Large-deviation theory for a Brownian particle on a ring: a WKB approach

Peer-reviewed author version

PROESMANS, Karel & Derrida, Bernard (2019) Large-deviation theory for a Brownian particle on a ring: a WKB approach. In: JOURNAL OF STATISTICAL MECHANICS-THEORY AND EXPERIMENT, (Art N° 023201).

DOI: 10.1088/1742-5468/aafa7e

Handle: http://hdl.handle.net/1942/28556

Large-deviation theory for a Brownian particle on a ring: a WKB approach

Karel Proesmans

E-mail: Karel.Proesmans@uhasselt.be
Hasselt University, B-3590 Diepenbeek, Belgium.
Collège de France, PSL, 11 place Marcelin Berthelot, F-75231 Paris Cedex 05, France

Bernard Derrida

Collège de France, PSL, 11 place Marcelin Berthelot, F-75231 Paris Cedex 05, France

Abstract. We study the large deviation function of the displacement of a Brownian particle confined on a ring. In the zero noise limit this large deviation function has a cusp at zero velocity given by the Freidlin-Wentzell theory. We develop a WKB approach to analyse how this cusp is rounded in the weak noise limit.

PACS numbers: 05.10.Gg,02.50.Ga

1. Introduction

Large deviations have a long history in the mathematical literature [1, 2, 3]. Over the last decades, they have also become a central part of non-equilibrium statistical mechanics [4, 5, 6], in particular in the context of the fluctuation theorem [7].

One of the simplest models one can consider in the context of large-deviation theory is the Brownian particle dragged through a periodic potential [8, 9, 10, 11, 12, 13]. In the long-time limit, the empirical velocity, v, of the Brownian particle satisfies a large-deviation principle:

$$I(v) = -\lim_{t\to\infty}\frac{1}{t}\ln P_t(x_t = vt), \tag{1}$$

where $P_t(x_t)$ is the probability distribution associated with the displacement x_t after a time t. A general, exact expression for this large-deviation function does not exist, but several approximations have been derived to get to a solution [14, 15, 16, 17, 18]. Furthermore, related studies have been done in the context of e.g. first-passage time distributions [19] and underdamped dynamics [20]. In the low-noise limit, one can tackle the problem using the Freidlin-Wentzell theory [21, 22, 23, 24, 25]. This method is based on the fact that, in the aforementioned limit, one can calculate the large deviation function associated with trajectories. One can subsequently contract this large-deviation function of trajectories to obtain $I(v)$. [26]

Near $v = 0$, something odd happens; a 'kink' appears in the Freidlin-Wentzell large-deviation function [7, 14, 18, 22, 27]. Therefore, to get a precise value of $I(v)$ in this neighbourhood, we need to look at higher order contributions of the noise. To do this, we use a tilted generator method [5]. This method focuses on finding the largest eigenvalue of a Schrödinger-like equation, which is generally hard to solve, but there exist methods known from quantum mechanics, such as diffusive Monte-Carlo methods [28, 29] and Rayleigh-Schrödinger perturbation theory [30], to obtain the lowest eigenvalue. Here, we will solve the equation in the low-but-finite-noise limit using a WKB approach. This approach allows us to understand how the kink of $I(v)$ is rounded in a weak-noise expansion.

We will start in section 2, by introducing the model and discussing some basic concepts of large-deviation theory. In section 3, we will review the Freidlin-Wentzell approach to derive $I(v)$ and discuss its limitations. We reproduce a number of existing results [18, 22], in order to connect them with our results of section 4. In particular, we will see that the large-deviation function of the velocity generally exhibits a cusp at zero velocity. In the main part of this paper (section 4) we use a WKB approach to calculate $I(v)$ or rather its Legendre transform $\mu(\lambda)$ in the case where the force vanishes nowhere i.e., the case where there is no metastable state. This will allow to analyse how the cusp in the large-deviation function is rounded by a small but finite noise. This analytic result is the main contribution of the present work. Finally, we end with conclusions and perspectives in section 5.

2. Model

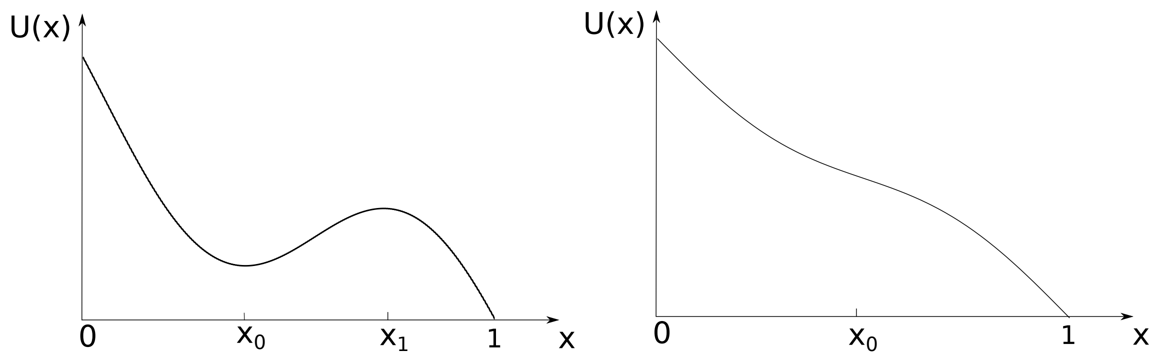

Figure 1. One period of $U(x)$. On the left hand side, a case with a single metastable state. On the right hand side, a case without metastable state.

The focus in this paper will be on a Brownian particle dragged through a periodic potential $V(x)$ (period 1) with a force f. For notational simplicity, we shall assume that $f \geq 0$ throughout this text. The particle 'feels' an effective force equal to

$$F(x) = f - V'(x), \qquad V(x+1) = V(x) \tag{2}$$

and an associated effective potential

$$U(x) = V(x) - fx, \tag{3}$$

cf. Fig. 1. In this section, we shall construct the steady state associated with the position of the particle, and discuss how one can derive the large-deviation function associated with the displacement of the particle. Throughout this paper, we will mainly focus on periodic potentials with no extrema such as the one drawn on the left panel of Fig. 1.

2.1. Steady-state distribution

The position $x(t)$ of the Brownian particle evolves on the infinite line according to an overdamped Langevin equation

$$\dot{x}(t) = -U'(x) + \eta(t), \tag{4}$$

where $\eta(t)$ is a Brownian motion,

$$\langle \eta(t) \rangle = 0, \qquad \langle \eta(t)\eta(t') \rangle = \epsilon\delta(t - t'), \tag{5}$$

and ϵ is a measure for the strength of the noise. Associated with this Langevin equation, one can write a Fokker-Planck equation, describing the time-evolution of the probability distribution, $p_t(x)$ associated with $x(t)$:

$$\frac{d}{dt}p_t(x) = -\frac{d}{dx}\left(F(x)p_t(x)\right) + \frac{\epsilon}{2}\frac{d^2}{dx^2}p_t(x). \tag{6}$$

Although the distribution $p_t(x)$ broadens and spreads out over the whole real axis, the distribution $P_t(x)$, projected on the ring,

$$P_t(x) = \sum_{n\in\mathbb{Z}} p_t(x+n), \tag{7}$$

has a steady-state solution, $p_{\rm ss}(x)$, that satisfies

$$-\frac{d}{dx}\left(F(x)p_{\rm ss}(x)\right) + \frac{\epsilon}{2}\frac{d^2}{dx^2}p_{\rm ss}(x) = 0. \tag{8}$$

