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Non Peer-reviewed author version

ANH-KHOA, Vo; Le Thi Phuong Ngoc & Nguyen Thanh Long (2019) Existence, blow-up and exponential decay of solutions for a porous-elastic system with damping and source terms. In: EVOLUTION EQUATIONS AND CONTROL THEORY, 8(2), p. 359-395.

DOI: 10.3934/eect.2019019
Handle: http://hdl.handle.net/1942/30222

doi:10.3934/xx.xx.xx.xx

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# EXISTENCE, BLOW-UP AND EXPONENTIAL DECAY OF SOLUTIONS FOR A POROUS-ELASTIC SYSTEM WITH DAMPING AND SOURCE TERMS

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(Communicated by the associate editor name)

Abstract. In this paper we consider a porous-elastic system consisting of nonlinear boundary/interior damping and nonlinear boundary/interior sources. Our interest lies in the theoretical understanding of the existence, finite time blow-up of solutions and their exponential decay using non-trivial adaptations of well-known techniques. First, we apply the conventional Faedo-Galerkin method with standard arguments of density on the regularity of initial conditions to establish two local existence theorems of weak solutions. Moreover, we detail the uniqueness result in some specific cases. In the second theme, we prove that any weak solution possessing negative initial energy has the latent blow-up in finite time. Finally, we obtain the so-called exponential decay estimates for the global solution under the construction of a suitable Lyapunov functional. In order to corroborate our theoretical decay, a numerical example is provided.

## 1. Introduction.

This paper is concerned with the following polynomially damped system of wave equations

$$\begin{cases} u_{tt} - u_{xx} + \lambda_1 \left|u_t\right|^{r_1-2} u_t = f_1\left(u,v\right) + F_1\left(x,t\right), \\ v_{tt} - v_{xx} + \lambda_2 \left|v_t\right|^{r_2-2} v_t = f_2\left(u,v\right) + F_2\left(x,t\right). \end{cases} \tag{1.1}$$

2010 *Mathematics Subject Classification.* 35L05, 35L15, 35L20, 35L55, 35L70.

*Key words and phrases.* System of nonlinear equations, Faedo-Galerkin method, local existence, global existence, blow up, exponential decay.

This research is funded by Vietnam National University Ho Chi Minh City (VNU-HCM) under Grant no. **B2017-18-04**. The work of the first author was partly supported by a postdoctoral fellowship of the Research Foundation-Flanders (FWO).

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This prototypical system of wave equations arises naturally within frameworks of material science and quantum field theory. Accounting for the Kirchhoff-Love plate theory in shear deformations, the system (1.1) is closely related to the Reissner-Mindlin plate equations (see [9]), structured by three coupled wave and wave-like equations involving the influence of nonlinear damping and source terms. Mathematically, systems of wave equations have been extensively studied by many authors, see [1, 4, 6, 14, 15, 16] and references therein where the existence, regularity and the asymptotic behavior of solutions are investigated.

In [6], Guo et al. considered the local and global well-posedness of a general system

$$\begin{cases} u_{tt} - u_{xx} + g_1\left(u_t\right) = f_1\left(u,v\right), \\ v_{tt} - v_{xx} + g_2\left(v_t\right) = f_2\left(u,v\right), \end{cases} \tag{1.2}$$

in a bounded domain of $\mathbb{R}^n$ with a nonlinear Robin boundary condition on $u$ and a zero boundary conditions on $v$. The nonlinearities $f_1\left(u,v\right)$ and $f_2\left(u,v\right)$ are supercritical exponents representing strong sources, while $g_1\left(u_t\right)$ and $g_2\left(v_t\right)$ act as damping. These damping terms are assumed to be continuous and monotone increasing functions vanishing at the origin and satisfying restrictions in growing up at infinity. By employing the nonlinear semigroups and the theory of monotone operators, the well-posedness of (1.2) is moderately investigated. An important result obtained in [6] and further in [14] is that every weak solution blows up in finite time, provided the initial energy is negative and the sources are more dominant than the damping involved in the system.

In [3, 4], Cavalcanti et al. studied the existence of global solutions, and showed the relation between the asymptotic behavior of the energy and the degenerate system of wave equations with boundary conditions of memory type. Constructing a suitable Lyapunov functional, the authors proved that the energy decays exponentially. The same method was also used in [15, 16] to study the asymptotic behavior of the solutions to a coupled system having integral convolutions as the memory terms. They proved that the solution decays uniformly in time with rates depending on the speed of decay of the convolutions kernel.

In recent years, various types of wave equations with linear or nonlinear damping and sources have been solved by using the Galerkin approximation (see, for instance, [2, 12, 14, 15, 17]). Based on *a priori* estimates, weak convergence, and compactness techniques, and via the construction of a suitable Lyapunov functional, the existence, regularity, blow-up, and exponential decay estimates of solutions for such typical wave equations have been proved in [2, 12, 17]. On the other side, the finite time blow-up of any weak solutions with negative initial energy is obtained in [17].

In light of the aforementioned works, we put ourselves into the study of the existence, blow-up, and exponential decay estimate for the system (1.1).

Let $\Omega = (0,1)$ and $Q_T = \Omega \times (0,T)$ for $T > 0$, a couple of real unknown functions $(u,v)$ is sought for $(x,t) \in \overline{Q_T} = [0,1] \times [0,T]$. The problem we consider here is made of (1.1) and the following nonlinear boundary conditions

$$\begin{cases} u\left(0,t\right) = 0, \ \ - u_x\left(1,t\right) + K_1\left|u\left(1,t\right)\right|^{p_1-2} u\left(1,t\right) = \mu_1\left|u_t\left(1,t\right)\right|^{q_1-2} u_t\left(1,t\right), \\ v_x\left(0,t\right) + K_2\left|v\left(0,t\right)\right|^{p_2-2} v\left(0,t\right) = \mu_2\left|v_t\left(0,t\right)\right|^{q_2-2} v_t\left(0,t\right), \ v\left(1,t\right) = 0, \end{cases} \tag{1.3}$$

and the initial conditions

$$\begin{cases} u(x,0) = \tilde{u}_0(x), \ u_t(x,0) = \tilde{u}_1(x), \\ v(x,0) = \tilde{v}_0(x), \ v_t(x,0) = \tilde{v}_1(x). \end{cases} \tag{1.4}$$

As short-hand explanation for physical parameters in this system, the constants $\lambda_i > 0$ $(i = 1, 2)$ are usually called as the friction terms, while the constants $r_i \geq 2$ play a central role in deciding the order of damped parts. The functions $f_i$ are known as the interior sources, while $F_i$ are the external functions. Moreover, we have on the boundary the presence of the constants $K_i > 0$, $\mu_i > 0$, $p_i \geq 2$, $q_i \geq 2$ $(i = 1, 2)$ as well as given functions $\tilde{u}_i$, $\tilde{v}_i$ $(i = 0, 1)$ that will be specified later.

It is worth mentioning that the nonlinear boundary condition (1.3) is the main difficulty we face, although the approach we use here is already analyzed in many simpler models. As far as we know, a nonlinear wave equation with the two-point boundary conditions has been considered in [12, 17]. Nevertheless, the circumstance for the coupled system (1.1)-(1.4) is still open and its rigorous treatment is technically demanding in this direction. In [14], the authors have solved the similar coupled system where the results are controlled not only by the damping orders $r_1$, $r_2$, but also by the involved parameters on the boundary. Note that compared again to [14] with the homogeneous Dirichlet boundary condition for $u$, our paper needs a careful adaptation to handle several different parameters at the same time. On top of that, we would like to see how necessary assumptions on such input data will be established. Thus, now is the moment we discover the answer.

## 2. **Preliminaries.**

2.1. **Abstract settings.** Let us denote the usual functional spaces used in this paper by $C^m\left(\overline{\Omega}\right)$, $W^{m,p} = W^{m,p}(\Omega)$, $L^p = W^{0,p}(\Omega)$, $H^m = W^{m,2}(\Omega)$ for $1 \leq p \leq \infty$ and $m \in \mathbb{N}$. Let $\langle \cdot, \cdot \rangle$ be either the scalar product in $L^2$ or the dual pairing of a continuous linear functional and an element of a functional space. The notation $\|\cdot\|$ stands for the norm in $L^2$ and we denote by $\|\cdot\|_X$ the norm in the Banach space $X$. We call $X'$ the dual space of $X$ and denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$ for the Banach space of the real functions $u : (0,T) \to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \left( \int_0^T \|u(t)\|_X^p \, dt \right)^{1/p} < \infty, \quad \text{for } 1 \leq p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \operatorname{ess} \sup_{0<t<T} \|u(t)\|_X, \quad \text{for } p = \infty.$$

Since the domain of interest is one-dimensional, let $u(t)$, $u' = u_t$, $u''(t) = u_{tt}(t)$, $\nabla u = u_x$, $\Delta u = u_{xx}$ denote $u(x,t)$, $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial t^2}$, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$, respectively. In $H^1$, we use the following norm:

$$\|u\|_{H^1} = \left( \|u\|^2 + \|u_x\|^2 \right)^{1/2}.$$

We define

$$\mathbb{V}_1 = \left\{ v \in H^1 : v(0) = 0 \right\}, \quad \mathbb{V}_2 = \left\{ v \in H^1 : v(1) = 0 \right\}$$

two closed subspaces of $H^1$. Moreover, the following standard lemmas read as the imbedding $H^1$ into $C^0\left(\overline{\Omega}\right)$, and the equivalence between the norms $\|v_x\|$ and $\|v\|_{H^1}$ in $\mathbb{V}_1$ and $\mathbb{V}_2$.

