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**DOCTORAL DISSERTATION**

# Gevrey asymptotic properties of invariant manifolds in slow-fast systems

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# Contents

# Introduction

This thesis considers singular perturbation problems, which arise in the study of slow-fast systems. In their standard form, slow-fast systems are represented by

$$\left\{\begin{array}{rcl} \dot{X}(t) & = & \varepsilon F(X, Z, \varepsilon) \\ \dot{Z}(t) & = & G(X, Z, \varepsilon) \end{array}\right., \tag{0.0.1}$$

where $(X, Z) \in \mathbb{R}^n \times \mathbb{R}^m$ and the parameter $\varepsilon$ is typically thought of as being a very small positive real number. The presence of $\varepsilon$ induces a time scale separation between the "slow" variables $X$ whose evolution is slow in comparison with that of the "fast" variables $Z$.
There are two different manners in which to reduce a slow-fast system in the singular limit $\varepsilon = 0$. Keeping the formulation of (0.0.1) in fast time $t$ and setting $\varepsilon = 0$ produces the fast subsystem, also called the layer equations,

$$\left\{\begin{array}{rcl} \dot{X}(t) & = & 0 \\ \dot{Z}(t) & = & G(X, Z, 0) \end{array}\right..$$

However one can rescale time to the slow time $\tau = \varepsilon t$ in which the system takes the form

$$\left\{\begin{array}{rcl} \dot{X}(\tau) & = & F(X, Z, \varepsilon) \\ \varepsilon\dot{Z}(\tau) & = & G(X, Z, \varepsilon) \end{array}\right..$$

Taking the singular limit here gives us the slow subsystem, which is a differential-algebraic equation,

$$\left\{\begin{array}{rcl} \dot{X}(\tau) & = & F(X, Z, 0) \\ 0 & = & G(X, Z, 0) \end{array}\right..$$

It is often assumed, as will be throughout this thesis, that the solution set of $0 = G(X, Z, 0)$, which is also the set of equilibria of the layer equations, is given by an $n$-dimensional manifold, called the critical manifold. The slow subsystem then generates a flow on this manifold, called the slow flow.

A large part of this thesis will be dedicated to the persistence of the critical manifold under small perturbations of $\varepsilon$. Said roughly, we investigate the existence and properties of an $\varepsilon$-family of locally invariant manifolds, $S_\varepsilon$, of the full system, which tend

to (a part of) the critical manifold for $\varepsilon \to 0$. Such a family of manifolds is called a slow manifold.
There are the classical results by Fenichel in [Fen79] concerning these slow manifolds. If a compact submanifold of the critical manifold is normally hyperbolic, meaning that, as equilibria of the fast subsystem/layer equations, all points of the submanifold are hyperbolic, a slow manifold is guaranteed to perturb from this set. However, much like center manifolds obtained through the center manifold theorem, the slow manifold will, in general, not be unique. Moreover, even in the case when the considered slow-fast system is real analytic, Fenichel's results can only guarantee the existence of slow manifolds up to any finite degree of smoothness.

One aim in this thesis is to improve the results of Fenichel in certain areas. We will achieve this by employing the theory of Gevrey asymptotic expansions. The basics of this theory were developed by Watson [Wat12a, Wat12b] and Nevanlinna [Nev18] as a means to associate a unique "sum" to a class of divergent series. The application of Gevrey expansions in differential equations was pioneered by Ramis, [Ram78, Ram80].

The approach is to start of from a formal point of view, constructing formal power series in the singular parameter $\varepsilon$, with as coefficients functions of the slow variables $X$, that are formally invariant under the flow of (0.0.1). As a first result we obtain that these formal manifolds exist, at points of the critical manifold where the differential $D_Z G$ is invertible (a fast-slow regular point to follow the terminology in [Kue15]). These series are, in general, not convergent but divergent of Gevrey type, this is not surprising as it generalizes a result achieved by Sibuya, [Sib90] for one slow variable, to an arbitrary amount of slow variables.
Our subsequent course of action depends on the type of point on the critical manifold around which we wish to perturb a slow manifold. If the point is not a singularity of the slow flow, we achieve, without imposing extra assumptions, the existence of a local slow manifold which has a Gevrey expansion. This is an improvement of the classical Fenichel results in the sense that we do not demand normal hyperbolicity but only fast-slow regularity. This includes the case where the fast spectrum is purely imaginary (the slow manifolds are then occasionally referred to as elliptic manifolds see for example [Van08, LZ11]). In particular we can handle the case where the critical manifold undergoes a change of stability through an elliptic point. In the terminology of [CDRSS00], these are overstable solutions. This result is again a generalization of a result of Sibuya for one slow variable, [Sib58]. Moreover, functions with a Gevrey expansion are $\mathcal{C}^\infty$ smooth (even a bit stronger) which is also an improvement over the classical result.
If we are interested in a singularity of the slow flow, we actually achieve stronger

results, but at the cost of having to impose extra conditions. More specifically we will assume that the fast variable is one dimensional and the linearisation of the slow flow around the singularity is either attractive or repelling (for real analytic systems). Under these conditions we achieve that the formal Gevrey manifold is summable in a direction. This means that, on top of all the properties that a manifold with Gevrey expansion has, the manifold is in a sense unique, there is a "best" manifold realizing the divergent series.
We do need to remark that, contrary to the classical Fenichel theory, our results are local in nature, except for the study of the formal series which can be conducted on compact sets.

In a second part of the thesis we present two results in slow-fast systems with one slow and one fast variable. A first result is on the saturation of summable slow manifolds along the critical curve while maintaining summability. A second one concerns the connection of summable slow manifolds over a turning point, where there occurs a change of stability of the critical curve.
We can apply these results to a system of one slow and one fast variable which satisfies a particular configuration. The critical curve has an attracting and repelling part, where the change of stability is through a turning point, and the slow flow is directed from the attracting to the repelling part. Moreover, on the attracting part the slow flow has a repelling equilibrium while the repelling part has an attracting equilibrium, i.e. both parts of the curve have a slow-fast saddle. By our local results, a summable slow manifold perturbs from each of these saddles. These manifolds are then saturated towards the turning point and, by introducing a parameter in our system, can be matched to each other over the turning point creating canard solutions connecting the two saddles. Moreover, these solutions will still have the summability property away from the turning point.

In a last part we consider a system of delay differential equations which models neuron interaction, found in [KT16]. We use this model as an example to corroborate that Gevrey asymptotic techniques are also viable to construct invariant manifolds in the more general setting of functional differential equations. We achieve quasi-solutions, approximating slow manifolds up to an exponentially small error. The step from the quasi-solutions to actual slow manifolds is not made in this thesis and could be a possible future topic of research.

The thesis is structured as follows.
In chapter 1 we introduce basic notions and results concerning Gevrey series and asymptotics.

Chapter 2 concerns the complete Gevrey analysis of slow manifolds. In section 2.1 the formal Gevrey slow manifolds are constructed for a very general class of systems.

Section 2.2 then details the construction of a Gevrey slow manifold around a regular point of the slow flow, still for the same class of systems. The construction essentially entails the application of a Borel-Laplace resummation procedure to the formal solution which gives a manifold which is invariant "up to an exponentially small error". One can then carefully remove this error by a procedure reminiscent of the proof of the Cauchy-Kowalevski theorem.
In the last section 2.3, summability is proven under extra conditions. This is done by relating summability of a formal series to the extension of its Borel transform to infinity. We prove such an extension exists by a fixed point argument. We also provide examples showing that summability fails when our imposed conditions are not satisfied.

In chapter 3, we start of by showing that, in a system with one slow and one fast variable, a summable slow manifold can be saturated along a normally hyperbolic part of the critical curve by means of the slow flow, while maintaining the summability. We remark that such a summable manifold does not necessarily need to arise from one of the cases described in section 2.3, it could for example also arise from another type of doubly singular equation as in [CDMFS07].
Secondly we show that the summable manifolds can be connected to each other in a blow up of the turning point. Combining the results of this chapter and section 2.3 gives rise to "canard heteroclinic saddle connections".

The last chapter 4 concerns delay equations and we employ a toy model to exhibit Gevrey techniques in these types of equations. An important aspect of this chapter is dedicated to proving that a "naive" characterization of slow manifolds in delay equations is actually correct.

The results in this thesis are a collection of those in [Ken16, DMK19, DMK].

# Chapter 1

# Preliminaries

We state, together with fixing some notation, the Cauchy inequalities for holomorphic functions of several variables. These inequalities will be used throughout the thesis on numerous occasions. Next we introduce Gevrey formal series and Gevrey asymptotic functions together with some basic results.

## 1.1 The Cauchy inequalities

We will use the following notations.

- For a metric space $(X, d)$, an $x \in X$ and $r > 0$ we denote the open and closed balls around $x$ with radius $r$ by

$$B(x, r), \quad \text{resp.} \quad \overline{B}(x, r).$$

- Let $n \in \mathbb{N}_0$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ a multi-index and $X = (x_1, \ldots, x_n) \in \mathbb{C}^n$. We denote

$$|\alpha| = \alpha_1 + \ldots + \alpha_n$$
$$\alpha! = \alpha_1! \cdot \ldots \cdot \alpha_n!$$
$$X^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}.$$

- For $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$ and $X \in \mathbb{C}^n$ we define the open and closed polydisks centred at $X$ with polyradius $r$ by

$$\mathbb{P}_n(X, r) = B(x_1, r_1) \times \ldots \times B(x_n, r_n)$$

resp.

$$\overline{\mathbb{P}}_n(X, r) = \overline{B}(x_1, r_1) \times \ldots \times \overline{B}(x_n, r_n).$$

If the centre is the origin, we denote the polydisks simply by $\mathbb{P}_n(r)$.

For $R > 0$ the notation $\mathbb{P}_n(X, R)$ means that we consider the polyradius $(R, \ldots, R)$.

- For $r$ a polyradius and $X \in \mathbb{C}^n$, $\partial_0 \mathbb{P}_n(X, r) = \partial B(x_1, r_1) \times \ldots \times \partial B(x_n, r_n)$ or equivalently

$$\partial_0 \mathbb{P}_n(X, r) = \{Y \in \mathbb{C}^n \mid |y_j - x_j| = r_j, \forall j = 1, \ldots, n\}.$$

- Let $V \subset \mathbb{C}^n$, not necessarily open, we say that $f \in \mathcal{O}(V)$ if there exists an open $W \subset \mathbb{C}^n$ with $V \subset W$ such that $f \colon W \to \mathbb{C}$ is holomorphic on $W$. If $V$ is open, one can of course take $W = V$.

- Let $s \in \mathbb{N}_0$ and $f = (f_1, \ldots, f_n) : V \to \mathbb{C}^s$. Then $f \in \mathcal{O}(V, \mathbb{C}^s)$ if and only if $f_j \in \mathcal{O}(V)$ for all $j = \{1, \ldots, n\}$.

We can now state the Cauchy inequalities, for a proof one can consult, for example, [Hor73].

**Lemma 1.1.1.** *Let $r \in \mathbb{R}^n_{>0}$, $\alpha \in \mathbb{N}^n$ and $X \in \mathbb{C}^n$, suppose that*

$$f \in \mathcal{O}(\mathbb{P}_n(X, r), \mathbb{C}^s) \cap \mathcal{C}\left(\overline{\mathbb{P}}_n(X, r), \mathbb{C}^s\right).$$

*If we denote by $\|\cdot\|_{\max}$ the maximum norm on $\mathbb{C}^s$,*

$$\left\| \frac{\partial^{|\alpha|} f}{\partial X^\alpha}(X) \right\|_{\max} \leqslant \frac{\alpha!}{r^\alpha} \sup_{Y \in \partial_0 \mathbb{P}_n(x,r)} \|f(Y)\|_{\max}.$$

## 1.2 A short introduction to Gevrey asymptotics

We introduce Gevrey asymptotic expansions and present some basic results concerning them. Throughout the literature, the definitions of Gevrey series and expansions are not uniform, several slight alterations on the definition we give here can be found.

### 1.2.1 Gevrey formal series

**Definition 1.2.1.** *Let $V \subset \mathbb{C}^\ell$ be open, $\ell \in \mathbb{N}_0$, and let $s \in \mathbb{N}_0$, $m \geqslant 0$, $B > 0$. Consider a formal series of the form*

$$\widehat{f}(X, \varepsilon) = \sum_{k=0}^{\infty} f_k(X) \varepsilon^k,$$

*with $f_k \in \mathcal{O}(V, \mathbb{C}^s)$, i.e. $\widehat{f} \in \mathcal{O}(V, \mathbb{C}^s)[\![\varepsilon]\!]$. We say that $\widehat{f}$ is Gevrey-$m$ of type $B$ in $\varepsilon$, uniformly for $X$ in $V$, if there exists $A > 0$ such that*

$$\sup_{X \in V} \|f_k(X)\|_{\max} \leqslant AB^k \Gamma(1 + mk).$$

*Here $\|\cdot\|_{\max}$ denotes the maximum norm on $\mathbb{C}^s$.*

**Remark 1.2.2.** *Notice that a Gevrey-0 series is convergent for $(X, \varepsilon) \in V \times B\left(0, \frac{1}{B}\right)$ and thus induces a holomorphic function on this subset.*

### 1.2.2 Gevrey asymptotic functions

**Notation 1.2.3.** *For $\theta \in [0, 2\pi[$, $\delta \in ]0, 2\pi[$ and $r > 0$ we denote the (open) sector in the direction $\theta$ with opening $\delta$ and radius $r$ by*

$$S(\theta, \delta, r) = \left\{ z \in \mathbb{C} \;\middle|\; 0 < |z| < r,\ \mathrm{Arg}(ze^{-i\theta}) \in \left] -\tfrac{\delta}{2}, \tfrac{\delta}{2} \right[ \right\}.$$

*The infinite sector $\bigcup_{r>0} S(\theta, \delta, r)$ in the direction $\theta$ is denoted by $S(\theta, \delta)$.*
*While we will only concern ourselves with sectors of opening smaller than $2\pi$, as above, sectors with a larger opening can be considered as subsets of the Riemann surface of the logarithm.*

**Definition 1.2.4.** *Consider some open sector $S$ and a subset $V \subset \mathbb{C}^\ell$, $\ell, s \in \mathbb{N}_0$ and $m \geqslant 0$. Let $\widehat{f}(X, \varepsilon) = \sum_{n=0}^{\infty} f_n(X) \varepsilon^n \in \mathcal{O}(V, \mathbb{C}^s)[\![\varepsilon]\!]$. We say that a function $f(X, \varepsilon)$, holomorphic on $V \times S$, is Gevrey-$m$ asymptotic to the formal series $\widehat{f}(X, \varepsilon)$, with respect to $\varepsilon$, uniformly for $X \in V$, if for every $\varepsilon \in S$ and every $N \in \mathbb{N}_0$ we have*

$$\sup_{X \in V} \left\| f(X, \varepsilon) - \sum_{n=0}^{N-1} f_n(X) \varepsilon^n \right\|_{\max} \leqslant C D^N \Gamma(1 + mN) |\varepsilon|^N$$

*for certain $C, D > 0$. We denote this by*

$$f(X, \varepsilon) \sim_m \widehat{f}(X, \varepsilon)$$

**Remark 1.2.5.** *It is not demanded, a priori, in the above definition that the formal series $\widehat{f}$ is Gevrey-1. However, this series gains the Gevrey property immediately from the fact that some function is Gevrey-1 asymptotic to it. Indeed, for all $X \in V$,*

$$\begin{aligned} \|f_k(X)\|_{\max} |\varepsilon|^k &\leqslant \left\| \sum_{n=0}^{k} f_n(X) \varepsilon^n - f(X, \varepsilon) \right\|_{\max} + \left\| f(X, \varepsilon) - \sum_{n=0}^{k-1} f_n(X) \varepsilon^n \right\|_{\max} \\ &\leqslant C D^{k+1} \Gamma(1 + m(k+1)) |\varepsilon|^{k+1} + C D^k \Gamma(1 + mk) |\varepsilon|^k. \end{aligned}$$

*Dividing both sides of the inequality by $|\varepsilon|^k$ and then setting $\varepsilon = 0$, shows that the coefficients all satisfy the Gevrey bounds.*

**Remark 1.2.6.** *If $f(X, \varepsilon) \sim_m \widehat{f}(X, \varepsilon)$ one can, for a fixed value $\varepsilon_* \in S$, approximate $f(X, \varepsilon_*)$ up to an exponential accuracy by a well-chosen truncation of the formal series $\widehat{f}$. Indeed, denote*

$$N_* = \left\lfloor \frac{(D |\varepsilon_*|)^{-1/m}}{m} \right\rfloor.$$

*Clearly*

$$\frac{(D |\varepsilon_*|)^{-1/m}}{m} - 1 < N_* \leqslant \frac{(D |\varepsilon_*|)^{-1/m}}{m},$$

*and by Stirling's formula, see [Sti],*

$$\Gamma(1 + z) = \sqrt{2\pi} z^{z + \frac{1}{2}} e^{-z} \left(1 + \mathrm{o}(1)\right), \text{ for } z \to \infty, |\arg(z)| < \pi.$$

*By definition of a Gevrey-$m$ asymptotic expansion we then have,*

$$\begin{aligned}
&\sup_{X\in V}\left\| f(X,\varepsilon)-\sum_{n=0}^{N_*-1} f_n(X)\,\varepsilon^n\right\|_{\max}\\
&\leqslant CD^{N_*}\Gamma(1+mN_*)\,|\varepsilon|^{N_*}\\
&=\sqrt{2}C\,(D\,|\varepsilon_*|)^{N_*}\,(mN_*)^{mN_*+\frac{1}{2}}\,e^{-mN_*}\,(1+\mathrm{o}(1))\\
&<\sqrt{2}C\,(D\,|\varepsilon_*|)^{N_*}\,(D\,|\varepsilon_*|)^{-N_*}\,(D\,|\varepsilon_*|)^{-\frac{1}{2m}}\,e^{m-\frac{1}{(D|\varepsilon_*|)^{1/m}}}\,(1+\mathrm{o}(1))\\
&=\sqrt{2}Ce^m\,(D\,|\varepsilon_*|)^{-\frac{1}{2m}}\,e^{-\frac{1}{(D|\varepsilon_*|)^{1/m}}}\,(1+\mathrm{o}(1))\,.
\end{aligned}$$

*Where* $\mathrm{o}(1)$ *is for* $N_*\to\infty$ *or equivalently* $|\varepsilon_*|\to 0$.

It is obvious that the class of functions with a Gevrey-$m$ expansion is closed under addition and scalar multiplication. In the following interpretation it is also closed under differentiation.

**Proposition 1.2.7.** *Suppose* $f(X,\varepsilon)$ *is defined on* $V\times S(\theta,2\delta_1,r_1)$ *with* $f\sim_m \hat{f}$.

- *Suppose* $W\subset V$ *with* $d\left(W,V^C\right)=R>0$, *where the distance is measured with the maximum metric. For all* $\alpha\in\mathbb{N}^\ell$ *we have* $\frac{\partial^{|\alpha|}f}{\partial X^\alpha}\sim_m\frac{\partial^{|\alpha|}\hat{f}}{\partial X^\alpha}$ *w.r.t.* $\varepsilon\in S(\theta,2\delta_1,r_1)$, *uniformly for* $X\in W$.

- *If* $0<\delta_2<\delta_1$, *there then exists an* $r_2<r_1$ *such that for all* $k\in\mathbb{N}$, $\frac{\partial^k f}{\partial\varepsilon^k}\sim_m\frac{\partial^k\hat{f}}{\partial\varepsilon^k}$ *w.r.t.* $\varepsilon\in S(\theta,2\delta_2,r_2)$, *uniformly for* $X\in V$

**Proof:**

- Due to the Cauchy inequalities, we have for all $X\in W$, $\varepsilon\in S(\theta,2\delta_1,r_1)$ and $N\in\mathbb{N}_0$,

$$\begin{aligned}
&\left\|\frac{\partial^{|\alpha|}f}{\partial X^\alpha}(X,\varepsilon)-\sum_{n=0}^{N-1}\frac{\partial^{|\alpha|}f_n}{\partial X^\alpha}(X)\,\varepsilon^n\right\|_{\max}\\
&=\left\|\frac{\partial^{|\alpha|}}{\partial X^\alpha}\left(f(X,\varepsilon)-\sum_{n=0}^{N-1}f_n(X)\,\varepsilon^n\right)\right\|_{\max}\\
&\leqslant\alpha!\left(\frac{2}{R}\right)^{|\alpha|}\sup_{X\in V}\left\|f(X,\varepsilon)-\sum_{n=0}^{N-1}f_n(X)\,\varepsilon^n\right\|_{\max}\\
&\leqslant\alpha!\left(\frac{2}{R}\right)^{|\alpha|}CD^N\Gamma(1+mN)\,|\varepsilon|^N\,.
\end{aligned}$$

- Denote $r_2=\frac{r_1}{1+\sin(\delta_1-\delta_2)}$, one can verify that for all $\varepsilon\in S(\theta,2\delta_2,r_2)$,

$$\overline{B}(\varepsilon,|\varepsilon|\sin(\delta_1-\delta_2))\subset S(\theta,2\delta_1,r_1)\,.$$

By the Cauchy inequalities and the definition of Gevrey asymptotics, this implies for all $X \in V$, $\varepsilon \in S(\theta, 2\delta_2, r_2)$ and $N \in \mathbb{N}_0$ that

$$
\begin{aligned}
& \left\| \frac{\partial^k f}{\partial \varepsilon^k}(X,\varepsilon) - \sum_{n=0}^{N-1} \frac{(n+k)!}{n!} f_{n+k}\varepsilon^n \right\|_{\max} \\
& \left\| \frac{\partial^k f}{\partial \varepsilon^k}(X,\varepsilon) - \sum_{n=k}^{N+k-1} \frac{n!}{(n-k)!} f_n \varepsilon^{n-k} \right\|_{\max} \\
&= \left\| \frac{\partial^k}{\partial \varepsilon^k} \left( f(X,\varepsilon) - \sum_{n=0}^{N+k-1} f_n(X)\,\varepsilon^n \right) \right\|_{\max} \\
&\leqslant \frac{k!}{|\varepsilon|^k \sin^k(\delta_1-\delta_2)} \max_{|w-\varepsilon| = |\varepsilon|\sin(\delta_1-\delta_2)} \left\| f(X,w) - \sum_{n=0}^{N+k-1} f_n(X)\, w^n \right\|_{\max} \\
&\leqslant \frac{k!}{|\varepsilon|^k \sin^k(\delta_1-\delta_2)} \max_{|w-\varepsilon| = |\varepsilon|\sin(\delta_1-\delta_2)} CD^{N+k}\Gamma(1+mk+mN)\,|w|^{N+k} \\
&\leqslant k! \left( D \frac{1+\sin(\delta_1-\delta_2)}{\sin(\delta_1-\delta_2)} \right)^k CD^N \Gamma(1+mk+mN)\,|\varepsilon|^N .
\end{aligned}
$$

Since there exists a constant $M > 0$ such that

$$\Gamma(1+mk+mN) \leqslant M^N \Gamma(1+mN),$$

which can be deduced from the Stirling formula, the result follows.

□

One can think of Gevrey functions as being $\mathcal{C}^\infty$ smooth at the vertex of the sector, $\varepsilon = 0$. We specify this a bit more.

**Proposition 1.2.8.** *Suppose $f(X,\varepsilon)$ is defined on $V \times S(\theta, 2\delta_1, r_1)$ with $f \sim_1 \widehat{f}$ and choose any $0 < \delta_2 < \delta_1$, there then exists an $0 < r_2 < r_1$ such that for all $n \in \mathbb{N}$*

$$\lim_{\substack{\varepsilon \to 0 \\ \varepsilon \in S(\theta, 2\delta_2, r_2)}} \sup_{X \in V} \left\| \frac{\partial^n f}{\partial \varepsilon^n}(X,\varepsilon) - n! f_n(X) \right\|_{\max} = 0$$

*and*

$$\lim_{\substack{\varepsilon \to 0 \\ \varepsilon \in S(\theta, 2\delta_2, r_2)}} \sup_{X \in V} \left\| \frac{\frac{\partial^n f}{\partial \varepsilon^n}(X,\varepsilon) - n! f_n(X)}{\varepsilon} - (n+1)! f_{n+1}(X) \right\|_{\max} = 0.$$

**Proof:** This follows from proposition 1.2.7. □

In general, a formal Gevrey series can be the Gevrey expansion of multiple functions, the difference of such functions however, can at most be exponentially small as the following result clarifies, a proof can be found in, for example [Ram78].

**Proposition 1.2.9.** *Let $f \in \mathcal{O}(V \times S, \mathbb{C}^s)$, then $f \sim_m 0$ if and only if $f$ is exponentially decaying w.r.t. $\varepsilon \in S$, uniformly for $X \in V$ i.e.*

$$\exists K, L > 0\colon \sup_{X \in V} \|f(X,\varepsilon)\|_{\max} \leqslant K e^{-L|\varepsilon|^{-1/m}}, \forall \varepsilon \in S.$$

One can also wonder of every formal Gevrey series is necessarily the Gevrey expansion of a function. The following theorem, which is a specification of the Borel-Ritt theorem (see for example [Was02]) to Gevrey asymptotics, affirms this but the functions realizing the series can only be guaranteed to exist on "small" sectors.

**Theorem 1.2.10** (Borel-Ritt-Gevrey theorem). *Consider* $V \subset \mathbb{C}^{\ell}$, $m > 0$ *and* $S(\theta, \delta, r)$ *a sector of opening* $\delta < m\pi$. *If* $\widehat{f}(X, \varepsilon)$ *is a Gevrey-m formal series in* $\varepsilon$, *uniformly for* $X \in V$ *there exists an* $f \in \mathcal{O}(V \times S(\theta, \delta, r), \mathbb{C}^s)$ *such that* $f \sim_m \widehat{f}$.

A proof of this theorem can be found in [Bal00].
An essential tool in the study of Gevrey asymptotic functions is the Ramis-Sibuya theorem. This result gives an equivalence between a function possessing a Gevrey expansion and it being part of a "function chain" covering a disk around the origin. This allows us to make statements about a function's Gevrey asymptotic properties without explicit calculation of the asymptotic bounds, or even without knowledge of the asymptotic series. We clarify this further now.

**Definition 1.2.11.** *Given a punctured disk* $B(0, r) \setminus \{0\} \subset \mathbb{C}$. *A good sectorial covering of the punctured disk is a finite ordered set of sectors* $S_j := S(\theta_j, \delta_j, r)$, $1 \leqslant j \leqslant n$ *such that*

- $\bigcup_{j=1}^{n} S_j = B(0, r) \setminus \{0\}$.
- $S_i \cap S_j \neq \varnothing$ *if and only if* $|(j - i) \bmod n| = 1$.

*See figure 1.1 for an illustrated example.*

**Theorem 1.2.12** (Ramis-Sibuya theorem). *Let* $V \subset \mathbb{C}^{\ell}$ *and* $m > 0$. *Suppose that we have sectors* $Sj$, $1 \leqslant j \leqslant n$, *forming a good sectorial covering of the punctured disk* $B(0, r)\setminus\{0\}$. *Given bounded functions* $f_j \in \mathcal{O}(V \times S_j, \mathbb{C}^s)$ *satisfying the following. There exist* $A, B > 0$ *such that for every* $1 \leqslant i, j \leqslant n$ *with* $S_i \cap S_j \neq \varnothing$

$$\sup_{X \in V} \| f_i(X, \varepsilon) - f_j(X, \varepsilon) \|_{\max} \leqslant A e^{-B|\varepsilon|^{-1/m}} \tag{1.2.1}$$

*for all* $\varepsilon \in S_i \cap S_j$.
*It then follows that all the functions* $f_j$ *are Gevrey-m asymptotic to a common Gevrey-m formal series.*

A proof of this theorem can be found in [FS13, RS89].
The following is an application of the Ramis-Sibuya theorem.

**Lemma 1.2.13.** *Consider open subsets* $V \subset C^{\ell}$, $U \subset \mathbb{C}^s$ *and an open sector* $S(\theta, \delta, r)$. *Let* $F \colon V \times U \times B(0, R) \to \mathbb{C}^k$ *be holomorphic for* $k \in \mathbb{N}_0$ *and* $R > 0$.

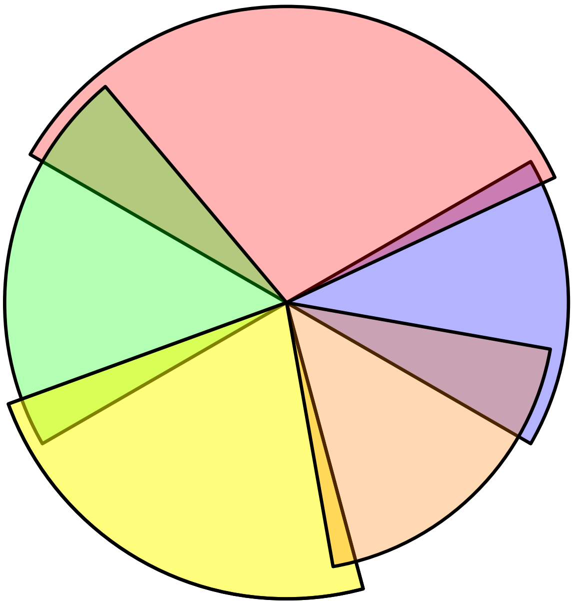

Figure 1.1: A good sectorial covering of the punctured disk by five sectors. The sectors are drawn with varying radii to help distinguish them.

*Suppose that* $f \sim_m \widehat{f}$*, w.r.t.* $\varepsilon \in S(\theta, \delta, r)$*, uniformly for* $X \in V$*, and* $\widehat{f}(X, 0) \in U$*. Then there exists an* $0 < r_1 \leqslant r$ *such that*

$$F(X, f(X, \varepsilon), \varepsilon) \sim_m F\left(X, \widehat{f}(X, \varepsilon), \varepsilon\right)$$

*w.r.t.* $\varepsilon \in S(\theta, \delta, r_1)$*, uniformly for* $X \in V$*.*

**Proof:** By the Borel-Ritt-Gevrey theorem 1.2.10, one can construct a good sectorial covering $S_1, \ldots, S_n$ of $B(0, r) \setminus \{0\}$ with corresponding functions $f_j \sim_m \widehat{f}$ such that $S_1 = S(\theta, \delta, r)$ and $f_1 = f$. By proposition 1.2.9 these functions satisfy inequalities (1.2.1).

Since $\widehat{f}(X, 0) \in U$, one can choose $r_1 \leqslant r$ and a compact set $\widehat{f}(X, 0) \in K \subset U$ such that for all $j$, $f_j(V \times (S_j \cap B(0, r_1))) \subset K$. One can then apply the mean value theorem and the Cauchy inequalities to show that there exist $\widetilde{A} > 0$ such that

$$\sup_{X \in V} \|F(X, f_i(X, \varepsilon), \varepsilon) - F(X, f_j(X, \varepsilon), \varepsilon)\|_{\max} \leqslant \widetilde{A} e^{-B|\varepsilon|^{-1/m}}$$

for all $\varepsilon \in S_i \cap S_j \cap B(0, r_1)$. The Ramis-Sibuya theorem then guarantees the existence of a formal series $\widetilde{G}(X, \varepsilon)$ such that (in particular) $F(X, f(X, \varepsilon), \varepsilon) \sim_m \widehat{G}(X, \varepsilon)$.

It remains to show that $\widehat{G}(X,\varepsilon) = F\left(X, \widehat{f}(X,\varepsilon), \varepsilon\right)$. Denote

$$\begin{aligned}
\widehat{f}(X,\varepsilon) &= \sum_{n=0}^{\infty} f_n(X)\,\varepsilon^n, \\
\widehat{G}(X,\varepsilon) &= \sum_{n=0}^{\infty} g_n(X)\,\varepsilon^n.
\end{aligned}$$

Notice that for any $N \in \mathbb{N}_0$ the coefficients of $F\left(X, \widehat{f}(X,\varepsilon), \varepsilon\right)$ for $\varepsilon^0, \ldots, \varepsilon^{N-1}$ coincide with those in the Taylor expansion of $F\left(X, \sum_{n=0}^{N-1} f_n(X)\,\varepsilon^n, \varepsilon\right)$. It is thus sufficient to show that $F\left(X, \sum_{n=0}^{N-1} f_n(X)\,\varepsilon^n, \varepsilon\right) - \sum_{n=0}^{N-1} g_n(X)\,\varepsilon^n = \mathrm{O}\left(\varepsilon^N\right)$ for $\varepsilon \to 0$, staying in $S(\theta,\delta,r)$. We have that

$$\begin{aligned}
&\left\| F\left(X, \sum_{n=0}^{N-1} f_n(X)\,\varepsilon^n, \varepsilon\right) - \sum_{n=0}^{N-1} g_n(X)\,\varepsilon^n \right\|_{\max} \\
&\leqslant \left\| F\left(X, \sum_{n=0}^{N-1} f_n(X)\,\varepsilon^n, \varepsilon\right) - F\left(X, f(X,\varepsilon), \varepsilon\right) \right\|_{\max} \\
&\quad + \left\| F\left(X, f(X,\varepsilon), \varepsilon\right) - \sum_{n=0}^{N-1} g_n(X)\,\varepsilon^n \right\|_{\max}. \qquad (1.2.2)
\end{aligned}$$

By the Cauchy inequalities and the definition of Gevrey expansions, there exists $C_0, C_1, D_1$ such that (1.2.2) is bounded, for all $|\varepsilon|$ sufficiently small, by

$$\begin{aligned}
&C_0 \left\| f(X,\varepsilon) - \sum_{n=0}^{N-1} f_n(X)\,\varepsilon^n \right\|_{\max} + \left\| F\left(X, f(X,\varepsilon), \varepsilon\right) - \sum_{n=0}^{N-1} g_n(X)\,\varepsilon^n \right\|_{\max} \\
&\leqslant C_1 D_1^N \Gamma(1+mN)\,|\varepsilon|^N.
\end{aligned}$$

□

Due to the Borel-Ritt-Gevrey theorem, remark 1.2.5 and the above lemma, the following is immediate.

**Corollary 1.2.14.** *Let $F$ and $\widehat{f}$ be as in lemma 1.2.13 with $\widehat{f}$ a Gevrey-$m$ formal series. The formal series $F\left(X, \widehat{f}(X,\varepsilon), \varepsilon\right)$ is also Gevrey-$m$.*

We now give a version of the implicit function theorem for Gevrey asymptotic functions.

**Theorem 1.2.15** (Gevrey implicit function theorem)**.** *Let $m > 0$ and*

$$\widehat{f}(a,\varepsilon) = \sum_{n=0}^{\infty} f_n(a)\varepsilon^n$$

*be a Gevrey-$m$ series in $\varepsilon$, uniformly for $a \in A$, with $A \subset \mathbb{C}$ open.*

*Suppose there are* $\theta \in [0, 2\pi[$, $\lambda, r > 0$ *and* $f \in \mathcal{O}(A \times S(\theta, \lambda, r))$ *such that* $f \sim_m \widehat{f}$. *If moreover there exists an* $a_0 \in A$ *with* $f_0(a_0) = 0$ *and* $f_0'(a_0) \neq 0$, *we can find an* $r_1 > 0$ *and a holomorphic function*

$$\widetilde{a} : S(\theta, \lambda, r_1) \to A$$

*such that* $\widetilde{a}(0) = a_0$ *and*

$$f(\widetilde{a}(\varepsilon), \varepsilon) = 0$$

*for all* $\varepsilon \in S(\theta, \lambda, r_1)$.
*The function* $\widetilde{a}$ *is also Gevrey-m asymptotic to a formal series*

$$\widehat{a}(\varepsilon) = \sum_{n=0}^{\infty} a_n \varepsilon^n.$$

**Proof:** Take $\delta, R > 0$ such that $\overline{B}(a_0, \delta + R) \subset A$ and consider the following map

$$F : B(a_0, \delta) \times (S(\theta, \lambda, r) \cup \{0\}) \to \mathbb{C}$$
$$(a, \varepsilon) \mapsto \begin{cases} f(a, \varepsilon) \text{ if } \varepsilon \neq 0 \\ f_0(a) \text{ if } \varepsilon = 0 \end{cases}.$$

This is clearly a continuous map. The first partial derivative is given by

$$\frac{\partial F}{\partial a}(a, \varepsilon) = \begin{cases} \frac{\partial f}{\partial a}(a, \varepsilon) \text{ if } \varepsilon \neq 0 \\ f_0'(a) \text{ if } \varepsilon = 0 \end{cases}. \tag{1.2.3}$$

We claim that the partial derivative is also continuous on $B(a_0, \delta) \times (S(\theta, \lambda, r) \cup \{0\})$. For a point $(a_*, \varepsilon_*)$ with $\varepsilon_* \neq 0$ the continuity is obvious since $f$ is holomorphic. Thus consider a point $(a_*, 0)$and let $\rho > 0$ be random. Take a neighbourhood, $U \subset B(a_0, \delta)$, of $a_*$ such that

$$|f_0'(a) - f_0'(a_*)| < \frac{\rho}{2}$$

for $a \in U$.
For $(a, \varepsilon) \in U \times \left(S\left(\theta, \lambda, \min\left\{\frac{R\rho}{2CD\Gamma(1+s)}, r\right\}\right) \cup \{0\}\right)$, where $C, D > 0$ are chosen such that

$$\sup_{a \in A} |f(a, \varepsilon) - f_0(a)| \leqslant CD\Gamma(1+s)\,|\varepsilon|,$$

we have the following.
If $\varepsilon = 0$, clearly

$$\left|\frac{\partial F}{\partial a}(a, \varepsilon) - \frac{\partial F}{\partial a}(a_*, 0)\right| = |f_0'(a) - f_0'(a_*)| < \frac{\rho}{2}.$$

Otherwise, using Cauchy's inequalities we have

$$\begin{aligned}
\left|\frac{\partial F}{\partial a}(a,\varepsilon)-\frac{\partial F}{\partial a}(a_*,0)\right| &= \left|\frac{\partial f}{\partial a}(a,\varepsilon)-f_0'(a_*)\right| \\
&\leqslant \left|\frac{\partial (f-f_0)}{\partial a}(a,\varepsilon)\right| + \left|f_0'(a)-f_0'(a_*)\right| \\
&< \frac{1}{R}\sup_{|w-a|=R}|f(w,\varepsilon)-f_0(w)| + \frac{\rho}{2} \\
&\leqslant \frac{1}{R}CD\Gamma(1+s)\,|\varepsilon| + \frac{\rho}{2} \\
&< \rho.
\end{aligned}$$

The map $F$ is thus continuous, with continuous first partial derivative and satisfying $F(a_0,0)=0$. This suffices to employ a continuous version of the implicit function theorem, see for example [LS14]. There thus exists an $r_0>0$ and a continuous map

$$\widehat{a}: S(\theta,\lambda,r_0)\cup\{0\}\to B(a_0,\delta)$$

with $\widehat{a}(0)=a_0$ and $F(\widehat{a}(\varepsilon),\varepsilon)=0$, where $\widehat{a}(\varepsilon)$ is the unique element solving this equation.

Since $\frac{\partial F}{\partial a}(a_0,0)=f_0'(a_0)\neq 0$ we can find an $0<r_1\leqslant r_0$ such that for all $\varepsilon\in S(\theta,\lambda,r_1)\cup\{0\}$

$$\frac{\partial F}{\partial a}(\widehat{a}(\varepsilon),\varepsilon)\neq 0.$$

Let $\varepsilon_*\in S(\theta,\lambda,r_1)$, then

$$\begin{aligned}
f(\widehat{a}(\varepsilon_*),\varepsilon_*) &= F(\widehat{a}(\varepsilon_*),\varepsilon_*)=0 \\
\frac{\partial f}{\partial a}(\widehat{a}(\varepsilon_*),\varepsilon_*) &= \frac{\partial F}{\partial a}(\widehat{a}(\varepsilon_*),\varepsilon_*)\neq 0.
\end{aligned}$$

By the holomorphic implicit function theorem there exists a holomorphic function $g_{\varepsilon_*}$ on an environment of $\varepsilon_*$ for which

$$f\left(g_{\varepsilon_*}(\varepsilon),\varepsilon\right)=0.$$

By uniqueness of $\widehat{a}$ we must have that $\widehat{a}(\varepsilon)=g_{\varepsilon_*}(\varepsilon)$ on this environment. Consequently, $\widehat{a}$ is analytic around $\varepsilon_*$ and thus it is analytic on $S(\theta,\lambda,r_1)$.

It remains to show that $\widehat{a}$ is Gevrey-$m$ asymptotic to a formal series.

For this, take a covering of $B(0,r)\backslash\{0\}$ by sectors $(S_i)_{i=1\ldots l}$ where $S_1=S(\theta,\lambda,r_1)$ and the other sectors have opening smaller than $m\pi$. By the Borel-Ritt-Gevrey theorem 1.2.10 there exist $g_i\in\mathcal{O}(A\times S_i)$ for $i=1,\ldots,l$ with $g_1=f$ and $g_i\sim_m\widehat{f}$. There exist $T,Q>0$ such that for $i,j$ with $S_i\cap S_j\neq\varnothing$ we have

$$\sup_{a\in A}|g_i(a,\varepsilon)-g_j(a,\varepsilon)|\leqslant Qe^{-\frac{T}{|\varepsilon|^{1/s}}},\ \forall\varepsilon\in S_i\cap S_j.$$

Using our previous assertions we find, for $i = 1, \ldots, l$, maps $\widehat{a}_i \in \mathcal{O}(S_i)$ (where the radius of $S_i$ could be diminished), with $\widehat{a}_1 = \widehat{a}$, satisfying

$$g_i(\widehat{a}_i(\varepsilon), \varepsilon) = 0.$$

For $i, j$ with $S_i \cap S_j \neq \varnothing$ and $\varepsilon \in S_i \cap S_j$ we get

$$\begin{aligned} 0 =& g_i(\widehat{a}_i(\varepsilon), \varepsilon) - g_j(\widehat{a}_j(\varepsilon), \varepsilon) \\ =& g_i(\widehat{a}_i(\varepsilon), \varepsilon) - g_i(\widehat{a}_j(\varepsilon), \varepsilon) + g_i(\widehat{a}_j(\varepsilon), \varepsilon) - g_j(\widehat{a}_j(\varepsilon), \varepsilon) \\ =& \int_0^1 \frac{\partial g_i}{\partial a} \left(\widehat{a}_i(\varepsilon) + v\left(\widehat{a}_j(\varepsilon) - \widehat{a}_i(\varepsilon)\right), \varepsilon\right) \mathrm{d}v \left(\widehat{a}_j(\varepsilon) - \widehat{a}_i(\varepsilon)\right) \\ &+ g_i(\widehat{a}_j(\varepsilon), \varepsilon) - g_j(\widehat{a}_j(\varepsilon), \varepsilon). \end{aligned} \tag{1.2.4}$$

Since

$$\begin{aligned} &\left| \int_0^1 \frac{\partial g_i}{\partial a} \left(\widehat{a}_i(\varepsilon) + v\left(\widehat{a}_j(\varepsilon) - \widehat{a}_i(\varepsilon)\right), \varepsilon\right) \mathrm{d}v - f_0'(a_0) \right| \\ &\leqslant \int_0^1 \left| \frac{\partial g_i}{\partial a} \left(\widehat{a}_i(\varepsilon) + v\left(\widehat{a}_j(\varepsilon) - \widehat{a}_i(\varepsilon)\right), \varepsilon\right) - f_0'(a_0) \right| \mathrm{d}v \end{aligned}$$

we can find by continuity of (1.2.3) and using $f_0'(a_0) \neq 0$, a $D > 0$ such that for $|\varepsilon|$ sufficiently small

$$\left| \int_0^1 \frac{\partial g_i}{\partial a} \left(\widehat{a}_i(\varepsilon) + v\left(\widehat{a}_j(\varepsilon) - \widehat{a}_i(\varepsilon)\right), \varepsilon\right) \mathrm{d}v \right| \geqslant D.$$

Combining this with (1.2.4) we find

$$\begin{aligned} |\widehat{a}_j(\varepsilon) - \widehat{a}_i(\varepsilon)| &= \left| \frac{g_i(\widehat{a}_j(\varepsilon), \varepsilon) - g_j(\widehat{a}_j(\varepsilon), \varepsilon)}{\int_0^1 \frac{\partial g_i}{\partial a} \left(\widehat{a}_i(\varepsilon) + v\left(\widehat{a}_j(\varepsilon) - \widehat{a}_i(\varepsilon)\right), \varepsilon\right) \mathrm{d}v} \right| \\ &\leqslant \frac{Q}{D} e^{-\frac{T}{|\varepsilon|^{1/s}}}. \end{aligned}$$

The Ramis-Sibuya theorem 1.2.12, thus guarantees that all $\widehat{a}_i$ are Gevrey-$s$ asymptotic to a common formal series. In particular, $\widehat{a}$ is Gevrey-$s$ asymptotic to a formal series on $S(\theta, \lambda, r_1)$, for some $r_1 > 0$. □

### 1.2.3 Summability

The Borel-Ritt-Gevrey theorem 1.2.10, guarantees for any formal Gevrey-$m$ series the existence of functions having this series as their Gevrey-$m$ expansion. These functions will however only be defined on "small" sectors, of opening less than $m\pi$ and will certainly not be unique but their difference is at most exponentially small, see lemma 1.2.9.

On "large" sectors, of opening larger than $m\pi$, the story is quite different. Given a "large" sector and a formal Gevrey series, there might not be any function defined on this sector, admitting the series as its Gevrey expansion. However when such a function does exist it is necessarily unique, said differently, on large sectors lemma 1.2.9 can be strengthened to what is known as Watson's lemma.

**Lemma 1.2.16** (Watson's lemma, [Wat12a])**.** *Suppose that $S$ is a sector of opening larger than $m\pi$ and $f \in \mathcal{V} \times \mathcal{S}$ such that $f \sim_m 0$ then $f = 0$.*

The above leads us to the following definition.