Due to the periodicity, one has $p_{\rm ss}(x) = p_{\rm ss}(x+1)$. This boundary condition fixes the solution of Eq. (8) [31]:

$$\begin{aligned} p_{\rm ss}(x) = C \exp\left(-\frac{2}{\epsilon}U(x)\right) \\ \times \left(\int_0^x dy \exp\left(\frac{2}{\epsilon}U(y)\right) + e^{\frac{2f}{\epsilon}} \int_x^1 dy \exp\left(\frac{2}{\epsilon}U(y)\right)\right), \end{aligned} \tag{9}$$

where C is a normalization constant. The average velocity of the particle is given by

$$\langle v \rangle = \frac{C\epsilon}{2}\left(e^{\frac{2f}{\epsilon}} - 1\right). \tag{10}$$

In the weak noise limit (ϵ small), the behaviour of the velocity depends on the strength of the external force and can be separated in two classes:

- If $f < \max V'(x)$, the effective potential $U(x)$ exhibits a local minimum and maximum, at $x = x_0$ and $x = x_1$ respectively (see the left panel of Fig. 1), leading to a meta-stable state for the particle at $x = x_0$. The particle generally spends most of its time in this metastable state and the average velocity is exponentially small [31],
$$\langle v \rangle \simeq \frac{\sqrt{-U''(x_0)U''(x_1)}}{2\pi} e^{\frac{2(U(x_0)-U(x_1))}{\epsilon}} . \tag{11}$$
Analysing the whole x range in Eq. (9), one can also see that $p_{\rm ss}(x)$ is exponentially peaked at $x = x_0$, and exhibits a large-deviation principle in terms of ϵ, with non-analytic points at the values of x where the two terms in Eq. (9) have the same magnitude [32, 23, 27].
- If $f > \max V'(x)$, there are no local minima in the effective potential. Therefore, the probability distribution associated with the position of the particle is much more spread out over the ring and the average velocity of the particle stays finite for arbitrary small noise:
$$\langle v \rangle \simeq \frac{1}{\int_0^1 dy F(y)^{-1}}. \tag{12}$$
As $f > 0$, the second term in Eq. (9) is dominant, leading to
$$p_{\rm ss}(x) \simeq \frac{C'}{F(x)}, \tag{13}$$
where C' again is a normalisation constant.

2.2. Large-deviation theory

In the long-time limit, the measured velocity of the Brownian particle will always converge to the average velocity, Eq. (11)-(12). All other velocities become exponentially unlikely. This behaviour is described by the associated large-deviation function:
$$I(v) = -\lim_{t\to\infty} \frac{1}{t} \ln P_t(x_t = vt), \tag{14}$$
where x_t is the total displacement of the Brownian particle after time t.

In the following it will be more convenient to work with the cumulant-generating function $\mu(\lambda)$ defined by

$$\mu(\lambda) = \lim_{t\to\infty} \frac{1}{t} \ln \left\langle e^{t\lambda v} \right\rangle . \tag{15}$$
From $\mu(\lambda)$, one can uncover all cumulants associated with the displacement, as the n-th derivative of $\mu(\lambda)$ evaluated at $\lambda = 0$ is equal to the n-th cumulant. The convexity of the large-deviation function allows one to extract it via a Legendre transform [5],
$$I(v) = \max_\lambda (\lambda v - \mu(\lambda)) \qquad ; \qquad \mu(\lambda) = \max_v (\lambda v - I(v))). \tag{16}$$
Therefore, one can determine the large-deviation function by first calculating the cumulant-generating function, $\mu(\lambda)$, and then doing a Legendre transform.

The cumulant-generating function can be found as the largest eigenvalue of a 'tilted' Fokker-Planck operator [33, 34], see also Appendix A,

$$\mu(\lambda)r(x) = \lambda F(x)r(x) - \frac{d}{dx}\left(F(x)r(x)\right) + \frac{\epsilon}{2}\left(\lambda^2 r(x) - 2\lambda\frac{d}{dx}r(x) + \frac{d^2}{dx^2}r(x)\right), \tag{17}$$

where $r(x)$ is the associated eigenvector, which satisfies the periodic boundary condition $r(x+1) = r(x)$. This equation can be simplified by introducing

$$s(x) = \exp\left(-\lambda x\right) r(x), \tag{18}$$

leading to

$$\mu(\lambda)s(x) = -\frac{d}{dx}\left(F(x)s(x)\right) + \frac{\epsilon}{2}\frac{d^2}{dx^2}s(x), \tag{19}$$

with boundary condition

$$s(x+1) = e^{-\lambda}s(x). \tag{20}$$

In this way, the eigenvalue equation, Eq. (19), does no longer explicitly depend on λ, which only appears via the boundary condition, Eq. (20). As $\mu(\lambda)$ is the largest eigenvalue of a tilted Fokker-Planck operator, it is also the largest eigenvalue of the adjoint operator,

$$\mu(\lambda)\ell(x) = \lambda F(x)\ell(x) + F(x)\frac{d\ell(x)}{dx} + \frac{\epsilon}{2}\left(\lambda^2\ell(x) + 2\lambda\frac{d\ell(x)}{dx} + \frac{d^2\ell(x)}{dx^2}\right), \tag{21}$$

which can also be simplified by defining $m(x) = \exp\left(\lambda x\right)\ell(x)$:

$$\mu(\lambda)m(x) = F(x)\frac{d}{dx}m(x) + \frac{\epsilon}{2}\frac{d^2}{dx^2}m(x). \tag{22}$$

Interestingly, the left and right eigenvector have a physical interpretation [35, 36, 33, 34]:

$$\ell(x)r(x) = m(x)s(x) \sim P(x|v = \mu'(\lambda)). \tag{23}$$

In words, this means that, up to a normalisation constant, the product of the left and right eigenvector is equal to the probability distribution associated with the position of the particle, conditioned to the average velocity $v = \mu'(\lambda)$.

3. Freidlin-Wentzell theory

One way to try to obtain the large-deviation function, $I(v)$, is via a Freidlin-Wentzell approach [21], where one determines the most likely trajectory leading to the average velocity v. In this section, we shall review this approach, which was earlier applied to the model under study in [22, 25]. Whenever this approach holds, the large-deviation function is given, up to leading order in the noise strength, by

$$I(v) \simeq \lim_{\tau\to\infty}\min_{\{x(t)\}}\frac{1}{2\epsilon\tau}\int_0^\tau dt\Big(\dot{x}(t) - F(x(t))\Big)^2, \tag{24}$$

with the boundary conditions

$$x(0) = 0, \qquad \frac{x(\tau)}{\tau} = v, \tag{25}$$

where one takes the limit $\tau \to \infty$. The optimal path in Eq. (24) can be obtained using Lagrangian techniques:

$$\dot{x}(t)^2 = F(x(t))^2 + K, \tag{26}$$

where K is an integration constant, which can be determined by the boundary condition, Eqs. (25). As K is a constant of motion, the velocity $\dot{x}(t)$ is a function of the position $x(t)$ only. Therefore, the time $\mathcal{T}$ for the particle to travel around the ring once is given by

$$\mathcal{T} = \int_0^1 \frac{dx}{|\dot{x}(t)|}. \tag{27}$$

This implies that the optimal trajectory is periodic, which leads to the following expression of $I(v)$ in a parametric form [22]:

$$I(v) \simeq \frac{v}{\epsilon} \int_0^1 dx \left(\frac{2F(x)^2 + K}{2\sqrt{F(x)^2 + K}} - F(x) \right), \tag{28}$$

