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CORRECTOR HOMOGENIZATION ESTIMATES FOR A NON-STATIONARY STOKES-NERNST-PLANCK-POISSON SYSTEM IN PERFORATED DOMAINS *

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Abstract. We consider a non-stationary Stokes-Nernst-Planck-Poisson system posed in perforated domains. Our aim is to justify rigorously the homogenization limit for the upscaled system derived by means of two-scale convergence in [29]. In other words, we wish to obtain the so-called corrector homogenization estimates that specify the error obtained when upscaling the microscopic equations. Essentially, we control in terms of suitable norms differences between the micro- and macro-concentrations and between the corresponding micro- and macro-concentration gradients. The major challenges that we face are the coupled flux structure of the system, the nonlinear drift terms and the presence of the microstructures. Employing various energy-like estimates, we discuss several scalings choices and boundary conditions.

Keywords. Stokes-Nernst-Planck-Poisson system; Variable scalings; Two-scale convergence; Perforated domain; Homogenization asymptotics; Corrector estimates.

AMS subject classifications. 35B27, 35C20, 35D30, 65M15

1. Introduction Colloidal dynamics is a relevant research topic of interest from both theoretical perspectives and modern industrial applications. Relevant technological applications include oil recovery and transport [37], drug-delivery design [24], motion of micro-organisms in biological suspensions [9], harvesting energy via solar cells [6], and also, sol-gel synthesis [7]. Typically, they all involve different phases of dispersed media (solid morphologies), which resemble at least remotely to homogeneous domains paved with arrays of contrasting microstructures that are distributed periodically. Mathematically, the interplay between populations of colloidal particles lead to work in the multiscale analysis of PDEs especially what concerns the Smoluchowski coagulation-fragmentation system and the Stokes-Nernst-Planck-Poisson system, which is our target here.

It is well known (cf. [8], e.g.) that many particles in colloidal chemistry are able to carry electrical charges (positive or negative) and, in some circumstances, they can be described using intensive quantities like the number density or ions concentration, say $c_\varepsilon^\pm$. Following [11], we consider such concentrations $c_\varepsilon^\pm$ of electrically charged colloidal particles to be involved as unknowns in the Nernst-Planck equations. These equations model the diffusion, deposition, convection and electrostatic interaction within a porous medium. The associated electrostatic potential, called here Φ_ε, is usually determined by a Poisson equation linearly coupled with the densities of charged species, describing the electric field formation inside the heterogeneous domain. Colloidal particles are always immersed in a background fluid. Here, we assume that the fluid velocity v_ε fulfills a suitable variant of the Stokes equations.

It is the aim of this paper to explore mathematically the upscaling of such non-stationary Stokes-Nernst-Planck-Poisson (SNPP) systems posed in a porous medium $\Omega^\varepsilon \subset \mathbb{R}^d$, where $\varepsilon \in (0,1)$ represents the scale parameter relative to the perforation (pore

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sizes) of the domain. To be more precise, we wish to justify the homogenization asymptotics for a class of SNPP systems developed by the group of Prof. P. Knabner in Erlangen, Germany, that fit well to the motion of charged colloidal particles through saturated soils.

As starting point of the discussion, we consider the following microscopic Stokes-Nernst-Planck-Poisson (SNPP) system:

$$-\varepsilon^2\Delta v_\varepsilon+\nabla p_\varepsilon=-\varepsilon^\beta\left(c_\varepsilon^+-c_\varepsilon^-\right)\nabla\Phi_\varepsilon \quad \text{in} \quad Q_T^\varepsilon:=(0,T)\times\Omega^\varepsilon, \tag{1.1}$$
$$\nabla\cdot v_\varepsilon=0 \quad \text{in} \quad Q_T^\varepsilon, \tag{1.2}$$
$$v_\varepsilon=0 \quad \text{on} \quad (0,T)\times(\Gamma^\varepsilon\cup\partial\Omega), \tag{1.3}$$
$$-\varepsilon^\alpha\Delta\Phi_\varepsilon=c_\varepsilon^+-c_\varepsilon^- \quad \text{in} \quad Q_T^\varepsilon, \tag{1.4}$$
$$\varepsilon^\alpha\nabla\Phi_\varepsilon\cdot\mathrm{n}=0 \quad \text{on} \quad (0,T)\times\partial\Omega, \tag{1.5}$$
$$\partial_t c_\varepsilon^\pm+\nabla\cdot\left(v_\varepsilon c_\varepsilon^\pm-\nabla c_\varepsilon^\pm\mp\varepsilon^\gamma c_\varepsilon^\pm\nabla\Phi_\varepsilon\right)=R_\varepsilon^\pm\left(c_\varepsilon^+,c_\varepsilon^-\right) \quad \text{in} \quad Q_T^\varepsilon, \tag{1.6}$$
$$-\left(v_\varepsilon c_\varepsilon^\pm-\nabla c_\varepsilon^\pm\mp\varepsilon^\gamma c_\varepsilon^\pm\nabla\Phi_\varepsilon\right)\cdot\mathrm{n}=0 \quad \text{on} \quad (0,T)\times(\Gamma^\varepsilon\cup\partial\Omega), \tag{1.7}$$
$$c_\varepsilon^\pm=c^{\pm,0} \quad \text{in} \quad \{t=0\}\times\Omega^\varepsilon. \tag{1.8}$$

We refer to (1.1)-(1.8) as (P^ε). The system (1.1)-(1.8) is endowed either with

$$\varepsilon^\alpha\nabla\Phi_\varepsilon\cdot\mathrm{n}=\varepsilon\sigma \quad \text{on} \quad (0,T)\times\Gamma_N^\varepsilon, \tag{1.9}$$

or with

$$\Phi_\varepsilon=\Phi_D \quad \text{on} \quad (0,T)\times\Gamma_D^\varepsilon. \tag{1.10}$$

We deliberately use variable scaling parameters α,β,γ for the ratio of the magnitudes of differently incorporated physical processes to weigh the effect a certain heterogeneity (morphology) has on effective transport coefficients. In view of the motivated application in soil and colloidal transport, our microscopic system (P^ε) stems from the non-dimensionalization procedure of the SNPP system in the colloidal dynamics with different length scales (cf. [28] for the detailed derivation of these systems). In this scaled setting, the parameter α and β should then be related to the presence of the so-called Debye screening length that describes the thickness of the double layer and the Reynolds number, respectively, while the Peclet and Strouhal numbers are linked to the value of γ.

A few additional remarks are in order: The background fluid (solvent) is assumed to be isothermal, incompressible and electrically neutral. The movement of this liquid at low Reynolds numbers decides the momentum equation behind our Stokes flow (see in (1.1)-(1.3)). The Stokes equation further couples to the mass balance equations of the involved colloidal species as described by the Nernst-Planck equations in (1.6)-(1.8). The initial charged densities $c^{\pm,0}$ are present cf. (1.8), while we remark that the involved reaction terms $R_\varepsilon^\pm$ linearly include positive and negative charged densities. Usually, such linear rates are necessary in describing the electric interaction for e.g. a simple mass-conserving reaction $X_1\rightleftharpoons X_2$ with rate coefficients equal to one, which essentially leads to the explicit form chosen in the assumption (A_3) made in Section 2. On the other hand, this choice is essentially fine to ensure the conservation of mass for the system under scrutiny. More complex nonlinear structures for $R_\varepsilon^\pm$ are used e.g. when multicomponent ionic flows are involved (e.g. when the Smoluchowski dynamics is assumed to take place). The Poisson-type equation points out an induced electric field acting on the liquid as well as on the charges carried by the colloidal species (see

$v_\varepsilon : Q_T^\varepsilon \to \mathbb{R}$	velocity
$p_\varepsilon : Q_T^\varepsilon \to \mathbb{R}$	pressure
$\Phi_\varepsilon : Q_T^\varepsilon \to \mathbb{R}$	electrostatic potential
$c_\varepsilon^\pm : Q_T^\varepsilon \to \mathbb{R}$	number densities
$c^{\pm,0} : \Omega^\varepsilon \to \mathbb{R}$	initial charged densities
$\sigma \in \mathbb{R}$	surface charge density
$\Phi_D \in \mathbb{R}$	ζ-potential
$R_\varepsilon^\pm : \mathbb{R}^2 \to \mathbb{R}$	reaction rates
$\alpha, \beta, \gamma \in \mathbb{R}$	variable choices of scalings

TABLE 1.1. *Physical unknowns and parameters arising in the microscopic problem* (P^ε).

in (1.4)-(1.5)). The surface charge density σ of the porous medium is prescribed as in (1.9), while we consider in (1.10) the surface potential related to the specification of the so-called zeta potential of the porous medium.

Although it can in principle introduce a boundary layer potentially interacting with the homogenization asymptotics, the magnitude of the ζ-potential Φ_D in (1.10) does not influence our theoretical results. Here, it only indicates the degree of electrostatic repulsion between charged colloidal particles within a dispersion. In fact, experiments provide that colloids with high ζ-potential (i.e. $\Phi_D \gg 1$ or $\Phi_D \ll -1$) are electrically stabilized while with low ζ-potential, they tend to coagulate or flocculate rapidly (see e.g. [18, 26] for a detailed calculation).

A glimpse on the structure of the selected microscopic system reveals that the current setting does not involve asymmetric multicomponent electrolytes, and hence, Faradaic processes at electrode surfaces are not included in the discussion (see appendix B in [5]). This can be a possible route for further work.

Specific scenarios for averaging Poisson-Nernst-Planck (PNP) systems as well as Stokes-Nernst-Planck-Poisson (SNPP) systems were discussed in a number of recent papers; see e.g. [12, 13, 15, 16, 33, 35]. The SNPP-type models are more difficult to handle mathematically mostly because of the oscillations introduced by the presence of the Stokes flow. The SNPP systems shown in [14, 29] are endowed with several scaling choices to cover various types of SNPP systems including the study of a stationary and linearized SNPP system by Allaire et al. cf. [3] and related to Schmuck's work cf. [33] where also an additional electric permittivity of the solid phase is taken into account. As main results, the global weak solvability of the respective models as well as their periodic homogenization limit procedures were obtained. We refer the reader to the *lit. cit.* also for the precise structure of the associated effective transport tensor parameters and upscaled equations. It is worth also mentioning that sometimes, like e.g. in [33–35], a classification of the upscaling results is done depending on the choice of boundary conditions for the Poisson equation.

The main theme of this paper is the derivation of corrector estimates quantifying the convergence rate of the periodic homogenization limit process leading to upscaled SNPP systems. This should be seen as a quantitative check of the quality of the two-scale averaging procedure. Getting grip on corrector estimates is a needed step in designing convergent multiscale finite element methods (see, e.g. [20]) and can play an important role also in studying multiscale inverse problems.

Our main results are reported in Theorem 4.1 in and Theorem 4.2. Here both the Neumann and Dirichlet boundary data for the electrostatic potential are considered.

The two types of boundary conditions for the electrostatic potential will lead to different structures of the upscaled systems, and hence, also the structure of the correctors will be different. To obtain these corrector estimates, we rely on the energy method combined with integral estimates for periodically oscillating functions as well as with appropriate macroscopic reconstructions, regularity results on limit and cell functions as well as the smoothness assumptions for the microscopic boundaries and data. It is worth mentioning that the corrector estimate for the closest model to ours, i.e. for the PNP equations in [34, Theorem 2.3], reveals already a class of possible assumptions on the cell functions (taken in $W^{1,\infty}$) as well as on the smoothness of the interior and exterior boundaries (taken in C^{∞}). Also, we borrowed ideas from both linear elliptic theory [1] as well as from the techniques behind the previously obtained corrector estimates [4, 21–23] for periodically perforated media. Concerning the locally periodic case, we refer the reader to [25] and references cited therein or to Zhang et al. [40]. In the latter paper, the authors have studied the homogenization of a steady reaction-diffusion system in a chemical vapor infiltration (CVI) process and have also deduced the convergence rate for the homogenization limit.

The reader should bear in mind that our way of deriving corrector estimates does not extend to the stochastic homogenization setting, but can cover, involving only minimal technical modifications, the locally periodic homogenization setting.

The corrector estimates we claim are the following:

Case 1: If the electrostatic potential Φ_ε satisfies the homogeneous Neumann boundary condition, then it holds

$$\left\|\tilde{\Phi}_\varepsilon-\tilde{\Phi}_0^\varepsilon\right\|_{L^2((0,T)\times\Omega^\varepsilon)}+\left\|c_\varepsilon^\pm-c_0^{\pm,\varepsilon}\right\|_{L^2((0,T)\times\Omega^\varepsilon)}$$
$$+\left\|\nabla\left(\tilde{\Phi}_\varepsilon-\tilde{\Phi}_1^\varepsilon\right)\right\|_{[L^2((0,T)\times\Omega^\varepsilon)]^d}\le C\max\left\{\varepsilon^{\frac{1}{2}},\varepsilon^{\frac{\mu}{2}}\right\}, \tag{1.11}$$

$$\left\|\nabla\left(c_\varepsilon^\pm-c_1^{\pm,\varepsilon}\right)\right\|_{[L^2((0,T)\times\Omega^\varepsilon)]^d}\le C\max\left\{\varepsilon^{\frac{1}{4}},\varepsilon^{\frac{\mu}{2}}\right\}, \tag{1.12}$$

$$\left\|v_\varepsilon-|Y_l|^{-1}\mathbb{D}v_0^\varepsilon-\varepsilon|Y_l|^{-1}\mathbb{D}v_1^\varepsilon\right\|_{[L^2((0,T)\times\Omega^\varepsilon)]^d}$$
$$+\|p_\varepsilon-p_0\|_{L^2(\Omega)/\mathbb{R}}\le C\left(\max\left\{\varepsilon^{\frac{1}{2}},\varepsilon^{\frac{\mu}{2}}\right\}+\varepsilon^{\frac{\lambda}{2}}+\varepsilon^{1-\frac{3\lambda}{2}}+\varepsilon^{\frac{1}{2}-\lambda}\right), \tag{1.13}$$

where $\mu\in\mathbb{R}_+$ and $\lambda\in(0,1)$.

Case 2: If the electrostatic potential Φ_ε satisfies the homogeneous Dirichlet boundary condition, then it holds

$$\left\|\tilde{\Phi}_\varepsilon-\tilde{\Phi}_0^\varepsilon\right\|_{L^2((0,T)\times\Omega^\varepsilon)}+\left\|c_\varepsilon^\pm-c_0^{\pm,\varepsilon}\right\|_{L^2((0,T)\times\Omega^\varepsilon)}$$
$$+\left\|\nabla\left(\tilde{\Phi}_\varepsilon-\tilde{\Phi}_0^\varepsilon\right)\right\|_{[L^2((0,T)\times\Omega^\varepsilon)]^d}+\left\|\nabla\left(c_\varepsilon^\pm-c_1^{\pm,\varepsilon}\right)\right\|_{[L^2((0,T)\times\Omega^\varepsilon)]^d}\le C\max\left\{\varepsilon^{\frac{1}{2}},\varepsilon^{\frac{\mu}{2}}\right\}, \tag{1.14}$$

$$\left\|v_\varepsilon-|Y_l|^{-1}\mathbb{D}v_0^\varepsilon-\varepsilon|Y_l|^{-1}\mathbb{D}v_1^\varepsilon\right\|_{[L^2((0,T)\times\Omega^\varepsilon)]^d}$$
$$+\|p_\varepsilon-p_0\|_{L^2(\Omega)/\mathbb{R}}\le C\left(\max\left\{\varepsilon^{\frac{1}{2}},\varepsilon^{\frac{\mu}{2}}\right\}+\varepsilon^{\frac{\lambda}{2}}+\varepsilon^{1-\frac{3\lambda}{2}}+\varepsilon^{\frac{1}{2}-\lambda}\right). \tag{1.15}$$

∎

The paper is organized as follows. In Section 2, the geometry of our perforated domains is introduced together with some notation and conventions. The list of assumptions on the data is also reported here. In the second part of the section, we present the classical concepts of the two-scale convergence on periodic domains and periodic interfaces and then provide the weak and strong formulations of all systems of PDEs mentioned in this framework (including the microscopic and macroscopic evolution systems, the cell problems). Section 4 is devoted to the statement of our main results and to the corresponding proofs. The remarks from Section 5 conclude the paper.

2. Technical preliminaries

2.1. A geometrical interpretation of porous media

Let Ω be a bounded and open domain in $\mathbb{R}^d$ with $\partial\Omega \in C^{0,1}$. Without loss of generality, we assume Ω to be the parallelepiped $(0,a_1)\times\ldots\times(0,a_d)$ for $a_i>0, i\in\{1,\ldots,d\}$.

Let Y be the unit cell defined by

$$Y := \left\{ \sum_{i=1}^{d} \lambda_i \vec{e}_i : 0<\lambda_i<1 \right\},$$

where $\vec{e}_i$ denotes the ith unit vector in $\mathbb{R}^d$. We suppose that Y consists of two open sets Y_l and Y_s which respectively represent the liquid part (the pore) and the solid part (the skeleton) such that $\bar{Y}_l \cup \bar{Y}_s = \bar{Y}$ and $Y_l \cap Y_s = \emptyset$, while $\bar{Y}_l \cap \bar{Y}_s = \Gamma$ has a non-zero $(d-1)$-dimensional Hausdorff measure. Additionally, we do not allow the solid part Y_s to touch the outer boundary ∂Y of the unit cell. As a consequence, the fluid part is connected (see Figure 2.1).

Let $Z \subset \mathbb{R}^d$ be a hypercube. For $X \subset Z$ we denote by X^k the shifted subset

$$X^k := X + \sum_{i=1}^{d} k_i \vec{e}_i,$$

where $k=(k_1,\ldots,k_d)\in\mathbb{Z}^d$ is a vector of indices.

Let $\varepsilon>0$ be a given scale factor. We assume that Ω is completely covered by a regular array of ε-scaled shifted cells. In porous media terminology, the solid part/pore skeleton is defined as the union of the cell regions εY_s^k, i.e.

$$\Omega_0^\varepsilon := \bigcup_{k\in\mathbb{Z}^d} \varepsilon Y_s^k,$$

while the fluid part, which is filling up the total space, is represented by

$$\Omega^\varepsilon := \bigcup_{k\in\mathbb{Z}^d} \varepsilon Y_l^k.$$

We denote the total pore surface of the skeleton by $\Gamma^\varepsilon := \partial\Omega_0^\varepsilon$. This description indicates that the porous medium we have in mind is saturated with the fluid.

Note that we use the subscripts N and D in (1.9)-(1.10) to distinguish, respectively, the case when the Neumann and Dirichlet conditions are applied across the pore surface. Furthermore, the assumption $\partial\Omega\cap\Gamma^\varepsilon=\emptyset$ holds.

In Figure 2.1, we show an admissible geometry mimicking a porous medium with periodic microstructures. We let $\mathrm{n}_\varepsilon := (n_1,\ldots,n_d)$ be the unit outward normal vector on the boundary Γ^ε. The representation of the periodic geometries is in line with the descriptions from [19, 22, 29] and the references cited therein.

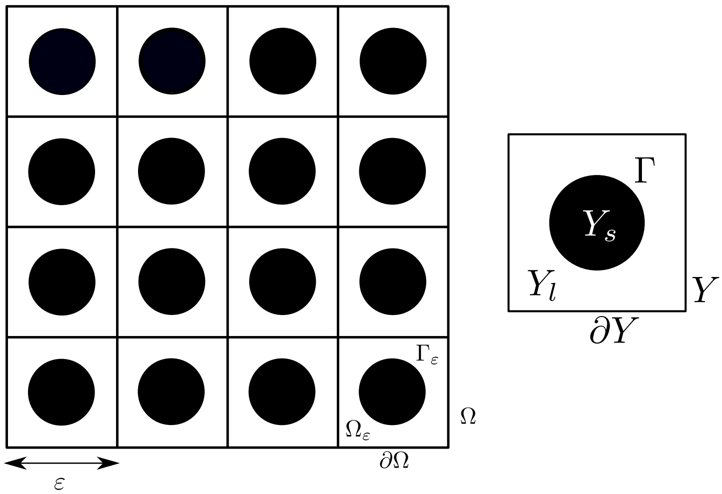

FIG. 2.1. *An admissible perforated domain. The perforations are referred here as microstructures.*

We denote by $x \in \Omega^\varepsilon$ the macroscopic variable and by $y = x/\varepsilon$ the microscopic variable representing fast variations at the microscopic geometry. In the following, the upper index ε thus denotes the corresponding quantity evaluated at $y = x/\varepsilon$. Suppose that our total pore space Ω^ε is bounded, connected and possesses $C^{0,1}$-boundary.