**Definition 1.2.17.** *Given a Gevrey-$1/k$ series*

$$\hat{f}\left(X,\varepsilon\right) = \sum_{n=0}^{\infty} f_n\left(X\right)\varepsilon^n.$$

*We say that $\hat{f}$ is Borel $k$-summable in a direction $\theta \in [0,2\pi[$ if there exist $r,\tau > 0$ and a function $f\left(X,\varepsilon\right)$ analytic on $V \times S\left(\theta, \frac{\pi}{k} + \tau, r\right)$ such that $f \sim_{1/k} \hat{f}$.*

**Definition 1.2.18.** *A Gevrey-$1/k$ series is called $k$-summable if it is $k$-summable in all but finitely many directions.*

In this thesis, we will almost exclusively deal with summability in directions, i.e. only with definition 1.2.17.
We will state a theorem that gives an equivalent definition for summability in a direction. For this we first need to introduce the following.

**Definition 1.2.19.** *Let, for $k > 0$, $\hat{f}\left(X,\varepsilon\right) = \sum_{n=1}^{\infty} f_n\left(X\right)\varepsilon^n$ be a Gevrey-$\frac{1}{k}$ series in $\varepsilon$ (without constant coefficient), uniformly for $X \in V \subset \mathbb{C}^\ell$. We define the formal Borel transform of order $k$ (with respect to $\varepsilon$) of this series to be*

$$\mathcal{B}_k(\hat{f})\left(X,\eta\right) = \sum_{n=1}^{\infty} \frac{f_n\left(X\right)}{\Gamma(1+\frac{(n-1)}{k})}\eta^{n-1}.$$

We see that the formal Borel transform of order $k$ of a type $B$ Gevrey-$\frac{1}{k}$ series is a convergent series for $(X,\eta) \in V \times B(0,1/B)$ since, for example, the following bound can be found (see [Bat08])

$$\frac{\Gamma(1+\frac{n}{k})}{\Gamma(1+\frac{(n-1)}{k})} < \sqrt{\frac{\pi}{e}}\left(\frac{n}{k}+\frac{1}{2}\right)^{\frac{1}{k}}.$$

The following theorem gives an equivalent definition for $k$-summability in a direction.

**Theorem 1.2.20.** *([Bal00]) Let $\hat{f}\left(X,\varepsilon\right) = \sum_{n=1}^{\infty} f_n\left(X\right)\varepsilon^n$ be a Gevrey-$\frac{1}{k}$ series, $k > 0$, uniformly for $X \in V \subset \mathbb{C}^\ell$. For every $\theta \in [0,2\pi[$, the following two statements are equivalent*

- *The series $\hat{f}\left(X,\varepsilon\right)$ is Borel $k$-summable in the direction $\theta$ with Borel sum $f\left(X,\varepsilon\right)$.*

- *There exists an infinite sector* $S(\theta, \tau)$ *for* $\tau > 0$ *such that* $\mathcal{B}_k\left(\hat{f}\right)(X, \eta)$ *admits a holomorphic continuation to* $S(\theta, \tau)$ *of exponential growth at most of order* $k$*, i.e. there exist* $M, \nu > 0$ *such that for all* $\eta \in S(\theta, \tau)$

$$\sup_{X \in V} \left|\mathcal{B}_k(\hat{f})(X, \eta)\right| \leqslant M e^{\nu|\eta|^k}.$$

*Moreover the function* $f$ *is unique in the case the statements are true.*

# Chapter 2

# Local Gevrey analysis of slow manifolds

As a general remark we point out that we work throughout this thesis with complex slow-fast systems. Our results can be applied to real analytic systems by simply limiting them to the real numbers. Moreover, the methods described here are robust enough that one can add parameters without altering the results. We have not added such parameters, mainly to not further complicate the notation.

In this chapter we study the local Gevrey asymptotic properties of slow manifolds in a broad class of holomorphic slow-fast systems. We commence this study from a formal point of view in section 2.1. The existence of power series in the singular parameter which are formally invariant under the flow of the system is demonstrated. These power series are then shown to be Gevrey-1. We then make a distinction depending on the behaviour of the slow flow.
Section 2.2 deals with regular points of the slow flow. In this case one can construct, by employing the Borel-Ritt-Gevrey theorem 1.2.10, manifolds which are Gevrey asymptotic to the formal solution but are only nearly invariant. By this we mean that the error is exponentially decaying w.r.t. the singular parameter. However, starting from such a manifold one can construct invariant manifolds, which inherit the Gevrey property. This approach has been already successfully employed in the case of one slow variable in [CDRSS00].
Afterwards, singular points of the slow flow are considered in section 2.3. Under certain conditions, a better result can be achieved. Here the formal solution is 1-summable in a direction.
Concretely, we will consider in this chapter slow-fast systems

$$\left\{ \begin{array}{lcl} \dot{X} & = & \varepsilon G_1(X, Z, \varepsilon) \\ \dot{Z} & = & G_2(X, Z, \varepsilon) \end{array} \right. , \tag{2.0.1}$$

where

$$X \in \mathbb{C}^n, Z \in \mathbb{C}^m, \varepsilon \in \mathbb{C}$$

and the functions are holomorphic on an open subset of $\mathbb{C}^n \times \mathbb{C}^m \times \mathbb{C}$. Suppose there is a point $(X_0, Z_0, 0)$ satisfying $G_2(X_0, Z_0, 0) = 0$ and $D_Z G_2(X_0, Z_0, 0)$ is an invertible matrix. Then there exists, by the implicit function theorem, locally around $X_0$ a holomorphic function $\Phi_0(X)$, satisfying

$$G_2\left(X, \Phi_0(X), 0\right) = 0$$

and $\Phi_0\left(X_0\right) = Z_0$ i.e. $Z = \Phi_0\left(X\right)$ is a critical manifold. Applying the transformation

$$\begin{aligned}\varepsilon Z_1 &= Z - \Phi_0\left(X\right) + \left(D_Z G_2\left(X, \Phi_0\left(X\right), 0\right)\right)^{-1} \cdot \\ &\qquad \left(\frac{\partial G_2}{\partial \varepsilon}\left(X, \Phi_0\left(X\right), 0\right) - D\Phi_0\left(X\right) G_1\left(X, \Phi_0\left(X\right), 0\right)\right), \\ X_1 &= X - X_0,\end{aligned}$$

and dropping the subscripts, brings the system into the form

$$\left\{\begin{array}{lcl} \dot{X} &=& \varepsilon F(X, Z, \varepsilon) \\ \dot{Z} &=& A(X)Z + \varepsilon H(X, Z, \varepsilon) \end{array}\right., \tag{2.0.2}$$

Where $A(X) = D_Z G_2(X + X_0, \Phi_0(X + X_0), 0)$, the critical manifold is now given by $Z = 0$ and the following assumptions hold

(i) $A \in \mathcal{O}\left(\mathbb{P}_n(R), \mathbb{C}^{m\times m}\right)$ and $A(X)$ is an invertible matrix for all $X \in \mathbb{P}_n(R)$,

(ii) $F \in \mathcal{O}\left(\mathbb{P}_n(R) \times \mathbb{P}_m(R) \times B(0, R), \mathbb{C}^n\right)$,

(iii) $H \in \mathcal{O}\left(\mathbb{P}_n(R) \times \mathbb{P}_m(R) \times B(0, R), \mathbb{C}^m\right)$,

for some $R > 0$.

## 2.1 Formal slow manifolds

### 2.1.1 Formal expansions in terms of the singular parameter

The slow manifold equation associated to (2.0.2) is given by

$$\varepsilon D_X Z(X, \varepsilon) F(X, Z(X, \varepsilon), \varepsilon) = A(X) Z(X, \varepsilon) + \varepsilon H(X, Z(X, \varepsilon), \varepsilon). \tag{2.1.1}$$

We start off by searching for a formal solution to the above equation.
For this we introduce the spaces of formal series $\mathcal{O}\left(\mathbb{P}_n(R), \mathbb{C}^s\right)[\![\varepsilon]\!]$, where $s$ may denote any natural number. These spaces can account for series where the coefficients are holomorphic functions taking values in the space of linear operators between products of $\mathbb{C}$. Indeed, for this one simply uses the canonical identifications of linear operators with complex valued matrices, and matrix spaces with finite product

spaces. We equip these with the following metric, Let $V, W \in \mathcal{O}(\mathbb{P}_n(R), \mathbb{C}^s)[\![\varepsilon]\!]$ then $d(V,W) = 2^{-K}$, where

$$K = \min\left\{\ell \in \mathbb{N} \;\middle|\; \Big(\text{ coefficient of } \varepsilon^\ell \text{ in } V - W\Big) \neq 0\right\}.$$

It is easily seen that $\mathcal{O}(\mathbb{P}_n(R), \mathbb{C}^s)[\![\varepsilon]\!]$ is a complete metric space. Moreover, the metric has the following additional properties:

- $d(V,W) = d(V-W,0)$.
- $d(U,W) \leqslant \max\{d(U,V), d(V,W)\}$ (so $d$ is an ultrametric).
- $d(V \cdot W, 0) \leqslant d(V,0)d(W,0)$ whenever the product is defined, as is for example the case when
$$V \in \mathcal{O}\left(\mathbb{P}_n(R), \mathbb{C}^{m\times n}\right)[\![\varepsilon]\!]$$
and
$$W \in \mathcal{O}(\mathbb{P}_n(R), \mathbb{C}^n)[\![\varepsilon]\!].$$
Combining this with the fact that the metric is bounded by 1, shows that in particular $d(V \cdot W, 0) \leqslant d(V,0)$ as well as $d(V \cdot W, 0) \leqslant d(W,0)$.

**Proposition 2.1.1.** *Equation (2.1.1) has a unique formal solution of the form*

$$\widetilde{Z}(X,\varepsilon) = \sum_{k=1}^{\infty} Z_k(X)\varepsilon^k,$$

*with* $Z_k \in \mathcal{O}(\mathbb{P}_n(R), \mathbb{C}^m)$.

**Proof:** Consider the map

$$\mathcal{T}\colon Z = Z(X,\varepsilon) \mapsto \varepsilon A(X)^{-1}\left(D_X Z F(X,Z,\varepsilon) - H(X,Z,\varepsilon)\right) \tag{2.1.2}$$

from $\overline{B}\left(0,\frac{1}{2}\right) \subset \mathcal{O}(\mathbb{P}_n(R), \mathbb{C}^m)[\![\varepsilon]\!]$ to itself. This map is well defined, one can see easily that $\mathcal{T}(Z) \in \mathcal{O}(\mathbb{P}_n(R), \mathbb{C}^m)[\![\varepsilon]\!]$ and $d(\mathcal{T}(Z), 0) \leqslant \frac{1}{2}$ simply due to the multiplication with $\varepsilon$.
Moreover, for $Z = Z(X,\varepsilon), W = W(X,\varepsilon) \in \mathcal{O}(\mathbb{P}_n(R), \mathbb{C}^m)[\![\varepsilon]\!]$ we have

$$\begin{aligned}\mathcal{T}(Z) - \mathcal{T}(W) = \varepsilon A(X)^{-1}\big(&D_X(Z-W)F(X,Z,\varepsilon)\\ &+ D_X W\left(F(X,Z,\varepsilon) - F(X,W,\varepsilon)\right)\\ &+ H(X,W,\varepsilon) - H(X,Z,\varepsilon)\big).\end{aligned}$$

Using $d(D_X(Z-W),0) \leqslant d(Z-W,0)$ and

$$\begin{aligned}F(X,Z,\varepsilon) - F(X,W,\varepsilon) &= \widetilde{F}_{Z,W}(X,\varepsilon)(Z-W),\\ H(X,Z,\varepsilon) - H(X,W,\varepsilon) &= \widetilde{H}_{Z,W}(X,\varepsilon)(Z-W),\end{aligned}$$

for certain formal series $\widetilde{F}_{Z,W}, \widetilde{H}_{Z,W}$, and combining this with the properties of the metric $d$, one sees that

$$d(\mathcal{T}(Z), \mathcal{T}(W)) = d(\mathcal{T}(Z) - \mathcal{T}(W), 0) \leqslant \frac{1}{2} d(Z - W, 0) = \frac{1}{2} d(Z, W).$$

Consequently there exists a fixed point of $\mathcal{T}$ in $\overline{B}\left(0, \frac{1}{2}\right)$, which is clearly a (formal) solution to (2.1.1) and due to the definition of the metric has no constant term. □

### 2.1.2 Gevrey growth of the formal expansion

One can not expect that the formal solution found proposition 2.1.1 is convergent in in a full neighbourhood of $\varepsilon = 0$. Indeed even for a very simple example,

$$\varepsilon \frac{\partial Z}{\partial x}(x, \varepsilon) = z - \varepsilon h(x),$$

the formal solution is calculated to be given by

$$\widetilde{Z}(x, \varepsilon) = \sum_{n=1}^{\infty} h^{(n-1)}(x)\, \varepsilon^n.$$

The coefficients will generally grow like (or are at least always be bounded by) $n!B^n$ which indicates that the series is of Gevrey-1 type.

In this section we show that the conclusion of the simple example above actually holds for all formal solutions to the general equation (2.1.1). The formal slow manifolds of (2.0.2) are Gevrey-1 series. The analysis will be done, locally around $X = 0$. At the end of this section, more specifically in corollary 2.1.11 we mention a more global result. The remainder of the section is devoted to proving

**Proposition 2.1.2.** *Let $0 < T < R$, the unique formal solution to equation (2.1.1) is Gevrey-1 w.r.t. $\varepsilon$ uniformly for $X \in \mathbb{P}_n(T)$ i.e. $\exists C_1, D_1 > 0$ such that $\forall k \in \mathbb{N}$*

$$\sup_{X \in \mathbb{P}_n(T)} \|Z_k(X)\|_{\max} \leqslant C_1 D_1^k k!.$$

Our approach is to introduce an auxiliary series whose coefficients are bounds on a well chosen family of norms of the $Z_k$. By proving convergence of the auxiliary series, the Gevrey property can be deduced. This approach is also taken in [CDRSS00] for a single slow variable (i.e. $X \in \mathbb{C}$), moreover, the techniques employed in the remainder of this section are an adaptation of the ones used in this article.
We consider a family of norms on $\mathcal{O}(\mathbb{P}_n(R), \mathbb{C}^s)$ which are multi-dimensional variants of the Nagumo norms, [Nag41].

**Definition 2.1.3.** *For $p \in \mathbb{N}$ and $\varphi \in \mathcal{O}(\mathbb{P}_n(R), \mathbb{C}^s)$ define*

$$\|\varphi\|_p = \sup_{X \in \mathbb{P}_n(R)} \|\varphi(X)\|_{\max} \left( \sum_{\ell=1}^{n} \frac{1}{R - |x_\ell|} \right)^{-p}.$$

*Here we have denoted $X = (x_1, \ldots, x_n)$.*

Notice that such a norm may be unbounded, when this is the case it will be denoted as being $+\infty$.

**Proposition 2.1.4.** *Let $\varphi \in \mathcal{O}(\mathbb{P}_n(R), \mathbb{C}^m)$, $\psi \in \mathcal{O}(\mathbb{P}_n(R), \mathbb{C}^n)$ and $p, q \in \mathbb{N}$, we have*

$$\|D\varphi \cdot \psi\|_{p+q+1} \leqslant e(p+1) \|\varphi\|_p \|\psi\|_q$$

**Proof:** The result is trivial when either $\|\varphi\|_p = +\infty$ or $\|\psi\|_q = +\infty$, we thus concentrate only on the case where both are finite.
Let $Y \in \mathbb{P}_n(R)$, then it is easily seen that

$$\begin{aligned}
&\|D\varphi(Y)\psi(Y)\|_{\max} \left( \sum_{\ell=1}^{n} \frac{1}{R - |y_\ell|} \right)^{-p-q-1} \\
&\leqslant \max_{j \in \{1,\ldots,m\}} \left\{ \sum_{i=1}^{n} \left| \frac{\partial \varphi_j}{\partial x_i}(Y) \right| \right\} \|\psi(Y)\|_{\max} \left( \sum_{\ell=1}^{n} \frac{1}{R - |y_\ell|} \right)^{-p-q-1} \\
&\leqslant \max_{j \in \{1,\ldots,m\}} \left\{ \sum_{i=1}^{n} \left| \frac{\partial \varphi_j}{\partial x_i}(Y) \right| \left( \sum_{\ell=1}^{n} \frac{1}{R - |y_\ell|} \right)^{-p-1} \right\} \|\psi\|_q . \qquad (2.1.3)
\end{aligned}$$

We now concentrate on bounding $\sum\limits_{i=1}^{n} \left| \frac{\partial \varphi_j}{\partial x_i}(Y) \right| \left( \sum_{\ell=1}^{n} \frac{1}{R-|y_\ell|} \right)^{-p-1}$. By Cauchy's inequalities,

$$\begin{aligned}
\left| \frac{\partial \varphi_j}{\partial x_i}(Y) \right| &\leqslant \frac{1}{R_i} \max_{X \in \partial_0 \mathbb{P}_n(Y,(R_1,\ldots,R_n))} |\varphi_j(X)| \\
&\leqslant \frac{1}{R_i} \max_{X \in \partial_0 \mathbb{P}_n(Y,(R_1,\ldots,R_n))} \|\varphi(X)\|_{\max},
\end{aligned}$$

where $R_1, \ldots, R_n$ can be any real numbers satisfying $0 < R_i < R - |y_i|$ for all $i = 1, \ldots, n$ and

$$\partial_0 \mathbb{P}_n(Y, (R_1, \ldots, R_n)) = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid |x_i - y_i| = R_i, \forall 1 \leqslant i \leqslant n\}.$$

Consequently we have

$$\sum_{i=1}^{n} \left| \frac{\partial \varphi_j}{\partial x_i}(Y) \right| \leqslant \left( \sum_{i=1}^{n} \frac{1}{R_i} \right) \max_{X \in \partial_0 \mathbb{P}_n(Y,(R_1,\ldots,R_n))} \|\varphi(X)\|_{\max}.$$

Using the definition of $\|\varphi\|_p$ we then get

$$\begin{aligned}
\sum_{i=1}^{n} \left| \frac{\partial \varphi_j}{\partial x_i}(Y) \right| &\leqslant \left( \sum_{i=1}^{n} \frac{1}{R_i} \right) \|\varphi\|_p \max_{X \in \partial_0 \mathbb{P}_n(Y,(R_1,\ldots,R_n))} \left( \sum_{\ell=1}^{n} \frac{1}{R - |x_\ell|} \right)^p \\
&\leqslant \left( \sum_{i=1}^{n} \frac{1}{R_i} \right) \|\varphi\|_p \left( \sum_{\ell=1}^{n} \frac{1}{R - |y_\ell| - R_\ell} \right)^p .
\end{aligned}$$

The last inequality follows from $|x_\ell| \leqslant |x_\ell - y_\ell| + |y_\ell| = R_\ell + |y_\ell|$.
By setting $R_i = \frac{R-|y_i|}{p+1}$ we get

$$\begin{aligned}
\sum_{i=1}^{n} \left|\frac{\partial \varphi_j}{\partial x_i}(Y)\right| &\leqslant \left(\sum_{i=1}^{n} \frac{p+1}{R-|y_i|}\right) \|\varphi\|_p \left(\sum_{\ell=1}^{n} \frac{1}{\left(1-\frac{1}{p+1}\right)(R-|y_\ell|)}\right)^p \\
&= (p+1)\left(1-\frac{1}{p+1}\right)^{-p} \|\varphi\|_p \left(\sum_{\ell=1}^{n} \frac{1}{R-|y_\ell|}\right)^{p+1} \\
&\leqslant (p+1)e\,\|\varphi\|_p \left(\sum_{\ell=1}^{n} \frac{1}{R-|y_\ell|}\right)^{p+1}.
\end{aligned}$$

We thus have

$$\sum_{i=1}^{n} \left|\frac{\partial \varphi_j}{\partial x_i}(Y)\right| \left(\sum_{\ell=1}^{n} \frac{1}{R-|y_\ell|}\right)^{-p-1} \leqslant (p+1)e\,\|\varphi\|_p\,.$$

Plugging this into (2.1.3) gives us that

$$\|D\varphi(Y)\psi(Y)\|_{\max} \left(\sum_{\ell=1}^{n} \frac{1}{R-|y_\ell|}\right)^{-p-q-1} \leqslant (p+1)e\,\|\varphi\|_p\,\|\psi\|_q\,.$$

Since this holds for all $Y \in \mathbb{P}_n(R)$ it immediately follows that

$$\|D\varphi \cdot \psi\|_{p+q+1} \leqslant e(p+1)\,\|\varphi\|_p\,\|\psi\|_q\,.$$

□

We now introduce the concept of majorizing series.

**Definition 2.1.5.** *Let* $\widetilde{\Phi}(X,\varepsilon) = \sum_{n=0}^{\infty} \varphi_n(X)\varepsilon^n$ *be in* $\mathcal{O}(\mathbb{P}_n(R), \mathbb{C}^s)[\![\varepsilon]\!]$ *and* $\widetilde{G}(v) = \sum_{n=0}^{\infty} g_n v^n$ *a formal series with coefficients* $g_n \in \mathbb{R}$ *i.e.* $\widetilde{G} \in \mathbb{R}[\![v]\!]$.
*We say that* $\widetilde{\Phi}(X,\varepsilon)$ *is* *1-majorized* *by* $\widetilde{G}(v)$*, uniformly in* $X$*, if, for all* $n \in \mathbb{N}$*,* $\|\varphi_n\|_n \leqslant n! g_n$*. This is denoted by*

$$\widetilde{\Phi} \ll_X^1 \widetilde{G}.$$

**Proposition 2.1.6.** *Let*

$$\begin{aligned}
&\widetilde{\Phi} \ll_X^1 \widetilde{A} \qquad \text{with } \widetilde{\Phi}(X,\varepsilon) \in \mathcal{O}(\mathbb{P}_n(R), \mathbb{C}^m)[\![\varepsilon]\!], \widetilde{A} \in \mathbb{R}[\![v]\!],\\
&\widetilde{\Psi} \ll_X^1 \widetilde{B} \qquad \text{with } \widetilde{\Psi}(X,\varepsilon) \in \mathcal{O}(\mathbb{P}_n(R), \mathbb{C}^n)[\![\varepsilon]\!], \widetilde{B} \in \mathbb{R}[\![v]\!].
\end{aligned}$$

*Then* $\varepsilon D_X \widetilde{\Phi} \cdot \widetilde{\Psi} \ll_X^1 ev\widetilde{A}\widetilde{B}$

**Proof:** We have

$$\left(\sum_{k=0}^{\infty} D\varphi_k(X)\varepsilon^{k+1}\right)\left(\sum_{\ell=0}^{\infty} \psi_\ell(X)\varepsilon^\ell\right) = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i} D\varphi_j(X)\psi_{i-j}(X)\right)\varepsilon^{i+1}.$$

By proposition 2.1.4 and the definition of 1-majorizing series,

$$\begin{aligned}\left\|\sum_{j=0}^{i} D\varphi_j(X)\psi_{i-j}(X)\right\|_{i+1} &\leqslant \sum_{j=0}^{i} e(j+1)\left\|\varphi_j\right\|_j \left\|\psi_{i-j}\right\|_{i-j} \\ &\leqslant \sum_{j=0}^{i} e(j+1)j!a_j(i-j)!b_{i-j} \\ &= \sum_{j=0}^{i} e(j+1)!a_j(i-j)!b_{i-j} \\ &\leqslant (i+1)!e\sum_{j=0}^{i} a_j b_{i-j},\end{aligned}$$

where we have used that the reciprocal of any binomial coefficient is bounded by 1. Consequently

$$\begin{aligned}\left(\sum_{k=0}^{\infty} D\varphi_k(X)\varepsilon^{k+1}\right)\left(\sum_{\ell=0}^{\infty}\psi_\ell(X)\varepsilon^{\ell}\right) &\ll_X^1 \sum_{i=0}^{\infty} e\left(\sum_{j=0}^{i} a_j b_{i-j}\right) v^{i+1} \\ &= ev\widetilde{A}(v)\widetilde{B}(v).\end{aligned}$$

□

The following properties are straightforward to prove.

**Properties 2.1.7.** *Assume that*

$$\begin{aligned}\widetilde{\Phi} &\ll_X^1 \widetilde{A}, &&\textit{with } \widetilde{\Phi}\in\mathcal{O}\left(\mathbb{P}_n(R),\mathbb{C}^m\right)[\![\varepsilon]\!], \widetilde{A}\in\mathbb{R}[\![v]\!],\\ \widetilde{\Psi}_1 &\ll_X^1 \widetilde{B}_1, &&\textit{with } \widetilde{\Psi}_1\in\mathcal{O}\left(\mathbb{P}_n(R),\mathbb{C}^s\right)[\![\varepsilon]\!], \widetilde{B}_1\in\mathbb{R}[\![v]\!],\\ \widetilde{\Psi}_2 &\ll_X^1 \widetilde{B}_2, &&\textit{with } \widetilde{\Psi}_2\in\mathcal{O}\left(\mathbb{P}_n(R),\mathbb{C}^s\right)[\![\varepsilon]\!], \widetilde{B}_2\in\mathbb{R}[\![v]\!],\end{aligned}$$

*and*

$$\sup_{X\in\mathbb{P}_n(R)}\left\|\Lambda(X)\right\|_{\mathrm{op}} = C, \qquad \textit{for some } \Lambda\in\mathcal{O}\left(\mathbb{P}_n(R),\mathbb{C}^{m\times m}\right)$$

*(where $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm). Then:*

*(i)* $\left(\widetilde{\Psi}_1+\widetilde{\Psi}_2\right) \ll_X^1 \left(\widetilde{B}_1+\widetilde{B}_2\right)$,

*(ii)* $\Lambda\cdot\widetilde{\Phi} \ll_X^1 C\widetilde{A}$,

*(iii)* $\widetilde{\Psi}_1\left(\widetilde{\Phi}\right)^{\alpha} \ll_X^1 \widetilde{B}_1\widetilde{A}^{|\alpha|}$, *for all* $\alpha\in\mathbb{N}^m$ .

To prove the Gevrey property of the formal solution found in proposition 2.1.1 we rewrite equation (2.1.1), by expanding the functions $F$ and $H$ in fitting Taylor expansions, as

$$\begin{aligned}Z(X,\varepsilon) =&\varepsilon A(X)^{-1}\Bigg[D_X Z(X,\varepsilon)\left(\sum_{\alpha\in\mathbb{N}^m}\left(\sum_{q=0}^{\infty}F_{\alpha q}(X)\varepsilon^q\right)Z^{\alpha}(X,\varepsilon)\right)\\ &-\sum_{\alpha\in\mathbb{N}^m}\left(\sum_{q=0}^{\infty}H_{\alpha q}(X)\varepsilon^q\right)Z^{\alpha}(X,\varepsilon)\Bigg].\end{aligned}\tag{2.1.4}$$

We intend to associate to this equation what we will call a majorant equation. To that end we introduce the notations $f_{\alpha q} = \frac{1}{q!}\left\|F_{\alpha q}\right\|_q$, $h_{\alpha q} = \frac{1}{q!}\left\|H_{\alpha q}\right\|_q$ and

$$\widetilde{A} = \sup_{X\in\mathbb{P}_n(R)} \left\|A(X)^{-1}\right\|_{\mathrm{op}}, \quad \widetilde{f}_\alpha(v) = \sum_{q=0}^{\infty} f_{\alpha q} v^q, \qquad \widetilde{h}_\alpha(v) = \sum_{q=0}^{\infty} h_{\alpha q} v^q.$$

Let us hence state the so-called majorant equation

$$V(v) = v\widetilde{A}\left(eV(v)\sum_{\alpha\in\mathbb{N}^m}\widetilde{f}_\alpha(v)V^{|\alpha|}(v) + \sum_{\alpha\in\mathbb{N}^m}\widetilde{h}_\alpha(v)V^{|\alpha|}(v)\right). \tag{2.1.5}$$

Before relating equation (2.1.5) to (2.1.4), we claim that this equation has a convergent solution.

**Proposition 2.1.8.** *Equation* (2.1.5) *has a unique formal solution of the form*

$$\widetilde{V}(v) = \sum_{k=1}^{\infty} c_k v^k$$

*where* $c_k \in \mathbb{R}$. *Moreover, this series is convergent.*

**Proof:** We consider the space $\mathbb{C}[\![v]\!]$ with a formal series metric as in proposition 2.1.1. Completely analogous as in the proof of proposition 2.1.1 one shows that the map

$$\mathcal{S}\colon V(v) \mapsto v\widetilde{A}\left(eV(v)\sum_{\alpha\in\mathbb{N}^m}\widetilde{f}_\alpha(v)V^{|\alpha|}(v) + \sum_{\alpha\in\mathbb{N}^m}\widetilde{h}_\alpha(v)V^{|\alpha|}(v)\right) \tag{2.1.6}$$

from $\overline{B}\left(0,\frac{1}{2}\right) \subset \mathbb{C}[\![v]\!]$ to itself is well defined and a contraction. Let us now deal with the convergence. Notice that since

$$F_{\alpha q}(X) = \frac{1}{\alpha!q!}\frac{\partial^{|\alpha|+q}F}{\partial Z^\alpha\partial\varepsilon^q}(X,0,0),$$

there exists, due to Cauchy's inequalities, an $M > 0$ such that

$$\sup_{X\in\mathbb{P}_n(R)} |F_{\alpha q}| \leqslant \frac{M}{R^{|\alpha|+q}}.$$

This implies that

$$f_{\alpha q} = \frac{1}{q!}\left\|F_{\alpha q}\right\|_q \leqslant \frac{M}{q!R^{|\alpha|+q}}\left(\frac{R}{n}\right)^q = \frac{M}{q!R^{|\alpha|}n^q}.$$

It explains that $\widetilde{f_\alpha}$ is an entire function satisfying

$$\left|\widetilde{f}_\alpha(v)\right| \leqslant \frac{Me^{|v|/n}}{R^{|\alpha|}}.$$

Since the expression $\sum_{\alpha\in\mathbb{N}^m}\widetilde{f}_\alpha(v)V^{|\alpha|}$ can be written as

$$\sum_{p=0}\sum_{\substack{\alpha\in\mathbb{N}^m\\|\alpha|=p}}\widetilde{f}_\alpha(v)V^p$$

and since

$$\left|\sum_{\substack{\alpha\in\mathbb{N}^m\\|\alpha|=p}} \tilde{f}_\alpha(v)V^p\right| \leqslant \sum_{\substack{\alpha\in\mathbb{N}^m\\|\alpha|=p}} \frac{Me^{|v|/n}}{R^p}|V|^p = Me^{|v|/n}\binom{p+m-1}{p}\left(\frac{|V|}{R}\right)^p,$$

the expression in (2.1.5) is actually convergent for $|V| < R$, taking into account that similar bounds as above also hold for the second term and using that

$$\sum_{p=0}^{\infty}\binom{p+m-1}{p}\left(\frac{|V|}{R}\right)^p = \frac{R}{R-|V|}.$$

One can thus apply the analytic implicit function theorem to (2.1.5), and it has a unique analytic solution which is 0 for $v = 0$. This implies that the formal series solution $\tilde{V}(v)$ is convergent. □

The following result shows that the name majorant equation is fitting for (2.1.5).

**Proposition 2.1.9.** *Given two formal series*

$$\tilde{\zeta}(X,\varepsilon) = \sum_{n=1}^{\infty}\zeta_n(X)\varepsilon^n \in \mathcal{O}\left(\mathbb{P}_n(R),\mathbb{C}^m\right)[\![\varepsilon]\!],$$
$$\tilde{\eta}(v) = \sum_{n=1}^{\infty}\eta_n v^n \in \mathbb{R}[\![v]\!],$$

*satisfying* $\tilde{\zeta} \ll_X^1 \tilde{\eta}$, *then*

$$\begin{aligned}\varepsilon A(X)^{-1}\Bigg[&D_X\tilde{\zeta}(X,\varepsilon)\sum_{\alpha\in\mathbb{N}^m}\left(\sum_{q=0}^{\infty}F_{\alpha q}(X)\varepsilon^q\right)\tilde{\zeta}^\alpha(X,\varepsilon)\\ &-\sum_{\alpha\in\mathbb{N}^m}\left(\sum_{q=0}^{\infty}H_{\alpha q}(X)\varepsilon^q\right)\tilde{\zeta}^\alpha(X,\varepsilon)\Bigg]\\ \ll_X^1 v\tilde{A}&\left(eV(v)\sum_{\alpha\in\mathbb{N}^m}\tilde{f}_\alpha(v)V^{|\alpha|}(v)+\sum_{\alpha\in\mathbb{N}^m}\tilde{h}_\alpha(v)V^{|\alpha|}(v)\right).\end{aligned}$$

**Proof:** Due to proposition 2.1.6 and the properties in 2.1.7, it suffices to show that

$$\sum_{\alpha\in\mathbb{N}^m}\left(\sum_{q=0}^{\infty}F_{\alpha q}(X)\varepsilon^q\right)\left(\sum_{k=1}^{\infty}\zeta_k(X)\varepsilon^k\right)^\alpha \ll_X^1 \sum_{\alpha\in\mathbb{N}^m}\tilde{f}_\alpha(v)\tilde{\eta}^{|\alpha|}(v),$$
$$\sum_{\alpha\in\mathbb{N}^m}\left(\sum_{q=0}^{\infty}H_{\alpha q}(X)\varepsilon^q\right)\left(\sum_{k=1}^{\infty}\zeta_k(X)\varepsilon^k\right)^\alpha \ll_X^1 \sum_{\alpha\in\mathbb{N}^m}\tilde{h}_\alpha(v)\tilde{\eta}^{|\alpha|}(v).$$

Since the proofs of both statements are identical, we concentrate on the first one. By construction $\sum_{q=0}^{\infty}F_{\alpha q}(X)\varepsilon^q \ll_X^1 \tilde{f}_\alpha(v)$. Let $\ell \geqslant 1$ since $\tilde{\zeta}$ has 0 as the coefficient of $\varepsilon^0$, the coefficient belonging to $\varepsilon^\ell$ in the formal series

$$\sum_{\alpha\in\mathbb{N}^m}\left(\sum_{q=0}^{\infty}F_{\alpha q}(X)\varepsilon^q\right)\left(\sum_{k=1}^{\infty}\zeta_k(X)\varepsilon^k\right)^\alpha$$

is equal to the one of the series

$$\sum_{|\alpha|\leqslant\ell}\left(\sum_{q=0}^{\infty}F_{\alpha q}(X)\varepsilon^q\right)\left(\sum_{k=1}^{\infty}\zeta_k(X)\varepsilon^k\right)^{\alpha}.$$

By the properties 2.1.7,

$$\sum_{|\alpha|\leqslant\ell}\left(\sum_{q=0}^{\infty}F_{\alpha q}(X)\varepsilon^q\right)\left(\sum_{k=1}^{\infty}\zeta_k(X)\varepsilon^k\right)^{\alpha} \ll_X^1 \sum_{|\alpha|\leqslant\ell}\widetilde{f}_\alpha(v)\widetilde{\eta}^{|\alpha|}(v).$$

Once again, due to the fact that $\widetilde{\eta}$ has no constant term, the coefficients of $v^\ell$ in the series $\sum_{|\alpha|\leqslant\ell}\widetilde{f}_\alpha(v)\widetilde{\eta}^{|\alpha|}(v)$ and $\sum_{\alpha\in\mathbb{N}^m}\widetilde{f}_\alpha(v)\widetilde{\eta}^{|\alpha|}(v)$ coincide.
We have thus proven that the $\ell$-th Nagumo norm of the coefficient of $\varepsilon^\ell$ in the series $\sum_{\alpha\in\mathbb{N}^m}\left(\sum_{q=0}^{\infty}F_{\alpha q}(X)\varepsilon^q\right)\left(\sum_{k=1}^{\infty}\zeta_k(X)\varepsilon^k\right)^{\alpha}$ is bounded by $\ell!$ times the coefficient of $v^\ell$ of the series $\sum_{\alpha\in\mathbb{N}^m}\widetilde{f}_\alpha(v)\widetilde{\eta}^{|\alpha|}(v)$, which means that the result holds. □

**Corollary 2.1.10.** *The formal solution $\widetilde{Z}(X,\varepsilon)$ to equation* (2.1.1) *is majorized by the formal solution $\widetilde{V}(v)$, to the majorant equation* (2.1.5)*, i.e.*

$$\widetilde{Z} \ll_X^1 \widetilde{V}.$$

**Proof:** Restating proposition 2.1.9 in terms of the maps (2.1.2) and (2.1.6) gives that for $\widetilde{\zeta}\in\overline{B}\left(0,\frac{1}{2}\right)\subset\mathcal{O}\left(\mathbb{P}_n(R),\mathbb{C}^m\right)[\![\varepsilon]\!]$ and $\widetilde{\eta}\in\overline{B}\left(0,\frac{1}{2}\right)\subset\mathbb{C}[\![v]\!]$ with $\widetilde{\zeta}\ll_X^1\widetilde{\eta}$ it holds that $\mathcal{T}\left(\widetilde{\zeta}\right)\ll_X^1\mathcal{S}\left(\widetilde{\eta}\right)$. Since $0\ll_X^1 0$, and $\widetilde{Z}=\lim_{n\to\infty}\mathcal{T}^n(0)$, $\widetilde{V}=\lim_{n\to\infty}\mathcal{S}^n(0)$ the result follows. □
Corollary 2.1.10 then immediately implies, together with the convergence of $\widetilde{V}$ (yielding $c_\ell\leqslant CD^\ell$), that for $0<T<R$,

$$\sup_{X\in\mathbb{P}_n(T)}\|Z_\ell(X)\|_{\max}\leqslant\|Z_\ell\|_\ell\left(\frac{n}{R-T}\right)^\ell\leqslant c_\ell\ell!\left(\frac{n}{R-T}\right)^\ell\leqslant C\left(\frac{nD}{R-T}\right)^\ell\ell!.$$

We have thus proven the proposition 2.1.2.

A more global version of this result can be easily deduced.

**Corollary 2.1.11.** *Let $\mathcal{K}\subset\mathbb{C}^n$ be compact and $\Omega\subset\mathbb{C}^n$ open with $\mathcal{K}\subset\Omega$. Suppose that the assumptions* (i) *to* (iii) *on* (2.0.2) *hold on $\Omega$ instead of locally around $(X_0,Z_0,0)$.*
*Then there exists a unique formal solution to* (2.0.2)*, $\widetilde{Z}(X,\varepsilon)\in\mathcal{O}\left(\mathcal{K},\mathbb{C}^m\right)[\![\varepsilon]\!]$, which is Gevrey-*1 *w.r.t. $\varepsilon$ uniformly for $X\in\mathcal{K}$.*

**Proof:** By proposition 2.1.2 such a solution exists locally around each point of $\mathcal{K}$, going to a finite subcover gives the result. □

## 2.2 Gevrey asymptotic slow manifolds at a regular point of the slow flow

We once again consider the system (2.0.2),

$$\left\{\begin{array}{lcl} \dot{X} & = & \varepsilon F(X,Z,\varepsilon) \\ \dot{Z} & = & A(X)Z+\varepsilon H(X,Z,\varepsilon) \end{array}\right. , \tag{2.2.1}$$

Together with assumptions (i)-(iii) on page 16.
In this section we impose the additional condition that the slow flow is nonsingular:

$$F\,(0,0,0) \neq 0.$$

We show that, under this condition, the formal solution obtained in the previous section can be realized as an actual solution of the slow manifold equation, more specifically, the following is proven throughout this section.

**Lemma 2.2.1.** *Let $\theta \in [0,2\pi[$, $\tau \in \left]0,\frac{\pi}{2}\right[$, there exists a solution to the slow manifold equation* (2.1.1)*, defined for $X$ in a neighbourhood, say $V$, of* 0 *and $\varepsilon \in S(\theta,2\delta,r)$, for a certain $r>0$, which is Gevrey-*1 *asymptotic, w.r.t. $\varepsilon$, uniformly for $X \in V$, to the unique, Gevrey-*1*, formal solution to the slow manifold equation, see proposition 2.1.2.*

We point out that a Gevrey solution on a small sector is the best one can hope for in general, i.e. in the absence of a singularity of the slow flow on the critical manifold there exist equations of the form (2.1.1) which do not admit a solution defined for $\varepsilon$ in an open sector of opening larger than $\pi$ or in other words, there is no direction in which a 1-summable solution exists. An example showing this is given in remark 3.1.4, the example is given in a setting of one slow and one fast variable but can be easily generalized to an arbitrary amount of slow variables. We have deferred this example to a later chapter since we have not yet introduced the relevant terminology.

The proof of lemma 2.2.1 consists of the following steps. First we will use the formal expansion of Gevrey type to identify a quasi-invariant manifold, i.e. where the invariance equation shows an error that is exponentially small in $\varepsilon$. We then rewrite the equation relative to this quasi-invariant manifold, and try to solve the rewritten equation. We attempt this by using a formal power series approach w.r.t. $x$ (one of the slow variables), aiming at proving convergence of this formal series by using a majorant method again. We will succeed in doing so, defining a majorant equation of PDE type. The section finishes by proving the presence of a convergent solution to the majorant equation.

### 2.2.1 Preparing the equation

Denote the formal solution to the slow manifold equation (2.1.1) by $\widetilde{Z} = \widetilde{Z}(X, \varepsilon)$. By the Borel-Ritt-Gevrey theorem 1.2.10, there exist $R, r > 0$ and a function $\widehat{Z} = \widehat{Z}(X, \varepsilon)$, holomorphic for $(X, \varepsilon) \in \mathbb{P}_n(R) \times S(\theta, 2\tau, r)$, Gevrey-1 asymptotic to $\widetilde{Z}$. We define the following error term

$$\mathcal{R}(X, \varepsilon) = \varepsilon D_X \widehat{Z} F\left(X, \widehat{Z}, \varepsilon\right) - A(X)\widehat{Z} - \varepsilon H\left(X, \widehat{Z}, \varepsilon\right). \tag{2.2.2}$$

Since $\widetilde{Z}$ formally solves the slow manifold equation, we have that $\mathcal{R} \sim_1 0$ by lemma 1.2.13, implying there exist $K, L > 0$ such that for all $\varepsilon \in S(\theta, 2\tau, r)$

$$\sup_{X \in \mathbb{P}_n(R)} |\mathcal{R}(X, \varepsilon)| \leqslant K e^{-\frac{L}{|\varepsilon|}}.$$

If there exists a solution to the equation

$$\begin{aligned} \varepsilon D_X \Delta F(X, \widehat{Z} + \Delta, \varepsilon) =& A(X)\Delta - \varepsilon D_X \widehat{Z}\left(F(X, \widehat{Z} + \Delta, \varepsilon) - F(X, \widehat{Z}, \varepsilon)\right) \\ &+ \varepsilon\left(H(X, \Delta + \widehat{Z}, \varepsilon) - H(X, \widehat{Z}, \varepsilon)\right) - \mathcal{R}(X, \varepsilon), \end{aligned} \tag{2.2.3}$$

exponentially small w.r.t. $\varepsilon$ on a sector contained in $S(\theta, 2\delta, r)$ and holomorphic on a subset of $\mathbb{P}_n(R) \times S(\theta, 2\tau, r)$, it induces a solution to (2.1.1) which is Gevrey-1 asymptotic to $\widetilde{Z}$ by setting $Z = \widehat{Z} + \Delta$.

**Remark 2.2.2.** *The exposition that is to follow is primarily aimed at systems with two or more slow variables, i.e. $n \geqslant 2$. In the case of one slow variable, the used method is still valid by essentially disregarding all variables $Y$, as defined below. However the case of one slow variable has already been treated and the result can be achieved in a slightly easier manner, this is done for example in section 6 of [CDRSS00].*

Let $F_1, \ldots, F_n$ denote the component functions of $F$, since we are assuming that $F(0,0,0) \neq 0$, there exists $k \in \{1, \ldots, n\}$ such that $F_k(0,0,0) \neq 0$. We now rename and reorder the variables $X = (X_1, \ldots, X_n)$ by setting $x = X_k$ and denoting the remaining variables by $Y = (Y_1, \ldots, Y_{n-1})$. With slight abuse of notation we might replace $X$ with $(x, Y)$, the expression $D_X \Delta F\left(X, \widehat{Z} + \Delta, \varepsilon\right)$ is then given by

$$F_k(x, Y, \widehat{Z} + \Delta, \varepsilon)\frac{\partial \Delta}{\partial x} + D_Y \Delta F_*(x, Y, \widehat{Z} + \Delta, \varepsilon),$$

where $F_*$ denotes $F$ with its $k$-th component function removed.
By noticing that

$$F\left(x, Y, \widehat{Z} + \Delta, \varepsilon\right) - F\left(x, Y, \widehat{Z}, \varepsilon\right) = \int_0^1 D_Z F\left(x, Y, \widehat{Z} + u\Delta, \varepsilon\right) \Delta \mathrm{d}u$$

and similarly for $H$ we can, by denoting

$$\begin{aligned}
\mathcal{S}\left(x,Y,\Delta,\varepsilon\right)\Delta &= \tfrac{-D_{(x,Y)}\hat{Z}\int_0^1 D_Z F\left(x,Y,\hat{Z}+u\Delta,\varepsilon\right)\Delta \mathrm{d}u+\int_0^1 D_Z H\left(x,Y,\hat{Z}+u\Delta,\varepsilon\right)\Delta \mathrm{d}u}{F_k\left(x,Y,\hat{Z}+\Delta,\varepsilon\right)},\\
\mathcal{F}(x,Y,\Delta,\varepsilon) &= \tfrac{F_*(x,Y,\Delta+\hat{Z},\varepsilon)}{F_k(x,Y,\Delta+\hat{Z},\varepsilon},\\
\mathcal{A}(x,Y,\Delta,\varepsilon) &= \tfrac{A(x,Y)}{F_k(x,Y,\Delta+\hat{Z},\varepsilon)}\\
\mathcal{R}_1(x,Y,\Delta,\varepsilon) &= \tfrac{\mathcal{R}(x,Y,\varepsilon)}{\varepsilon F_k(x,Y,\Delta+\hat{Z},\varepsilon)},
\end{aligned}$$

rewrite (2.2.3) as

$$\frac{\partial \Delta}{\partial x} = -D_Y\Delta\mathcal{F} + \frac{\mathcal{A}}{\varepsilon}\Delta + \mathcal{S}\Delta - \mathcal{R}_1. \tag{2.2.4}$$

### 2.2.2 Formal expansions in terms of $x$

**Proposition 2.2.3.** *Consider equation (2.2.4). There exists a unique formal solution of the form*

$$\widetilde{\Delta}\left(x,Y,\varepsilon\right) = \sum_{k=1}^{\infty}\delta_k\left(Y,\varepsilon\right)x^k \tag{2.2.5}$$

*with* $\delta_k \in \mathcal{O}\left(\mathbb{P}_{n-1}(R)\times S\left(\theta,2\tau,r\right),\mathbb{C}^m\right)$.