with

$$v^{-1} = \mathcal{T} = \int_0^1 \frac{dx}{\sqrt{F(x)^2 + K}}, \tag{29}$$

for $v > 0$ and

$$I(v) \simeq -\frac{v}{\epsilon} \int_0^1 dx \left(\frac{2F(x)^2 + K}{2\sqrt{F(x)^2 + K}} + F(x) \right) \tag{30}$$

$$v^{-1} = -\int_0^1 \frac{dx}{\sqrt{F(x)^2 + K}} \tag{31}$$

for $v < 0$. Using Eq. (16), one can also determine the cumulant generating function:

$$\lambda = I'(v) \simeq \frac{1}{\epsilon} \int_0^1 dx \left(\pm\sqrt{F(x)^2 + K} - F(x) \right), \tag{32}$$

$$\mu(\lambda) = \lambda v - I(v) = \frac{K}{2\epsilon} \tag{33}$$

which gives an implicit equation for $\mu(\lambda)$:

$$\epsilon\lambda = -f \pm \int_0^1 dx \sqrt{2\epsilon\mu(\lambda) + F(x)^2}, \tag{34}$$

where the sign associated with the integral is everywhere equal to the sign of v.

There is a peculiarity about this solution. Clearly for the square roots in the above equations (28-34) to be defined, one needs that $K \geq -\min_x F(x)^2$ so that

$$\mu(\lambda) \geq -\frac{F(x)^2}{2\epsilon} \qquad \text{for all} \;\; x \; . \tag{35}$$

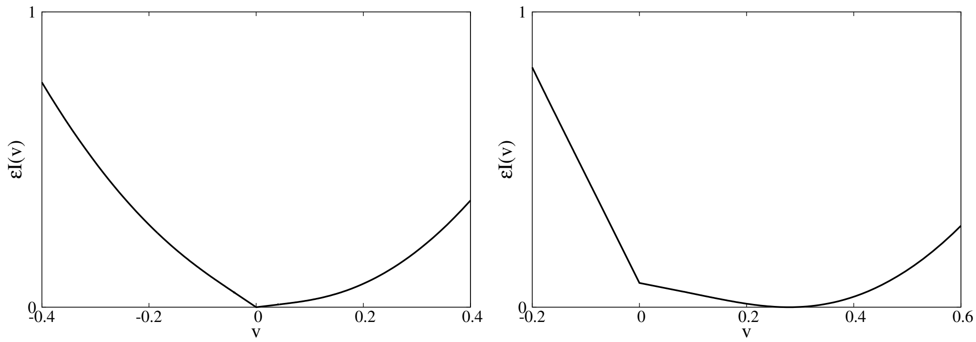

Figure 2. Freidlin-Wentzell large-deviation function with $F(x) = \cos(2\pi x) + f$, with a) $f = 1/2$ and b)$f = 2$. One sees on the left pannel that in the presence of metastable states $I(0) = 0$. In both cases a cusp appears at $v = 0$.

Therefore the above expression (34) is only valid outside the following range for λ

$$-f - \int_0^1 dx \sqrt{F(x)^2 - F(x^*)^2} < \epsilon\lambda < -f + \int_0^1 dx \sqrt{F(x)^2 - F(x^*)^2}, \tag{36}$$

where $F(x^*)^2$ is the minimal value of $F(x)^2$. If $F(x^*) = 0$, i.e., in the presence of metastable states, this unreachable range simplifies to

$$-f - \int_0^1 dx\, |F(x)| < \epsilon\lambda < -f + \int_0^1 dx\, |F(x)| \ . \tag{37}$$

In this range, $\mu(\lambda)$ will be exponentially small, as discussed in [18]. This also manifests itself in the large-deviation function, which has a 'cusp' around $v = 0$, cf. Fig. 2. Indeed, one sees from Eqs. (28-31) that $K \to -F(x^*)^2$ as $v \to 0$ so that

$$I(0^+) = I(0^-) = \frac{F(x^*)^2}{2\epsilon}, \tag{38}$$

and from Eq. (32) [18, 22]

$$\begin{aligned} \epsilon I'(0^-) = &-f - \int_0^1 dx \sqrt{F(x)^2 - F(x^*)^2} \\ \neq &-f + \int_0^1 dx \sqrt{F(x)^2 - F(x^*)^2} = \epsilon I'(0^+). \end{aligned} \tag{39}$$

To explore the range (36) or (37) one needs to study more carefully the limit $\mu \to -\frac{F(x^*)^2}{2\epsilon}$ and this will be done in the next section using a WKB approach.

It is clear from Eq. (26) that $|\dot{x}| = \sqrt{F^2(x) + K}$. As the time spent near position x is proportional to $|\dot{x}|^{-1}$, the probability $P(x|v)$ of finding the particle in x, conditioned on a certain value of the empirical velocity v, is given by

$$P(x|v) \simeq \frac{v}{\sqrt{F^2(x) + K}}, \tag{40}$$

where we used Eqs. (29)-(31) to find the normalisation constant. We will come back to this below (c.f., Eq. (49)). This equation of course reduces to Eq. (13) in the limit $\lambda \to 0$ (i.e., $\mu \to 0$ and $K \to 0$).

Finally, we note (see Eqs. (28,30)) that the large-deviation function satisfies the fluctuation theorem [7, 18, 37]:

$$I(v) = I(-v) - \frac{2v}{\epsilon} \int_0^1 dx\, F(x). \tag{41}$$

4. WKB approach when there is no metastable state

In this section, we obtain $\mu(\lambda)$ by solving the eigenvalue equation, Eq. (19), in the low-but-finite noise limit. To do this, we look for an eigenvector, in a WKB form

$$s(x) \simeq g(x) \exp\left(\frac{h(x)}{\epsilon}\right), \tag{42}$$

where $g(x)$ and $h(x)$ are unknown functions, independent of ϵ.

As we expect from Eq. (33) that $\mu(\lambda) = O(\epsilon^{-1})$ plugging in Eq. (42) into Eq. (19) one gets:

$$\frac{2\mu\epsilon + 2F(x)h'(x) - h'(x)^2}{2\epsilon} + \frac{2F'(x)g(x) - g(x)h''(x) + 2F(x)g'(x) - 2g'(x)h'(x)}{2g(x)} = O(\epsilon). \tag{43}$$

Solving this equation gives us a solution for $s(x)$ (up to zero-th order in ϵ in the prefactors):

$$s(x) = C_+\ s_+(x)\ +\ C_-\ s_-(x) \tag{44}$$

where

$$s_\pm(x) = \sqrt{1 \pm \frac{F(x)}{\sqrt{2\epsilon\mu + F(x)^2}}}\, \exp\left[\frac{1}{\epsilon}\int_0^x dy\, \left(F(y) \pm \sqrt{2\epsilon\mu + F(y)^2}\right)\right] \tag{45}$$

Writing that $s(x)$ and its derivatives satisfy the boundary condition, Eq. (20), implies that one of the two constants C_+ or C_- vanishes and fixes the value of λ

$$\epsilon\lambda = -\left[f \pm \int_0^1 dx \sqrt{2\epsilon\mu(\lambda) + F(x)^2}\right]. \tag{46}$$

One recovers that way the result from the previous section, Eq. (34).