**Lemma 2.1.** *The imbedding $H^1 \hookrightarrow C\left(\overline{\Omega}\right)$ is compact and the following inequality holds*

$$\|v\|_{C(\overline{\Omega})} \le \sqrt{2}\,\|v\|_{H^1} \quad \text{for all } v \in H^1.$$

**Lemma 2.2.** *On $\mathbb{V}_1$ and $\mathbb{V}_2$ two norms $v \mapsto \|v_x\|$ and $v \mapsto \|v\|_{H^1}$ are equivalent. Furthermore, it holds*

$$\|v\|_{C(\overline{\Omega})} \le \|v_x\| \quad \text{for all } v \in \mathbb{V}_1 \text{ and } \mathbb{V}_2.$$

For the sake of simplicity, we refer $(P)$ to the problem (1.1) endowed with the conditions (1.3)-(1.4). In addition, we denote the damping terms and also possibly related functions by $\Psi_r(z) = |z|^{r-2} z$ where $r$ is a given constant.

2.2. **Weak formulation of** $(P)$**.** The weak formulation of the initial-boundary value problem $(P)$ can be given in the following manner:

For $T > 0$, find a pair of real unknown solutions $(u, v)$ belonging to the following functional space

$$\mathbb{W} = \{(u,v) \in L^\infty\left(0,T;\left(\mathbb{V}_1 \cap H^2\right) \times \left(\mathbb{V}_2 \cap H^2\right)\right) : (u_t, v_t) \in L^\infty(0,T;\mathbb{V}_1 \times \mathbb{V}_2), \\ (u_{tt}, v_{tt}) \in L^\infty\left(0,T;L^2 \times L^2\right)\},$$

such that $(u, v)$ satisfies the variational equations

$$\begin{cases} \langle u_{tt}(t), \phi\rangle + \langle u_x(t), \phi_x\rangle + \lambda_1\langle \Psi_{r_1}(u_t(t)), \phi\rangle + \mu_1 \Psi_{q_1}(u_t(1,t))\,\phi(1) \\ \qquad = K_1 \Psi_{p_1}(u(1,t))\,\phi(1) + \langle f_1(u,v), \phi\rangle + \langle F_1(t), \phi\rangle, \\ \langle v_{tt}(t), \tilde{\phi}\rangle + \langle v_x(t), \tilde{\phi}_x\rangle + \lambda_2\langle \Psi_{r_2}(v_t(t)), \tilde{\phi}\rangle + \mu_2 \Psi_{q_2}(v_t(0,t))\,\tilde{\phi}(0) \\ \qquad = K_2 \Psi_{p_2}(v(0,t))\,\tilde{\phi}(0) + \langle f_2(u,v), \tilde{\phi}\rangle + \langle F_2(t), \tilde{\phi}\rangle, \end{cases} \tag{2.1}$$

for all $(\phi, \tilde{\phi}) \in \mathbb{V}_1 \times \mathbb{V}_2$ and for almost all $t \in (0,T)$. This system is endowed with the initial conditions

$$(u(0), u_t(0)) = (\tilde{u}_0, \tilde{u}_1), \quad (v(0), v_t(0)) = (\tilde{v}_0, \tilde{v}_1). \tag{2.2}$$

3. **The existence and uniqueness of a weak solution.** We now pose the following assumptions:

**$(\mathbf{A_1})$** $(\tilde{u}_0, \tilde{u}_1) \in \left(\mathbb{V}_1 \cap H^2\right) \times \mathbb{V}_1$ and $(\tilde{v}_0, \tilde{v}_1) \in \left(\mathbb{V}_2 \cap H^2\right) \times \mathbb{V}_2$;

**$(\mathbf{A_2})$** $F_1, F_2 \in L^2(Q_T)$ such that $F_1', F_2' \in L^1\left(0,T;L^2\right)$;

**$(\mathbf{A_3})$** there exists a $C^2$-function $\mathcal{F} : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{\partial \mathcal{F}}{\partial u}(u,v) = f_1(u,v), \quad \frac{\partial \mathcal{F}}{\partial v}(u,v) = f_2(u,v), \tag{2.3}$$

and there also exists the constants $\alpha$, $\beta > 2$ and $C > 0$ such that

$$\mathcal{F}(u,v) \le C\left(1 + |u|^\alpha + |v|^\beta\right), \quad \text{for all } u, v \in \mathbb{R}; \tag{2.4}$$

**Remark 3.1.** There are several examples in which the functions $f_1$ and $f_2$ satisfy **$(\mathbf{A_3})$**, see e.g. [1, 14]. In particular, the authors in [1] considered

$$\mathcal{F}(u,v) = \alpha\,|u+v|^{p+1} + 2\beta\,|uv|^{\frac{p+1}{2}},$$

where $p \ge 3$, $\alpha > 1$ and $\beta > 0$. In [14], the authors exploited another type

$$\mathcal{F}(u,v) = \gamma_1\left(|u|^\alpha + |v|^\beta\right) + \gamma_2\,|u|^{\frac{\alpha}{2}}\,|v|^{\frac{\beta}{2}}, \tag{2.5}$$

where $\alpha$, $\beta$, $\gamma_1$ and $\gamma_2$ are positive constants with $\gamma_2 < 2\gamma_1$.

In the following, we claim the existence and uniqueness of a weak solution. Depending on data assumptions, we consider the weak setting (2.1)-(2.2) to show that the problem $(P)$ has a solution in some given function spaces with a small time length $T$. In some subcases of $q_1, q_2$ and $p_1, p_2$, we obtain the uniqueness result in such a small time.

**Theorem 3.1.** *Suppose that* $\mathbf{(A_1)}$*-*$\mathbf{(A_3)}$ *hold and the initial data obey the compatibility relation*

$$\begin{cases} -\tilde{u}_{0x}(1) + K_1 \Psi_{p_1}(\tilde{u}_0(1)) = \mu_1 \Psi_{q_1}(\tilde{u}_1(1)), \\ \tilde{v}_{0x}(0) + K_2 \Psi_{p_2}(\tilde{v}_0(0)) = \mu_2 \Psi_{q_2}(\tilde{v}_1(0)). \end{cases} \tag{3.1}$$

*If* $p_1, p_2, q_1, q_2$ *are such that*

$$\begin{cases} p_1,\ p_2 \geq 2, \\ 2 \leq q_1,\ q_2 \leq 4, \end{cases} \quad or \quad \begin{cases} p_1,\ p_2 \in \{2\} \cup [3, \infty), \\ q_1,\ q_2 > 4, \end{cases}$$

*and* $r_1,\ r_2 \geq 2$*, then there exists a local weak solution* $(u, v)$ *of the problem* $(P)$ *such that*

$$\begin{cases} (u, v) \in L^\infty\left(0, T_*; \left(\mathbb{V}_1 \cap H^2\right) \times \left(\mathbb{V}_2 \cap H^2\right)\right), \\ (u_t, v_t) \in L^\infty\left(0, T_*; \mathbb{V}_1 \times \mathbb{V}_2\right), \\ (u_{tt}, v_{tt}) \in L^\infty\left(0, T_*; L^2 \times L^2\right), \\ |u_t|^{\frac{r_1}{2}-1} u_t,\ \ |v_t|^{\frac{r_2}{2}-1} v_t \in H^1\left(Q_{T_*}\right), \\ |u_t(1,\cdot)|^{\frac{q_1}{2}-1} u_t(1,\cdot),\ \ |v_t(0,\cdot)|^{\frac{q_2}{2}-1} v_t(0,\cdot) \in H^1(0, T_*), \end{cases} \tag{3.2}$$

*for* $T_* > 0$ *sufficiently small. Furthermore, if* $q_1 = q_2 = 2$ *and* $p_1, p_2 \geq 2$*, the obtained solution is unique.*

*Proof of Theorem 3.1.*
**Step 1. The Faedo-Galerkin approximation.** Let $\{(\phi_i, \tilde{\phi}_j)\}$ be a denumerable base of $\left(\mathbb{V}_1 \cap H^2\right) \times \left(\mathbb{V}_2 \cap H^2\right)$. The approximate solution of $(P)$ is a sequence $\{(u_m, v_m)\}_{m \in \mathbb{N}}$ structured as

$$u_m(t) = \sum_{j=1}^{m} c_{mj}(t)\, \phi_j, \quad v_m(t) = \sum_{j=1}^{m} d_{mj}(t)\, \tilde{\phi}_j,$$

where the time-dependent coefficient functions $(c_{mj}, d_{mj})$ satisfy the following system

$$\begin{cases} \langle u_m''(t), \phi_j\rangle + \langle u_{mx}(t), \phi_{jx}\rangle + \lambda_1 \langle \Psi_{r_1}(u_m'(t)), \phi_j\rangle + \mu_1 \Psi_{q_1}(u_m'(1,t))\, \phi_j(1) \\ \qquad\qquad = K_1 \Psi_{p_1}(u_m(1,t))\, \phi_j(1) + \langle f_1(u_m, v_m), \phi_j\rangle + \langle F_1(t), \phi_j\rangle, \\ \langle v_m''(t), \tilde{\phi}_j\rangle + \langle v_{mx}(t), \tilde{\phi}_{jx}\rangle + \lambda_2 \langle \Psi_{r_2}(v_m'(t)), \tilde{\phi}_j\rangle + \mu_2 \Psi_{q_2}(v_m'(0,t))\, \tilde{\phi}_j(0) \\ \qquad\qquad = K_2 \Psi_{p_2}(v_m(0,t))\, \tilde{\phi}_j(0) + \langle f_2(u_m, v_m), \tilde{\phi}_j\rangle + \langle F_2(t), \tilde{\phi}_j\rangle, \\ (u_m(0), u_m'(0)) = (\tilde{u}_0, \tilde{u}_1),\ \ (v_m(0), v_m'(0)) = (\tilde{v}_0, \tilde{v}_1), \end{cases} \tag{3.3}$$

for $1 \leq j \leq m$. A combination of assumptions of this theorem is the direct argument to gain the existence of solution $(u_m, v_m)$ for the system (3.3) on an interval $[0, T_m] \subset [0, T]$.