In the sequel, all the constants C are independent of the homogenization parameter ε, but their precise values may differ from line to line and may change even within a single chain of estimates. Throughout this paper, we use the superscript ε to emphasize the dependence of the material on the heterogeneity characterized by the homogenization parameter. In the following, we use dS_ε to indicate the surface measure of oscillating surfaces (boundary of microstructures). In addition, depending on the context, by $|\cdot|$ we denote either the volume measure of a domain or the absolute value of a function domain.

When writing the superscript $\pm$ or $\mp$ in e.g. $c_\varepsilon^\pm$, we mean both the positive c_ε^+ and negative densities c_ε^-.

Due to our choice of microstructures, the interior extension from $H^1(\Omega^\varepsilon)$ into $H^1(\Omega)$ exists and the extension constant is independent of ε (see [19, Lemma 5]).

2.2. Assumptions on the data To ensure the weak solvability of our SNPP system, we need essentially several assumptions on the involved data and parameters.

(A_1) The initial data of charged densities are non-negative and bounded independently of ε, i.e. there exists an ε-independent constant $C_0 > 0$ such that

$$0 \le c^{\pm,0}(x) \le C_0 \quad \text{for a.e. } x \in \Omega.$$

(A_2) The initial data of charged densities satisfy the compatibility condition:

$$\int_{\Omega^\varepsilon} \left(c^{+,0} - c^{-,0}\right) dx = \int_{\Gamma^\varepsilon} \sigma dS_\varepsilon.$$

(A_3) The chemical reaction rates are structured as $R_\varepsilon^\pm(c_\varepsilon^+, c_\varepsilon^-) = \mp(c_\varepsilon^+ - c_\varepsilon^-)$.
(A_4) The surface charge density σ and the ζ-potential Φ_D are constants.

(A_5) The electrostatic potential Φ_ε has zero mean value in the fluid part, i.e. it satisfies

$$\int_{\Omega^\varepsilon} \Phi_\varepsilon dx = 0.$$

(A_6) The pressure p_ε has zero mean value in the fluid part, i.e. it satisfies

$$\int_{\Omega^\varepsilon} p_\varepsilon(t,x)dx = 0 \quad \text{for all } t \geq 0.$$

REMARK 2.1. *Assumption (A_1) implies that at the initial moment, our charged colloidal particles are either neutral or positive in the macroscopic domain and their maximum voltage is known. Based on (A_2), if the surface charge density is static (i.e. $\sigma = 0$), then we obtain the so-called global charge neutrality which means that the charge density of our colloidal particles $c_\varepsilon^\pm$ is initially in neutrality. This global electroneutrality condition is particularly helpful in the analysis work (well-posedness, upscaling approach and numerical scheme) of related systems as stated in e.g. [29, 30, 35]. Nevertheless, it is not used in the derivation of the corrector estimates in this work. Cf. (A_3), the reaction rates are linear and ensure the conservation of mass for the concentration fields.*

3. Weak settings of SNPP models

3.1. Preliminary results

In this subsection, we present the definition of two-scale convergence as well as related compactness arguments (cf. [2,27]). We also recall the results on the weak solvability and periodic homogenization of the problem (P^ε), which are derived rigorously in [28,29], e.g.

DEFINITION 3.1. ***Two-scale convergence***

Let (u^ε) be a sequence of functions in $L^2((0,T)\times\Omega)$ with Ω being an open set in $\mathbb{R}^d$, then it two-scale converges to a unique function $u^0 \in L^2((0,T)\times\Omega\times Y)$, denoted by $u^\varepsilon \overset{2}{\rightharpoonup} u^0$, if for any $\varphi \in C_0^\infty\left((0,T)\times\Omega; C_\#^\infty(Y)\right)$ we have

$$\lim_{\varepsilon\to 0}\int_0^T \int_\Omega u^\varepsilon(t,x)\varphi\left(t,x,\frac{x}{\varepsilon}\right)dxdt = \frac{1}{|Y|}\int_0^T\int_\Omega\int_Y u^0(t,x,y)\varphi(t,x,y)dydxdt.$$

THEOREM 3.1. **Two-scale compactness**

- *Let (u^ε) be a bounded sequence in $L^2((0,T)\times\Omega)$. Then there exists a function $u^0 \in L^2((0,T)\times\Omega\times Y)$ such that, up to a subsequence, u^ε two-scale converges to u^0.*
- *Let (u^ε) be a bounded sequence in $L^2\left(0,T;H^1(\Omega)\right)$, then up to a subsequence, we have the two-scale convergence in gradient $\nabla u^\varepsilon \overset{2}{\rightharpoonup} \nabla_x u^0 + \nabla_y u^1$ for $u^0 \in L^2((0,T)\times\Omega\times Y)$ and $u^1 \in L^2\left((0,T)\times\Omega; H_\#^1(Y)/\mathbb{R}\right)$.*

DEFINITION 3.2. ***Two-scale convergence for ε-periodic hypersurfaces***

Let (u^ε) be a sequence of functions in $L^2((0,T)\times\Gamma^\varepsilon)$, then u^ε two-scale converges to a limit $u^0 \in L^2((0,T)\times\Omega\times\Gamma)$ if for any $\varphi \in C_0^\infty\left((0,T)\times\Omega; C_\#^\infty(\Gamma)\right)$ we have

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\Gamma^\varepsilon}\varepsilon u^\varepsilon(t,x)\varphi\left(t,x,\frac{x}{\varepsilon}\right)dS_\varepsilon dt = \frac{1}{|Y|}\int_0^T\int_\Omega\int_\Gamma u^0(t,x,y)\varphi(t,x,y)dS_y dxdt.$$

Remark 3.1. *The two-scale compactness on surfaces is the following: for each bounded sequence* (u^ε) *in* $L^2((0,T)\times\Gamma^\varepsilon)$*, one can extract a subsequence which two-scale converges to a limit* $u^0\in L^2((0,T)\times\Omega\times\Gamma)$*. Furthermore, if* (u^ε) *is bounded in* $L^\infty((0,T)\times\Gamma^\varepsilon)$*, it then two-scale converges to a limit function* $u^0\in L^\infty((0,T)\times\Omega\times\Gamma)$*.*

Definition 3.3. ***Weak formulation of*** (P^ε)

The vector $(v_\varepsilon,p_\varepsilon,\Phi_\varepsilon,c_\varepsilon^\pm)$ *satisfying*

$$v_\varepsilon\in L^\infty\left(0,T;H_0^1(\Omega^\varepsilon)\right),p_\varepsilon\in L^\infty\left(0,T;L^2(\Omega^\varepsilon)\right),\Phi_\varepsilon\in L^\infty\left(0,T;H^1(\Omega^\varepsilon)\right),$$

$$c_\varepsilon^\pm\in L^\infty\left(0,T;L^2(\Omega^\varepsilon)\right)\cap L^2\left(0,T;H^1(\Omega^\varepsilon)\right),\partial_t c_\varepsilon^\pm\in L^2\left(0,T;\left(H^1(\Omega^\varepsilon)\right)'\right),$$

is a weak solution to (P^ε) *provided that*

$$\int_{\Omega^\varepsilon}\left(\varepsilon^2\nabla v_\varepsilon\cdot\nabla\varphi_1-p_\varepsilon\nabla\cdot\varphi_1\right)dx=-\int_{\Omega^\varepsilon}\varepsilon^\beta\left(c_\varepsilon^+-c_\varepsilon^-\right)\nabla\Phi_\varepsilon\cdot\varphi_1dx, \tag{3.1}$$

$$\int_{\Omega^\varepsilon}v_\varepsilon\cdot\nabla\psi dx=0, \tag{3.2}$$

$$\int_{\Omega^\varepsilon}\varepsilon^\alpha\nabla\Phi_\varepsilon\cdot\nabla\varphi_2dx-\int_{\Gamma^\varepsilon}\varepsilon^\alpha\nabla\Phi_\varepsilon\cdot n\varphi_2dS_\varepsilon=\int_{\Omega^\varepsilon}\left(c_\varepsilon^+-c_\varepsilon^-\right)\varphi_2dx, \tag{3.3}$$

$$\begin{aligned}\left\langle\partial_t c_\varepsilon^\pm,\varphi_3\right\rangle_{(H^1(\Omega^\varepsilon))',H^1(\Omega^\varepsilon)}&+\int_{\Omega^\varepsilon}\left(-v_\varepsilon c_\varepsilon^\pm+\nabla c_\varepsilon^\pm\pm\varepsilon^\gamma c_\varepsilon^\pm\nabla\Phi_\varepsilon\right)\cdot\nabla\varphi_3dx\\&=\int_{\Omega^\varepsilon}R_\varepsilon^\pm\left(c_\varepsilon^+,c_\varepsilon^-\right)\varphi_3dx.\end{aligned} \tag{3.4}$$

for all $(\varphi_1,\varphi_2,\varphi_3,\psi)\in\left[H_0^1(\Omega^\varepsilon)\right]^d\times H^1(\Omega^\varepsilon)\times H^1(\Omega^\varepsilon)\times H^1(\Omega^\varepsilon)$*.*

Theorem 3.2. **Existence and uniqueness of solutions**

Assume (A_1)*-*(A_6)*. For each* $\varepsilon>0$*, the microscopic problem* (P^ε) *admits a unique weak solution* $(v_\varepsilon,p_\varepsilon,\Phi_\varepsilon,c_\varepsilon^\pm)$ *in the sense of Definition 3.3.*

The proof of Theorem 3.2 can be found in [29] (see Theorem 3.7) and [28].

Theorem 3.3. **Effective transport tensors. Cell problems**

The averaged macroscopic permittivity/diffusion tensor $\mathbb{D}=(D_{ij})_{1\le i,j\le d}$ *is defined by*

$$D_{ij}:=\int_{Y_l}(\delta_{ij}+\partial_{y_i}\varphi_j(y))dy,$$

where $\varphi_j=\varphi_j(y)$ *for* $1\le j\le d$ *are unique weak solutions in* $H^1(Y_l)$ *of the following family of cell problems*

$$\begin{cases}-\Delta_y\varphi_j(y)=0 & \text{in } Y_l,\\ \nabla_y\varphi_j(y)\cdot n=-e_j\cdot n & \text{on } \Gamma,\\ \varphi_j \text{ periodic in } y.\end{cases} \tag{3.5}$$

Furthermore, the averaged macroscopic permeability tensor $\mathbb{K}=(K_{ij})_{1\le i,j\le d}$ *is defined by*

$$K_{ij}:=\int_{Y_l}w_j^i dy,$$

where $w_j = w_j(y)$ together with $\pi_j = \pi_j(y)$ for $1 \le j \le d$ are unique weak solutions, respectively, in $H^1(Y_l)$ and $L^2(Y_l)$ of the following family of cell problems

$$\begin{cases} -\Delta_y w_j + \nabla_y \pi_j = e_j & \text{in } Y_l, \\ \nabla_y \cdot w_j = 0 & \text{in } Y_l, \\ w_j = 0 & \text{in } \Gamma, \\ w_j, \pi_j \text{ periodic in } y. \end{cases} \tag{3.6}$$

Also, we define the following cell problem

$$\begin{cases} -\Delta_y \varphi(y) = 1 & \text{in } Y_l, \\ \varphi(y) = 0 & \text{on } \Gamma, \\ \varphi \text{ periodic in } y, \end{cases} \tag{3.7}$$

which admits a unique weak solution in $H^1(Y_l)$.

Note that δ_{ij} denotes the Kronecker symbol and e_j is the jth unit vector of $\mathbb{R}^d$.

The proof of Theorem 3.3 can be found in [29] (see Definition 4.4) and [28].

Remark 3.2. *Fundamental results for elliptic equations provide that the problems (3.5) and (3.7) admit a unique weak solution in $H^1(Y_l)$ (cf. [4]). Similarly, the solutions w_j^i and π_j $(1 \le i,j \le d)$ of (3.6) are in $H^1(Y_l)$ and $L^2(Y_l)$, respectively. Particularly, for every $s \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ it follows from Theorem 4 and Theorem 7 in [32] that for $1 \le i,j \le d$,*

$$\varphi_j^i \in H^{1+s}(Y_l) \text{ and } w_j^i \in H^{1+s}(Y_l), \pi_j \in H^s(Y_l)$$

are unique weak solution to (3.5) and (3.6), respectively.

The permeability tensor $\mathbb{K}$ is symmetric and positive definite (cf. [31, Proposition 2.2, Chapter 7]), whilst the same properties of the permittivity tensor $\mathbb{D}$ are proven in [4].

3.2. Neumann condition for the electrostatic potential

In this section, we study the effect the choice of the Neumann boundary condition on the electrostatic potential has on the corrector estimates.

Theorem 3.4. **Positivity and Boundedness of solution**

Assume (A_1)-(A_4). Let $(v_\varepsilon, p_\varepsilon, \Phi_\varepsilon, c_\varepsilon^\pm)$ be a weak solution of the microscopic problem (P^ε) with the Neumann condition (1.9) in the sense of Definition 3.3. Then the concentration fields $c_\varepsilon^\pm$ are non-negative and essentially bounded from above uniformly in ε.

The proof of Theorem 3.4 can be found in [29] (see Theorems 3.3 and 3.4) and [28].

Theorem 3.5. ***A priori* estimates**

Assume (A_1)-(A_6). The following a priori *estimates hold:*

For the electrostatic potential, we have

$$\varepsilon^\alpha \|\Phi_\varepsilon\|_{L^2(0,T;H^1(\Omega^\varepsilon))} \le C. \tag{3.8}$$

If $\beta \ge \alpha$, it holds

$$\|v_\varepsilon\|_{L^2((0,T)\times\Omega^\varepsilon)} + \varepsilon \|\nabla v_\varepsilon\|_{L^2((0,T)\times\Omega^\varepsilon)} \le C, \tag{3.9}$$

and additionally, if $\gamma \ge \alpha$, it holds

$$\max_{t\in[0,T]} \|c_\varepsilon^-\|_{L^2(\Omega^\varepsilon)} + \max_{t\in[0,T]} \|c_\varepsilon^+\|_{L^2(\Omega^\varepsilon)} + \|\nabla c_\varepsilon^-\|_{L^2((0,T)\times\Omega^\varepsilon)} + \|\nabla c_\varepsilon^+\|_{L^2((0,T)\times\Omega^\varepsilon)}$$

$$+\left\|\partial_t c_\varepsilon^-\right\|_{L^2(0,T;(H^1(\Omega^\varepsilon))')}+\left\|\partial_t c_\varepsilon^+\right\|_{L^2(0,T;(H^1(\Omega^\varepsilon))')}\le C. \tag{3.10}$$

The proof of Theorem 3.5 can be found in [29] (see Theorem 3.5) and [28].

THEOREM 3.6. **Homogenization of (P_N^ε)**

Let the a priori *estimates* (3.8)-(3.10) *of Theorem 3.5 be valid. Taking* $\tilde{\Phi}_\varepsilon := \varepsilon^\alpha \Phi_\varepsilon$, *there exist functions* $\tilde{\Phi}_0 \in L^2\left(0,T;H^1(\Omega)\right)$ *and* $\tilde{\Phi}_1 \in L^2\left((0,T)\times\Omega;H^1_\#(Y)\right)$ *such that, up to a subsequence, we have*

$$\tilde{\Phi}_\varepsilon \overset{2}{\rightharpoonup} \tilde{\Phi}_0,$$
$$\nabla\tilde{\Phi}_\varepsilon \overset{2}{\rightharpoonup} \nabla_x\tilde{\Phi}_0+\nabla_y\tilde{\Phi}_1.$$

If $\beta\ge\alpha$, *then there exist functions* $v_0\in L^2\left((0,T)\times\Omega;H^1_\#(Y)\right)$ *and* $p_0\in L^2((0,T)\times\Omega\times Y)$ *such that, up to a subsequence, we have*

$$v_\varepsilon \overset{2}{\rightharpoonup} v_0,$$
$$\varepsilon\nabla v_\varepsilon \overset{2}{\rightharpoonup} \nabla_y v_0,$$
$$p_\varepsilon \overset{2}{\rightharpoonup} p_0.$$

Moreover, the convergence for the pressure is strong in $L^2(\Omega)/\mathbb{R}$.

If $\gamma\ge\alpha$, *then there exist functions* $c_0^\pm\in L^2\left(0,T;H^1(\Omega)\right)$ *and* $c_1^\pm\in L^2\left((0,T)\times\Omega;H^1_\#(Y)\right)$ *such that, up to a subsequence, we have*

$$c_\varepsilon^\pm \to c_0^\pm \ \textit{strongly in}\ L^2((0,T)\times\Omega),$$
$$\nabla c_\varepsilon^\pm \overset{2}{\rightharpoonup} \nabla_x c_0^\pm+\nabla_y c_1^\pm.$$

THEOREM 3.7. **Strong formulation of the macroscopic problem in the Neumann case - (P_N^0)**

Let $(v_\varepsilon,p_\varepsilon,\Phi_\varepsilon,c_\varepsilon^\pm)$ *be a weak solution of* (P^ε) *in the sense of Definition 3.3. According to Theorem 3.6, we have the following results:*

Let $\tilde{\Phi}_0$ *be the two-scale limit of the electrostatic potential* $\tilde{\Phi}_\varepsilon$, *it then satisfies the following macroscopic system:*

$$\begin{cases}-\nabla_x\cdot\left(\mathbb{D}\nabla_x\tilde{\Phi}_0(t,x)\right)=\bar{\sigma}+|Y_l|\left(c_0^+(t,x)-c_0^-(t,x)\right) & \textit{in } (0,T)\times\Omega,\\ \mathbb{D}\nabla_x\tilde{\Phi}_0(t,x)\cdot n=0 & \textit{on } (0,T)\times\partial\Omega,\end{cases}$$

where $\bar{\sigma}:=\int_\Gamma\sigma dS_y$ *and the permittivity/diffusion tensor* $\mathbb{D}$ *is defined in Theorem 3.3.*

Let v_0 *be the two-scale limit of the velocity field* v_ε. *With additionally* $\beta\ge\alpha$, *it then satisfies the following macroscopic system:*

$$\begin{cases}\bar{v}_0(t,x)+\mathbb{K}\nabla_x p_0(t,x)=-\mathbb{K}\left(c_0^+-c_0^-\right)\nabla_x\tilde{\Phi}_0(t,x) & \textit{in } (0,T)\times\Omega,\ \textit{if } \beta=\alpha,\\ \bar{v}_0(t,x)+\mathbb{K}\nabla_x p_0(t,x)=0 & \textit{in } (0,T)\times\Omega,\ \textit{if } \beta>\alpha,\\ \nabla_x\cdot\bar{v}_0(t,x)=0 & \textit{in } (0,T)\times\Omega,\\ \bar{v}_0(t,x)\cdot n=0 & \textit{on } (0,T)\times\partial\Omega,\end{cases}$$

where $\bar{v}_0(t,x)=\int_{Y_l}v_0(t,x,y)\,dy$ *and the permeability tensor* $\mathbb{K}$ *is defined in Theorem 3.3.*