**Proof:** Analogous as in section 2.1.2 and in particular the proof of proposition 2.1.1, we consider the formal series spaces $\mathcal{O}\left(\mathbb{P}_{n-1}(R)\times S\left(\theta,2\tau,r\right),\mathbb{C}^s\right)[\![x]\!]$, equipped with the formal series metric. One can then show that the map, given by

$$\begin{aligned}
&\mathcal{V}\left(\Delta\left(x,Y,\varepsilon\right)\right)\\
&= \int_0^x -D_Y\Delta\left(u,Y,\varepsilon\right)\mathcal{F}\left(u,Y,\Delta\left(u,Y,\varepsilon\right),\varepsilon\right)\\
&\quad+\left(\frac{\mathcal{A}\left(u,Y,\Delta\left(u,Y,\varepsilon\right),\varepsilon\right)}{\varepsilon}+\mathcal{S}\left(u,Y,\varepsilon,\Delta\left(u,Y,\varepsilon\right)\right)\right)\Delta\left(u,Y,\varepsilon\right)\\
&\qquad-\mathcal{R}_1\left(u,Y,\Delta\left(u,Y,\varepsilon\right),\varepsilon\right)\mathrm{d}u,
\end{aligned} \tag{2.2.6}$$

from $\mathcal{O}\left(\mathbb{P}_{n-1}(R)\times S\left(\theta,2\tau,r\right),\mathbb{C}^m\right)[\![x]\!]$ to itself is a contraction on $\overline{B}\left(0,\frac{1}{2}\right)$ and the fixed point of this contraction is the desired formal solution. ☐

We will continue this subsection by setting up a majorant equation for the formal series.

**Definition 2.2.4.** *Given formal series*

$$\begin{aligned}
G\left(x,Y,\varepsilon\right) &= \sum_{k=0}^{\infty}G_k\left(Y,\varepsilon\right)x^k\\
g(x,y,\beta) &= \sum_{k=0}^{\infty}g_k\left(y,\beta\right)x^k
\end{aligned}$$

*with* $G_k \in \mathcal{O}\left(\mathbb{P}_{n-1}(R)\times S\left(\theta,2\tau,r\right),\mathbb{C}^s\right)$, *where* $s\in\mathbb{N}_0$, $R,r>0$ *and*

$$g_k\colon \Omega\times\,]0,r[\,\to\mathbb{C}$$

*with* $\Omega \subset \mathbb{C}$ *an open neighbourhood of* $0$*, such that for all* $\beta \in ]0, r[$*,* $g_k(\cdot,\cdot,\beta) \in \mathcal{O}(\Omega, \mathbb{C})$*.*
*We say that* $G$ *is* *differentiably majorized* *by* $g$*, denoted by* $G \ll_D g$*, if for all* $k \in \mathbb{N}$*,* $q \in \mathbb{N}^{n-1}$ *and all* $\varepsilon \in S(\theta, 2\tau, r)$*,*

$$\left\| D_Y^q G_k(0,\varepsilon) \right\|_{\mathrm{op}} \leqslant \frac{\partial^q g_k}{\partial y^q}(0, |\varepsilon|).$$

**Property 2.2.5.** *Using the notations of the above definition, suppose that* $G \ll_D g$ *and there exist* $0 < T < R$*,* $0 < r_1 < r$ *and* $K, L > 0$ *such that for all* $\beta \in ]0, r_1[$ *we have* $g(\cdot,\cdot,\beta) \in \mathcal{O}(\mathbb{P}_2(T), \mathbb{C})$ *satisfying*

$$\max_{|x|,|y|\leqslant T} |g(x,y,\beta)| \leqslant K e^{-\frac{L}{\beta}}.$$

*Then, for any* $0 < c < T$*,* $G \in \mathcal{O}(B(0,T-c) \times \mathbb{P}_{n-1}(T-c) \times S(\theta, 2\tau, r_1), \mathbb{C}^s)$ *with*

$$\sup_{|x|,\|Y\|_{\max} < T-c} \|G(x,Y,\varepsilon)\|_{\max} \leqslant K_1 e^{-\frac{L}{|\varepsilon|}}$$

*for a certain* $K_1 > 0$ *(which depends on* $c$*).*

**Proof:** Choose any $0 < c < T$, by the Cauchy inequalities we have that

$$\left| \frac{\partial^q g_k}{\partial y^q}(0,\beta) \right| = \frac{1}{k!} \left| \frac{\partial^{q+k} g}{\partial x^k y^q}(0,0,\beta) \right| \leqslant \frac{q!}{\left(T - \frac{c}{2}\right)^{k+q}} K e^{-\frac{L}{\beta}}$$

and thus

$$\begin{aligned}
\sum_{k=0}^{\infty} \|G_k(Y,\varepsilon)\|_{\max} |x|^k &\leqslant \sum_{k=0}^{\infty} \sum_{q=0}^{\infty} \frac{1}{q!} \left\| D_Y^q G_k(0,\varepsilon) \right\|_{\mathrm{op}} \|Y\|_{\max}^q |x|^k \\
&\leqslant \sum_{k=0}^{\infty} \sum_{q=0}^{\infty} \frac{\|Y\|_{\max}^q |x|^k}{\left(T - \frac{c}{2}\right)^{k+q}} K e^{-\frac{L}{|\varepsilon|}} \\
&= \frac{\left(T - \frac{c}{2}\right)^2}{\left(T - \frac{c}{2} - \|Y\|_{\max}\right)\left(T - \frac{c}{2} - |x|\right)} K e^{-\frac{L}{|\varepsilon|}}.
\end{aligned}$$

□

**Proposition 2.2.6.** *Let* $G, H, g, h$ *be formal series such that*

$$G \ll_D g$$
$$H \ll_D h$$

*and* $D_Y G(x,Y,\varepsilon) H(x,Y,\varepsilon)$ *is defined, then*

$$D_Y G \cdot H \ll_D \frac{\partial g}{\partial y} \cdot h$$

**Proof:** We have

$$\begin{aligned} D_Y G(x,Y,\varepsilon) H(x,Y,\varepsilon) &= \left(\sum_{k=0}^{\infty} D_Y G_k(Y,\varepsilon) x^k\right)\left(\sum_{k=0}^{\infty} H_k(Y) x^k\right) \\ &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} D_Y G_\ell(Y,\varepsilon) H_{k-\ell}(Y) x^k. \end{aligned}$$

Fix $k, q \in \mathbb{N}$ and denote for all $j \in \{1,\dots,q\}$

$$T_j^q = \{V \subset \{1,\dots,q\} \mid |V| = j\},$$

where we assume that the elements of $T_j^q$ are ordered sets w.r.t. $<$. Then

$$\begin{aligned} & D^q\left(Y \mapsto \sum_{\ell=0}^{k} D_Y G_\ell(Y,\varepsilon) H_{k-\ell}(Y,\varepsilon)\right)(Z) \\ &= \sum_{\ell=0}^{k}\sum_{j=0}^{q}\sum_{V\in T_j^q} D_Y^{q-j+1} G_\ell(Z,\varepsilon) P_{V^c} D_Y^j H_{k-\ell}(Z,\varepsilon) P_V. \end{aligned}$$

Here, if we denote $V = \{v_1,\dots,v_j\}$, $V^c = \{w_1,\dots,w_{q-j}\}$,

$$\begin{aligned} & \left(D_Y^{q-j+1} G_\ell(Z,\varepsilon) P_{V^c} D_Y^j H_{k-\ell}(Z,\varepsilon) P_V\right)(h_1,\dots,h_q) \\ &= D_Y^{q-j+1} G_\ell(Z,\varepsilon)\left(h_{w_1},\dots,h_{w_{q-j}}\right) D_Y^j H_{k-\ell}(Z,\varepsilon)\left(h_{v_1},\dots,h_{v_j}\right). \end{aligned}$$

Using that $G \ll_D g$ and $H \ll_D h$, we then get

$$\begin{aligned} & \left\| D^q\left(Y \mapsto \sum_{\ell=0}^{k} D_y G_\ell(Y,\varepsilon) H_{k-\ell}(Y,\varepsilon)\right)(0)\right\|_{\mathrm{op}} \\ &\leqslant \sum_{\ell=0}^{k}\sum_{j=0}^{q}\sum_{V\in T_j^q} \left\|D_Y^{q-j+1} G_\ell(0,\varepsilon)\right\|_{\mathrm{op}} \left\|D_Y^j H_{k-\ell}(0,\varepsilon)\right\|_{\mathrm{op}} \\ &\leqslant \sum_{\ell=0}^{k}\sum_{j=0}^{q}\sum_{V\in T_j^q} \frac{\partial^{q-j+1} g_\ell}{\partial y^{q-j+1}}(0,|\varepsilon|)\frac{\partial^j h_{k-\ell}}{\partial y^j}(0,|\varepsilon|) \\ &= \sum_{\ell=0}^{k}\sum_{j=0}^{q}\binom{q}{j}\frac{\partial^{q-j+1} g_\ell}{\partial y^{q-j+1}}(0,|\varepsilon|)\frac{\partial^j h_{k-\ell}}{\partial y^j}(0,|\varepsilon|) \\ &= \sum_{\ell=0}^{k}\frac{\partial}{\partial y^q}\left(\frac{\partial g_\ell}{\partial y}\cdot h_{k-\ell}\right)(0,|\varepsilon|) = \frac{\partial}{\partial y^q}\left(\sum_{\ell=0}^{k}\frac{\partial g_\ell}{\partial y}\cdot h_{k-\ell}\right)(0,|\varepsilon|). \end{aligned}$$

Consequently

$$\begin{aligned} D_Y G(x,Y,\varepsilon) H(x,Y,\varepsilon) &\ll_D \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\frac{\partial g_\ell}{\partial y}(y,\beta) h_{k-\ell}(y,\beta) x^k \\ &= \frac{\partial g}{\partial y}(x,y,\beta) h(x,y,\beta). \end{aligned}$$

□

The following properties can be easily derived from the definitions.

**Properties 2.2.7.** *Let $G, H_1, H_2, g, h_1, h_2$ be formal series such that $G \ll_D g$, $H_{1,2} \ll_D h_{1,2}$. The following properties hold:*

*(i)* $H_1 + H_2 \ll_D h_1 + h_2$,

*(ii)* $H_1 \cdot G \ll_D h_1 g$, *when the product is defined,*

*(iii)* $H_1 G^\alpha \ll_D h_1 g^{|\alpha|}$,

*(iv)* $\int_0^x G(u, Y, \varepsilon)\,\mathrm{d}u \ll_D \int_0^x g(u, y, \beta)\,\mathrm{d}u$.

Consider again equation (2.2.4) in its fixed point form (2.2.6), which we repeat here for the sake of convenience.

$$\begin{aligned}\Delta(x, Y, \varepsilon) = \int_0^x & -D_Y \Delta(u, Y, \varepsilon)\,\mathcal{F}(u, Y, \Delta(u, Y, \varepsilon), \varepsilon) \\ & + \left(\frac{\mathcal{A}(u, Y, \Delta(u, Y, \varepsilon), \varepsilon)}{\varepsilon} + \mathcal{S}(u, Y, \Delta(u, Y, \varepsilon), \varepsilon)\right) \Delta(u, Y, \varepsilon) \\ & - \sum_{\alpha \in \mathbb{N}^m} \mathcal{R}_1(u, Y, \Delta(u, Y, \varepsilon), \varepsilon)\,\mathrm{d}u.\end{aligned} \tag{2.2.7}$$

Expanding the functions in appropriate Taylor series, this can be rewritten as

$$\begin{aligned}\Delta(x, Y, \varepsilon) = \int_0^x & -D_Y \Delta(u, Y, \varepsilon) \sum_{\alpha \in \mathbb{N}^m} \mathcal{F}_\alpha(u, Y, \varepsilon)\,\Delta^\alpha(u, Y, \varepsilon) \\ & + \left(\sum_{\alpha \in \mathbb{N}^m} \left(\frac{\mathcal{A}_\alpha(u, Y, \varepsilon)}{\varepsilon} + \mathcal{S}_\alpha(u, Y, \varepsilon)\right) \Delta^\alpha(u, Y, \varepsilon)\right) \Delta(u, Y, \varepsilon) \\ & - \sum_{\alpha \in \mathbb{N}^m} \mathcal{R}_{1,\alpha}(u, Y, \varepsilon)\,\Delta^\alpha(u, Y, \varepsilon)\,\mathrm{d}u.\end{aligned}$$

We may assume, by if necessary reducing $R$ slightly, that the functions $\mathcal{F}$, $\mathcal{A}$, $\mathcal{S}$ and $\mathcal{R}_1$ are all holomorphic on $B(0, R) \times \mathbb{P}_{n-1}(R) \times \mathbb{P}_m(R) \times S(\theta, 2\tau, r)$ with a continuous extension to the closure of this set. Moreover we can assume that $\mathcal{F}$, $\mathcal{A}$, $\mathcal{S}$ are uniformly bounded on this closure by $M \geqslant 0$ and there exist $K, L \geqslant 0$ such that for all $\varepsilon \in S(\theta, 2\tau, r)$,

$$\max_{\overline{B}(0,R) \times \overline{\mathbb{P}}_{n-1}(R) \times \overline{\mathbb{P}}_m(R)} \|\mathcal{R}_1(x, Y, \Delta, \varepsilon)\|_{\max} \leqslant K e^{-\frac{L}{|\varepsilon|}}$$

Using this one sees that for

$$\mathcal{F}_\alpha(x, Y, \varepsilon) = \sum_{\ell=0}^\infty \mathcal{F}_{\alpha\ell}(Y, \varepsilon)\,x^\ell$$

we have $\|\mathcal{F}_{\alpha\ell}(Y, \varepsilon)\| \leqslant \frac{M}{R^{|\alpha|+\ell}}$ and from this it follows that

$$\|D_Y^q \mathcal{F}_{\alpha\ell}(0)\|_{\mathrm{op}} \leqslant q! \frac{M}{R^{q+|\alpha|+\ell}} \binom{q+n-2}{q}$$

such that

$$\mathcal{F}_\alpha(x,Y,\varepsilon) \ll_D \sum_{\ell=0}^{\infty}\sum_{q=0}^{\infty} \frac{M}{R^{q+|\alpha|+\ell}} \binom{q+n-2}{q} y^q x^\ell$$
$$= \frac{M}{R^{|\alpha|}} \sum_{\ell=0}^{\infty} \left(1-\frac{y}{R}\right)^{1-n} \left(\frac{x}{R}\right)^{\ell} = \frac{M}{R^{|\alpha|}} \left(1-\frac{y}{R}\right)^{1-n} \left(1-\frac{x}{R}\right)^{-1}$$

and similarly for the other functions.

One then constructs a majorant equation to (2.2.7),

$$V(x,y,\beta) = \int_0^x \left(\sum_{\alpha\in\mathbb{N}^m} \frac{M}{R^{|\alpha|}} \left(1-\frac{y}{R}\right)^{1-n} \left(1-\frac{u}{R}\right)^{-1} V^{|\alpha|}(u,y)\right) \frac{\partial V}{\partial y}(u,y)$$
$$+ \left(\sum_{\alpha\in\mathbb{N}^m} \frac{M\left(\frac{1}{\beta}+1\right)}{R^{|\alpha|}} \left(1-\frac{y}{R}\right)^{1-n} \left(1-\frac{u}{R}\right)^{-1} V^{|\alpha|}(u,y)\right) V(u,y)$$
$$+ \left(\sum_{\alpha\in\mathbb{N}^m} \frac{Ke^{-\frac{L}{\beta}}}{R^{|\alpha|}} \left(1-\frac{y}{R}\right)^{1-n} \left(1-\frac{u}{R}\right)^{-1} V^{|\alpha|}(u,y)\right) \mathrm{d}u,$$

which is rewritten as

$$V = \int_0^x \frac{M\frac{\partial V}{\partial y}(u,y) + M\left(\frac{1}{\beta}+1\right)V(u,y) + Ke^{-\frac{L}{\beta}}}{Q(u,y,V(u,y))} \mathrm{d}u \tag{2.2.8}$$

where

$$Q(u,y,V) := \left(1-\frac{y}{R}\right)^{n-1} \left(1-\frac{u}{R}\right) \left(1-\frac{V}{R}\right)^{m}.$$

### 2.2.3 Convergence of the solution to the majorant equation

**Lemma 2.2.8.** *There exists a unique formal solution of the form*

$$V(x,y,\beta) = \sum_{k=1}^{\infty} V_k(y,\beta)\, x^k$$

*to* (2.2.8)*, where the functions* $V_k$ *are defined on* $B(0,R) \times ]0,r[$*. Moreover, for the unique formal solution,* $\widetilde{\Delta}$*, to* (2.2.4) *in proposition 2.2.3 it holds that* $\widetilde{\Delta}(x,Y,\varepsilon) \ll_D V(x,y,\beta)$

**Proof:** The proof is analogous to a combination of the proofs of proposition 2.1.9 and corollary 2.1.10. □

We now want to employ property 2.2.5, to show that $\widetilde{\Delta}(x,Y,\varepsilon)$ is holomorphic and exponentially decaying w.r.t. $\varepsilon$. The remainder of this section will thus be devoted to proving the following,

**Lemma 2.2.9.** *There exists an* $r_1 > 0$ *and* $0 < T < R$ *such that for all* $\beta \in ]0, r_1[$*,* $V(\cdot,\cdot,\beta) \in \mathcal{O}(\mathbb{P}_2(T), \mathbb{C})$ *with* $V$ *the unique formal solution to* (2.2.8)*. Moreover there exist* $K, L > 0$ *such that*

$$\sup_{(x,y)\in\mathbb{P}_2(T)} |V(x,y,\beta)| \leqslant Ke^{-\frac{L}{\beta}}.$$

The system of characteristic equations, with $\beta$ as a parameter, of the partial differential equation associated to (2.2.8) is given by

$$\left\{\begin{array}{rcl}\dot{x} &=& \left(1-\frac{x}{R}\right)\left(1-\frac{y}{R}\right)^{n-1}\left(1-\frac{V}{R}\right)^{m}\\ \dot{y} &=& -M\\ \dot{V} &=& M\left(\frac{1}{\beta}+1\right)V+Ke^{-\frac{L}{\beta}}\end{array}\right. \tag{2.2.9}$$

$$x(0,s)=0; y(0,s)=s; V(0,s)=0.$$

We see that the $\dot{V}$ is independent of $x$ and $y$ and by direct calculation

$$V(t,\beta)=\frac{Ke^{-\frac{L}{\beta}}}{M\left(\frac{1}{\beta}+1\right)}\left(e^{M\left(\frac{1}{\beta}+1\right)t}-1\right).$$

Notice the independence of $V$ on $s$. The function $V$ is, for each parameter value $\beta$, an entire function and if we fix a $0<c<\frac{L}{M}$ it is exponentially decaying w.r.t $\beta$, uniformly for $|t|\leqslant\frac{L}{M}-c$.
Using this, the $(\dot{x},\dot{y})$ equations are uniformly Lipschitz for $|t|\leqslant\frac{L}{M}-c$ and all $\beta$. By Picard's theorem, the solutions $x(t,s)$ and $y(t,s)$ (depending on $\beta$ ) are guaranteed to exist for $(t,s)$ in a neighbourhood of the origin, which is independent of $\beta$. Denote this neighbourhood by $A_0$ and define the map

$$h\colon A_0\times]0,\infty[\to\mathbb{C}^2\times]0,\infty[$$

given by

$$h(t,s,\beta)=(x(t,s,\beta),y(t,s,\beta),\beta)$$

With these notations, lemma 2.2.9 is then equivalent to the following.

**Lemma 2.2.10.** *There exists an $r_1>0$, $0<T<R$ and a neighbourhood $A_1\subset A_0$ of $(0,0)$ such that for all $\beta\in]0,r_1[$, $h(\cdot,\cdot,\beta)$ is a biholomorphism on $A_1$ with inverse defined on a set containing $\mathbb{P}_2(T)$.*

**Proof:** One can show that $h$ can be extended in a $\mathcal{C}^1$ manner to the set $A_0\times[0,\infty[$ and the differential of this extension in $(0,0,0)$ is given by $\begin{pmatrix}1&0&0\\M&1&0\\0&0&1\end{pmatrix}$. The inverse function theorem thus guarantees that (the extension of) $h$ is invertible on a neighbourhood $A_1\times[0,r_1[$. Denote this inverse by $g$
For each $\beta\in]0,r_1[$, $h(\cdot,\cdot,\beta)$ is a holomorphic map (solutions of a holomorphic ODE are holomorphic). This automatically implies that its inverse function, $g(\cdot,\cdot,\beta)$, is also holomorphic, see for example [Ros82].
We finish the proof by noticing that $h(0,0,0)=(0,0,0)$ and $g$ is thus necessarily defined on a neighbourhood of the origin. □

## 2.3 Summability of slow manifolds at singular points of slow flow

We once again consider the system (2.0.2), which we repeat here

$$\begin{cases} \dot{X} &= \varepsilon F(X, Z, \varepsilon) \\ \dot{Z} &= A(X) Z + \varepsilon H(X, Z, \varepsilon) \end{cases},$$

again making the assumptions (i)-(iii) on page 16.
Where in section 2.2 the slow manifold was investigated around regular points of the slow flow, we will now focus on equilibria of the slow flow. We will make two further restrictions on the class of systems we will treat.
The first one is on the dimensions of the system, we will assume that there is only one fast variable. Furthermore, at the equilibrium of the slow flow, the eigenvalues of the linearised slow flow lie in the Poincaré domain, or formulated differently, all eigenvalues lie in an open sector of opening at most $\pi$. Concretely we make the following additional assumptions.

(i) $Z \in \mathbb{C}$ (notice that this means $A(X) \in \mathbb{C}$ and assumption (i) on page 16 just reads $A(X) \neq 0$).

(ii) There exists an $X_0$ for which $F(X_0, 0, 0) = 0$.

(iii) There exists an open sector, $S$, of opening at most $\pi$ such that the eigenvalues of $D_X F(X_0, 0, 0)$ all lie in this sector.

Under these conditions we prove.

**Theorem 2.3.1.** *There exists a direction $\beta \in [0, 2\pi[$ and a neighbourhood $W$ of $X_0$ such that (2.0.2) has an invariant manifold $z = \Psi(X, \varepsilon)$ that is Borel-1 summable in the direction $\theta$ (uniformly for $X \in W$).*
*There are constraints on the possible directions $\beta$, these are elaborated upon in lemma 2.3.9. We specify the directions of summability, that can be obtained from this lemma, in a few special cases that could be of interest in the setting of real analytic systems of equations.*
*Denote the eigenvalues of $D_X F(X_0, 0, 0)$ by $\lambda_1, \ldots, \lambda_n$.*

*(i) In the case of 1 slow and 1 fast variable, when there is a slow-fast saddle point, meaning $\lambda_1 A(0) < 0$, summability can be obtained in all directions lying in the strict right half-plane i.e. $\beta \in \left]-\frac{\pi}{2}, -\frac{\pi}{2}\right[$.*

*(ii) In the case of 2 slow and 1 fast variable:*

- *The slow dynamics has a unstable hyperbolic node on the normally attracting critical manifold, meaning that we have $A(0,0) < 0$ and $\lambda_1, \lambda_2 \in \mathbb{R}_+$. The summability can be obtained in all directions lying in the strict right half-plane.*
- *The slow dynamics has an unstable hyperbolic focus on the normally attracting critical manifold, meaning that we have $A(0,0) < 0$ and $\lambda_{1,2} = a \pm ib$ with $a, b > 0$. Summability can be obtained in directions close to the positive real axis, where the size of the possible deviation is inversely proportional to the size $b/a$ i.e. there exists a function $\gamma$ satisfying $\gamma(0) = \frac{\pi}{2}$ and $\gamma(x) \to 0$ for $x \to \infty$, such that $\beta \in \left]-\gamma\left(\frac{b}{a}\right), \gamma\left(\frac{b}{a}\right)\right[$. In particular, 1-summability in the positive real direction is guaranteed.*

By the following remark, the conditions on the eigenvalues are essential to achieve summability of the slow manifold, some terminology and properties used in this remark will be introduced later on in sections 2.3.2 and 2.3.3.

**Remark 2.3.2.** *As the following example shows, it is not possible, in general, to find solutions that are 1-summable (i.e. 1-summable in all but finitely many directions) when our assumption on the positions of $\lambda_1, \ldots, \lambda_n$ holds. Moreover when this assumption is violated, thus when no sector of opening less than $\pi$ contains all $\lambda_j$, there is not one direction in which 1-summability is guaranteed.*
*Consider the slow manifold equation*

$$\varepsilon D_X z(X,\varepsilon)\Lambda X = z(X,\varepsilon) - \varepsilon \prod_{j=1}^{n} f(x_j)$$

*where $\Lambda$ is the diagonal matrix with entries $\lambda_1, \ldots, \lambda_n$ and $f$ is holomorphic on the unit disc but non-continuable to the boundary of the disc, one can take for example*

$$f(z) = \sum_{n=0}^{\infty} z^{n!}.$$

*Suppose that the above equation has a solution which is 1-summable in a certain direction. Applying the Borel transform of order 1 and denoting by $Z$ the Borel transform of the solutions shows that*

$$(1 * D_X Z)(X,\eta)\Lambda X = Z(X,\eta) - \prod_{j=1}^{n} f(x_j)$$

*must hold. One can check that this implies that*

$$Z(X,\eta) = \prod_{j=1}^{n} f\left(x_j e^{\lambda_j \eta}\right).$$

*Due to the 1-summability in a certain direction, $Z$ is defined, by theorem 1.2.20, for $\eta$ in an (open) infinite sector around this direction. We denote this sector by $S_\eta$.*

*Suppose that our assumptions on $\Lambda$ hold and that $\overline{S}(\theta_0, 2\delta) \setminus \{0\}$, with $\delta < \frac{\pi}{2}$, is the smallest sector which contains all $\lambda_j$. It is clear that all exponentials $e^{\lambda_j \eta}$ should remain bounded on $S_\eta$ which is equivalent to*

$$S_\eta \subset \bigcap_{j=1}^{n} \overline{S}(\pi - \arg(\lambda_j), \pi) = \overline{S}(\pi - \theta_0, \pi - 2\delta),$$

*consequently, summability can not be obtained in a direction not contained in*

$$S(\pi - \theta_0, \pi - 2\delta).$$

*Suppose now that the assumption is violated. Let $\tau$ be the bisecting direction of $S_\eta$, it must then hold in particular that $e^{i\tau}\lambda_j \in \overline{S}(\pi, \pi)$ or equivalently $\lambda_j \in \overline{S}(\pi - \tau, \pi)$ for all $j = 1, \ldots, n$. If $\lambda_j \in S(\pi - \tau, \pi)$ for all $j = 1, \ldots, n$, then $\Lambda$ does satisfy our assumption, which is a contradiction. Otherwise there are $\lambda_{j_1}$ and $\lambda_{j_2}$ for which* $\arg(\lambda_{j_2}) = \arg(\lambda_{j_1}) + \pi$ *such that it should hold that* $S_\eta \subset \overline{S}(\pi - \arg(\lambda_{j_1}), \pi) \cap \overline{S}(-\arg(\lambda_{j_1}), \pi)$ *which is of course impossible.*

Also when there is more than one fast variable, summability can not be guaranteed.

**Example 2.3.3.** *Consider the slow manifold equation*

$$\begin{aligned} \varepsilon x \frac{\partial z_1}{\partial x} &= z_2 - \varepsilon f(x) \\ \varepsilon x \frac{\partial z_2}{\partial x} &= z_1 \end{aligned}$$

*where $f$ is as in remark 2.3.2. If this equation has a solution that is 1-summable in a direction, the functions $Z_1(X, \eta), Z_2(X, \eta)$ satisfying*

$$\begin{aligned} 1 * x \frac{\partial Z_1}{\partial x} &= Z_2 - f(x) \\ 1 * x \frac{\partial Z_2}{\partial x} &= Z_1 \end{aligned}$$

*should both be defined on the the same infinite sector $S_\eta$. One can check that*

$$Z_1(X, \eta) = \frac{1}{2}\left(f(xe^{\eta}) - f\left(xe^{-\eta}\right)\right), \quad Z_2(X, \eta) = \frac{1}{2}\left(f(xe^{\eta}) + f\left(xe^{-\eta}\right)\right),$$

*implying that $S_\eta \subset \overline{S}(0, \pi) \cap \overline{S}(\pi, \pi)$ which is clearly impossible.*

It is plausible that by imposing certain conditions on $A(X)$ in (2.0.2) that summability results could be achieved for more than one fasts variable, perhaps even employing the techniques that we will use in what is to follow. We have not pursued this any further in this thesis.

We now commence with the proof of theorem 2.3.1.
Using assumptions (i)-(iii) on page i, we will first bring the system into a simpler form.

### 2.3.1 Simplifying the system

Consider a system

$$\left\{\begin{array}{rcl}\dot{X} & = & \varepsilon F(X,z,\varepsilon)\\ \dot{z} & = & \varphi(X)z+\varepsilon H(X,z,\varepsilon)\end{array}\right.$$

where $\varphi\in\mathcal{O}\left(\overline{\mathbb{P}}_n(R)\right)$, $F,H\in\mathcal{O}\left(\overline{\mathbb{P}}_{n+2}(R),\mathbb{C}^n\right)$ and there exists an $X_0$ such that $F(X_0,0,0)=0$, $\varphi(X_0)\neq 0$. Moreover if we denote the eigenvalues of $D_XF(X_0,0,0)$ by $\lambda_1,\ldots,\lambda_n$ (repeated by their multiplicity) we assume that there exists an open sector $S$, of opening less than $\pi$ such that $\lambda_j\in S$ for all $j$. Since the sector is open this implies in particular that $\lambda_j\neq 0$ for all $j=1,\ldots,n$.

We now make a series of transformations, simplifying the above system. It is important that throughout these transformations, whenever we denote $\lambda_j$, these are the eigenvalues specified above.

Firstly, set $X_1=X-X_0$ and $z=\varepsilon\left(z_1-\frac{H(X,0,0)}{\varphi(X)}\right)$, this transforms the system into

$$\left\{\begin{array}{rcl}\dot{X}_1 & = & \varepsilon F\left(X_1+X_0,\varepsilon z_1-\varepsilon\frac{H(X,0,0)}{\varphi(X)},\varepsilon\right)\\ \dot{z}_1 & = & \varphi(X_1+X_0)z_1+\ H\left(X_1+X_0,\varepsilon\left(z_1-\frac{H(X,0,0)}{\varphi(X)}\right),\varepsilon\right)\\ & & -H(X_1+X_0,0,0)\\ & & +\varepsilon D_X\left(\frac{H(X,0,0)}{\varphi(X)}\right)F\left(X_1+X_0,\varepsilon z_1-\varepsilon\frac{H(X,0,0)}{\varphi(X)},\varepsilon\right)\end{array}\right..$$

Which is, by dropping the subscripts, of the form

$$\left\{\begin{array}{rcl}\dot{X} & = & \varepsilon F(X,\varepsilon z,\varepsilon)\\ \dot{z} & = & \varphi(X)z+\varepsilon H(X,z,\varepsilon)\end{array}\right.$$

where $F(0,0,0)=0$, $\varphi(0)\neq 0$ and the eigenvalues of $D_XF(0,0,0)$ are given by $\lambda_j$, $j=1,\ldots,n$.

Now

$$F(X,\varepsilon z,\varepsilon)=F(X,0,0)+\varepsilon zF_1(X,\varepsilon z,\varepsilon)+\varepsilon F_2(X,\varepsilon z,\varepsilon),$$

for certain functions $F_1,F_2$, and

$$F(X,0,0)=D_XF(0,0,0)X+\mathrm{O}(X^2).$$

There exists a matrix $P$ such that $P^{-1}D_XF(0,0,0)P$ is in Jordan normal form, which we denote by $\Lambda+U$, setting $X_1=P^{-1}X$ gives us a system of the form (writing $X_1=X$)

$$\left\{\begin{array}{rcl}\dot{X} & = & \varepsilon(\Lambda+U)X+\varepsilon A(X)+\varepsilon^2V(X,z,\varepsilon)\\ \dot{z} & = & \varphi(X)z+\varepsilon H(X,z,\varepsilon)\end{array}\right., \tag{2.3.1}$$

such that $A\in\mathcal{O}\left(\overline{\mathbb{P}}_n(R),\mathbb{C}^n\right)$, $\varphi\in\mathcal{O}\left(\overline{\mathbb{P}}_n(R)\right)$, $H,V\in\mathcal{O}\left(\overline{\mathbb{P}}_{n+2}(R),\mathbb{C}^n\right)$. We furthermore have that

- $\varphi(0)\neq 0$.

- $A = \mathrm{O}(X^2)$.
- $\Lambda$ is a diagonal matrix with its diagonal entries given by $\lambda = (\lambda_1, \ldots, \lambda_n)$.
- There exists an open sector $S$, with vertex at the origin of opening less than $\pi$ such that $\lambda_i \in S$, $\forall i \in \{1, \ldots, n\}$ .
- The matrix $U$ has only non-zero entries on its superdiagonal and if such an entry is not zero, it is equal to 1, we denote these entries with $\zeta_{k,k+1}$ for $k = 1, \ldots, n-1$. One can be more stringent in when the $\zeta$ are 0 or 1 but we will not need this. In section 2.3.8 we do use the fact that $\Lambda + U$ is a Jordan normal form.

The proof of the theorem 2.3.1 involves solving the slow manifold equation, given in this simplified system by

$$\begin{aligned}\varepsilon D_X z \cdot \Lambda X - \varphi(X)\, z =& \varepsilon H(X, z, \varepsilon) - \varepsilon D_X z \cdot UX \\ & - \varepsilon D_X z \cdot A(X) - \varepsilon^2 D_X z V(X, z, \varepsilon).\end{aligned} \tag{2.3.2}$$

By the results in section 2.1.2, we already know that this equation has a Gevrey-1 formal solution, $\tilde{z}(X, \varepsilon)$. Our strategy for improving this result towards summability is inspired by theorem 1.2.20; we will thus search for a holomorphic continuation to an infinite sector of the Borel transform of $\tilde{z}$. We now introduce the spaces in which this continuation will be found.

### 2.3.2 Setting up Banach spaces

Let $\mu > 0$, $\bar{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$, and $S$ some infinite sector. We define

$$\mathcal{G}^\mu := \left\{ h \in \mathcal{O}(S) \;\middle|\; \|h\|_{\mu,S} = \sup_{\eta \in S} |h(\eta)|\,(1 + \mu^2 |\eta|^2) e^{-\mu|\eta|} < \infty \right\}.$$

and

$$\begin{aligned}&\mathcal{G}^\mu_{\bar{r}}\{X\} \\ &:= \left\{ F(X, \eta) = \sum_{\gamma \in \mathbb{N}^n} F_\gamma(\eta) X^\gamma \;\middle|\; F_\gamma \in \mathcal{G}^\mu \text{ and } {}_{\bar{r}}\|F\|_{\mu,S} < \infty \right\}\end{aligned}$$

with

$${}_{\bar{r}}\|F\|_{\mu,S} = \sum_{\gamma \in \mathbb{N}^n} \|F_\gamma\|_{\mu,S}\, \bar{r}^\gamma.$$

Clearly an element $F \in \mathcal{G}^\mu_{\bar{r}}\{X\}$ satisfies

$$F \in \mathcal{O}(\mathbb{P}_n(\bar{r}) \times S)$$

such that $F$ is of exponential growth of order at most 1. The following is obvious.

**Property 2.3.4.** *If $\tilde{\mu} \geqslant \mu$ then $\|f\|_{\tilde{\mu},S} \leqslant \|f\|_{\mu,S}$ and thus $\mathcal{G}^{\mu} \subset \mathcal{G}^{\tilde{\mu}}$. Consequently if also $s_i \leqslant r_i$ for all $i = 1, \ldots, n$, we have that ${}_{\bar{s}}\|F\|_{\tilde{\mu},S} \leqslant {}_{\bar{r}}\|F\|_{\mu,S}$ such that $\mathcal{G}^{\mu}_{\bar{r}}\{X\} \subset \mathcal{G}^{\tilde{\mu}}_{\bar{s}}\{X\}$.*

We now show that $\mathcal{G}^{\mu}_{\bar{r}}\{X\}$ is a Banach space, for this we first need that $\mathcal{G}^{\mu}$ is one.

**Lemma 2.3.5.** *For all $\mu > 0$, $\mathcal{G}^{\mu}$ equipped with the norm $\|\cdot\|_{\mu,S}$ is a Banach space.*

**Proof:** It is a straightforward verification that $\left(\mathcal{G}^{\mu}, \|\cdot\|_{\mu,S}\right)$ is a normed vector space.
Let $(g_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\left(\mathcal{G}^{\mu}, \|\cdot\|_{\mu,S}\right)$. Since for all $\eta \in S$ and $p, q \in \mathbb{N}$,

$$|g_p(\eta) - g_q(\eta)| \leqslant \|g_p - g_q\|_{\mu,S} \frac{e^{\mu|\eta|}}{1 + \mu^2 |\eta|^2},$$

it is clear that $(g_n)_{n\in\mathbb{N}}$ is a uniform Cauchy sequence on all compact subsets of $S$. This implies that there exists a $g \in \mathcal{O}(S)$ such that $g_n \to g$ in the standard Fréchet space topology on $\mathcal{O}(S)$ i.e. $g_n \to g$ uniformly on all compact subsets of $S$, see for example [Mos02].
It remains to show that $g_n \to g$ in $\mathcal{G}^{\mu}$. Let $\tau > 0$ and choose $N \in \mathbb{N}$ such that $\|g_m - g_N\|_{\mu,S} < \frac{\tau}{2}$ for all $m \geqslant N$. Fix any $\eta \in S$, since in particular $g_n \to g$ point wise over $S$, we can choose an $m \geqslant N$ such that

$$|g(\eta) - g_m(\eta)| < \frac{\tau e^{\mu|\eta|}}{2\left(1 + \mu^2 |\eta|^2\right)}$$

and thus

$$\begin{aligned}
&|g(\eta) - g_N(\eta)| \left(1 + \mu^2 |\eta|^2\right) e^{-\mu|\eta|} \\
&\leqslant \left(|g(\eta) - g_m(\eta)| + |g_m(\eta) - g_N(\eta)|\right) \left(1 + \mu^2 |\eta|^2\right) e^{-\mu|\eta|} \\
&< \frac{\tau}{2} + \|g_m - g_N\|_{\mu,S} \\
&< \tau.
\end{aligned}$$

Clearly this implies that $\|g - g_N\|_{\mu,S} < \tau$, proving that $g \in \mathcal{G}^{\mu}$ and $g_n \to g$ in this space. □

**Lemma 2.3.6.** *For every $\mu > 0$ and $\bar{r} \in \mathbb{R}^n_{>0}$, $\mathcal{G}^{\mu}_{\bar{r}}\{X\}$, equipped with the norm ${}_{\bar{r}}\|\cdot\|_{\mu,S}$, is a Banach space.*

**Proof:** Let $(F_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{G}^{\mu}_{\bar{r}}\{X\}$, each $F_n$ is of the form

$$F_n = \sum_{\gamma} f^n_{\gamma}(\eta) X^{\gamma}.$$

For each $\tau > 0$, there exists an $N \in \mathbb{N}$ such that for all $k, l \geqslant N$

$$\sum_{\gamma} \left\| f_\gamma^k - f_\gamma^l \right\|_{\mu,S} \bar{r}^\gamma < \tau.$$

It is thus clear that for all $\gamma \in \mathbb{N}^n$, $(f_\gamma^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{G}^\mu$. Due to the completeness of $\mathcal{G}^\mu$, there exists an $f_\gamma \in \mathcal{G}^\mu$ with $f_\gamma^n \xrightarrow{n\to\infty} f_\gamma$.
Put

$$F(X,\eta) = \sum_{\gamma} f_\gamma(\eta) X^\gamma,$$

we show that $F \in \mathcal{G}_{\bar{r}}^\mu \{X\}$ and $F_n \xrightarrow{n\to\infty} F$. Choose a random $\sigma > 0$. Take $n_0$ such that for all $n \geqslant n_0$

$$\sum_{\gamma} \left\| f_\gamma^n - f_\gamma^{n_0} \right\|_{\mu,S} \bar{r}^\gamma < \frac{\sigma}{4}$$

and

$$\left\| f_0 - f_0^{n_0} \right\|_{\mu,S} < \frac{\sigma}{4}.$$

Now pick successively, for each $l \geqslant 1$, an $n_l \geqslant n_{l-1}$ such that for all $n \geqslant n_l$

$$\sum_{\gamma} \left\| f_\gamma^n - f_\gamma^{n_l} \right\|_{\mu,S} \bar{r}^\gamma < \frac{\sigma}{2^{l+1}} \tag{2.3.3}$$

and

$$\sum_{|\gamma|=l} \left\| f_\gamma - f_\gamma^{n_l} \right\|_{\mu,S} \bar{r}^\gamma < \frac{\sigma}{2^{l+2}}. \tag{2.3.4}$$

We have that

$$\begin{aligned}
\sum_{\gamma} \left\| f_\gamma - f_\gamma^{n_0} \right\|_{\mu,S} \bar{r}^\gamma &= \sum_{l=0}^{\infty} \sum_{|\gamma|=l} \left\| f_\gamma - f_\gamma^{n_0} \right\|_{\mu,S} \bar{r}^\gamma \\
&\leqslant \sum_{l=0}^{\infty} \sum_{|\gamma|=l} \left\| f_\gamma - f_\gamma^{n_l} \right\|_{\mu,S} \bar{r}^\gamma + \sum_{l=1}^{\infty} \sum_{|\gamma|=l} \left\| f_\gamma^{n_l} - f_\gamma^{n_0} \right\|_{\mu,S} \bar{r}^\gamma \\
&< \sum_{l=0}^{\infty} \frac{\sigma}{2^{l+2}} + \sum_{l=1}^{\infty} \sum_{|\gamma|=l} \left\| f_\gamma^{n_l} - f_\gamma^{n_0} \right\|_{\mu,S} \bar{r}^\gamma
\end{aligned}$$

where the last inequality is due to (2.3.4). Noticing that $f_\gamma^{n_l} - f_\gamma^{n_0} = \sum_{k=1}^{l} f_\gamma^{n_k} - f_\gamma^{n_{k-1}}$ we can further estimate the last expression by

$$\begin{aligned}
&\sum_{l=0}^{\infty} \frac{\sigma}{2^{l+2}} + \sum_{l=1}^{\infty} \sum_{k=1}^{l} \sum_{|\gamma|=l} \left\| f_\gamma^{n_k} - f_\gamma^{n_{k-1}} \right\|_{\mu,S} \bar{r}^\gamma \\
&= \frac{\sigma}{2} + \sum_{k=1}^{\infty} \sum_{l=k}^{\infty} \sum_{|\gamma|=l} \left\| f_\gamma^{n_k} - f_\gamma^{n_{k-1}} \right\|_{\mu,S} \bar{r}^\gamma.
\end{aligned}$$

Because we have chosen $n_k \geqslant n_{k-1}$ we get by (2.3.3) that

$$\sum_\gamma \left\| f_\gamma - f_\gamma^{n_0} \right\|_{\mu,S} \bar{r}^\gamma < \frac{\sigma}{2} + \sum_{k=1}^{\infty} \frac{\sigma}{2^{k+1}} = \sigma.$$

It follows that $F \in \mathcal{G}^\mu_{\bar{r}} \{X\}$ and because $\sigma$ was random we also get that $F_n \xrightarrow{n\to\infty} F$ in $\mathcal{G}^\mu_{\bar{r}} \{X\}$. □

For functions $F(X,\eta), G(X,\eta)$, we define the convolution product as follows

$$(F * G)(X,\eta) = \int_0^\eta f(X,s)g(X,\eta - s)\mathrm{d}s,$$

at least when this integral is well defined. It is important to notice that the convolution product satisfies the following properties, all of which are easy to check.

- The convolution product is commutative i.e. $F * G = G * F$.
- It is associative i.e. $(F * G) * H = F * (G * H)$.
- Convolution is distributive w.r.t. addition i.e. $F * (G + H) = F * G + F * H$.
- It is compatible with scalar multiplication, for $\beta \in \mathbb{C}$, $(\beta F) * G = \beta\,(F * G)$.

The following property of $\mathcal{G}^\mu$ and $\mathcal{G}^\mu_{\bar{r}} \{X\}$ will be essential in what is to follow. A proof can be found in [BDM08], in this proof the extra factor $1 + \mu^2 \left|\eta\right|^2$ of the norm plays an important role and this is the reason why it is added.

**Property 2.3.7.** *Let $\|\cdot\|$ denote any of the two norms $\mathcal{G}^\mu$, $\mathcal{G}^\mu_{\bar{r}} \{X\}$ and $f, g$ functions in the corresponding space. It holds that*

$$\|f * g\| \leqslant \frac{4\pi}{\mu} \|f\| \|g\|.$$

*Consequently $f * g \in \mathcal{G}^\mu, \mathcal{G}^\mu_{\bar{r}} \{X\}$.*

### 2.3.3 The equation in the Borel plane

Let $\tilde{z}\,(X,\varepsilon)$ be the formal, Gevrey-1 solution to (2.3.2). We have, as formal series, that

$$\begin{aligned} \varepsilon D_X \tilde{z} \cdot \Lambda X - \varphi\,(X)\,\tilde{z} =& \varepsilon H\,(X, \tilde{z}, \varepsilon) - \varepsilon D_X \tilde{z} \cdot UX \\ & - \varepsilon D_X \tilde{z} \cdot A\,(X) - \varepsilon^2 D_X \tilde{z} V\,(X, \tilde{z}, \varepsilon)\,. \end{aligned} \tag{2.3.5}$$

The summability of $\tilde{z}$ in a certain direction is by theorem 1.2.20 equivalent to the existence of a continuation of the formal Borel transform, $\mathcal{B}_1\,(\tilde{z})$, to an infinite sector in this direction.