**Step 2. A priori estimates.**
*Step 2.1. The first estimate.* Multiplying the $j$-th system of (3.3) (specifically, we multiply the first equation by $c_{mj}'(t)$ and the second equation by $d_{mj}'(t)$), summing

up to $m$ with respect to $j$, and then integrating the resulting equation with respect to the time variable from 0 to $t$, we obtain the following equation

$$\begin{aligned}
&\mathcal{S}_m(t) \qquad (3.4)\\
&= \mathcal{S}_m(0) + K_1\int_0^t \Psi_{p_1}(u_m(1,s))\,u_m'(1,s)\,ds + K_2\int_0^t \Psi_{p_2}(v_m(0,s))\,v_m'(0,s)\,ds\\
&+ 2\int_0^t \left[\langle \frac{\partial \mathcal{F}}{\partial u}(u_m(s), v_m(s)), u_m'(s)\rangle + \langle \frac{\partial \mathcal{F}}{\partial v}(u_m(s), v_m(s)), v_m'(s)\rangle\right] ds\\
&+ 2\int_0^t \left[\langle F_1(s), u_m'(s)\rangle + \langle F_2(s), v_m'(s)\rangle\right] ds\\
&= \mathcal{S}_m(0) + \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4,
\end{aligned}$$

where we have denoted by

$$\begin{aligned}
&\mathcal{S}_m(t) \qquad (3.5)\\
&:= \|u_m'(t)\|^2 + \|v_m'(t)\|^2 + \|u_{mx}(t)\|^2 + \|v_{mx}(t)\|^2 + 2\lambda_1\int_0^t \|u_m'(s)\|_{L^{r_1}}^{r_1}\,ds\\
&+ 2\lambda_2\int_0^t \|v_m'(s)\|_{L^{r_2}}^{r_2}\,ds + 2\mu_1\int_0^t |u_m'(1,s)|^{q_1}\,ds + 2\mu_2\int_0^t |v_m'(0,s)|^{q_2}\,ds.
\end{aligned}$$

By the structures of (3.5) and the third equation in (3.3), there exists a positive constant $\mathcal{S}_0$ such that that for all $m \in \mathbb{N}$

$$\mathcal{S}_m(0) = \|\tilde{u}_1\|^2 + \|\tilde{v}_1\|^2 + \|\tilde{u}_{0x}\|^2 + \|\tilde{v}_{0x}\|^2 \equiv \mathcal{S}_0.$$

From now on, we estimate from above the integrals $\mathcal{I}_k$ for $k = \overline{1,4}$ in (3.4). To do this, we need the following elementary inequalities.

**Remark 3.2** (Young-type inequality)**.** Let $\delta > 0$ and $a, b \geq 0$ be arbitrarily real numbers and given $q, q' > 1$ real constants which are Hölder conjugates of each other. The following inequality holds

$$ab \leq \frac{1}{q}\delta^q a^q + \frac{1}{q'}\delta^{-q'} b^{q'}. \qquad (3.6)$$

**Remark 3.3.** Given $N = \frac{1}{2}\max\left\{2; \alpha; \beta; \frac{q_1(p_1-1)}{q_1-1}; \frac{q_2(p_2-1)}{q_2-1}\right\}$, then for all $s \geq 0$, the inequality $s^\gamma \leq 1 + s^N$ holds for all $\gamma \in (0, N]$.

Observe that the estimates for $\mathcal{I}_1$ and $\mathcal{I}_2$ are similar when using the above inequalities. By choosing $\delta > 0$ in such a way that

$$\delta = \min\left\{\sqrt[q_1]{\frac{\mu_1 q_1}{2K_1}}; \sqrt[q_2]{\frac{\mu_2 q_2}{2K_2}}\right\},$$

it naturally arises that for $C_T > 0$ only dependent of $T$, it holds

$$\mathcal{I}_1 + \mathcal{I}_2 \leq \frac{1}{2}\mathcal{S}_m(t) + C_T\int_0^t \left(1 + \mathcal{S}_m^N(s)\right) ds. \qquad (3.7)$$

Additionally, for $C_0 > 0$ depending only on the initial data $\tilde{u}_0$, $\tilde{v}_0$, $\tilde{u}_1$, $\tilde{v}_1$, and the constants $\alpha$, $\beta$, we can prove that

$$\|u_m(t)\|_{L^\alpha}^\alpha + \|v_m(t)\|_{L^\beta}^\beta \leq C_0 + (\alpha + \beta)\int_0^t \left(1 + \mathcal{S}_m^N(s)\right) ds,$$

and then in the same spirit of [14] the integral $\mathcal{I}_3$ is essentially estimated by

$$\mathcal{I}_3 \leq 2 \sup_{|y|,|z|\leq\sqrt{C_0}} |\mathcal{F}(y,z)| + 2C_1 + 2C_1\left[C_0 + (\alpha+\beta)\int_0^t \left(1+\mathcal{S}_m^N(s)\right)ds\right] \quad (3.8)$$
$$\leq C_0 + C_0\int_0^t \left(1+\mathcal{S}_m^N(s)\right)ds,$$

where we have also used the assumption $(\mathbf{A_3})$. The last integral can be bounded by the standard Cauchy-Schwartz inequality, i.e.

$$\mathcal{I}_4 \leq \|F_1\|_{L^2(Q_T)}^2 + \|F_2\|_{L^2(Q_T)}^2 + \int_0^t \mathcal{S}_m(s)\,ds \leq C_T + \int_0^t \left(1+\mathcal{S}_m^N(s)\right)ds. \quad (3.9)$$

We hence obtain from (3.7)-(3.9) that for $0 \leq t \leq T_m$

$$\mathcal{S}_m(t) \leq C_T + C_T\int_0^t \left(1+\mathcal{S}_m^N(s)\right)ds.$$

Cf. [10], it should be noted that there exists a constant $T_* > 0$ depending on $T$ (but independent of $m$) such that

$$\mathcal{S}_m(t) \leq C_T,\ \forall m \in \mathbb{N},\ \forall t \in [0,T_*]\,. \quad (3.10)$$

Consequently, this result allows us to take $T_m = T_*$ for all $m$.
*Step 2.2. The second estimate.* Consider the first equation of (3.3). Letting $t \to 0^+$, then multiplying the equation by $c''_{mj}(0)$ with summing up to $m$ with respect to $j$, and using the first compatibility relation (3.1), we thus obtain

$$\|u''_m(0)\|^2 - \langle \Delta u_m(0), u''_m(0)\rangle + \lambda_1\langle |\tilde{u}_1|^{r_1-2}\tilde{u}_1, u''_m(0)\rangle$$
$$= \langle f_1(\tilde{u}_0,\tilde{v}_0), u''_m(0)\rangle + \langle F_1(0), u''_m(0)\rangle.$$

Relying on the classical inequalities, we have

$$\|u''_m(0)\| \leq \|\Delta\tilde{u}_0\| + \lambda_1\left\||\tilde{u}_1|^{r_1-1}\right\| + \|f_1(\tilde{u}_0,\tilde{v}_0)\| + \|F_1(0)\|\,, \quad (3.11)$$

then state that there exists $\mathcal{C}_1 > 0$ such that $\|u''_m(0)\| \leq \mathcal{C}_1$ for all $m \in \mathbb{N}$.

For the second equation of (3.3), one also proves without difficulty using similar arguments that there exists $\mathcal{C}_2 > 0$ in which it bounds $\|v''_m(0)\|$, i.e. $\|v''_m(0)\| \leq \mathcal{C}_2$ for all $m \in \mathbb{N}$.