Let $c_0^{\pm}$ be the two-scale limits of the concentration fields $c_{\varepsilon}^{\pm}$. With $\gamma=\alpha$, they satisfy the following macroscopic system:

$$\begin{cases} |Y_l|\partial_t c_0^{\pm}(t,x)+\nabla_x\cdot\left[c_0^{\pm}(t,x)\left(\bar{v}_0\mp\mathbb{D}\nabla_x\tilde{\Phi}_0\right)-\mathbb{D}\nabla_x c_0^{\pm}(t,x)\right] & \\ =|Y_l|R_0^{\pm}\left(c_0^{+}(t,x),c_0^{-}(t,x)\right) & \text{in } (0,T)\times\Omega, \\ \left(c_0^{\pm}(t,x)\left(\bar{v}_0(t,x)\mp\mathbb{D}\nabla_x\tilde{\Phi}_0(t,x)\right)-\mathbb{D}\nabla_x c_0^{\pm}(t,x)\right)\cdot n=0 & \text{on } (0,T)\times\partial\Omega, \end{cases}$$

while with $\gamma>\alpha$, they satisfy

$$\begin{cases} |Y_l|\partial_t c_0^{\pm}(t,x)+\nabla_x\cdot\left[c_0^{\pm}(t,x)\bar{v}_0(t,x)-\mathbb{D}\nabla_x c_0^{\pm}(t,x)\right] & \\ =|Y_l|R_0^{\pm}\left(c_0^{+}(t,x),c_0^{-}(t,x)\right) & \text{in } (0,T)\times\Omega, \\ \left(c_0^{\pm}(t,x)\bar{v}_0(t,x)-\mathbb{D}\nabla_x c_0^{\pm}(t,x)\right)\cdot n=0 & \text{on } (0,T)\times\partial\Omega. \end{cases}$$

REMARK 3.3. *Observe that when multiplying the homogenized system in Theorem 3.7 by $|Y|^{-1}$, one obtains the fraction $\frac{|Y_l|}{|Y|}$ as the well-known volumetric porosity. Note that the mathematical results with and without the presence of such $|Y|^{-1}$ are the same. We pursue the results originally obtained in [29] and continue to work on those structures of the homogenized systems. Due to the* a priori *estimate* (3.8) *for the electrostatic potential in Theorem 3.5, Φ_{ε} and its gradient $\nabla\Phi_{\varepsilon}$ converge to zero when $\alpha<0$. In Theorem 3.7, the number densities $c_0^{\pm}$ in the macroscopic Poisson equations with permittivity tensor $\mathbb{D}$ positions itself as forcing terms. Similarly, the forcing terms in the macroscopic Stokes equations with the case $\beta=\alpha$ dwell in the part of the electrostatic potential $\tilde{\Phi}_0$ and the distribution of the number densities $c_0^{\pm}$. Clearly, the macroscopic Nernst-Planck equations in the case $\gamma=\alpha$ yield the fully coupled system of partial differential equations, whilst with $\gamma>\alpha$ it reduces to a convection-diffusion-reaction system due to also the structure of the reaction terms $R_0^{\pm}$.*

Let us define the function space

$$H_N^1(\Omega):=\left\{v\in H^1(\Omega):-\mathbb{D}\nabla_x v\cdot \mathrm{n}=0 \text{ on } \partial\Omega\right\},$$

which is a closed subspace of $H^1(\Omega)$. This Hilbert space plays a role when writing the weak formulation of the macroscopic systems in Theorem 3.8 and Theorem 3.12.

THEOREM 3.8. **Weak formulation of** (P_N^0)

Let the quadruple of functions $\left(v_0,p_0,\tilde{\Phi}_0,c_0^{\pm}\right)$ be defined as in Theorem 3.7. Then $\left(v_0,p_0,\tilde{\Phi}_0,c_0^{\pm}\right)$ satisfies

$$\bar{v}_0\in L^2((0,T)\times\Omega),\, p_0\in L^2((0,T)\times\Omega),$$
$$\tilde{\Phi}_0\in L^2\left(0,T;H^1(\Omega)\right),\, c_0^{\pm}\in L^2\left(0,T;H^1(\Omega)\right),\, \partial_t c_0^{\pm}\in L^2\left(0,T;\left(H^1(\Omega)\right)'\right)$$

and becomes a weak solution to $\left(P_N^0\right)$ provided that

$$\int_{\Omega}\left(\bar{v}_0\varphi_1-\mathbb{K}p_0\nabla\cdot\varphi_1\right)dx=-\mathbb{K}\int_{\Omega}\left(c_0^{+}-c_0^{-}\right)\nabla\tilde{\Phi}_0\cdot\varphi_1 dx \;\; \text{if } \beta=\alpha,$$
$$\int_{\Omega}\left(\bar{v}_0\varphi_1-\mathbb{K}p_0\nabla\cdot\varphi_1\right)dx=0 \;\; \text{if } \beta>\alpha,$$
$$\int_{\Omega}\bar{v}_0\cdot\nabla\psi dx=0,$$

$$\int_\Omega |Y_l|^{-1}\mathbb{D}\nabla\tilde{\Phi}_0\cdot\nabla\varphi_2 dx - |Y_l|^{-1}\bar{\sigma}\int_\Omega \varphi_2 dx = \int_\Omega \left(c_0^+ - c_0^-\right)\varphi_2 dx,$$

$$\left\langle \partial_t c_0^\pm, \varphi_3\right\rangle_{(H^1)',H^1} + \int_\Omega |Y_l|^{-1}\left(-c_0^\pm\left(\bar{v}_0 \mp \mathbb{D}\nabla\tilde{\Phi}_0\right) + \mathbb{D}\nabla c_0^\pm\right)\cdot\nabla\varphi_3 dx$$
$$= \int_\Omega R_0^\pm\left(c_0^+, c_0^-\right)\varphi_3 dx \quad \text{if } \gamma = \alpha,$$

$$\left\langle \partial_t c_0^\pm, \varphi_3\right\rangle_{(H^1)',H^1} + \int_\Omega |Y_l|^{-1}\left(-c_0^\pm\bar{v}_0 + \mathbb{D}\nabla c_0^\pm\right)\cdot\nabla\varphi_3 dx$$
$$= \int_\Omega R_0^\pm\left(c_0^+, c_0^-\right)\varphi_3 dx \quad \text{if } \gamma > \alpha,$$

for all $(\varphi_1,\varphi_2,\varphi_3,\psi)\in\left[H_0^1(\Omega)\right]^d\times H_N^1(\Omega)\times H^1(\Omega)\times H^1(\Omega)$.

The proof of Theorems 3.6, 3.7 and 3.8 are collected from Theorems 4.5–4.10 in [29] and can also be found in [28].

3.3. Dirichlet condition for the electrostatic potential In this section, we study the effect the choice of the Dirichlet boundary condition on the electrostatic potential has on the corrector estimates.

REMARK 3.1. *What concerns Theorem 3.4, the proof (as mentioned in [29, Theorem 3.3, Theorem 3.4]) consists in suitable choices of test functions, based on the energy-estimates arguments. Nevertheless, for the case where the Dirichlet boundary condition (1.10) is prescribed, the volume additivity constraint* $c_\varepsilon^+ + c_\varepsilon^- = 1$ *is required to guarantee the* ε*-independent boundedness of the concentration fields.*

DEFINITION 3.4. *Assume* (A_1)-(A_4). *Let* Φ_ε *be a solution of the microscopic problem* (P^ε) *in the sense of Definition 3.3. Then the transformed electrostatic potential* $\Phi_\varepsilon^{hom} := \Phi_\varepsilon - \Phi_D$ *satifies the following system:*

$$-\varepsilon^\alpha\Delta\Phi_\varepsilon^{hom} = c_\varepsilon^+ - c_\varepsilon^- \quad \text{in } Q_T^\varepsilon,$$
$$\Phi_\varepsilon^{hom} = 0 \quad \text{in } (0,T)\times\Gamma_D^\varepsilon,$$
$$\varepsilon^\alpha\nabla\Phi_\varepsilon^{hom}\cdot n = 0 \quad \text{in } (0,T)\times\partial\Omega.$$

THEOREM 3.9. ***A priori* estimates**

Assume (A_1)-(A_4). *The following* a priori *estimates hold:*

For the electrostatic potential, we have

$$\varepsilon^{\alpha-2}\left\|\Phi_\varepsilon^{hom}\right\|_{L^2((0,T)\times\Omega^\varepsilon)} + \varepsilon^{\alpha-1}\left\|\nabla\Phi_\varepsilon^{hom}\right\|_{L^2((0,T)\times\Omega^\varepsilon)} \le C. \tag{3.11}$$

If $\beta \ge \alpha - 1$, *it holds*

$$\|v_\varepsilon\|_{L^2((0,T)\times\Omega^\varepsilon)} + \varepsilon\|\nabla v_\varepsilon\|_{L^2((0,T)\times\Omega^\varepsilon)} \le C, \tag{3.12}$$

and additionally if $\gamma \ge \alpha - 1$, *it holds*

$$\max_{t\in[0,T]}\left\|c_\varepsilon^-\right\|_{L^2(\Omega^\varepsilon)} + \max_{t\in[0,T]}\left\|c_\varepsilon^+\right\|_{L^2(\Omega^\varepsilon)} + \left\|\nabla c_\varepsilon^-\right\|_{L^2((0,T)\times\Omega^\varepsilon)} + \left\|\nabla c_\varepsilon^+\right\|_{L^2((0,T)\times\Omega^\varepsilon)}$$
$$+ \left\|\partial_t c_\varepsilon^-\right\|_{L^2(0,T;(H^1(\Omega^\varepsilon))')} + \left\|\partial_t c_\varepsilon^+\right\|_{L^2(0,T;(H^1(\Omega^\varepsilon))')} \le C. \tag{3.13}$$

The proof of Theorem 3.9 can be found in [29] (see Theorem 3.6); see also [28].

THEOREM 3.10. **Homogenization of (P_D^ε)**

Let the a priori *estimates* (3.11)-(3.13) *of Theorem 3.9 be valid. Let Φ_ε^{hom} be as defined in Definition 3.4. Taking $\tilde{\Phi}_\varepsilon := \varepsilon^{\alpha-2}\Phi_\varepsilon^{hom}$, then it satisfies the following system:*

$$\begin{aligned}
&-\varepsilon^2\Delta\tilde{\Phi}_\varepsilon = c_\varepsilon^+ - c_\varepsilon^- \text{ in } Q_T^\varepsilon,\\
&\tilde{\Phi}_\varepsilon = 0 \text{ in } (0,T)\times\Gamma_\varepsilon,\\
&\varepsilon^2\nabla\tilde{\Phi}_\varepsilon\cdot n = 0 \text{ in } (0,T)\times\partial\Omega.
\end{aligned}$$

Therefore, we can find a function $\tilde{\Phi}_0 \in L^2\left((0,T)\times\Omega; H^1_\#(Y)\right)$ such that, up to a subsequence,

$$\begin{aligned}
&\tilde{\Phi}_\varepsilon \stackrel{2}{\rightharpoonup} \tilde{\Phi}_0,\\
&\varepsilon\nabla\tilde{\Phi}_\varepsilon \stackrel{2}{\rightharpoonup} \nabla_y\tilde{\Phi}_0.
\end{aligned}$$

If, additionally, $\beta \geq \alpha - 1$, then there exist functions $v_0 \in L^2\left((0,T)\times\Omega; H^1_\#(Y)\right)$ and $p_0(t,x,y) \in L^2((0,T)\times\Omega\times Y)$ such that, up to a subsequence, we have

$$\begin{aligned}
&v_\varepsilon \stackrel{2}{\rightharpoonup} v_0,\\
&\varepsilon\nabla v_\varepsilon \stackrel{2}{\rightharpoonup} \nabla_y v_0,\\
&p_\varepsilon \stackrel{2}{\rightharpoonup} p_0.
\end{aligned}$$

Furthermore, there exist functions $c_0^\pm \in L^2\left(0,T; H^1(\Omega)\right)$ and $c_1^\pm \in L^2\left((0,T)\times\Omega; H^1_\#(Y)\right)$ such that, up to a subsequence, we have

$$\begin{aligned}
&c_\varepsilon^\pm \to c_0^\pm \text{ strongly in } L^2((0,T)\times\Omega),\\
&\nabla c_\varepsilon^\pm \stackrel{2}{\rightharpoonup} \nabla_x c_0^\pm + \nabla_y c_1^\pm.
\end{aligned}$$

THEOREM 3.11. **Strong formulation of the macroscopic problem in the Dirichlet case -** (P_D^0)

Let $(v_\varepsilon, p_\varepsilon, \Phi_\varepsilon, c_\varepsilon^\pm)$ be a weak solution of (P^ε) in the sense of Definition 3.3. According to Theorem 3.10, we have the following results:

Let $\tilde{\Phi}_0$ be the two-scale limit of the electrostatic potential $\tilde{\Phi}_\varepsilon$, it then satisfies the macroscopic equation:

$$\overline{\tilde{\Phi}}_0(t,x) = \left(\int_{Y_l}\varphi(y)\,dy\right)\left(c_0^+(t,x) - c_0^-(t,x)\right),$$

where $\overline{\tilde{\Phi}}_0(t,x) = \int_{Y_l}\tilde{\Phi}_0(t,x,y)\,dy$ and φ is the solution of the cell problem (3.7).

Let v_0 be the two-scale limit of the velocity field v_ε. With $\beta \geq \alpha - 1$, it then satisfies the following macroscopic system:

$$\begin{cases}
\bar{v}_0(t,x) + \mathbb{K}\nabla_x p_0(t,x) = 0 & \text{in } (0,T)\times\Omega,\\
\nabla_x\cdot\bar{v}_0(t,x) = 0 & \text{in } (0,T)\times\Omega,\\
\bar{v}_0(t,x)\cdot n = 0 & \text{on } (0,T)\times\partial\Omega,
\end{cases}$$

where $\bar{v}_0(t,x) = \int_{Y_l} v_0(t,x,y)\,dy$ *and the permeability tensor* $\mathbb{K}$ *is defined in Theorem 3.3.*

Let $c_0^\pm$ *be the two-scale limits of the concentration fields* $c_\varepsilon^\pm$*. With* $\gamma \geq \alpha - 1$*, they satisfy the following macroscopic system:*

$$\begin{cases} |Y_l|\partial_t c_0^\pm(t,x) + \nabla_x \cdot \left[c_0^\pm(t,x)\bar{v}_0(t,x) - \mathbb{D}\nabla_x c_0^\pm(t,x)\right] & \\ = |Y_l| R_0^\pm\left(c_0^+(t,x), c_0^-(t,x)\right) & \text{in } (0,T)\times\Omega, \\ \left(c_0^\pm(t,x)\bar{v}_0(t,x) - \mathbb{D}\nabla_x c_0^\pm(t,x)\right)\cdot n = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases}$$

where the permittivity/diffusion tensor $\mathbb{D}$ *is defined in Theorem 3.3.*

REMARK 3.4. *Due to the* a priori *estimate for the electrostatic potential in Theorem 3.9,* Φ_ε *converges to* Φ_D *as* $\alpha < 2$*. Moreover, in the case* $\alpha < 1$ *we obtain the convergence of* Φ_ε *and its gradient* $\nabla\Phi_\varepsilon$ *to the* ζ*-potential* Φ_D *and zero, respectively. When* $\alpha = 2$*, then* $\tilde{\Phi}_\varepsilon = \Phi_\varepsilon^{hom} := \Phi_\varepsilon - \Phi_D$ *holds, we compute that*

$$\bar{\Phi}_0(t,x) = \int_{Y_l}\left(\Phi_0^{hom}(t,x,y) + \Phi_D\right)dy = \left(\int_{Y_l}\varphi(y)\,dy\right)\left(c_0^+(t,x) - c_0^-(t,x)\right) + |Y_l|\Phi_D. \tag{3.14}$$

In Theorem 3.11, we see that in contrast to Theorem 3.7, the electrostatic potential is not present in the macroscopic Stokes and Nernst-Planck equations. In addition, the macroscopic Poisson system for the electrostatic potential reduces from the partial differential equations in the Neumann case to the macroscopic "representation" in the Dirichlet case. Both cases are all coupled with the concentration fields $c_0^\pm$*. Note that in both Neumann and Dirichlet cases, we need the strong convergence of the concentration fields, i.e.* $c_\varepsilon^\pm \to c_0^\pm$ *in* $L^2((0,T)\times\Omega)$*, to derive the macroscopic systems for the electrostatic potential, the fluid flow as well as for the pressure, respectively.*

THEOREM 3.12. **Weak formulation of** (P_D^0)

Let the quadruple of functions $\left(v_0, p_0, \tilde{\Phi}_0, c_0^\pm\right)$ *be defined as in Theorem 3.11. Then, it satisfies*

$$\begin{aligned} &\bar{v}_0 \in L^2((0,T)\times\Omega), p_0 \in L^2((0,T)\times\Omega), \\ &\tilde{\Phi}_0 \in L^2((0,T)\times\Omega), c_0^\pm \in L^2\left(0,T;H^1(\Omega)\right), \partial_t c_0^\pm \in L^2\left(0,T;\left(H^1(\Omega)\right)'\right) \end{aligned}$$

and is a weak solution to $\left(P_D^0\right)$ *provided that*

$$\begin{aligned} &\int_\Omega \left(\bar{v}_0\varphi_1 - \mathbb{K}p_0\nabla\cdot\varphi_1\right)dx = 0, \\ &\int_\Omega \bar{v}_0\cdot\nabla\psi\,dx = 0, \\ &\int_\Omega \bar{\tilde{\Phi}}_0\varphi_2\,dx = \left(\int_{Y_l}\varphi(y)\,dy\right)\int_\Omega\left(c_0^+ - c_0^-\right)\varphi_2\,dx, \\ &\left\langle \partial_t c_0^\pm, \varphi_3\right\rangle_{(H^1)',H^1} + \int_\Omega |Y_l|^{-1}\left(-c_0^\pm\bar{v}_0 + \mathbb{D}\nabla c_0^\pm\right)\cdot\nabla\varphi_3\,dx = \int_\Omega R_0^\pm\left(c_0^+, c_0^-\right)\varphi_3\,dx, \end{aligned}$$

for all $(\varphi_1,\varphi_2,\varphi_3,\psi) \in \left[H_0^1(\Omega)\right]^d \times H_N^1(\Omega)\times H^1(\Omega)\times H^1(\Omega)$.

The proof of Theorems 3.10, 3.11 and 3.12 are collected from Theorems 4.11–4.16 in [29] and can also be found in [28].

3.4. Discussions According to the proofs of the macroscopic systems in Theorems 4.6, 4.8, 4.10, 4.12, 4.14 and 4.16 cf. [28], we formulate here the first-order limit functions of the systems (P_N^0) and (P_D^0), respectively.

When the electric potential satisfies the Neumann condition on the micro-surface, we deduce that $\tilde{\Phi}_1$ can be formulated by

$$\tilde{\Phi}_1(t,x,y) = \sum_{j=1}^{d} \varphi_j(y)\,\partial_{x_j}\tilde{\Phi}_0(t,x),$$

with φ_j being solutions of the cell problems (3.5). We also remark that the limit function p_0 for the pressure is proved to be independent of y, i.e. $p_0(t,x,y) = p_0(t,x)$, due to the structure of the Stokes equation, see Theorem 3.6. Accordingly, the representation of the limit function v_0 for the fluid flow is given by

$$v_0(t,x,y) = \begin{cases} -\sum_{j=1}^{d} w_j(y)\left[\left(c_0^+ - c_0^-\right)\partial_{x_j}\tilde{\Phi}_0(t,x) + \partial_{x_j}p_0(t,x)\right] & \text{if } \beta = \alpha, \\ -\sum_{j=1}^{d} w_j(y)\,\partial_{x_j}p_0(t,x) & \text{if } \beta > \alpha, \end{cases}$$

where $w_j = w_j(y)$ for $1 \le j \le d$ are the solutions of the cell problems (3.6). We are able to determine the (extended) macroscopic Darcy's law by the following pressure:

$$\tilde{p}_1(t,x,y) = p_1(t,x,y) + \left(c_0^+(t,x) - c_0^-(t,x)\right)\tilde{\Phi}_1(t,x,y),$$

where with $\pi_j = \pi_j(y)$ for $1 \le j \le d$ are the solutions of the cell problems (3.6), we compute that

$$p_1(t,x,y) = \begin{cases} -\sum_{j=1}^{d} \pi_j(y)\left[\left(c_0^+ - c_0^-\right)\partial_{x_j}\tilde{\Phi}_0(t,x) + \partial_{x_j}p_0(t,x)\right] & \text{if } \beta = \alpha, \\ -\sum_{j=1}^{d} \pi_j(y)\,\partial_{x_j}p_0(t,x) & \text{if } \beta > \alpha. \end{cases}$$

On the other hand, the representation of the first-order functions $c_1^\pm$ is

$$c_1^\pm(t,x,y) = \begin{cases} \sum_{j=1}^{d} \left(\varphi_j(y)\,\partial_{x_j}c_0^\pm(t,x) \mp c_0^\pm(t,x)\,\partial_{x_j}\tilde{\Phi}_0(t,x)\right) & \text{if } \gamma = \alpha, \\ \sum_{j=1}^{d} \varphi_j(y)\,\partial_{x_j}c_0^\pm(t,x) & \text{if } \gamma > \alpha, \end{cases}$$

where $\varphi_j = \varphi_j(y)$ for $1 \le j \le d$ are the solutions of the cell problems (3.5).