To search for such a continuation we transform (2.3.5) into an equivalent expression concerning the Borel transform. For this we need the following results.

**Proposition 2.3.8.** *Let $\widehat{f}(X,\varepsilon)$, $\widehat{g}(X,\varepsilon)$ be Gevrey-1 formal series w.r.t. $\varepsilon$, uniformly in $X$, with image in $\mathbb{C}$ and both without a constant term.*

*(i)* $\mathcal{B}_1\left(\widehat{f}+\widehat{g}\right)=\mathcal{B}_1\left(\widehat{f}\right)+\mathcal{B}_1\left(\widehat{g}\right)$

*(ii)* $\mathcal{B}_1\left(\widehat{f}\cdot\widehat{g}\right)=\mathcal{B}_1\left(\widehat{f}\right)*\mathcal{B}_1\left(\widehat{g}\right)$.

*(iii) Let $z\in\mathbb{C}$ and $H(X,z,\varepsilon)$ be holomorphic on a neighbourhood of the origin, with Taylor series given by*

$$H(X,z,\varepsilon)=\sum_{k=0}^{\infty}H_k(X,\varepsilon)z^k=\sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}H_{kn}(X)\varepsilon^n\right)z^k. \tag{2.3.6}$$

*The formal series $H\left(X,\widehat{f}(X,\varepsilon),\varepsilon\right)$ is Gevrey-1 and*

$$\mathcal{B}_1\left(\varepsilon H\left(X,\widehat{f},\varepsilon\right)\right)=\sum_{k=0}^{\infty}\mathcal{B}_1\left(\varepsilon H_k\right)*\mathcal{B}_1\left(\widehat{f}\right)^{*k}$$

*where we denoted, for $k\geqslant 1$, $\widehat{f}^{*k}=\underbrace{\widehat{f}*\ldots*\widehat{f}}_{k\ times}$ and for $k=0$, $\widehat{f}^{*0}$ is the identity element for the convolution i.e. $G*\widehat{f}^{*0}=G$.*

**Proof:**

(i) Trivial

(ii) This is a straightforward computation involving the Cauchy product of power series and using the fact that $1^{*k}=\frac{\eta^{k-1}}{(k-1)!}$.

(iii) From corollary 1.2.14 we already know that $H\left(X,\widehat{f},\varepsilon\right)$ is Gevrey-1.

We can assume, by the Cauchy inequalities, that there exist $C,D\geqslant 0$ such that $\sup_X|H_{kn}(X)|\leqslant CD^{k+n}$. Let furthermore $A,B\geqslant 0$ be such that $\sup_X|f_n(X)|\leqslant AB^n n!$ where $\widehat{f}(X,\varepsilon)=\sum_{n=1}^{\infty}f_n(X)\varepsilon^n$. We may assume that $AD<1$ by, if necessary, enlarging $B$.

For all $K\in\mathbb{N}_0$ it is an easy calculation that the coefficients of $\sum_{k=0}^{K-1}\varepsilon H_k\widehat{f}^k$ are bounded by

$$C\sum_{k=0}^{K-1}(AD)^k\underbrace{\max\{D,B\}^n}_{M^n}n!=C\frac{1-(AD)^K}{1-AD}M^n n!\leqslant\frac{C}{1-AD}M^n n!.$$

If we denote the Gevrey bounds of $\varepsilon H\left(X,\widehat{f},\varepsilon\right)$ by $UV^n n!$ we get immediately that the coefficients of $\varepsilon H\left(X,\widehat{f},\varepsilon\right)-\sum_{k=0}^{K-1}\varepsilon H_k\widehat{f}^k$ are bounded by

$$\left(U+\frac{C}{1-AD}\right)\max\{M,V\}^n n!.$$

Denote $E = U + \frac{C}{1-AD}$ and $F = \max\{M, V\}$, noticing that

$$\varepsilon H\left(X, \widehat{f}(X,\varepsilon), \varepsilon\right)$$

and

$$\sum_{k=0}^{K-1} \varepsilon H_k (X,\varepsilon) \widehat{f}^k (X,\varepsilon)$$

have equal coefficients for at least $\varepsilon^1, \ldots, \varepsilon^K$, it is an easy calculation that

$$\begin{aligned}
&\left| \mathcal{B}_1\left(\varepsilon H\left(X, \widehat{f}, \varepsilon\right)\right) - \sum_{k=0}^{K-1}\left(\mathcal{B}_1(\varepsilon H_k) * \mathcal{B}_1(\widehat{f})^{*k}\right)\right| \\
&= \left|\mathcal{B}_1\left(\varepsilon H\left(X, \widehat{f}, \varepsilon\right) - \sum_{k=0}^{K-1} \varepsilon H_k \widehat{f}^k\right)\right| \\
&\leqslant EF\frac{(F|\eta|)^K(1+K(1-F|\eta|))}{(1-F|\eta|)^2}.
\end{aligned}$$

which convergences for $K \to \infty$ for $\eta$ in a sufficiently small neighbourhood of 0.

□

Applying the formal Borel transform of order 1 to (2.3.5) and using the above results gives us

$$\begin{aligned}
(1 * D_X \mathcal{B}_1(\widetilde{z}))\Lambda X = &\varphi(X)\mathcal{B}_1(\widetilde{z}) + \sum_{k=0}^{\infty} \mathcal{B}_1(\varepsilon H_k) * \mathcal{B}_1(\widetilde{z})^{*k} \\
&- (1 * D_X\mathcal{B}_1(\widetilde{z}))UX - (1 * D_X\mathcal{B}_1(\widetilde{z}))A(X) \\
&- (1 * D_X\mathcal{B}_1(\widetilde{z})) * \sum_{k=0}^{\infty} \mathcal{B}_1(\varepsilon V_k) * \mathcal{B}_1(\widetilde{z})^{*k}.
\end{aligned} \tag{2.3.7}$$

By theorem 1.2.20, we can prove theorem 2.3.1 by showing that $\mathcal{B}_1(\widetilde{z})$ has a holomorphic continuation to an infinite sector, which is of exponential growth at most of order 1.

Our method of showing the existence of such a continuation is twofold. Firstly we will prove the following lemma.

**Lemma 2.3.9.** *Let* $\theta \in \left[0, 2\pi\right[$, $\rho \in \left]0, \frac{\pi}{2}\right[$ *be such that* $\frac{\lambda_j}{\varphi(0)} \in S(\theta, 2\rho)$, *for all* $j = 1, \ldots, n$, *which is possible due to our assumptions. Let* $\alpha \in \left]0, \frac{\pi}{2} - \rho\right[$ *and denote*

$$S = S(-\theta + \pi, 2\alpha).$$

*For a suitable choice of*

$$\mathcal{G}_{\bar{r}}^{\mu}\{X\} \subset \mathcal{O}(\mathbb{P}_n(\bar{r}) \times S),$$

*there exists, in $\mathcal{G}_{\tilde{r}}^{\mu}\{X\}$, a solution to the equation*

$$\begin{aligned}(1 * D_X Z)\,\Lambda X =&\varphi(X)\,Z + \sum_{k=0}^{\infty} \mathcal{B}_1(\varepsilon H_k) * Z^{*k}\\ &- (1 * D_X Z)\,UX - (1 * D_X Z)\,A(X)\\ &- (1 * D_X Z) * \sum_{k=0}^{\infty} \mathcal{B}_1(\varepsilon V_k) * Z^{*k}.\end{aligned} \tag{2.3.8}$$

However due to functions in $\mathcal{G}_{\tilde{r}}^{\mu}\{X\}$ lacking being holomorphic at the origin $\eta = 0$, it is not immediate that a solution of (2.3.8) is indeed a continuation of $\mathcal{B}_1(\tilde{z})$ and thus theorem 1.2.20 can not be directly applied. We will actually prove directly that the Laplace transform, which we define later, of the solution we have found is Gevrey asymptotic to the formal solution of (2.3.5) on a large sector. This will be done in section 2.3.8.

Let $Z(X,\eta) = \sum_{\gamma\in\mathbb{N}^n} Z_\gamma(\eta)X^\gamma$, if we denote for $k = 1, \ldots n-1$,

$$d_k = \left(0, \ldots, 0, \underset{k-\text{th}}{1}, -1, 0, \ldots, 0\right),$$

equation (2.3.8) can be written as

$$\begin{aligned}\sum_{\gamma\in\mathbb{N}^n} \langle\lambda,\gamma\rangle\,(1 * Z_\gamma)\,X^\gamma =&\varphi(0)\,Z - \sum_{k=1}^{n-1} \sum_{\substack{\gamma\in\mathbb{N}^n\\ \gamma_{k+1}\geqslant 1}} \zeta_{k,k+1}\,(\gamma_k + 1)\,(1 * Z_{\gamma+d_k})\,X^\gamma\\ &+ (\varphi(X) - \varphi(0))\,Z + \sum_{k=0}^{\infty} \mathcal{B}_1(\varepsilon H_k) * Z^{*k}\\ &- (1 * D_X Z)\,A(X)\\ &- (1 * D_X Z) * \sum_{k=0}^{\infty} \mathcal{B}_1(\varepsilon V_k) * Z^{*k}.\end{aligned} \tag{2.3.9}$$

Here $\langle\lambda,\gamma\rangle = \sum_{j=1}^{n} \lambda_j\gamma_j$.

To find a solution of this equation, we will proceed in the following manner.

- Show that there exists solution operators solving, in $\mathcal{G}^{\mu}$, the affine equation

$$\langle\lambda,\gamma\rangle\,(1 * Z_\gamma) = \varphi(0)\,Z_\gamma + F$$

  for all $|\gamma| \geqslant 1$.

- Using these operators and an induction argument, prove the existence of a solution to the "recursive" affine equation

$$\sum_{\gamma\in\mathbb{N}^n} \langle\lambda,\gamma\rangle\,(1 * Z_\gamma)\,X^\gamma = \varphi(0)\,Z - \sum_{k=1}^{n-1} \sum_{\substack{\gamma\in\mathbb{N}^n\\ \gamma_{k+1}\geqslant 1}} \zeta_{k,k+1}\,(\gamma_k + 1)\,(1 * Z_{\gamma+d_k})\,X^\gamma + F.$$

- Construct a solution in $\mathcal{G}_{\tilde{r}}^{\mu}\{X\}$ to (2.3.9) by a fixed point argument.

### 2.3.4 Some preparatory results

In this section we collect some results which will be employed in the next section 2.3.5. We do this to not burden these simple proofs with an overcomplicated notation. We would like to remind that the spaces $\mathcal{G}^\mu$ are defined as subsets of $\mathcal{O}(S)$ for an infinite sector $S$.

**Lemma 2.3.10.** *Let $\beta \in \mathbb{C}$ such that there exists $b > 0$ for which* $\operatorname{Re}(\beta\eta) \leqslant -b|\eta|$ *for all $\eta \in S$.*

*(i)* $e^{\beta\eta} \in \mathcal{G}^\mu$ *for all $\mu > 0$ and* $\left\|e^{\beta\eta}\right\|_{\mu,S} = 1$.

*(ii)* $1 * e^{\beta\eta} = \frac{e^{\beta\eta}-1}{\beta}$.

*(iii) If $f \in \mathcal{G}^\mu$ then $f * e^{\beta\eta} \in \mathcal{G}^\mu$ and*

$$\left\|f * e^{\beta\eta}\right\|_{\mu,S} \leqslant \frac{\|f\|_{\mu,S}}{b}.$$

**Proof:**

(i) Both the kernel function $(1+\mu^2|\eta|^2)e^{-\mu|\eta|}$ that is used in the definition of the norm and the bounding function $|e^{\beta\eta}| \leqslant e^{-b|\eta|}$ are decreasing as $|\eta|$ is increased. So the norm is simply the modulus of the function evaluated at the origin $\eta = 0$.

(ii) Straightforward since

$$\int_0^\eta e^{\beta s} = \frac{e^{\beta\eta}-1}{\beta}.$$

(iii) We start off by remarking that property 2.3.7, combined with (i) gives us immediately that $\left\|f * e^{\beta\eta}\right\|_{\mu,S} \leqslant \frac{4\pi\|f\|_{\mu,S}}{\mu}$, this however turns out to be an insufficient bound to prove our later results. We will effectively need this "improved" bound.

We have, for all $\eta \in S$,

$$\begin{aligned}
\left|\left(f * e^{\beta\eta}\right)\right| &= \left|\int_0^\eta f(s)\, e^{\beta(\eta-s)}\mathrm{d}s\right| \\
&= |\eta| \left|\int_0^1 f(t\eta)\, e^{(1-t)\beta\eta}\mathrm{d}t\right| \\
&\leqslant |\eta| \int_0^1 |f(t\eta)|\, e^{-(1-t)b|\eta|}\mathrm{d}t \\
&\leqslant |\eta| \sup_z |f(z)| \int_0^1 e^{-(1-t)b|\eta|}\mathrm{d}t \\
&= \frac{\sup_z |f(z)|}{b}\left(1-e^{-b|\eta|}\right) \leqslant \frac{\sup_z |f(z)|}{b}
\end{aligned}$$

where the sup is taken for $z \in S, |z| \leqslant |\eta|$.

Because $(1+\mu^2|\eta|^2)e^{-\mu|\eta|}$ is decreasing with respect to $|\eta|$, we get

$$\begin{aligned}\left|\left(f * e^{\beta\eta}\right)\right|(1+\mu^2|\eta|^2)e^{-\mu|\eta|} &\leqslant \frac{\sup_z|f(z)|}{b}(1+\mu^2|\eta|^2)e^{-\mu|\eta|}\\ &\leqslant \frac{1}{b}\sup_z|f(z)|(1+\mu^2|z|^2)e^{-\mu|z|}\\ &\leqslant \frac{1}{b}\|f\|_{\mu,S},\end{aligned}$$

which gives the requested improved bound on $\left\|f * e^{\beta\eta}\right\|_{\mu,S}$ after taking the supremum over $\eta$.

□

**Lemma 2.3.11.** *Let $\beta$ be as in lemma 2.3.10 and consider the map*

$$T\colon \mathcal{G}^\mu \to \mathcal{G}^\mu : g \mapsto (\beta-\delta) * g$$

*where $\delta$ denotes the identity element for the convolution. The map $T$ is linear and continuous with a continuous linear inverse given by*

$$T^{-1}\colon \mathcal{G}^\mu \to \mathcal{G}^\mu : f \mapsto -\left(\delta+\beta e^{\beta\eta}\right) * f.$$

*Moreover $\left\|T^{-1}\right\| \leqslant 1+\frac{|\beta|}{b}$.*

**Proof:** The linearity and continuity of both $T$ and $T^{-1}$ are clear, for the bound on $\left\|T^{-1}\right\|$ one uses lemma 2.3.10 (iii). Remains to check that the two maps are indeed each others inverse. We have

$$\begin{aligned}T\left(T^{-1}(f)\right) &= -(\beta-\delta) * \left(\delta+\beta e^{\beta\eta}\right) * f\\ &= -\left(\beta+\beta^2\left(1 * e^{\beta\eta}\right)-\delta-\beta e^{\beta\eta}\right) * f\\ &= -\left(\beta+\beta e^{\beta\eta}-\beta-\delta-\beta e^{\beta\eta}\right) * f\\ &= f,\end{aligned}$$

where we have used lemma 2.3.10 (ii). Due to commutativity of the convolution product, also $T^{-1}\circ T = \mathrm{Id}$. □

### 2.3.5 The termwise affine equations

By our assumptions on equation (2.3.1) there exist $\theta \in [0,2\pi[$, $\rho \in \left]0,\frac{\pi}{2}\right[$ such that for all $j=1,\dots,n$, $\frac{\lambda_j}{\varphi(0)} \in S(\theta,2\rho)$. It is easily seen that this implies that $\frac{\langle\lambda,\gamma\rangle}{\varphi(0)} \in S(\theta,2\rho)$ for all $|\gamma| \geqslant 1$ and by denoting

$$|\lambda| = \min_j\left\{\left|\mathrm{Re}\left(\lambda_j e^{-i(\theta+\arg(\varphi(0)))}\right)\right|\right\}, \tag{2.3.10}$$

which is non zero, we have

$$|\langle\lambda,\gamma\rangle| = \left|\left\langle \lambda e^{-i(\theta+\arg(\varphi(0)))},\gamma\right\rangle\right| \geqslant \left|\left\langle \mathrm{Re}\left(\lambda e^{-i(\theta+\arg(\varphi(0)))}\right),\gamma\right\rangle\right| \geqslant |\gamma|\,|\lambda|\,. \quad (2.3.11)$$

Take care to notice that $|\cdot|$ is here differently defined for respectively $\gamma$ and $\lambda$ since $|\gamma| = \gamma_1 + \ldots + \gamma_n$.
From here on out we will denote

$$S = S\left(-\theta+\pi, 2\alpha\right) \quad (2.3.12)$$

where $\alpha \in \left]0, \frac{\pi}{2} - \rho\right[$. This implies for $\eta \in S$ that

$$\left|\mathrm{Arg}\left(\frac{\langle\lambda,\gamma\rangle}{\varphi(0)}\eta\right) - \pi\right| < \rho + \alpha < \frac{\pi}{2}.$$

Consequently we have that

$$\mathrm{Re}\left(\frac{\langle\lambda,\gamma\rangle}{\varphi(0)}\eta\right) \leqslant -\cos(\rho+\alpha)\left|\frac{\langle\lambda,\gamma\rangle}{\varphi(0)}\right||\eta| \leqslant \cos(\rho+\alpha)\frac{|\gamma|\,|\lambda|}{|\varphi(0)|}|\eta|\,. \quad (2.3.13)$$

Let $F \in \mathcal{G}^\mu$, in this section we are concerned with solving equations of the form

$$\langle\lambda,\gamma\rangle\,(1 * Z) = \varphi(0)\,Z + F \quad (2.3.14)$$

or written alternatively

$$T_\gamma(Z) = F, \quad \text{with} \quad T_\gamma(Z) := \varphi(0)\left(\left(\frac{\langle\lambda,\gamma\rangle}{\varphi(0)} - \delta\right) * Z\right).$$

**Lemma 2.3.12.** *The linear operator $T_\gamma$ has a continuous linear inverse given by*

$$T_\gamma^{-1} : \mathcal{G}^\mu \to \mathcal{G}^\mu : f \mapsto -\frac{1}{\varphi(0)}\left(\left(\delta + \frac{\langle\lambda,\gamma\rangle}{\varphi(0)} e^{\frac{\langle\lambda,\gamma\rangle}{\varphi(0)}\eta}\right) * f\right).$$

*We have that*

$$\left\|T_\gamma^{-1}\right\| \leqslant \frac{1}{|\varphi(0)|}\left(1 + \frac{1}{\cos(\rho+\alpha)}\right).$$

*Moreover for all $f \in \mathcal{G}^\mu$,*

$$\left\|T_\gamma^{-1}(1 * f)\right\|_{\mu,S} \leqslant \frac{\|f\|_{\mu,S}}{|\gamma|\,|\lambda|\cos(\rho+\alpha)}$$

*where $|\lambda|$ is as in* (2.3.11).

**Proof:** By lemma 2.3.11 and the estimate (2.3.13) it is immediate that $T_\gamma^{-1}$ is indeed the inverse of $T_\gamma$ and is linear, continuous with the given bound for the operator norm. For the second part of the lemma, one checks that, by lemma 2.3.10 (ii),

$$T_\gamma^{-1}(1 * f) = \frac{-1}{\varphi(0)}\left(e^{\frac{\langle\lambda,\gamma\rangle}{\varphi(0)}\eta} * f\right).$$

The result then follows by lemma 2.3.10 (iii) and (2.3.13). □

We remark that for the above lemma to hold, our choice of bisecting direction and opening of $S$ in (2.3.12) is essential.

### 2.3.6 The recursive affine equation

We now turn our attention to the equation

$$\sum_{\gamma\in\mathbb{N}^n}\langle\lambda,\gamma\rangle\left(1*Z_\gamma\right)X^\gamma=\varphi\left(0\right)Z-\sum_{k=1}^{n-1}\sum_{\substack{\gamma\in\mathbb{N}^n\\ \gamma_{k+1}\geqslant 1}}\zeta_{k,k+1}\left(\gamma_k+1\right)\left(1*Z_{\gamma+d_k}\right)X^\gamma+F \tag{2.3.15}$$

where $F=\sum_{(\gamma)\in\mathbb{N}^2}F_\gamma X^\gamma\in\mathcal{G}^\mu[\![X]\!]$. We do not demand that $F\in\mathcal{G}^\mu_{\bar r}\{X\}$ since we will want to solve (2.3.15) for a slightly broader class of functions. We remind that we denoted $d_k=\left(0,\ldots 0,\underset{k-\text{th}}{1},-1,0,\ldots,0\right)$.
We first need to introduce some definitions

**Definition 2.3.13.** *Denote for $k=1,\ldots,n-1$,*

$$u_k=\left(0,\ldots,0,\underset{k-\text{th}}{-1},1,0,\ldots,0\right).$$

*Let $\sigma,\gamma\in\mathbb{N}^n$, $\ell\in\mathbb{N}_0$ and $k_1,\ldots,k_\ell\in\{1,\ldots,n-1\}$. We call the finite sequence $(k_1,\ldots,k_\ell)$ a path from $\sigma$ to $\gamma$ if*

$$\gamma=\sigma+\sum_{j=1}^{\ell}u_{k_j}.$$

*We furthermore denote by $c\left(\gamma\right)$ the set of all multi-indices for which there exists a path towards $\gamma$, i.e.*

$$c\left(\gamma\right):=\left\{\sigma\in\mathbb{N}^n\,\middle|\,\exists\ell\in\mathbb{N}_0,\exists k_1,\ldots,k_\ell\in\{1,\ldots,n-1\},\gamma=\sigma+\sum_{j=1}^{\ell}u_{k_j}\right\}.$$

**Property 2.3.14.** *If there exists a path from $\sigma$ to $\gamma$ it must hold that $|\gamma|=\gamma_1+\gamma_2+\ldots+\gamma_n$ is equal to $|\sigma|$.*

**Proof:** Clearly $|u_k|=0$ from which the result immediately follows. ☐

**Remark 2.3.15.** *It is possible for multiple paths to exist between two points. For example as a path from $(2,1,1)$ to $(1,1,2)$ one can take the $(1,2)$ path given by*

$$(2,1,1)\to(1,2,1)\to(1,1,2)$$

*or the $(2,1)$ path given by*

$$(2,1,1)\to(2,0,2)\to(1,1,2),$$

*see figure 2.1. The following proposition however shows that all paths between two points are closely related.*

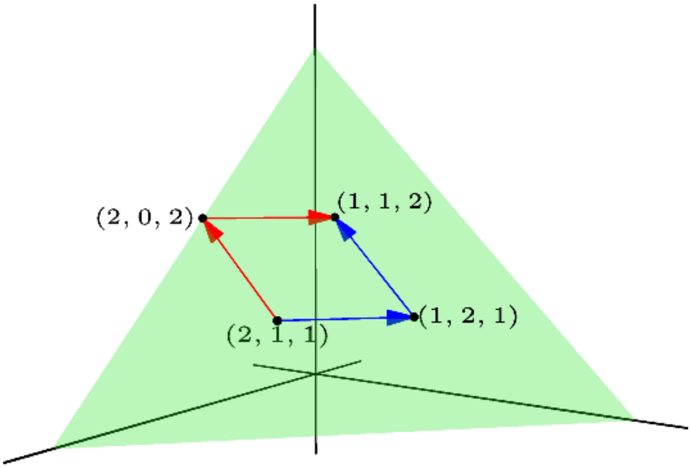


Figure 2.1: The paths $(1, 2)$ and $(2, 1)$ in blue resp. red. Both paths lie on the hyperplane given by $|\gamma| = 4$.

**Proposition 2.3.16.** *Given two points $\sigma, \gamma \in \mathbb{N}^n$ and a path $(k_1, \ldots, k_\ell)$ from $\sigma$ to $\gamma$. Define for $m = 1, \ldots, n-1$,*

$$\ell_m = \#\{j \in \{1, \ldots, \ell\} \mid k_j = m\}.$$

*The values $\ell_m$ are invariant amongst all paths from $\sigma$ to $\gamma$, in particular, all paths have the same length $\ell = \ell_1 + \ldots + \ell_{n-1}$.*

**Proof:** Take any path form $\sigma$ to $\gamma$, by reordering the terms one sees that

$$\sum_{j=1}^{\ell} u_{k_j} = \sum_{m=1}^{n-1} \ell_m u_m = (-\ell_1, \ell_1 - \ell_2, \ldots, \ell_{n-2} - \ell_{n-1}, \ell_{n-1}).$$

By our definition of a path it must hold that

$$(-\ell_1, \ell_1 - \ell_2, \ldots, \ell_{n-2} - \ell_{n-1}, \ell_{n-1}) = \gamma - \sigma$$

and it is then easily seen that

$$\ell_m = \sum_{j=1}^{m} \sigma_j - \gamma_j$$

which is clearly independent of the specific path. $\square$

**Definition 2.3.17.** *Suppose $\gamma \in \mathbb{N}^n$ and $\sigma \in c(\gamma)$. We call the numbers*

$$\ell_1(\sigma,\gamma), \ldots, \ell_{n-1}(\sigma,\gamma)$$

*defined in proposition 2.3.16 the* <u>*intrinsic steps*</u> *associated to $(\sigma,\gamma)$.*
*For any $\sigma \in \mathbb{N}^n$ we define*

$$v(\sigma) := \left\{ (\ell_1, \ldots, \ell_{n-1}) \in \mathbb{N}^{n-1} \,\middle|\, \ell_1 + \ldots + \ell_{n-1} \geqslant 1 \text{ and } \sigma + \sum_{m=1}^{n-1} \ell_m u_m \in \mathbb{N}^n \right\}.$$

*In other words, the set $v(\sigma)$ consists of all $(n-1)$-tuples which are the intrinsic steps of a path starting at $\sigma$.*

**Remark 2.3.18.** *For any given $\sigma$ there are many $(n-1)$-tuples which are not an element of $v(\sigma)$. For example $(3,0)$ are not intrinsic steps of $(1,1,1)$ since $(1,1,1)+(-3,3,0)=(-2,4,1)\notin\mathbb{N}^3$.*

**Definition 2.3.19.** *For $\sigma,\gamma$ we denote by $\ell(\sigma,\gamma)$ the length of all paths from $\sigma$ to $\gamma$, if no paths exists we set it equal to $0$. Due to proposition 2.3.16 this is well defined. We also denote*

$$p(\sigma,\gamma) := \\ \left\{ \left(k_1, \ldots, k_{\ell(\sigma,\gamma)}\right) \in \{1,\ldots,n-1\}^{\ell(\sigma,\gamma)} \,\middle|\, \left(k_1,\ldots,k_{\ell(\sigma,\gamma)}\right) \text{ is a path from } \sigma \text{ to } \gamma \right\}$$

*(this set can be empty).*

Turning our attention back to equation (2.3.15), we see that by denoting for $k = 2, \ldots, n$

$$\chi_k : \mathbb{N}^n \to \{0,1\} : \gamma \mapsto \begin{cases} 0 \text{ if } \gamma_k = 0 \\ 1 \text{ if } \gamma_k \neq 0 \end{cases},$$

we can equate the coefficients of corresponding powers of $X$ as follows

$$\langle \lambda, \gamma \rangle (1 * Z_\gamma) - \varphi(0) Z_\gamma = F_\gamma - \sum_{k=1}^{n-1} \chi_{k+1}(\gamma)\, \zeta_{k,k+1}(\gamma_k + 1)(1 * Z_{\gamma+d_k}). \quad (2.3.16)$$

We will construct a solution to (2.3.16) using the following operators.
Given $k \in \{1,\ldots,n-1\}$ and $\tau \in \mathbb{N}^n$ with $|\tau| \geqslant 1$, we define

$$\mathcal{K}(k,\tau) : \mathcal{G}^\mu \to \mathcal{G}^\mu : f \mapsto -\zeta_{k,k+1} \cdot (\tau_k + 1)\, T_\tau^{-1}(1 * f)$$

**Property 2.3.20.** *The maps $\mathcal{K}(k,\tau)$ are all linear and continuous with operator norm bounded by $\frac{\tau_k+1}{|\tau||\lambda|\cos(\rho+\alpha)}$.*

**Proof:** This is immediate by lemma 2.3.12. ☐

**Lemma 2.3.21.** *The solution to* (2.3.16) *is given by*

$$Z_{\bar{0}} = -\frac{F_{\bar{0}}}{\varphi(0)}$$

*and for* $|\gamma| \geqslant 1$*,*

$$Z_\gamma = T_\gamma^{-1}(F_\gamma) + \sum_{\sigma\in c(\gamma)} \sum_{\bar{k}\in p(\sigma,\gamma)} \left( \prod_{j=1}^{\ell(\sigma,\gamma)} \mathcal{K}\left(k_j, \sigma + \sum_{i=1}^{j} u_{k_i}\right)\right) T_\sigma^{-1}(F_\sigma). \qquad (2.3.17)$$

**Proof:** Consider any $C \in \mathbb{N}_0$, we will show that (2.3.17) holds for all $|\gamma| = C$ and since $C$ is random, this is sufficient.

We start of by defining a total order on the hyperplane $|\gamma| = C$. Given $\sigma, \gamma$, we say that $\sigma \leqslant \gamma$ if in the difference $\gamma - \sigma$, the first non zero value, starting from the right, is positive (or of course of $\sigma = \gamma$). We thus have

$$\begin{aligned}(C, 0, \ldots, 0) &< (C-1, 1, 0, \ldots, 0) < \ldots < (1, C-1, 0, \ldots, 0) \\ &< (0, C, 0, \ldots, 0) < \ldots < (0, 0, \ldots, 0, C).\end{aligned}$$

We now prove the result by induction on this order.

Due to property 2.3.14, it is clear that $c((C, 0, \ldots, 0)) = \varnothing$ and thus (2.3.17) reads $Z_{(C,0,\ldots,0)} = T^{-1}_{(C,0,\ldots,0)}\left(F_{(C,0,\ldots,0)}\right)$. Since the summation disappears in (2.3.16) for this index, it is clearly a solution by lemma 2.3.12.

Suppose now that $\gamma$ is such that (2.3.17) holds for all $\sigma < \gamma$. We remark the following (which is quite obvious), for $m = 1, \ldots, n-1$ it holds that $\chi_{m+1}(\gamma) = 1$ if and only if $\gamma + d_m \in c(\gamma)$. This allows us to rewrite (2.3.17) as

$$\begin{aligned} Z_\gamma =& T_\gamma^{-1}(F_\gamma) + \sum_{m=1}^{n-1} \chi_{m+1}(\gamma)\, \mathcal{K}(m, \gamma + d_m + u_m)\, T^{-1}_{\gamma+d_m}(F_{\gamma+d_m}) \\ &+ \sum_{m=1}^{n-1} \chi_{m+1}(\gamma)\, \mathcal{K}(m, \gamma + d_m + u_m) \\ &\qquad \sum_{\sigma\in c(\gamma+d_m)} \sum_{\bar{k}\in p(\sigma,\gamma+d_m)} \left( \prod_{j=1}^{\ell(\sigma,\gamma+d_m)} \mathcal{K}\left(k_j, \sigma + \sum_{i=1}^{j} u_{k_i}\right)\right) T_\sigma^{-1}(F_\sigma). \end{aligned}$$

Directing our attention again to (2.3.16), we notice that for all $m \in \{1, \ldots, n-1\}$, $\gamma + d_m < \gamma$ (at least when $\gamma + d_m$ exists). Consequently, we can apply the induction hypothesis, giving us that

$$\begin{aligned} Z_\gamma =& T_\gamma^{-1}(F_\gamma) - \sum_{m=1}^{n-1} \chi_{m+1}(\gamma)\, \zeta_{m,m+1}(\gamma_m + 1)\, T_\gamma^{-1}\left(1 * T^{-1}_{\gamma+d_m}(F_{\gamma+d_m})\right) \\ &- \sum_{m=1}^{n-1} \chi_{m+1}(\gamma)\, \zeta_{m,m+1}(\gamma_m + 1)\, T_\gamma^{-1} \\ &\qquad \left(1 * \sum_{\sigma\in c(\gamma+d_m)} \sum_{\bar{k}\in p(\sigma,\gamma+d_m)} \left( \prod_{j=1}^{\ell(\sigma,\gamma+d_m)} \mathcal{K}\left(k_j, \sigma + \sum_{i=1}^{j} u_{k_i}\right)\right) T_\sigma^{-1}(F_\sigma)\right). \end{aligned}$$

Looking at the definition of the maps $\mathcal{K}$, this expression of $Z_\gamma$ is clearly equal to the one above, proving the result. $\square$

**Lemma 2.3.22.** *As formal power series the equation* (2.3.15) *is solved by the linear operator*

$$\mathcal{L}\colon \mathcal{G}^\mu[\![X]\!] \to \mathcal{G}^\mu[\![X]\!]$$

*where $\mathcal{L}(F)$ is given by,*

$$-\frac{F_{\bar{0}}}{\varphi(\bar{0})} + \sum_{|\gamma|\geqslant 1} T_\gamma^{-1}(F_\gamma) X^\gamma$$
$$+\sum_{|\gamma|\geqslant 1}\sum_{\sigma\in c(\gamma)}\sum_{\bar{k}\in p(\sigma,\gamma)}\left(\prod_{j=1}^{\ell(\sigma,\gamma)}\mathcal{K}\left(k_j,\sigma+\sum_{i=1}^{j}u_{k_i}\right)\right)T_\sigma^{-1}(F_\sigma)X^\gamma.$$

*Denote* $\mathcal{M} = \frac{1}{|\varphi(\bar{0})|}\left(1+\frac{1}{\cos(\rho+\alpha)}\right)$ *and* $\mathcal{N} = |\lambda|\cos(\rho+\alpha)$, *if for* $\bar{r} = (r_1,\dots,r_n)$, $\sum_{m=1}^{n-1}\frac{r_{m+1}}{r_m} \leqslant \frac{\mathcal{N}}{2}$ *we have the following cases.*

*(i) If $F \in \mathcal{G}^\mu_{\bar{r}}\{X\}$ then $\mathcal{L}(F) \in \mathcal{G}^\mu_{\bar{r}}\{X\}$ and*

$$_{\bar{r}}\|\mathcal{L}(F)\|_{\mu,S} \leqslant 2\mathcal{M}\ _{\bar{r}}\|F\|_{\mu,S}\,.$$

*(ii) If $F = (1 * D_X f)\, g(X)$ with $g \in \mathcal{O}\left(\overline{\mathbb{P}}_n(\bar{r}),\mathbb{C}^n\right)$ and $f \in \mathcal{G}^\mu_{\bar{r}}\{X\}$ then $\mathcal{L}(F) \in \mathcal{G}^\mu_{\bar{r}}\{X\}$ and*

$$_{\bar{r}}\|\mathcal{L}(F)\|_{\mu,S} \leqslant \frac{4}{\mathcal{N}}\left(\sum_{m=1}^{n}\frac{1}{r_m}\right)\ _{\bar{r}}\|f\|_{\mu,S}\sum_{\gamma}\|g_\gamma\|_{\max}\bar{r}^\gamma.$$

*(iii) If $F = 1 * D_X f * g$ where $f \in \mathcal{G}^\mu_{\bar{r}}\{X\}$ and $g = (g_1,\dots,g_n) \in \mathcal{G}^\mu_{\bar{r}}\{X\}^n$ then $\mathcal{L}(F) \in \mathcal{G}^\mu_{\bar{r}}\{X\}$ and*

$$_{\bar{r}}\|\mathcal{L}(F)\|_{\mu,S} \leqslant \frac{16\pi}{\mathcal{N}\mu}\left(\sum_{m=1}^{n}\frac{1}{r_m}\right)\ _{\bar{r}}\|f\|_{\mu,S}\sum_{j=1}^{n}\ _{\bar{r}}\|g_j\|_{\mu,S}\,.$$

**Proof:** By lemma 2.3.21 $\mathcal{L}(F)$ is formally a solution, it remains to check the convergence in the three cases.

(i) It is immediate by lemma 2.3.12 that $-\frac{F_{\bar{0}}}{\varphi(\bar{0})} + \sum_{|\gamma|\geqslant 1} T_\gamma^{-1}(F_\gamma)X^\gamma \in \mathcal{G}^\mu\{X\}$ and

$$_{\bar{r}}\left\|-\frac{F_{\bar{0}}}{\varphi(\bar{0})} + \sum_{|\gamma|\geqslant 1} T_\gamma^{-1}(F_\gamma)X^\gamma\right\|_{\mu,S} \leqslant \mathcal{M}\ _{\bar{r}}\|F\|_{\mu,S}\,. \qquad (2.3.18)$$

By property 2.3.20 and lemma 2.3.12

$$\begin{aligned}
&\left\| \left( \prod_{j=1}^{\ell(\sigma,\gamma)} \mathcal{K}\left(k_j, \sigma + \sum_{i=1}^{j} u_{k_i}\right)\right) T_\sigma^{-1}\left(F_\sigma\right)\right\|_{\mu,S} \\
&\leqslant \mathcal{M} \frac{\prod_{j=1}^{\ell(\sigma,\gamma)} \left(\left(\sigma + \sum_{i=1}^{j} u_{k_i}\right)_{k_j} + 1\right)}{\left(|\sigma| \mathcal{N}\right)^{\ell(\sigma,\gamma)}} \left\|F_\sigma\right\|_{\mu,S} \\
&= \mathcal{M} \frac{\prod_{j=1}^{\ell(\sigma,\gamma)} \left(\sigma + \sum_{i=1}^{j-1} u_{k_i}\right)_{k_j}}{\left(|\sigma| \mathcal{N}\right)^{\ell(\sigma,\gamma)}} \left\|F_\sigma\right\|_{\mu,S}.
\end{aligned}$$

By property 2.3.14 we have that $\left|\sigma + \sum_{i=1}^{j-1} u_{k_i}\right| = |\sigma|$ and thus

$$\left(\sigma + \sum_{i=1}^{j-1} u_{k_i}\right)_{k_j} \leqslant |\sigma|,$$

implying that we can further estimate

$$\begin{aligned}
&\left\| \left( \prod_{j=1}^{\ell(\sigma,\gamma)} \mathcal{K}\left(k_j, \sigma + \sum_{i=1}^{j} u_{k_i}\right)\right) T_\sigma^{-1}\left(F_\sigma\right)\right\|_{\mu,S} \\
&\leqslant \mathcal{M} \frac{1}{\mathcal{N}^{\ell(\sigma,\gamma)}} \left\|F_\sigma\right\|_{\mu,S}.
\end{aligned}$$

Moreover there is a one-to-one correspondence between $p(\sigma,\gamma)$, and strings of the numbers $1,\dots,n-1$ where each number $j = 1,\dots,n-1$ appears exactly $l_j(\sigma,\gamma)$ times. These strings form the set of permutations of the multiset $\{l_1(\sigma,\gamma)\cdot 1,\dots,l_{n-1}(\sigma,\gamma)\cdot(n-1)\}$ and thus

$$\#p(\sigma,\gamma) = \frac{\ell(\sigma,\gamma)!}{\ell_1(\sigma,,\gamma)!\ell_2(\sigma,\gamma)!\dots\ell_{n-1}(\sigma,\gamma)!},$$

see [Bru18]. We thus have that

$$\begin{aligned}
&\left\| \sum_{\sigma\in c(\gamma)} \sum_{\bar{k}\in p(\sigma,\gamma)} \left( \prod_{j=1}^{\ell(\sigma,\gamma)} \mathcal{K}\left(k_j, \sigma + \sum_{i=1}^{j} u_{k_i}\right)\right) T_\sigma^{-1}\left(F_\sigma\right)\right\|_{\mu,S} \\
&\leqslant \mathcal{M} \sum_{\sigma\in c(\gamma)} \frac{\ell(\sigma,\gamma)!}{\ell_1(\sigma,\gamma)!\ell_2(\sigma,\gamma)!\dots\ell_{n-1}(\sigma,\gamma)!} \frac{\left\|F_\sigma\right\|_{\mu,S}}{\mathcal{N}^{\ell(\sigma,\gamma)}}.
\end{aligned}$$

Consequently we get that

$$\begin{aligned}
&\sum_{|\gamma|\geqslant 1} \left\| \sum_{\sigma\in c(\gamma)} \sum_{\bar{k}\in p(\sigma,\gamma)} \left( \prod_{j=1}^{\ell(\sigma,\gamma)} \mathcal{K}\left(k_j, \sigma + \sum_{i=1}^{j} u_{k_i}\right)\right) T_\sigma^{-1}\left(F_\sigma\right)\right\|_{\mu,S} \bar{r}^\gamma \\
&\leqslant \mathcal{M} \sum_{|\gamma|\geqslant 1} \sum_{\sigma\in c(\gamma)} \frac{\ell(\sigma,\gamma)!}{\ell_1(\sigma,\gamma)!\ell_2(\sigma,\gamma)!\dots\ell_{n-1}(\sigma,\gamma)!} \frac{\left\|F_\sigma\right\|_{\mu,S}}{\mathcal{N}^{\ell(\sigma,\gamma)}} \bar{r}^\gamma \\
&= \mathcal{M} \sum_{|\gamma|\geqslant 1} \sum_{\sigma\in c(\gamma)} \frac{\ell(\sigma,\gamma)!}{\ell_1(\sigma,\gamma)!\ell_2(\sigma,\gamma)!\dots\ell_{n-1}(\sigma,\gamma)!} \frac{\left\|F_\sigma\right\|_{\mu,S}}{\mathcal{N}^{\ell(\sigma,\gamma)}} \bar{r}^\sigma \bar{r}^{\sum_{m=1}^{n-1} \ell_m(\sigma,\gamma) u_m}.
\end{aligned}$$

By the proof of proposition 2.3.16

$$\sum_{m=1}^{n-1} \ell_m(\sigma,\gamma)\, u_m$$
$$= \left(-\ell_1(\sigma,\gamma), \ell_1(\sigma,\gamma) - \ell_2(\sigma,\gamma), \ldots, \ell_{n-2}(\sigma,\gamma) - \ell_{n-1}(\sigma,\gamma), \ell_{n-1}(\sigma,\gamma)\right)$$

and thus

$$\bar{r}^{\sum_{m=1}^{n-1} \ell_m(\sigma,\gamma) u_m} = \prod_{m=1}^{n-1} \left(\frac{r_{m+1}}{r_m}\right)^{\ell_m(\sigma,\gamma)}.$$

Consequently

$$\sum_{|\gamma|\geqslant 1} \left\| \sum_{\sigma\in c(\gamma)} \sum_{\bar{k}\in p(\sigma,\gamma)} \left(\prod_{j=1}^{\ell(\sigma,\gamma)} \mathcal{K}\left(k_j, \sigma + \sum_{i=1}^{j} u_{k_i}\right)\right) T_\sigma^{-1}(F_\sigma) \right\|_{\mu,S} \bar{r}^\gamma$$
$$\leqslant \mathcal{M} \sum_{|\gamma|\geqslant 1} \sum_{\sigma\in c(\gamma)} \frac{\ell(\sigma,\gamma)!}{\ell_1(\sigma,\gamma)!\ell_2(\sigma,\gamma)!\ldots\ell_{n-1}(\sigma,\gamma)!} \prod_{m=1}^{n-1}\left(\frac{r_{m+1}}{\mathcal{N} r_m}\right)^{\ell_m(\sigma,\gamma)} \|F_\sigma\|_{\mu,S}\,\bar{r}^\sigma$$
$$= \mathcal{M} \sum_{|\sigma|\geqslant 1} \sum_{(\ell_1,\ldots,\ell_{n-1})\in v(\sigma)} \frac{(\ell_1+\ldots+\ell_{n-1})!}{\ell_1!\ldots\ell_{n-1}!} \prod_{m=1}^{n-1}\left(\frac{r_{m+1}}{\mathcal{N} r_m}\right)^{\ell_m} \|F_\sigma\|_{\mu,S}\,\bar{r}^\sigma$$
$$\leqslant \mathcal{M} \sum_{|\sigma|\geqslant 1} \sum_{\ell_1+\ldots+\ell_{n-1}\geqslant 1} \frac{(\ell_1+\ldots+\ell_{n-1})!}{\ell_1!\ldots\ell_{n-1}!} \prod_{m=1}^{n-1}\left(\frac{r_{m+1}}{\mathcal{N} r_m}\right)^{\ell_m} \|F_\sigma\|_{\mu,S}\,\bar{r}^\sigma$$
$$= \mathcal{M} \frac{\sum_{m=1}^{n-1}\frac{r_{m+1}}{r_m}}{\mathcal{N} - \sum_{m=1}^{n-1}\frac{r_{m+1}}{r_m}} \sum_{|\sigma|\geqslant 1} \|F_\sigma\|_{\mu,S}\,\bar{r}^\sigma$$
$$\leqslant \mathcal{M}\ {}_{\bar{r}}\|F\|_{\mu,S}\,.$$

Where we used for the last inequality our assumption that $\sum_{m=1}^{n-1}\frac{r_{m+1}}{r_m} \leqslant \frac{\mathcal{N}}{2}$. Combining this bound with (2.3.18) proves the result.