Next, we differentiate (3.3) with respect to $t$. This way the first equation becomes

$$\begin{aligned}&\langle u'''_m(t), \phi_j\rangle + \langle u'_{mx}(t), \phi_{jx}\rangle \quad (3.12)\\ &\quad + \lambda_1\langle \Psi'_{r_1}(u'_m(t))\,u''_m(t), \phi_j\rangle + \mu_1\Psi'_{q_1}(u'_m(1,t))\,u''_m(1,t)\,\phi_j(1)\\ &\quad = K_1\Psi'_{p_1}(u_m(1,t))\,u'_m(1,t)\,\phi_j(1)\\ &\quad + \langle \frac{\partial^2\mathcal{F}}{\partial u^2}(u_m,v_m)\,u'_m + \frac{\partial^2\mathcal{F}}{\partial u\partial v}(u_m,v_m)\,v'_m, \phi_j\rangle + \langle F'_1(t), \phi_j\rangle,\end{aligned}$$

and for $1 \leq j \leq m$, the second equation is

$$\begin{aligned}&\langle v'''_m(t), \tilde{\phi}_j\rangle + \langle v'_{mx}(t), \tilde{\phi}_{jx}\rangle \quad (3.13)\\ &\quad + \lambda_2\langle \Psi'_{r_2}(v'_m(t))\,v''_m(t), \tilde{\phi}_j\rangle + \mu_2\Psi'_{q_2}(v'_m(0,t))\,v''_m(0,t)\,\tilde{\phi}_j(0)\\ &\quad = K_2\Psi'_{p_2}(v_m(0,t))\,v'_m(0,t)\,\tilde{\phi}_j(0)\\ &\quad + \langle \frac{\partial^2\mathcal{F}}{\partial v\partial u}(u_m,v_m)\,u'_m + \frac{\partial^2\mathcal{F}}{\partial v^2}(u_m,v_m)\,v'_m, \tilde{\phi}_j\rangle + \langle F'_2(t), \tilde{\phi}_j\rangle.\end{aligned}$$

Multiplying the $j$-th equation of (3.12) and (3.13), respectively, by $c''_{mj}(t)$ and $d''_{mj}(t)$, summing with respect to $j$ up to $m$, and then integrating with respect to the time variable from 0 to $t$, we obtain

$$\begin{aligned}
&\mathcal{P}_m(t) \qquad (3.14)\\
&= \mathcal{P}_m(0) + 2\int_0^t \langle \frac{\partial^2 \mathcal{F}}{\partial u^2}(u_m, v_m) u'_m(s) + \frac{\partial^2 \mathcal{F}}{\partial u \partial v}(u_m, v_m) v'_m(s), u''_m(s)\rangle ds\\
&+ 2\int_0^t \langle \frac{\partial^2 \mathcal{F}}{\partial v \partial u}(u_m, v_m) u'_m(s) + \frac{\partial^2 \mathcal{F}}{\partial v^2}(u_m, v_m) v'_m(s), v''_m(s)\rangle ds\\
&+ 2\int_0^t \left[\langle F'_1(s), u''_m(s)\rangle + \langle F'_2(s), v''_m(s)\rangle\right] ds\\
&+ 2\int_0^t \left[K_1 \Psi'_{p_1}(u_m(1,s)) u'_m(1,s) u''_m(1,s) + K_2 \Psi'_{p_2}(v_m(0,s)) v'_m(0,s) v''_m(0,s)\right] ds\\
&= \mathcal{P}_m(0) + \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4,
\end{aligned}$$

where we have denoted by

$$\begin{aligned}
\mathcal{P}_m(t) := {} & \|u''_m(t)\|^2 + \|v''_m(t)\|^2 + \|u'_{mx}(t)\|^2 + \|v'_{mx}(t)\|^2 \qquad (3.15)\\
&+ \frac{8\lambda_1(r_1-1)}{r_1^2}\int_0^t \left\|\frac{\partial}{\partial s}\left(|u'_m(s)|^{\frac{r_1}{2}-1} u'_m(s)\right)\right\|^2 ds\\
&+ \frac{8\lambda_2(r_2-1)}{r_2^2}\int_0^t \left\|\frac{\partial}{\partial s}\left(|v'_m(s)|^{\frac{r_2}{2}-1} v'_m(s)\right)\right\|^2 ds\\
&+ \frac{8\mu_1(q_1-1)}{q_1^2}\int_0^t \left|\frac{\partial}{\partial s}\left(|u'_m(1,s)|^{\frac{q_1}{2}-1} u'_m(1,s)\right)\right|^2 ds\\
&+ \frac{8\mu_2(q_2-1)}{q_2^2}\int_0^t \left|\frac{\partial}{\partial s}\left(|v'_m(0,s)|^{\frac{q_2}{2}-1} v'_m(0,s)\right)\right|^2 ds.
\end{aligned}$$

Combining the arguments from the boundedness of $\|u''_m(0)\|$, $\|v''_m(0)\|$ and the third equation of (3.3) to (3.15), there exists a positive constant $\mathcal{P}_0$ that bounds $\mathcal{P}_m(0)$ for all $m \in \mathbb{N}$, i.e.

$$\mathcal{P}_m(0) = \|u''_m(0)\|^2 + \|v''_m(0)\|^2 + \|\tilde{u}_{1x}\|^2 + \|\tilde{v}_{1x}\|^2 \le \mathcal{P}_0. \qquad (3.16)$$

Here the constant $\mathcal{P}_0$ is dependent of the initial data $\tilde{u}_0$, $\tilde{v}_0$, $\tilde{u}_1$, $\tilde{v}_1$, the interior sources $f_1$, $f_2$, the functions $F_1$, $F_2$ and the given constants $r_1$, $r_2$, $\lambda_1$, $\lambda_2$.

Estimating $\mathcal{P}_m(t)$ is almost similar to what have been done in [14], so we claim that by putting $\mathcal{K}(T, \mathcal{F}) = \sup_{|y|,\ |z| \le \sqrt{C_T},\ |\alpha|=2} |D^\alpha \mathcal{F}(y,z)|$, there are possibilities to estimate the first three integrals, i.e. one can show that

$$\mathcal{J}_1 + \mathcal{J}_2 \le C_T + \int_0^t \mathcal{P}_m(s)\, ds, \qquad (3.17)$$

and by the standard Cauchy-Schwartz inequality, we obtain

$$\mathcal{J}_3 \le C_T + \int_0^t \left(\|F'_1(s)\| + \|F'_2(s)\|\right) \mathcal{P}_m(s)\, ds. \qquad (3.18)$$

We can go through the last integral $\mathcal{J}_4$ by the following lemma.

**Lemma 3.2.** *If one of the following cases is valid, which is*

$$\begin{cases} p_1,\ p_2 \geq 2, \\ 2 \leq q_1,\ q_2 \leq 4, \end{cases} \quad or \quad \begin{cases} p_1,\ p_2 \in \{2\} \cup [3,\infty), \\ q_1,\ q_2 > 4, \end{cases}$$

*the integral $\mathcal{J}_4$ given by (3.14) can be bounded by*

$$\mathcal{J}_4 \leq C_T + \frac{1}{2}\mathcal{P}_m(t). \tag{3.19}$$

*Proof of Lemma 3.2.* Since $\mathcal{J}_4$ can be divided into two integrals and those can be separately estimated, we put ourselves into the following cases:
**Case 1**: $2 \leq q_1,\ q_2 \leq 4,\ p_1 = p_2 = 2$ and $1 \leq q_1,\ q_2 \leq 4,\ p_1,\ p_2 > 2$.
**Case 2**: $q_1,\ q_2 > 4,\ p_1,\ p_2 \geq 3$ and $q_1,\ q_2 > 4,\ p_1 = p_2 = 2$.
Accordingly, our proof is presented below case by case.
**Case 1.1:** $2 \leq q_1,\ q_2 \leq 4,\ p_1 = p_2 = 2$. We have

$$\begin{aligned} \mathcal{J}_4 \leq\ & \frac{4K_1}{q_1}\int_0^t |u_m'(1,s)|^{2-\frac{q_1}{2}} \left| \frac{\partial}{\partial s}\left(|u_m'(1,s)|^{\frac{q_1}{2}-1} u_m'(1,s)\right)\right| ds \\ &+ \frac{4K_2}{q_2}\int_0^t |v_m'(0,s)|^{2-\frac{q_2}{2}} \left| \frac{\partial}{\partial s}\left(|v_m'(0,s)|^{\frac{q_2}{2}-1} v_m'(0,s)\right)\right| ds. \end{aligned} \tag{3.20}$$