When the electric potential satisfies the Dirichlet condition on the micro-surface, we obtain a different scenario. In fact, the macroscopic electrostatic potential $\tilde{\Phi}_0$ is in this case dependent of y and it can be computed by the averaged term $\overline{\tilde{\Phi}}_0$ (see Theorem 3.11 and the special case in (3.14)). We obtain the same manner with the macroscopic velocity v_0 in Theorem 3.11. However, the limit function p_0 for the pressure remains independent of y. As a consequence, the representation of the first-order functions $c_1^{\pm}$ is

$$c_1^{\pm}(t,x,y) = \begin{cases} \sum_{j=1}^{d} \left(\varphi_j(y)\partial_{x_j} c_0^{\pm}(t,x) \mp c_0^{\pm}(t,x)\tilde{\Phi}_0(t,x)\right) & \text{if } \gamma = \alpha - 1, \\ \sum_{j=1}^{d} \varphi_j(y)\partial_{x_j} c_0^{\pm}(t,x) & \text{if } \gamma > \alpha - 1, \end{cases}$$

where $\varphi_j = \varphi_j(y)$ for $1 \le j \le d$ are the solutions of the cell problems (3.5).

It is worth mentioning that upscaling the microscopic system (P^{ε}) is done by the two-scale convergence method. This approach, which aims to derive the limit system, does not require the derivation of the first-order macroscopic velocity, denoted by v_1 herein. To gain the corrector for the oscillating pressure arising in the Stokes equation, we use the same procedures as in [36], and thus, we need the structure of v_1.

Following [31], we have in the Neumann case for the electrostatic potential that

$$v_1\left(t,x,\frac{x}{\varepsilon}\right) = \begin{cases} -\sum_{i,j=1}^{d} r_{ij}\left(\frac{x}{\varepsilon}\right)\partial_{x_i}\left(\left(c_0^{+} - c_0^{-}\right)\partial_{x_j}\tilde{\Phi}_0(t,x) + \partial_{x_j} p_0(t,x)\right) & \text{if } \beta = \alpha, \\ -\sum_{i,j=1}^{d} r_{ij}\left(\frac{x}{\varepsilon}\right)\partial^2_{x_i x_j} p_0(t,x) & \text{if } \beta > \alpha, \end{cases}$$

where $r_{ij} \in H^1(Y_l)$ for $1 \le i,j \le d$ is the solution for the following cell problem

$$\begin{cases} \nabla_y \cdot r_{ij} + w_j^i = |Y_l|^{-1} K_{ij} & \text{in } Y_l, \\ r_{ij} = 0 & \text{on } \Gamma, \\ r_{ij} \text{ periodic in } y. \end{cases} \tag{3.15}$$

It holds

$$v_1\left(t,x,\frac{x}{\varepsilon}\right) = -\sum_{i,j=1}^{d} r_{ij}\left(\frac{x}{\varepsilon}\right)\partial^2_{x_i x_j} p_0(t,x),$$

provided the electrostatic potential satisfies the Dirichlet boundary data on the micro-surfaces.

3.5. Auxiliary estimates Here, let Y_l and Ω^{ε} as defined in Subsubsection 2.1.

LEMMA 3.1. *(cf. [23]) Let $p^{\varepsilon}(x) := p(x/\varepsilon) \in H^1(\Omega^{\varepsilon})$ satisfy*

$$\bar{p} := \frac{1}{|Y_l|}\int_{Y_l} p(y)\,dy,$$

then the following estimate holds:

$$\|p^{\varepsilon} - \bar{p}\|_{L^2(\Omega^{\varepsilon})} \le C\varepsilon^{\frac{1}{2}} \|p^{\varepsilon}\|_{H^1(\Omega^{\varepsilon})}.$$

LEMMA 3.2. *Assume $\partial\Omega \in C^k$ for $k \ge 4$ holds. Then, there exist $\delta_0 > 0$ and a function $\eta^{\delta} \in \left[C^{k-1}\left(\overline{\Omega}\right)\right]^d$ such that $\eta^{\delta} = \bar{v}_0$ on $\partial\Omega$ with $\bar{v}_0$ being the averaged macroscopic velocity* *defined in Theorem 3.7, $\nabla_x \cdot \eta^\delta = 0$ in Ω and for any $1 \le q \le \infty$ and $0 \le \ell \le k-1$, the following estimate holds:*

$$\left\| \nabla^\ell \eta^\delta \right\|_{L^q(\Omega)} \le C\delta^{\frac{1}{q}-\ell} \quad \text{for } \delta \in (0,\delta_0]. \tag{3.16}$$

Proof. We adapt the notation from [36] (see Lemma 1) to our proof here. It is well known from [17, Lemma 14.16] that there exists an ε-independent $\gamma > 0$ such that the distance function $z(x) = \text{dist}(x, \partial\Omega)$ belongs to $C^k(\mathcal{S}_\gamma)$ where

$$\mathcal{S}_\gamma := \left\{ x \in \overline{\Omega} : \text{dist}(x, \partial\Omega) \le \gamma \right\}. \tag{3.17}$$

By definition, we have

$$\partial\Omega := \left\{ x \in \mathbb{R}^d : z(x) = 0 \right\} \text{ and } \mathrm{n} := -\frac{\nabla z}{|\nabla z|} \quad \text{for } x \in \mathcal{S}_\gamma.$$

If we define a function $V(z,\xi)$ by

$$V(z,\xi) := -\frac{\bar{v}_0(x)}{|\nabla z(x)|} \quad \text{for } x = x(z,\xi) \in \mathcal{S}_\gamma \tag{3.18}$$

where ξ is the tangential component of z along $\partial\Omega$. We observe that $|\nabla z| > 0$ for $x \in \mathcal{S}_\gamma$ and the trace $V(0,\xi)$ is well-defined as a function in $C^k(\mathcal{S}_\gamma)$.

Following the same spirit of the argument as in Temam [39] in e.g. Proposition 2.3, we aim to take η^δ as curlψ, where ψ is chosen in such a way that

$$\frac{\partial \psi}{\partial \tau} = 0 \quad \text{on } \partial\Omega,$$

where we denote by τ the tangential component of ψ, and

$$\nabla\psi \cdot \mathrm{n} = \bar{v}_0 \cdot \tau \quad \text{on } \partial\Omega.$$

Note from the structure of the macroscopic Stokes system (cf. Theorem 3.7 and Theorem 3.11) that $\bar{v}_0 \cdot \mathrm{n} = 0$ on $\partial\Omega$ and from the fact that the tangential component is different from 0 in principle. We aim to choose $\psi = 0$ on $\partial\Omega$. Based on the function $V(z,\xi)$, defined in (3.18), we choose

$$\psi(x) = z(x) \exp\left(-\frac{z(x)}{\delta} \right) V(0,\xi) \cdot \tau(x).$$

Due to the presence of z, it is clear that $\psi = 0$ on $\partial\Omega$. Furthermore, we can check that

$$\nabla\psi \cdot \mathrm{n} = -\frac{\nabla z}{|\nabla z|} \cdot \left(\nabla z \frac{\partial \psi}{\partial z} \right) = -|\nabla z| \left(1 - \frac{z}{\delta} \right) \exp\left(-\frac{z}{\delta} \right) V(0,\xi) \cdot \tau(x) = \bar{v}_0 \cdot \tau$$

holds on $\partial\Omega$.

Therefore, we are now allowed to take $\eta^\delta = \text{curl}\psi$ in $\mathcal{S}_\gamma$.

We can now complete the proof of the lemma. Indeed, we estimate that

$$\|\nabla\psi\|_{L^q(\mathcal{S}_\gamma)}^q \le C \int_{\mathcal{S}_\gamma} \left(\left| \left(1 - \frac{z}{\delta} \right) \exp\left(-\frac{z}{\delta} \right) V(0,\xi) \right|^2 + \left| z \exp\left(-\frac{z}{\delta} \right) \frac{\partial V}{\partial \xi}(0,\xi) \right|^2 \right)^{\frac{q}{2}} dx$$

$$\leq C\delta.$$

Owning to the C^k-smoothness of $\partial\Omega$, we can proceed as above to obtain the following high-order estimate:

$$\left\|\nabla^{\ell+1}\psi\right\|_{L^q(\mathcal{S}_\gamma)} \leq C\delta^{\frac{1}{q}-\ell} \quad \text{for } 0 \leq \ell \leq k-1.$$

Hence, for $\delta \ll \gamma$ the function ψ is exponentially small at $\bar{\mathcal{S}}_\gamma = \left\{x \in \overline{\Omega} : \text{dist}\,(x,\partial\Omega) = \gamma\right\}$ and we can extend it to a function, which is denoted again by ψ, in $C^k\left(\overline{\Omega}\right)$ such that it satisfies $\eta^\delta = \text{curl}\psi$ and thus the estimate (3.16). ◻

By Lemma 3.2, we can introduce a cut-off function $m^\varepsilon \in \mathcal{D}\left(\overline{\Omega}\right)$ corresponding to $\partial\Omega$, satisfying

$$m^\varepsilon(x) = \begin{cases} 0 & \text{if dist}\,(x,\partial\Omega) \leq \varepsilon, \\ 1 & \text{if dist}\,(x,\partial\Omega) \geq 2\varepsilon, \end{cases} \quad \text{and} \quad \left\|\nabla^\ell m^\varepsilon\right\|_{L^\infty(\Omega)} \leq C\varepsilon^{-\ell} \text{ for } \ell \in [0,2].$$

As a consequence, one can also show that

$$\|1 - m^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq C\varepsilon^{\frac{1}{2}}, \quad \varepsilon\|\nabla m^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq C\varepsilon^{\frac{1}{2}}. \tag{3.19}$$

Lemma 3.3. *(cf. [31, Lemma 1, Appendix]) For any $u \in H_0^1(\Omega^\varepsilon)$, it holds*

$$\|u\|_{L^2(\Omega^\varepsilon)} \leq C\varepsilon\|\nabla u\|_{[L^2(\Omega^\varepsilon)]^d}.$$

4. Macroscopic reconstructions and corrector estimates

In this section, we begin by introducing the so-called macroscopic reconstructions and provide supplementary estimates needed for the proof of our main results stated in Theorem 4.1 and Theorem 4.2. Our working methodology was used in [10] and successfully applied to derive the corrector estimates for a thermo-diffusion system in a uniformly periodic medium (cf. [23]) and an advection-diffusion-reaction system in a locally-periodic medium (cf. [25]). In principle, the asymptotic expansion can be justified by estimating the differences of the solutions of the microscopic model (P^ε) and macroscopic reconstructions which can be defined from the macroscopic models $\left(P_N^0\right)$ and $\left(P_D^0\right)$.

Our main results correspond to two cases:

Case 1: The electric potential satisfies the Neumann boundary condition at the boundary of the perforations

Case 2: The electric potential satisfies the Dirichlet boundary condition at the boundary of the perforations

Remark 4.1. *To gain the structure of the corrector estimates, we require more regularity assumptions on the involved functions as well as the smoothness of the boundaries of the macroscopic domain; compare with the assumptions employed when upscaling (P^ε). In fact, it is worth pointing out that in Theorem 4.1 and Theorem 4.2 we require regularity properties on the limit functions, postulated in Theorem 3.8 for Case 1 and in Theorem 3.12 for Case 2, as follows:*

$$\tilde{\Phi}_0, c_0^\pm \in W^{1,\infty}(\Omega^\varepsilon) \cap H^2(\Omega^\varepsilon), \bar{v}_0 \in L^\infty(\Omega^\varepsilon). \tag{4.1}$$

The cell functions φ_j for $1 \le j \le d$ solving the family of cell problems (3.5) are supposed to fulfill

$$\varphi_j \in W^{1+s,2}(Y_l) \text{ for } s > d/2. \tag{4.2}$$

Moreover, the cell functions w_j^i, π_j and r_{ij} for $1 \le i,j \le d$ solving the cell problems (3.6) and (3.15), respectively, satisfy

$$w_j^i \in W^{2+s,2}(Y_l), \pi_j \in W^{1+s,2}(Y_l) \text{ and } r_{ij} \in W^{1+s,2}(Y_l) \text{ for } s > d/2. \tag{4.3}$$

In addition, we stress that the corrector estimates for the Stokes equation can be gained if we take $\partial\Omega \in C^4$. This assumption is only needed to handle Lemma 3.2.

4.1. Main results

Theorem 4.1. **Corrector estimates for Case 1**

Assume $(A_1)-(A_6)$. Let the quadruples $(v_\varepsilon, p_\varepsilon, \Phi_\varepsilon, c_\varepsilon^\pm)$ and $(v_0, p_0, \Phi_0, c_0^\pm)$ be weak solutions to (P^ε) and (P_N^0) in the sense of Definition 3.3 and Theorem 3.8, respectively. Furthermore, we assume that the limit solutions satisfy the regularity property (4.1). Let φ_j for $1 \le j \le d$ be the cell functions solving the family of cell problems (3.5) and satisfy (4.2). Assume that the initial homogenization limit is of the rate

$$\left\| c_\varepsilon^{\pm,0} - c_0^{\pm,0} \right\|_{L^2(\Omega^\varepsilon)}^2 \le C\varepsilon^\mu \quad \text{for some } \mu \in \mathbb{R}_+.$$

Then the following corrector estimates hold:

$$\begin{aligned}
&\|v_\varepsilon - \bar{v}_0^\varepsilon\|_{L^2((0,T)\times\Omega^\varepsilon)} \le C\varepsilon^{\frac{1}{2}},\\
&\left\|\tilde{\Phi}_\varepsilon - \tilde{\Phi}_0^\varepsilon\right\|_{L^2((0,T)\times\Omega^\varepsilon)} + \left\|c_\varepsilon^\pm - c_0^{\pm,\varepsilon}\right\|_{L^2((0,T)\times\Omega^\varepsilon)}\\
&+ \left\|\nabla\left(\tilde{\Phi}_\varepsilon - \tilde{\Phi}_1^\varepsilon\right)\right\|_{[L^2((0,T)\times\Omega^\varepsilon)]^d} \le C\max\left\{\varepsilon^{\frac{1}{2}}, \varepsilon^{\frac{\mu}{2}}\right\},\\
&\left\|\nabla\left(c_\varepsilon^\pm - c_1^{\pm,\varepsilon}\right)\right\|_{[L^2((0,T)\times\Omega^\varepsilon)]^d} \le C\max\left\{\varepsilon^{\frac{1}{4}}, \varepsilon^{\frac{\mu}{2}}\right\},
\end{aligned}$$

where $\bar{v}_0^\varepsilon$, Φ_0^ε, $c_0^{\pm,\varepsilon}$, $\tilde{\Phi}_1^\varepsilon$, $c_1^{\pm,\varepsilon}$ are the macroscopic reconstructions defined in (4.4)-(4.8).

Let w_j^i, π_j and r_{ij} for $1 \le i,j \le d$ be the cell functions solving the cell problems (3.6) and (3.15), respectively, and satisfy (4.3). If we further assume that

$$\tilde{\Phi}_0 \in H^4(\Omega^\varepsilon), c_0^\pm \in W^{2,\infty}(\Omega^\varepsilon), p_0 \in H^4(\Omega^\varepsilon),$$

then for any $\lambda \in (0,1)$, the following corrector estimates hold:

$$\begin{aligned}
&\left\|v_\varepsilon - |Y_l|^{-1}\mathbb{D}v_0^\varepsilon - \varepsilon|Y_l|^{-1}\mathbb{D}v_1^\varepsilon\right\|_{[L^2((0,T)\times\Omega^\varepsilon)]^d} \le C\left(\max\left\{\varepsilon^{\frac{1}{2}}, \varepsilon^{\frac{\mu}{2}}\right\} + \varepsilon^{\frac{\lambda}{2}} + \varepsilon^{1-\frac{3\lambda}{2}} + \varepsilon^{\frac{1}{2}-\lambda}\right),\\
&\|p_\varepsilon - p_0\|_{L^2(\Omega)/\mathbb{R}} \le C\left(\max\left\{\varepsilon^{\frac{1}{2}}, \varepsilon^{\frac{\mu}{2}}\right\} + \varepsilon^{\frac{\lambda}{2}} + \varepsilon^{1-\frac{3\lambda}{2}} + \varepsilon^{\frac{1}{2}-\lambda}\right),
\end{aligned}$$

where v_0^ε and v_1^ε are defined in (4.9) and (4.10), respectively.

Theorem 4.2. **Corrector estimates for Case 2**

Assume $(A_1)-(A_4)$. Let the quadruples $(v_\varepsilon, p_\varepsilon, \Phi_\varepsilon, c_\varepsilon^\pm)$ and $(v_0, p_0, \Phi_0, c_0^\pm)$ be weak solutions to (P^ε) and (P_D^0) in the sense of Definition 3.3 and Theorem 3.12, respectively. Furthermore, we assume that the limit solutions satisfy the regularity property (4.1). Let

φ_j for $1\le j\le d$ be the cell functions solving the family of cell problems (3.5) and satisfy (4.2). Assume that the initial homogenization limit is of the rate

$$\left\|c_\varepsilon^{\pm,0}-c_0^{\pm,0}\right\|_{L^2(\Omega^\varepsilon)}^2\le C\varepsilon^\mu \quad \text{for some } \mu\in\mathbb{R}_+.$$

Then the following corrector estimates hold:

$$\begin{aligned}&\left\|\tilde\Phi_\varepsilon-\tilde\Phi_0^\varepsilon\right\|_{L^2((0,T)\times\Omega^\varepsilon)}+\left\|\nabla\left(\tilde\Phi_\varepsilon-\tilde\Phi_0^\varepsilon\right)\right\|_{[L^2((0,T)\times\Omega^\varepsilon)]^d}+\left\|c_\varepsilon^\pm-c_0^{\pm,\varepsilon}\right\|_{L^2((0,T)\times\Omega^\varepsilon)}\\&+\left\|\nabla\left(c_\varepsilon^\pm-c_1^{\pm,\varepsilon}\right)\right\|_{[L^2((0,T)\times\Omega^\varepsilon)]^d}+\left\|\tilde\Phi_\varepsilon-\overline{\tilde\Phi}_0^\varepsilon\right\|_{L^2((0,T)\times\Omega^\varepsilon)}\le C\max\left\{\varepsilon^{\frac12},\varepsilon^{\frac{\mu}{2}}\right\},\end{aligned}$$

where $c_0^{\pm,\varepsilon}$, $c_1^{\pm,\varepsilon}$, Φ_0^ε, $\overline{\tilde\Phi}_0^\varepsilon$ are the macroscopic reconstructions defined in (4.55)-(4.56) and (4.57)-(4.58).