Similarly one can write the solution of Eq. (22) for the left eigenvector in a WKB form

$$m(x) = C'_+\ m_+(x)\ +\ C'_-\ m_-(x) \tag{47}$$

where

$$m_\pm(x) = \sqrt{1 \mp \frac{F(x)}{\sqrt{2\epsilon\mu + F(x)^2}}}\, \exp\left[-\frac{1}{\epsilon}\int_0^x dy\, \left(F(y) \pm \sqrt{2\epsilon\mu + F(y)^2}\right)\right] \tag{48}$$

and again the boundary condition $m(x+1) = e^\lambda m(x)$ forces one of the two constants C'_+ or C'_- to be zero and fixes the value of λ as in Eq. (46). Using Eq. (23) one sees that the probability of finding the particle in x, conditioned on the velocity v is

$$P(x|v = \mu'(\lambda)) \sim l(x)r(x) = m(x)s(x) \sim \frac{C}{\sqrt{2\epsilon\mu(\lambda) + F(x)^2}}. \tag{49}$$

This is exactly what was obtained in Eq. (40) with the Freidlin-Wentzell approach. Note that in contrast to the left and right eigenvector, Eqs. (44)-(47), the probability distribution in Eq. (49) does not have any exponential factor, implying that the distribution is not heavily peaked at a certain value, but is relatively spread out over the entire ring, as was earlier pointed out in [18].

All the above calculations are valid as long as

$$\mu + \frac{F(x^*)^2}{2\epsilon} = O\left(1\right),$$

where $F(x^*)^2 = \min_x F(x)^2$. This can be seen as the prefactors in Eqs. (45) and (48) diverge in the limit $x \to x^*$ and $\epsilon\mu \to -\frac{F(x^*)^2}{2}$.

In order to understand this limit, we consider now the case where $F(x)$ does not vanish on the ring and has a single quadratic minimum F_0 at some position x_0

$$F(x) \simeq F_0 + F_1(x - x_0)^2 + O\Big((x - x_0)^2\Big) \tag{50}$$

and we set

$$\mu = -\frac{F_0^2}{2\epsilon} + \sqrt{2F_0F_1}\left(\nu - \frac{1}{2}\right) \tag{51}$$

where $\nu - \frac{1}{2}$ is of order 1 (or smaller) in the limit $\epsilon \to 0$

In this range of values of μ, to solve the eigenvalue problem Eq. (19), we decompose the ring into three regions

- Region I : $0 < x < x_0$ and $x_0 - x \gg \sqrt{\epsilon}$
- Region II : $x_0 - x = O(\sqrt{\epsilon})$
- Region III : $x_0 < x < 1$ and $x - x_0 \gg \sqrt{\epsilon}$

In regions I and III, one can use solutions analogous to Eqs. (44,45) for the eigenvector $s(x)$ solution of Eq. (19) (see Eq. (B.1)), whereas in region II the solution takes a scaling form

$$s = \exp\left[\frac{F_0(x - x_0)}{\epsilon} + \sqrt{\frac{F_0F_1}{2}}\frac{(x - x_0)^2}{\epsilon}\right] G\left((2F_0F_1)^{1/4}\ \frac{(x - x_0)}{\sqrt{\epsilon}}\right). \tag{52}$$

When this form is injected into Eq. (19), one gets that G should satisfy

$$\nu\, G = \frac{d}{dz}(zG) + \frac{1}{2}\frac{d^2G}{dz^2}\ . \tag{53}$$

Our task then is to choose pairs of constants C_+ and C_- of Eq. (44) in regions I and III and the appropriate solution of the Hermite equation, Eq. (53) for the asymptotics of Eq. (52) in region II to match with those of the solutions in regions I and III in the range $\sqrt{\epsilon} \ll |x - x_0| \ll 1$. This is what we do in Appendix B where we show that

$$\mu + \frac{F_0^2}{2\epsilon} + \frac{\sqrt{2F_0F_1}}{2} \simeq \frac{(2F_0F_1)^{\frac{3}{4}}}{\sqrt{\epsilon\pi}}\left(e^{\lambda - D_4 + D_2} + e^{-\lambda + D_3 - D_1}\right) \tag{54}$$

where the constants D_1, D_2, D_3, D_4 are given by Eqs. (B.3,B.4,B.7,B.8). We see that as λ varies in the range (36), the λ-dependence of $\mu(\lambda)$ is exponentially small, as was pointed out earlier in [18].

At the boundaries $\epsilon\lambda = -\left(f \pm \int_0^1 dx \sqrt{F(x)^2 - 2\epsilon F_0}\right) + O(\epsilon)$, one can connect the two results, Eqs. (46) and (54), via the formulas

$$-\lambda + D_3 - D_1 = \ln\left(\frac{2^\nu\sqrt{\pi}}{\Gamma(\nu)}\left(\frac{2F_0F_1}{\epsilon^2}\right)^{\frac{\nu}{2}-\frac{1}{4}}\right), \tag{55}$$

near $\epsilon\lambda \approx -f - \int_0^1 dx\sqrt{2\epsilon\mu(\lambda) + F(x)^2}$, and

$$\lambda - D_4 + D_2 = \ln\left(\frac{2^\nu\sqrt{\pi}}{\Gamma(\nu)}\left(\frac{2F_0F_1}{\epsilon^2}\right)^{\frac{\nu}{2}-\frac{1}{4}}\right), \tag{56}$$

near $\epsilon\lambda \approx -f + \int_0^1 dx\sqrt{2\epsilon\mu(\lambda) + F(x)^2}$. One can check that these equations are in agreement with Eqs. (46) and (54) in the appropriate limit [25].

Finally, we return to the large-deviation function $I(v)$. As discussed in the previous section, the large-deviation function away from $v \approx 0$, can be described by Eq. (28-31). Near $v = 0$ (in particular for $|v| \ll \epsilon$), one can now use Eq. (54) to determine the large deviation function. This gives,

$$\begin{aligned} I(v) \simeq \frac{F_0^2}{2\epsilon} + \frac{\sqrt{2F_0F_1}}{2} + (D_3 - D_1)\,v - \sqrt{v^2 + \frac{4(2F_0F_1)^{\frac{3}{2}}}{\epsilon\pi} e^{-D_1+D_2+D_3-D_4}} \\ - v\ln\left(\frac{\sqrt{\pi\epsilon}\left(\sqrt{v^2 + \frac{4(2F_0F_1)^{\frac{3}{2}}}{\epsilon\pi}e^{-D_1+D_2+D_3-D_4}} - v\right)}{2\,(2F_0F_1)^{\frac{3}{4}}}\right) \end{aligned} \tag{57}$$

In the limit where $v > 0$ and $\ln v \gg -1/\epsilon$, this becomes

$$\epsilon I(v) \simeq \frac{F_0^2}{2} + v\int_0^1 dx\,\left(-F(x) + \sqrt{F(x)^2 - F_0^2}\right) + O(\epsilon), \tag{58}$$

while $v < 0$ and $\ln(-v) \gg -1/\epsilon$ leads to

$$\epsilon I(v) \simeq \frac{F_0^2}{2} + v\int_0^1 dx\,\left(-F(x) - \sqrt{F(x)^2 - F_0^2}\right) + O(\epsilon). \tag{59}$$

In these two ranges one recovers the low-velocity limit in the Freidlin-Wentzell large-deviation function, Eqs. (38-39). Therefore, one concludes that the results can be smoothly connected to each other.

