c}-\\0\end{array}\right)$ for an arbitrary $g(\cdot)$ function, $I_r(t)$ becomes

$$
I_r(t)=\int_0^{\infty}x^{c-1}\exp\{sx-(\delta_r\,x)^c\}dx=\int_0^{\infty}x^{v-1}\,\mathrm{e}^{sx}\,G_{0,1}^{1,0}\left(\delta_r^c\,x^c\middle|\begin{array}{c}-\\0\end{array}\right)dx.
$$

We now assume that $c=p/q$, where $p\geq 1$ and $q\geq 1$ are co-prime integers. Note that this condition for calculating the integral $I_r(t)$ is not restrictive since every real number can be approximated by a rational number. Using equation (2.24.1.1) in [22] (volume3), we have

$$
I_r(t)=\frac{p^{p/q-1/2}(-t)^{-p/q}}{(2\pi)^{(p+q)/2-1}}\,G_{q,p}^{p,q}\left(\frac{\delta_r^q p^{p+q}}{(-t)^p q^{2q}}\middle|\begin{array}{c}\frac{q-p}{pq},\frac{2q-p}{pq},\ldots,\frac{pq-p}{pq}\\0,\frac{1}{q},\ldots,\frac{q-1}{q}\end{array}\right). \qquad (5.12)
$$

Using (5.8), (5.9) and (5.12), we can obtain $M(t)$ for the ECCW distribution.

A second general formula for $M(t)$ can be derived from (4.2) as

$$
M(t)=\sum_{j,k=0}^{\infty}[\lambda(j+k+1)]\,\omega_{j,k}\,\rho(t,\lambda(j+k+1)-1), \qquad (5.13)
$$

where $\rho(t,a)$ can be determined from the underlying qf $Q_G(x)$ by

$$\rho(t,a) = \int_{-\infty}^{\infty} \mathrm{e}^{t\,x}\, G(x)^a\, g(x) dx = \int_0^1 \exp\{t\, Q_G(u)\}\; u^a du. \tag{5.14}$$

An alternative expression for $\rho(t,a)$ in terms of the coefficients of the G qf follows using the power series for the exponential function and (5.4) and then integrating the result. We can obtain

$$\rho(t,a) = \sum_{n,i=0}^{\infty} \frac{c_{n,i}\; t^n}{(a+i+1)\, n!}. \tag{5.15}$$

We can derive the mgfs for several ECC-G distributions from equations (5.8) and (5.13), the last one combining with (5.14) or (5.15). Equations (5.8) and (5.13) are the main results of this section.

5.4. Moments. Here, we provide two general formulae for the nth moment of X. The first one is obtained from (4.2) as

$$\mu'_n = E(X^n) = \sum_{j,k=0}^{\infty} \omega_{j,k}\, E(Y_{j,k}^n) = \sum_{j,k=0}^{\infty} \omega_{j,k} \int_{-\infty}^{\infty} x^n\, h_{\lambda(j+k+1)}(x; \boldsymbol{\xi}). \tag{5.16}$$

Expressions for moments of some exponentiated distributions are given by [18]. They can be used to obtain μ'_n. We now provide an application of (5.16) for the ECCW distribution discussed in Section 3.2, where $G(x) = 1 - e^{-(\beta\, x)^c}$, $c > 0$ is a shape parameter and $\beta > 0$ a scale parameter. The corresponding exp-Weibull (exp-W) density function with power parameter $\lambda(j+k+1)$ is given by

$$h_{\lambda(j+k+1)}(x; \beta, c) = \lambda(j+k+1)\, c\, \beta^c\, x^{c-1}\, \mathrm{e}^{-(\beta x)^c}\, [1 - \mathrm{e}^{-(\beta x)^c}]^{\lambda(j+k+1)-1}. \tag{5.17}$$

The nth moment of (5.17), say $\rho_{j,k}^{(n)}$, can be obtained from [4] as

$$\rho_{j,k}^{(n)} = \frac{\Gamma(n/c+1)}{\beta^n} \sum_{r=0}^{\infty} \frac{w_{j,k}^{(r)}}{(r+1)^{n/c}}, \tag{5.18}$$

where

$$w_{j,k}^{(r)} = \frac{[\lambda(j+k+1)]}{(r+1)} \sum_{i=0}^{\infty} (-1)^{i+r} \binom{[\lambda(j+k+1)](i+1)-1}{r}.$$

Combining equations (5.16) and (5.18), we can write μ'_n as

$$\mu'_n = \frac{\Gamma(n/c+1)}{\beta^n} \sum_{k,j,r,i=0}^{\infty} \frac{(-1)^{i+r}\, [\lambda(j+k+1)]\, \omega_{j,k}}{(r+1)^{n/c+1}} \binom{[\lambda(j+k+1)](i+1)-1}{r}.$$

Next, we provide two more examples from (5.16). First, for the ECCPa distribution, where the underlying cdf is $G(x) = 1 - (1+x)^{-\nu}$ and $\nu > 0$, we obtain (for ν real non integer)

$$\mu'_n = \sum_{k,j,m=0}^{\infty} (-1)^{n+m}\, [\lambda(j+k+1)]\, B(\lambda(j+k+1)-1, 1-m\nu^{-1})\, \omega_{j,k} \binom{n}{m}.$$

Second, for the ECCSL distribution, where $G(x) = (1+\mathrm{e}^{-x})^{-1}$, we can write using a result by [22] (Section 2.6.13, equation 4) (for $t < 1$)

$$\mu'_n = \sum_{k,j=0}^{\infty} [\lambda(j+k+1)]\, \omega_{j,k} \left(\frac{\partial}{\partial t}\right)^n B(t+\lambda(j+k+1), 1-t)\bigg|_{t=0}.$$

A second general formula for μ'_n follows from (4.2) and $Q_G(u)$ as

$$\mu'_n = \sum_{k,j=0}^{\infty} (\alpha+k+j)\, \omega_{j,k}\, \tau(n, \lambda(j+k+1)-1), \tag{5.19}$$

where $\tau(n,a)$ is given by

$$\tau(n,a) = \int_{-\infty}^{\infty} x^n\, G(x)^a\, g(x) dx = \int_0^1 Q_G(u)^n\, u^a du.$$

Inserting (5.4) in the last equation and integrating, we obtain

$$\tau(n,a) = \sum_{i=0}^{\infty} \frac{c_{n,i}}{(a+1)}, \tag{5.20}$$

where the quantities $c_{n,i}$ can be determined from (5.5).