Due to (3.5) and (3.15) which read as

$$\mathcal{S}_m(t) \geq 2\mu_1 \int_0^t |u_m'(1,s)|^{q_1} ds + 2\mu_2 \int_0^t |v_m'(0,s)|^{q_2} ds, \tag{3.21}$$

$$\begin{aligned} \mathcal{P}_m(t) \geq\ & \frac{8\mu_1(q_1-1)}{q_1^2}\int_0^t \left| \frac{\partial}{\partial s}\left(|u_m'(1,s)|^{\frac{q_1}{2}-1} u_m'(1,s)\right)\right|^2 ds \\ &+ \frac{8\mu_2(q_2-1)}{q_2^2}\int_0^t \left| \frac{\partial}{\partial s}\left(|v_m'(0,s)|^{\frac{q_2}{2}-1} v_m'(0,s)\right)\right|^2 ds, \end{aligned} \tag{3.22}$$

together with (3.10), we bound $\mathcal{J}_4$ by

$$\begin{aligned} \mathcal{J}_4 \leq\ & \frac{2K_1}{q_1}\int_0^t \left[\frac{1}{\delta_1}|u_m'(1,s)|^{4-q_1} + \delta_1 \left| \frac{\partial}{\partial s}\left(|u_m'(1,s)|^{\frac{q_1}{2}-1} u_m'(1,s)\right)\right|^2\right] ds \\ &+ \frac{2K_2}{q_2}\int_0^t \left[\frac{1}{\delta_2}|v_m'(0,s)|^{4-q_2} + \delta_2 \left| \frac{\partial}{\partial s}\left(|v_m'(0,s)|^{\frac{q_2}{2}-1} v_m'(0,s)\right)\right|^2\right] ds \\ \leq\ & \frac{2K_1}{q_1\delta_1}\int_0^t \left[1 + |u_m'(1,s)|^{q_1}\right] ds + \frac{2K_2}{q_2\delta_2}\int_0^t \left[1 + |v_m'(0,s)|^{q_2}\right] ds \\ &+ \left(\frac{\delta_1 K_1 q_1}{4\mu_1(q_1-1)} + \frac{\delta_2 K_2 q_2}{4\mu_2(q_2-1)}\right)\mathcal{P}_m(t) \\ \leq\ & \frac{2K_1}{q_1\delta_1}\left[T + \frac{1}{2\mu_1}\mathcal{S}_m(t)\right] + \frac{2K_2}{q_2\delta_2}\left[T + \frac{1}{2\mu_2}\mathcal{S}_m(t)\right] \\ &+ \left(\frac{\delta_1 K_1 q_1}{4\mu_1(q_1-1)} + \frac{\delta_2 K_2 q_2}{4\mu_2(q_2-1)}\right)\mathcal{P}_m(t) \\ \leq\ & \left(\frac{1}{\delta_1} + \frac{1}{\delta_2}\right) C_T + \left(\frac{\delta_1 K_1 q_1}{4\mu_1(q_1-1)} + \frac{\delta_2 K_2 q_2}{4\mu_2(q_2-1)}\right)\mathcal{P}_m(t). \end{aligned} \tag{3.23}$$

We use $a^{4-q} \leq 1 + a^q$ for all $a \geq 0$, $2 \leq q \leq 4$ and $2ab \leq \delta a^2 + \delta^{-1} b^2$ for all $a$, $b \geq 0$ and $\delta > 0$. Thus, to deduce (3.19) from (3.23) we choose

$$\delta_1 = \delta_2 \leq \frac{2\mu_1\mu_2 (q_1 - 1)(q_2 - 1)}{K_1\mu_2 q_1 (q_2 - 1) + K_2 \mu_1 q_2 (q_1 - 1)}.$$

**Case 1.2:** $2 \leq q_1,\ q_2 \leq 4,\ p_1,\ p_2 > 2$. By simple computations, we have

$$\begin{aligned}
&\mathcal{J}_4 \\
&= 2K_1 (p_1 - 1) \int_0^t |u_m (1,s)|^{p_1 - 2} |u_m' (1,s)|^{1 - \frac{q_1}{2}} u_m' (1,s) |u_m' (1,s)|^{\frac{q_1}{2} - 1} u_m'' (1,s)\, ds \\
&+ 2K_2 (p_2 - 1) \int_0^t |v_m (0,s)|^{p_2 - 2} |v_m' (0,s)|^{1 - \frac{q_2}{2}} v_m' (0,s) |v_m' (0,s)|^{\frac{q_2}{2} - 1} v_m'' (0,s)\, ds \\
&\leq \frac{4}{q_1} K_1 (p_1 - 1) C_T^{\frac{p_1}{2} - 1} \int_0^t |u_m' (1,s)|^{2 - \frac{q_1}{2}} \left| \frac{\partial}{\partial s} \left( |u_m' (1,s)|^{\frac{q_1}{2} - 1} u_m' (1,s) \right) \right| ds \\
&+ \frac{4}{q_2} K_2 (p_2 - 1) C_T^{\frac{p_2}{2} - 1} \int_0^t |v_m' (0,s)|^{2 - \frac{q_2}{2}} \left| \frac{\partial}{\partial s} \left( |v_m' (0,s)|^{\frac{q_2}{2} - 1} v_m' (0,s) \right) \right| ds,
\end{aligned}$$

using the fact that $|u_m (1,s)| + |v_m (0,s)| \leq \sqrt{C_T}$ (cf. Lemma 2.2), (3.5) and (3.10). This way gives us back to (3.20) in the previous case. Thus, (3.19) holds.

**Case 2.1:** $q_1,\ q_2 > 4,\ p_1,\ p_2 \geq 3$. We start by using the integration by parts

$$\begin{aligned}
\mathcal{J}_4 &= K_1 (p_1 - 1) |u_m (1,t)|^{p_1 - 2} |u_m' (1,t)|^2 - K_1 (p_1 - 2) |\tilde{u}_0 (1)|^{p_1 - 2} \tilde{u}_1^2 (1) \\
&- K_1 (p_1 - 1)(p_1 - 2) \int_0^t |u_m (1,s)|^{p_1 - 4} u_m (1,s) (u_m' (1,s))^3\, ds \\
&+ K_2 (p_2 - 1) |v_m (0,t)|^{p_2 - 2} |v_m' (0,t)|^2 - K_2 (p_2 - 2) |\tilde{v}_0 (0)|^{p_2 - 2} \tilde{v}_1^2 (0) \\
&- K_2 (p_2 - 1)(p_2 - 2) \int_0^t |v_m (0,s)|^{p_2 - 4} v_m (0,s) (v_m' (0,s))^3\, ds.
\end{aligned}$$

Based on (3.5) and (3.10), we first estimate $\mathcal{J}_4$ as follows:

$$\begin{aligned}
\mathcal{J}_4 &\leq K_1 (p_1 - 1) \mathcal{S}_m^{\frac{p_1}{2} - 1} |u_m' (1,t)|^2 + K_1 (p_1 - 1)(p_1 - 2) \int_0^t \mathcal{S}_m^{\frac{p_1 - 3}{2}} (s) |u_m' (1,s)|^3\, ds \\
&+ K_2 (p_2 - 1) \mathcal{S}_m^{\frac{p_2}{2} - 1} |v_m' (0,t)|^2 + K_2 (p_2 - 1)(p_2 - 2) \int_0^t \mathcal{S}_m^{\frac{p_2 - 3}{2}} (s) |v_m' (0,s)|^3\, ds \\
&\leq C_T \left( |u_m' (1,t)|^2 + |v_m' (0,t)|^2 \right) + C_T \int_0^t \left[ |u_m' (1,s)|^3 + |v_m' (0,s)|^3 \right] ds.
\end{aligned}$$

Secondly, it is immediate to see that using the inequality $a^3 \leq 1 + a^q$ for all $a \geq 0$ and $q \geq 3$, together with (3.10) and (3.21), we arrive at

$$\int_0^t \left[ |u_m' (1,s)|^3 + |v_m' (0,s)|^3 \right] ds \leq T + \frac{1}{2} \left( \frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \mathcal{S}_m (t) \leq C_T.$$

One also deduces from

$$\begin{aligned}
&\left|u_m'(1,t)\right|^{\frac{q_1}{2}-1} u_m'(1,t) \\
&= \left|\tilde{u}_{1m}(1)\right|^{\frac{q_1}{2}-1} \tilde{u}_{1m}(1) + \int_0^t \frac{\partial}{\partial s}\left(\left|u_m'(1,s)\right|^{\frac{q_1}{2}-1} u_m'(1,s)\right) ds, \\
&\left|v_m'(0,t)\right|^{\frac{q_2}{2}-1} v_m'(0,t) \\
&= \left|\tilde{v}_{1m}(0)\right|^{\frac{q_2}{2}-1} \tilde{v}_{1m}(0) + \int_0^t \frac{\partial}{\partial s}\left(\left|v_m'(0,s)\right|^{\frac{q_2}{2}-1} v_m'(0,s)\right) ds,
\end{aligned}$$

and from the elementary inequality $(a+b)^2 \le 2\left(a^2+b^2\right)$ for all $a,\ b \ge 0$ with Hölder's inequality and (3.22) that

$$\begin{aligned}
\left|u_m'(1,t)\right|^{q_1} &\le 2\left|\tilde{u}_{1m}(1)\right|^{q_1} + 2t\int_0^t \left|\frac{\partial}{\partial s}\left(\left|u_m'(1,s)\right|^{\frac{q_1}{2}-1} u_m'(1,s)\right)\right|^2 ds \\
&\le 2\left|\tilde{u}_{1m}(1)\right|^{q_1} + \frac{q_1^2 T}{4\mu_1(q_1-1)}\mathcal{P}_m(t).
\end{aligned}$$

In the same vein, we get

$$\left|v_m'(0,t)\right|^{q_2} \le 2\left|\tilde{v}_{1m}(0)\right|^{q_2} + \frac{q_2^2 T}{4\mu_2(q_2-1)}\mathcal{P}_m(t).$$