(ii) Let $f = \sum_\gamma f_\gamma X^\gamma$ and $g = \sum_\gamma g_\gamma X^\gamma$ where $g_\gamma = (g_{\gamma,1},\ldots,g_{\gamma,n}) \in \mathbb{C}^n$. If we denote $s_m = (0,\ldots 0,1,0,\ldots,0)$ with 1 at position $m$ we have that

$$(1 * D_X f)\, g = \sum_{\gamma\in\mathbb{N}^n} \sum_{\alpha+\beta=\gamma} \sum_{m=1}^{n} (\alpha_m+1)(1 * f_{\alpha+s_m})\, g_{\beta,m} X^\gamma.$$

By lemma 2.3.12 one sees that

$$\left\| T_\gamma^{-1}\left(\sum_{\alpha+\beta=\gamma}\sum_{m=1}^{n} (\alpha_m+1)(1 * f_{\alpha+s_m})\, g_{\beta,m}\right)\right\|_{\mu,S}$$
$$\leqslant \frac{1}{\mathcal{N}} \sum_{\alpha+\beta=\gamma}\sum_{m=1}^{n} \frac{\alpha_m+1}{|\gamma|} \|f_{\alpha+s_m}\|_{\mu,S} |g_{\beta,m}|\,.$$

Since $\alpha_m \leqslant |\gamma|$ a further estimate is given by

$$\frac{2}{\mathcal{N}} \sum_{\alpha+\beta=\gamma}\sum_{m=1}^{n} \|f_{\alpha+s_m}\|_{\mu,S} \|g_\beta\|_{\max} \leqslant \frac{2}{\mathcal{N}} \sum_{m=1}^{n} \sum_{\alpha+\beta=\gamma+s_m} \|f_\alpha\|_{\mu,S} \|g_\beta\|_{\max}\,.$$

By repeating the proof of (i) with the above bound one finds that

$$_{\bar{r}}\|\mathcal{L}\left(\left(1 * D_X f\right) g\right)\|_{\mu,S} \leqslant \frac{4}{\mathcal{N}} \sum_{\gamma\in\mathbb{N}^n} \sum_{m=1}^{n} \sum_{\alpha+\beta=\gamma+s_m} \|f_\alpha\|_{\mu,S} \|g_\beta\|_{\max} \bar{r}^{\gamma}.$$

Finally we have that

$$\begin{aligned}
&\frac{4}{\mathcal{N}} \sum_{\gamma\in\mathbb{N}^n} \sum_{m=1}^{n} \sum_{\alpha+\beta=\gamma+s_m} \|f_\alpha\|_{\mu,S} \|g_\beta\|_{\max} \bar{r}^{\gamma} \\
&= \frac{4}{\mathcal{N}} \sum_{m=1}^{n} \frac{1}{r_m} \sum_{\gamma\in\mathbb{N}^n} \sum_{\alpha+\beta=\gamma+s_m} \|f_\alpha\|_{\mu,S} \|g_\beta\|_{\max} \bar{r}^{\gamma+s_m} \\
&\leqslant \frac{4}{\mathcal{N}} \sum_{m=1}^{n} \frac{1}{r_m} \sum_{\gamma\in\mathbb{N}^n} \sum_{\alpha+\beta=\gamma} \|f_\alpha\|_{\mu,S} \|g_\beta\|_{\max} \bar{r}^{\gamma} \\
&= \frac{4}{\mathcal{N}} \sum_{m=1}^{n} \frac{1}{r_m} {}_{\bar{r}}\|f\|_{\mu,S} \sum_{\gamma} \|g_\gamma\|_{\max} \bar{r}^{\gamma}.
\end{aligned}$$

(iii) Let $f = \sum_\gamma f_\gamma X^\gamma$ and $g = \sum_\gamma g_\gamma X^\gamma$ where $g_\gamma = (g_{\gamma,1}, \ldots, g_{\gamma,n})$ with all $g_{\gamma,j} \in \mathcal{G}^\mu$. We have that

$$1 * D_X f * g = \sum_{\gamma\in\mathbb{N}^n} \sum_{\alpha+\beta=\gamma} \sum_{m=1}^{n} \left(\alpha_m + 1\right)\left(1 * f_{\alpha+s_m} * g_{\beta,m}\right) X^{\gamma}.$$

Using lemma 2.3.12 and property 2.3.7 one can find, similarly as in (ii), that

$$\begin{aligned}
&\left\| T_\gamma^{-1} \left( \sum_{\alpha+\beta=\gamma} \sum_{m=1}^{n} \left(\alpha_m + 1\right)\left(1 * f_{\alpha+s_m} * g_{\beta,m}\right) \right) \right\|_{\mu,S} \\
&\leqslant \frac{8\pi}{\mathcal{N}\mu} \sum_{m=1}^{n} \sum_{\alpha+\beta=\gamma+s_m} \|f_\alpha\|_{\mu,S} \max_{1\leqslant j\leqslant n} \left\{ \|g_{\beta,j}\|_{\mu,S} \right\} \\
&\leqslant \frac{8\pi}{\mathcal{N}\mu} \sum_{m=1}^{n} \sum_{\alpha+\beta=\gamma+s_m} \|f_\alpha\|_{\mu,S} \left( \sum_{j=1}^{n} \|g_{\beta,j}\|_{\mu,S} \right)
\end{aligned}$$

Repeating then the steps in the proof of (ii) gives the desired estimate.

□

### 2.3.7 The complete equation

We will find a solution to equation (2.3.9), as a fixed point of the map

$$\begin{aligned}
F \mapsto \mathcal{L}\Bigg( &\left(\varphi\left(X\right) - \varphi\left(0\right)\right) Z + \sum_{k=0}^{\infty} \mathcal{B}_1\left(\varepsilon H_k\right) * Z^{*k} \\
&- \left(1 * D_X Z\right) A\left(X\right) - \left(1 * D_X Z\right) * \sum_{k=0}^{\infty} \mathcal{B}_1\left(\varepsilon V_k\right) * Z^{*k} \Bigg).
\end{aligned} \tag{2.3.19}$$

Here $\mathcal{L}$ is the map specified in lemma 2.3.22.

We now collect some results in preparation of proving that the above map has a fixed point.

**Proposition 2.3.23.** *Let $F \in \mathcal{G}^{\mu}_{\bar{r}}\{X\}$ and $g(X) \in \mathcal{O}\left(\overline{\mathbb{P}}_n(\bar{r})\right)$, then*

$$_{\bar{r}}\|g \cdot F\|_{\mu,S} \leqslant \left(\sum_{\gamma\in\mathbb{N}^n} |g_\gamma|\, \bar{r}^{\gamma}\right) {}_{\bar{r}}\|F\|_{\mu,S}\,.$$

**Proof:** This is immediate since

$$\begin{aligned} _{\bar{r}}\|g \cdot F\|_{\mu,S} &= \sum_{\gamma} \left\| \sum_{\alpha+\beta=\gamma} F_\alpha g_\beta \right\|_{\mu,S} \bar{r}^{\gamma} \\ &\leqslant \sum_{\gamma} \sum_{\alpha+\beta=\gamma} \|F_\alpha\|_{\mu,S} |g_\beta|\, \bar{r}^{\gamma} = \left(\sum_{\gamma\in\mathbb{N}^n} |g_\gamma|\, \bar{r}^{\gamma}\right) {}_{\bar{r}}\|F\|_{\mu,S}\,. \end{aligned}$$

□

**Proposition 2.3.24.** *Given any $R > 0$, there exists a $\mathcal{U}(R) > 1/R$ such that for all $\mu \geqslant \mathcal{U}(R)$,*

$$\sup_{\eta\in S} \left(1 + \mu^2 |\eta|^2\right) e^{\left(\frac{1}{R}-\mu\right)|\eta|} \leqslant 1.$$

**Proof:** One can calculate that, as a function of $|\eta|$, the derivative of

$$\left(1 + \mu^2 |\eta|^2\right) e^{\left(\frac{1}{R}-\mu\right)|\eta|}$$

has two zeroes, both of the form $\frac{1}{\mu}\left(1 + \mathrm{o}(1)\right)$, as $\mu \to \infty$.

This implies that the maximum of the function is of the form $\left(2 + \mathrm{o}(1)\right) e^{\left(\frac{1}{R\mu}-1\right)\left(1+\mathrm{o}\left(1\right)\right)}$ and thus converges to $2e^{-1} < 1$ for $\mu \to \infty$. Consequently $\left(1 + \mu^2 |\eta|^2\right) e^{\left(\frac{1}{R}-\mu\right)|\eta|} \leqslant 1$ for $\mu$ greater than a certain value $\mathcal{U}(R)$. □

**Lemma 2.3.25.** *Let*

$$g(X, \varepsilon) = \sum_{n=0}^{\infty} \sum_{\gamma\in\mathbb{N}^n} g_{\gamma n} X^{\gamma} \varepsilon^n$$

*be holomorphic on $\overline{\mathbb{P}}_{n+1}(R)$. If $\mu \geqslant \mathcal{U}(R)$, where $\mathcal{U}$ is the function from proposition 2.3.24, and $r_j \leqslant R/2$ for $j = 1, \dots, n$. Then $\mathcal{B}_1(\varepsilon g) \in \mathcal{G}^{\mu}_{\bar{r}}\{X\}$ and*

$$_{\bar{r}}\|\mathcal{B}_1(\varepsilon g)\|_{\mu,S} \leqslant 2^n \max_{(X,\varepsilon)\in\overline{\mathbb{P}}_{n+1}(R)} |g(X, \varepsilon)|\,.$$

**Proof:** Denote $\|g\| = \max_{(X,\varepsilon)\in\overline{\mathbb{P}}_{n+1}(R)} |g(X, \varepsilon)|$. The Borel transform is given by

$$\mathcal{B}_1(\varepsilon g) = \sum_{\gamma} \left(\sum_{n=0}^{\infty} \frac{g_{\gamma n}}{n!} \eta^n\right) X^{\gamma}.$$

Using the Cauchy inequalities one sees

$$\left\|\sum_{n=0}^{\infty}\frac{g_{\gamma n}}{n!}\eta^n\right\|_{\mu,S}\leqslant\frac{\|g\|}{R^{|\gamma|}}\sup_{\eta\in S}\left(1+\mu^2|\eta|^2\right)e^{\left(\frac{1}{R}-\mu\right)|\eta|}\leqslant\frac{\|g\|}{R^{|\gamma|}},$$

where the last inequality holds since $\mu\geqslant\mathcal{U}(R)$. Consequently

$${}_{\bar{r}}\|\mathcal{B}_1(\varepsilon g)\|_{\mu,S}\leqslant\|g\|\sum_{\gamma}\frac{\bar{r}^{\gamma}}{R^{|\gamma|}}=\|g\|\prod_{j=1}^{n}\frac{R}{R-r_j}\leqslant 2^n\|g\|.$$

□

Before stating the next result we want to remind that in (2.3.19), the functions $V_k$ are $\mathbb{C}^n$ valued and we can thus consider the component functions $V_{k,j}$ for $1\leqslant j\leqslant n$.

**Corollary 2.3.26.** *There exists a $C_0>0$ independent of $\mu,\bar{r}$, such that for any choice of $r_j\leqslant R/2$ and any $C>0$ there exist large enough $\mu$ such that the maps*

$$F\mapsto\sum_{k=0}^{\infty}\mathcal{B}_1(\varepsilon H_k)*F^{*k},\qquad F\mapsto\sum_{k=0}^{\infty}\mathcal{B}_1(\varepsilon V_{k,j})*F^{*k}$$

*are all well defined for $\overline{B}(0,C)\subset\mathcal{G}^{\mu}_{\bar{r}}\{X\}\to\overline{B}(0,C_0)$.*
*Moreover, these maps are then Lipschitz continuous, with a Lipschitz constant that is $\mathrm{O}(\mu^{-1})$ for $\mu\to\infty$.*

**Proof:** We give the proof for $H$, it is identical for the other functions.
Denote

$$M=\max_{\overline{\mathbb{P}}_{n+2}(R)}\left\{|H(X,z,\varepsilon)|\,\|V(X,z,\varepsilon)\|_{\max}\right\}.$$

By the Cauchy inequalities and proposition 2.3.25 we can assume that

$${}_{\bar{r}}\|\mathcal{B}_1(\varepsilon H_k)\|_{\mu,S}\leqslant 2^n MR^{-k}$$

and thus for sufficiently large $\mu$,

$$\begin{aligned}{}_{\bar{r}}\left\|\sum_{k=0}^{\infty}\mathcal{B}_1(\varepsilon H_k)*F^{*k}\right\|_{\mu,S}&\leqslant 2^nM+2^nM\sum_{k=1}^{\infty}\left(\frac{4\pi}{R\mu}\right)^k{}_{\bar{r}}\|F\|_{\mu,S}{}^k\\&=2^nM\frac{R\mu}{R\mu-4\pi\,{}_{\bar{r}}\|F\|_{\mu,S}}.\end{aligned}$$

This proves the first part of the result by setting $C_0=2^{n+1}M$.
Regarding the Lipschitz continuity, we must bound

$${}_{\bar{r}}\left\|\sum_{k=1}^{\infty}\mathcal{B}_1(\varepsilon H_k)*\left(F_1^{*k}-F_2^{*k}\right)\right\|_{\mu,S}$$

for $F_1,F_2\in\overline{B}(0,C)$. A first estimate is given by

$$\frac{2^{n+2}\pi M}{\mu}\sum_{k=1}^{\infty}R^{-k}\,{}_{\bar{r}}\left\|F_1^{*k}-F_2^{*k}\right\|_{\mu,S}.$$

Since $F_1^{*k} - F_2^{*k} = (F_1 - F_2) * \sum_{j=1}^{k} F_1^{*(k-j)} * F_2^{*(j-1)}$ we have

$$\begin{aligned} {}_{\bar{r}}\left\| F_1^{*k} - F_2^{*k} \right\|_{\mu,S} &\leqslant \left(\frac{4\pi}{\mu}\right)^{k-1} \sum_{j=1}^{k} C^{k-j} C^{j-1} \; {}_{\bar{r}}\| F_1 - F_2 \|_{\mu,S} \\ &= k \left(\frac{4\pi C}{\mu}\right)^{k-1} \; {}_{\bar{r}}\| F_1 - F_2 \|_{\mu,S}, \end{aligned}$$

implying that the Lipschitz constant is bounded by

$$\frac{2^{n+2}\pi M}{R\mu} \sum_{k=1}^{\infty} k \left(\frac{4\pi C}{R\mu}\right)^{k-1} = \frac{2^{n+2}\pi M R\mu}{(R\mu - 4\pi C)^2}$$

which is clearly $\mathrm{O}\left(\mu^{-1}\right)$ for $\mu \to \infty$. □

**Lemma 2.3.27.** *Denote the map* (2.3.19) *by* $\mathcal{V}(F)$. *For sufficiently large* $\mu$ *and sufficiently small* $r_j$, $\mathcal{V}$ *is well defined as a map from a closed ball around* 0 *in* $\mathcal{G}_{\bar{r}}^{\mu}\{X\}$ *to itself. Moreover this map is a contraction. Consequently there exists a unique* $Z \in \mathcal{G}_{\bar{r}}^{\mu}\{X\}$ *solving equation* (2.3.8).

**Proof:** By lemma 2.3.22, $\mathcal{V}$ is already well defined as a map from $\mathcal{G}_{\bar{r}}^{\mu}\{X\}$ to itself. Set, for the constants used in lemma 2.3.22, $L = \max\left\{2\mathcal{M}, \frac{16\pi}{\mathcal{N}}\right\}$ and denote $C = 2LC_0$ where $C_0$ is as in corollary 2.3.26. Since $\varphi(X) - \varphi(0) = \mathrm{O}(X)$ and $A = \mathrm{O}(X^2)$ we have by lemma 2.3.22 and proposition 2.3.23 that

$$\begin{aligned} {}_{\bar{r}}\| \mathcal{L}\left(\left(\varphi(X) - \varphi(0)\right) F\right) \|_{\mu,S} &\leqslant L \cdot \mathrm{O}(\bar{r}) \; {}_{\bar{r}}\| F \|_{\mu,S} \\ {}_{\bar{r}}\| \mathcal{L}\left(\left(1 * D_X F\right) A(X)\right) \|_{\mu,S} &\leqslant L \cdot \mathrm{O}(\bar{r}) \; {}_{\bar{r}}\| F \|_{\mu,S} \end{aligned}$$

where in both inequalities, $\mathrm{O}(\bar{r})$ is independent of $F$, one can thus assume that both are smaller than $1/(8L)$ by diminishing $\bar{r}$.
Lastly, we have by lemma 2.3.22 and corollary 2.3.26 that for $F \in \overline{B}(0, C)$ and sufficiently large $\mu$,

$$\begin{aligned} {}_{\bar{r}}\left\| \mathcal{L}\left(\sum_{k=0}^{\infty} \mathcal{B}_1(\varepsilon H_k) * F^{*k}\right) \right\|_{\mu,S} &\leqslant LC_0, \\ {}_{\bar{r}}\left\| \mathcal{L}\left(1 * D_X F * \left(\sum_{k=0}^{\infty} \mathcal{B}_1(\varepsilon V_k) * F^{*k}\right)\right) \right\|_{\mu,S} &\leqslant \frac{L}{\mu} \left(\sum_{m=1}^{n} \frac{1}{r_m}\right) n C_0^2. \end{aligned}$$

Consequently

$$\begin{aligned} {}_{\bar{r}}\| \mathcal{V}(F) \|_{\mu,S} &\leqslant L \left(\frac{C}{4L} + C_0 + \frac{1}{\mu} \left(\sum_{m=1}^{n} \frac{1}{r_m}\right) n C_0^2\right) \\ &= \left(\frac{1}{4} + \frac{1}{2} + \frac{n}{2\mu} \left(\sum_{m=1}^{n} \frac{1}{r_m}\right) C_0\right) C. \end{aligned}$$

This clearly implies that $\mathcal{V}\colon \overline{B}(0,C) \to \overline{B}(0,C)$ for sufficiently large $\mu$.
To prove that $\mathcal{V}$ is a contraction we notice the following,

$$
\begin{aligned}
{}_{\bar{r}}\|\mathcal{L}\left(\left(\varphi(X)-\varphi(0)\right)\left(F_1-F_2\right)\right)\|_{\mu,S} &\leqslant L\cdot \mathrm{O}(\bar{r})\ {}_{\bar{r}}\|F_1-F_2\|_{\mu,S}\,,\\
{}_{\bar{r}}\|\mathcal{L}\left(\left(1*D_X\left(F_1-F_2\right)\right)A(X)\right)\|_{\mu,S} &\leqslant L\cdot \mathrm{O}(\bar{r})\ {}_{\bar{r}}\|F_1-F_2\|_{\mu,S}\,,
\end{aligned}
$$

similarly as above, and

$$
{}_{\bar{r}}\left\|\mathcal{L}\left(\sum_{k=0}^{\infty}\mathcal{B}_1\left(\varepsilon H_k\right)*\left(F_1^{*k}-F_2^{*k}\right)\right)\right\|_{\mu,S} \leqslant L\,\mathrm{O}(\mu^{-1})\ {}_{\bar{r}}\|F_1-F_2\|_{\mu,S}
$$

by corollary 2.3.26. Finally we have that

$$
\begin{aligned}
&\mathcal{L}\left(1*D_XF_1*\left(\sum_{k=0}^{\infty}\mathcal{B}_1\left(\varepsilon V_k\right)*F_1^{*k}\right)-1*D_XF_2*\left(\sum_{k=0}^{\infty}\mathcal{B}_1\left(\varepsilon V_k\right)*F_2^{*k}\right)\right)\\
&=\mathcal{L}\left(1*D_X\left(F_1-F_2\right)*\left(\sum_{k=0}^{\infty}\mathcal{B}_1\left(\varepsilon V_k\right)*F_1^{*k}\right)\right)\\
&\quad+\mathcal{L}\left(1*D_XF_2*\left(\sum_{k=0}^{\infty}\mathcal{B}_1\left(\varepsilon V_k\right)*\left(F_1^{*k}-F_2^{*k}\right)\right)\right).
\end{aligned}
$$

The norm of which is bounded by

$$
\frac{L}{\mu}\left(\sum_{m=1}^{n}\frac{1}{r_m}\right)\left(nC_0+n\,\mathrm{O}(\mu^{-1})\right)\ {}_{\bar{r}}\|F_1-F_2\|_{\mu,S}\,.
$$

Putting all this together it is clear that $\mathcal{V}$ is a contraction by choosing $\mu$ sufficiently large. $\square$

### 2.3.8 Gevrey asymptotics for the Laplace transform

We start of by recollecting results already achieved.
By lemma 2.3.9 there exists a solution, say $Z(X,\eta)$, to (2.3.8) i.e.

$$
\begin{aligned}
\left(1*D_XZ\right)\Lambda X =&\varphi(X)\,Z+\sum_{k=0}^{\infty}\mathcal{B}_1\left(\varepsilon H_k\right)*Z^{*k}-\left(1*D_XZ\right)UX\\
&-\left(1*D_XZ\right)A(X)-\left(1*D_XZ\right)*\sum_{k=0}^{\infty}\mathcal{B}_1\left(\varepsilon V_k\right)*Z^{*k}.
\end{aligned}
\tag{2.3.20}
$$

where $Z$ is defined and holomorphic for $(X,\eta)\in\overline{\mathbb{P}}_n(\bar{r})\times S(-\theta+\pi,2\alpha)$ for certain $\bar{r}\in\mathbb{R}^n_{>0}$, $0<\alpha<\frac{\pi}{2}-\rho$.
Furthermore, by the results of section 2.1.2 there exists a formal, Gevrey-1, series solution, $\tilde{z}(X,\varepsilon)$, to the slow manifold equation (2.3.2). The Borel transform, $\mathcal{B}_1(\tilde{z})(X,\eta)$, of this formal solution also satisfies equation 2.3.20 but only for $\eta$ in a

ball around the origin. It is, a priori, not necessary that $Z$ is a holomorphic continuation of $\mathcal{B}_1(\tilde{z})$. A direct application of theorem 1.2.20 is thus not possible to infer the 1-summability of $\tilde{z}$.

We will not prove directly that $Z$ is a holomorphic continuation of $\tilde{z}$. Instead we construct, from $Z$, a function that is Gevrey-1 asymptotic to the formal solution of (2.3.2) on a "large" region. For this construction we use the Laplace transform.

**Definition 2.3.28.** *Let $F \in \mathcal{O}(V \times S(\sigma, 2\beta))$ for $V \subset \mathbb{C}^n$ and $\beta \in \left]0, \frac{\pi}{2}\right[$ such that there exist $K, \mu > 0$ for which*

$$\sup_{X \in V} |F(X, \eta)| \leqslant K e^{\mu |\eta|}.$$

*The Laplace transform of order 1 of $F$ is defined as*

$$\mathcal{L}_1(F)(X, \varepsilon) = \int_0^{\infty(\sigma)} F(X, \eta)\, e^{-\frac{\eta}{\varepsilon}} \mathrm{d}\eta$$

*where the integration is taken along the ray $se^{i\sigma}$, $s > 0$.*

**Remark 2.3.29.** *The Laplace transform of order 1 can be seen as the inverse of the formal Borel transform of order 1 in the class of formal series that are 1-summable in a certain direction, see for example [Bal00].*

**Proposition 2.3.30.**

- *For every $0 < \tilde{\beta} < \beta$, there exists $R > 0$ such that $\mathcal{L}_1(F)(X, \varepsilon) \in \mathcal{O}\left(V \times S\left(\sigma, \pi + 2\tilde{\beta}, R\right)\right)$.*
- *If $F_n \to F$ in $\mathcal{G}^{\mu}_{\bar{r}}\{X\}$ (over the sector $S(\sigma, 2\beta)$) then $\mathcal{L}_1(F_n) \to \mathcal{L}_1(F)$ in $\mathcal{O}\left(\overline{\mathbb{P}}_n(\bar{r}) \times S\left(\sigma, \pi + 2\tilde{\beta}, R\right)\right)$.*
- *$\mathcal{L}_1(F * G) = \mathcal{L}_1(F) \cdot \mathcal{L}_1(G)$.*

**Proof:**

- Since on the ray $se^{i\sigma}$, $s > 0$, $\left|e^{-\frac{\eta}{\varepsilon}}\right| = e^{-\frac{|\eta|}{|\varepsilon|}\cos(\sigma - \arg(\varepsilon))}$ it is clear that the Laplace transform is well defined on a bounded sector in the direction $\sigma$ with opening slightly less than $\pi$.

  Due to the integral being invariant under deformations of the path, one can see that the function is also defined on rotations of this sector. In this manner, the Laplace transform can be defined on a "large" sector. For a more detailed proof one can consult for example [Bal00].

- If $F_n \to F$ we have by definition that

$$\sup_{X \in \overline{\mathbb{P}}_n(\bar{r})} |F_n(X, \eta) - F(X, \eta)| \leqslant {}_{\bar{r}}\|F_n - F\|_{\mu, S}\, e^{\mu|\eta|}, \forall \eta.$$

From this it follows readily that $\mathcal{L}_1(F_n) \to \mathcal{L}_1(F)$ uniformly for $X \in \overline{\mathbb{P}}_n(\bar{r})$ and $\varepsilon$ in a bounded sector in the direction $\sigma$ with opening slightly less than $\pi$. Once again by the independence of path, this can be extended to the "large" sector.

- This is a straightforward application of the Fubini theorem.

□

Denote the Laplace transform of $Z$ by $\Psi$, since

$$\sum_{k=0}^{n} \mathcal{B}_1(\varepsilon H_k) * Z^{*k} \to \sum_{k=0}^{\infty} \mathcal{B}_1(\varepsilon H_k) * Z^{*k}$$

in $\mathcal{G}_{\bar{r}}^{\mu}\{X\}$ for $n \to \infty$ we have that

$$\begin{aligned}\varepsilon H(X, \Psi, \varepsilon) &= \lim_{n\to\infty} \sum_{k=0}^{n} \varepsilon H_k \Psi^k \\ &= \lim_{n\to\infty} \mathcal{L}_1\left(\sum_{k=0}^{n} \mathcal{B}_1(\varepsilon H_k) * Z^{*k}\right) = \mathcal{L}_1\left(\sum_{k=0}^{\infty} \mathcal{B}_1(\varepsilon H_k) * Z^{*k}\right).\end{aligned}$$

Thus applying the Laplace transform to (2.3.20) we have

$$\varepsilon D_X \Psi((\Lambda + U) X + A(X) + \varepsilon V(X, \Psi, \varepsilon)) = \varphi(X) \Psi + \varepsilon H(X, \Psi, \varepsilon).$$

Moreover $\Psi$ is defined and holomorphic for

$$(X, \varepsilon) \in \overline{\mathbb{P}}_n(\bar{r}) \times S(-\theta + \pi, \pi + 2\alpha, R),$$

for a certain $R > 0$ (and the opening should actually be slightly less than $\pi + 2\alpha$ but we do not reflect this in the notation).

By the Borel-Ritt-Gevrey theorem there exists a function $\gamma(X, \varepsilon)$, defined on $\overline{\mathbb{P}}_n(\bar{r}) \times S(-\theta, \pi - 2\sigma, R)$, where we can take $0 < \sigma < \alpha$, satisfying

$$\varepsilon D_X \gamma((\Lambda + U) X + A(X) + \varepsilon V(X, \gamma, \varepsilon)) = \varphi(X) \gamma + \varepsilon H(X, \gamma, \varepsilon) + \mathcal{R}(X, \varepsilon).$$

Here $\mathcal{R}$ is defined and holomorphic on the same domain as $\gamma$ and this function is exponentially decaying w.r.t. $\varepsilon$, uniformly for $X$. Moreover $\gamma$ is Gevrey-1 asymptotic to the formal solution of (2.3.2).

Since we take $\sigma < \alpha$, $S(-\theta + \pi, \pi + 2\alpha, R) \cap S(-\theta, \pi - 2\sigma, R) \neq \varnothing$, more specifically, the intersection is given by

$$S\left(-\theta - \frac{\pi}{2} + \frac{\alpha + \sigma}{2}, \frac{\alpha - \sigma}{2}, R\right) \cup S\left(-\theta + \frac{\pi}{2} - \frac{\alpha + \sigma}{2}, \frac{\alpha - \sigma}{2}, R\right),$$

which we will denote by $S_1 \cup S_2$. On this set we can thus define the difference $\Delta = \Psi - \gamma$, which satisfies

$$\varepsilon D_X \Delta\left((\Lambda + U) X + A(X) + \varepsilon \widetilde{V}(X, \varepsilon)\right) = \left(\varphi(X) + \varepsilon \widetilde{H}(X, \varepsilon)\right) \Delta - \mathcal{R}(X, \varepsilon). \tag{2.3.21}$$

where we denoted

$$\widetilde{V}(X,\varepsilon) = V(X,\Psi(X,\varepsilon),\varepsilon)$$

and

$$\mathcal{H}(X,\varepsilon) = \int_0^1 \frac{\partial H}{\partial z}(X, u\Psi(X,\varepsilon) + (1-u)\gamma(X,\varepsilon),\varepsilon) \\ + \varepsilon D_X \gamma(X,\varepsilon)\frac{\partial V}{\partial z}(X,(1-u)\Psi(X,\varepsilon) + u\gamma(X,\varepsilon),\varepsilon)\,\mathrm{d}u.$$

If we can show that $\Delta$ is exponentially decaying w.r.t. $\varepsilon$ in $S_1 \cup S_2$, the Ramis-Sibuya theorem 1.2.12, guarantees that $\Psi$ is Gevrey-1 asymptotic to the formal solution, this in turn proves theorem 2.3.1.

Before continuing we give a few remarks.

- In what follows it will often be necessary to shrink the radii in the $X$ domain $\mathbb{P}_n(\bar{r})$ or $\varepsilon$ domain $S_1 \cup S_2$. We will not reflect this in the notation.
- It is easily seen that both $\Psi$ and $\gamma$ tend to 0 for $\varepsilon \to 0$ i.e.

$$\lim_{\delta\to 0} \sup_{\substack{X\in\mathbb{P}_n(\bar{r})\\ |\varepsilon|\leqslant\delta}} |\Psi(X,\varepsilon)| = 0,$$

$$\lim_{\delta\to 0} \sup_{\substack{X\in\mathbb{P}_n(\bar{r})\\ |\varepsilon|\leqslant\delta}} |\gamma(X\varepsilon)| = 0,$$

  where $\varepsilon$ is to remain in the sector corresponding to the function.

- Using the previous point we may assume, by shrinking the radius of $S_1 \cup S_2$ and employing the Cauchy inequalities (for $H$ and $V$), that

$$\sup_{X,\varepsilon} |\mathcal{H}(X,\varepsilon)|, \qquad \sup_{X,\varepsilon} \left\| \widetilde{V}(X,\varepsilon) \right\|_{\max} < M, \tag{2.3.22}$$

  for a certain $M > 0$ and where the supremum is taken over $\mathbb{P}_n(\bar{r}) \times (S_1 \cup S_2)$.

- Since for all the diagonal elements $\lambda_j$ of $\Lambda$,

$$\lambda_j \in S(\theta + \arg(\varphi(0,0)), 2\rho)$$

  it holds that

$$\varepsilon\lambda_j \in S\left(\arg(\varphi(0,0)) - \frac{\pi}{2} + \frac{\alpha+\sigma}{2}, \alpha - \sigma + 2\rho\right),$$

  for all $\varepsilon \in S_1$ and

$$\varepsilon\lambda_j \in S\left(\arg(\varphi(0,0)) + \frac{\pi}{2} - \frac{\alpha-\sigma}{2}, \alpha - \sigma + 2\rho\right),$$

  for all $\varepsilon \in S_2$.

We now show that $\Delta$ is exponentially decaying w.r.t $\varepsilon \in S_1$, the result for $S_2$ is completely analogous.
Notice that

$$S\left(-\arg\left(\varphi(0,0)\right) - \frac{\pi}{2} - \frac{\alpha+\sigma}{2}, \pi - \alpha + \sigma - 2\rho\right) \cap S\left(-\arg\left(\varphi(0,0)\right), \pi\right) \neq \emptyset.$$

Let $\tau \in [0, 2\pi[$ be a direction in this intersection, by shrinking the radius of $\mathbb{P}_n(\bar{r})$ and $S_1$ if necessary, we may assume that there exists a $0 < d_\tau < \frac{\pi}{2}$ such that

$$\left|\arg\left(\varepsilon\lambda_j e^{i\tau}\right) - \pi\right| \leqslant \frac{\pi}{2} - d_\tau, \tag{2.3.23}$$

$$\left|\arg\left(\left(\varphi(X) + \varepsilon\mathcal{H}(X,\varepsilon)\right)e^{i\tau}\right)\right| \leqslant \frac{\pi}{2} - d_\tau, \tag{2.3.24}$$

$$\left|\varphi(X) + \varepsilon\mathcal{H}(X,\varepsilon)\right| \geqslant \frac{|\varphi(0)|}{2}, \tag{2.3.25}$$

for $j = 1, \ldots, n$ and $(X, \varepsilon) \in \mathbb{P}_n(\bar{r}) \times S_1$ (we use in the second and third inequalities the continuity of $\varphi$ and that $\mathcal{H}$ is bounded by an $M$).
Due to (2.3.23), there exists an $\nu > 0$ such that for all $j = 1, \ldots, n$, $t > 0$,

$$\left|e^{t\varepsilon e^{i\tau}\lambda_j}\right| \leqslant e^{-|\varepsilon|t\nu}. \tag{2.3.26}$$

Fix $X_0, \varepsilon$ and consider the following ODE

$$\dot{X}_\varepsilon = e^{i\tau}\varepsilon\left((\Lambda+U)X_\varepsilon + A(X_\varepsilon) + \varepsilon\widetilde{V}(X_\varepsilon,\varepsilon)\right) \tag{2.3.27}$$

$$X_\varepsilon(0) = X_0$$

**Proposition 2.3.31.** *There exists an $R > 0$ sufficiently small, such that $R < r_j, \forall j = 1, \ldots, n$, and a sufficiently small radius of $S_1$ such that the solutions of (2.3.27) with $\|X_0\|_{\max} \leqslant R$ exist for $t \in [0, \infty[$ and remain in $\overline{\mathbb{P}}_n(\bar{r})$.*

**Proof:** We first remark that we may assume, without loss of generality, that $\Lambda + U$ is a Jordan matrix.
Let $R_0 = \min_{j=1,\ldots,n}\{r_j\}$. Since $A = \mathrm{O}(X^2)$ and $V$ bounded, we can reduce $\bar{r}$ and the radius of $S_1$ such that

$$\max_{\|X\|_{\max} \leqslant R_0} \|A(X) + \varepsilon V(X,\varepsilon)\|_{\max} < \frac{\nu^2 R_0}{2(\nu+1)}.$$

We prove the result if we show that for $\|X_0\|_{\max} \leqslant \frac{\nu R_0 e^{1-\nu}}{2}$ the solution satisfies for all $t \geqslant 0$, $\|X_\varepsilon(t)\|_{\max} \leqslant R_0$.
The solution to (2.3.27) satisfies

$$X_\varepsilon(t) = e^{t\varepsilon e^{i\tau}(\Lambda+U)}X_0 + \varepsilon e^{i\tau}\int_0^t e^{(t-s)\varepsilon e^{i\tau}(\Lambda+U)}\left(A(X_\varepsilon(s)) + \varepsilon\widetilde{V}(X_\varepsilon(s),\varepsilon)\right)\mathrm{d}s.$$

Since $\Lambda + U$ is a Jordan matrix one can see rather easily that by (2.3.26) we have for all $t \geqslant 0$,

$$\left\|e^{t\varepsilon e^{i\tau}(\Lambda+U)}\right\| \leqslant (1 + t|\varepsilon|)\, e^{-|\varepsilon|t\nu} \leqslant \frac{e^{\nu-1}}{\nu}.$$

This last estimate follows easily by inspecting the derivative w.r.t. $t$.
Suppose now, by contradiction, that there exists a $t_* \in \, ]0, \infty[$ such that $\|X_\varepsilon(t_*)\|_{\max} = R_0$ and $\|X_\varepsilon(t)\|_{\max} < R_0$ for $t \in [0, t_*[$, then

$$\|X_\varepsilon(t_*)\|_{\max} < \frac{e^{\nu-1}}{\nu}\|X_0\|_{\max} + \frac{\nu^2 R_0}{2(\nu+1)}|\varepsilon|\int_0^{t_*}(1+(t-s)|\varepsilon|)\,e^{-|\varepsilon|(t-s)\nu}\mathrm{d}s.$$

A straightforward calculation learns that

$$|\varepsilon|\int_0^{t_*}(1+(t-s)|\varepsilon|)\,e^{-|\varepsilon|(t-s)\nu}\mathrm{d}s = \frac{1}{\nu} - \frac{1+t_*|\varepsilon|}{\nu}e^{-|\varepsilon|t_*\nu} + \frac{1-e^{-|\varepsilon|t_*\nu}}{\nu^2} \leqslant \frac{\nu+1}{\nu^2}$$

and thus $\|X_\varepsilon(t_*)\|_{\max} < R_0$, giving us the desired contradiction. ☐
Denote

$$f_\varepsilon(t) = \Delta(X_\varepsilon(t), \varepsilon).$$

Since $\Delta$ satisfies (2.3.21) it is immediate that

$$f_\varepsilon'(t) = (\varphi(X_\varepsilon(t)) + \varepsilon\mathcal{H}(X_\varepsilon(t),\varepsilon))\,f_\varepsilon(t) - \mathcal{R}(X_\varepsilon(t),\varepsilon)$$

and thus

$$\begin{aligned} f_\varepsilon(t) &= f_\varepsilon(0)e^{\int_0^t \varphi(X_\varepsilon(\alpha))+\varepsilon\mathcal{H}(X_\varepsilon(\alpha),\varepsilon)\mathrm{d}\alpha} \\ &\quad - \int_0^t e^{\int_s^t \varphi(X_\varepsilon(\alpha))+\varepsilon\mathcal{H}(X_\varepsilon(\alpha),\varepsilon)\mathrm{d}\alpha}\mathcal{R}(X_\varepsilon(s),\varepsilon)\,\mathrm{d}s, \end{aligned}$$

implying

$$\begin{aligned} & f_\varepsilon(t)e^{-\int_0^t \varphi(X_\varepsilon(\alpha))+\varepsilon\mathcal{H}(X_\varepsilon(\alpha),\varepsilon)\mathrm{d}\alpha} \\ &= f_\varepsilon(0) - \int_0^t e^{-\int_0^s \varphi(X_\varepsilon(\alpha))+\varepsilon\mathcal{H}(X_\varepsilon(\alpha),\varepsilon)\mathrm{d}\alpha}\mathcal{R}(X_\varepsilon(s),\varepsilon)\,\mathrm{d}s. \end{aligned} \tag{2.3.28}$$

Using (2.3.24) and (2.3.25) we get

$$\left|e^{-\int_0^t \varphi(X_\varepsilon(\alpha))+\varepsilon\mathcal{H}(X_\varepsilon(\alpha),\varepsilon)\mathrm{d}\alpha}\right| \leqslant e^{-t\frac{|\varphi(0)|\sin(d_\mathcal{T})}{2}},$$

combining this with the exponential decay of $\mathcal{R}$ i.e. $|\mathcal{R}| \leqslant Ke^{-\frac{L}{|\varepsilon|}}$ leads to

$$\begin{aligned} & \left|\int_0^t e^{-\int_0^s \varphi(X_\varepsilon(\alpha))+\varepsilon\mathcal{H}(X_\varepsilon(\alpha),\varepsilon)\mathrm{d}\alpha}\mathcal{R}(X_\varepsilon(s),\varepsilon)\,\mathrm{d}s\right| \\ &\leqslant \frac{2}{|\varphi(0)|\sin(d_\mathcal{T})}\left(1-e^{-t\frac{|\varphi(0)|\sin(d_\mathcal{T})}{2}}\right)Ke^{-\frac{L}{|\varepsilon|}}. \end{aligned}$$

Finally taking the limit $t \to \infty$ in both sides of (2.3.28) and noticing that $f_\varepsilon(t)$ remains bounded for all $t \in [0, \infty[$ shows that

$$\Delta(X_0,\varepsilon) = f_\varepsilon(0) = \int_0^\infty e^{-\int_0^s \varphi(X_\varepsilon(\alpha))+\varepsilon\mathcal{H}(X_\varepsilon(\alpha),\varepsilon)\mathrm{d}\alpha}\mathcal{R}(X_\varepsilon(s),\varepsilon)\,\mathrm{d}s$$

and thus

$$|\Delta(X_0,\varepsilon)| \leqslant \frac{2}{|\varphi(0)|\sin(d_\mathcal{T})}Ke^{-\frac{L}{|\varepsilon|}}.$$

# Chapter 3

# Canard-heteroclinic saddle connections

In this chapter we focus ourselves on real analytic slow-fast systems with 1 slow and 1 fast variable, with an additional parameter $a$,

$$\begin{cases} \dot{x} &= \varepsilon f(x, y, a, \varepsilon) \\ \dot{y} &= g(x, y, a, \varepsilon) \end{cases}, \tag{3.0.1}$$

where the following set of assumptions is satisfied.

- There are $x_a, x_r \in \mathbb{R}$ with $x_a < x_r$ such that there is (for a parameter value $a = a_0$) a real analytic critical curve of the form $y = \psi_0(x)$ present, defined for $x \in [x_a, x_r]$. The function $\psi_0$ is thus holomorphic in a complex neighbourhood of $[x_a, x_r]$ and satisfies $g(x, \psi_0(x), a_0, 0) = 0$.

- There is a point $x_t \in\, ]x_a, x_r[$ splitting the critical curve in a normally attracting part for $x < x_t$ and a normally repelling part for $x > x_t$. By this it is meant that

$$\frac{\partial g}{\partial y}(x, \psi_0(x), a_0, 0) < 0, \forall x \in [x_a, x_t[\,,$$
$$\frac{\partial g}{\partial y}(x, \psi_0(x), a_0, 0) > 0, \forall x \in\, ]x_t, x_r]\,,$$
$$\frac{\partial g}{\partial y}(x_t, \psi_0(x_t), a_0, 0) = 0.$$

  We call such a point a turning point.

- The points $x_a$ and $x_r$ are slow-fast saddles with the slow dynamics directed from the attracting to the repelling part of the critical curve, this is characterized by

$$f(x_*, \psi_0(x_*), a_0, 0) = 0, \text{ for } x_* = x_a, x_r;$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[x \mapsto f(x, \psi_0(x), 0)](x_*)\frac{\partial g}{\partial y}(x_*, \psi_0(x_*), 0) < 0, \text{ for } x_* = x_a, x_r;$$
$$f(x, \psi_0(x), a_0, 0) > 0, \forall x \in\, ]x_a, x_r[\,.$$

- The last assumption is more technical in nature, saying that locally around the turning point $x_t$ there is a holomorphic transformation bringing the system into a specific form. We elaborate more on this form later on in section 3.2.

By corollary 2.1.11 there exists, for any compact subinterval of $[x_a, x_t[$, a formal slow manifold of (3.0.1) which is Gevrey-1 w.r.t. $\varepsilon$ uniformly for $x$ in (a neighbourhood of) the compact subinterval and $a$ close to $a_0$. Moreover, by theorem 2.3.1 (i) this formal slow manifold is 1-summable in the positive real direction, locally around $x_a$. Concretely there exist $r, \sigma > 0$ and a function $\psi(x, a, \varepsilon)$ holomorphic on $B(x_a, r) \times B(a_0, r) \times S(0, \pi + \sigma, r)$ and Gevrey-1 asymptotic, w.r.t. $\varepsilon$, uniformly in $(x, a)$ to the formal slow manifold such that $y = \psi(x, a, \varepsilon)$ is an invariant manifold of (3.0.1).
An identical statement holds for $x_r$.

**Remark 3.0.1.** *As was already remarked at the start of chapter 2, this thesis does not treat the case where parameters are present, when conducting the Gevrey analysis of slow manifolds. Above, we have used the parameter dependent versions of the results in chapter 2. A version of the proof of existence of summable slow manifolds where a parameter is explicitly included can be found in [Ken16], but only the (simpler) case of one slow variable is studied.*

In the remainder of this chapter, two additional results are shown. First, in section 3.1, we prove that a local, 1-summable, slow manifold can be saturated along normally hyperbolic parts of the critical curve, retaining the summability.

Next, in section 3.2, the situation is considered where two slow manifolds are present around points on the normally attracting resp. repelling part of the critical curve, with a turning point in between them (and the system has a specific form). It is shown that these two manifolds can be connected across the turning point, forming what will be called a "canard curve". For this matching to occur, the presence of an additional parameter is needed. The matching parameter will retain a form of summability but the canard curve itself will not exhibit any Gevrey properties at the turning point. For more details on the, quite delicate, behaviour of the canard curve at the turning point, we refer to theorem 3.2.1 and the discussion following the proof of this theorem near the end of section 3.2.

While both results are essentially self-contained, they can be combined and applied to the setting of system (3.0.1) to give the following, informally formulated, result.

**Theorem 3.0.2.** *A heteroclinic saddle connection between two persistent slow-fast saddles on a slow manifold of a real analytic planar slow-fast system is summable (in the positive real direction) w.r.t. (a root of) the singular parameter , uniformly for $x$ in compact subsets of the domain of the critical curve not including the turning point.*

For a precise statement we refer to theorem 3.2.13. One possible class of systems where this theorem can be applied is those of the form

$$\begin{cases} \dot{x} &= \varepsilon(x-c)(d-x) \\ \dot{y} &= a + x^{m-1}y + \varepsilon F(x,y,\varepsilon,a) \end{cases}$$

where $c < 0 < d$ and $m$ is even.
Theorem 3.2.13 then shows the existence of a $m$-summable control curve $a = \mathcal{A}(\varepsilon^{1/m})$ along which the equation has a $m$-summable solution $y = y(x, \varepsilon^{1/m})$ w.r.t. $\varepsilon^{1/m}$, uniformly on arbitrary compact subsets of $[c, d]$ which do not include the turning point $x = 0$.

## 3.1 Tracing summability along the critical curve

The question that is answered in this section is whether or not the 1-summability of a formal slow manifold at a given location $x = x_0$ implies the 1-summability of this formal slow manifold at another location. In other words, is the summability information carried along the slow curve? The answer is given by the next theorem. An additional parameter $a$ is added in view of its necessity later on in the matching of slow manifolds across a turning point, but its inclusion does not influence the results or proofs in this section.
We remark that the "cause" of summability is irrelevant in this section. It could, for example, be achieved from slow-fast saddles as in theorem 2.3.1 (i) but at the same time, we can refer to [CDMFS07] to conclude that equations of the form

$$\varepsilon x^{r+1}\frac{dy}{dx} = \lambda(x)y + O(y^2, \varepsilon),$$

with $r > 0$, enjoy similar results: when $\lambda(0) < 0$, then it is easily seen from the monomial summability (w.r.t. the monomial $\varepsilon x^r$) proved in [CDMFS07] that for a sufficiently small neighbourhood of a compact interval, lying close to 0, on the strictly positive real axis, the equation has a solution that is 1-summable w.r.t. $\varepsilon$ in directions close to the real axis.