5. Conclusion

In this paper, we have calculated in the low noise limit the large-deviation and cumulant-generating functions associated with the velocity of a Brownian particle on a ring. In all cases the large deviation function exhibits a cusp at zero velocity in the limit of zero noise (see Figures 2). Our main progress is to calculate the leading order of the cumulant-generating function. Away from the region given by Eq. (36), this corresponds to the well-known Freidlin-Wentzell result [22]. Inside the region of Eq. (36), we have shown that the cumulant-generating function is given by Eq. (54). Furthermore, there exist boundaries between these two regions, where the cumulant-generating function is described by Eqs. (55-56). By doing a Legendre transform, we are able to show that the associated large-deviation function is smooth near $v = 0$, in contrast to the lowest-order Freidlin-Wentzell large-deviation function, Eq. (28).

We limited our analysis to the case of a periodic force with no metastable state, in contrast to most previous works [14, 18, 22, 23], which mainly focused on potentials with metastable states . Our analysis can be extended to those cases. For example in the case of a single metastable state as in the left panel of figure 1, one would need to consider 5 regions: $x < x_0$, x close to x_0, $x_0 < x < x_1$, x close to x_1 and $x_1 < x < 1$ and one would calculate μ by matching the asymptotics very much as we did in section 4 and Appendix B.

Acknowledgment

KP was supported by the Flemish Science Foundation (FWO-Vlaanderen) travel grant V436217N and post-doctoral grant 12J2819N. We also thank Bertrand Eynard for very useful discussions on the WKB method.

Appendix A. The large deviation of the current and the deformed Fokker-Planck equation

In this appendix, we show how to derive Eq. (17). Similar equations have appeared in a number of earlier works (see [5] and the references therein). We briefly explain here the derivation.

After a short time interval Δt one has

$$x(t + \Delta t) = x(t) + F(x(t))\Delta t + B \tag{A.1}$$

where B is a Gausian random variable satisfying

$$\langle B\rangle = 0 \quad ; \quad \langle B^2\rangle = \epsilon\Delta t \tag{A.2}$$

If $p_t(x|x_0)$ is the probability that $x_t = x$, given x_0, one has

$$p_{t+\Delta t}(x|x_0) = \int dx'\, \delta\Big(x - x' - F(x')\Delta t - B\Big)\, P_t(x', Q'|x_0) \tag{A.3}$$

and

$$p_{t+\Delta t}(x|x_0) = \int dx'_0\, \delta\Big(x'_0 - x_0 - F(x_0)\Delta t - B\Big) P_t(x, Q'|x'_0) \tag{A.4}$$

Taking the limit $\Delta t \to 0$ these equations become

$$\frac{dp(x|x_0)}{dt} = -\frac{d[F(x)p(x|x_0)]}{dx} + \frac{\epsilon}{2}\frac{d^2 p(x|x_0)}{dx^2} \ , \tag{A.5}$$

and

$$\frac{dp(x|x_0)}{dt} = F(x_0)\frac{dp(x|x_0)}{dx_0} + \frac{\epsilon}{2} + \frac{d^2 p(x|x_0)}{dx_0^2} \ , \tag{A.6}$$

with the initial condition

$$p_0(x|x_0) = \delta(x - x_0). \tag{A.7}$$

If one introduces the generating function

$$\widetilde{P}_t(x|x_0) = \sum_n e^{\lambda(x-x_0+n)} p(x+n|x_0) \tag{A.8}$$

it satisfies

$$\begin{aligned}\frac{d\widetilde{P}(x|x_0)}{dt} &= \lambda F(x)\,\widetilde{P}(x|x_0) - \frac{d[F(x)\widetilde{P}(x|x_0)]}{dx} \\ &\quad + \frac{\epsilon}{2}\left(\lambda^2 \widetilde{P}(x|x_0) - 2\lambda\,\frac{d\widetilde{P}(x|x_0)}{dx} + \frac{d^2\widetilde{P}(x|x_0)}{dx^2}\right) \ .\end{aligned} \tag{A.9}$$

and

$$\begin{aligned}\frac{d\widetilde{P}(x|x_0)}{dt} &= \lambda F(x_0)\,\widetilde{P}(x|x_0) + F(x_0)\frac{d\widetilde{P}(x|x_0)}{dx_0} \\ &+\frac{\epsilon}{2}\left(\lambda^2\,\widetilde{P}(x|x_0) + 2\lambda\,\frac{d\widetilde{P}(x|x_0)}{dx_0} + \frac{d^2\widetilde{P}(x|x_0)}{dx_0^2}\right) \ .\end{aligned} \tag{A.10}$$

with the initial condition

$$\widetilde{P}_0(x|x_0) = \delta(x - x_0). \tag{A.11}$$

In the long time limit

$$\widetilde{P}_0(x|x_0) \sim e^{\mu(\lambda)\,t}\ r(x)\ \ell(x_0). \tag{A.12}$$

$r(x)$ and $\ell(x)$ are the right and left eigenfunctions solution of the eigenvalue problem

$$\begin{aligned}\mu(\lambda)\,r(x) &= \lambda F(x)\,r(x) - \frac{d[F(x)r(x)]}{dx} \\ &\quad + \frac{\epsilon}{2}\left(\lambda^2 r(x) - 2\lambda\frac{dr(x)}{dx} + \frac{d^2 r(x)}{dx^2}\right) \ ,\end{aligned} \tag{A.13}$$

$$\mu(\lambda)\,\ell(x) = \lambda F(x)\,\ell(x) + F(x)\frac{d\ell(x)}{dx} + \frac{\epsilon}{2}\left(\lambda^2\ell(x) + 2\lambda\frac{d\ell(x)}{dx} + \frac{d^2\ell(x)}{dx^2}\right) \ . \tag{A.14}$$

Appendix B. The matching of the asymptotics

In this appendix, we analyse the situation Eq. (51) and we derive connection formulas between the expressions of the solution $s(x)$ in the various regions.

- In Region I ($0 < x < x_0$) one can write the solution $s(x)$ of Eq. (19) as (see Eqs. (44,45))

$$\begin{aligned} s_I(x) =& c_1 \sqrt{\frac{F(x)}{\sqrt{F(x)^2 - F_0^2}} + 1} \, \exp\left[\int_0^x dy \left(\frac{F(y) + \sqrt{F(y)^2 - F_0^2}}{\epsilon} + \frac{(\nu - \frac{1}{2})\sqrt{2F_0F_1}}{\sqrt{F(y)^2 - F_0^2}}\right)\right] \\ &+ c_2 \sqrt{\frac{F(x)}{\sqrt{F(x)^2 - F_0^2}} - 1} \, \exp\left[\int_0^x dy \left(\frac{F(y) - \sqrt{F(y)^2 - F_0^2}}{\epsilon} - \frac{(\nu - \frac{1}{2})\sqrt{2F_0F_1}}{\sqrt{F(y)^2 - F_0^2}}\right)\right] \end{aligned} \tag{B.1}$$