Using the inequalities

$$\begin{aligned}
&(a+b)^{\frac{2}{q}} \le a^{\frac{2}{q}} + b^{\frac{2}{q}}, \quad \forall a,b \ge 0,\ \forall q \ge 2, \\
&ab \le \left(1-\frac{2}{q}\right)\delta^{-\frac{q}{q-2}} a^{\frac{q}{q-2}} + \frac{2}{q}\delta^{\frac{q}{2}} b^{\frac{q}{2}}, \quad \forall a,b \ge 0,\ \forall q > 2,\ \delta > 0,
\end{aligned}$$

we therefore obtain

$$\begin{aligned}
&C_T\left(\left|u_m'(1,t)\right|^2 + \left|v_m'(0,t)\right|^2\right) \\
&\le C_T\left(2\left|\tilde{u}_{1m}(1)\right|^{q_1} + \frac{q_1^2 T}{4\mu_1(q_1-1)}\mathcal{P}_m(t)\right)^{\frac{2}{q_1}} \\
&+ C_T\left(2\left|\tilde{v}_{1m}(0)\right|^{q_2} + \frac{q_2^2 T}{4\mu_2(q_2-1)}\mathcal{P}_m(t)\right)^{\frac{2}{q_2}} \\
&\le C_T\left[2^{\frac{2}{q_1}}\left|\tilde{u}_{1m}(1)\right|^2 + 2^{\frac{2}{q_2}}\left|\tilde{v}_{1m}(0)\right|^2\right] \\
&+ C_T\left[\left(\frac{q_1^2 T}{4\mu_1(q_1-1)}\right)^{\frac{2}{q_1}}\mathcal{P}_m^{\frac{2}{q_1}}(t) + \left(\frac{q_2^2 T}{4\mu_2(q_2-1)}\right)^{\frac{2}{q_2}}\mathcal{P}_m^{\frac{2}{q_2}}(t)\right] \\
&\le C_0 + C_T\left[\left(1-\frac{2}{q_1}\right)\delta_1^{-\frac{q_1}{q_1-2}}\left(\frac{q_1^2 T}{4\mu_1(q_1-1)}\right)^{\frac{4}{q_1(q_1-2)}} + \frac{2}{q_1}\delta_1^{\frac{q_1}{2}}\mathcal{P}_m(t)\right] \\
&+ C_T\left[\left(1-\frac{2}{q_2}\right)\delta_2^{-\frac{q_2}{q_2-2}}\left(\frac{q_2^2 T}{4\mu_2(q_2-1)}\right)^{\frac{4}{q_2(q_2-2)}} + \frac{2}{q_2}\delta_2^{\frac{q_2}{2}}\mathcal{P}_m(t)\right] \\
&\le C_T(\delta_1,\delta_2) + 2\left(\frac{\delta_1^{\frac{q_1}{2}}}{q_1} + \frac{\delta_2^{\frac{q_2}{2}}}{q_2}\right)\mathcal{P}_m(t).
\end{aligned}$$

Hence, to deduce (3.19) we choose $\delta = \delta_1 = \delta_2 > 0$ such that $q_2\delta^{\frac{q_1}{2}} + q_1\delta^{\frac{q_2}{2}} \le \frac{q_1 q_2}{2}$.

**Case 2.2:** $q_1,\ q_2 > 4$, $p_1 = p_2 = 2$. By the same arguments exploited in the previous case, here we can state that

$$\begin{aligned}\mathcal{J}_4 &= K_1 \int_0^t \frac{d}{ds}\left(\left|u_m'(1,s)\right|^2\right) ds + K_2 \int_0^t \frac{d}{ds}\left(\left|v_m'(0,s)\right|^2\right) ds \\ &= K_1\left(\left|u_m'(1,t)\right|^2 - \tilde{u}_{1m}^2(1)\right) + K_2\left(\left|v_m'(0,t)\right|^2 - \tilde{v}_{1m}^2(0)\right) \\ &\le K_1\left|u_m'(1,t)\right|^2 + K_2\left|v_m'(0,t)\right|^2,\end{aligned}$$

also leads to (3.19).

Hence, we complete the proof of Lemma 3.2. □

Now, combining (3.17), (3.18), and (3.19) we are in a great position to obtain

$$\mathcal{P}_m(t) \le 2\mathcal{P}_0 + 6C_T + 2\int_0^t \left(1 + \|F_1'(s)\| + \|F_2'(s)\|\right)\mathcal{P}_m(s)\,ds.$$

Thanks to Gronwall's inequality, we conclude that

$$\mathcal{P}_m(t) \le (2\mathcal{P}_0 + 6C_T)\exp\left[\int_0^t \left(1 + \|F_1'(s)\| + \|F_2'(s)\|\right) ds\right] \le C_T, \tag{3.24}$$

for all $m \in \mathbb{N}$ and $t \in [0, T_*]$.

*Step 3. Passing to the limit.* The existence of solution in the interval $[0, T_*]$ is now approaching. To summarize, using the Banach-Alaoglu theorem (see, e.g., [5]), the uniform bounds with respect to $m$, as stated in the above results (3.5), (3.10), (3.15), and (3.24), imply that one can extract a further subsequence (which we relabel with the index $m$ if necessary) such that

$$(u_m, v_m) \to (u, v) \text{ weak-* in } L^\infty(0, T_*; \mathbb{V}_1 \times \mathbb{V}_2), \tag{3.25}$$

$$(u_m', v_m') \to (u', v') \text{ weak in } L^{r_1}(Q_{T_*}) \times L^{r_2}(Q_{T_*}), \text{weak-* in } L^\infty(0, T_*; \mathbb{V}_1 \times \mathbb{V}_2), \tag{3.26}$$

$$(u_m'', v_m'') \to (u'', v'') \text{ weak-* in } L^\infty\left(0, T_*; L^2 \times L^2\right), \tag{3.27}$$

$$(u_m(1,\cdot), v_m(0,\cdot)) \to (u(1,\cdot), v(0,\cdot)) \text{ weak in } W^{1,q_1}(0, T_*) \times W^{1,q_2}(0, T_*), \tag{3.28}$$

$$(u_m'(1,\cdot), v_m'(0,\cdot)) \to (u'(1,\cdot), v'(0,\cdot)) \text{ weak in } L^{q_1}(0, T_*) \times L^{q_2}(0, T_*), \tag{3.29}$$

$$\left(\left|u_m'(1,\cdot)\right|^{\frac{q_1}{2}-1} u_m'(1,\cdot), \left|v_m'(0,\cdot)\right|^{\frac{q_2}{2}-1} v_m'(0,\cdot)\right) \to (\chi_1, \chi_2) \text{ weak in } [H^1(0, T_*)]^2, \tag{3.30}$$

$$\left(\frac{\partial}{\partial t}\left(\left|u_m'\right|^{\frac{r_1}{2}-1} u_m'\right), \frac{\partial}{\partial t}\left(\left|v_m'\right|^{\frac{r_2}{2}-1} v_m'\right)\right) \to (\chi_3, \chi_4) \text{ weak in } [L^2(Q_{T_*})]^2. \tag{3.31}$$

Furthermore, by the Aubin-Lions compactness theorem in combination with the imbeddings $H^2(0, T_*) \hookrightarrow C^1([0, T_*])$, $H^1(0, T_*) \hookrightarrow C([0, T_*])$, $W^{1,q_1}(0, T_*) \hookrightarrow C([0, T_*])$, $W^{1,q_2}(0, T_*) \hookrightarrow C([0, T_*])$, it is straightforward to go on extracting from the weak convergence results (3.25)-(3.31) a subsequence $\{(u_m, v_m)\}$ such that

$$(u_m, v_m) \to (u, v) \text{ strong in } [L^2(Q_{T_*})]^2 \text{ and almost everywhere in } Q_{T_*}, \tag{3.32}$$

$$(u_m', v_m') \to (u', v') \text{ strong in } [L^2(Q_{T_*})]^2 \text{ and almost everywhere in } Q_{T_*}, \tag{3.33}$$

$$(u_m(1,\cdot), v_m(0,\cdot)) \to (u(1,\cdot), v(0,\cdot)) \text{ strong in } [C([0, T_*])]^2, \tag{3.34}$$

$$\left(\left|u_m'(1,\cdot)\right|^{\frac{q_1}{2}-1} u_m'(1,\cdot), \left|v_m'(0,\cdot)\right|^{\frac{q_2}{2}-1} v_m'(0,\cdot)\right) \to (\chi_1, \chi_2) \text{ strong in } [C([0, T_*])]^2. \tag{3.35}$$

Now we have to show the convergence of the nonlinear terms including damping and interior sources. In fact, using the continuity of $f_1$, one deduces that

$$f_1\left(u_m, v_m\right) \rightarrow f_1\left(u, v\right) \text{ almost everywhere in } Q_{T_*}.$$

Obverse that $\left\|f_1\left(u_m, v_m\right)\right\|_{L^2\left(Q_{T_*}\right)}$ is bounded by $\sqrt{T_*}\sup_{|y|,\ |z|\leq\sqrt{C_T}}\left|f_1\left(y, z\right)\right|$ which cannot go to infinity, and together with [11, Lemma 1.3], one continues to obtain the following

$$f_1\left(u_m, v_m\right) \rightarrow f_1\left(u, v\right) \text{ weak in } L^2\left(Q_{T_*}\right), \tag{3.36}$$

and

$$f_2\left(u_m, v_m\right) \rightarrow f_2\left(u, v\right) \text{ weak in } L^2\left(Q_{T_*}\right). \tag{3.37}$$