**Theorem 3.1.1.** *Consider the real analytic slow-fast family of vector fields*

$$\begin{cases} \dot{x} &= \varepsilon f(x,y,a,\varepsilon) \\ \dot{y} &= g(x,y,a,\varepsilon), \end{cases} \tag{3.1.1}$$

*with a real analytic critical curve given by the graph $y = \psi_0(x)$ (for $a = a_0$), $x \in [x_0, x_1] \subset \mathbb{R}$. Suppose that the unperturbed vector field is normally hyperbolically attracting at points of the critical curve, which means*

$$\frac{\partial g}{\partial y}(x, \psi_0(x), a_0, 0) < 0, \qquad \forall x \in [x_0, x_1].$$

*Assume furthermore that* $f(x, \psi_0(x), a_0, 0) > 0$ *for all* $x \in [x_0, x_1]$ *(in other words the slow dynamics is regular along the critical curve and directed from left to right).*
*Suppose the formal slow manifold is* 1*-summable in the real positive direction, w.r.t.* $\varepsilon$*, uniformly around* $(x_0, a_0)$*, i.e. there exist* $r, \sigma > 0$ *and a holomorphic function* $\Psi(x, a, \varepsilon)$ *defined on*

$$B(x_0, r) \times B(a_0, r) \times S(0, \pi + \sigma, r),$$

*such that* $y = \Psi(x, a, \varepsilon)$ *is an invariant manifold of* (3.1.1)*. Then the formal slow manifold is* 1*-summable in the real positive direction w.r.t.* $\varepsilon$*, uniformly around* $[x_0, x_1] \times \{a_0\}$ *meaning there exists an open* $V \subset \mathbb{C}$ *with* $[x_0, x_1] \subset V$ *and* $0 < \sigma' \leqslant \sigma$*,* $0 < r' < r$ *such that* $\Psi(x, a, \varepsilon)$ *can be extended to*

$$V \times B(a_0, r) \times S\left(0, \pi + \sigma', r'\right).$$

**Remark 3.1.2.** *Readers who are familiar with the terminology of complex relief functions (see [Wal91] for example) can see that the normally attracting nature of the critical curve and the fact that the theorem is stated on a compact* real *interval means that the straight path from* $x_0$ *to* $x_1$ *is a descending path according to the complex relief function associated with the slow-fast vector field. It is hence well-known that points close to* $x_0$ *and for* $\varepsilon > 0$ *can be easily integrated towards* $x_1$ *without straying from the critical curve. Up to the knowledge of the author, the literature does not contain a statement that carries summability information along a descending path.*

It is not hard (in fact this is the topic of the next subsection) to translate the question in theorem 3.1.1 to a question regarding analytic differential equations of the form

$$\varepsilon \frac{dy}{dx} = y + \varepsilon H(x, y, a, \varepsilon), \tag{3.1.2}$$

defined for $(x, y, a, \varepsilon)$ in a complex neighbourhood of $[x_0, x_1] \times \{0\} \times \{a_0\} \times \{0\}$. Using this reduction, theorem 3.1.1 is a direct consequence of the next theorem. We will elaborate a bit on this in a minute.

**Theorem 3.1.3.** *Given the analytic equation (3.1.2) defined for* $(x, y, a, \varepsilon)$ *in a complex neighbourhood of* $[X_1, 0] \times \{0\} \times \{a_0\} \times \{0\}$*, and with* $X_1 < 0$*.*
*Then* 1*-summability of the formal solution in the direction* 0*, w.r.t.* $\varepsilon$*, uniformly around* $(0, a_0)$ *implies the* 1*-summability of the formal solution in the direction* 0*, w.r.t.* $\varepsilon$*, uniformly around* $[X_1, 0] \times \{a_0\}$*.*

**Remark 3.1.4.** *In general, equations of the form (3.1.2) will not have a* 1*-summable solution (not even in isolated directions).*
*Consider, for example, an entire function* $h$ *whose set of zeroes is given by*

$$\bigcup_{k=1}^{\infty} \bigcup_{j=0}^{k-1} \left\{ k e^{i \frac{2j}{k} \pi} \right\}.$$

*Such a function exists by the Weierstrass theorem, see [Kra12].*
*We claim that the equation*

$$\varepsilon\frac{\mathrm{d}y}{\mathrm{d}x} = y + \frac{\varepsilon}{h(x)}$$

*has no* 1*-summable solution in any direction. Indeed, assuming that such a solution does exist, would imply that the Borel transformed equation*

$$1 * \frac{\partial Y}{\partial x} = Y + \frac{1}{h(x)}$$

*has a solution,* $Y(x,\eta)$*, which is defined for* $\eta$ *in some infinite sector. One can see easily that the unique solution of the above equation is given by* $\frac{-1}{h(x+\eta)}$*. This function is clearly, by construction of* $h$*, not defined on any infinite sector.*

### 3.1.1 Theorem 3.1.3 implies Theorem 3.1.1

Under the conditions of Theorem 3.1.1, we can make a time rescaling to reduce (3.1.1) to

$$\begin{cases} \dot{x} &= \varepsilon \\ \dot{y} &= G(x,y,a,\varepsilon), \qquad \text{with } G(x,y,a,\varepsilon) := \frac{g(x,y,a,\varepsilon)}{f(x,y,a,\varepsilon)}. \end{cases}$$

From the conditions imposed on $\frac{\partial g}{\partial y}$ easily follows $\lambda_0(x) := \frac{\partial G}{\partial y}(x,\psi_0(x),a_0,0) < 0$ for all $x \in [x_0,x_1]$. Let us now extend the critical curve defined for $a = a_0$ to critical curves for nearby values of $a$, using the implicit function theorem: there exists a unique analytic $\psi(x,a)$ such that $G(x,\psi(x,a),a,0) = 0$ and $\psi(x,a_0) = \psi_0(x)$. After writing $y = \tilde{y} + \psi(x,a)$, we find

$$\begin{cases} \dot{x} &= \varepsilon \\ \dot{\tilde{y}} &= \lambda(x,a)\tilde{y} + O(\tilde{y}^2) + O(\varepsilon), \end{cases}$$

where $\lambda(x,a) = \frac{\partial G}{\partial y}(x,\psi(x,a),a,0)$. Note that $\lambda(x,a) = \lambda_0(x) + O(|a-a_0|)$, meaning that we may assume that $\lambda(x,a)$ has a strictly negative real part. Now define

$$u(x,a) = \int_{x_0}^{x} \lambda(s,a)\mathrm{d}s,$$

where we limit this function to a sufficiently small (and simply connected) neighbourhood of $[x_0,x_1]$ and $a$ near $a_0$. Writing $\tilde{x} = u(x,a)$, we obtain after yet another time rescaling and reversal

$$\begin{cases} \dot{\tilde{x}} &= \varepsilon \\ \dot{\tilde{y}} &= \tilde{y} + O(\tilde{y}^2) + O(\varepsilon), \end{cases}$$

Denote

$$X_1 := u(x_1,a_0) = \int_{x_0}^{x_1} \lambda(s,a_0)\mathrm{d}s < 0.$$

One can see that the mapping $(x,a) \mapsto (u(x,a),a)$ is analytic with an analytic inverse on an environment of $[x_0,x_1] \times \{a_0\}$ mapping this last set onto $[X_1,0] \times \{a_0\}$. Since

the result in theorem 3.1.3 is obtained on an environment of $[X_1, 0] \times \{a_0\}$, going back to the original variables will yield a result on an environment of $[x_0, x_1] \times \{a_0\}$, which is indeed the goal in theorem 3.1.1.
Dropping the tildes, invariant manifolds of the above system of differential equations are solution curves of

$$\varepsilon \frac{dy}{dx} = y + y^2 C(x, y, a, \varepsilon) + \varepsilon D(x, y, a, \varepsilon),$$

for some analytic functions $C$ and $D$. We can now further reduce to a more elementary form with $C = 0$ by applying a singular transformation $y = \varepsilon Y$:

$$\varepsilon \frac{dY}{dx} = Y + \varepsilon Y^2 C(x, y, a, \varepsilon) + D(x, \varepsilon Y, a, \varepsilon) = Y + D(x, 0, a, 0) + O(\varepsilon).$$

The equation in Theorem 3.1.3 is obtained after a final translation in the $Y$ direction: $Y \mapsto Y + D(x, 0, a, 0)$.

### 3.1.2 Proof of Theorem 3.1.3

We may make the following assumptions about equation (3.1.2), which we repeat here for the sake of convenience:

$$\varepsilon \frac{\mathrm{d}y}{\mathrm{d}x} = y + \varepsilon H(x, y, a, \varepsilon). \tag{3.1.3}$$

($H_1$) $H$ is bounded and analytic on $U \times B(0, r) \times B(a_0, r) \times B(0, r)$ for some $r > 0$ and some open complex neighbourhood $U$ of $[X_1, 0]$.

($H_2$) Equation (3.1.3) has an $(a, \varepsilon)$-family of bounded analytic solutions $G(x, a, \varepsilon)$ defined for $(x, a, \varepsilon)$ in $B(0, s) \times B(a_0, s) \times S(0, \pi + \sigma, s)$ for some $s > 0$ and some $\sigma > 0$. This assumption is a consequence of the assumption formulated in Theorem 3.1.3 regarding the 1-summability w.r.t. $\varepsilon$ of a solution of the ODE near 0.

The proof of Theorem 3.1.3 essentially contains two steps. In a first step, we analytically continue the initial solution $G(x, a, \varepsilon)$ defined near 0 towards $X_1$ (actually a bit further) by using the ODE. This will provide a solution near $[X_1, 0]$ and for $\varepsilon$ in some sector of opening angle a bit larger than $\pi$. In the second and final step, we construct an other solution of the ODE near $[X_1, 0]$ but on a complementary complex sector for $\varepsilon$ and describe the relation with the analytically continued solution from step 1. We finally apply the Ramis-Sibuya theorem 1.2.12 to conclude the 1-summability of the analytically continued solution. This method has been used before in the literature, for example in [FS03].
Note that $G$ is a solution to equation (3.1.3) thus $G(x, a, \varepsilon) = \mathcal{O}(\varepsilon)$ and we may assume, by choosing $s$ sufficiently small

($H_3$) $|G(x, a, \varepsilon)| < \frac{r}{2}$, for all $(x, a, \varepsilon) \in B(0, s) \times B(a_0, s) \times S(0, \pi + \sigma, s)$.

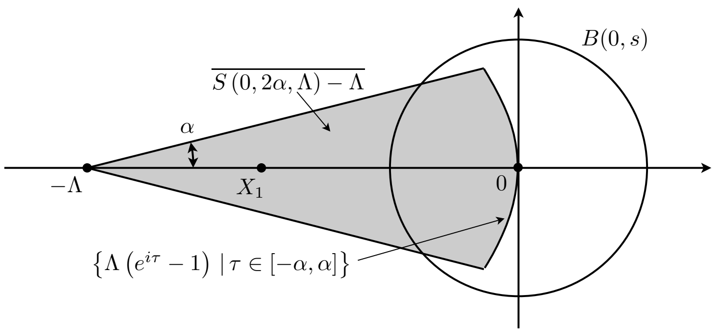


Figure 3.1: $\overline{S(0,2\alpha,\Lambda)-\Lambda}$

**Analytic continuation of the initial solution**

We continue with the notations introduced in hypotheses $(H_1)$ and $(H_2)$ above and specify the set on which we want to find a solution to (3.1.3).
Choose some $-\Lambda < X_1$ (thus $-\Lambda \in \mathbb{R}$) such that $[-\Lambda, 0] \subset U$. There then exists a small enough half-opening angle $\alpha < \frac{\pi}{2}$ such that

$$\overline{S(0,2\alpha,\Lambda)-\Lambda} \subset U.$$

(see figure 3.1.) We furthermore assume that

$$\left\{\Lambda\left(e^{i\tau}-1\right) \mid \tau \in [-\alpha,\alpha]\right\} \subset B(0,s).$$

(In other words, the terminating arc of the sector $S(0,2\alpha,\Lambda)-\Lambda$ with vertex $-\Lambda$ lies inside $B(0,s)$, again see figure 3.1.) Our aim is to analytically continue the initial solution provided in $(H_2)$ on $B(0,s)$ to the domain $S(0,2\alpha,\Lambda)-\Lambda$.

**Proposition 3.1.5.** *Let $a, z \in \mathbb{C}$. If $|a| < |z|$ then*

$$|\mathrm{Arg}(z+a) - \mathrm{Arg}(z)| \leqslant \sin^{-1}\frac{|a|}{|z|}.$$

**Lemma 3.1.6.** *Let $(H_1)$, $(H_2)$ and $(H_3)$ be satisfied. Let $0 < \sigma' < \max\{\sigma, \alpha\}$ be fixed. The initial solution $y = G(x,a,\varepsilon)$ of (3.1.3) can be analytically continued to a solution defined on*

$$\overline{S(0,2\alpha,\Lambda)-\Lambda} \times B(a_0,s) \times S\left(0,\pi+\sigma',s'\right),$$

*for sufficiently small $s' > 0$. Moreover this continued solution is bounded by $r/2$.*

**Proof:** Define

$$M = \sup_{x,y,a,\varepsilon} |H(x,y,a,\varepsilon)|,$$

where the supremum is taken for $x \in \overline{S(0, 2\alpha, \Lambda) - \Lambda}$, $|y| < r$, $|a - a_0| < r$ and $\varepsilon \in S(0, \pi + \sigma, s)$. We will define conditions on $s'$ so that for any given $\varepsilon \in S(0, \pi + \sigma', s')$, any given $a \in B(a_0, s)$ and any given $z \in \overline{S(0, 2\alpha, \Lambda) - \Lambda}$ it is possible to integrate (3.1.3) along a well-chosen path towards $z$. Independence of path and analytic dependence on parameters and initial conditions ensures that this method yields an analytic solution on the required domain.

In the remainder of the proof we hence fix $z$, $a$ and $\varepsilon$. The integration path is the linear path from $z_0$ to $z$, where $z_0 := \Lambda(e^{i\beta} - 1)$ ($\beta$ still to be specified, $|\beta| \leqslant \alpha$, which is located on the terminating arc of the sector $\overline{S(0, 2\alpha, \Lambda) - \Lambda}$) and which lies inside the definition domain of the initial solution defined in $(H_2)$. The ODE, restricted to the path from $z_0$ to $z$, parametrized by $p(t) = (1-t)z_0 + tz$ is given by:

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{z - z_0}{\varepsilon}\left(\gamma + \varepsilon H(p(t), \gamma, a, \varepsilon)\right) \tag{3.1.4a}$$

$$\gamma(0) = G(z_0, a, \varepsilon). \tag{3.1.4b}$$

It suffices to show that this equation has a maximal solution defined on an interval $]t_2, t_1[$ with $t_1 > 1$. Suppose by contradiction that $t_1 \leqslant 1$. Clearly the right hand side of (3.1.4a) is defined (for all parameters $(a, \varepsilon)$), for $(t, \gamma)$ in the compact set $[0, 1] \times \overline{B}(0, r/2)$. If we prove that $|\gamma(t)| \leqslant r/2$ for all $t \in [0, t_1[$ we thus get a contradiction, since $t_1 \leqslant 1$. Since $|\gamma(0)| = |G(z_0, a, \varepsilon)| < \frac{r}{2}$ by assumption, we prove this by showing that if there exists an $t_* \in ]0, 1[$ with $|\gamma(t_*)| = r/2$ we must have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t \mapsto |\gamma(t)|^2\right)(t_*) < 0$$

which implies what we are aiming for. After some calculations one finds that this derivative is given by

$$2\,\mathrm{Re}\left(\frac{z - z_0}{\varepsilon}\overline{\gamma(t_*)}\left(\gamma(t_*) + \varepsilon H(p(t_*), \gamma(t_*), a, \varepsilon)\right)\right).$$

Consequently it is sufficient to show that

$$\left|\arg\left(\frac{z - z_0}{\varepsilon}\overline{\gamma(t_*)}\left(\gamma(t_*) + \varepsilon H(\sigma(t_*), \gamma(t_*), a, \varepsilon)\right)\right) - \pi\right| < \frac{\pi}{2}.$$

Now define $\rho = \frac{1}{2}(\alpha - \sigma') > 0$ and choose $s' < s$ such that $s' < (r/2M)\sin\rho$ which implies that the next inequality is satisfied (remember that $\varepsilon \in S(0, \pi + \sigma', s')$):

$$|\varepsilon H(p(t_*), \gamma(t_*), a, \varepsilon)| \leqslant s'M \leqslant \frac{r}{2}\sin\rho.$$

By proposition 3.1.5 we then have, since

$$\arg\left(\overline{\gamma(t_*)}\right) = -\arg\left(\gamma(t_*)\right) \text{ and } |\gamma(t_*)| = \frac{r}{2},$$

that

$$\begin{aligned}&\left|\arg\left(\frac{z-z_0}{\varepsilon}\overline{\gamma(t_*)}\,(\gamma(t_*)+\varepsilon H(p(t_*),\gamma(t_*),a,\varepsilon))\right)-\pi\right|\\&<\left|\arg\frac{z-z_0}{\varepsilon}-\pi\right|+\rho=\left|\arg\frac{z_0-z}{\varepsilon}\right|+\rho.\end{aligned}$$

Given that $z$ lies in a sector with opening angle $\alpha$ and that $z_0$ can be chosen freely on the ending arc, it is easy to see that the argument of $z_0-z$ can be freely chosen between $-\alpha$ and $\alpha$.

When the argument of $\varepsilon$ is non-negative we choose

$$\frac{\alpha}{2}<\arg(z_0-z)<\frac{1}{2}(\pi+\sigma'-\alpha),$$

while for $\arg(\varepsilon)<0$ we take

$$-\frac{1}{2}(\pi+\sigma'-\alpha)<\arg(z_0-z)<-\frac{\alpha}{2}.$$

One can see that such a choice can be made by the assumptions $\alpha<\frac{\pi}{2}$, $0<\sigma'$ and that they guarantee that $\left|\arg\frac{z_0-z}{\varepsilon}\right|<\frac{1}{2}(\pi+\sigma'-\alpha)$. It follows that we get

$$\begin{aligned}&\left|\arg\left(\frac{z-z_0}{\varepsilon}\overline{\gamma(t_*)}\,(\gamma(t_*)+\varepsilon H(p(t_*),\gamma(t_*),a,\varepsilon))\right)-\pi\right|\\&<\frac{1}{2}(\pi+\sigma'-\alpha)+\rho=\frac{\pi}{2}\end{aligned}$$

given the definition of $\rho$ in this proof. □

### Gevrey asymptotics of the extension

We are now quite close to showing Theorem 3.1.3. It remains to show that the analytic continuation provided in Lemma 3.1.6 is 1-summable w.r.t. $\varepsilon$ uniformly for $x$ near $[X_1,0]$.

Let $y=G(x,a,\varepsilon)$ be the continuation provided by Lemma 3.1.6. We will define a second solution $y=G'(x,a,\varepsilon)$ defined for $x$ near $[-\Lambda,0]$, but for $\varepsilon$ on a different sector. We will then consider the difference $G-G'$ for $\varepsilon$ in overlapping sectors and show that it is exponentially small w.r.t. $|\varepsilon|$ as $\varepsilon\to 0$. By the Ramis-Sibuya theorem 1.2.12 it can then be concluded that both $G$ and $G'$ are Gevrey-1 asymptotic to the same formal power series $\hat{G}(x,a,\varepsilon)$, uniformly for $(x,a)$ given near $[X_1,0]\times\{a_0\}$. Furthermore, since the $\varepsilon$-sector of $G$ has opening angle larger than $\pi$, $G$ will be 1-summable w.r.t. $\varepsilon$ in the bisecting direction.

**Lemma 3.1.7.** *Assume $(H_1)$, $(H_2)$ and $(H_3)$ are satisfied. Let $0<\tau<\frac{\pi}{2}$ be fixed. The solution of*

$$\begin{aligned}\varepsilon\frac{\mathrm{d}y}{\mathrm{d}x}&=y+\varepsilon H(x,y,a,\varepsilon)\\y(-\Lambda,a,\varepsilon)&=0\end{aligned}$$

*is defined and analytic on* $V \times B(a_0, s) \times S(\pi, \pi - \tau, s'')$ *for some* $s'' > 0$ *and* $V$ *a neighbourhood of* $[-\Lambda, 0]$*. We may assume that the solution is bounded by* $r/2$*.*

**Proof:** The proof is completely analogous to the proof of Lemma 3.1.6. Note that when comparing the situation described in Lemma 3.1.6 with the one here, it is relevant to see that the real part of $\varepsilon$ is negative here, and hence exponential attraction is experienced while continuing the solution at $x = -\Lambda$ to values of $x$ in $[-\Lambda, 0]$ which in essence lie to the right of $-\Lambda$ in the complex plane. □

The following lemma finishes the proof of theorem 3.1.3.

**Lemma 3.1.8.** *Using the notations and assumptions from lemma 3.1.6, lemma 3.1.7, together with the extra assumption* $\frac{\sigma'}{2} < \tau < \sigma'$*, we have the following.*

*Denote* $\nu = \min\{s', s''\}$*. The solution from lemma 3.1.6, limited to*

$$\widetilde{V} \times B(a_0, s) \times S(0, \pi + \sigma', \nu),$$

*with* $\widetilde{V}$ *a neighbourhood of* $[X_1, 0]$*, is Gevrey-*1 *asymptotic, in* $\varepsilon$*, to a formal series, uniformly for* $(x, a)$ *and thus it is* 1*-summable.*

**Remark 3.1.9.** *It is possible to prove the above result on (almost) the entire domain of the* $x$ *variable which was found in lemma 3.1.6, this is however not necessary for our goal and would make the proof more convoluted.*

**Proof:** [Proof of lemma 3.1.8] Denote $G(x, a, \varepsilon)$ the solution found in lemma 3.1.6 and $G'(x, a, \varepsilon)$ the solution from the above lemma. If we put

$$\Delta(x, a, \varepsilon) = G(x, a, \varepsilon) - G'(x, a, \varepsilon),$$

it satisfies the following equation

$$\varepsilon \frac{\mathrm{d}\Delta}{\mathrm{d}x} = \Delta + \varepsilon \left(H(x, G(x, a, \varepsilon), a, \varepsilon) - H(x, G'(x, a, \varepsilon), a, \varepsilon)\right)$$
$$\Delta(-\Lambda, a, \varepsilon) = G(-\Lambda, a, \varepsilon).$$

Since

$$\begin{aligned} &H(x, G(x, a, \varepsilon), a, \varepsilon) - H(x, G'(x, a, \varepsilon), a, \varepsilon) \\ &= \underbrace{\int_0^1 \frac{\partial H}{\partial y}\left(x, (1-s)G'(x, a, \varepsilon) + sG(x, a, \varepsilon), a, \varepsilon\right) \mathrm{d}s}_{\mathcal{R}(x,a,\varepsilon)} \Delta(x, a, \varepsilon) \end{aligned}$$

it must hold that

$$\Delta(x, a, \varepsilon) = G(-\Lambda, a, \varepsilon) e^{\int_{-\Lambda}^{x} \mathcal{R}(w,a,\varepsilon)\mathrm{d}w} e^{\frac{x+\Lambda}{\varepsilon}}.$$

Denote

$$\tilde{M} = \sup_{x,y,a,\varepsilon} |H(x, y, a, \varepsilon)|,$$

where the supremum is taken for $x \in \overline{S(0, 2\alpha, \Lambda) - \Lambda}$, $|y| < r$, $|a - a_0| < r$, $\varepsilon \in S\left(0, \pi + \sigma', \nu\right)$
By Cauchy's inequalities it holds that (remember that both $G$ and $G'$ are bounded by $r/2$)

$$|\mathcal{R}(x, a, \varepsilon)| \leqslant \frac{3}{r}\tilde{M}$$

for all $(x, a, \varepsilon)$ in its domain. It follows that

$$\begin{aligned} |\Delta(x, a, \varepsilon)| &\leqslant |G(-\Lambda, a, \varepsilon)|\, e^{\left|\int_{-\Lambda}^{x} \mathcal{R}(w,a,\varepsilon)\mathrm{d}w\right|} e^{\mathrm{Re}\left(\frac{x+\Lambda}{\varepsilon}\right)} \\ &\leqslant \frac{r}{2} e^{\frac{3\tilde{M}|x+\Lambda|}{r}} e^{\left|\frac{x+\Lambda}{\varepsilon}\right| \cos\left(\arg\left(\frac{x+\Lambda}{\varepsilon}\right)\right)} \\ &\leqslant \frac{r}{2} e^{\frac{3\tilde{M}\Lambda}{r}} e^{\left|\frac{x+\Lambda}{\varepsilon}\right| \cos\left(\arg\left(\frac{x+\Lambda}{\varepsilon}\right)\right)}. \end{aligned}$$

To make further estimates we will restrict ourselves to the following domain for the $x$ variable. Define

$$\widetilde{V} := \left(V \cap \overline{S\left(0, \sigma' - \tau, \Lambda\right) - \Lambda}\right) \backslash B\left(-\Lambda, \frac{X_1 + \Lambda}{2}\right).$$

Notice that for $x \in \widetilde{V}$ we have $|x + \Lambda| \geqslant \frac{X_1+\Lambda}{2}$. It is furthermore cumbersome but easy to check that for $x \in \tilde{V}$ and $\varepsilon \in S(0, \pi + \sigma', \nu) \cap S(\pi, \pi - \tau, \nu)$,

$$\arg\left(\frac{x + \Lambda}{\varepsilon}\right) \in \left]\frac{\pi}{2} - \frac{\sigma'}{2} + \tau, \frac{3\pi}{2} + \frac{\sigma'}{2} - \tau\right[.$$

Consequently we have

$$|\Delta(x, a, \varepsilon)| \leqslant \frac{r}{2} e^{\frac{3\tilde{M}\Lambda}{r}} e^{-\frac{X_1+\Lambda}{2|\varepsilon|} \cos\left(\frac{\pi}{2} + \frac{\sigma'}{2} - \tau\right)}$$

for all $(x, a, \varepsilon) \in \widetilde{V} \times B(a_0, s) \times S(0, \pi + \sigma', \nu) \cap S(\pi, \pi - \tau, \nu)$.
The Ramis-Sibuya theorem 1.2.12 guarantees the existence of a formal Gevrey-1 series

$$\widehat{G}(x, a, \varepsilon) = \sum_{n=0}^{\infty} g_n(x, a)\varepsilon^n,$$

where the $g_n$ are analytic on $\widetilde{V} \times B(a_0, s)$, such that $G \sim_1 \widehat{G}$ w.r.t $\varepsilon \in S(0, \pi + \sigma', \nu)$ uniformly for $(x, a) \in \widetilde{V} \times B(a_0, s)$. Thus $G$ is 1-summable. □

## 3.2 Connection across the turning point

In this section we will limit ourselves to slow-fast systems with a turning point which can be transformed, locally around the turning point, into a system of the form

$$\left\{\begin{array}{rcl} \dot{x} &=& \varepsilon \\ \dot{y} &=& px^{p-1}y + \varepsilon H(x, y, a, \varepsilon) \\ \dot{\varepsilon} &=& 0. \end{array}\right. \tag{3.2.1}$$

Where $H$ is analytic and satisfies $H(0,0,a_0,0) = 0$, $\frac{\partial H}{\partial a}(0,0,a_0,0) \neq 0$, notice that $p$ has to be an even number for $x = 0$ to be a turning point, i.e. for the stability of the critical curve, $y = 0$, to change through $x = 0$.
It is shown in [FS13] that every system of the form

$$\begin{cases} \dot{x} &= \varepsilon \\ \dot{y} &= \varphi(x)y + \varepsilon H(x,y,a,\varepsilon) \\ \dot{\varepsilon} &= 0, \end{cases}$$

with $\varphi(x)$ a real analytic function with a zero of order $p-1$ at $x = 0$ and $H(0,0,a_0,0) = 0$, $\frac{\partial H}{\partial a}(0,0,a_0,0) \neq 0$, can be transformed into this form. The authors also give some conditions on more general systems, such that the necessary transformation exists.

Setting $u = \varepsilon^{1/p}$, using the branch of the $p$-th root for which $1^{1/p} = 1$, we prove the following theorem

**Theorem 3.2.1.** *Suppose $H(x,y,a,\varepsilon)$ is a bounded analytic function on*

$$B(0,r) \times B(0,r) \times B(a_0,r) \times B(0,r)$$

*with $H(0,0,a_0,0) = 0$, $\frac{\partial H}{\partial a}(0,0,a_0,0) \neq 0$. Moreover let there exist invariant manifolds of system (3.2.1), $G_1(x,a,\varepsilon)$ and $G_2(x,a,\varepsilon)$, 1-summable in the real direction and defined on*

$$B(\mp\lambda, s) \times B(a_0,r) \times S(0, \pi+\sigma, r)$$

*for certain $\lambda, s > 0$.*
*Then there exists a function $a(u)$, $p$-summable in the real direction, such that the system*

$$\begin{cases} \dot{x} &= u^p \\ \dot{y} &= px^{p-1}y + u^p H(x,y,a(u),u^p) \\ \dot{u} &= 0, \end{cases} \tag{3.2.2}$$

*has an invariant manifold of the form $y = G(x,u)$, defined for $x \in [-\lambda,\lambda] \times ]0,r_0]$ for some $r_0 > 0$ and extending the manifolds $G_1$ and $G_2$.*

**Remark 3.2.2.** *Notice that by the results from the previous sections the existence of such invariant manifolds is guaranteed if there are slow-fast saddles present on both the attracting and repelling part of the critical curve.*

### 3.2.1 Extension of invariant manifolds to 0

The general idea of the proof is to further extend the invariant manifolds until they reach $x = 0$ and then search for conditions on the parameter $a$ guaranteeing that the two extensions are matched. The continuation of these manifolds will be done under two transformations which resemble, using the terminology of blow-up maps in real variables, the phase-directional rescaling and family rescaling chart, see for example [DR01]. We note that the transformations used here actually arise as charts from a blow-up procedure in complex variables, the construction of which is slightly different than in the real case, see for example [BM88]. It is not necessary to introduce the complex blow-up procedure as we can work directly with the charts.

**Phase-directional rescaling chart**

The first chart we concentrate ourselves upon is a phase-directional rescaling chart, given by

$$x = v$$
$$y = v\overline{y}$$
$$u = v\overline{u}$$

which is clearly an analytic map with an analytic inverse between a domain and its image, provided that the domain does not contain any points where $v = 0$. Applying this transformation to the system (3.2.2) gives

$$\begin{cases} \dot{v} &= v^p \overline{u}^p \\ \dot{\overline{y}} &= v^{p-1}\left(p - \overline{u}^p\right)\overline{y} + v^{p-1}\overline{u}^p H\left(v, v\overline{y}, a, (v\overline{u})^p\right) \\ \dot{\overline{u}} &= -v^{p-1}\overline{u}^{p+1}. \end{cases}$$

Dividing by the common factor $v^{p-1}$ we arrive at

$$\begin{cases} \dot{v} &= v\overline{u}^p \\ \dot{\overline{y}} &= \left(p - \overline{u}^p\right)\overline{y} + \overline{u}^p H\left(v, v\overline{y}, a, (v\overline{u})^p\right) \\ \dot{\overline{u}} &= -\overline{u}^{p+1}. \end{cases} \tag{3.2.3}$$

Since invariant manifolds of the second system will also be invariant manifolds of the first system, we may focus on the second one.
In the following lemma we use the notations, by which we described the domain where equation (3.2.2) holds.

**Proposition 3.2.3.** *Let $p$ be even, $k \in \{0, \ldots, p-1\}$, $\rho, \theta_1, \theta_2, \Delta > 0$ satisfying $\rho + \theta_1 + \theta_2 + \Delta < \frac{\pi}{2}$, $v_0 \in B(0,r)\backslash\{0\}$, $0 < R < r$ and $K \in \mathbb{C}$ with $|K| < R$.*
*There exists a $U > 0$ such that for*

$$\overline{u}_1 \in S\left(\frac{2\pi k}{p}, \frac{\pi}{p} - \frac{2}{p}\left(\rho + \theta_1 + \theta_2 + \Delta\right), \sqrt[p]{U}\right)$$
$$v_1 \in \left(\left(v_0^p + S\left(\pi + \arg\left(v_0^p\right), 2\theta_1\right)\right) \cap S\left(\arg\left(v_0^p\right), 2\theta_2\right)\right)^{\frac{1}{p}} = \Omega\left(v_0, \theta_1, \theta_2\right),$$

*where the branch of the p-th root is chosen such that $\left(v_0^p\right)^{\frac{1}{p}} = v_0$ and the branch line lies opposite to the point $v_0^p$, we have the following. (See figure 3.2 for an example of an $\Omega\left(v_0, \theta_1, \theta_2\right)$.)*

*The solution of the initial value problem given by equation (3.2.3) supplemented with*

$$v(0) = v_0\,;\, \overline{y}(0) = \frac{K}{v_0}\,;\, \overline{u}(0) = \frac{v_1\overline{u}_1}{v_0}$$

*is defined on* $\left[0, \frac{v_1^p - v_0^p}{p(\overline{u}_1 v_1)^p}\right]$ *with the endpoint given by*

$$\left(v_1, \overline{y}\left(\frac{v_1^p - v_0^p}{p(\overline{u}_1 v_1)^p}\right), \overline{u}_1\right).$$

*Moreover* $\left|\overline{y}\left(a, \frac{v_1^p - v_0^p}{p(\overline{u}_1 v_1)^p}\right)\right| \leqslant \frac{R}{|v_0|}$.

**Proof:** Two calculations will be deferred until after the proof, they will be labelled (C1), (C2).
Let $M = \sup_{|x|,|y|,|a-a_0|,|\varepsilon|<r} |H(x,y,a,\varepsilon)|$ and choose $0 < U < \min\left\{\frac{r}{|v_0|^p}, p\sin(\Delta)\right\}$ sufficiently small such that

$$\frac{U}{p-U} < \frac{R}{M|v_0|}\sin(\rho)$$

holds.
We start off by looking at the solutions of

$$\begin{cases} \dot{v} &= v\overline{u}^p \\ \dot{\overline{u}} &= -\overline{u}^{p+1}. \end{cases}$$

$$v(0) = v_0\,;\, \overline{u}(0) = \frac{v_1\overline{u}_1}{v_0}$$

Clearly these are given by

$$v(t) = v_1\overline{u}_1\left(pt + \left(\frac{v_0}{v_1\overline{u}_1}\right)^p\right)^{\frac{1}{p}}$$

$$\overline{u}(t) = \left(pt + \left(\frac{v_0}{v_1\overline{u}_1}\right)^p\right)^{-\frac{1}{p}}$$

where in both expressions the branch of the $p$-th root given by $1^{\frac{1}{p}} = e^{i\frac{2\pi k}{p}}$ (and thus $(\overline{u}_1^p)^{\frac{1}{p}} = \overline{u}_1$) with branch line the negative real axis.
Notice that $v(t)\overline{u}(t)$ is a constant function equal to $v_1\overline{u}_1$, which also follows by calculating that the derivative of $v(t)\overline{u}(t)$ is 0.
By defining

$$T_{v_1,\overline{u}_1} = \frac{v_1^p - v_0^p}{p(\overline{u}_1 v_1)^p},$$

we get

$$v(T_{v_1,\overline{u}_1}) = v_1\,;\, \overline{u}(T_{v_1,\overline{u}_1}) = \overline{u}_1.$$

One can compute that both solutions are well defined on $[0, T_{v_1,\overline{u}_1}]$. We remark that, due to how we defined $\Omega(v_0, \theta_1, \theta_2)$,

$$|\arg(T_{v_1,\overline{u}_1}) - \pi + \arg(\overline{u}_1^p)| < \theta_1 + \theta_2. \tag{3.2.4}$$

We now concentrate on showing that the initial value problem

$$\dot{\overline{y}} = (p - (\overline{u}(t))^p)\,\overline{y} + (\overline{u}(t))^p\, H\left(v(t), v(t)\overline{y}, a, (v_1\overline{u}_1)^p\right)$$
$$\overline{y}(0) = \frac{K}{v_0}$$

has a solution on an environment of $[0, T_{v_1,\overline{u}_1}]$. By denoting $\gamma(s) = \overline{y}(sT_{v_1,\overline{u}_1})$ it suffices to show that

$$\begin{aligned}\frac{\mathrm{d}\gamma}{\mathrm{d}s} =& T_{v_1,\overline{u}_1}\Bigg( \left(p - \overline{u}\left(sT_{v_1,\overline{u}_1}\right)^p\right)\gamma \\ &+ \overline{u}\left(sT_{v_1,\overline{u}_1}\right)^p \underbrace{H\left(v\left(sT_{v_1,\overline{u}_1}\right), v\left(sT_{v_1,\overline{u}_1}\right)\gamma, a, (v_1\overline{u}_1)^p\right)}_{H_{v_1,\overline{u}_1}(s,\gamma,a)}\Bigg)\\ \gamma(0) =& \frac{K}{v_0}\end{aligned}$$

has a maximal solution on $]s_2, s_1[$ with $s_1 > 1$.
Assume by contradiction that $s_1 \leqslant 1$. Since for all $s \in ]0, s_1[$

$$\left|v\left(sT_{v_1,\overline{u}_1}\right)\right| < |v_0| < r, \tag{C1}$$
$$\left|\overline{u}\left(sT_{v_1,\overline{u}_1}\right)\right| < \sqrt[p]{U}, \tag{C2}$$

we arrive at a contradiction when we show $|\gamma(s)| \leqslant \frac{R}{|v_0|}$.
Thus, suppose that there exists an $s_* \in ]0, s_1[$ with $|\gamma(s_*)| = \frac{R}{|v_0|}$. We show that $\frac{\mathrm{d}}{\mathrm{d}s}\left(s \mapsto |\gamma(s)|^2\right)(s_*) < 0$. Since

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(s \mapsto |\gamma(s)|^2\right)(s) = 2\,\mathrm{Re}\left(\overline{\gamma(s)}\frac{\mathrm{d}\gamma}{\mathrm{d}s}(s)\right)$$

it suffices to show

$$\begin{aligned}\Bigg| &\pi - \arg\left(T_{v_1,\overline{u}_1}\left(p - \overline{u}\left(s_*T_{v_1,\overline{u}_1}\right)^p\right)\right) \\ &- \arg\left(\overline{\gamma(s_*)}\left(\gamma(s_*) + \frac{\overline{u}\left(s_*T_{v_1,\overline{u}_1}\right)^p}{p - \overline{u}\left(s_*T_{v_1,\overline{u}_1}\right)^p} H_{v_1,\overline{u}_1}(s_*, \gamma(s_*), a)\right)\right)\Bigg| < \frac{\pi}{2}\end{aligned} \tag{3.2.5}$$

since this would imply that $\frac{\mathrm{d}}{\mathrm{d}s}\left(s \mapsto |\gamma(s)|^2\right)(s_*) < 0$.
By proposition 3.1.5

$$\left|\arg\left(\overline{\gamma(s_*)}\left(\gamma(s_*) + \frac{\overline{u}\left(s_*T_{v_1,\overline{u}_1}\right)^p}{p - \overline{u}\left(s_*T_{v_1,\overline{u}_1}\right)^p} H_{v_1,\overline{u}_1}(s_*, \gamma(s_*), a)\right)\right)\right| < \rho$$

if

$$\left|\frac{\overline{u}\left(s_*T_{v_1,\overline{u}_1}\right)^p}{p - \overline{u}\left(s_*T_{v_1,\overline{u}_1}\right)^p} H_{v_1,\overline{u}_1}(s_*, \gamma(s_*), a)\right| < \frac{R}{|v_0|}\sin(\rho)$$

and this is the case when

$$\left|\frac{\overline{u}\left(s_*T_{v_1,\overline{u}_1}\right)^p}{p - \overline{u}\left(s_*T_{v_1,\overline{u}_1}\right)^p}\right| < \frac{R}{M\,|v_0|}\sin(\rho).$$

By (C2) and using $U < p\sin(\Delta) < p$, we have

$$\left|\frac{\overline{u}\left(s_*T_{v_1,\overline{u}_1}\right)^p}{p-\overline{u}\left(s_*T_{v_1,\overline{u}_1}\right)^p}\right| < \frac{U}{p-U} < \frac{R}{M\left|v_0\right|}\sin(\rho).$$

Thus (3.2.5) holds if

$$\left|\pi - \arg\left(T_{v_1,\overline{u}_1}\left(p-\overline{u}\left(s_*T_{v_1,\overline{u}_1}\right)^p\right)\right)\right| < \frac{\pi}{2} - \rho.$$

Now

$$\begin{aligned}&\left|\pi - \arg\left(T_{v_1,\overline{u}_1}\left(p-\overline{u}\left(s_*T_{v_1,\overline{u}_1}\right)^p\right)\right)\right|\\ &\leqslant \left|\pi - \arg\left(T_{v_1,\overline{u}_1}\right) - \arg\left(\overline{u}_1^p\right)\right| + \left|\arg\left(\overline{u}_1^p\right) - \arg\left(p-\overline{u}\left(s_*T_{v_1,\overline{u}_1}\right)^p\right)\right|\end{aligned}$$

Such that by (3.2.4)

$$\begin{aligned}&\left|\pi - \arg\left(T_{v_1,\overline{u}_1}\left(p-\overline{u}\left(s_*T_{v_1,\overline{u}_1}\right)^p\right)\right)\right|\\ &< \theta_1 + \theta_2 + \left|\arg\left(\overline{u}_1^p\right) - \arg\left(p-\overline{u}\left(s_*T_{v_1,\overline{u}_1}\right)^p\right)\right|\\ &\leqslant \theta_1 + \theta_2 + \frac{\pi}{2} - \left(\rho+\theta_1+\theta_2+\Delta\right) + \left|\arg(p) - \arg\left(p-\overline{u}\left(s_*T_{v_1,\overline{u}_1}\right)^p\right)\right|\end{aligned}$$

and this implies by proposition 3.1.5, (C2), and our choice of $U$ that

$$\begin{aligned}&\left|\pi - \arg\left(T_{v_1,\overline{u}_1}\left(p-\overline{u}\left(s_*T_{v_1,\overline{u}_1}\right)^p\right)\right)\right|\\ &< \frac{\pi}{2} - (\rho+\Delta) + \Delta = \frac{\pi}{2} - \rho.\end{aligned}$$

We have thus proven that equation (3.2.5) holds. □

**Proof:** [Proof of (C1)] It is easily seen that

$$\left|v\left(sT_{v_1,\overline{u}_1}\right)\right| = \left|v_0\right|\left|s\left(\frac{v_1}{v_0}\right)^p + (1-s)\right|^{\frac{1}{p}}.$$

To prove (C1) it thus suffices to show $\left|s\left(\frac{v_1}{v_0}\right)^p + (1-s)\right| < 1$ which follows if $\left|\frac{v_1}{v_0}\right|^p < 1$.