For $x \to x_0$ in this region I this leads to the following asymptotics and

$$\begin{aligned} s_I(x) \simeq \left(\frac{F_0}{2F_1}\right)^{\frac{1}{4}} \Bigg(& c_1 \, (x_0 - x)^{-\nu} \exp\left[D_1 - \frac{F_0(x_0 - x)}{\epsilon} - \sqrt{\frac{F_0F_1}{2}} \frac{(x_0 - x)^2}{\epsilon}\right] \\ & + c_2 \, (x_0 - x)^{\nu - 1} \exp\left[D_2 - \frac{F_0(x_0 - x)}{\epsilon} + \sqrt{\frac{F_0F_1}{2}} \frac{(x_0 - x)^2}{\epsilon}\right]\Bigg) \end{aligned} \tag{B.2}$$

where

$$D_1 = \left(\nu - \frac{1}{2}\right) \log x_0 + \int_0^{x_0} dy \left(\frac{F(y) + \sqrt{F(y)^2 - F_0^2}}{\epsilon} + \frac{\left(\nu - \frac{1}{2}\right) \sqrt{2F_0F_1}}{\sqrt{F(y)^2 - F_0^2}} - \frac{\nu - \frac{1}{2}}{x_0 - y}\right) \tag{B.3}$$

and

$$D_2 = -\left(\nu - \frac{1}{2}\right) \log x_0 + \int_0^{x_0} dy \left(\frac{F(y) - \sqrt{F(y)^2 - F_0^2}}{\epsilon} - \frac{\left(\nu - \frac{1}{2}\right) \sqrt{2F_0F_1}}{\sqrt{F(y)^2 - F_0^2}} + \frac{\nu - \frac{1}{2}}{x_0 - y}\right). \tag{B.4}$$

- Similarly in Region III ($x_0 < x < 1$)

$$\begin{aligned} s_{III}(x) =& c_3 \sqrt{\frac{F(x)}{\sqrt{F(x)^2 - F_0^2}} + 1} \, \exp\left[-\int_x^1 dy \left(\frac{F(y) + \sqrt{F(y)^2 - F_0^2}}{\epsilon} + \frac{(\nu - \frac{1}{2})\sqrt{2F_0F_1}}{\sqrt{F(y)^2 - F_0^2}}\right)\right] \\ &+ c_4 \sqrt{\frac{F(x)}{\sqrt{F(x)^2 - F_0^2}} - 1} \, \exp\left[-\int_x^1 dy \left(\frac{F(y) - \sqrt{F(y)^2 - F_0^2}}{\epsilon} - \frac{(\nu - \frac{1}{2})\sqrt{2F_0F_1}}{\sqrt{F(y)^2 - F_0^2}}\right)\right] \end{aligned} \tag{B.5}$$

which gives as $x \to x_0$

$$s_{III}(x) \simeq \left(\frac{F_0}{2F_1}\right)^{\frac{1}{4}} \left(c_3\ (x-x_0)^{\nu-1} \exp\left[D_3 + \frac{F_0(x-x_0)}{\epsilon} + \sqrt{\frac{F_0F_1}{2}}\frac{(x_0-x)^2}{\epsilon}\right] \right.$$
$$\left. + c_4\ (x_0-x)^{-\nu} \exp\left[D_4 + \frac{F_0(x-x_0)}{\epsilon} - \sqrt{\frac{F_0F_1}{2}}\frac{(x_0-x)^2}{\epsilon}\right]\right) \tag{B.6}$$

where

$$D_3 = -\left(\nu - \frac{1}{2}\right)\log(1-x_0) - \int_{x_0}^1 dy \left(\frac{F(y)+\sqrt{F(y)^2-F_0^2}}{\epsilon} + \frac{\left(\nu-\frac{1}{2}\right)\sqrt{2F_0F_1}}{\sqrt{F(y)^2-F_0^2}} - \frac{\nu-\frac{1}{2}}{y-x_0}\right) \tag{B.7}$$

and

$$D_4 = \left(\nu - \frac{1}{2}\right)\log(1-x_0) - \int_{x_0}^1 dy \left(\frac{F(y)-\sqrt{F(y)^2-F_0^2}}{\epsilon} - \frac{\left(\nu-\frac{1}{2}\right)\sqrt{2F_0F_1}}{\sqrt{F(y)^2-F_0^2}} + \frac{\nu-\frac{1}{2}}{y-x_0}\right). \tag{B.8}$$

- Finally in Region II ($x - x_0 = O(\sqrt{\epsilon})$) the solution is of the form Eq. (52) with the following asymptotics (see Eqs. (C.2,C.3)):
for $(x-x_0)/\sqrt{\epsilon} \to -\infty$

$$s_{II} \simeq \exp\left[\frac{F_0(x-x_0)}{\epsilon}\right]\left(V \left(\frac{2F_0F_1}{\epsilon^2}\right)^{\frac{\nu-1}{4}} (x_0-x)^{\nu-1} \exp\left[\sqrt{\frac{F_0F_1}{2}}\frac{(x-x_0)^2}{\epsilon}\right]\right.$$
$$\left. + W \left(\frac{2F_0F_1}{\epsilon^2}\right)^{-\frac{\nu}{4}} (x_0-x)^{-\nu} \exp\left[-\sqrt{\frac{F_0F_1}{2}}\frac{(x-x_0)^2}{\epsilon}\right]\right) \tag{B.9}$$

and for $(x-x_0)/\sqrt{\epsilon} \to +\infty$

$$s_{II} \simeq \exp\left[\frac{F_0(x-x_0)}{\epsilon}\right]\left(V' \left(\frac{2F_0F_1}{\epsilon^2}\right)^{\frac{\nu-1}{4}} (x-x_0)^{\nu-1} \exp\left[\sqrt{\frac{F_0F_1}{2}}\frac{(x-x_0)^2}{\epsilon}\right]\right.$$
$$\left. + W'(x-x_0)^{-\nu} \left(\frac{2F_0F_1}{\epsilon^2}\right)^{-\frac{\nu}{4}} \exp\left[-\sqrt{\frac{F_0F_1}{2}}\frac{(x-x_0)^2}{\epsilon}\right]\right). \tag{B.10}$$

Now using the boundary condition Eq. (20) one has

$$c_3 = c_1 e^{-\lambda} \quad ; \quad c_4 = c_2 e^{-\lambda} \tag{B.11}$$

and matching the asymptotics, on the one hand Eqs. (B.2) and (B.9) and on the other hand Eqs. (B.6) and (B.10), one gets using Eq. (C.4) that λ should satisfy

$$e^{2\lambda} - e^{\lambda}\left(\frac{X(\nu)}{\Gamma(\nu)}e^{D_4-D_2} + \frac{\Gamma(\nu)(1-Z^2)}{X(\nu)}e^{D_3-D_1}\right) + e^{D_4+D_3-D_1-D_2} = 0 \tag{B.12}$$

where

$$X(\nu) = 2^{\nu}\sqrt{\pi}\left(\frac{2F_0F_1}{\epsilon^2}\right)^{\frac{\nu}{2}-\frac{1}{4}}. \tag{B.13}$$

For ϵ small, one has $D_4 - D_2 \gg D_3 - D_1$. This, combined with $Z = 1 + O(\nu)$ and $\nu \ll \epsilon^{-1}$ one can see that the term containing Z becomes negligible over the entire range the range (36), i.e.,

$$D_3 - D_1 < \lambda < D_4 - D_2.$$

This simplifies the above equation to

$$\Gamma(\nu) = \frac{X(\nu)}{e^{\lambda - D_4 + D_2} + e^{-\lambda + D_3 - D_1}}. \tag{B.14}$$

Generally, $\lambda - D_4 + D_2$ and $-\lambda + D_3 - D_1$ are of the order ϵ^{-1}, and in this regime the above equation can only be satisfied for $\nu \ll 1$, leading to

$$\nu = \frac{(2F_0F_1)^{\frac{1}{4}}}{\sqrt{\pi\epsilon}}\left(e^{\lambda - D_4 + D_2} + e^{-\lambda + D_3 - D_1}\right). \tag{B.15}$$

One can see that for $\lambda \approx D_3 - D_1$ or $\lambda \approx D_4 - D_2$ this simplification does no longer hold. In these regimes, one gets

$$-\lambda + D_3 - D_1 = \ln\left(\frac{X(\nu)}{\Gamma(\nu)}\right). \tag{B.16}$$

and

$$\lambda - D_4 + D_2 = \ln\left(\frac{X(\nu)}{\Gamma(\nu)}\right), \tag{B.17}$$

respectively. For $\nu \gg 1$ this simplifies to

$$\lambda + \frac{\int_0^1 dy (F(y) + \sqrt{F(y)^2 - F_0^2})}{\epsilon} = \nu \log(\nu\epsilon) \tag{B.18}$$

and

$$\lambda - \frac{\int_0^1 dy (F(y) - \sqrt{F(y)^2 - F_0^2})}{\epsilon} = -\nu \log(\nu\epsilon). \tag{B.19}$$

This result can be verified by taking the limit to the boundary of Eqs. (46), which leads to exactly the same result [25].