The central moments (μ_n) and cumulants (κ_n) of X can be determined from (5.16) or (5.19) as

$$\mu_n = \sum_{k=0}^{r} (-1)^k \binom{n}{k} \mu_1'^n\, \mu'_{n-k} \quad \text{and} \quad \kappa_n = \mu'_n - \sum_{k=1}^{n-1} \binom{n-1}{k-1} \kappa_k\, \mu'_{n-k},$$

respectively, where $\kappa_1 = \mu'_1$. Thus, $\kappa_2 = \mu'_2 - \mu_1'^2$, $\kappa_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu_1'^3$, $\kappa_4 = \mu'_4 - 4\mu'_3\mu'_1 - 3\mu_2'^2 + 12\mu'_2\mu_1'^2 - 6\mu_1'^4$, etc. The skewness $\gamma_1 = \kappa_3/\kappa_2^{3/2}$ and kurtosis $\gamma_2 = \kappa_4/\kappa_2^2$ can be calculated from the third and fourth standardized cumulants.

5.5. Incomplete moments. The answers to many important questions in economics require more than just knowing the mean of a distribution, but its shape as well. This is obvious not only in the study of econometrics (for example, asymmetric error terms cannot be generated by the commonly assumed spherical distributions) and income distribution, but in other areas as well. Incomplete moments of the income distribution form natural building blocks for measuring inequality: for example, the Lorenz and Bonferroni curves and Pietra and Gini measures of inequality all depend upon the incomplete moments of the income distribution.

The nth incomplete moment of X is defined as $m_n(y) = E(X^n | X < y) = \int_{-\infty}^{y} x^r\, f(x) dx$. Here, we propose two methods to determine the incomplete moments of the new family. First, the nth incomplete moment of X can be expressed as

$$m_n(y) = \sum_{j,k=0}^{\infty} [\lambda(j+k+1)]\, \omega_{j,k} \int_0^{G(y;\boldsymbol{\xi})} Q_G(u)^n\, u^{\lambda(j+k+1)}\, du. \tag{5.21}$$

The integral in (5.21) can be computed at least numerically for most underlying distributions. A second method to obtain the incomplete moments of X follows from (5.21) using equations (5.4) and (5.5). We obtain

$$m_n(y) = \sum_{j,k,i=0}^{\infty} \frac{[\lambda(j+k+1)]\,\omega_{j,k}\,c_{n,i}}{[\lambda(j+k+1)+i]}\,G(y;\boldsymbol{\xi})^{\lambda(j+k+1)+i}. \tag{5.22}$$

5.6. Mean deviations. The mean deviations about the mean $\delta_1 = E(|X-\mu_1'|)$ and about the median $\delta_2(X) = E(|X-M|)$ of X can be expressed as

$$\delta_1 = 2\mu_1' F(\mu_1') - 2T(\mu_1') \quad \text{and} \quad \delta_2 = \mu_1' - 2T(M),$$

respectively, where $\mu_1' = E(X)$, $M = Median(X)$ denotes the median, $F(\mu_1')$ comes from equation (1.1) and $T(z) = \int_{-\infty}^{z} x f(x)\mathrm{d}x$. The median M follows from equation (1.1) as

$$M = Q_G\left[\left(\frac{1-2^{-1/\alpha}}{1-p\,2^{-1/\alpha}}\right)^{1/\lambda}\right].$$

Then, using ordinary and incomplete moments, we can easily obtain δ_1 and δ_2.

5.7. Quantile measure. The effects of the shape parameters a and b on the skewness and kurtosis can be based on quantile measures. The shortcomings of the classical kurtosis measure are well-known. The Bowley skewness [14] is one of the earliest skewness measures defined by the average of the quartiles minus the median, divided by half the interquartile range, namely

$$B = \frac{Q\left(\frac{3}{4}\right) + Q\left(\frac{1}{4}\right) - 2Q\left(\frac{1}{2}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)}.$$

Since only the middle two quartiles are considered and the other two quartiles are ignored, this adds robustness to the measure. The Moors kurtosis [16] is based on octiles

$$M = \frac{Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right) + Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)}.$$

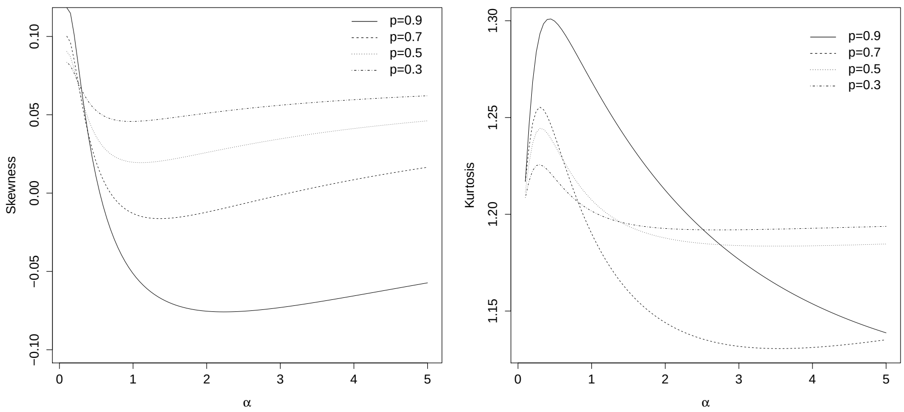
These measures are less sensitive to outliers and they exist even for distributions without moments. Plots of the skewness and kurtosis for the distributions ECCW and ECCN (discussed in Section 3) and selected parameter values are displayed in Figures 7 and 8, respectively. These plots indicate how both measures B and M vary depending on the values of the shape parameters.