By our choice of $\Omega\left(v_0,\theta_1,\theta_2\right)$ we have

$$v_1^p \in \left(v_0^p + S\left(\pi + \arg\left(v_0^p\right), 2\theta_1\right)\right) \cap S\left(\arg\left(v_0^p\right), 2\theta_2\right).$$

This implies that

$$\left(\frac{v_1}{v_0}\right)^p \in \left(1 + S\left(\pi, 2\theta_1\right)\right) \cap S\left(0, 2\theta_2\right).$$

The boundaries of these sectors intersect in the points $\frac{\tan(\theta_1)}{\tan(\theta_1)+\tan(\theta_2)} \pm i\frac{\tan(\theta_1)\tan(\theta_2)}{\tan(\theta_1)+\tan(\theta_2)}$. One can check that the modulus of these points is strictly smaller than 1, due to the convexity of the disk of radius 1, this shows us that $\left|\frac{v_1}{v_0}\right|^p < 1$. □

**Proof:** [Proof of (C2)] We have

$$\left|\overline{u}\left(sT_{v_1,\overline{u}_1}\right)\right| = \frac{\left|\overline{u}_1\right|}{\left|s + (1-s)\left(\frac{v_0}{v_1}\right)^p\right|^{\frac{1}{p}}}.$$

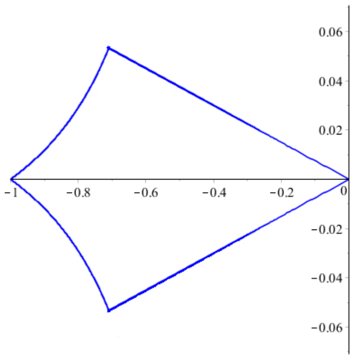


Figure 3.2: $\Omega(-1, 0.1, 0.3)$

Since $\left|\arg\left(\frac{v_0^p - v_1^p}{v_1^p}\right)\right| < \frac{\pi}{2}$ (by definition of $\Omega(v_0, \theta_1, \theta_2)$) we have

$$\begin{aligned}\left|s + (1-s)\left(\frac{v_0}{v_1}\right)^p\right| &\geqslant s + (1-s)\operatorname{Re}\left(\left(\frac{v_0}{v_1}\right)^p\right)\\ &= s + (1-s) + (1-s)\operatorname{Re}\left(\frac{v_0^p - v_1^p}{v_1^p}\right)\\ &= 1 + (1-s)\operatorname{Re}\left(\frac{v_0^p - v_1^p}{v_1^p}\right)\\ &> 1\end{aligned}$$

and consequently

$$|\overline{u}\left(sT_{v_1,\overline{u}_1}\right)| < |\overline{u}_1| < \sqrt[p]{U}.$$

□

Consider the invariant manifolds $y = G_1(x, a, u^p)$ and $y = G_2(x, a, u^p)$ of system (3.2.2). By restricting them to

$$B(\mp\lambda) \times B(a_0, r) \times S\left(0, \frac{\pi + \sigma}{p}, s\right)$$

for $s > 0$ sufficiently small, we may assume that $|G_{1,2}(x, a, u^p)| < R$ (we use the notations from proposition 3.2.3). Choose furthermore an $\alpha > 0$ such that $2p\alpha < \pi$ and $\left\{\mp\lambda e^{i\beta} \,|\, \beta \in [-\alpha, \alpha]\right\} \subset B(\mp\lambda)$

**Corollary 3.2.4.** *We reuse the notations from proposition 3.2.3. The extra demand* $0 < p\alpha - \rho - \theta_1 - \Delta < \frac{\sigma}{2}$ *is also needed.*
*Under these conditions, there exists a* $U > 0$ *such that the system*

$$\begin{aligned}\dot{v} &= v\overline{u}^p\\ \dot{\overline{y}} &= (p - \overline{u}^p)\,\overline{y} + \overline{u}^p H\left(v, v\overline{y}, a, (v\overline{u})^p\right)\\ \dot{\overline{u}} &= -\overline{u}^{p+1}.\end{aligned}$$

*has two analytic invariant manifolds. The first,* $(v, \Upsilon_1(v,a,\overline{u}), \overline{u})$, *is defined for* $(v,a,\overline{u})$ *in*

$$\bigcup_{\beta\in[-\alpha,\alpha]} \Omega\left(-\lambda e^{i\beta},\theta_1,\theta_2\right) \times B\left(a_0,r\right) \times S\left(\pi, \frac{\pi}{p} - \frac{2}{p}\left(\rho+\theta_1+\theta_2+\Delta\right), \sqrt[p]{U}\right)$$

*The second,* $(v, \Upsilon_2(v,A,\overline{u}), \overline{u})$, *is defined for* $(v,a,\overline{u})$ *in*

$$\bigcup_{\beta\in[-\alpha,\alpha]} \Omega\left(\lambda e^{i\beta},\theta_1,\theta_2\right) \times B\left(a_0,r\right) \times S\left(0, \frac{\pi}{p} - \frac{2}{p}\left(\rho+\theta_1+\theta_2+\Delta\right), \sqrt[p]{U}\right)$$

*Moreover, both* $|\Upsilon_1(v,a,\overline{u})|$ *and* $|\Upsilon_2(v,a,\overline{u})|$ *are bounded by* $\frac{R}{\lambda}$

**Proof:** Since the proof is analogous for both invariant manifolds, we prove the existence of the manifold $\Upsilon_1$.
Let $(v_1,\overline{u}_1)$ be elements from the domain specified in the lemma. Consider the following initial value problem

$$\begin{cases} \dot{v} = v\overline{u}^p \\ \dot{\overline{y}} = \left(p-\overline{u}^p\right)\overline{y} + \overline{u}^p H\left(v, v\overline{y}, a, (v\overline{u})^p\right) \\ \dot{\overline{u}} = -\overline{u}^{p+1}. \end{cases} \tag{3.2.6a}$$

$$v(0) = -\lambda\,;\; \overline{y}(0) = -\frac{G_1\left(-\lambda, a, (v_1\overline{u}_1)^p\right)}{\lambda}\,;\; \overline{u}(0) = -\frac{v_1\overline{u}_1}{\lambda}. \tag{3.2.6b}$$

We have that

$$\begin{aligned} v(t) &= v_1\overline{u}_1\left(pt + \left(\frac{\lambda}{v_1\overline{u}_1}\right)^p\right)^{\frac{1}{p}} \\ \overline{y}(t) &= \frac{G_1\left(v(t), a, (v_1\overline{u}_1)^p\right)}{v(t)} \\ \overline{u}(t) &= \left(pt + \left(\frac{\lambda}{v_1\overline{u}_1}\right)^p\right)^{-\frac{1}{p}} \end{aligned}$$

is a solution to the above problem and it is defined for $t$ in a neighbourhood of

$$\left\{\frac{\lambda^p}{p(v_1\overline{u}_1)^p}\left(e^{pi\beta}-1\right) \mid \beta\in[-\alpha,\alpha]\right\}.$$

Using proposition 3.2.3 we also know that the system consisting of (3.2.6a) with initial values

$$v(0) = -\lambda e^{i\tilde{\beta}}\,;\; \overline{y}(0) = -\frac{G_1\left(-\lambda e^{i\tilde{\beta}}, a, (v_1\overline{u}_1)^p\right)}{\lambda e^{i\tilde{\beta}}}\,;\; \overline{u}(0) = -\frac{v_1\overline{u}_1}{\lambda e^{i\tilde{\beta}}},$$

with $\tilde{\beta}$ chosen such that $v_1 \in \Omega\left(-\lambda e^{i\tilde{\beta}},\theta_1,\theta_2\right)$, has a solution defined for $t$ in a neighbourhood of $\left[0, \frac{v_1^p - \left(\lambda e^{i\tilde{\beta}}\right)^p}{p(v_1\overline{u}_1)^p}\right]$.

Consequently the solution to the problem (3.2.6a) with initial values (3.2.6b) is defined for $t$ in a neighbourhood of some path between 0 and $\frac{v_1^p - \lambda^p}{p(v_1\overline{u}_1)^p}$. By analytic dependence upon initial values we have that the general solution to (3.2.6a); (3.2.6b),

$$v(v_1\overline{u}_1;t)\,,\,\overline{y}(v_1\overline{u}_1;t)\,,\,\overline{u}(v_1\overline{u}_1;t),$$

consists of analytic functions in the variables $v_1\overline{u}_1$ and $t$. It follows that the map

$$(v_1,\overline{u}_1)\mapsto\left(v_1,\overline{y}\left(v_1\overline{u}_1;\frac{v_1^p-\lambda^p}{p(v_1\overline{u}_1)^p}\right),\overline{u}_1\right)$$

is also analytic. The inequality follows readily from proposition 3.2.3. □

**Lemma 3.2.5.** *Take $\tilde{\lambda}>0$ sufficiently small such that $S\left(\pi,2\left(\alpha+\frac{\theta_2}{p}\right),\tilde{\lambda}\right)$ is contained in $\bigcup_{\beta\in[-\alpha,\alpha]}\Omega\left(-\lambda e^{i\beta},\theta_1,\theta_2\right)$. The function $\Upsilon_1(v,a,\overline{u})$ from corollary 3.2.4 is Gevrey-$\frac{1}{p}$ asymptotic to a formal series for $v\in S\left(\pi,2\left(\alpha+\frac{\theta_2}{p}\right),\tilde{\lambda}\right)$, uniformly for $(a,\overline{u})$. An analogous statement holds for $\Upsilon_2(v,a,\overline{u})$.*

**Proof:** The proof can be given in a nearly identical manner as the proofs of proposition 6.24 and theorem 6.25 in [DM03]. □

**Proposition 3.2.6.** *Let*

$$\hat{f}_{1,2}(v,a,\overline{u})=\sum_{n=0}^{\infty}f_n^{1,2}(a,\overline{u})v^n$$

*be the formal series associated to $\Upsilon_1$ resp. $\Upsilon_2$ as in lemma 3.2.5. The coefficient of $v^0$ is given by*

$$-\frac{H(0,0,a,0)}{p}\int_1^{\infty}z^{\frac{1}{p}-1}e^{\frac{1-z}{\overline{u}^p}}\,\mathrm{d}z$$

*for both formal series*

**Proof:** Since the proof is exactly the same for $\Upsilon_1$ and $\Upsilon_2$ we only treat $\Upsilon_1$. Since $(v,\Upsilon_1,\overline{u})$ is an invariant manifold of system (3.2.3) it must hold that

$$\begin{aligned}v\overline{u}^p\frac{\partial\Upsilon_1}{\partial v}(v,a,\overline{u})-\overline{u}^{p+1}\frac{\partial\Upsilon_1}{\partial\overline{u}}(v,a,\overline{u})&\\&=(p-\overline{u}^p)\Upsilon_1(v,a,\overline{u})+\overline{u}^pH\left(v,v\Upsilon_1(v,a,\overline{u}),a,(v\overline{u})^p\right).\end{aligned}$$

Since $\Upsilon_1(v,a,\overline{u})\sim_{\frac{1}{p}}\hat{f}_1(v,a,\overline{u})$ w.r.t. $v$ it follows that

$$\begin{aligned}\Upsilon_1(v,a,\overline{u})&\xrightarrow{v\to0}f_0^1(a,\overline{u})\\\frac{\partial\Upsilon_1}{\partial v}(v,a,\overline{u})&\xrightarrow{v\to0}f_1^1(a,\overline{u})\\\frac{\partial\Upsilon_1}{\partial\overline{u}}(v,a,\overline{u})&\xrightarrow{v\to0}\frac{\partial f_0^1}{\partial\overline{u}}(a,\overline{u}).\end{aligned}$$

Consequently we must have

$$-\overline{u}^{p+1}\frac{\partial f_0^1}{\partial \overline{u}}(a,\overline{u}) = (p-\overline{u}^p) f_0^1(a,\overline{u}) + \overline{u}^p H(0,0,a,0),$$

moreover by proposition 3.2.3 it must hold that $\lim_{\overline{u}\to 0} f_0^1(a,\overline{u}) = 0$. This implies that the following identity holds

$$\begin{aligned} f_0^1(a,\overline{u}) &= -H(0,0,a,0)\int_0^{\overline{u}} t^{-1} e^{\int_t^{\overline{u}} \frac{s^p-p}{s^{p+1}}\mathrm{d}s}\mathrm{d}t \\ &= \overline{u}H(0,0,a,0)\int_{\overline{u}}^0 t^{-2} e^{\overline{u}^{-p}-t^{-p}}\mathrm{d}t. \end{aligned}$$

Using the path $\gamma(z) = \overline{u}z^{-\frac{1}{p}}$ with $z \in [1,\infty[$ we get

$$f_0^1(a,\overline{u}) = -\frac{H(0,0,a,0)}{p}\int_1^{\infty} z^{\frac{1}{p}-1} e^{\frac{1-z}{\overline{u}^p}}\mathrm{d}z.$$

□

**Family rescaling chart**

To let our two manifolds actually meet each other we will have to switch to another chart, which resembles the family rescaling chart, this is given by

$$\begin{aligned} x &= wX \\ y &= wY \\ u &= w \end{aligned}$$

which is an analytic map with analytic inverse on domains which do not contain $w = 0$. Applying this transformation to our system (3.2.2), which we repeat for the sake of convenience,

$$\begin{cases} \dot{x} &= u^p \\ \dot{y} &= px^{p-1}y + u^p H(x,y,a,u^p) \\ \dot{u} &= 0 \end{cases}$$

brings us, after dividing by a common factor $w^p$, to the system

$$\begin{cases} \dot{X} &= 1 \\ \dot{Y} &= pX^{p-1}Y + H(wX,wY,a,w^p) \\ \dot{w} &= 0. \end{cases} \tag{3.2.7}$$

By corollary 3.2.4 and lemma 3.2.5, there exist two invariant manifolds of this system

$$\left(X, X\Upsilon_1\left(wX,a,X^{-1}\right), w\right)$$

defined and holomorphic on

$$\left\{(X,w)\;\middle|\; X\in S\left(\pi,\frac{\pi}{p}-\frac{2}{p}(\rho+\theta_1+\theta_2+\Delta)\right)\Big\backslash\overline{B}\left(0,\frac{1}{\sqrt[p]{U}}\right),\right.$$
$$\left. wX\in S\left(\pi,2\left(\alpha+\frac{\theta_2}{p}\right),\tilde{\lambda}\right)\right\}\times B(a_0,r)$$

and

$$\left(X,X\Upsilon_2\left(wX,a,X^{-1}\right),w\right)$$

defined and holomorphic on

$$\left\{(X,w)\;\middle|\; X\in S\left(0,\frac{\pi}{p}-\frac{2}{p}(\rho+\theta_1+\theta_2+\Delta)\right)\Big\backslash\overline{B}\left(0,\frac{1}{\sqrt[p]{U}}\right),\right.$$
$$\left. wX\in S\left(0,2\left(\alpha+\frac{\theta_2}{p}\right),\tilde{\lambda}\right)\right\}\times B(a_0,r).$$

Moreover if we take some $X_0\in S\left(0,\frac{\pi}{p}-\frac{2}{p}(\rho+\theta_1+\theta_2+\Delta)\right)\backslash\overline{B}\left(0,\frac{1}{\sqrt[p]{U}}\right)$, both $-X_0\Upsilon_1\left(-wX_0,a,-X_0^{-1}\right)$ and $X_0\Upsilon_2\left(wX_0,a,X_0^{-1}\right)$ are Gevrey-$\frac{1}{p}$ asymptotic to a formal series for $w\in S\left(-\arg(X_0),2\left(\alpha+\frac{\theta_2}{p}\right),\frac{\tilde{\lambda}}{|X_0|}\right)$.

**Proposition 3.2.7.** *For every*

$$X_0\in S\left(0,\frac{\pi}{p}-\frac{2}{p}(\rho+\theta_1+\theta_2+\Delta)\right)\Big\backslash\overline{B}\left(0,\frac{1}{\sqrt[p]{U}}\right)$$

*there exist $\delta>0$ such that the solution to*

$$\frac{\mathrm{d}Y}{\mathrm{d}X}=pX^{p-1}Y+H(wX,wY,a,w^p)$$
$$Y(X_0,a,w)=X_0\Upsilon_2\left(wX_0,a,X_0^{-1}\right)$$

*is defined and analytic on*

$$[X_0,0]\times B(a_0,r)\times S\left(-\arg(X_0),2\left(\alpha+\frac{\theta_2}{p}\right),\delta\right).$$

*Furthermore, for the same $X_0$, the solution to*

$$\frac{\mathrm{d}Y}{\mathrm{d}X}=pX^{p-1}Y+H(wX,wY,a,w^p)$$
$$Y(-X_0,a,w)=-X_0\Upsilon_1\left(-wX_0,a,-X_0^{-1}\right)$$

*is also defined and analytic on*

$$[-X_0,0]\times B(a_0,r)\times S\left(-\arg(X_0),2\left(\alpha+\frac{\theta_2}{p}\right),\delta\right).$$

**Proof:** We prove the result for the first initial value problem

$$\frac{\mathrm{d}Y}{\mathrm{d}X}=pX^{p-1}Y+H(wX,wY,a,w^p)$$
$$Y(X_0,a,w)=X_0\Upsilon_2\left(wX_0,a,X_0^{-1}\right).$$

The argument for the other one is analogous.
Denote

$$M = \max \left\{ \sup_{\substack{v \in S\left(0, 2\left(\alpha + \frac{\theta_2}{p}\right), \delta\right) \\ |a| < r}} \left| \Upsilon_2 \left(v, a, X_0^{-1}\right) \right|, \sup_{|x|, |y|, |a - a_0|, |\varepsilon|} \left| H\left(x, y, a, \varepsilon\right) \right| \right\}$$

and set $\delta = \frac{r}{8|X_0|M}$.
It suffices, by holomorphic dependence of solutions on parameters, to show that for fixed values of $(a, w) \in B(a_0, r) \times S\left(-\arg(X_0), 2\left(\alpha + \frac{\theta_2}{p}\right), \delta\right)$ the solution to the initial value problem satisfies $|wY(X)| \leqslant \frac{r}{2}$ for all $[X_0, 0]$. Suppose by contradiction that there exists an $X_*$ for which $|wY(X_*)| = \frac{r}{2}$ and $|wY(X)| < \frac{r}{2}$ for all $[X_0, X_*]$. Since the solution satisfies

$$\begin{aligned} Y\left(X, a, w\right) = & X_0 \Upsilon_2 \left(w X_0, a, X_0^{-1}\right) e^{X^p - X_0^p} \\ & + \int_{X_0}^{X} H\left(ws, wY\left(s, a, w\right), a, w^p\right) e^{X^p - s^p} \mathrm{d}s \end{aligned}$$

this would imply that, if we denote $X_* = cX_0$ with $c \in ]0, 1[$,

$$\left| Y\left(X_*, a, w\right) \right| \leqslant \left| X_0 \right| M e^{(c^p - 1)\operatorname{Re}\left(X_0^p\right)} + M \left| X_0 \right| \int_0^c e^{(1-t)(c^p - 1)\operatorname{Re}\left(X_0^p\right)} \mathrm{d}t.$$

Noticing that $\operatorname{Re}\left(X_0^p\right) > 0$ shows that

$$\left| wY\left(X_*\right) \right| \leqslant \delta 2M \left| X_0 \right| = \frac{r}{4} < \frac{r}{2}$$

which is the contradiction we wanted to achieve. □

**Remark 3.2.8.** *It is clear from the above proof that a $\delta$ associated to a $X_0$ as in the proposition, this same $\delta$ will also allow us to prove the result for any other*

$$\tilde{X}_0 \in S\left(0, \frac{\pi}{p} - \frac{2}{p}\left(\rho + \theta_1 + \theta_2 + \Delta\right)\right) \Big\backslash \overline{B}\left(0, \frac{1}{\sqrt[p]{U}}\right)$$

*provided that* $\left|\tilde{X}_0\right| = |X_0|$.

We now show that the saturations of the invariant manifolds from proposition 3.2.7 above, can be connected to each other at 0, for a good choice of the parameter $a$. For this we will employ the Gevrey implicit function theorem 1.2.15.
Consider some $\tilde{X} > \frac{1}{\sqrt[p]{U}}$.

**Lemma 3.2.9.** *Let $\beta \in ]-\frac{\pi}{2p} + \frac{1}{p}(\rho + \theta_1 + \theta_2 + \Delta), \frac{\pi}{2p} - \frac{1}{p}(\rho + \theta_1 + \theta_2 + \Delta)[$, there exists an analytic function $a_\beta(w)$ defined for $w \in S\left(-\beta, 2\left(\alpha + \frac{\theta_2}{p}\right), \omega_\beta\right)$, for some $\omega_\beta > 0$, with $a_\beta(w) = a_0$ such that $Y_1^\beta(0, a_\beta(w), w) = Y_2^\beta(0, a_\beta(w), w)$. Here $Y_1^\beta$ and $Y_2^\beta$ are the solutions associated to $\tilde{X}e^{i\beta}$ as in proposition 3.2.7.*
*Moreover $a_\beta(w)$ is a Gevrey-$\frac{1}{p}$ function.*

**Proof:** We have

$$\begin{aligned}Y_1^\beta(0,a,w) =& -\tilde{X}e^{i\beta}e^{-(\tilde{X}e^{i\beta})^p}\Upsilon_1\left(-w\tilde{X}e^{i\beta},a,-\left(\tilde{X}e^{i\beta}\right)^{-1}\right)\\ &+\int_{-\tilde{X}e^{i\beta}}^{0} H\left(wz,wY_1^\beta(z,a,w),a,w^p\right)e^{-z^p}\mathrm{d}z\end{aligned}$$

and

$$\begin{aligned}Y_2^\beta(0,a,w) =&\tilde{X}e^{i\beta}e^{-(\tilde{X}e^{i\beta})^p}\Upsilon_2\left(w\tilde{X}e^{i\beta},a,\left(\tilde{X}e^{i\beta}\right)^{-1}\right)\\ &+\int_{\tilde{X}e^{i\beta}}^{0} H\left(wz,wY_2^\beta(z,a,w),a,w^p\right)e^{-z^p}\mathrm{d}z.\end{aligned}$$

Consider the time-$(-\tilde{X}e^{i\beta})$ and time-$(\tilde{X}e^{i\beta})$ mappings associated to the analytic differential equation

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = pX^{p-1}Y + H(wX,wY,a,w^p).$$

The above expressions are the images of $-\tilde{X}e^{i\beta}\Upsilon_1(-w\tilde{X}e^{i\beta},a,-(\tilde{X}e^{i\beta})^{-1})$ resp. $\tilde{X}e^{i\beta}\Upsilon_2(w\tilde{X}e^{i\beta},a,(\tilde{X}e^{i\beta})^{-1})$ under these mappings. Theorem 1.2.13 thus shows that these expressions are Gevrey-$\frac{1}{p}$, uniformly in $a$, for $w \in S(-\beta, 2(\alpha+\frac{\theta_2}{p}),\delta_2)$.

By proposition 3.2.6 we have

$$\begin{aligned}&\lim_{w\to 0} Y_2^\beta(0,a,w) - Y_1^\beta(0,a,w)\\ &= H(0,0,a,0)\left(-\frac{2\tilde{X}e^{i\beta}e^{-(\tilde{X}e^{i\beta})^p}}{p}\int_1^\infty z^{\frac{1}{p}-1}e^{(1-z)(\tilde{X}e^{i\beta})^p}\mathrm{d}z\right.\\ &\qquad\left.+\int_{\tilde{X}e^{i\beta}}^{-\tilde{X}e^{i\beta}} e^{-z^p}\mathrm{d}z\right)\\ &= H(0,0,a,0)\left(-\frac{2\tilde{X}e^{i\beta}}{p}\int_1^\infty z^{\frac{1}{p}-1}e^{-z(\tilde{X}e^{i\beta})^p}\mathrm{d}z + \int_{\tilde{X}e^{i\beta}}^{-\tilde{X}e^{i\beta}} e^{-z^p}\mathrm{d}z\right)\end{aligned}$$

from which it follows that the coefficient of $w^0$ of the formal series associated to the Gevrey-$\frac{1}{p}$ function $Y_2^\beta(0,a,w) - Y_1^\beta(0,a,w)$ is given by the expression above. Using the Gevrey implicit function theorem 1.2.15, we prove the result if we can show that

$$H(0,0,a_0,0)\left(-\frac{2\tilde{X}e^{i\beta}}{p}\int_1^\infty z^{\frac{1}{p}-1}e^{-z(\tilde{X}e^{i\beta})^p}\mathrm{d}z + \int_{\tilde{X}e^{i\beta}}^{-\tilde{X}e^{i\beta}} e^{-z^p}\mathrm{d}z\right) = 0$$

and

$$\frac{\partial H}{\partial a}(0,0,a_0,0)\left(-\frac{2\tilde{X}e^{i\beta}}{p}\int_1^\infty z^{\frac{1}{p}-1}e^{-z(\tilde{X}e^{i\beta})^p}\mathrm{d}z + \int_{\tilde{X}e^{i\beta}}^{-\tilde{X}e^{i\beta}} e^{-z^p}\mathrm{d}z\right) \neq 0.$$

Using our assumption in theorem 3.2.1 that $H(0,0,a_0,0) = 0$ and $\frac{\partial H}{\partial a}(0,0,a_0,0) \neq 0$, it clearly suffices to check that

$$-\frac{2\tilde{X}e^{i\beta}}{p}\int_1^\infty z^{\frac{1}{p}-1}e^{-z(\tilde{X}e^{i\beta})^p}\mathrm{d}z + \int_{\tilde{X}e^{i\beta}}^{-\tilde{X}e^{i\beta}} e^{-z^p}\mathrm{d}z \neq 0.$$

One can calculate that

$$\begin{aligned}
-\frac{2\tilde{X}e^{i\beta}}{p}\int_1^\infty z^{\frac{1}{p}-1}e^{-z\left(\tilde{X}e^{i\beta}\right)^p}\mathrm{d}z + \int_{\tilde{X}e^{i\beta}}^{-\tilde{X}e^{i\beta}} e^{-z^p}\mathrm{d}z \\
&= -\frac{2\tilde{X}e^{i\beta}}{p}\int_0^\infty z^{\frac{1}{p}-1}e^{-z\left(\tilde{X}e^{i\beta}\right)^p}\mathrm{d}z \\
&= -\frac{2}{p}\int_0^{\infty(p\beta)} z^{\frac{1}{p}-1}e^{-z}\mathrm{d}z \\
&= -\frac{2}{p}\int_0^{\infty} z^{\frac{1}{p}-1}e^{-z}\mathrm{d}z \\
&= -\frac{2}{p}\Gamma\left(\frac{1}{p}\right) \neq 0.
\end{aligned}$$

□

**Corollary 3.2.10.** *The functions $a_\beta$ in lemma 3.2.9 are all analytic continuations of each other. Together they form a p-summable function.*

**Proof:** We first prove that the functions are all continuations of each other. Suppose that $\beta_1$ and $\beta_2$ are such that

$$S\left(-\beta_1, 2\left(\alpha+\frac{\theta_2}{p}\right), \omega_{\beta_1}\right) \cap S\left(-\beta_2, 2\left(\alpha+\frac{\theta_2}{p}\right), \omega_{\beta_2}\right) \neq \varnothing,$$

this intersection is then again a sector. By reducing the opening of this sector slightly one can see that $\Upsilon_1(-wX, a, -X^{-1})$ and $\Upsilon_2(wX, a, X^{-1})$ are defined for $w$ in this sector and $X$ in some neighbourhood of $\left\{\tilde{X}e^{i\alpha} \mid \alpha \in [\beta_1, \beta_2]\right\}$. One then sees, using the uniqueness of solutions for analytic initial value problems, that both

$$Y_1^{\beta_1}(0, a_{\beta_1}(w), w) = Y_2^{\beta_1}(0, a_{\beta_1}(w), w)$$

and

$$\begin{aligned}
Y_1^{\beta_1}(0, a_{\beta_2}(w), w) &= Y_1^{\beta_2}(0, a_{\beta_2}(w), w) \\
&= Y_2^{\beta_2}(0, a_{\beta_2}(w), w) = Y_2^{\beta_1}(0, a_{\beta_2}(w), w)
\end{aligned}$$

hold.

Using the uniqueness part in the Gevrey implicit function theorem we get that $a_{\beta_1}$ and $a_{\beta_2}$ are analytic continuations of each other.

To prove the summability property it suffices to show that a finite union of sectors of the form $S\left(-\beta, 2\left(\alpha+\frac{\theta_2}{p}\right), \omega_\beta\right)$ covers a sector with opening larger than $\frac{\pi}{p}$. Since $\beta$ can be any value in $\left]-\frac{\pi}{2p}+\frac{1}{p}(\rho+\theta_1+\theta_2+\Delta), \frac{\pi}{2p}-\frac{1}{p}(\rho+\theta_1+\theta_2+\Delta)\right[$, it is quickly checked that a finite union of sectors can be found to cover all directions in any compact subset of $\left]-\frac{\pi+2(p\alpha-\rho-\theta_1-\Delta)}{2p}, \frac{\pi+2(p\alpha-\rho-\theta_1-\Delta)}{2p}\right[$. In corollary 3.2.4 it

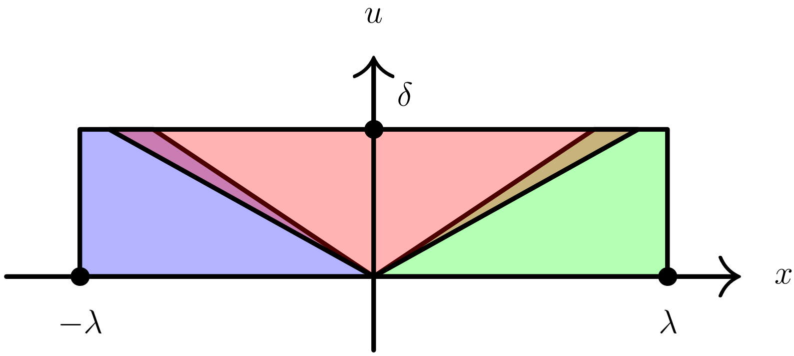


Figure 3.3: The domains of $\Upsilon_1\left(x, a, \frac{u}{x}\right)$ (in blue), $\Upsilon_1\left(x, a, \frac{u}{x}\right)$ (in green) and $Y\left(\frac{x}{u}, a\left(u\right), u\right)$ (in red)

was assumed that $p\alpha - \rho - \theta_1 - \Delta > 0$ and thus the opening of the union can indeed be taken larger than $\frac{\pi}{p}$. $\square$

Combining the previous results allows us to prove theorem 3.2.1.

**Proof:** [Proof of theorem 3.2.1] By the assumptions of theorem 3.2.1 there exist invariant manifolds $y = G_1\left(x, a, \varepsilon\right)$ and $y = G_2\left(x, a, \varepsilon\right)$ around $x = -\lambda$ resp. $x = \lambda$.

In corollary 3.2.4 it is shown that these manifolds can be extended, in the "phase-directional" coordinates, to invariant manifolds $\overline{y} = \Upsilon_1\left(v, a, \overline{u}\right)$ and $\overline{y} = \Upsilon_2\left(v, a, \overline{u}\right)$. One checks that the domain of definition of $\Upsilon_1$ contains the set $[-\lambda, 0[ \times B\left(a_0, r\right) \times \left]-\sqrt[p]{U}, 0\right[$ where both intervals are part of the real line. Rewriting this in the original coordinates shows that $y = G_1\left(x, a, u^p\right)$ can be extended with the function $y = x\Upsilon_1\left(x, a, \frac{u}{x}\right)$ to the domain where $a \in B\left(a_0, r\right)$ and $(x, u)$ satisfy $x \in [-\lambda, 0[$ and $u \in \left]0, -x\sqrt[p]{U}\right[$.

Similarly $y = G_2\left(x, a, u^p\right)$ can be extended by $y = x\Upsilon_2\left(x, a, \frac{u}{x}\right)$ to the domain where $a \in B\left(a_0, r\right)$ and $(x, u)$ satisfy $x \in ]0, \lambda]$ and $u \in \left]0, x\sqrt[p]{U}\right[$.

By proposition 3.2.7, lemma 3.2.9 and corollary 3.2.10 one can choose an $X_0 \in \left]\frac{1}{\sqrt[p]{U}}, +\infty\right[$ such that in the "family rescaling" coordinates the above extension can be further extended by a function $Y\left(X, a\left(w\right), w\right)$ which is defined for $(X, w) \in [-X_0, X_0] \times ]0, \delta[$ for a certain $\delta > 0$. Rewriting in the original coordinates, this extension is of the form $y = uY\left(\frac{x}{u}, a\left(u\right), u\right)$ where $(x, u)$ satisfies $u \in ]0, \delta[$ and $x \in [-uX_0, uX_0]$. $\square$

In figure 3.3 the domains of the extensions are depicted.

To further illustrate the result in theorem 3.2.1 we introduce two notions from [FS03].

**Definition 3.2.11.** *A (local)* <u>*canard*</u> *solution of* (3.2.2) *is a function* $\Phi\left(x, u\right)$ *defined* *and bounded on* $]-d,d[\times]0,u_0]$ *for* $d,u_0>0$ *such that for each fixed value of* $u$, $x\mapsto\Phi(x,u)$ *is an invariant manifold of* (3.2.2).

**Definition 3.2.12.** *Let* $D\subset\mathbb{C}$ *be a simply connected domain containing* 0, *and* $S$ *an open sector. A function* $\Phi(x,u)$ *defined and bounded on* $D\times S$ *is called a (local)* *overstable* *solution if for each fixed value of* $u$, $x\mapsto\Phi(x,u)$ *is an invariant manifold of* (3.2.2).

In theorem 3.2.1 we only achieve the existence of canard solutions. Our results do not give overstable solutions, indeed, examining corollary 3.2.4 the extensions given by $y=x\Upsilon_{1,2}\left(x,a(u),\frac{u}{x}\right)$ are already not defined for $x$ in a complete neighbourhood of 0 but only on (deformed) sectors around part of the negative real axis (for $\Upsilon_1$) or a part of the positive real axis (for $\Upsilon_2$). The other possible form of the extensions is $y=uY\left(\frac{x}{u},a(u),u\right)$ where the domain of definition is described in proposition 3.2.7. This description is rather convoluted. One does see that to remain in the domain, for a fixed $x\in\mathbb{C}\backslash\{0\}$, $\frac{x}{u}$ should be bounded for $u\to0$, which is of course impossible.

The statement regarding the summability in theorem 3.2.1 is limited to the control curve $a(u)$. A natural question to ask is whether the canard curve itself also has summability properties. The answer hereto is negative, as the following example shows. Consider the system

$$\begin{cases}\dot{x} &= u^4\\ \dot{y} &= 4x^3y+u^4(a-x)\end{cases}.$$

Clearly the assumptions of theorem 3.2.1 are satisfied and thus the existence of a control curve $a(u)$ is guaranteed, together with a canard solution $y=G(x,u)$ (for the above system with $a$ replaced by $a(u)$). If $G$ was summable w.r.t. the variable $u$, uniformly in a neighbourhood of the turning point $x=0$ (or even just Gevrey asymptotic), the asymptotic expansions $\widehat{a}=\sum_{n=0}^{\infty}a_nu^n$ and $\widehat{G}=\sum_{n=0}^{\infty}g_n(x)u^n$ associated to resp $a(u)$ and $G(x,u)$ would formally satisfy the equation

$$u^4\frac{\partial\widehat{G}}{\partial x}(x,u)=4x^3\widehat{G}(x,u)+u^4\left(\widehat{a}(u)-x\right).$$

It is then straightforward to calculate that necessarily

$$4x^3g_4(x)=x-a_0,$$

which is impossible without introducing a pole at the origin for $g_4$. This result is consistent with that in [DM07] where a similar study is done but for Gevrey asymptotics on “narrow” regions instead of summability.

Collecting the results of theorems 2.3.1 (i), 3.1.1 and 3.2.1 we arrive at the following conclusion.

**Theorem 3.2.13.** *Consider a real analytic slow-fast family of vector fields*

$$\begin{cases} \dot{x} &= \varepsilon f(x,y,a,\varepsilon) \\ \dot{y} &= g(x,y,a,\varepsilon), \end{cases}$$

*with points $x_a, x_t, x_r \in \mathbb{R}$ such that $x_t$, a turning point, lies in between the two other points, we may assume without loss of generality that $x_a < x_t < x_r$. We furthermore make the following assumptions.*

- *There exists a critical curve given by the graph $y = \varphi_0(x)$ (for $a = a_0$), $x \in [x_a, x_r]$ which is hyperbolically attracting to the left of $x_t$ and repelling to the right of this point i.e.*

$$\frac{\partial g}{\partial y}(x, \varphi_0(x), a_0, 0) < 0, x \in [x_a, x_t[,$$
$$\frac{\partial g}{\partial y}(x, \varphi_0(x), a_0, 0) > 0, x \in ]x_t, x_r],$$
$$\frac{\partial g}{\partial y}(x_t, \varphi_0(x_t), a_0, 0) = 0.$$

- *The points $x_a$ and $x_r$ are slow-fast saddle points with the slow dynamics directed from the attracting to the repelling part of the critical curve, which is characterized by*

$$f(x_*, \varphi_0(x_*), a_0, 0) = 0; x_* = x_a, x_r,$$
$$\left(\frac{\partial g}{\partial x}\cdot\frac{\partial f}{\partial y} - \frac{\partial g}{\partial y}\cdot\frac{\partial f}{\partial x}\right)(x_*, \varphi_0(x_*), a_0, 0) > 0; x_* = x_a, x_r,$$
$$f(x, \varphi_0(x), a_0, 0) > 0; x \in ]x_a, x_r[.$$

- *Locally around the turning point there exists an analytic transformation which transforms the system into the form* (3.2.1)*.*

*Under these assumptions there exists a function $a(u)$, $p$-summable in the real direction such that the system*

$$\begin{cases} \dot{x} &= u^p f(x,y,a(u),u^p) \\ \dot{y} &= g(x,y,a(u),u^p), \end{cases}$$

*has an invariant manifold $y = G(x,u)$ defined for $[x_a, x_r]$ which is $p$-summable in the real direction in $u$, uniformly for $x$ in compact sets of $[x_a, x_r]$ which do not include the turning point $x_t$.*

Let us conclude by remarking that an alternative method of proving this theorem could have used the technique of combined asymptotic developments, developed in [FS13].

# Chapter 4

# Gevrey series in delay equations

In this chapter we consider the following system of singularly perturbed delay differential equations,

$$\begin{cases} \dot{x}(t) &= \varepsilon\left(a-\gamma x(t)\right) \\ \dot{y}(t) &= \left(1+J\right)y(t)-Jy(t-\tau)+x(t)-\frac{y^3(t)}{3} \end{cases}, \tag{4.0.1}$$

with $a \in \mathbb{R}$, $\gamma \in \mathbb{R}_0$, $J, \tau \in \mathbb{R}_0^+$. This model can be encountered in mathematical neuroscience, see [KT16]. This is very much a toy model, allowing us to exhibit the use of Gevrey expansion techniques in the study of delay equations.
We are interested in slow manifolds of system (4.0.1), or equivalently center manifolds of the extended system

$$\begin{cases} \dot{x}(t) &= \varepsilon(t)\left(a-\gamma x(t)\right) \\ \dot{y}(t) &= \left(1+J\right)y(t)-Jy(t-\tau)+x(t)-\frac{y^3(t)}{3} \\ \dot{\varepsilon}(t) &= 0 \end{cases}. \tag{4.0.2}$$

## 4.1 Setting up a slow manifold equation

We will use the characterization of center manifolds found in [HVL93]. To this end we rewrite (4.0.2) into a more appropriate form. Let, for $\alpha > 0$, $\varphi \in \mathcal{C}\left([-\tau,\alpha,],\mathbb{R}^3\right)$, then we define for for each $t \in [0,\alpha]$,

$$\varphi_t\colon\ [-\tau,0] \to \mathbb{R}^3 \colon \theta \to \varphi\left(t+\theta\right).$$

Clearly $\varphi_t \in \mathcal{C}\left([-\tau,0],\mathbb{R}^3\right)$.

**Definition 4.1.1.** *Given a function $F\colon D \subset \mathcal{C}\left([-\tau,0],\mathbb{R}^3\right) \to \mathbb{R}^3$. Denoting by $\dot{X}\left(t\right)$ the right sided derivative, the relation*

$$\dot{X}\left(t\right) = F\left(X_t\right) \tag{4.1.1}$$

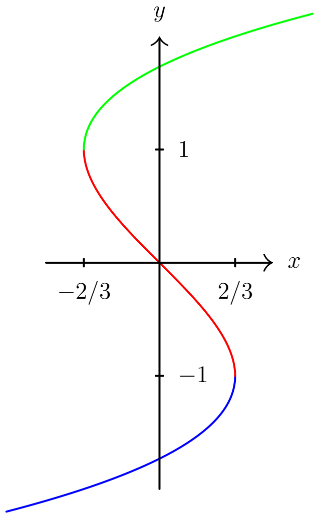


Figure 4.1: Equilibria of system (4.0.2) in the plane $\varepsilon = 0$, divided into the curves $f_-$, $f_0$, $f_+$.

*is called an autonomous retarded functional differential equation. A function $X$ is a solution to* (4.1.1) *on* $[-\tau, \alpha[$ *if* $X \in \mathcal{C}\left([-\tau, \alpha, ], \mathbb{R}^3\right)$, $X_t \in D, \forall t \in [0, \alpha]$ *and* $X(t)$ *satisfies* (4.1.1) *for* $t \in [0, \alpha[$.

Using this definition, (4.0.2) can be rewritten as $(\dot{x}(t), \dot{y}(t), \dot{\varepsilon}(t)) = F(x_t, y_t, \varepsilon_t)$ with the function

$$\begin{aligned} F \colon \mathcal{C}\left([-\tau, 0], \mathbb{R}^3\right) &\to \mathbb{R}^3 \\ : (\varphi_1, \varphi_2, \varphi_3) &\mapsto \begin{pmatrix} \varphi_3(0)\left(a - \gamma\varphi_1(0)\right) \\ (1 + J)\varphi_2(0) - J\varphi_2(-\tau) + \varphi_1(0) - \frac{\varphi_2^3(0)}{3} \\ 0 \end{pmatrix}. \end{aligned}$$

There is clearly a curve of equilibria of of $F$ given by $\{p_b : b \in \mathbb{R}\}$, with

$$p_b := \left(-b + \frac{b^3}{3}, b, 0\right),$$

see figure 4.1. One calculates that $DF(p_b)$ is given by

$$\begin{aligned} DF(p_b) \colon \mathcal{C}\left([-\tau, 0], \mathbb{R}^3\right) &\to \mathbb{R}^3 \\ : (\psi_1, \psi_2, \psi_3) &\mapsto \begin{pmatrix} \psi_3(0)\left(a + \gamma\left(b - \frac{b^3}{3}\right)\right) \\ \left(1 - b^2\right)\psi_2(0) + J\left(\psi_2(0) - \psi_2(-\tau)\right) + \psi_1(0) \\ 0 \end{pmatrix} \end{aligned}$$

(as a norm on $\mathbb{R}^3$ we use the maximum norm and on $\mathcal{C}\left([-\tau,0],\mathbb{R}^3\right)$ the supremum norm).

Similarly as in the ODE case, a value $\lambda \in \mathbb{C}$ is a characteristic root of the linear equation

$$\dot{X}(t) = DF(p_b)\,X_t \tag{4.1.2}$$

if there exists a non-zero vector $V \in \mathbb{R}^3$ such that $Ve^{\lambda t}$ is a solution to (4.1.2). Setting $X(t) = Ve^{\lambda t}$ in (4.1.2) gives us that $\lambda$ must satisfy

$$\left(\lambda\,\mathrm{Id} - \begin{pmatrix} 0 & 0 & a+\gamma\left(b-\frac{b^3}{3}\right) \\ 1 & J\left(1-e^{-\lambda\tau}\right)+\left(1-b^2\right) & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) Ve^{\lambda t} = 0.$$

From this we see that $\lambda$ is a solution to the characteristic equation

$$\lambda^2\left(\lambda - \left(1-b^2\right) - J\left(1-e^{-\lambda\tau}\right)\right) = 0.$$

In the ODE setting ($\tau = 0$), the curve of singular points is normally hyperbolic almost at all points $p_b$ (except for $b = \pm 1$) meaning that almost everywhere $\lambda = 0$ is a root of order 2 and there is one nonzero root. Also in the DDE setting, $p_{\pm 1}$ splits the curve of equilibria in three parts, each of which is a graph where 0 is a root of order 2. Let us denote these graphs by $f_-$, $f_0$, $f_+$ where $f_-(x) < -1 < f_0(x) < 1 < f_+(x)$, see figure 4.1. For all points on $f_0(x)$, $\lambda = 0$ is the only characteristic root on the imaginary axis. On $f_\pm$ there is a possibility for an extra pair of complex conjugated characteristic roots of an equilibrium to lie on the imaginary axis. This, however, can only happen in a finite number of points and it is not necessary for such points to even exist. If, for example, $J\tau \leqslant 1$, extra characteristic roots on the imaginary axis do not appear.

From here on out, we focus on one of the three graphs and denote it for simplicity by $f(x)$, we give the important remark that $f$ is a holomorphic function and thus has an extension to a subset of the complex plane. Choose any $x_0$ in the domain of $f$ for which $\lambda = 0$ is the only root on the imaginary axis. Translating the graph to the $x$ axis and $(x_0, f(x_0))$ to the origin brings system (4.0.2) in the form

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & \varepsilon(t)\left(a-\gamma x_0-\gamma x(t)\right) \\ \dot{y}(t) & = & J\left(f\left(x(t)+x_0\right)-f\left(x(t-\tau)+x_0\right)\right) \\ & & +\left(1+J-f^2\left(x(t)+x_0\right)\right)y(t)-Jy(t-\tau) \\ & & -f\left(x(t)+x_0\right)y^2(t)-\frac{y^3(t)}{3} \\ & & -\varepsilon(t)f'\left(x(t)+x_0\right)\left(a-\gamma x_0-\gamma x(t)\right) \\ \dot{\varepsilon}(t) & = & 0 \end{array}\right. . \tag{4.1.3}$$

One can calculate directly that the solution to the first equation satisfies

$$x(t) = x(0)e^{-\varepsilon\gamma t} + \frac{a-\gamma x_0}{\gamma}\left(1-e^{-\varepsilon\gamma t}\right),$$

from which it can be derived that

$$x(t-\tau) = x(t) + \left(x(t) - \frac{a-\gamma x_0}{\gamma}\right)\left(e^{\varepsilon\gamma\tau} - 1\right).$$

Thinking naively, one could then assume that a solution to the following equation,

$$\begin{aligned}
&\varepsilon(a-\gamma x_0-\gamma x)\frac{\partial Y}{\partial x}(x,\varepsilon)\\
&= J\left(f\left(x+x_0\right) - f\left(x + \left(x - \frac{a-\gamma x_0}{\gamma}\right)\left(e^{\varepsilon\gamma\tau}-1\right) + x_0\right)\right)\\
&\quad + \left(1 + J - f^2\left(x+x_0\right)\right)Y(x,\varepsilon) - JY\left(x + \left(x - \frac{a-\gamma x_0}{\gamma}\right)\left(e^{\varepsilon\gamma\tau}-1\right), \varepsilon\right)\\
&\quad - f\left(x+x_0\right)Y^2(x,\varepsilon) - \frac{Y^3(x,\varepsilon)}{3} - \varepsilon f'\left(x+x_0\right)\left(a-\gamma x_0-\gamma x\right),
\end{aligned} \tag{4.1.4}$$

satisfying $Y(x,0)=0$,would induce a center manifold of system (4.1.3).
We show that this naive intuition is indeed correct.

## 4.2 Characterizing a center manifold

We use the following definition of a center manifold due to [HVL93].