Let w_j^i, π_j and r_{ij} for $1\le i,j\le d$ be the cell functions solving the cell problems (3.6) and (3.15), respectively, and satisfy (4.3). If we further assume that $p_0\in H^4(\Omega^\varepsilon)$, then for any $\lambda\in(0,1)$, the following corrector estimates hold:

$$\begin{aligned}&\left\|v_\varepsilon-|Y_l|^{-1}\mathbb{D}v_0^\varepsilon-\varepsilon|Y_l|^{-1}\mathbb{D}v_1^\varepsilon\right\|_{[L^2((0,T)\times\Omega^\varepsilon)]^d}\le C\left(\max\left\{\varepsilon^{\frac12},\varepsilon^{\frac{\mu}{2}}\right\}+\varepsilon^{\frac{\lambda}{2}}+\varepsilon^{1-\frac{3\lambda}{2}}+\varepsilon^{\frac12-\lambda}\right),\\&\|p_\varepsilon-p_0\|_{L^2(\Omega)/\mathbb{R}}\le C\left(\max\left\{\varepsilon^{\frac12},\varepsilon^{\frac{\mu}{2}}\right\}+\varepsilon^{\frac{\lambda}{2}}+\varepsilon^{1-\frac{3\lambda}{2}}+\varepsilon^{\frac12-\lambda}\right),\end{aligned}$$

where v_0^ε and v_1^ε are defined in (4.53) and (4.54), respectively.

4.2. Proof of Theorem 4.1

To study the homogenization limit, the existence of asymptotic expansions

$$\begin{aligned}v_\varepsilon(t,x)&=v_0\left(t,x,\frac{x}{\varepsilon}\right)+\varepsilon v_1\left(t,x,\frac{x}{\varepsilon}\right)+\ldots\\p_\varepsilon(t,x)&=p_0\left(t,x,\frac{x}{\varepsilon}\right)+\varepsilon p_1\left(t,x,\frac{x}{\varepsilon}\right)+\ldots\\\tilde\Phi_\varepsilon(t,x)&=\tilde\Phi_0\left(t,x,\frac{x}{\varepsilon}\right)+\varepsilon\tilde\Phi_1\left(t,x,\frac{x}{\varepsilon}\right)+\ldots\\c_\varepsilon^\pm(t,x)&=c_0^\pm\left(t,x,\frac{x}{\varepsilon}\right)+\varepsilon c_1^\pm\left(t,x,\frac{x}{\varepsilon}\right)+\ldots,\end{aligned}$$

is assumed and some terms (e.g. $v_0,p_0,\tilde\Phi_0,c_0^\pm$) have been determined in the previous section. Since the route to derive the corrector for Stokes' equation is different from the usual construction of corrector estimates for the other equations, we shall postpone for a moment the proof of the corrector for the pressure.

We define the macroscopic reconstructions, as follows:

$$\bar v_0^\varepsilon(t,x):=|Y_l|^{-1}\bar v_0(t,x),\tag{4.4}$$

$$\tilde\Phi_0^\varepsilon(t,x):=\tilde\Phi_0(t,x),\tag{4.5}$$

$$\tilde\Phi_1^\varepsilon(t,x):=\tilde\Phi_0^\varepsilon(t,x)+\varepsilon\sum_{j=1}^d\varphi_j\left(\frac{x}{\varepsilon}\right)\partial_{x_j}\tilde\Phi_0^\varepsilon(t,x),\tag{4.6}$$

$$c_0^{\pm,\varepsilon}(t,x):=c_0^\pm(t,x),\tag{4.7}$$

$$c_1^{\pm,\varepsilon}(t,x):=c_0^{\pm,\varepsilon}(t,x)+\varepsilon\sum_{j=1}^d\varphi_j\left(\frac{x}{\varepsilon}\right)\partial_{x_j}c_0^{\pm,\varepsilon}(t,x),\tag{4.8}$$

$$v_0^\varepsilon(t,x) := v_0\left(t,x,\frac{x}{\varepsilon}\right), \tag{4.9}$$
$$v_1^\varepsilon(t,x) := v_1\left(t,x,\frac{x}{\varepsilon}\right). \tag{4.10}$$

Lemma 3.1 ensures the following estimate:

$$\|v_\varepsilon - \bar{v}_0^\varepsilon\|_{L^2((0,T)\times\Omega^\varepsilon)} \le C\varepsilon^{\frac{1}{2}}, \tag{4.11}$$

where Definition 3.3 and Theorem 3.2 guarantee the regularity for v_ε.

Let us now consider the correctors for the electrostatic potential and the concentrations. We take the difference of the microscopic and macroscopic Poisson equations in Definition 3.3 and Theorem 3.7, respectively, with the test function $\varphi_2 \in H^1(\Omega^\varepsilon)$ and thus obtain

$$\begin{aligned}\int_{\Omega^\varepsilon}\left(\nabla\tilde{\Phi}_\varepsilon - |Y_l|^{-1}\mathbb{D}\nabla\tilde{\Phi}_0\right)\cdot\nabla\varphi_2 dx + |Y_l|^{-1}\bar{\sigma}\int_{\Omega^\varepsilon}\varphi_2 dx - \varepsilon\int_{\Gamma^\varepsilon}\sigma\varphi_2 dS_\varepsilon \\ = \int_{\Omega^\varepsilon}\left(c_\varepsilon^+ - c_0^+ + c_0^- - c_\varepsilon^-\right)\varphi_2 dx,\end{aligned} \tag{4.12}$$

where we recall that $\tilde{\Phi}_\varepsilon = \varepsilon^\alpha\Phi_\varepsilon$ cf. Theorem 3.6.

Similarly, for $\varphi_3 \in H^1(\Omega^\varepsilon)$ we also find the difference equations for the Nernst-Planck equations, as follows:

$$\begin{aligned}\left\langle\partial_t\left(c_\varepsilon^\pm - c_0^\pm\right),\varphi_3\right\rangle_{(H^1)',H^1} + \int_{\Omega^\varepsilon}\left(\nabla c_\varepsilon^\pm - |Y_l|^{-1}\mathbb{D}\nabla c_0^\pm\right)\cdot\nabla\varphi_3 dx \\ + \int_{\Omega^\varepsilon}\left[|Y_l|^{-1}c_0^\pm\left(\bar{v}_0 \mp \mathbb{D}\nabla\tilde{\Phi}_0\right) - c_\varepsilon^\pm\left(v_\varepsilon \mp \nabla\tilde{\Phi}_\varepsilon\right)\right]\cdot\nabla\varphi_3 dx \\ = \int_{\Omega^\varepsilon}\left(R_\varepsilon^\pm\left(c_\varepsilon^+,c_\varepsilon^-\right) - R_0^\pm\left(c_0^+,c_0^-\right)\right)\varphi_3 dx.\end{aligned} \tag{4.13}$$

We start the investigation of these corrector justifications by the following choice of test functions:

$$\varphi_2(t,x) := \tilde{\Phi}_\varepsilon(t,x) - \left(\tilde{\Phi}_0^\varepsilon(t,x) + \varepsilon m^\varepsilon(x)\sum_{j=1}^d \varphi_j\left(\frac{x}{\varepsilon}\right)\partial_{x_j}\tilde{\Phi}_0(t,x)\right), \tag{4.14}$$

$$\varphi_3(t,x) := c_\varepsilon^\pm(t,x) - \left(c_0^{\pm,\varepsilon}(t,x) + \varepsilon m^\varepsilon(x)\sum_{j=1}^d \varphi_j\left(\frac{x}{\varepsilon}\right)\partial_{x_j}c_0^\pm(t,x)\right). \tag{4.15}$$

To get the estimates from (4.12) and (4.13), we denote the following terms just for ease of presentation:

$$\mathcal{J}_1 := \int_{\Omega^\varepsilon}\left(\nabla\tilde{\Phi}_\varepsilon - |Y_l|^{-1}\mathbb{D}\nabla\tilde{\Phi}_0\right)\cdot\nabla\varphi_2 dx,$$
$$\mathcal{J}_2 := |Y_l|^{-1}\bar{\sigma}\int_{\Omega^\varepsilon}\varphi_2 dx - \varepsilon\int_{\Gamma^\varepsilon}\sigma\varphi_2 dS_\varepsilon,$$
$$\mathcal{J}_3 := \int_{\Omega^\varepsilon}\left(c_\varepsilon^+ - c_0^+ + c_0^- - c_\varepsilon^-\right)\varphi_2 dx,$$
$$\mathcal{K}_1 := \left\langle\partial_t\left(c_\varepsilon^\pm - c_0^\pm\right),\varphi_3\right\rangle_{(H^1)',H^1} = \int_{\Omega^\varepsilon}\partial_t\left(c_\varepsilon^\pm - c_0^\pm\right)\varphi_3 dx,$$

$$\mathcal{K}_2 := \int_{\Omega^\varepsilon} \left(\nabla c_\varepsilon^\pm - |Y_l|^{-1} \mathbb{D} \nabla c_0^\pm \right) \cdot \nabla \varphi_3 dx,$$

$$\mathcal{K}_3 := \int_{\Omega^\varepsilon} \left[|Y_l|^{-1} c_0^\pm \left(\bar{v}_0 \mp \mathbb{D} \nabla \tilde{\Phi}_0 \right) - c_\varepsilon^\pm \left(v_\varepsilon \mp \nabla \tilde{\Phi}_\varepsilon \right) \right] \cdot \nabla \varphi_3 dx,$$

$$\mathcal{K}_4 := \int_{\Omega^\varepsilon} \left(R_\varepsilon^\pm \left(c_\varepsilon^+, c_\varepsilon^- \right) - R_0^\pm \left(c_0^+, c_0^- \right) \right) \varphi_3 dx.$$

Using the representation

$$\nabla \tilde{\Phi}_\varepsilon - |Y_l|^{-1} \mathbb{D} \nabla \tilde{\Phi}_0 = \nabla \left(\tilde{\Phi}_\varepsilon - \tilde{\Phi}_1^\varepsilon \right) + \nabla \tilde{\Phi}_1^\varepsilon - |Y_l|^{-1} \mathbb{D} \nabla \tilde{\Phi}_0,$$

the term $\mathcal{J}_1$ thus becomes

$$\mathcal{J}_1 = \int_{\Omega^\varepsilon} \nabla \left(\tilde{\Phi}_\varepsilon - \tilde{\Phi}_1^\varepsilon \right) \cdot \nabla \varphi_2 dx + \int_{\Omega^\varepsilon} \left(\nabla \tilde{\Phi}_1^\varepsilon - |Y_l|^{-1} \mathbb{D} \nabla \tilde{\Phi}_0 \right) \cdot \nabla \varphi_2 dx.$$

With the choice of φ_2 in (4.14), we have

$$\int_{\Omega^\varepsilon} \nabla \left(\tilde{\Phi}_\varepsilon - \tilde{\Phi}_1^\varepsilon \right) \cdot \nabla \varphi_2 dx \geq C \left\| \nabla \left(\tilde{\Phi}_\varepsilon - \tilde{\Phi}_1^\varepsilon \right) \right\|_{[L^2(\Omega^\varepsilon)]^d}^2 - C\varepsilon^2 \left\| \nabla \left((1-m^\varepsilon) \sum_{j=1}^d \varphi_j^\varepsilon \partial_{x_j} \tilde{\Phi}_0 \right) \right\|_{[L^2(\Omega^\varepsilon)]^d}^2. \tag{4.16}$$

To estimate the second term on the right-hand side of (4.16), we assume that $\tilde{\Phi}_0 \in W^{1,\infty}(\Omega^\varepsilon) \cap H^2(\Omega^\varepsilon)$ and $\varphi_j \in W^{1+s,2}(Y_l)$ for $s > d/2$ and $1 \leq j \leq d$. Using the Sobolev embedding $W^{1+s,2}(Y_l) \subset C^1\left(\bar{Y}_l\right)$ together with the inequalities in (3.19), we estimate that

$$\begin{aligned}
\varepsilon \left\| \nabla \left((1-m^\varepsilon) \sum_{j=1}^d \varphi_j^\varepsilon \partial_{x_j} \tilde{\Phi}_0 \right) \right\|_{[L^2(\Omega^\varepsilon)]^d} &\leq \varepsilon \| \nabla m^\varepsilon \|_{[L^2(\Omega^\varepsilon)]^d} \left\| \tilde{\Phi}_0 \right\|_{W^{1,\infty}(\Omega^\varepsilon)} \sum_{j=1}^d \| \varphi_j \|_{C(\bar{Y}_l)} \\
&\quad + \| 1 - m^\varepsilon \|_{L^2(\Omega^\varepsilon)} \left\| \tilde{\Phi}_0 \right\|_{W^{1,\infty}(\Omega^\varepsilon)} \sum_{j=1}^d \| \nabla_y \varphi_j \|_{[C(\bar{Y}_l)]^d} \\
&\quad + \varepsilon \left\| \tilde{\Phi}_0 \right\|_{H^2(\Omega^\varepsilon)} \sum_{j=1}^d \| \varphi_j \|_{C(\bar{Y}_l)} \\
&\leq C \left(\varepsilon + \varepsilon^{\frac{1}{2}} \right).
\end{aligned}$$

Taking into account the explicit computation of $\nabla \tilde{\Phi}_1^\varepsilon$, which reads

$$\nabla \tilde{\Phi}_1^\varepsilon = \nabla_x \tilde{\Phi}_0 + (\nabla_y \bar{\varphi})^\varepsilon \nabla_x \tilde{\Phi}_0 + \varepsilon \bar{\varphi}^\varepsilon \nabla_x \nabla \tilde{\Phi}_0 \quad \text{for } \bar{\varphi} = (\varphi_j)_{j=\overline{1,d}},$$

we can write

$$\nabla \tilde{\Phi}_1^\varepsilon - |Y_l|^{-1} \mathbb{D} \nabla \tilde{\Phi}_0 = \nabla \tilde{\Phi}_0 + (\nabla_y \bar{\varphi})^\varepsilon \nabla_x \tilde{\Phi}_0 - |Y_l|^{-1} \mathbb{D} \nabla \tilde{\Phi}_0 + \varepsilon \bar{\varphi}^\varepsilon \nabla_x \nabla \tilde{\Phi}_0. \tag{4.17}$$

Due to the smoothness of the involved functions, the fourth term in (4.17) is bounded in L^2-norm by

$$\varepsilon \left\| \bar{\varphi}^\varepsilon \nabla_x \nabla \tilde{\Phi}_0 \right\|_{[L^2(\Omega^\varepsilon)]^d} \leq C\varepsilon \| \bar{\varphi} \|_{[C(\bar{Y}_l)]^d} \left\| \tilde{\Phi}_0 \right\|_{H^2(\Omega^\varepsilon)}. \tag{4.18}$$

On the other hand, from the structure of the cell problem (3.5) we see that $\mathcal{G} := \mathbb{I} + \nabla_y \bar{\varphi} - |Y_l|^{-1}\mathbb{D}$ is divergence-free with respect to y. In parallel with that, its average also vanishes in the sense that

$$\int_{Y_l} \mathcal{G} dy = 0.$$

Consequently, the function $\mathcal{G}$ possesses a vector potential $\mathbf{V}$ which is skew-symmetric and satisfies $\mathcal{G} = \nabla_y \mathbf{V}$. Note that the choice of this potential is not unique in general, but $\mathbf{V}$ can be chosen in such a way that it solves a Poisson equation $\Delta_y \mathbf{V} = f(y) \nabla_y \mathcal{G}$ for some constant f only dependent of the cell's dimension. Therefore, to determine $\mathbf{V}$ uniquely, we associate this Poisson equation with the periodic boundary condition at Γ and the vanishing cell average. Using the simple relation $\nabla_y = \varepsilon\nabla - \varepsilon\nabla_x$, we arrive at

$$\mathcal{G}^{\varepsilon} \nabla \tilde{\Phi}_0 = \varepsilon \nabla \cdot \left(\mathbf{V}^{\varepsilon} \nabla \tilde{\Phi}_0 \right) - \varepsilon \mathbf{V}^{\varepsilon} \Delta \tilde{\Phi}_0. \tag{4.19}$$

Due to the skew-symmetry of $\mathbf{V}$, the first term on the right-hand side of (4.19) is divergence-free and its boundedness in $L^2(\Omega^{\varepsilon})$ is thus of the order of ε. Since $\bar{\varphi} \in \left[W^{1+s,2}(Y_l) \right]^d$ for $s > d/2$, it yields from the Poisson equation for $\mathbf{V}$ that

$$\|\mathbf{V}\|_{W^{1+s,2}(Y_l)} \le C \|\mathcal{G}\|_{W^{s,2}(Y_l)}.$$

Applying again the compact embedding $W^{s,2}(Y_l) \subset C\left(\bar{Y}_l\right)$ for $s > d/2$, we obtain $\mathbf{V} \in C\left(\bar{Y}_l\right)$ and it enables us to get the boundedness of the second term on the right-hand side of (4.19). In fact, it gives

$$\varepsilon \left\| \mathbf{V}^{\varepsilon} \Delta \tilde{\Phi}_0 \right\|_{L^2(\Omega^{\varepsilon})} \le \varepsilon \|\mathbf{V}\|_{C(\bar{Y}_l)} \left\| \tilde{\Phi}_0 \right\|_{H^2(\Omega^{\varepsilon})}.$$

Combining this inequality with (4.17), (4.18) and using the Hölder's inequality, we conclude that

$$\int_{\Omega^{\varepsilon}} \left(\nabla \tilde{\Phi}_1^{\varepsilon} - |Y_l|^{-1} \mathbb{D} \nabla \tilde{\Phi}_0 \right) \cdot \nabla \varphi_2 dx \le C\varepsilon.$$

This step completes the estimates for $\mathcal{J}_1$. More precisely, we obtain

$$\mathcal{J}_1 \ge C \left\| \nabla \left(\tilde{\Phi}_{\varepsilon} - \tilde{\Phi}_1^{\varepsilon} \right) \right\|^2_{[L^2(\Omega^{\varepsilon})]^d} - C\left(\varepsilon^2 + \varepsilon\right). \tag{4.20}$$

In the same vein, we can estimate the term $\mathcal{K}_2$ with the aid of the *a priori* regularity $c_0^{\pm} \in W^{1,\infty}(\Omega^{\varepsilon}) \cap H^2(\Omega^{\varepsilon})$ and $\varphi_j \in W^{1+s,2}(Y_l)$ for $s > d/2$ and $1 \le j \le d$. We thus get

$$\mathcal{K}_2 \ge C \left\| \nabla \left(c_{\varepsilon}^{\pm} - c_1^{\pm,\varepsilon} \right) \right\|^2_{[L^2(\Omega^{\varepsilon})]^d} - C\left(\varepsilon^2 + \varepsilon\right). \tag{4.21}$$

We now turn our attention to the estimates for $\mathcal{J}_2$ and $\mathcal{J}_3$. Noticing $\bar{\sigma} := \int_{\Gamma} \sigma dS_y$ which implies that

$$|Y_l|^{-1} \int_{Y_l} \bar{\sigma} dy = \int_{\Gamma} \sigma dS_y,$$

we then apply [25, Lemma 5.2] to gain

$$|\mathcal{J}_2| \le C\varepsilon \|\varphi_2\|_{H^1(\Omega^{\varepsilon})}.$$

Note that due to the choice of φ_2 in (4.14), we have

$$\begin{aligned}\|\varphi_2\|_{H^1(\Omega^\varepsilon)} &\le \left\|\tilde{\Phi}_\varepsilon - \tilde{\Phi}_0\right\|_{L^2(\Omega^\varepsilon)} + \left\|\nabla\left(\tilde{\Phi}_\varepsilon - \tilde{\Phi}_1^\varepsilon\right)\right\|_{[L^2(\Omega^\varepsilon)]^d} \\ &\quad + \left\|\nabla\left(\tilde{\Phi}_1^\varepsilon - \tilde{\Phi}_0\right)\right\|_{[L^2(\Omega^\varepsilon)]^d} + \varepsilon\left\|m^\varepsilon\bar{\varphi}\cdot\nabla_x\tilde{\Phi}_0\right\|_{H^1(\Omega^\varepsilon)} \\ &\le \left\|\tilde{\Phi}_\varepsilon - \tilde{\Phi}_0\right\|_{L^2(\Omega^\varepsilon)} + \left\|\nabla\left(\tilde{\Phi}_\varepsilon - \tilde{\Phi}_1^\varepsilon\right)\right\|_{[L^2(\Omega^\varepsilon)]^d} + C\left(1+\varepsilon+\varepsilon^{\frac{1}{2}}\right), \end{aligned} \tag{4.22}$$

where we use the inequalities (3.19) with the regularity assumptions on $\bar{\varphi}$ and $\tilde{\Phi}_0$, and the following bound:

$$\left\|\nabla\left(\tilde{\Phi}_1^\varepsilon - \tilde{\Phi}_0\right)\right\|_{[L^2(\Omega^\varepsilon)]^d} \le \|\nabla_y\bar{\varphi}\|_{C(\bar{Y}_l)}\left\|\tilde{\Phi}_0\right\|_{W^{1,\infty}(\Omega^\varepsilon)} + \varepsilon\|\bar{\varphi}\|_{C(\bar{Y}_l)}\left\|\tilde{\Phi}_0\right\|_{H^2(\Omega^\varepsilon)}.$$