5.8. Entropies. An entropy is a measure of variation or uncertainty of a random variable X. Two popular entropy measures are the Rényi and Shannon entropies [24, 23]. The Rényi entropy of a random variable with pdf $f(x)$ is defined as

$$I_R(c) = \frac{1}{1-c}\log\left(\int_0^{\infty} f^c(x)dx\right),$$

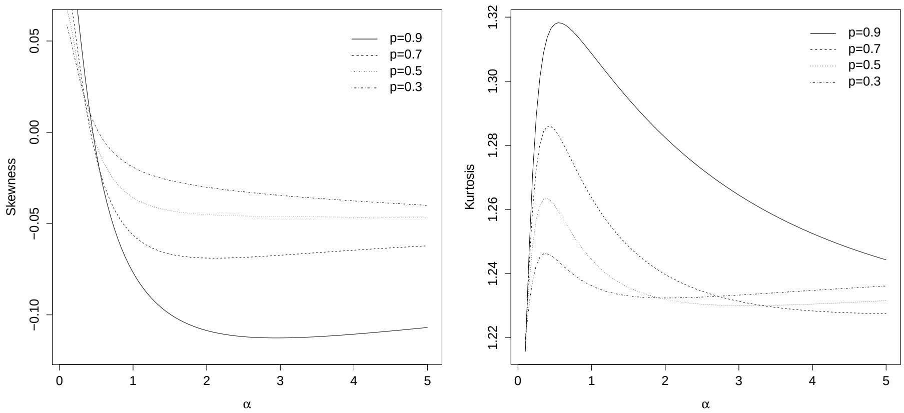
for $c > 0$ and $c \neq 1$. The Shannon entropy of a random variable X is defined by $E\{-\log[f(X)]\}$. It is the special case of the Rényi entropy when $c \uparrow 1$. Direct calculation gives

$$\begin{aligned}
\mathrm{E}\{-\log[f(X)]\} &= -\log[\alpha\lambda(1-p)] - \mathrm{E}\{\log[g(X;\boldsymbol{\xi})]\} - (\lambda-1)\,\mathrm{E}\{\log[G(x;\boldsymbol{\xi})]\} \\
&- (\alpha-1)\,\mathrm{E}\left\{\log\left[1-G(x;\boldsymbol{\xi})^{\lambda}\right]\right\} + (\alpha+1)\,\mathrm{E}\left\{\log\left[1-p\,G(x;\boldsymbol{\xi})^{\lambda}\right]\right\}
\end{aligned}$$



(a) $\lambda = 1$ and $\beta = c = 2$ (b) $\lambda = 1$ and $\beta = c = 2$

Figure 7. Skewness (a) and Kurtosis (b) of the ECCW distribution.



(a) $\lambda = 1$, $\mu = -2$ and $\sigma = 0.5$ (b) $\lambda = 1$, $\mu = -2$ and $\sigma = 0.5$

Figure 8. Skewness (a) and Kurtosis (b) of the ECCN distribution.

After some algebraic manipulations, we obtain:

5.3. Proposition. *Let X be a random variable with pdf* (2.1). *Then,*

$$\mathrm{E}\left\{\log\left[G(X)\right]\right\} = \frac{\alpha(1-p)}{\lambda}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j+1}p^{i}\binom{-\alpha-1}{i}\binom{\alpha-1}{j}}{\left[i+j+1\right]^{2}},$$

$$\mathrm{E}\left\{\log\left[1-G(X)^{\lambda}\right]\right\} = \alpha(1-p)\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}p^{i}\binom{-\alpha-1}{i}\left[\frac{\partial}{\partial t}\binom{\alpha+t-1}{j}\big|_{t=0}\right]}{i+j+1},$$

$$\mathrm{E}\left\{\log\left[1-p\,G(X)^{\lambda}\right]\right\}=\alpha(1-p)\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}p^{i}\left[\frac{\partial}{\partial t}\binom{t-\alpha-1}{i}\big|_{t=0}\right]\binom{-\alpha-1}{i}}{i+j+1},$$

where $\psi(\cdot)$ *is the digamma function.*

The simplest formula for the entropy of X is given by

$$\begin{aligned}
\mathrm{E}\left\{-\log[f(X)]\right\} &= -\log[\alpha\lambda(1-p)]-\mathrm{E}\left\{\log[g(X;\boldsymbol{\xi})]\right\}\\
&+\frac{\alpha(1-\lambda)(1-p)}{\lambda}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j+1}p^{i}\binom{-\alpha-1}{i}\binom{\alpha-1}{j}}{[i+j+1]^{2}},\\
&+\alpha(1-\alpha)(1-p)\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}p^{i}\binom{-\alpha-1}{i}\left[\frac{\partial}{\partial t}\binom{\alpha+t-1}{j}\big|_{t=0}\right]}{i+j+1},\\
&+\alpha(\alpha+1)(1-p)\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}p^{i}\left[\frac{\partial}{\partial t}\binom{t-\alpha-1}{i}\big|_{t=0}\right]\binom{-\alpha-1}{i}}{i+j+1}
\end{aligned}$$

After some algebraic developments, we obtain an alternative expression for $I_R(c)$

$$I_R(c)\;=\frac{c}{1-c}\log\left[\alpha\lambda(1-p)\right]+\frac{1}{1-c}\log\left[\sum_{i,j=0}^{\infty}w_{i,j}^{*}\,\mathrm{E}_{Y_{i,j}}\left(g^{c-1}[G^{-1}(Y)]\right)\right],$$

where $Y_{i,j}\sim B(\gamma(\lambda-1)+\lambda(i+j)+1,1)$ and

$$w_{i,j}^{*}=\frac{(-1)^{i+j}p^{i}\binom{-\gamma(\alpha+1)}{i}\binom{\gamma(\alpha-1)}{j}}{\gamma(\lambda-1)+\lambda(i+j)+1}.$$