**Definition 4.2.1.** *Given an autonomous retarded functional differential equation*

$$\dot{X}(t) = F\left(X_t\right) \tag{4.2.1}$$

*and suppose $F$ is continuously differentiable. If $0$ is an equilibrium point of $F$, there is a direct sum decomposition*

$$\mathcal{C}\left(\left[-\tau,0\right],\mathbb{R}^3\right) = U \oplus N \oplus S$$

*where $U$ is finite dimensional and corresponds to the span of the generalized eigenvectors of the characteristic roots of $DF(0)$ with positive real part and $N$ is finite dimensional and corresponds to the span of the generalized eigenvectors of the characteristic roots of $DF(0)$ with zero real part.*
*For a neighbourhood $V$ of $0 \in \mathcal{C}\left(\left[-\tau,0\right],\mathbb{R}^3\right)$, a local center manifold $W^c_{\mathrm{loc}}(0)$ is a $\mathcal{C}^1$ submanifold that is a graph over $V \cap N$ in $\mathcal{C}\left(\left[-\tau,0\right],\mathbb{R}^3\right)$, tangent to $N$ at $0$, and locally invariant under the flow of (4.2.1). Said differently*

$$W^c_{\mathrm{loc}}(0) = \left\{\psi \in \mathcal{C}\left(\left[-\tau,0\right],\mathbb{R}^3\right) \mid \psi = \varphi + h(\varphi), \varphi \in N \cap V\right\}$$

*where $h: N \to U \oplus S$ is a $\mathcal{C}^1$ mapping with $h(0)=0$, $Dh(0)=0$. Moreover, every orbit that starts on $W^c_{\mathrm{loc}}(0)$ remains in this set as long as it stays in $V$.*

Denoting the right hand side of (4.1.3) once again by $F(x_t, y_t, \varepsilon_t)$ we find that

$$DF(0,0,0)(\varphi_1,\varphi_2,\varphi_3) = \begin{pmatrix} \varphi_3(0)(a-\gamma x_0) \\ \\ Jf'(x_0)(\varphi_1(0)-\varphi_1(-\tau)) + \left(1-f^2(x_0)\right)\varphi_2(0) \\ +J(\varphi_2(0)-\varphi_2(-\tau)) - \varphi_3(0) f'(x_0)(a-\gamma x_0) \\ \\ 0 \end{pmatrix}.$$

The characteristic equation associated to this linear operator has, with our assumptions on $x_0$, 0 as a characteristic root of order 2 and no other characteristic roots on the imaginary axis.
The generalized eigenspace of the 0 root is two dimensional and given by the null space of $A^2$ with the linear operator

$$A\colon \mathcal{D}(A) \to \mathcal{C}\left([-\tau,0], \mathbb{R}^3\right) : \varphi \mapsto \frac{\mathrm{d}\varphi}{\mathrm{d}\theta}$$

where

$$\mathcal{D}(A) = \left\{\varphi \in \mathcal{C}\left([-\tau,0],\mathbb{R}^3\right) \;\middle|\; \frac{\mathrm{d}\varphi}{\mathrm{d}\theta} \in \mathcal{C}\left([-\tau,0],\mathbb{R}^3\right), \frac{\mathrm{d}\varphi}{\mathrm{d}\theta}(0) = DF(0,0,0)(\varphi)\right\}.$$

**Remark 4.2.2.** *The operator $A$ is the infinitesimal generator of the semigroup of solution operators associated to the equation*

$$\dot{X}(t) = DF(0,0,0)(X_t).$$

*For an elaborate treatment of the theory of these infinitesimal generators, invariant manifold theory in delay equations and more, one can consult the literature, for example [HVL93], [DvGVLW95].*

One can check that the generalized eigenspace, when $a - \gamma x_0 \neq 0$, of the zero characteristic root of the linearisation of system (4.1.3) at $(0,0,0)$ is given by

$$\left\{ \begin{pmatrix} (a-\gamma x_0)(A+B\theta) \\ B(a-\gamma x_0)\frac{f'(x_0)}{1-f^2(x_0)}(1-J\tau) \\ B \end{pmatrix} \;\middle|\; A, B \in \mathbb{R} \right\}.$$

For $a - \gamma x_0 = 0$ the generalized eigenspace is given by

$$\left\{ \begin{pmatrix} A \\ 0 \\ B \end{pmatrix} \;\middle|\; A, B \in \mathbb{R} \right\}.$$

This case does not essentially differ from when $a - \gamma x_0 \neq 0$ and we will thus not detail it any further.

We now show that a solution to (4.1.4) induces a center manifold to (4.1.3). Define $h_1 \colon \mathbb{R}^2 \to \mathcal{C}([-\tau,0],\mathbb{R})$ where $h_1(A,B)(\theta)$ is given by

$$(a-\gamma x_0)\,A\left(e^{-\gamma B\theta}-1\right)-\frac{a-\gamma x_0}{\gamma}\left(e^{-\gamma B\theta}-1+\gamma B\theta\right)$$

and $\widetilde{h}_1 \colon \mathbb{R}^2 \to \mathcal{C}([-\tau,0],\mathbb{R})$ where $\widetilde{h}_1(A,B)(\theta)$ is given by

$$(a-\gamma x_0)\,Ae^{-\gamma B\theta}-\frac{a-\gamma x_0}{\gamma}\left(e^{-\gamma B\theta}-1\right).$$

The function is nothing more than a shorthand notation and is given by

$$\widetilde{h}_1(A,B)(\theta)=(a-\gamma x_0)(A+B\theta)+h_1(A,B)(\theta).$$

Furthermore define $\widetilde{h}_2 \colon G\subset\mathbb{R}^2 \to \mathcal{C}([-\tau,0],\mathbb{R})$ as given by

$$\widetilde{h}_2(A,B)(\theta)=Y\left(\widetilde{h}_1(A,B)(\theta),B\right),$$

where $Y$ is a solution to (4.1.4) and $G$ is a sufficiently small neighbourhood of $(0,0)$. One calculates that

$$\begin{aligned}
\widetilde{h}_1(A,B)'(t)&=B\left(a-\gamma x_0-\gamma\widetilde{h}_1(A,B)(t)\right)\\
\widetilde{h}_1(A,B)(t+\theta)&=\widetilde{h}_1\left(\frac{1}{\gamma}+\left(A-\frac{1}{\gamma}\right)e^{-\gamma Bt},B\right)(\theta)\\
\widetilde{h}_1(A,B)(\theta)+\left(\widetilde{h}_1(A,B)(\theta)-\frac{a-\gamma x_0}{\gamma}\right)\left(e^{\gamma B\tau}-1\right)&=\widetilde{h}_1(A,B)(\theta-\tau).
\end{aligned}$$

This implies that supplementing system (4.1.3) with initial conditions

$$\begin{aligned}
x_0(\theta)&=\widetilde{h}_1(A,B)(\theta)\\
y_0(\theta)&=\widetilde{h}_2(A,B)(\theta)\\
\varepsilon_0(\theta)&=B,
\end{aligned}$$

has solution given by

$$\begin{aligned}
x_t(\theta)&=\widetilde{h}_1\left(\frac{1}{\gamma}+\left(A-\frac{1}{\gamma}\right)e^{-\gamma Bt},B\right)(\theta)\\
y_t(\theta)&=\widetilde{h}_2\left(\frac{1}{\gamma}+\left(A-\frac{1}{\gamma}\right)e^{-\gamma Bt},B\right)(\theta)\\
\varepsilon_t(\theta)&=B.
\end{aligned}$$

Moreover, if we define

$$h_2(A,B)=\widetilde{h}_2(A,B)-B(a-\gamma x_0)\frac{f'(x_0)}{1-f^2(x_0)}(1-J\tau),$$

we have

$$h_1(0,0)=0,h_2(0,0)=0,Dh_1(0,0)=0,Dh_2(0,0)=0.$$

The first two assertions are immediate, we elaborate a bit on the differentials

- For all $\theta \in [-\tau, 0]$,

$$|h_1(A,B)(\theta)| \leqslant \left( |a-\gamma x_0| \gamma\tau |AB| + \frac{|a-\gamma x_0|}{\gamma} (\gamma\tau)^2 |B|^2 \right) e^{\gamma\tau|B|}$$

and thus $h_1(A,B) = \mathrm{O}\big((A,B)^2\big)$, for $(A,B) \to 0$, from which it follows that $Dh_1(0,0) = 0$.

- Using the above we have that

$$\widetilde{h}_2(A,B)(\theta) = \frac{\partial Y}{\partial x}(0,0)(a-\gamma x_0)(A+B\theta) + \frac{\partial Y}{\partial \varepsilon}(0,0)B + \mathrm{O}\big((A,B)^2\big).$$

Since $Y(x,0) = 0$, also $\frac{\partial Y}{\partial x}(0,0) = 0$. By setting $x = 0$ and taking the derivative w.r.t. $\varepsilon$ in (4.1.4), we find that

$$\begin{aligned}
&(a-\gamma x_0)\left(\varepsilon \frac{\partial^2 Y}{\partial x \partial \varepsilon}(0,\varepsilon) + \frac{\partial Y}{\partial x}(0,\varepsilon)\right) \\
&= J\tau(a-\gamma x_0) f'\left(x_0 - \frac{(a-\gamma x_0)}{\gamma}\left(e^{\varepsilon\gamma\tau}-1\right)\right) + \left(1+J-f^2(x_0)\right)\frac{\partial Y}{\partial \varepsilon}(0,\varepsilon) \\
&\quad + J\tau(a-\gamma x_0)\frac{\partial Y}{\partial x}\left(-\frac{(a-\gamma x_0)}{\gamma}\left(e^{\varepsilon\gamma\tau}-1\right), \varepsilon\right) \\
&\quad - J\frac{\partial Y}{\partial \varepsilon}\left(-\frac{(a-\gamma x_0)}{\gamma}\left(e^{\varepsilon\gamma\tau}-1\right), \varepsilon\right) \\
&\quad - \left(2f(x_0)Y(0,\varepsilon) + Y^2(0,\varepsilon)\right)\frac{\partial Y}{\partial \varepsilon}(0,\varepsilon) - f'(x_0)(a-\gamma x_0).
\end{aligned}$$

Setting $\varepsilon = 0$, it is seen that $\frac{\partial Y}{\partial \varepsilon}(0,0) = (1-J\tau)(a-\gamma x_0)\frac{f'(x_0)}{1-f^2(x_0)}$. By the definition of $h_2$ this implies that $h_2(A,B) = \mathrm{O}\big((A,B)^2\big)$, for $(A,B) \to 0$.

We have thus proven.

**Lemma 4.2.3.** *If $Y(x,\varepsilon)$ is a $\mathcal{C}^1$ solution to* (4.1.4) *with $Y(x,0) = 0$, the map*

$$h \colon G \subset \mathbb{R}^2 \to \mathcal{C}\left([-\tau,0], \mathbb{R}^3\right) : (A,B) \mapsto (h_1(A,B), h_2(A,B), B)$$

*is a center manifold of* (4.1.3) *at the equilibrium point* $(0,0,0)$. *Moreover, it inherits the smoothness of $Y$.*

## 4.3 Formal Gevrey analysis of the slow manifold

Now that we know that solutions (4.1.4) give rise to a slow manifold we analyse this equation further. We will prove the following.

**Theorem 4.3.1.** *There exists a unique formal series of the form*

$$\widehat{Y}(x,\varepsilon) = \sum_{n=1}^{\infty} y_n(x)\varepsilon^n,$$

*where all coefficients $y_n$ are holomorphic on a neighbourhood of* 0*, which formally solves equation* (4.1.4)*.*
*Moreover for any open sector $S \subset \mathbb{C}$ of opening less than $\pi$, there exists a function $\widetilde{Y}(x,\varepsilon)$, Gevrey-*1 *asymptotic to $\widehat{Y}$ w.r.t. $\varepsilon$, uniformly for $x$ in a neighbourhood of* 0*, which satisfies equation* (4.1.4) *up to an exponentially small error i.e. there exists $K, L > 0$ such that*

$$\sup_x \Bigg| \varepsilon(a-\gamma x_0-\gamma x)\frac{\partial \widetilde{Y}}{\partial x}(x,\varepsilon) \\ - J\left(f\left(x+x_0\right) - f\left(x+\left(x-\frac{a-\gamma x_0}{\gamma}\right)\left(e^{\varepsilon\gamma\tau}-1\right)+x_0\right)\right) \\ - \left(1+J-f^2\left(x+x_0\right)\right)\widetilde{Y}(x,\varepsilon) + J\widetilde{Y}\left(x+\left(x-\frac{a-\gamma x_0}{\gamma}\right)\left(e^{\varepsilon\gamma\tau}-1\right),\varepsilon\right) \\ + f\left(x+x_0\right)\widetilde{Y}^2(x,\varepsilon) + \frac{\widetilde{Y}^3(x,\varepsilon)}{3} + \varepsilon f'\left(x+x_0\right)\left(a-\gamma x_0-\gamma x\right)\Bigg| \leqslant K e^{-\frac{L}{|\varepsilon|}}.$$

**Remark 4.3.2.** *While our results will be local in nature, they can be easily applied to any compact subset of a normally hyperbolic part of the curve of equilibria.*

### 4.3.1 Formal solution

Since $f$ is a holomorphic function at $x_0$, there exists an $R > 0$ such that $f(x+x_0)$ is holomorphic on $B(0,R)$.

**Proposition 4.3.3.** *There exists a unique formal series solution to* (4.1.4) *of the form $\widehat{Y}(x,\varepsilon) = \sum_{n=1}^{\infty} y_n(x)\varepsilon^n$ with $y_n \in \mathcal{O}\left(B\left(0,R\right)\right)$.*

**Proof:** Plugging the formal series $\widehat{Y}(x,\varepsilon) = \sum_{n=1}^{\infty} y_n(x)\varepsilon^n$ into equation (4.1.4), expanding $f\left(x+\left(x-\frac{a-\gamma x_0}{\gamma}\right)\left(e^{\varepsilon\gamma\tau}-1\right)+x_0\right)$ in its Taylor series around $x+x_0$ and similarly expanding $y_n\left(x+\left(x-\frac{a-\gamma x_0}{\gamma}\right)\left(e^{\varepsilon\gamma\tau}-1\right)\right)$ around $x$ we can arrive at

$$\begin{aligned} &\sum_{n=1}^{\infty}\left(1-f^2(x+x_0)\right)y_n(x)\varepsilon^n \\ &= \varepsilon f'\left(x+x_0\right)\left(a-\gamma x_0-\gamma x\right) + \sum_{k=1}^{\infty}\frac{J\left(x-\frac{a-\gamma x_0}{\gamma}\right)^k}{k!}f^{(k)}(x+x_0)\left(e^{\varepsilon\gamma\tau}-1\right)^k \\ &\quad + \sum_{n=1}^{\infty}(a-\gamma x_0-\gamma x)y_n'(x)\varepsilon^{n+1} + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{J\left(x-\frac{a\gamma x_0}{\gamma}\right)^k}{k!}y_n^{(k)}(x)\left(e^{\varepsilon\gamma\tau}-1\right)^k\varepsilon^n \\ &\quad + f\left(x+x_0\right)\left(\sum_{n=1}^{\infty}y_n(x)\varepsilon^n\right)^2 + \frac{1}{3}\left(\sum_{n=1}^{\infty}y_n(x)\varepsilon^n\right)^3. \end{aligned} \tag{4.3.1}$$

The expansion in $\varepsilon$ powers of $e^{\varepsilon\gamma\tau} - 1$ has no constant term. Thus for $n \geqslant 1$ the coefficient of $\varepsilon^{n+1}$ on the RHS (right hand side) of (4.3.1) only depends on the functions $y_1, \ldots, y_n, f$ and their derivatives. Together with $1 - f^2(x + x_0) \neq 0$, indeed this follows immediately from $f(x) - \frac{f^3(x)}{3} + x = 0$, we can thus recursively determine the coefficients of our formal solution.
Notice that since $f(x + x_0) \in \mathcal{O}(B(0, R))$, the same holds for the coefficients $y_n$. □

### 4.3.2 Gevrey property

We aim to prove that the formal solution found in the previous section is Gevrey-1 w.r.t. $\varepsilon$ uniformly for $x$ in a neighbourhood of $x_0$ i.e. there exist $C, D > 0$ such that

$$\sup_{|x|<S} |y_n(x)| \leqslant CD^n n!$$

for $0 < S < R$.
This is achieved analogously as in section 2.1.2, we will repeat some results, in a slightly different formulation, but only elaborate on results not yet treated in this previous section. For convenience we repeat,

**Definition 4.3.4.** *Let $p \in \mathbb{N}$ and $g \in \mathcal{O}(B(0, R))$, the $p$-th Nagumo norm of $g$ is given by*

$$\|g\|_p := \sup_{|x|<R} (R - |x|)^p |g(x)|.$$

Nagumo norms have the following properties.

- $\|g_1 + g_2\|_p \leqslant \|g_1\|_p + \|g_2\|_p$.
- $\|hg_2\|_p \leqslant \sup_{|x|<R} |h(x)| \|g_2\|_q$ if $h$ is a bounded function on $B(0, R)$.
- $\|g_1 g_2\|_{p+q} \leqslant \|g_1\|_p \|g_2\|_q$.
- $\|g'\|_{p+1} \leqslant e(p + 1) \|g\|_p$.

**Definition 4.3.5.** *For formal series $\widehat{g}(x, \varepsilon) = \sum_{n=0}^{\infty} g_n(x)\varepsilon^n$ and $\widehat{h}(z) = \sum_{n=0}^{\infty} h_n z^n$ we say that $\widehat{g}$ is majorized by $\widehat{h}$, denoted $\widehat{g} \ll \widehat{h}$, if and only if*

$$\|g_n\|_n \leqslant n! h_n \text{ for all } n \in \mathbb{N}.$$

The following relations hold.

**Proposition 4.3.6.** *If $\widehat{g} \ll \widehat{h}$ then*

$$\sum_{m=1}^{\infty} g'_{m-1}(x)\varepsilon^m = \varepsilon \frac{\mathrm{d}}{\mathrm{d}x}\widehat{g} \ll ez\widehat{h},$$

$$\sum_{m=k}^{\infty} g^{(k)}_{m-k}(x)\varepsilon^m = \varepsilon^k \frac{\mathrm{d}^k}{\mathrm{d}x^k}\widehat{g} \ll e^k z^k \widehat{h}, \text{ for all } k \geqslant 2,$$

*and if* $\widehat{g}_1 \ll \widehat{h}_1$, $\widehat{g}_2 \ll \widehat{h}_2$ *then*

$$\begin{aligned}\widehat{g}_1 + \widehat{g}_2 &\ll \widehat{h}_1 + \widehat{h}_2,\\ \widehat{g}_1\widehat{g}_2 &\ll \widehat{h}_1\widehat{h}_2.\end{aligned}$$

We will need one more relation which has not yet been treated in section 2.1.2.

**Proposition 4.3.7.** *Let* $\widehat{g} \ll \widehat{h}$ *and denote*

$$\widehat{E}(z) = \sup_{|x|<R}\left|x - \frac{a-\gamma x_0}{\gamma}\right| \sum_{l=0}^{\infty} \frac{|\gamma\tau|^{l+1} R^l}{(l+1)!l!} z^l,$$

*then*

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=1}^{\infty} \frac{\left(x - \frac{a-\gamma x_0}{\gamma}\right)^k}{k!} g_n^{(k)}(x)\left(e^{\varepsilon\gamma\tau}-1\right)^k \varepsilon^n\\ &\ll \sum_{k=1}^{\infty} \frac{\widehat{E}^k(z)e^k z^k}{k!}\widehat{h}(z) = \left(e^{\widehat{E}(z)ez}-1\right)\widehat{h}(z).\end{aligned}$$

**Proof:** We start off by rearranging the summation

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=1}^{\infty} \frac{\left(x - \frac{a-\gamma x_0}{\gamma}\right)^k}{k!} g_n^{(k)}(x)\left(e^{\varepsilon\gamma\tau}-1\right)^k \varepsilon^n\\ &= \sum_{k=1}^{\infty} \frac{\left(x - \frac{a-\gamma x_0}{\gamma}\right)^k}{k!}\left(\frac{e^{\varepsilon\gamma\tau}-1}{\varepsilon}\right)^k \sum_{n=0}^{\infty} g_n^{(k)}(x)\varepsilon^{n+k}\\ &= \sum_{k=1}^{\infty} \frac{\left(x - \frac{a-\gamma x_0}{\gamma}\right)^k}{k!}\left(\frac{e^{\varepsilon\gamma\tau}-1}{\varepsilon}\right)^k \sum_{m=k}^{\infty} g_{m-k}^{(k)}(x)\varepsilon^{m}.\end{aligned}$$

Let $p \in \mathbb{N}$, the coefficient belonging to $\varepsilon^p$ for the last series above is equal to the coefficient of $\varepsilon^p$ in the series

$$\sum_{k=1}^{p} \frac{\left(x - \frac{a-\gamma x_0}{\gamma}\right)^k}{k!}\left(\frac{e^{\varepsilon\gamma\tau}-1}{\varepsilon}\right)^k \sum_{m=k}^{\infty} g_{m-k}^{(k)}(x)\varepsilon^{m}.$$

Using the relations for $\ll$, that $\left(x - \frac{a-\gamma x_0}{\gamma}\right)\left(\frac{e^{\varepsilon\gamma\tau}-1}{\varepsilon}\right) \ll \widehat{E}(z)$ , and that the above series is a finite sum of formal series it holds that it is majorized by

$$\sum_{k=1}^{p} \frac{\widehat{E}^k(z)e^k z^k}{k!}\widehat{h}(z).$$

The coefficient associated to $z^p$ in the above series coincides with the coefficient in the series

$$\sum_{k=1}^{\infty} \frac{\widehat{E}^k(z)e^k z^k}{k!}\widehat{h}(z)$$

which proves the result. $\square$

We are now equipped to show that the formal series solution from proposition 4.3.3 is a Gevrey-1 series. Rewrite equation (4.3.1) as

$$\begin{aligned}
&\sum_{n=1}^{\infty} y_n(x)\varepsilon^n \\
&= \frac{1}{1-f^2(x+x_0)}\left((a-\gamma x_0-\gamma x)f'(x+x_0)\varepsilon\right. \\
&+ \sum_{k=1}^{\infty} \frac{J\left(x-\frac{a-\gamma x_0}{\gamma}\right)^k}{k!} f^{(k)}(x+x_0)\left(e^{\varepsilon\gamma\tau}-1\right)^k \\
&+ \sum_{n=1}^{\infty}(a-\gamma x_0-\gamma x)y_n'(x)\varepsilon^{n+1} + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{J\left(x-\frac{a-\gamma x_0}{\gamma}\right)^k}{k!} y_n^{(k)}(x)\left(e^{\varepsilon\gamma\tau}-1\right)^k\varepsilon^n \\
&+ \left. f(x+x_0)\left(\sum_{n=1}^{\infty} y_n(x)\varepsilon^n\right)^2 + \frac{1}{3}\left(\sum_{n=1}^{\infty} y_n(x)\varepsilon^n\right)^3\right).
\end{aligned} \tag{4.3.2}$$

Denote

$$\begin{aligned}
\mathcal{F}_0 &= \sup_{|x|<R} |f(x+x_0)|, \\
M_1 &= \sup_{|x|<R} \left|\frac{1}{1-f^2(x+x_0)}\right|, \\
M_2 &= \sup_{|x|<R} \left|x-\frac{a-\gamma x_0}{\gamma}\right|,
\end{aligned}$$

we may assume that all the values are finite, by if necessary decreasing $R$ slightly.
We call

$$\begin{aligned}
v(z) =& M_1\left(M_2 e\mathcal{F}_0 z + \left(e^{\widehat{E}(z)ez}-1\right)\mathcal{F}_0\right. \\
&+ M_2 e v(z) z + |J|\left(e^{\widehat{E}(z)ez}-1\right)v(z) \\
&+ \left.\mathcal{F}_0 v^2(z) + \frac{1}{3}v^3(z)\right)
\end{aligned} \tag{4.3.3}$$

the majorant equation.

This is a fitting name, after all, given $\widehat{g} \ll \widehat{h}$ then plugging $\widehat{g}$ into the RHS of (4.3.2) and $\widehat{h}$ into the RHS of (4.3.3) yields two new formal series say $\widetilde{g}$ and $\widetilde{h}$ with $\widetilde{g} \ll \widetilde{h}$, which follows directly from the relations on $\ll$.

In a manner that is very similar to the proof of proposition 4.3.3, equation (4.3.3) has a unique formal solution of the form $\widehat{V}(z) = \sum_{n=1}^{\infty} v_n z^n$, notice that the constant term is zero. However, (4.3.3) is actually a holomorphic equation in the variables $v$ and

$z$ and as such we can apply the holomorphic implicit function theorem to find that there exists (around $z = 0$) a unique holomorphic solution, $V(z)$, of (4.3.3), satisfying $V(0) = 0$. The Taylor series of this holomorphic solution necessarily coincides with $\widehat{V}$ which implies that the coefficients, $v_n$, of $\widehat{V}$ are bounded by $CD^n$ for certain $C, D > 0$. We can employ the fact that $\widehat{V}$ converges to show that $\widehat{Y}$ is a Gevrey-1 formal series. Put $\widehat{Y}_0 = 0$ and $\widehat{V}_0 = 0$, clearly $\widehat{Y}_0 \ll \widehat{h}_0$ we can recursively define formal series

$$\begin{aligned}\widehat{Y}_{n+1} &= \text{RHS}(4.3.2)\left(\widehat{Y}_n\right)\\ \widehat{V}_{n+1} &= \text{RHS}(2.1.5)\left(\widehat{V}_n\right)\end{aligned}$$

with $\widehat{Y}_n \ll \widehat{V}_n$ and $\widehat{Y}_n$ has all coefficient up to $\varepsilon^n$ in common with our formal solution $\widehat{Y}$ found in proposition 4.3.3. Furthermore the sequence $\widehat{V}_n$ converges to the unique formal solution of (4.3.3), $\widehat{V}$, and thus $\widehat{Y} \ll \widehat{V}$.
Finally, we thus have $\|y_n\|_n \leqslant v_n n! \leqslant CD^n n!$ which implies for all $|x| \leqslant R$ that $|y_n(x)| \leqslant CD^n(R - |x|)^{-n} n!$. Consequently for $0 < T < R$

$$\sup_{|x| \leqslant T} |y_n(x)| \leqslant C\left(\frac{D}{R-T}\right)^n n!.$$

To surmise.

**Lemma 4.3.8.** *Given the unique formal solution of the form*

$$\widehat{Y}(x,\varepsilon) = \sum_{n=1}^{\infty} y_n(x)\varepsilon$$

*to equation* (4.1.4)*, with $y_n \in \mathcal{O}(B(0,R))$. For $0 < T < R$, $\widehat{Y}(x,\varepsilon)$ is a Gevrey-*1 *series in $\varepsilon$, uniformly for $x \in \overline{B}(0,T)$. More specifically there exist $C_1, D_1 > 0$ such that*

$$\sup_{|x| \leqslant T} |y_n(x)| \leqslant C_1 D_1^n n!.$$

## 4.4 Constructing quasi-solutions

Given any sector $S$ of opening less than $\pi$, we can apply the Borel-Ritt-Gevrey theorem 1.2.10 to the formal series solution $\widehat{Y}$ and find a function $\widetilde{Y}$, Gevrey-1 asymptotic to it. We prove theorem 4.3.1 if we can show that

$$\begin{aligned}&\varepsilon(a - \gamma x_0 - \gamma x)\frac{\partial \widetilde{Y}}{\partial x}(x,\varepsilon)\\ &\quad - J\left(f(x + x_0) - f\left(x + \left(x - \frac{a - \gamma x_0}{\gamma}\right)\left(e^{\varepsilon\gamma\tau} - 1\right) + x_0\right)\right)\\ &\quad - \left(1 + J - f^2(x + x_0)\right)\widetilde{Y}(x,\varepsilon) + J\widetilde{Y}\left(x + \left(x - \frac{a - \gamma x_0}{\gamma}\right)\left(e^{\varepsilon\gamma\tau} - 1\right), \varepsilon\right)\\ &\quad + f(x + x_0)\widetilde{Y}^2(x,\varepsilon) + \frac{\widetilde{Y}^3(x,\varepsilon)}{3} + \varepsilon f'(x + x_0)(a - \gamma x_0 - \gamma x)\end{aligned}$$

is Gevrey-1 asymptotic to the zero series. For this it suffices, by lemma 1.2.13, to prove that $\widetilde{Y}\left(x+\left(x-\frac{a-\gamma x_0}{\gamma}\right)\left(e^{\varepsilon\gamma\tau}-1\right),\varepsilon\right) \sim_1 \widehat{Y}\left(x+\left(x-\frac{a-\gamma x_0}{\gamma}\right)\left(e^{\varepsilon\gamma\tau}-1\right),\varepsilon\right)$. For this, let $T$ be as in lemma 4.3.8, choose any $T_1 < T$ and let $s > 0$ be sufficiently small such that $\left(T_1+\frac{|a-\gamma x_0|}{\gamma}\right)\left(e^{s\gamma\tau}-1\right) \leqslant \frac{T-T_1}{2}$. By the Borel-Ritt-Gevrey theorem there exist functions $\widetilde{Y}_1,\ldots,\widetilde{Y}_m$ and a good sectorial covering $S_1,\ldots,S_m$ of $B(0,s)\setminus\{0\}$ such that $\widetilde{Y}_j$ is defined on $\overline{B}(0,T)\times S_j$ and $\widetilde{Y}_j \sim_1 \widehat{Y}$. We may assume that $\widetilde{Y}=\widetilde{Y}_1$ and $S=S_1$. This implies that there exist $K,L>0$ such that

$$\sup_{|x|\leqslant T}\left|\widetilde{Y}_i(x,\varepsilon)-\widetilde{Y}_j(x,\varepsilon)\right| \leqslant Ke^{-\frac{L}{|\varepsilon|}} \tag{4.4.1}$$

for $\varepsilon \in S_i \cap S_j \neq \varnothing$.

Due to our choices of $T_1$ and $s$ the functions $\widetilde{Y}_j\left(x+\left(x-\frac{a-\gamma x_0}{\gamma}\right)\left(e^{\varepsilon\gamma\tau}-1\right),\varepsilon\right)$ are defined for $(x,\varepsilon)\in\overline{B}(0,T_1)\times S_j$. We have that

$$\begin{aligned}&\widetilde{Y}_j\left(x+\left(x-\frac{a-\gamma x_0}{\gamma}\right)\left(e^{\varepsilon\gamma\tau}-1\right),\varepsilon\right)\\&=\sum_{k=0}^{\infty}\frac{1}{k!}\frac{\partial^k\widetilde{Y}_j}{\partial x^k}(x,\varepsilon)\left(x-\frac{a-\gamma x_0}{\gamma}\right)^k\left(e^{\varepsilon\gamma\tau}-1\right)^k\end{aligned}$$

and thus

$$\widetilde{Y}_i\left(x+\left(x-\frac{a-\gamma x_0}{\gamma}\right)\left(e^{\varepsilon\gamma\tau}-1\right),\varepsilon\right)-\widetilde{Y}_j\left(x+\left(x-\frac{a-\gamma x_0}{\gamma}\right)\left(e^{\varepsilon\gamma\tau}-1\right),\varepsilon\right)$$

is given by

$$\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\partial^k\widetilde{Y}_i}{\partial x^k}(x,\varepsilon)-\frac{\partial^k\widetilde{Y}_j}{\partial x^k}(x,\varepsilon)\right)\left(x-\frac{a-\gamma x_0}{\gamma}\right)^k\left(e^{\varepsilon\gamma\tau}-1\right)^k.$$

By the Cauchy inequalities and (4.4.1),

$$\sup_{|x|\leqslant T_1}\left|\frac{\partial^k\widetilde{Y}_i}{\partial x^k}(x,\varepsilon)-\frac{\partial^k\widetilde{Y}_j}{\partial x^k}(x,\varepsilon)\right| \leqslant k!\left(\frac{1}{T-T_1}\right)^k Ke^{-\frac{L}{|\varepsilon|}},$$

implying that for all $|x|\leqslant T_1$ and $\varepsilon\in S_i\cap S_j$,

$$\left|\widetilde{Y}_i\left(x+\left(x-\frac{a-\gamma x_0}{\gamma}\right)\left(e^{\varepsilon\gamma\tau}-1\right),\varepsilon\right)-\widetilde{Y}_j\left(x+\left(x-\frac{a-\gamma x_0}{\gamma}\right)\left(e^{\varepsilon\gamma\tau}-1\right),\varepsilon\right)\right|$$

is bounded by

$$Ke^{-\frac{L}{|\varepsilon|}}\sum_{k=0}^{\infty}\frac{1}{2^k}=2Ke^{-\frac{L}{|\varepsilon|}}.$$

The Ramis-Sibuya theorem 1.2.12 then guarantees that, in particular,

$$\widetilde{Y}\left(x+\left(x-\frac{a-\gamma x_0}{\gamma}\right)\left(e^{\varepsilon\gamma\tau}-1\right),\varepsilon\right)$$

is Gevrey-1 asymptotic to some formal series. The proof that this series is given by $\widehat{Y}\left(x+\left(x-\frac{a-\gamma x_0}{\gamma}\right)\left(e^{\varepsilon\gamma\tau}-1\right),\varepsilon\right)$ is analogous to the second part of the proof of lemma 1.2.13 and we will not detail it further.

**Remark 4.4.1.** *Now that we have a quasi-solution, a logical next step would be to construct an actual solution to the slow manifold equation from the quasi-solution, similarly as was done in section 2.2. This however seems to be quite delicate since smoothness issues w.r.t. the singular parameter $\varepsilon$ arise, similar as what is encountered in [HT97]. A future topic of research could be to adapt, if possible, the techniques in [HT97] for the construction of smooth slow manifolds.*

# Overview

This thesis focuses on Gevrey asymptotic properties of slow manifolds in slow-fast dynamical systems.

A large part, specifically chapter 2, is dedicated to the existence of formal slow manifolds and the requirements for these manifolds to induce actual slow manifolds.

The formal study is carried out in section 2.1. Here it is shown that general holomorphic slow-fast systems,

$$\begin{cases} \dot{X} &= \varepsilon F(X, Z, \varepsilon) \\ \dot{Z} &= G(X, Z, \varepsilon) \end{cases},$$

under the mild condition of slow-fast regularity at an equilibrium of the fast subsystem, meaning $G(X_0, Z_0, 0) = 0$ and $\det D_Z(X_0, Z_0, 0) \neq 0$, have a unique formal solution to the associated slow manifold equation

$$\varepsilon D_X Z \cdot F(X, Z, \varepsilon) = G(X, Z, \varepsilon).$$

Moreover this solution is a formal Gevrey-1 series, see proposition 2.1.1 and proposition 2.1.2. In the case of one slow variable ($X \in \mathbb{C}$) this result is already well known, see for example [Sib90]. The technique we use, i.e. a majorant method employing the Nagumo norms, is essentially identical to that in [CDRSS00], however they only treat the case of one slow variable.

The next question is whether there also exists, besides a formal slow manifold, an actual slow manifold of the slow-fast system. Moreover, if such a slow manifold exists, what are its asymptotic properties w.r.t. the formal manifold? Our answer depends on the regularity of the slow flow.

The case of a regular point of the slow flow, $F(X_0, Z_0, 0) \neq 0$, is consider in section 2.2. We show there, imposing no other conditions on the slow-fast systems besides the already assumed slow-fast regularity, the existence (locally around the considered regular point of the slow flow) of slow manifolds which are Gevrey-1 asymptotic to the formal slow manifold. These manifolds can be defined for any narrow sector, of

opening less than $\pi$, with direction to be chosen freely, see lemma 2.2.1. Once again, this result is already known for one slow variable, see for example [Sib58].
Our approach in generalizing this result to systems with an arbitrary amount of slow variables is to realize the formal manifold as a function by the Borel-Ritt-Gevrey theorem, and search for an actual slow manifold as having an exponentially small (w.r.t. $\varepsilon \to 0$) difference with this realization. This reduces the existence of a slow manifold to finding a solution to a PDE, (2.2.3). The idea of this approach coincides with that of [CDRSS00], where the case of one slow variable is treated. A crucial difference however is that for one slow variable, one does not need to solve a PDE but an ODE. The ODE can be solved, for example, by employing the Gronwall lemma. This method can not be used for the PDE, moreover due to the presence of the singular parameter the classical result on existence of PDE solutions, the Cauchy-Kowalevski theorem (see for example [Fol95]), is not directly applicable. We construct a solution be once again a majorant method.
The results in this section can be seen as a generalization and improvement on the classical results of Fenichel on normally hyperbolic slow manifolds, see [Fen79]. The generalization is in the sense that our condition of slow-fast regularity is weaker than that of normal hyperbolicity. The improvements are with respect to the smoothness (w.r.t. $\varepsilon$) of the manifolds, in the theory of Fenichel only finite smoothness up to any degree can be guaranteed. Our Gevrey manifolds are in particular $\mathcal{C}^{\infty}$. We do have to remark that the results of Fenichel can be global in nature while our results are only local.
A possible area where our results could find application is in the study of elliptic slow manifolds ([Van08, LZ11]). As an example, in [Wir04] a system of equations

$$\begin{cases} \dot{X} &= \varepsilon F(X,Z) \\ \dot{Z} &= -LZ + \varepsilon G(X,Z) \end{cases}$$

is considered. Here $L$ is a real, invertible, skew-symmetric (constant) matrix. Since all eigenvalues of this matrix are purely imaginary, Fenichel theory can not be applied but by our results a slow manifold $Z(X,\varepsilon)$ exists, Gevrey-1 asymptotic to the formal slow manifold, $\sum_{n=1}^{\infty} Z_n(X)\varepsilon^n$, of the above system. In [Wir04] it is shown that for a fixed $\varepsilon_0 \in ]0,+\infty[$, there exists an $N_{\varepsilon_0} \in \mathbb{N}$ such that

$$Z_{\varepsilon_0}(X,\varepsilon) = \sum_{n=1}^{N_{\varepsilon_0}} Z_n(X)\varepsilon^n$$

is a slow manifold up to an exponentially small remainder. By this it is meant that the error

$$R_{\varepsilon_0}(X) \coloneqq \varepsilon D_X Z_{\varepsilon_0}(X,\varepsilon) F(X, Z_{\varepsilon_0}(X,\varepsilon)) + LZ_{\varepsilon_0}(X,\varepsilon) - \varepsilon G(X, Z_{\varepsilon_0}(X,\varepsilon))$$

is exponentially decaying of order 1/3 w.r.t. $\varepsilon$ (see proposition 1.2.9 for the definition of exponential decay). Using the actual slow manifold and its Gevrey asymptotic expansion

$$\left\| Z(X,\varepsilon) - \sum_{n=1}^{k-1} Z_n(X)\varepsilon^n \right\| \leqslant AB^k k!\,|\varepsilon|^k$$

this result can easily be replicated. Indeed setting $N_{\varepsilon_0} = \left\lfloor \frac{1}{B\varepsilon_0} \right\rfloor - 1$ one can calculate, as in remark 1.2.6, that

$$\left\| Z(X,\varepsilon) - \sum_{n=1}^{N_{\varepsilon_0}} Z_n(X)\varepsilon^n \right\| \leqslant Ae^2 e^{-\frac{1}{B\varepsilon_0}} (B\varepsilon_0)^{-\frac{1}{2}}\,.$$

It is then rather easy to show that also $R_{\varepsilon_0}(X)$ is exponentially decaying, and even with order 1.

Where section 2.2 concerned regular points of the slow flow, section 2.3 deals with equilibria of the slow flow. In this section we narrowed down the class of slow-fast system under consideration by imposing more stringent conditions. Simply said, apart from slow-fast regularity, the extra conditions are that only one fast variable is present and the eigenvalues of the linearisation of the slow flow lie in the Poincaré domain. We showed in theorem 2.3.1 and lemma 2.3.9 that the formal slow manifold is 1-summable in a certain set of directions. Furthermore, through examples in remark 2.3.2 and example 2.3.3 we show that our assumptions are necessary in the sense that there exist systems, not satisfying the assumptions, whose formal slow manifolds are not 1-summable in any direction.
We would like to note that one particular class of systems (or rather their associated slow manifold equations) for which our results are applicable are given by

$$\varepsilon x \frac{\mathrm{d}z}{\mathrm{d}x} = \varphi(x)\,z + \varepsilon f(x,y,\varepsilon)$$

with $\varphi(0) \neq 0$ and $x, z \in \mathbb{C}$. In [CDMFS07, BMF02] equations of the form

$$\varepsilon x^{r+1} \frac{\mathrm{d}z}{\mathrm{d}x} = \varphi(x)\,z + \varepsilon f(x,y,\varepsilon)$$

with $r > 0$ are studied. The results in these papers amount to the existence of monomially summable formal solutions i.e. summable w.r.t. to the monomial $\varepsilon x^r$. They do not treat the case where $r = 0$.

After concluding the local Gevrey analysis of slow manifolds in slow-fast systems of arbitrary dimension in chapter 2, we turn our attention, in chapter 3, to slow-fast systems with one slow and one fast variable. We are motivated by considering systems where the critical curve connect an attracting to a repelling slow-fast saddle, where along this connection there is a change of stability through a turning point in between the two saddles.

By the results in section 2.3 there exist, locally around the slow-fast saddles, slow manifolds, 1-summable in the positive real direction. The main result in this chapter is that the two manifolds can be saturated towards the turning point and even be connected to each other over it, with the help of an additional parameter.
The first step in the proof is to show that any 1-summable, in the positive real direction, slow manifold can be saturated alongside normally attracting parts of the critical curve, see theorem 3.1.1, while retaining the summability property. The fact that the saturation exists is well known, see for example [Wal91], our contribution lies in the retention of the summability along the saturation.
In the second step we focus ourselves on an environment of the turning point which we assume can be brought into the form

$$\left\{ \begin{array}{lcl} \dot{x} & = & \varepsilon \\ \dot{y} & = & px^{p-1}y + \varepsilon H(x,y,a,\varepsilon) \end{array} \right. . \tag{4.4.2}$$

When two 1-summable slow manifolds are present to the "left" and "right" of the turning point, the existence of a control curve $a\left(\varepsilon^{1/p}\right)$ such that the two manifolds can be extended towards the turning point and match each other there forming a canard solution, see theorem 3.2.1. Moreover the control curve is $p$-summable w.r.t. the variable $\varepsilon^{1/p}$, but the canard solution does not enjoy summability properties at the turning point. The results are consistent with those in [DM07], where a similar study is done but for Gevrey asymptotics on "narrow" regions and our work can be seen as an extension of this.

For the concluding chapter 4 we make a short foray into the world of delay differential equations. Using a model from mathematical neuroscience to experiment on, we show that the center manifold of this system can be characterized by an equation which is formally solved by a Gevrey-1 formal series. Moreover a function that is Gevrey-1 asymptotic to this formal solution satisfies the center manifold equation up to an exponentially small error (in $\varepsilon$), see theorem 4.3.1. Clearly a lot of obvious questions are still left unanswered. For example, the class of systems for which the result is formulated is very restrictive and the result should be able to be extended to a much broader class. A perhaps more difficult problem is the realization of an actual solution from the formal one, and its related smoothness properties. It would seem that a good choice of function spaces to work in plays an important role here.

# Nederlandstalige samenvatting

In deze thesis worden singulier verstoorde problemen, die voorkomen in de studie van snel-trage systemen, bestudeerd. In hun standaardvorm zijn deze systemen gegeven door,

$$\begin{cases} \dot{X}(t) & = & \varepsilon F(X, Z, \varepsilon) \\ \dot{Z}(t) & = & G(X, Z, \varepsilon) \end{cases}$$

De kritieke variëteit van zulk systeem wordt gegeven door (een deel van) de nulpuntsverzameling van de vergelijking $G(X, Z, 0) = 0$.

Een eerste groot deel van de thesis besteedt aandacht aan het blijven bestaan van de kritieke variëteit, als een invariante variëteit, onder kleine verstoringen van de singuliere parameter $\varepsilon$. Meer bepaald onderzoeken we het bestaan en de eigenschappen van een $\varepsilon$-familie van lokaal invariante variëteiten van een snel-traag systeem, dewelke naar de kritieke variëteit streven voor $\varepsilon \to 0$. Zulke familie variëteiten wordt een trage variëteit genoemd.

Het is algemeen bekend dat onder de veronderstelling van het normaal hyperbolisch zijn van de kritieke variëteit, de trage variëteit bestaat. Echter is deze in het algemeen niet uniek en bovendien, zelfs wanneer het snel-trage systeem reëel analytisch is, kan het enkel gegarandeerd worden dat de trage variëteit een gladheid van een willekeurige, maar eindige, graad heeft.

Het doel is om deze klassieke resultaten op verschillende punten te verbeteren. We zullen dit bereiken door het gebruik van de theorie over Gevrey asymptotische functies.

Onze aanpak start vanuit een formeel standpunt. Onder voorwaarde van snel-trage regulariteit, construeren we formele machtreeksen in de de singuliere parameter die, formeel gezien, invariant zijn in het snel-trage systeem. We bekomen verder dat deze reeksen in het algemeen niet convergent zijn maar divergent van Gevrey type.

Afhankelijk van het type punt, van de kritieke variëteit, waarrond we een trage variëteit willen construeren, onderscheiden we twee gevallen. Wanneer het punt geen evenwichtspunt van het trage vectorveld is, tonen we aan, zonder het opleggen van verdere veronderstellingen, dat er een trage variëteit bestaat dewelke een Gevrey asymptotische expansie bezit. Dit is een verbetering ten opzichte van het klassieke resultaat aangezien we enkel snel-trage regulariteit eisen, wat een zwakkere eis is dan normaal hyperbolisch zijn. Bovendien zijn de trage variëteiten die we bekomen in het bijzonder $\mathcal{C}^\infty$ glad.

Wanneer we kijken rond een evenwichtspunt van het trage vectorveld, bekomen we zelfs sterkere resultaten maar hiervoor moeten bijkomende veronderstellingen gemaakt worden. Meer bepaald zullen we veronderstellen dat er slechts één snelle variable is en bovendien moet het evenwichtspunt van het trage vectorveld ofwel aantrekkend ofwel afstotend zijn. Onder deze voorwaarden bekomen we dat de formele oplossing sommeerbaar is in een richting. Dit betekent dat, bovenop alle eigenschappen die een variëteit met Gevrey expansie heeft, de variëteit in zekere zin uniek is.

Een tweede onderwerp in de thesis betreft globale dynamica in snel-trage systemen. We beschouwen een systeem, met één trage en één snelle veranderlijke, dat voldoet aan een specifieke configuratie waarbij die kritieke kromme opgedeeld is in een aantrekkend en afstotend deel en op ieder van die delen zich een snel-traag zadel bevindt. Vanwege onze eerdere, lokale, resultaten bestaan er sommeerbare trage variëteiten rond deze zadels. We tonen aan dat deze verder gezet kunnen worden langsheen de kritieke kromme, met behoud van sommeerbaarheid. Vervolgens wordt nagegaan dat twee sommeerbare variëteiten aaneengesloten kunnen worden, met behulp van een extra parameter, overheen een punt waar de stabiliteit van de kritieke curve verandert. Op deze manier construeren we canard oplossingen.

Als laatste worden "delay differential equations" onderzocht. Dit doen we aan de hand van een specifiek model, dat gebruikt wordt bij het modelleren van neuronen activiteit. Ons hoofddoel is het aantonen dat Gevrey asymptotische technieken ook hun waarde kunnen hebben binnen het meer algemene gebied van functionele differentiaalvergelijkingen. In onze resultaten tonen we het bestaan van quasi-oplossingen aan, die een trage variëteit tot op een exponentieel kleine fout na benaderen. De stap van quasi-oplossing naar echte oplossing wordt niet gemaakt en is een mogelijk toekomstig onderwerp voor onderzoek.

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