It remains to see the weak convergence of damping terms. Thanks to the inequality

$$\left|\Psi_r\left(z_1\right)-\Psi_r\left(z_2\right)\right| \leq (r-1)\, C^{r-2}\left|z_1-z_2\right|, \quad \forall z_1, z_2 \in[-C, C],\ C>0,\ r \geq 2 \tag{3.38}$$

in accordance with (3.15), (3.24) and (3.33), one easily obtains

$$\left(\Psi_{r_1}\left(u_m'\right), \Psi_{r_2}\left(v_m'\right)\right) \rightarrow \left(\Psi_{r_1}\left(u'\right), \Psi_{r_2}\left(v'\right)\right) \text{ strong in } [L^2\left(Q_{T_*}\right)]^2. \tag{3.39}$$

It is then worthwhile to mention that (3.34) gives

$$\left(\Psi_{p_1}\left(u_m\left(1, \cdot\right)\right), \Psi_{p_2}\left(v_m\left(0, \cdot\right)\right)\right) \rightarrow \left(\Psi_{p_1}\left(u\left(1, \cdot\right)\right), \Psi_{p_2}\left(v\left(0, \cdot\right)\right)\right) \text{ strong in } \left[C\left([0, T_*]\right)\right]^2, \tag{3.40}$$

by the continuity of $\Psi_{p_i}$ for $i = 1, 2$, and (3.35) provides that

$$\left(u_m'\left(1, \cdot\right), v_m'\left(0, \cdot\right)\right) \rightarrow \left(\left|\chi_1\right|^{\frac{2}{q_1}-1}\chi_1, \left|\chi_2\right|^{\frac{2}{q_2}-1}\chi_2\right) \text{ strong in } \left[C\left([0, T_*]\right)\right]^2. \tag{3.41}$$

Thus, we take (3.29) and (3.41) to gain

$$\left(\left|\chi_1\right|^{\frac{2}{q_1}-1}\chi_1, \left|\chi_2\right|^{\frac{2}{q_2}-1}\chi_2\right) = \left(u'\left(1, \cdot\right), v'\left(0, \cdot\right)\right) \tag{3.42}$$

by virtue of the uniqueness of convergence.

In the same vein, we claim

$$\left(\Psi_{q_1}\left(u_m'\left(1, \cdot\right)\right), \Psi_{q_2}\left(v_m'\left(0, \cdot\right)\right)\right) \rightarrow \left(\Psi_{q_1}\left(u'\left(1, \cdot\right)\right), \Psi_{q_2}\left(v'\left(0, \cdot\right)\right)\right) \text{ strong in } \left[C\left([0, T_*]\right)\right]^2 \tag{3.43}$$

from (3.41) and (3.42).

From here on, combining (3.25)-(3.27), (3.32)-(3.37), (3.39), (3.40), (3.43) is sufficient to pass to the limit in (3.3) and then to show that $(u, v)$ satisfies the problem $(P)$. In addition, one can use (3.25)-(3.27), (2.1) and $\mathbf{(A_2)}$ to prove that

$$\begin{cases} u_{xx} = u_{tt} + \lambda_1\Psi_{r_1}\left(u_t\right) - f_1\left(u, v\right) - F_1 \in L^\infty\left(0, T_*; L^2\right), \\ v_{xx} = v_{tt} + \lambda_2\Psi_{r_2}\left(v_t\right) - f_2\left(u, v\right) - F_2 \in L^\infty\left(0, T_*; L^2\right), \end{cases}$$

which verifies $(u, v) \in L^\infty\left(0, T_*; \left(\mathbb{V}_1 \cap H^2\right) \times \left(\mathbb{V}_2 \cap H^2\right)\right)$ and completes the proof of the existence of a local weak solution.

*Step 4. Uniqueness of the solution.* Suppose $(u_1, v_1)$ and $(u_2, v_2)$ are two solutions to $(P)$ in the interval $[0, T_*]$, which is devoted to the case $q_1 = q_2 = 2$ and $p_1$, $p_2 \geq 2$. Going along with the same initial data $(\tilde{u}_0, \tilde{u}_1)$ and $(\tilde{v}_0, \tilde{v}_1)$, we prove that these solutions must be equal.

Define $(u,v) := (u_1 - u_2, v_1 - v_2)$ and based on (2.1) and (2.2), these quantities satisfy the following system:

$$\langle u''(t), \phi\rangle + \langle u_x(t), \phi_x\rangle + \lambda_1 \langle \Psi_{r_1}(u_1'(t)) - \Psi_{r_1}(u_2'(t)), \phi\rangle + \mu_1 u'(1,t)\phi(1) \tag{3.44}$$
$$= K_1 [\Psi_{p_1}(u_1(1,t)) - \Psi_{p_1}(u_2(1,t))]\phi(1) + \langle f_1(u_1,v_1) - f_2(u_2,v_2), \phi\rangle,$$

$$\langle v''(t), \tilde{\phi}\rangle + \langle v_x(t), \tilde{\phi}_x\rangle + \lambda_2 \langle \Psi_{r_2}(v_1'(t)) - \Psi_{r_2}(v_2'(t)), \tilde{\phi}\rangle + \mu_2 v'(0,t)\tilde{\phi}(0) \tag{3.45}$$
$$= K_2 [\Psi_{p_2}(v_1(0,t)) - \Psi_{p_2}(v_2(0,t))]\tilde{\phi}(0) + \langle f_2(u_1,v_1) - f_2(u_2,v_2), \tilde{\phi}\rangle,$$

for all $(\phi, \tilde{\phi}) \in \mathbb{V}_1 \times \mathbb{V}_2$. We endow this system with the initial conditions

$$u(0) = v(0) = u'(0) = v'(0) = 0.$$

Taking into account $(\phi, \tilde{\phi}) = (u', v')$ in (3.44) and (3.45), then integrating with respect to $t$, we obtain the following:

$$\mathcal{W}(t) \tag{3.46}$$
$$= 2\int_0^t \langle f_1(u_1,v_1) - f_1(u_2,v_2), u'(s)\rangle ds + 2\int_0^t \langle f_2(u_1,v_1) - f_2(u_2,v_2), v'(s)\rangle ds$$
$$+ 2K_1 \int_0^t [\Psi_{p_1}(u_1(1,s)) - \Psi_{p_1}(u_2(1,s))] u'(1,s)\, ds$$
$$+ 2K_2 \int_0^t [\Psi_{p_2}(v_1(0,s)) - \Psi_{p_2}(v_2(0,s))] v'(0,s)\, ds$$
$$= \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3 + \mathcal{K}_4,$$

where we have denoted by

$$\mathcal{W}(t) := \|u'(t)\|^2 + \|v'(t)\|^2 + \|u_x(t)\|^2 + \|v_x(t)\|^2 \tag{3.47}$$
$$+ 2\lambda_1 \int_0^t \langle \Psi_{r_1}(u_1'(s)) - \Psi_{r_1}(u_2'(s)), u'(s)\rangle ds + 2\mu_1 \int_0^t |u'(1,s)|^2 ds$$
$$+ 2\lambda_2 \int_0^t \langle \Psi_{r_2}(v_1'(s)) - \Psi_{r_2}(v_2'(s)), v'(s)\rangle ds + 2\mu_2 \int_0^t |v'(0,s)|^2 ds.$$

Essentially, our procedure below is similar to the above parts: attempt to estimate $\mathcal{K}_i$ for $i = 1,4$ to derive the uniform boundedness of $\mathcal{W}(t)$ for which we can use Gronwall's inequality, then the proof of uniqueness is self-contained. To handle this, we first state the following inequality: for all $r \geq 2$, there exists $\bar{C}_r > 0$ such that

$$(\Psi_r(z_1) - \Psi_r(z_2))(z_1 - z_2) \geq \bar{C}_r |z_1 - z_2|^r, \quad \forall z_1, z_2 \in \mathbb{R}.$$

It therefore leads to the fact that

$$\mathcal{W}(t) \geq \|u'(t)\|^2 + \|v'(t)\|^2 + \|u_x(t)\|^2 + \|v_x(t)\|^2 \tag{3.48}$$
$$+ 2\bar{C}_{r_1}\lambda_1 \int_0^t \|u'(s)\|_{L^{r_1}}^{r_1} ds + \bar{C}_{r_2}\lambda_2 \int_0^t \|v'(s)\|_{L^{r_2}}^{r_2} ds$$
$$+ 2\mu_1 \int_0^t |u'(1,s)|^2 ds + 2\mu_2 \int_0^t |v'(0,s)|^2 ds.$$

Second, we introduce

$$M = \max_{i=1,2} \left( \|u_{ix}\|_{L^\infty(0,T_*;H^1)} + \|v_{ix}\|_{L^\infty(0,T_*;H^1)} \right),$$

$$\tilde{\mathcal{C}}\left(M\right)=\max_{i=1,2}\sup_{|y|,|z|\leq M}\left(\left|\frac{\partial f_i}{\partial y}\left(y,z\right)\right|+\left|\frac{\partial f_i}{\partial z}\left(y,z\right)\right|\right),\quad i=1,2.$$