6. Estimation

Here, we determine the maximum likelihood estimates (MLEs) of the model parameters of the new family from complete samples only. Let $x_1,\ldots,x_n$ be observed values from the ECC-G distribution with parameters p,α,λ and $\boldsymbol{\xi}$. Let $\Theta=(p,\alpha,\lambda,\boldsymbol{\xi})^{\top}$ be the $r\times 1$ parameter vector. The total log-likelihood function for Θ is given by

$$\begin{aligned}
\ell_n=\ell_n(\Theta) &= n\,\log\alpha+n\,\log\lambda+n\,\log(1-p)+\sum_{i=1}^{n}\log\left[g(x;\boldsymbol{\xi})\right]+(\lambda-1)\sum_{i=1}^{n}\log\left[G(x;\boldsymbol{\xi})\right]\\
&+(\alpha-1)\sum_{i=1}^{n}\log\left[1-G(x;\boldsymbol{\xi})^{\lambda}\right]-(\alpha+1)\sum_{i=1}^{n}\log\left[1-p\,G(x;\boldsymbol{\xi})^{\lambda}\right]. \qquad (6.1)
\end{aligned}$$

The log-likelihood function can be maximized either directly by using the SAS (PROC NLMIXED) or the Ox program (sub-routine MaxBFGS) [8] or by solving the nonlinear likelihood equations obtained by differentiating (6.1). The components of the score function $U_n(\Theta) = (\partial\ell_n/\partial p, \partial\ell_n/\partial\alpha, \partial\ell_n/\partial\lambda, \partial\ell_n/\partial\boldsymbol{\xi})^\top$ are

$$\frac{\partial\ell_n}{\partial p} = (\alpha+1)\sum_{i=1}^{n}\frac{G(x;\boldsymbol{\xi})^\lambda}{1-p\,G(x;\boldsymbol{\xi})^\lambda} - \frac{n}{1-p},$$

$$\frac{\partial\ell_n}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left[1-G(x;\boldsymbol{\xi})^\lambda\right] - \sum_{i=1}^{n}\log\left[1-p\,G(x;\boldsymbol{\xi})^\lambda\right],$$

$$\frac{\partial\ell_n}{\partial\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}\log\left[G(x;\boldsymbol{\xi})\right] - (\alpha-1)\sum_{i=1}^{n}\frac{G(x;\boldsymbol{\xi})^\lambda\log\left[G(x;\boldsymbol{\xi})\right]}{1-G(x;\boldsymbol{\xi})^\lambda}$$
$$+p\,(\alpha+1)\sum_{i=1}^{n}\frac{G(x;\boldsymbol{\xi})^\lambda\log\left[G(x;\boldsymbol{\xi})\right]}{1-p\,G(x;\boldsymbol{\xi})^\lambda} \qquad \text{and}$$

$$\frac{\partial\ell_n}{\partial\boldsymbol{\xi}} = p\,\lambda\,(\alpha+1)\sum_{i=1}^{n}\frac{G(x;\boldsymbol{\xi})^{\lambda-1}}{[1-p\,G(x;\boldsymbol{\xi})^\lambda]}G^{(\xi)}(x;\boldsymbol{\xi}) - \lambda(\alpha-1)\sum_{i=1}^{n}\frac{G(x;\boldsymbol{\xi})^{\lambda-1}}{[1-G(x;\boldsymbol{\xi})^\lambda]}G^{(\xi)}(x;\boldsymbol{\xi})$$
$$+\sum_{i=1}^{n}\frac{g^{(\xi)}(x;\boldsymbol{\xi})}{g(x;\boldsymbol{\xi})} + (\lambda-1)\sum_{i=1}^{n}\frac{G^{(\xi)}(x;\boldsymbol{\xi})}{G(x;\boldsymbol{\xi})},$$

where $h^{(\xi)}(\cdot)$ means the derivative of the function h with respect to ξ. For interval estimation on the model parameters, we require the observed information matrix

$$J_n(\Theta) = -\begin{pmatrix} U_{pp} & U_{p\alpha} & U_{p\lambda} & | & U_{p\boldsymbol{\xi}}^\top \\ U_{\alpha p} & U_{\alpha\alpha} & U_{\alpha\lambda} & | & U_{\alpha\boldsymbol{\xi}}^\top \\ U_{\lambda p} & U_{\lambda\alpha} & U_{\lambda\lambda} & | & U_{\lambda\boldsymbol{\xi}}^\top \\ -- & -- & -- & -- & -- \\ U_{\boldsymbol{\xi}p} & U_{\boldsymbol{\xi}\alpha} & U_{\boldsymbol{\xi}\lambda} & | & U_{\boldsymbol{\xi}\boldsymbol{\xi}} \end{pmatrix},$$

whose elements are listed in Appendix A. Let $\widehat{\Theta}$ be the MLE of Θ. Under standard regularity conditions [7] that are fulfilled for the proposed model whenever the parameters are in the interior of the parameter space, we can approximate the distribution of $\sqrt{n}(\widehat{\Theta}-\Theta)$ by the multivariate normal $N_r(0, K(\Theta)^{-1})$, where $K(\Theta) = \lim_{n\to\infty} J_n(\Theta)$ is the unit information matrix and r is the number of parameters of the new distribution.

Often with lifetime data and reliability studies, one encounters censoring. A very simple random censoring mechanism very often realistic is one in which each individual i is assumed to have a lifetime X_i and a censoring time C_i, where X_i and C_i are independent random variables. Suppose that the data consist of n independent observations $x_i = \min(X_i, C_i)$ and $\delta_i = I(X_i \le C_i)$ is such that $\delta_i = 1$ if X_i is a time to event and $\delta_i = 0$ if it is right censored for $i = 1, \ldots, n$. The censored likelihood $L(\Theta)$ for the model parameters is

$$L(\Theta) \propto \prod_{i=1}^{n}[f(x_i;p,\alpha,\lambda,\boldsymbol{\xi})]^{\delta_i}\,[S(x_i;p,\alpha,\lambda,\boldsymbol{\xi})]^{1-\delta_i},$$

where $f(x;p,\alpha,\lambda,\boldsymbol{\xi})$ is given by (2.1) and $S(x;p,\alpha,\lambda,\boldsymbol{\xi})$ is the survival function which comes from (1.1).