Appendix C. On the asymptotics of the solution of Eq. (53)

In this appendix we discuss some aspects of the connection formula of the asymptotics at $z \to +\infty$ and at $z \to -\infty$ of a solution G of

$$\nu G = \frac{d}{dz}(zG) + \frac{1}{2}\frac{d^2G}{dz^2} \tag{C.1}$$

For large z one expects either $G \sim z^{\nu-1}$ or $G \sim e^{-z^2} z^{-\nu}$ and our goal is to relate between the pair V, W to the pair V', W' which characterize the asymptotics at $\pm\infty$

$$G \simeq V(-z)^{\nu-1}\Big(1+\frac{(\nu-1)(\nu-2)}{4\,z^2}+\cdots\Big)+W\,\frac{e^{-z^2}}{(-z)^{\nu}}\Big(1-\frac{\nu(\nu+1)}{4\,z^2}+\cdots\Big) \quad \text{as} \quad z\to-\infty \tag{C.2}$$

and

$$G \simeq V' z^{\nu-1}\Big(1+\frac{(\nu-1)(\nu-2)}{4\,z^2}+\cdots\Big)+W'\,\frac{e^{-z^2}}{z^{\nu}}\Big(1-\frac{\nu(\nu+1)}{4\,z^2}+\cdots\Big) \quad \text{as} \quad z\to+\infty \tag{C.3}$$

The goal of this appendix is to show that

$$V' = -Z\,V+\frac{2^{\nu}\sqrt{\pi}}{\Gamma(\nu)}W \qquad ; \qquad W' = \frac{(1-Z^2)\,\Gamma(\nu)}{2^{\nu}\sqrt{\pi}}V+Z\,W \tag{C.4}$$

where

$$Z = \cos(\pi\nu). \tag{C.5}$$

By expanding around $z=0$, a general solution of Eq. (C.1) can be written as

$$G = g\,G_3+g'\,G_4 \tag{C.6}$$

where G_3 and G_4 are the even and the odd solutions

$$G_3=\sum_{n\geq 0}(-)^n z^{2n}\frac{\Gamma(2n-\nu)\,\Gamma(-\frac{\nu}{2})}{\Gamma(-\nu)\,\Gamma(n-\frac{\nu}{2})\,(2n)!} \;=\; 1+(\nu-1)z^2+\frac{(\nu-1)(\nu-3)}{6}z^4+\cdots$$

and

$$G_4=\sum_{n\geq 0}(-)^n z^{2n+1}\frac{2^{2n}\,\Gamma(n+1-\frac{\nu}{2})}{\Gamma(1-\frac{\nu}{2})\,(2n+1)!} \;=\; z+\frac{\nu-2}{3}z^3+\frac{(\nu-2)(\nu-4)}{30}z^5+\cdots$$

If one defines (assuming that ν is not an integer or half an integer)

$$G_1=\int_{-\infty+i0}^{\infty} e^{-z^2+tz-\frac{t^2}{4}}\,t^{\nu-1}dt$$

$$G_2=\int_{-\infty-i0}^{\infty} e^{-z^2+tz-\frac{t^2}{4}}\,t^{\nu-1}dt$$

one has

$$g_1=\Gamma\left(\frac{\nu}{2}\right)\left(1-e^{i\pi\nu}\right)2^{\nu-1} \qquad ; \qquad g_1'=\Gamma\left(\frac{\nu+1}{2}\right)\left(1+e^{i\pi\nu}\right)2^{\nu}$$

and

$$g_2=\Gamma\left(\frac{\nu}{2}\right)\left(1-e^{-i\pi\nu}\right)2^{\nu-1} \qquad ; \qquad g_2'=\Gamma\left(\frac{\nu+1}{2}\right)\left(1+e^{-i\pi\nu}\right)2^{\nu}$$

Therefore

$$G_3 = \frac{1}{2^\nu\,\Gamma\left(\frac{\nu}{2}\right)\,(1-e^{i\pi\nu})}\Big(G_1 - e^{i\pi\nu}G_2\Big) \tag{C.7}$$

$$G_4 = \frac{1}{2^{\nu+1}\,\Gamma\left(\frac{\nu+1}{2}\right)\,(1+e^{i\pi\nu})}\Big(G_1 + e^{i\pi\nu}G_2\Big). \tag{C.8}$$

Because

$$G_2 - G_1 = \int_{-\infty-i0}^{-\infty+i0} e^{-z^2+tz-\frac{t^2}{4}}t^{\nu-1}dt \tag{C.9}$$

and because this integral is dominated for large z by the neighborhood of $t=0$ one has the following asymptotics for $z\to+\infty$

$$G_2 - G_1 \sim \left(e^{i\pi\nu} - e^{-i\pi\nu}\right)\ e^{-z^2}\ \left(\frac{\Gamma(\nu)}{z^\nu} - \frac{\Gamma(\nu+2)}{4z^{\nu+2}} + \cdots\right) \tag{C.10}$$

On the other hand for large positive z a saddle point calculation leads to

$$G_1 \sim G_2 \sim 2^\nu\sqrt{\pi}z^{\nu-1}\left(1 + \frac{(\nu-1)(\nu-2)}{4z^2} + \cdots\right).$$

Then from Eqs. (C.7,C.8) one gets for large positive z

$$G_3 \simeq \frac{\sqrt{\pi}}{\Gamma(\frac{\nu}{2})}\,z^{\nu-1} \qquad ; \qquad G_4 \simeq \frac{\sqrt{\pi}}{2\,\Gamma(\frac{\nu+1}{2})}\,z^{\nu-1} \tag{C.11}$$

and from Eq. (C.9)

$$\Gamma\left(\frac{\nu}{2}\right)2^{\nu-1}G_3 - 2^\nu\,\Gamma\left(\frac{\nu+1}{2}\right)G_4 \simeq \Gamma(\nu)\frac{e^{-z^2}}{z^\nu}$$

So if one postulates that for $z\to+\infty$

$$G_3 = \frac{\sqrt{\pi}}{\Gamma(\frac{\nu}{2})}\left[z^{\nu-1}\left(1+\frac{(\nu-1)(\nu-2)}{4z^2}+\cdots\right) + \frac{\beta}{2^{\nu-1}}\frac{e^{-z^2}}{z^\nu}\,(1+\cdots)\right] \tag{C.12}$$

$$G_4 = \frac{\sqrt{\pi}}{2\Gamma(\frac{\nu+1}{2})}\left[z^{\nu-1}\left(1+\frac{(\nu-1)(\nu-2)}{4z^2}+\cdots\right) + \frac{\gamma}{2^{\nu-1}}\frac{e^{-z^2}}{z^\nu}\,(1+\cdots)\right] \tag{C.13}$$

one should have

$$\beta - \gamma = \frac{1}{\sqrt{\pi}}\Gamma(\nu) \tag{C.14}$$

In the above expressions β and γ are factors of subdominant terms and they are a priori ill defined unless one specifies how the dominant divergent series is resummed.