Therefore, we can write that

$$|\mathcal{J}_2| \le C\varepsilon\left(\left\|\tilde{\Phi}_\varepsilon - \tilde{\Phi}_0\right\|_{L^2(\Omega^\varepsilon)} + \left\|\nabla\left(\tilde{\Phi}_\varepsilon - \tilde{\Phi}_1^\varepsilon\right)\right\|_{[L^2(\Omega^\varepsilon)]^d} + 1\right). \tag{4.23}$$

The estimate for $\mathcal{J}_3$ can be derived by the Hölder inequality, which reads

$$|\mathcal{J}_3| \le C\left(\|c_\varepsilon^+ - c_0^+\|_{L^2(\Omega^\varepsilon)} + \|c_\varepsilon^- - c_0^-\|_{L^2(\Omega^\varepsilon)}\right)\|\varphi_2\|_{L^2(\Omega^\varepsilon)},$$

and then leads to

$$|\mathcal{J}_3| \le C\left(\|c_\varepsilon^+ - c_0^+\|_{L^2(\Omega^\varepsilon)} + \|c_\varepsilon^- - c_0^-\|_{L^2(\Omega^\varepsilon)}\right)\left(\left\|\tilde{\Phi}_\varepsilon - \tilde{\Phi}_0\right\|_{L^2(\Omega^\varepsilon)} + 1\right). \tag{4.24}$$

Let us now consider the term $\mathcal{K}_1$ and $\mathcal{K}_4$. Note that $\mathcal{K}_1$ can be rewritten as

$$\begin{aligned}\int_{\Omega^\varepsilon}\partial_t\left(c_\varepsilon^\pm - c_0^\pm\right)&\left[c_\varepsilon^\pm - \left(c_0^{\pm,\varepsilon}(t,x) + \varepsilon m^\varepsilon\bar{\varphi}^\varepsilon\cdot\nabla_x c_0^\pm\right)\right]dx \\ &= \frac{1}{2}\frac{d}{dt}\|c_\varepsilon^\pm - c_0\|_{L^2(\Omega^\varepsilon)}^2 - \varepsilon\int_{\Omega^\varepsilon}\partial_t\left(c_\varepsilon^\pm - c_0^\pm\right)m^\varepsilon\bar{\varphi}\cdot\nabla_x c_0^\pm dx, \end{aligned} \tag{4.25}$$

while from the structure of the reaction in (A_3), we have the similar result for $\mathcal{K}_4$ (to $\mathcal{J}_3$), i.e.

$$|\mathcal{K}_4| \le C\left(\|c_\varepsilon^+ - c_0^+\|_{L^2(\Omega^\varepsilon)} + \|c_\varepsilon^- - c_0^-\|_{L^2(\Omega^\varepsilon)}\right)\left(\|c_\varepsilon^\pm - c_0^\pm\|_{L^2(\Omega^\varepsilon)} + 1\right). \tag{4.26}$$

The estimate for $\mathcal{K}_3$ relies on the following decomposition:

$$\begin{aligned}|Y_l|^{-1}c_0^\pm\left(\bar{v}_0 \mp \mathbb{D}\nabla\tilde{\Phi}_0\right) - c_\varepsilon^\pm\left(v_\varepsilon \mp \nabla\tilde{\Phi}_\varepsilon\right) &= \left(c_0^\pm - c_\varepsilon^\pm\right)\left(|Y_l|^{-1}\bar{v}_0 \mp |Y_l|^{-1}\mathbb{D}\nabla\tilde{\Phi}_0\right) \\ &\quad + c_\varepsilon^\pm\left(|Y_l|^{-1}\bar{v}_0 - v_\varepsilon\right) \mp c_\varepsilon^\pm\left(|Y_l|^{-1}\mathbb{D}\nabla\tilde{\Phi}_0 - \nabla\tilde{\Phi}_\varepsilon\right).\end{aligned}$$

Clearly, if $\bar{v}_0 \in L^\infty(\Omega^\varepsilon)$ and since $\tilde{\Phi}_0 \in W^{1,\infty}(\Omega^\varepsilon)\cap H^2(\Omega^\varepsilon)$, we can estimate, by Hölder's inequality, that

$$\int_{\Omega^\varepsilon}\left(c_0^\pm - c_\varepsilon^\pm\right)\left(|Y_l|^{-1}\bar{v}_0 \mp |Y_l|^{-1}\mathbb{D}\nabla\tilde{\Phi}_0\right)\cdot\nabla\varphi_3 dx \le C\|c_\varepsilon^\pm - c_0^\pm\|_{L^2(\Omega^\varepsilon)}\|\nabla\varphi_3\|_{[L^2(\Omega^\varepsilon)]^d}. \tag{4.27}$$

By using the same arguments in estimating the norm $\|\varphi_2\|_{H^1(\Omega^\varepsilon)}$ in (4.22), we get from (4.27) that

$$\int_{\Omega^\varepsilon}\left(c_0^\pm - c_\varepsilon^\pm\right)\left(|Y_l|^{-1}\bar{v}_0 \mp |Y_l|^{-1}\mathbb{D}\nabla\tilde{\Phi}_0\right)\cdot\nabla\varphi_3 dx$$
$$\le C\left\|c_\varepsilon^\pm - c_0^\pm\right\|_{L^2(\Omega^\varepsilon)}\left(\left\|\nabla\left(c_\varepsilon^\pm - c_1^{\pm,\varepsilon}\right)\right\|_{[L^2(\Omega^\varepsilon)]^d}+1\right). \tag{4.28}$$

Next, we observe that

$$\int_{\Omega^\varepsilon} c_\varepsilon^\pm\left(|Y_l|^{-1}\bar{v}_0 - v_\varepsilon\right)\cdot\nabla\varphi_3 dx \le C\|v_\varepsilon - \bar{v}_0^\varepsilon\|_{L^2(\Omega^\varepsilon)}\left(\left\|\nabla\left(c_\varepsilon^\pm - c_1^{\pm,\varepsilon}\right)\right\|_{[L^2(\Omega^\varepsilon)]^d}+1\right)$$
$$\le C\varepsilon^{\frac{1}{2}}\left(\left\|\nabla\left(c_\varepsilon^\pm - c_1^{\pm,\varepsilon}\right)\right\|_{[L^2(\Omega^\varepsilon)]^d}+1\right), \tag{4.29}$$

which is a direct result of (4.11) and of the fact that all the microscopic solutions are bounded from above uniformly in the choice of ε (see Theorem 3.4).

Using again Theorem 3.4, we estimate that

$$\int_{\Omega^\varepsilon} c_\varepsilon^\pm\left(|Y_l|^{-1}\mathbb{D}\nabla\tilde{\Phi}_0 - \nabla\tilde{\Phi}_\varepsilon\right)\cdot\nabla\varphi_3 dx$$
$$\le C\left(\left\|\nabla\left(\tilde{\Phi}_\varepsilon - \tilde{\Phi}_1^\varepsilon\right)\right\|_{[L^2(\Omega^\varepsilon)]^d} + \left\|\nabla\left(\tilde{\Phi}_1^\varepsilon - |Y_l|^{-1}\mathbb{D}\tilde{\Phi}_0\right)\right\|_{[L^2(\Omega^\varepsilon)]^d}\right)$$
$$\times\left(\left\|\nabla\left(c_\varepsilon^\pm - c_1^{\pm,\varepsilon}\right)\right\|_{[L^2(\Omega^\varepsilon)]^d}+1\right)$$
$$\le C\left(\left\|\nabla\left(\tilde{\Phi}_\varepsilon - \tilde{\Phi}_1^\varepsilon\right)\right\|_{[L^2(\Omega^\varepsilon)]^d}+\varepsilon\right)\left(\left\|\nabla\left(c_\varepsilon^\pm - c_1^{\pm,\varepsilon}\right)\right\|_{[L^2(\Omega^\varepsilon)]^d}+1\right), \tag{4.30}$$

which also completes the estimates for $\mathcal{K}_3$.

Combining (4.20), (4.21), (4.23), (4.24), (4.26), (4.28), (4.29) and (4.30), we obtain, after some rearrangements, that

$$\left\|\nabla\left(\tilde{\Phi}_\varepsilon - \tilde{\Phi}_1^\varepsilon\right)\right\|^2_{[L^2(\Omega^\varepsilon)]^d} + \varepsilon\left\|\nabla\left(c_\varepsilon^\pm - c_1^{\pm,\varepsilon}\right)\right\|^2_{[L^2(\Omega^\varepsilon)]^d}$$
$$\le C\left(\varepsilon^2+\varepsilon\right) + C\varepsilon^{\frac{3}{2}}\left(\left\|\nabla\left(c_\varepsilon^\pm - c_1^{\pm,\varepsilon}\right)\right\|_{[L^2(\Omega^\varepsilon)]^d}+1\right)$$
$$+ C\varepsilon\left(\left\|\tilde{\Phi}_\varepsilon - \tilde{\Phi}_0\right\|_{L^2(\Omega^\varepsilon)} + \left\|\nabla\left(\tilde{\Phi}_\varepsilon - \tilde{\Phi}_1^\varepsilon\right)\right\|_{[L^2(\Omega^\varepsilon)]^d}\right)$$
$$+ C\left\|c_\varepsilon^\pm - c_0^\pm\right\|_{L^2(\Omega^\varepsilon)}\left(\left\|\tilde{\Phi}_\varepsilon - \tilde{\Phi}_0\right\|_{L^2(\Omega^\varepsilon)}+1\right)$$
$$+ C\varepsilon\left(\left\|\nabla\left(\tilde{\Phi}_\varepsilon - \tilde{\Phi}_1^\varepsilon\right)\right\|_{[L^2(\Omega^\varepsilon)]^d}+\varepsilon\right)\left(\left\|\nabla\left(c_\varepsilon^\pm - c_1^{\pm,\varepsilon}\right)\right\|_{[L^2(\Omega^\varepsilon)]^d}+1\right)$$
$$+ C\varepsilon\left\|c_\varepsilon^\pm - c_0^\pm\right\|_{L^2(\Omega^\varepsilon)}\left(\left\|\nabla\left(c_\varepsilon^\pm - c_1^{\pm,\varepsilon}\right)\right\|_{[L^2(\Omega^\varepsilon)]^d}+1\right). \tag{4.31}$$

It now remains to estimate the second term on the right-hand side of (4.25). In fact, integrating the right-hand side of (4.25) by parts gives

$$\int_0^t\int_{\Omega^\varepsilon} m^\varepsilon\partial_t\left(c_\varepsilon^\pm - c_0^\pm\right)\bar{\varphi}\cdot\nabla_x c_0^\pm dxds = \int_{\Omega^\varepsilon} m^\varepsilon\left(c_\varepsilon^\pm - c_0^\pm\right)\bar{\varphi}\cdot\nabla_x c_0^\pm dx\Big|_{s=0}^{s=t}$$
$$-\int_0^t\int_{\Omega^\varepsilon} m^\varepsilon\left(c_\varepsilon^\pm - c_0^\pm\right)\bar{\varphi}\cdot\nabla_x\partial_t c_0^\pm dxds,$$

and we also have

$$\varepsilon\left|\int_{\Omega^\varepsilon} m^\varepsilon\left[\left(c_\varepsilon^\pm - c_0^\pm\right) - \left(c_\varepsilon^\pm(0) - c_0^\pm(0)\right)\right]\bar{\varphi}\cdot\nabla_x c_0^\pm dx\right|$$
$$\le C\varepsilon\left(\left\|c_\varepsilon^\pm - c_0^\pm\right\|_{L^2(\Omega^\varepsilon)} + \left\|c_\varepsilon^{\pm,0} - c_0^{\pm,0}\right\|_{L^2(\Omega^\varepsilon)}\right). \quad (4.32)$$

At this moment, if we set

$$w_1(t) = \left\|\tilde{\Phi}_\varepsilon(t) - \tilde{\Phi}_0(t)\right\|^2_{L^2(\Omega^\varepsilon)} + \left\|c_\varepsilon^\pm(t) - c_0^\pm(t)\right\|^2_{L^2(\Omega^\varepsilon)},$$
$$w_2(t) = \left\|\nabla\left(\tilde{\Phi}_\varepsilon - \tilde{\Phi}_1^\varepsilon\right)(t)\right\|^2_{[L^2(\Omega^\varepsilon)]^d} + \varepsilon\left\|\nabla\left(c_\varepsilon^\pm - c_1^{\pm,\varepsilon}\right)(t)\right\|^2_{[L^2(\Omega^\varepsilon)]^d},$$
$$w_0 = \left\|c_\varepsilon^{\pm,0} - c_0^{\pm,0}\right\|^2_{L^2(\Omega^\varepsilon)},$$

then, after integrating (4.31) and (4.25) from 0 to t, we are led to the following Gronwall-like estimate:

$$w_1(t) + \int_0^t w_2(s)\,ds \le C\left(\varepsilon + (1+\varepsilon)w_0 + \int_0^t w_1(s)\,ds\right),$$

which provides that

$$w_1(t) + \int_0^t w_2(s)\,ds \le C\left(\varepsilon + (1+\varepsilon)w_0\right) \quad \text{for } t\in[0,T].$$

Assuming

$$\left\|c_\varepsilon^{\pm,0} - c_0^{\pm,0}\right\|^2_{L^2(\Omega^\varepsilon)} \le C\varepsilon^\mu \quad \text{for some } \mu\in\mathbb{R}_+, \quad (4.33)$$

we thus obtain

$$\left\|\tilde{\Phi}_\varepsilon - \tilde{\Phi}_0\right\|^2_{L^2((0,T)\times\Omega^\varepsilon)} + \left\|c_\varepsilon^\pm - c_0^\pm\right\|^2_{L^2((0,T)\times\Omega^\varepsilon)} + \left\|\nabla\left(\tilde{\Phi}_\varepsilon - \tilde{\Phi}_1^\varepsilon\right)\right\|^2_{[L^2((0,T)\times\Omega^\varepsilon)]^d}$$
$$+\varepsilon\left\|\nabla\left(c_\varepsilon^\pm - c_1^{\pm,\varepsilon}\right)\right\|^2_{[L^2((0,T)\times\Omega^\varepsilon)]^d} \le C\max\{\varepsilon,\varepsilon^\mu\}. \quad (4.34)$$

Since the obtained estimate for $\left\|\nabla\left(\tilde{\Phi}_\varepsilon - \tilde{\Phi}_1^\varepsilon\right)\right\|_{[L^2((0,T)\times\Omega^\varepsilon)]^d}$ is of the order of $\mathcal{O}(\max\{\varepsilon,\varepsilon^\mu\})$, we can also increase the rate of $\left\|\nabla\left(c_\varepsilon^\pm - c_1^{\pm,\varepsilon}\right)\right\|_{[L^2((0,T)\times\Omega^\varepsilon)]^d}$. Indeed, let us consider the estimate (4.28) and (4.30) for $\left\|c_\varepsilon^\pm - c_0^\pm\right\|_{L^2((0,T)\times\Omega^\varepsilon)}$ and $\left\|\nabla\left(\tilde{\Phi}_\varepsilon - \tilde{\Phi}_1^\varepsilon\right)\right\|_{[L^2((0,T)\times\Omega^\varepsilon)]^d}$, respectively. Then, we combine again (4.21), (4.26), (4.28), (4.29), (4.30) and (4.32) to get another Gronwall-like estimate:

$$\left\|\nabla\left(c_\varepsilon^\pm - c_1^{\pm,\varepsilon}\right)(t)\right\|^2_{[L^2(\Omega^\varepsilon)]^d} \le C\left(\varepsilon^{\frac{1}{2}} + \max\{\varepsilon,\varepsilon^\mu\} + \varepsilon\int_0^t \left\|\nabla\left(c_\varepsilon^\pm - c_1^{\pm,\varepsilon}\right)(s)\right\|^2_{[L^2(\Omega^\varepsilon)]^d} ds\right).$$

As a result, we have

$$\left\|\nabla\left(c_\varepsilon^\pm - c_1^{\pm,\varepsilon}\right)\right\|^2_{[L^2((0,T)\times\Omega^\varepsilon)]^d} \le C\max\left\{\varepsilon^{\frac{1}{2}},\varepsilon^\mu\right\}. \quad (4.35)$$

Note that for $\gamma > \alpha$, the drift term in the macroscopic Nernst-Planck system is not present. Thus, this term does not appear in (4.28) and (4.30). Due to the *a priori* estimate that $\left\|\tilde{\Phi}_{\varepsilon}\right\|_{L^2(0,T;H^1(\Omega^{\varepsilon}))} \le C$ (cf. Theorem 3.5) in combination with the boundedness of $c_{\varepsilon}^{\pm}$ (cf. Theorem 3.4), it is straightforward to get the same corrector estimate as (4.34). Moreover, if $\alpha < 0$, the corrector becomes of the order $\mathcal{O}(\max\{\varepsilon, \varepsilon^{-\alpha}, \varepsilon^{\mu}\})$. This explicitly illustrates the effect of the scaling parameter α on the rate of the convergence.

For the time being, it only remains to come up with the corrector estimates for the Stokes equation. At this point, we must pay a regularity price[1] concerning the smoothness of the boundaries to make use of Lemma 3.2. With $\partial\Omega \in C^4$, we adapt the ideas of [36] to define the following velocity corrector:

$$\begin{aligned}\mathcal{V}^{\varepsilon,\delta}(t,x) := &-\sum_{j=1}^{d} w_j\left(\frac{x}{\varepsilon}\right)\left[\left(c_0^+ - c_0^-\right)\partial_{x_j}\tilde{\Phi}_0(t,x) + \partial_{x_j}p_0(t,x) + \left(\mathbb{K}^{-1}\eta^{\delta}\right)_j\right]\\ &-\varepsilon\sum_{i,j=1}^{d} r_{ij}\left(\frac{x}{\varepsilon}\right)(1-m^{\varepsilon})\partial_{x_i}\left(\left(c_0^+ - c_0^-\right)\partial_{x_j}\tilde{\Phi}_0(t,x) + \partial_{x_j}p_0(t,x) + \left(\mathbb{K}^{-1}\eta^{\delta}\right)_j\right),\end{aligned} \tag{4.36}$$

and the pressure corrector:

$$\begin{aligned}\mathcal{P}^{\varepsilon,\delta}(t,x) := &\, p_0(t,x)\\ &-\varepsilon\sum_{j=1}^{d}\pi_j\left(\frac{x}{\varepsilon}\right)\left[\left(c_0^+ - c_0^-\right)\partial_{x_j}\tilde{\Phi}_0(t,x) + \partial_{x_j}p_0(t,x) + \left(\mathbb{K}^{-1}\eta^{\delta}\right)_j\right],\end{aligned} \tag{4.37}$$

where w_j, π_j and r_{ij} are solutions of the problems (3.5) and (3.15), respectively, for $1 \le i,j \le d$; and η^{δ} is a function defined in Lemma 3.2.