An easy way to validate the approximate normal distribution for $\widehat{\Theta}$ is by simulating a specific distribution of the new family of distribution. Here, the

ECCW model is selected as an example. We use equation (2.3) to simulate the ECCW($\beta = 0.5, c = 2, \alpha = 0.5, 2.0, \lambda = 0.2, p = 0.9, 0.1$) model by taking u as a uniform random variable in $(0, 1)$ for n = 50, 150 and 300. For each sample size, we evaluate the MLEs of the parameters. Then, we repeat this process 1,000 times and compute the averages of the estimates (AEs), biases and mean squared errors (MSEs). The simulation results are listed in Table 2.

Table 2. The AEs, biases and MSEs based on 1,000 simulations of the ECCW distribution when $\beta = 0.5$, $c = 2$, $\alpha = 0.5, 2.0$, $\lambda = 0.2$ and p=0.9,0.1, and n=50, 150 and 300.

		$\alpha = 2.0$ and $p = 0.9$				$\alpha = 2.0$ and $p = 0.3$		
n	Θ	AE	Bias	MSE	Θ	AE	Bias	MSE
50	β	0.5249	0.0249	0.0776	β	0.5352	0.0352	0.0257
	c	3.1990	1.1990	4.4361	c	2.0607	0.0607	0.0304
	α	2.6291	0.6291	6.4462	α	2.0393	0.0393	0.0534
	λ	0.2056	0.0056	0.0449	λ	0.2017	0.0017	0.0017
	p	0.7791	-0.1209	0.1169	p	0.2798	-0.0202	0.0491
150	β	0.5108	0.0108	0.0276	β	0.5048	0.0048	0.0067
	c	2.4565	0.4565	1.0313	c	2.0279	0.0279	0.0126
	α	2.3201	0.3201	2.0399	α	2.0570	0.0570	0.0248
	λ	0.1972	-0.0028	0.0062	λ	0.1998	-0.0002	0.0005
	p	0.8614	-0.0386	0.0125	p	0.3096	0.0096	0.0140
300	β	0.5041	0.0041	0.0133	β	0.5008	0.0008	0.0029
	c	2.2180	0.2180	0.3185	c	2.0181	0.0181	0.0086
	α	2.2463	0.2463	1.3088	α	2.0560	0.0560	0.0133
	λ	0.2014	0.0014	0.0032	λ	0.2001	0.0001	0.0002
	p	0.8826	-0.0174	0.0032	p	0.3111	0.0111	0.0064
		$\alpha = 0.5$ and $p = 0.9$				$\alpha = 0.5$ and $p = 0.3$		
n	Θ	AE	Bias	MSE	Θ	AE	Bias	MSE
50	β	0.4965	-0.0035	0.0060	β	0.4914	-0.0086	0.0069
	c	2.0099	0.0099	0.0286	c	2.0531	0.0531	0.0584
	α	0.5505	0.0505	0.0217	α	0.5467	0.0467	0.0167
	λ	0.2720	0.0720	0.0187	λ	0.2261	0.0261	0.0089
	p	0.8745	-0.0255	0.0026	p	0.2761	-0.0239	0.0550
150	β	0.4937	-0.0063	0.0029	β	0.4838	-0.0162	0.0022
	c	2.0132	0.0132	0.0124	c	2.0452	0.0452	0.0305
	α	0.5377	0.0377	0.0088	α	0.5344	0.0344	0.0056
	λ	0.2396	0.0396	0.0063	λ	0.2062	0.0062	0.0020
	p	0.8839	-0.0161	0.0009	p	0.3053	0.0053	0.0221
300	β	0.4967	-0.0033	0.0025	β	0.4881	-0.0119	0.0016
	c	2.0094	0.0094	0.0065	c	2.0380	0.0380	0.0203
	α	0.5326	0.0326	0.0057	α	0.5263	0.0263	0.0026
	λ	0.2342	0.0342	0.0041	λ	0.2031	0.0031	0.0010
	p	0.8855	-0.0145	0.0006	p	0.3080	0.0080	0.0123

The figures in Table 2 indicate that the MSEs and the biases of the estimated parameters decay toward zero when the sample size increases for all settings, as expected under first-under asymptotic theory. When n increases, the AEs of the parameters tend to be closer to the true parameter values. This fact supports that the asymptotic normal distribution provides an adequate approximation to the finite sample distribution of the MLEs.

7. Empirical illustrations

In this section, we compare the fits of some special models of the ECC-G family by means of two real data sets to show the potentiality of the new family. In order to estimate the parameters of these special models, we adopt the maximum likelihood method. All the computations were done using the subroutine `NLMixed` of the `SAS` software.

The first data set consists of fracture toughness from the silicon nitride. The data taken from the web-site `https://goo.gl/UMx3h9` was already studied by [19]. The ECC-G model used in the first application is defined by equation (3.2) with $\boldsymbol{\theta_1} = (\alpha, \beta, \lambda, c, p)$. Further, the *extended Cordeiro and de Castro-exponential* (ECCE) density function is given by (for $x > 0$)

$$f_2(x; \boldsymbol{\theta_2}) = \alpha\, \beta\, \lambda\, (1-p) \frac{[1-\exp(-\beta x)]^{\lambda-1}}{\exp(\beta x)} \frac{\left\{1-[1-\exp(-\beta x)]^{\lambda}\right\}^{\alpha-1}}{\left\{1-p\,[1-\exp(-\beta x)]^{\lambda}\right\}^{\alpha+1}},$$

where $\boldsymbol{\theta_2} = (\alpha, \beta, \lambda, p)$. These ECC-G models are compared with the Kumaraswamy Weibull (KwW) and beta Weibull (BW) models with corresponding densities (both for $x > 0$)

$$f_3(x; \boldsymbol{\theta_3}) = \alpha\, \lambda\, c\, \beta^c\, x^{c-1}\, \mathrm{e}^{-(\beta x)^c} \left[1-\mathrm{e}^{-(\beta x)^c}\right]^{\lambda-1} \left\{1-\left[1-\mathrm{e}^{-(\beta x)^c}\right]^{\lambda}\right\}^{\alpha-1},$$

and

$$f_4(x; \boldsymbol{\theta_4}) = \frac{c\, \lambda^c}{B(a,b)}\, x^{c-1} \exp\left[-b(\lambda x)^c\right] \left[1-\mathrm{e}^{-(\lambda x)^c}\right]^{a-1},$$

where $\boldsymbol{\theta_3} = (\alpha, \beta, \lambda, c)$ and $\boldsymbol{\theta_4} = (a, b, c, \lambda)$.