A general solution of Eq. (C.1) can always be written as

$$G = x\,G_3 + y\,G_4$$

Then one has (see Eqs. (C.12,C.13))

$$V = \frac{\sqrt{\pi}}{\Gamma(\frac{\nu}{2})}x - \frac{\sqrt{\pi}}{2\Gamma(\frac{\nu+1}{2})}y \qquad ; \qquad W = \frac{2\sqrt{\pi}}{2^\nu\Gamma(\frac{\nu}{2})}x\,\beta - \frac{2\sqrt{\pi}}{2^{\nu+1}\Gamma(\frac{\nu+1}{2})}y\,\gamma$$

$$V' = \frac{\sqrt{\pi}}{\Gamma(\frac{\nu}{2})}x + \frac{\sqrt{\pi}}{2\Gamma(\frac{\nu+1}{2})}y \quad ; \quad W' = \frac{2\sqrt{\pi}}{2^{\nu}\Gamma(\frac{\nu}{2})}x\,\beta + \frac{2\sqrt{\pi}}{2^{\nu+1}\Gamma(\frac{\nu+1}{2})}y\,\gamma.$$

Eliminating x and y one gets Eq. (C.4) where

$$Z = \frac{\sqrt{\pi}}{\Gamma(\mu)}(\beta + \gamma). \tag{C.15}$$

So far Z is undetermined, and as mentionned earlier it depends on the way the dominant contribution is resummed in Eqs. (C.12,C.13). This is related to Stokes phenomenon [38].

As for real positive z the solutions G_1 and G_2 are complex conjugates one can consider that their real part is by definition the resummed dominant contribution of the large z asymptotics. This implies (see Eq. (C.10))

$$G_2 \simeq 2\sqrt{\pi}z^{\nu-1}\Big(1+\cdots\Big) \quad + \quad \frac{\Gamma(\nu)}{2}\left(e^{i\pi\mu} - e^{-i\pi\nu}\right)\frac{e^{-z^2}}{z^{\nu}}$$

$$G_1 \simeq 2\sqrt{\pi}z^{\nu-1}\Big(1+\cdots\Big) \quad - \quad \frac{\Gamma(\nu)}{2}\left(e^{i\pi\nu} - e^{-i\pi\nu}\right)\frac{e^{-z^2}}{z^{\nu}}.$$

This gives Eqs. (C.7,C.8,C.12,C.13)

$$\beta = \frac{\Gamma(\nu)\Big(1+\cos(\pi\nu)\Big)}{2\sqrt{\pi}} \quad ; \quad \gamma = \frac{\Gamma(\nu)\Big(-1+\cos(\pi\nu)\Big)}{2\sqrt{\pi}}$$

so that (see Eq. (C.15))

$$Z = \cos(\pi\nu)$$

as in Eq. (C.5).

Bibliography

[1] Donsker f D and Varadhan S S 1975 Communications on Pure and Applied Mathematics **28** 1–47
[2] Ellis R S 1988 The Annals of Probability 1496–1508
[3] Den Hollander F 2008 Large deviations vol 14 (American Mathematical Soc.)
[4] Derrida B 2007 Journal of Statistical Mechanics: Theory and Experiment **2007** P07023
[5] Touchette H 2009 Physics Reports **478** 1–69
[6] Bertini L, De Sole A, Gabrielli D, Jona-Lasinio G and Landim C 2015 Reviews of Modern Physics **87** 593
[7] Lebowitz J L and Spohn H 1999 Journal of Statistical Physics **95** 333–365
[8] Derrida B 1983 Journal of statistical physics **31** 433–450
[9] Faucheux L P, Stolovitzky G and Libchaber A 1995 Physical Review E **51** 5239
[10] Speck T, Blickle V, Bechinger C and Seifert U 2007 EPL (Europhysics Letters) **79** 30002
[11] Maes C, Netočnỳ K and Wynants B 2008 Physica A: Statistical Mechanics and its Applications **387** 2675–2689
[12] Chernyak V Y, Chertkov M, Malinin S V and Teodorescu R 2009 Journal of Statistical Physics **137** 109
[13] Masharian S 2018 Physica A: Statistical Mechanics and its Applications **501** 126–133

[14] Mehl J, Speck T and Seifert U 2008 Physical Review E **78** 011123
[15] Lacoste D and Mallick K 2009 Physical Review E **80** 021923
[16] Nemoto T and Sasa S i 2011 Physical Review E **83** 030105
[17] Chetrite R and Touchette H 2015 Journal of Statistical Mechanics: Theory and Experiment **2015** P12001
[18] Nyawo P T and Touchette H 2016 Physical Review E **94** 032101
[19] Saito K and Dhar A 2016 EPL (Europhysics Letters) **114** 50004
[20] Fischer L P, Pietzonka P and Seifert U 2018 Physical Review E **97** 022143
[21] Freidlin M I and Wentzell A D 1994 Random perturbations of Hamiltonian systems vol 523 (American Mathematical Soc.)
[22] Speck T, Engel A and Seifert U 2012 Journal of Statistical Mechanics: Theory and Experiment **2012** P12001
[23] Faggionato A, Gabrielli D et al. 2012 A representation formula for large deviations rate functionals of invariant measures on the one dimensional torus Annales de l'Institut Henri Poincaré, Probabilités et Statistiques vol 48 (Institut Henri Poincaré) pp 212–234
[24] Bouchet F and Reygner J 2016 Generalisation of the eyring–kramers transition rate formula to irreversible diffusion processes Annales Henri Poincaré vol 17 (Springer) pp 3499–3532
[25] Tizón-Escamilla N, Lecomte V and Bertin E 2018 arXiv preprint arXiv:1807.06438
[26] Graham R 1987 Macroscopic potentials, bifurcations and noise in dissipative systems Fluctuations and Stochastic Phenomena in Condensed Matter (Springer) pp 1–34
[27] Baek Y and Kafri Y 2015 Journal of Statistical Mechanics: Theory and Experiment **2015** P08026
[28] Lecomte V and Tailleur J 2007 Journal of Statistical Mechanics: Theory and Experiment **2007** P03004
[29] Ray U, Chan G K L and Limmer D T 2018 Phys. Rev. Lett. **120**(21) 210602
[30] Baiesi M, Maes C and Netočný K 2009 Journal of statistical physics **135** 57–75
[31] Risken H 1996 Fokker-planck equation The Fokker-Planck Equation (Springer) pp 63–95
[32] Graham R 1995 Fluctuations in the steady state 25 Years of Non-Equilibrium Statistical Mechanics (Springer) pp 125–134
[33] Touchette H 2017 Physica A: Statistical Mechanics and its Applications ISSN 0378-4371
[34] Derrida B and Sadhu T 2018 arXiv preprint arXiv:1807.06543
[35] Jack R L and Sollich P 2010 Progress of Theoretical Physics Supplement **184** 304–317
[36] Chetrite R and Touchette H 2015 Nonequilibrium markov processes conditioned on large deviations Annales Henri Poincaré vol 16 (Springer) pp 2005–2057
[37] Gallavotti G and Cohen E G D 1995 Physical Review Letters **74** 2694
[38] Temme N M 2015 Asymptotic methods for integrals (World Scientific)