From (4.36), one can structure the divergence of the corrector $\mathcal{V}^{\varepsilon,\delta}$. In fact, by definition of the function η^{δ} and the structure of the macroscopic system for the velocity in Theorem 3.7, the divergence of the first term of vanishes (4.36) itself. Therefore, one computes that

$$\begin{aligned}\nabla\cdot\mathcal{V}^{\varepsilon,\delta} = &-\sum_{i,j=1}^{d}\left(w_j^i\left(\frac{x}{\varepsilon}\right) - |Y_l|^{-1}K_{ij}\right)(1-m^{\varepsilon})\partial_{x_i}\left[\left(c_0^+ - c_0^-\right)\partial_{x_j}\tilde{\Phi}_0 + \partial_{x_j}p_0 + \left(\mathbb{K}^{-1}\eta^{\delta}\right)_j\right]\\ &-\varepsilon\sum_{i,j=1}^{d} r_{ij}\left(\frac{x}{\varepsilon}\right)(1-m^{\varepsilon})\nabla\cdot\left[\partial_{x_i}\left(\left(c_0^+ - c_0^-\right)\partial_{x_j}\tilde{\Phi}_0 + \partial_{x_j}p_0 + \left(\mathbb{K}^{-1}\eta^{\delta}\right)_j\right)\right]\\ &+\varepsilon\sum_{i,j=1}^{d} r_{ij}\left(\frac{x}{\varepsilon}\right)\nabla m^{\varepsilon}\partial_{x_i}\left[\left(c_0^+ - c_0^-\right)\partial_{x_j}\tilde{\Phi}_0 + \partial_{x_j}p_0 + \left(\mathbb{K}^{-1}\eta^{\delta}\right)_j\right],\end{aligned}$$

where we also use the structure of the cell problem (3.15).

Taking into account that

$$-\sum_{i,j=1}^{d} K_{ij}\partial_{x_i}\left(\left(c_0^{\pm} - c_0^-\right)\partial_{x_j}\tilde{\Phi}_0 + \partial_{x_j}p_0\right) = 0,$$

[1]Compare to the two-scale convergence method when deriving the structure of the macroscopic system in [29].

$$\sum_{i,j=1}^{d} K_{ij}\partial_{x_i}\left(\mathbb{K}^{-1}\eta^{\delta}\right)_j = 0,$$

hold (see again the macroscopic system for the velocity in Theorem 3.7 as well as the properties of η^{δ} in Lemma 3.2), the estimate for the divergence of $\mathcal{V}^{\varepsilon,\delta}$ in L^2-norm

$$\left\|\nabla\cdot\mathcal{V}^{\varepsilon,\delta}\right\|_{L^2(\Omega^{\varepsilon})} \le C\left(\varepsilon^{\frac{1}{2}}\delta^{-1} + \varepsilon\delta^{-\frac{3}{2}} + \varepsilon^{\frac{1}{q}}\delta^{-\frac{1}{2}-\frac{1}{q}}\right) \quad \text{for } q\in[2,\infty],$$

is directly obtained from Lemma 3.2 and the inequalities in (3.19).

At this stage, if we choose $q=2$ and $\delta \gg \varepsilon$, we get

$$\left\|\nabla\cdot\mathcal{V}^{\varepsilon,\delta}\right\|_{L^2(\Omega^{\varepsilon})} \le C\left(\varepsilon\delta^{-\frac{3}{2}} + \varepsilon^{\frac{1}{2}}\delta^{-1}\right), \tag{4.38}$$

and hence,

$$\left\|\nabla\cdot\mathcal{V}^{\varepsilon,\delta}\right\|_{L^2((0,T)\times\Omega^{\varepsilon})} \le C\left(\varepsilon\delta^{-\frac{3}{2}} + \varepsilon^{\frac{1}{2}}\delta^{-1}\right).$$

Next, we introduce the following function:

$$\Psi^{\varepsilon}(t,x) := \Delta\mathcal{V}^{\varepsilon,\delta}(t,x) - \varepsilon^{-2}\nabla\mathcal{P}^{\varepsilon,\delta} - \left(c_0^{+}(t,x) - c_0^{-}(t,x)\right)\nabla\tilde{\Phi}_0(t,x).$$

Thus, for any $\varphi_1 \in \left[H_0^1(\Omega^{\varepsilon})\right]^d$ we have, after direct computations, that

$$\begin{aligned}
&\left\langle \Psi^{\varepsilon},\varphi_1\right\rangle_{([H^1]^d)',[H^1]^d} \qquad (4.39)\\
&= -\sum_{j=1}^{d}\int_{\Omega^{\varepsilon}}\left(\Delta w_j\left(\frac{x}{\varepsilon}\right) - \varepsilon^{-1}\nabla\pi_j\left(\frac{x}{\varepsilon}\right)\right)\left[\left(c_0^{+}-c_0^{-}\right)\partial_{x_j}\tilde{\Phi}_0 + \partial_{x_j}p_0 + \left(\mathbb{K}^{-1}\eta^{\delta}\right)_j\right]\varphi_1 dx\\
&\quad -\varepsilon^{-2}\int_{\Omega^{\varepsilon}}\left(\nabla p_0 + \left(c_0^{+}-c_0^{-}\right)\nabla\tilde{\Phi}_0\right)\varphi_1 dx\\
&\quad -\sum_{j=1}^{d}\int_{\Omega^{\varepsilon}}\left(2\nabla w_j\left(\frac{x}{\varepsilon}\right) - \varepsilon^{-1}\pi\left(\frac{x}{\varepsilon}\right)\mathbb{I}\right)\nabla\left[\left(c_0^{+}-c_0^{-}\right)\partial_{x_j}\tilde{\Phi}_0 + \partial_{x_j}p_0 + \left(\mathbb{K}^{-1}\eta^{\delta}\right)_j\right]\varphi_1 dx\\
&\quad -\sum_{j=1}^{d}\int_{\Omega^{\varepsilon}} w_j\left(\frac{x}{\varepsilon}\right)\Delta\left[\left(c_0^{+}-c_0^{-}\right)\partial_{x_j}\tilde{\Phi}_0 + \partial_{x_j}p_0 + \left(\mathbb{K}^{-1}\eta^{\delta}\right)_j\right]\varphi_1 dx\\
&\quad -\varepsilon\sum_{j=1}^{d}\int_{\Omega^{\varepsilon}}\nabla\left[r_{ij}\left(\frac{x}{\varepsilon}\right)(1-m^{\varepsilon})\partial_{x_j}\left(\left(c_0^{+}-c_0^{-}\right)\partial_{x_j}\tilde{\Phi}_0 + \partial_{x_j}p_0 + \left(\mathbb{K}^{-1}\eta^{\delta}\right)_j\right)\right]\cdot\nabla\varphi_1 dx\\
&:= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5. \qquad (4.40)
\end{aligned}$$

Note that $\mathbb{I}$ here stands for the identity matrix. From now on, to get the estimate for Ψ^{ε} in $\left(H^1\right)'$-norm, we need bounds on $\mathcal{I}_i$ for $1\le i\le 5$. Indeed, with the help of Lemma 3.3 applied to the test function φ_1, and the estimates of the involved functions, one immediately obtains from the Hölder's inequality that

$$|\mathcal{I}_3| + |\mathcal{I}_4| \le C\left(\delta^{-\frac{1}{2}} + \varepsilon\delta^{-\frac{3}{2}}\right)\|\nabla\varphi_1\|_{[L^2(\Omega^{\varepsilon})]^d}, \tag{4.41}$$

where we also apply again the estimate of η^{δ} in Lemma 3.2.

To estimate $\mathcal{I}_5$, we notice

$$|\mathcal{I}_5| \le C\left(\delta^{-\frac{1}{2}} + \varepsilon\delta^{-\frac{3}{2}}\right) \|\nabla\varphi_1\|_{[L^2(\Omega^\varepsilon)]^d}, \tag{4.42}$$

where we also employ the estimates (3.19) on m^ε.

In addition, we have

$$\begin{aligned} |\mathcal{I}_1 + \mathcal{I}_2| \le & \left| \int_{\Omega^\varepsilon} \varepsilon^{-2} \left[-\sum_{j=1}^{d} \left(\left(c_0^+ - c_0^-\right) \partial_{x_j} \tilde{\Phi}_0 + \partial_{x_j} p_0 + \left(\mathbb{K}^{-1}\eta^\delta\right)_j \right) \right. \right. \\ & \left. \left. + \nabla p_0 + \left(c_0^+ - c_0^-\right) \nabla\tilde{\Phi}_0 \right] \varphi_1 dx \right| \\ & \le C\varepsilon^{-1}\delta^{\frac{1}{2}} \|\nabla\varphi_1\|_{[L^2(\Omega^\varepsilon)]^d}. \end{aligned} \tag{4.43}$$

Consequently, collecting (4.40)-(4.43) and according to the definition of the $\left(H^1\right)'$-norm, we arrive at

$$\begin{aligned} \|\Psi^\varepsilon\|_{\left([H^1(\Omega^\varepsilon)]^d\right)'} = & \sup_{\varphi_1 \in [H^1(\Omega^\varepsilon)]^d, \|\varphi_1\|_{[H^1(\Omega^\varepsilon)]^d} \le 1} \langle \Psi^\varepsilon, \varphi_1 \rangle_{\left([H^1]^d\right)', [H^1]^d} \\ & \le C\left(\varepsilon^{-1}\delta^{\frac{1}{2}} + \delta^{-\frac{1}{2}} + \varepsilon\delta^{-\frac{3}{2}}\right) \|\nabla\varphi_1\|_{[L^2(\Omega^\varepsilon)]^d}. \end{aligned} \tag{4.44}$$

Now, we have available a couple of estimates related to the correctors $\mathcal{V}^{\varepsilon,\delta}$ and $\mathcal{P}^{\varepsilon,\delta}$. To go on, we consider the differences

$$\mathcal{D}_1^\varepsilon := v_\varepsilon - |Y_l|^{-1} \mathbb{D}\mathcal{V}^{\varepsilon,\delta}, \quad \mathcal{D}_2^\varepsilon := p_\varepsilon - |Y_l|^{-1} \mathbb{D}\mathcal{P}^{\varepsilon,\delta},$$

and observe that the equation

$$-\varepsilon^2 \Delta\mathcal{D}_1^\varepsilon + \nabla\mathcal{D}_2^\varepsilon = \varepsilon^2 \left[|Y_l|^{-1} \mathbb{D}\Psi^\varepsilon - \varepsilon^{-2} \left(\left(c_\varepsilon^+ - c_\varepsilon^-\right) \nabla\tilde{\Phi}_\varepsilon - \left(c_0^+ - c_0^-\right) |Y_l|^{-1} \mathbb{D}\nabla\tilde{\Phi}_0 \right) \right] \tag{4.45}$$

holds a.e. in Ω^ε.

It remains to estimate the second term on the right-hand side of the equation (4.48) in $\left(H^1\right)'$-norm. This estimate fully relies on the corrector estimate for the electrostatic potentials in (4.34), the boundedness of concentration fields in Theorem 3.4 with the assumption that $c_0^\pm \in W^{1,\infty}(\Omega^\varepsilon) \cap H^2(\Omega^\varepsilon)$. In fact, the estimate resembles very much the one in (4.30), viz.

$$\begin{aligned} \left\langle \left(c_\varepsilon^+ - c_\varepsilon^-\right) \nabla\tilde{\Phi}_\varepsilon - \left(c_0^+ - c_0^-\right) |Y_l|^{-1} \mathbb{D}\nabla\tilde{\Phi}_0, \varphi_1 \right\rangle_{\left([H^1]^d\right)', [H^1]^d} & \\ \le C \left\| \nabla\tilde{\Phi}_\varepsilon - |Y_l|^{-1} \mathbb{D}\nabla\tilde{\Phi}_0 \right\|_{[L^2(\Omega^\varepsilon)]^d} \|\varphi_1\|_{L^2(\Omega^\varepsilon)} & \\ \le C \max\left\{ \varepsilon^{\frac{3}{2}}, \varepsilon^{\frac{\mu}{2}+1} \right\} \|\nabla\varphi_1\|_{[L^2(\Omega^\varepsilon)]^d}, & \end{aligned} \tag{4.46}$$

for all $\varphi_1 \in \left[H_0^1(\Omega^\varepsilon)\right]^d$ and where we also use Lemma 3.3.

For ease of presentation, we put

$$\mathcal{L}^\varepsilon := \varepsilon^{-2} \left(\left(c_\varepsilon^+ - c_\varepsilon^-\right) \nabla\tilde{\Phi}_\varepsilon - \left(c_0^+ - c_0^-\right) |Y_l|^{-1} \mathbb{D}\nabla\tilde{\Phi}_0 \right).$$

The corrector for the pressure can be obtained by the use of the following results which are deduced from [38] and [36]:

- there exists an extension $E(\mathcal{D}_2^\varepsilon)\in L^2(\Omega)/\mathbb{R}$ of $\mathcal{D}_2^\varepsilon$ such that

$$\|E(\mathcal{D}_2^\varepsilon)\|_{L^2(\Omega)/\mathbb{R}}\le C\varepsilon\left(\|\Psi^\varepsilon-\mathcal{L}^\varepsilon\|_{\left([H^1(\Omega^\varepsilon)]^d\right)'}+\|\nabla\mathcal{D}_1^\varepsilon\|_{[L^2(\Omega^\varepsilon)]^{d^2}}\right), \tag{4.47}$$

- the following estimates hold:

$$\|\nabla\mathcal{D}_1^\varepsilon\|_{[L^2(\Omega^\varepsilon)]^{d^2}}\le C\left(\|\Psi^\varepsilon-\mathcal{L}^\varepsilon\|_{\left([H^1(\Omega^\varepsilon)]^d\right)'}+\varepsilon^{-1}\left\|\nabla\cdot\mathcal{V}^{\varepsilon,\delta}\right\|_{L^2(\Omega^\varepsilon)}\right), \tag{4.48}$$

$$\|\mathcal{D}_1^\varepsilon\|_{[L^2(\Omega^\varepsilon)]^d}\le C\left(\varepsilon\|\Psi^\varepsilon-\mathcal{L}^\varepsilon\|_{\left([H^1(\Omega^\varepsilon)]^d\right)'}+\left\|\nabla\cdot\mathcal{V}^{\varepsilon,\delta}\right\|_{L^2(\Omega^\varepsilon)}\right). \tag{4.49}$$

Collecting (4.44) and (4.46), we get

$$\|\Psi^\varepsilon-\mathcal{L}^\varepsilon\|_{\left([H^1(\Omega^\varepsilon)]^d\right)'}\le C\left(\varepsilon^{-1}\delta^{\frac{1}{2}}+\delta^{-\frac{1}{2}}+\varepsilon\delta^{-\frac{3}{2}}+\max\left\{\varepsilon^{-\frac{1}{2}},\varepsilon^{\frac{\mu}{2}-1}\right\}\right)\|\nabla\varphi_1\|_{[L^2(\Omega^\varepsilon)]^d}. \tag{4.50}$$

We thus observe from (4.49), (4.38) and (4.50) that

$$\|\mathcal{D}_1^\varepsilon\|_{[L^2(\Omega^\varepsilon)]^d}\le C\left(\delta^{\frac{1}{2}}+\varepsilon\delta^{-\frac{1}{2}}+\varepsilon^2\delta^{-\frac{3}{2}}+\max\left\{\varepsilon^{\frac{1}{2}},\varepsilon^{\frac{\mu}{2}}\right\}+\varepsilon\delta^{-\frac{3}{2}}+\varepsilon^{\frac{1}{2}}\delta^{-1}\right).$$

Since $\delta\gg\varepsilon$, we can take $\delta=\varepsilon^\lambda$ for $\lambda\in(0,1)$ to obtain

$$\begin{aligned}\|\mathcal{D}_1^\varepsilon\|_{[L^2(\Omega^\varepsilon)]^d}&\le C\left(\varepsilon^{\frac{\lambda}{2}}+\varepsilon^{1-\frac{\lambda}{2}}+\varepsilon^{2-\frac{3\lambda}{2}}+\varepsilon^{1-\frac{3\lambda}{2}}+\varepsilon^{\frac{1}{2}-\lambda}+\max\left\{\varepsilon^{\frac{1}{2}},\varepsilon^{\frac{\mu}{2}}\right\}\right)\\&\le C\left(\max\left\{\varepsilon^{\frac{1}{2}},\varepsilon^{\frac{\mu}{2}}\right\}+\varepsilon^{\frac{\lambda}{2}}+\varepsilon^{1-\frac{3\lambda}{2}}+\varepsilon^{\frac{1}{2}-\lambda}\right).\end{aligned}$$

On the other hand, the optimal value for λ is $1/3$ which leads to the following estimate:

$$\|\mathcal{D}_1^\varepsilon\|_{[L^2(\Omega^\varepsilon)]^d}\le C\max\left\{\varepsilon^{\frac{1}{6}},\varepsilon^{\frac{\mu}{2}}\right\}. \tag{4.51}$$

Hereafter, it follows from (4.51), (4.47), (4.48) and (4.50) that

$$\begin{aligned}\|E(\mathcal{D}_2^\varepsilon)\|_{L^2(\Omega)/\mathbb{R}}&\le C\left(\varepsilon\|\Psi^\varepsilon-\mathcal{L}^\varepsilon\|_{\left([H^1(\Omega^\varepsilon)]^d\right)'}+\left\|\nabla\cdot\mathcal{V}^{\varepsilon,\delta}\right\|_{[L^2(\Omega^\varepsilon)]^{d^2}}\right)\\&\le C\left(\max\left\{\varepsilon^{\frac{1}{2}},\varepsilon^{\frac{\mu}{2}}\right\}+\varepsilon^{\frac{\lambda}{2}}+\varepsilon^{1-\frac{3\lambda}{2}}+\varepsilon^{\frac{1}{2}-\lambda}\right).\end{aligned}$$

This indicates the following estimate:

$$\|p_\varepsilon-p_0\|_{L^2(\Omega)/\mathbb{R}}\le C\left(\max\left\{\varepsilon^{\frac{1}{2}},\varepsilon^{\frac{\mu}{2}}\right\}+\varepsilon^{\frac{\lambda}{2}}+\varepsilon^{1-\frac{3\lambda}{2}}+\varepsilon^{\frac{1}{2}-\lambda}\right). \tag{4.52}$$

Finally, we gather (4.11), (4.34), (4.35), (4.51) and (4.52) to conclude the proof of Theorem 4.1.

4.3. Proof of Theorem 4.2

We turn the attention to the Dirichlet boundary condition for the electrostatic potential on the micro-surface. Based on Theorem 3.11, we observe that the structure of the macroscopic systems for the Stokes and Nernst-Planck equations are the same as the corresponding systems in the Neumann case (see Theorem 3.7). Therefore, the corrector estimates for these systems remain unchanged in Theorem 4.1. Also, some regularity properties are not needed in this case. We derive first the corrector estimates for the velocity and pressure and then the corrector estimates of the concentration fields.