As a second application, we consider a real data set on the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England, see [26]. We fit the ECCW and ECCE models to these data. These models are compared with the BW model and beta Birnbaum-Saunders (BBS) model (for $x > 0$) defined by

$$f_5(x; \boldsymbol{\theta_5}) = \frac{\kappa(\alpha,\beta)}{B(a,b)}\, x^{-3/2}(x+\beta) \exp\left[-\tau(x/\beta)/(2\alpha^2)\right] \Phi(\nu)^2 \left[1-\Phi(\nu)\right]^{b-1},$$

where $\nu = \alpha^{-1}\rho(x/\beta)$, $\rho(z) = z^{1/2}+z^{-1/2}$, $\kappa(\alpha,\beta) = \exp(\alpha^{-2})/(2\alpha\sqrt{2\pi\beta})$, $\tau(z) = z+z^{-1}$, $\Phi(\cdot)$ is the standard normal cumulative function and $\boldsymbol{\theta_5} = (a, b, \alpha, \beta)$. We also compere, in both applications, the results by fitting standard distributions such as Weibull ($\boldsymbol{\theta_6} = (c, \beta)$), gamma ($\boldsymbol{\theta_7} = (a, b)$) and log-logistic ($\boldsymbol{\theta_8} = (a, b)$) distributions as well as the beta exponentiated Weibull (BEW) ($\boldsymbol{\theta_9} = (\lambda, c, \alpha, a, b)$) distribution [5], which is defined by (for $x > 0$)

$$f_9(x; \boldsymbol{\theta_9}) = \frac{\alpha\, c\, \lambda^c}{B(a,b)} x^{c-1} \mathrm{e}^{-(\lambda x)^c} (1-\mathrm{e}^{-(\lambda x)^c})^{a\alpha-1} \{1-(1-\mathrm{e}^{-(\lambda x)^c})^{\alpha}\}^{b-1}.$$

The MLEs of the parameters and their standards errors are given in Table 3. We also perform formal goodness-of-fit tests in order to verify which distribution fits better to these data. We apply the Cramér-von Mises (W^*) and Arderson-Darling (A^*) statistics. The W^* and A^* statistics are described in details in [3]. In general, the smaller the values of W^* and A^*, the better the fit to the data. Table 4 gives the values of these statistics for the first and second data sets. According to them, the ECCW model fits the first data set better than the others competing models.

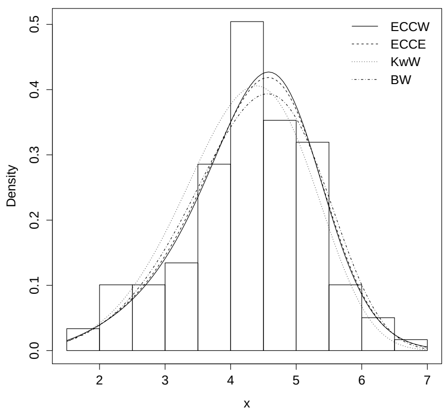
The figures in Table 3 for the second data set indicate that the ECCW model is a very competitive model to the other fitted models to these data, although it does not give the smallest AIC. However, the smallest values of the W^* and A^* statistics in Table 4 indicate that the ECCW model provides a more adequate fit to these data than the other distributions. Overall, these results illustrate the potentiality of the ECCW model for lifetime data and the importance of its additional parameters.

Table 3. Estimates (^{a} denotes standard errors).

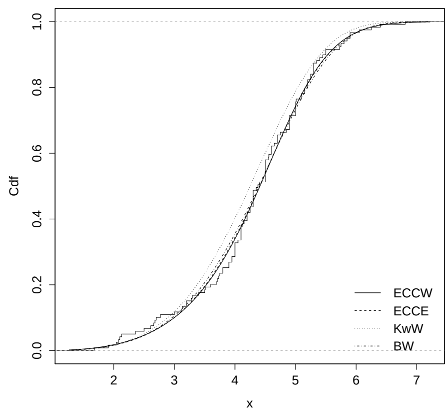
Data set	Distribution	Estimates
1	ECCW	$\widehat{\boldsymbol{\theta}}_1 = (1.9217, 0.5422, 4.3091, 1.5526, 0.9589)$
$(n = 119)$		$(3.7100, 1.3219, 17.9943, 2.6544, 0.1602)^a$
	ECCE	$\widehat{\boldsymbol{\theta}}_2 = (3.0949, 1.2853, 16.8832, 0.9848)$
		$(2.2243, 0.2272, 10.6914, 0.0137)^a$
	KwW	$\widehat{\boldsymbol{\theta}}_3 = (7.0242, 0.1450, 0.8329, 5.8042)$
		$(2.4542, 0.1655, 0.5797, 3.4765)^a$
	BW	$\widehat{\boldsymbol{\theta}}_4 = (5.6663, 0.1634, 0.8054, 3.4077)$
		$(0.5568, 0.3708, 0.0067, 3.5823)^a$
	Weibull	$\widehat{\boldsymbol{\theta}}_6 = (4.9909, 0.2121)$
		$(0.3576, 0.0040)^a$
	gamma	$\widehat{\boldsymbol{\theta}}_7 = (15.5335, 3.5913)$
		$(1.9925, 0.4681)^a$
	log-logistic	$\widehat{\boldsymbol{\theta}}_8 = (4.3226, 7.0597)$
		$(0.0959, 0.5531)^a$
	BEW	$\widehat{\boldsymbol{\theta}}_9 = (0.0964, 5.7762, 0.9848, 0.8015, 66.5370)$
		$(0.1366, 16.7231, 3.1581, 0.4255, 623.5624)^a$
2	ECCW	$\widehat{\boldsymbol{\theta}}_1 = (0.9367, 0.8276, 0.6341, 3.6621, 0.9420)$
$(n = 51)$		$(0.7763, 0.2411, 0.7032, 2.2503, 0.1053)^a$
	ECCE	$\widehat{\boldsymbol{\theta}}_2 = (3.6519, 4.5125, 11.1218, 0.9978)$
		$(2.7515, 0.7429, 22.1433, 0.0049)^a$
	BW	$\widehat{\boldsymbol{\theta}}_4 = (7.0127, 0.9199, 0.4493, 0.0496)$
		$(0.1867, 0.0484, 0.8872, 0.1522)^a$
	BBS	$\widehat{\boldsymbol{\theta}}_5 = (0.3638, 7857.5658, 1.0505, 30.4783)$
		$(0.1517, 2558.5670, 0.2506, 18.1233)^a$
	Weibull	$\widehat{\boldsymbol{\theta}}_6 = (5.2655, 0.6394)$
		$(0.5648, 0.0178)^a$
	gamma	$\widehat{\boldsymbol{\theta}}_7 = (16.2574, 11.2742)$
		$(3.1869, 2.2445)^a$
	log-logistic	$\widehat{\boldsymbol{\theta}}_8 = (1.4571, 7.5383)$
		$(0.04564, 0.9253)^a$
	BEW	$\widehat{\boldsymbol{\theta}}_9 = (0.7971, 5.7669, 0.0109, 48.8659, 0.2552)$
		$(0.0205, 0.0213, 0.0156, 71.7566, 0.0538)^a$

Table 4. Goodness-of-fit tests

Data set	Model	Statistics	
		W^*	A^*
1	ECCW	0.0469	0.3116
	ECCE	0.0577	0.5629
	KwW	0.0784	0.6064
	BW	0.1967	1.3748
	Weibull	0.0916	0.5616
	gamma	0.3883	2.3501
	log-logistic	0.3547	2.2304
	BEW	0.0829	0.5020
2	ECCW	0.1824	1.1636
	ECCE	0.2350	1.3730
	BW	0.2390	1.3750
	BBS	0.3651	1.9727
	Weibull	0.2362	1.2984
	gamma	0.5683	3.1173
	log-logistic	0.4969	2.7488
	BEW	0.2719	1.5476



(a)



(b)

Figure 9. Estimated (a) pdf and (b) cdf for the ECCW, ECCE, KwW and BW models for the first data set.

8. Concluding remarks

We define a new family of distributions, called the *extended Cordeiro and de Castro* (ECC-G) family of distributions, which generalizes several well-known distributions in the statistical literature such as the normal, Weibull and beta distributions by adding three shape parameters. We provide a mathematical treatment

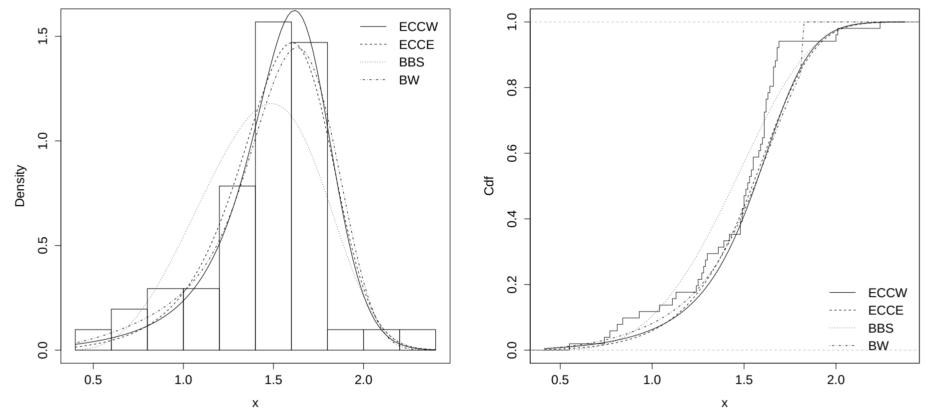


Figure 10. Estimated (a) pdf and (b) cdf for the ECCW, ECCE, BBS and BW models for the second data set.

of the new family including expansions for the density function, moments, generating function and incomplete moments. The ECC-G density function can be expressed as a linear combination of exponentiated density functions. This property is important to obtain several other structural results. We derive a power series for the quantile function of this family. Our formulas related with the ECC-G model are manageable, and with the use of modern computer resources with analytic and numerical capabilities, they may turn into adequate tools comprising the arsenal of applied statisticians. Some special models are studied in some detail. The estimation of the model parameters is approached by the method of maximum likelihood. The observed information matrix is derived. Finally, we fit the ECC-G models to two real data sets to demonstrate the potentiality of the proposed family.