Thereby, the corrector for the electrostatic potential can also be obtained. Here, the macroscopic reconstructions are defined as follows:

$$v_0^\varepsilon(t,x) := v_0\left(t,x,\frac{x}{\varepsilon}\right), \tag{4.53}$$

$$v_1^\varepsilon(t,x) := v_1\left(t,x,\frac{x}{\varepsilon}\right), \tag{4.54}$$

$$c_0^{\pm,\varepsilon}(t,x) := c_0^{\pm}(t,x), \tag{4.55}$$

$$c_1^{\pm,\varepsilon}(t,x) := c_0^{\pm,\varepsilon}(t,x) + \varepsilon\sum_{j=1}^{d}\varphi_j\left(\frac{x}{\varepsilon}\right)\partial_{x_j}c_0^{\pm,\varepsilon}(t,x). \tag{4.56}$$

Recall $\tilde{\Phi}_\varepsilon := \varepsilon^{\alpha-2}\Phi_\varepsilon^{\text{hom}}$. By Theorem 3.10, $\tilde{\Phi}_\varepsilon$ obeys the weak formulation

$$\int_{\Omega^\varepsilon}\varepsilon^2\nabla\tilde{\Phi}_\varepsilon\cdot\nabla\varphi_2 dx = \int_{\Omega^\varepsilon}\left(c_\varepsilon^+ - c_\varepsilon^-\right)\varphi_2 dx \quad \text{for all } \varphi_2 \in H_0^1(\Omega^\varepsilon).$$

Therefore, we define the following macroscopic reconstructions:

$$\tilde{\Phi}_0^\varepsilon(t,x) := \tilde{\Phi}_0\left(t,x,\frac{x}{\varepsilon}\right), \tag{4.57}$$

$$\overline{\tilde{\Phi}}_0^\varepsilon(t,x) := |Y_l|^{-1}\overline{\tilde{\Phi}}_0(t,x), \tag{4.58}$$

and recall that the strong formulation for $\tilde{\Phi}_0$ (see [29, Theorem 4.12]) is given by

$$\begin{aligned}
&-\Delta_y\tilde{\Phi}_0(t,x,y) = c_0^{\pm}(t,x) - c_0^-(t,x) \text{ in } (0,T)\times\Omega\times Y_l,\\
&\tilde{\Phi}_0 = 0 \text{ in } (0,T)\times\Omega\times\Gamma.
\end{aligned}$$

Consequently, the difference equation for the Poisson equation can be written as

$$-\varepsilon^2\Delta\tilde{\Phi}_\varepsilon + \left(\Delta_y\tilde{\Phi}_0\right)^\varepsilon = \left(c_\varepsilon^+ - c_0^+\right) + \left(c_0^- - c_\varepsilon^-\right).$$

Choosing the test function $\varphi_2 = \tilde{\Phi}_\varepsilon - \tilde{\Phi}_0^\varepsilon$, let us now estimate the following integral:

$$\int_{\Omega^\varepsilon}\left(\Delta_y\tilde{\Phi}_0\right)^\varepsilon\varphi_2 dx.$$

Using the simple relation $\nabla_y = \varepsilon(\nabla - \nabla_x)$ and the decomposition

$$\left(\Delta_y\tilde{\Phi}_0\right)^\varepsilon = (1-m^\varepsilon)\left(\Delta_y\tilde{\Phi}_0\right)^\varepsilon + \varepsilon m^\varepsilon\nabla\cdot\left(\nabla_y\left(\tilde{\Phi}_0\right)^\varepsilon\right) - \varepsilon m^\varepsilon\left(\nabla_x\cdot\left(\nabla_y\tilde{\Phi}_0\right)\right)^\varepsilon,$$

and we obtain, after integrating by parts the term $\nabla\cdot\left(\nabla_y\left(\tilde{\Phi}_0\right)^\varepsilon\right)$, that

$$\begin{aligned}
\int_{\Omega^\varepsilon}\left(\Delta_y\tilde{\Phi}_0\right)^\varepsilon\varphi_2 dx = \int_{\Omega^\varepsilon}&\Big[(1-m^\varepsilon)\left(\Delta_y\tilde{\Phi}_0\right)^\varepsilon\\
&- \varepsilon m^\varepsilon\left(\nabla_x\cdot\left(\nabla_y\tilde{\Phi}_0\right)\right)^\varepsilon - \varepsilon\nabla m^\varepsilon\cdot\nabla_y\left(\tilde{\Phi}_0\right)^\varepsilon\Big]\varphi_2 dx\\
&+ \varepsilon\int_{\Omega^\varepsilon}(1-m^\varepsilon)\nabla_y\left(\tilde{\Phi}_0\right)^\varepsilon\cdot\nabla\varphi_2 dx - \varepsilon\int_{\Omega^\varepsilon}\nabla_y\left(\tilde{\Phi}_0\right)^\varepsilon\cdot\nabla\varphi_2 dx\\
&:= \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3.
\end{aligned} \tag{4.59}$$

The first and second integrals on the right-hand side of (4.59) can be estimated by

$$\begin{aligned}|\mathcal{F}_1|+|\mathcal{F}_2| \le C\Big(&\|1-m^\varepsilon\|_{L^2(\Omega^\varepsilon)}\left\|\Delta_y\tilde{\Phi}_0\right\|_{L^\infty(\Omega^\varepsilon;C(Y_l))}\\ &+\varepsilon\left\|\nabla_x\cdot\left(\nabla_y\tilde{\Phi}_0\right)\right\|_{L^2(\Omega^\varepsilon;C(Y_l))}\Big)\|\varphi_2\|_{L^2(\Omega^\varepsilon)}\\ &+C\varepsilon\|\nabla m^\varepsilon\|_{L^2(\Omega^\varepsilon)}\left\|\nabla_y\tilde{\Phi}_0\right\|_{L^\infty(\Omega^\varepsilon;C(Y_l))}\|\varphi_2\|_{L^2(\Omega^\varepsilon)}\\ &+C\varepsilon\|1-m^\varepsilon\|_{L^2(\Omega^\varepsilon)}\left\|\nabla_y\tilde{\Phi}_0\right\|_{L^\infty(\Omega^\varepsilon;C(Y_l))}\|\nabla\varphi_2\|_{L^2(\Omega^\varepsilon)},\end{aligned}$$

where we assume that $\tilde{\Phi}_0\in L^\infty\left(\Omega^\varepsilon;W^{2+s,2}(Y_l)\right)\cap H^1\left(\Omega^\varepsilon;W^{1+s,2}(Y_l)\right)$ and make use of the compact embeddings $W^{2+s,2}(Y_l)\subset C^2(Y_l)$, $W^{1+s,2}(Y_l)\subset C^1(Y_l)$ for $s>d/2$. Applying the inequalities (3.19), we thus have

$$|\mathcal{F}_1|+|\mathcal{F}_2|\le C\left(\varepsilon+\varepsilon^{\frac{1}{2}}\right)\|\varphi_2\|_{L^2(\Omega^\varepsilon)}+C\varepsilon^{\frac{3}{2}}\|\nabla\varphi_2\|_{L^2(\Omega^\varepsilon)}. \tag{4.60}$$

It now remains to estimate the following integral:

$$\int_{\Omega^\varepsilon}\varepsilon^2\nabla\tilde{\Phi}_\varepsilon\cdot\nabla\varphi_2dx=\int_{\Omega^\varepsilon}\varepsilon\nabla\tilde{\Phi}_\varepsilon\cdot\varepsilon\nabla\left(\tilde{\Phi}_\varepsilon-\tilde{\Phi}_0^\varepsilon\right)dx.$$

Its right-hand side can be estimated by

$$\int_{\Omega^\varepsilon}\varepsilon\nabla\tilde{\Phi}_\varepsilon\cdot\varepsilon\nabla\left(\tilde{\Phi}_\varepsilon-\tilde{\Phi}_0^\varepsilon\right)dx\le C\varepsilon\left\|\nabla\left(\tilde{\Phi}_\varepsilon-\tilde{\Phi}_0^\varepsilon\right)\right\|_{[L^2(\Omega^\varepsilon)]^d}, \tag{4.61}$$

where we use the fact that $\varepsilon\left\|\nabla\tilde{\Phi}_\varepsilon\right\|_{L^2(\Omega^\varepsilon)}\le C$ in Theorem 3.9.

Based on the corrector estimates for the concentration fields $c_\varepsilon^\pm$, we see that

$$\int_{\Omega^\varepsilon}\left[\left(c_\varepsilon^+-c_0^+\right)+\left(c_0^--c_\varepsilon^-\right)\right]\varphi_2dx\le C\left\|c_\varepsilon^\pm-c_0^\pm\right\|_{L^2(\Omega^\varepsilon)}\left\|\tilde{\Phi}_\varepsilon-\tilde{\Phi}_0^\varepsilon\right\|_{L^2(\Omega^\varepsilon)}. \tag{4.62}$$

Setting

$$\begin{aligned}w_1(t)&:=\left\|\tilde{\Phi}_\varepsilon(t)-\tilde{\Phi}_0^\varepsilon(t)\right\|^2_{L^2(\Omega^\varepsilon)}+\left\|c_\varepsilon^\pm(t)-c_0^\pm(t)\right\|^2_{L^2(\Omega^\varepsilon)},\\ w_2(t)&:=\left\|\nabla\left(\tilde{\Phi}_\varepsilon-\tilde{\Phi}_0^\varepsilon\right)(t)\right\|^2_{[L^2(\Omega^\varepsilon)]^d}+\left\|\nabla\left(c_\varepsilon^\pm-c_1^{\pm,\varepsilon}\right)(t)\right\|^2_{[L^2(\Omega^\varepsilon)]^d},\\ w_0&:=\left\|c_\varepsilon^{\pm,0}-c_0^{\pm,0}\right\|^2_{L^2(\Omega^\varepsilon)},\end{aligned}$$

the combination of the estimates (4.60)-(4.62) with the respective estimates for the concentration fields (which are similar to the Neumann case) and the application of suitable Hölder-like inequalities give

$$w_1(t)+\int_0^t w_2(s)ds\le C\left(\varepsilon+(1+\varepsilon)w_0+\int_0^t w_1(s)ds\right).$$

Using Gronwall's inequality yields

$$w_1(t)+\int_0^t w_2(s)ds\le C(\varepsilon+(1+\varepsilon)w_0).$$

As a consequence, we obtain

$$\begin{aligned}&\left\|\tilde{\Phi}_{\varepsilon}-\tilde{\Phi}_{0}^{\varepsilon}\right\|_{L^{2}((0,T)\times\Omega^{\varepsilon})}+\left\|\nabla\left(\tilde{\Phi}_{\varepsilon}-\tilde{\Phi}_{0}^{\varepsilon}\right)\right\|_{[L^{2}((0,T)\times\Omega^{\varepsilon})]^{d}}\\&+\left\|c_{\varepsilon}^{\pm}-c_{0}^{\pm,\varepsilon}\right\|_{L^{2}((0,T)\times\Omega^{\varepsilon})}+\left\|\nabla\left(c_{\varepsilon}^{\pm}-c_{1}^{\pm,\varepsilon}\right)\right\|_{[L^{2}((0,T)\times\Omega^{\varepsilon})]^{d}}\le C\max\left\{\varepsilon^{\frac{1}{2}},\varepsilon^{\frac{\mu}{2}}\right\}\text{ for }\mu\in\mathbb{R}_{+},\end{aligned}$$

where we have used (4.33).

Finally, we apply Lemma 3.1 to get

$$\begin{aligned}\left\|\tilde{\Phi}_{\varepsilon}-\overline{\tilde{\Phi}}_{0}^{\varepsilon}\right\|_{L^{2}((0,T)\times\Omega^{\varepsilon})}&\le\left\|\tilde{\Phi}_{\varepsilon}-\tilde{\Phi}_{0}^{\varepsilon}\right\|_{L^{2}((0,T)\times\Omega^{\varepsilon})}+\left\|\tilde{\Phi}_{0}^{\varepsilon}-\overline{\tilde{\Phi}}_{0}^{\varepsilon}\right\|_{L^{2}((0,T)\times\Omega^{\varepsilon})}\\&\le C\max\left\{\varepsilon^{\frac{1}{2}},\varepsilon^{\frac{\mu}{2}}\right\}.\end{aligned}$$

This completes the proof of Theorem 4.2.

5. Conclusions

In [29], the two-scale convergence method has discovered possible macroscopic structures of a non-stationary SNPP model coupled with various scaling factors and different boundary conditions. In this paper, we have justified such homogenization limits by deriving several corrector estimates (cf. Theorem 4.1 and Theorem 4.2). Although we always rely on the ε-independent constant C in every step of proofs, it is worth noting that such corrector estimates are exponentially controlled with respect to time due to the aid of the Gronwall argument. The techniques we have presented here are mainly based on the construction of suitable macroscopic reconstructions and on a number of energy-like estimates. The employed methodology is applicable to more complex scenarios, where coupled systems of partial differential equations posed in perforated media are involved. As indicated in [5], a quite interesting aspect meriting additional mathematical study is the coupling of the electrochemical dynamics to the background fluid, targeting applications concerning microfluidic devices. In such contexts, one may also wonder whether a super fast electrophoresis is actually possible, while attempting to devise validity regimes for an eventual dilute solution approximation.

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REFERENCES

[1] S. Agmon, A. Douglis, and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary value conditions I. *Communications on Pure and Applied Mathematics*, 12:623–727, 1959. 1

[2] G. Allaire. Homogenization and two-scale convergence. *SIAM Journal on Mathematical Analysis*, 23:1482–1518, 1992. 3.1

[3] G. Allaire, A. Mikelíc, and A. Piatnitski. Homogenization of the linearized ionic transport equations in rigid periodic porous media. *Journal of Mathematical Physics*, 51(12), 2010. 123103. 1

[4] D. Cioră nescu and J. Saint Jean Paulin. *Homogenization of Reticulated Structures*. Springer, 1999. 1, 3.2

[5] M. Z. Bazant, K. Thornton, and A. Ajdari. Diffuse-charge dynamics in electrochemical systems. *Physical Review E*, 70:021506, 2004. 1, 5

[6] D. Boda and D. Gillespie. Steady-state electrodiffusion from the Nernst-Planck equation coupled to local equilibrium Monte Carlo simulations. *Journal of Chemical Theory and Computation*, 8(3):824–829, 2012. 1

[7] C.J. Brinker and G.W. Scherer. *Sol-Gel Science: The Physics and Chemistry of Sol-Gel Processing*. Elsevier, 1990. 1

[8] T.G.M. Van de Ven. *Colloidal Hydrodynamics*. Academic Press, 1989. 1

[9] D.J. Earl. Modeling microscopic swimmers at low Reynolds number. *The Journal of Chemical Physics*, 126, 2007. 064703. 1

[10] C. Eck. Homogenization of a phase field model for binary mixtures. *Multiscale Modeling and Simulation*, 3:1–27, 2004. 4

[11] M. Elimelech, J. Gregory, X. Jia, and R. Williams. *Particle Deposition & Aggregation: Measurement, Modelling and Simulation*. Elsevier, 1998. 1

[12] K. Fellner and V.A. Kovtunenko. A discontinuous Poisson-Boltzmann equation with interfacial jump: homogenisation and residual error estimate. *Applicable Analysis*, 95(12):2661–2682, 2016. 1

[13] F. Frank. *Numerical Studies of Models for Electrokinetic Flow and Charged Solute Transport in Periodic Porous Media*. PhD thesis, University of Erlangen-Nuremberg, 2013. 1

[14] F. Frank, N. Ray, and P. Knabner. Numerical investigation of homogenized Stokes-Nernst-Planck-Poisson systems. *Computing and Visualization in Science*, 14(8):385–400, 2011. 1

[15] G. Gagneux and O. Millet. Homogenization of the Nernst-Planck-Poisson system by two-scale convergence. *Journal of Elasticity*, 114(1):69–84, 2014. 1

[16] G. Gagneux and O. Millet. A survey on properties of Nernst-Planck-Poisson system. Application to ionic transport in porous media. *Applied Mathematical Modelling*, 40:846–858, 2016. 1

[17] D. Gilbarg and N. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, 1983. 3.5

[18] R. Greenwood and K. Kendall. Electroacoustic studies of moderately concentrated colloidal suspensions. *Journal of the European Ceramic Society*, 19(4):479–488, 1999. 1

[19] U. Hornung and W. Jäger. Diffusion, convection, adsorption, and reaction of chemicals in porous media. *Journal of Differential Equations*, 92:199–225, 1991. 2.1, 2.1

[20] T. Hou, X. H. Wu, and Z. Cai. Convergence of a multiscale finite element method for elliptic problems with rapidly oscillating coefficients. *Mathematics of Computation*, 68(227):913–943, 1999. 1

[21] V.A. Khoa. A high-order corrector estimate for a semi-linear elliptic system in perforated domains. *Comptes Rendus Mécanique*, 345:337–343, 2017. 1

[22] V.A. Khoa and A. Muntean. Asymptotic analysis of a semi-linear elliptic system in perforated domains: Well-posedness and corrector for the homogenization limit. *Journal of Mathematical Analysis and Applications*, 439:271–295, 2016. 1, 2.1

[23] V.A. Khoa and A. Muntean. Correctors justification for a Smoluchowski-Soret-Dufour model posed in perforated domains. arXiv:1704.01790, 2017. 1, 3.1, 4

[24] S. Mitragotri and J. Lahann. Physical approaches to biomaterial design. *Nature Materials*, 8:15–23, 2009. 1

[25] A. Muntean and T.L. van Noorden. Corrector estimates for the homogenization of a locally-periodic medium with areas of low and high diffusivity. *European Journal of Applied Mathematics*, 24(5):657–677, 2012. 1, 4, 4.2

[26] J. Newman and K.E. Thomas-Alyea. *Electrochemical Systems*. Wiley, 3rd edition, 2004. 1

[27] G. Nguetseng. A general convergence result for a functional related to the theory of homogenization. *SIAM Journal on Mathematical Analysis*, 20:608–623, 1989. 3.1

[28] N. Ray. *Colloidal Transport in Porous Media Modeling and Analysis*. PhD thesis, University of Erlangen-Nuremberg, 2013. 1, 3.1, 3.1, 3.1, 3.2, 3.2, 3.2, 3.3, 3.3, 3.4

[29] N. Ray, A. Muntean, and P. Knabner. Rigorous homogenization of a Stokes-Nernst-Planck-Poisson system. *Journal of Mathematical Analysis and Applications*, 390(1):374–393, 2012. (document), 1, 2.1, 2.1, 3.1, 3.1, 3.1, 3.2, 3.2, 3.3, 3.2, 3.1, 3.3, 3.3, 1, 4.3, 5

[30] E. Samson, J. Marchand, J.-L. Robert, and J.-P. Bournazel. Modelling ion diffusion mechanisms in porous media. *International Journal for Numerical Methods in Engineering*, 46(12):2043–2060, 1999. 2.1

[31] E. Sanchez-Palencia. *Non-Homogeneous Media and Vibration Theory*, volume 127. Springer, 1980. Lecture Notes in Physics. 3.2, 3.4, 3.3, 38

[32] G. Savaré. Regularity results for elliptic equations in Lipschitz domains. *Journal of Functional Analysis*, 152:176–201, 1998. 3.2

[33] M. Schmuck. Modeling and deriving porous media Stokes-Poisson-Nernst-Planck equations by a multiple-scale approach. *Communications in Mathematical Sciences*, 3(9):685–710, 2011. 1

[34] M. Schmuck. First error bounds for the porous media approximation of the Poisson-Nernst-Planck equations. *ZAMM Journal of applied mathematics and mechanics: Zeitschrift für angewandte*, 92(4):304–319, 2012. 1

[35] M. Schmuck and M.Z. Bazant. Homogenization of the Poisson-Nernst-Planck equations for ion transport in charged porous media. *SIAM Journal on Applied Mathematics*, 75(3):1369–1401, 2015. 1, 2.1

[36] E. Marušić-Paloka and A. Mikelíc. An error estimate for correctors in the homogenization of the Stokes and the Navier-Stokes equations in a porous medium. *Bollettino dell'Unione Matematica Italiana*, 10(3):661–671, 1996. 3.4, 3.5, 4.2, 4.2

[37] G.-Q. Tang and N.R. Morrow. Influence of brine composition and fines migration on crude oil/brine/rock interactions and oil recovery. *Journal of Petroleum Science & Engineering*, 24(2–4):99–111, 1999. 1

[38] L. Tartar. *Convergence of the Homogenization Process*, volume 127. Springer, Berlin, 1980. In: Appendix of [31]. 4.2

[39] R. Temam. *Navier–Stokes Equations*, volume 2 of *Studies in Mathematics and Its Applications*. North Holland, revised edition, 1979. 3.5

[40] C. Zhang, Y. Bai, S. Xu, and X. Yue. Homogenization for chemical vapor infiltration process. *Communications in Mathematical Sciences*, 15(4):1021–1040, 2017. 1