Appendix A. Observed information matrix

The elements of the $r \times r$ observed information matrix $J_n(\Theta)$ are

$$U_{pp} = (\alpha+1)\sum_{i=1}^{n} \frac{G(x_i;\boldsymbol{\xi})^{2\lambda}}{\left[1-p\,G(x_i;\boldsymbol{\xi})^{\lambda}\right]^2} - \frac{n}{(1-p)^2}, \quad U_{p\alpha} = \sum_{i=1}^{n} \frac{G(x_i;\boldsymbol{\xi})^{\lambda}}{1-p\,G(x_i;\boldsymbol{\xi})^{\lambda}},$$

$$U_{p\lambda} = (\alpha+1)\sum_{i=1}^{n} \frac{G(x_i;\boldsymbol{\xi})^{\lambda}\log G(x_i;\boldsymbol{\xi})}{\left[1-p\,G(x_i;\boldsymbol{\xi})^{\lambda}\right]^2}, \quad U_{p\xi} = \lambda(\alpha+1)\sum_{i=1}^{n} \frac{G(x_i;\boldsymbol{\xi})^{\lambda-1}\,G^{(\xi)}(x;\boldsymbol{\xi})}{\left[1-p\,G(x_i;\boldsymbol{\xi})^{\lambda}\right]^2},$$

$$U_{\alpha\alpha} = -\frac{n}{\alpha^2}, \quad U_{\alpha\lambda} = p\sum_{i=1}^{n} \frac{G(x_i;\boldsymbol{\xi})^{\lambda}\log G(x_i;\boldsymbol{\xi})}{1-p\,G(x_i;\boldsymbol{\xi})^{\lambda}} - p\sum_{i=1}^{n} \frac{G(x_i;\boldsymbol{\xi})^{\lambda}\log G(x_i;\boldsymbol{\xi})}{1-G(x_i;\boldsymbol{\xi})^{\lambda}},$$

$$U_{\alpha\boldsymbol{\xi}} = \lambda\,p\sum_{i=1}^{n} \frac{G(x_i;\boldsymbol{\xi})^{\lambda-1}\,G^{(\xi)}(x;\boldsymbol{\xi})}{1-p\,G(x_i;\boldsymbol{\xi})^{\lambda}} - \lambda\sum_{i=1}^{n} \frac{G(x_i;\boldsymbol{\xi})^{\lambda-1}\,G^{(\xi)}(x;\boldsymbol{\xi})}{1-G(x_i;\boldsymbol{\xi})^{\lambda}},$$

$$
\begin{aligned}
U_{\lambda\lambda} &= -\frac{n}{\lambda^2} - \lambda(\alpha-1)\sum_{i=1}^{n}\frac{G(x_i;\boldsymbol{\xi})^{\lambda-1}\,G^{(\xi)}(x;\boldsymbol{\xi})}{\left[1-G(x_i;\boldsymbol{\xi})^{\lambda}\right]^2}\log G(x_i;\boldsymbol{\xi})\\
&\quad + \lambda\,p\,(\alpha+1)\sum_{i=1}^{n}\frac{G(x_i;\boldsymbol{\xi})^{\lambda-1}\,G^{(\xi)}(x;\boldsymbol{\xi})}{\left[1-p\,G(x_i;\boldsymbol{\xi})^{\lambda}\right]^2}\log G(x_i;\boldsymbol{\xi}),\\
U_{\lambda\boldsymbol{\xi}} &= \sum_{i=1}^{n}\frac{G^{(\xi)}(x;\boldsymbol{\xi})}{G(x_i;\boldsymbol{\xi})} - (\alpha-1)\sum_{i=1}^{n}\frac{G(x_i;\boldsymbol{\xi})^{\lambda-1}\,G^{(\xi)}(x;\boldsymbol{\xi})\left[1-G(x_i;\boldsymbol{\xi})^{\lambda}+\log G(x_i;\boldsymbol{\xi})^{\lambda}\right]}{\left[1-G(x_i;\boldsymbol{\xi})^{\lambda}\right]^2}\\
&\quad + p\,(\alpha+1)\sum_{i=1}^{n}\frac{G(x_i;\boldsymbol{\xi})^{\lambda-1}\,G^{(\xi)}(x;\boldsymbol{\xi})\left[1-G(x_i;\boldsymbol{\xi})^{\lambda}+\log G(x_i;\boldsymbol{\xi})^{\lambda}\right]}{\left[1-p\,G(x_i;\boldsymbol{\xi})^{\lambda}\right]^2}
\end{aligned}
$$

and

$$
\begin{aligned}
U_{\boldsymbol{\xi\xi}} &= (\lambda-1)\sum_{i=1}^{n}\frac{G(x_i;\boldsymbol{\xi})\,G^{(2\xi)}(x;\boldsymbol{\xi})}{G^{(\xi)}(x_i;\boldsymbol{\xi})^2} + \sum_{i=1}^{n}\frac{g(x_i;\boldsymbol{\xi})\,g^{(2\xi)}(x;\boldsymbol{\xi})}{g^{(\xi)}(x_i;\boldsymbol{\xi})^2} - n\lambda\\
&\quad + \lambda\,(\alpha-1)\sum_{i=1}^{n}\frac{G(x_i;\boldsymbol{\xi})^{\lambda-2}\,G^{(\xi)}(x;\boldsymbol{\xi})^2}{\left[1-G(x_i;\boldsymbol{\xi})^{\lambda}\right]^2} - \sum_{i=1}^{n}\frac{G(x_i;\boldsymbol{\xi})^{\lambda-2}\,G^{(\xi)}(x;\boldsymbol{\xi})^2}{1-G(x_i;\boldsymbol{\xi})^{\lambda}}\left(\frac{G(x_i;\boldsymbol{\xi})}{G^{(\xi)}(x;\boldsymbol{\xi})}\right)^{(\xi)}\\
&\quad - \lambda\,p\,(\alpha+1)\sum_{i=1}^{n}\frac{G(x_i;\boldsymbol{\xi})^{\lambda-2}\,G^{(\xi)}(x;\boldsymbol{\xi})^2}{\left[1-p\,G(x_i;\boldsymbol{\xi})^{\lambda}\right]^2} - \sum_{i=1}^{n}\frac{G(x_i;\boldsymbol{\xi})^{\lambda-2}\,G^{(\xi)}(x;\boldsymbol{\xi})^2}{1-p\,G(x_i;\boldsymbol{\xi})^{\lambda}}\left(\frac{G(x_i;\boldsymbol{\xi})}{G^{(\xi)}(x;\boldsymbol{\xi})}\right)^{(\xi)},
\end{aligned}
$$

where $h^{(2\xi)}(\cdot)$ denotes the second derivative of the function h with respect to ξ.

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