Then it is sufficient to estimate $\mathcal{K}_i$ for $i=\overline{1,4}$. Indeed, we apply the Cauchy-Schwartz inequality to have

$$\begin{aligned}
&\mathcal{K}_1+\mathcal{K}_2 \\
&\leq 2\int_0^t \left[\left\|f_1\left(u_1,v_1\right)-f_1\left(u_2,v_2\right)\right\|\left\|u'\left(s\right)\right\|+\left\|f_2\left(u_1,v_1\right)-f_2\left(u_2,v_2\right)\right\|\left\|v'\left(s\right)\right\|\right]ds \\
&\leq 2\int_0^t \left(\left\|u\left(s\right)\right\|+\left\|v\left(s\right)\right\|\right)\left(\tilde{\mathcal{C}}\left(M\right)\left\|u'\left(s\right)\right\|+\tilde{\mathcal{C}}\left(M\right)\left\|v'\left(s\right)\right\|\right)ds \\
&\leq 2\int_0^t \left(\left\|u_x\left(s\right)\right\|+\left\|v_x\left(s\right)\right\|\right)\left(\tilde{\mathcal{C}}\left(M\right)\left\|u'\left(s\right)\right\|+\tilde{\mathcal{C}}\left(M\right)\left\|v'\left(s\right)\right\|\right)ds \\
&\leq 4\tilde{\mathcal{C}}\left(M\right)\int_0^t \mathcal{W}\left(s\right)ds.
\end{aligned}\tag{3.49}$$

To estimate $\mathcal{K}_3$ and $\mathcal{K}_4$, we only need to consider two cases, $p_1=p_2=2$ and $p_1$, $p_2>2$. First, one may easily show that for $p_1=p_2=2$,

$$\begin{aligned}
&\mathcal{K}_3+\mathcal{K}_4 \\
&=2K_1\int_0^t u\left(1,s\right)u'\left(1,s\right)ds+2K_2\int_0^t v\left(0,s\right)v'\left(0,s\right)ds \\
&\leq \frac{K_1^2}{\mu_1}\int_0^t u^2\left(1,s\right)ds+\mu_1\int_0^t\left|u'\left(1,s\right)\right|^2ds+\frac{K_2^2}{\mu_2}\int_0^t v^2\left(0,s\right)ds+\mu_2\int_0^t\left|v'\left(0,s\right)\right|^2ds \\
&\leq \left(\frac{K_1^2}{\mu_1}+\frac{K_2^2}{\mu_2}\right)\int_0^t\mathcal{W}\left(s\right)ds+\frac{1}{2}\mathcal{W}\left(t\right),
\end{aligned}\tag{3.50}$$

where we have followed from (3.48) the inequality

$$\mathcal{W}\left(t\right)\geq u^2\left(1,t\right)+v^2\left(0,t\right)+2\left(\mu_1\int_0^t\left|u'\left(1,s\right)\right|^2ds+\mu_2\int_0^t\left|v'\left(0,s\right)\right|^2ds\right).$$

For $p_1$, $p_2>2$, we have

$$\begin{aligned}
\mathcal{K}_3+\mathcal{K}_4 &\leq 2\left[K_1\left(p_1-1\right)M^{p_1-2}\int_0^t\left|u\left(1,s\right)\right|\left|u'\left(1,s\right)\right|ds\right. \\
&\qquad\left.+K_2\left(p_2-1\right)M^{p_2-2}\int_0^t\left|v\left(0,s\right)\right|\left|v'\left(0,s\right)\right|ds\right] \\
&\leq \left[\frac{K_1^2}{\mu_1}\left(p_1-1\right)^2M^{2p_1-4}+\frac{K_2^2}{\mu_2}\left(p_2-1\right)^2M^{2p_2-4}\right]\int_0^t\mathcal{W}\left(s\right)ds+\frac{1}{2}\mathcal{W}\left(t\right),
\end{aligned}\tag{3.51}$$

where we have recalled the inequality (3.38).

Combining (3.50) and (3.51), we claim that there exists $\eta=\eta\left(p_1,p_2\right)>0$ depending on $p_1$, $p_2$ such that

$$\mathcal{K}_3+\mathcal{K}_4\leq\eta\left(p_1,p_2\right)\int_0^t\mathcal{W}\left(s\right)ds+\frac{1}{2}\mathcal{W}\left(t\right).\tag{3.52}$$

Therefore, (3.49) and (3.52) together with (3.46) – (3.48) imply that

$$\mathcal{W}\left(t\right)\leq 2\left[4\tilde{\mathcal{C}}\left(M\right)+\eta\left(p_1,p_2\right)\right]\int_0^t\mathcal{W}\left(s\right)ds.$$

Thanks to Gronwall's inequality, we have $\mathcal{W}(t) \equiv 0$ that indicates the uniqueness of solution. Hence, this completes the proof of the theorem. □

**Remark 3.4.** It is worth noting in the above theorem that the existence of a strong solution can be obtained from the regularity of weak solutions. In fact, (3.2) allows us to show that there exists a pair of strong solutions $(u, v)$ to the problem $(P)$, which satisfies

$$\begin{cases} (u,v) \in L^\infty\left(0,T_*;\left(\mathbb{V}_1\cap H^2\right)\times\left(\mathbb{V}_2\cap H^2\right)\right)\cap C^0\left([0,T_*];\mathbb{V}_1\times\mathbb{V}_2\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cap C^1\left([0,T_*];L^2\times L^2\right), \\ (u',v') \in L^\infty\left(0,T_*;\mathbb{V}_1\times\mathbb{V}_2\right)\cap C^0\left([0,T_*];L^2\times L^2\right), \\ (u'',v'') \in L^\infty\left(0,T;L^2\times L^2\right), \\ |u'|^{\frac{r_1}{2}-1}u', \quad |v'|^{\frac{r_2}{2}-1}v' \in H^1\left(Q_{T_*}\right), \\ |u'(1,\cdot)|^{\frac{q_1}{2}-1}u'(1,\cdot), \quad |v'(0,\cdot)|^{\frac{q_2}{2}-1}v'(0,\cdot) \in H^1\left(0,T_*\right). \end{cases} \tag{3.53}$$

Set the following assumptions:
**$(\mathbf{B_1})$** $(\tilde{u}_0,\tilde{u}_1) \in \mathbb{V}_1 \times L^2$ and $(\tilde{v}_0,\tilde{v}_1) \in \mathbb{V}_2 \times L^2$;
**$(\mathbf{B_2})$** $F_1, F_2 \in L^2(Q_T)$.

In the following theorem, the existence and uniqueness of a local weak solution are also obtainable using **$(\mathbf{B_1})$-$(\mathbf{B_2})$** instead of **$(\mathbf{A_1})$-$(\mathbf{A_2})$**. In this scenario, we remark that the regularity of initial data is lower (compared to **$(\mathbf{A_1})$**), while the external functions lack of information (compared to **$(\mathbf{A_2})$**).

**Theorem 3.3.** *Let $q_1 = q_2 = 2$ and $p_1, p_2 \geq 2$ in the problem $(P)$. Assume that **$(\mathbf{A_3})$** and **$(\mathbf{B_1})$-$(\mathbf{B_2})$** hold. Moreover, the initial data obey the compatibility conditions (3.1). Then the problem $(P)$ admits a unique local solution $(u, v)$ such that*

$$\begin{cases} (u,v) \in C^0\left([0,T_*];\mathbb{V}_1\times\mathbb{V}_2\right)\cap C^1\left([0,T_*];L^2\times L^2\right), \\ (u',v') \in L^{r_1}\left(Q_{T_*}\right)\times L^{r_2}\left(Q_{T_*}\right), u(1,\cdot),\ v(0,\cdot) \in H^1\left(0,T_*\right), \end{cases}$$

*for $T_* > 0$ sufficiently small.*

*Proof of Theorem 3.3.* In this proof, we establish several sequences $\{(u_{0m}, u_{1m})\} \in C_0^\infty\left(\overline{\Omega}\right)\times C_0^\infty\left(\overline{\Omega}\right)$, $\{(v_{0m}, v_{1m})\} \in C_0^\infty\left(\overline{\Omega}\right)\times C_0^\infty\left(\overline{\Omega}\right)$ and $\{(F_{1m}, F_{2m})\} \subset C_0^\infty\left(\overline{Q_T}\right)\times C_0^\infty\left(\overline{Q_T}\right)$ satisfying

$$(u_{0m}, u_{1m}) \to (\tilde{u}_0,\tilde{u}_1) \text{ strong in } \mathbb{V}_1\times L^2,$$
$$(v_{0m}, v_{1m}) \to (\tilde{v}_0,\tilde{v}_1) \text{ strong in } \mathbb{V}_2\times L^2,$$
$$(F_{1m}, F_{2m}) \to (F_1,F_2) \text{ strong in } [L^2(Q_T)]^2.$$

Note that the sequences $\{(u_{0m}, u_{1m})\}$ and $\{(v_{0m}, v_{1m})\}$, as a result, satisfy themselves the compatibility relation for all $m \in \mathbb{N}$. So what we can deduce next is the existence of a pair of unique functions $(u_m, v_m)$ for each $m$ that makes the conditions in the aforementioned theorem self-propelling. Thus, one easily verifies that such functions $(u_m, v_m)$ for each $m$ satisfy the variational problem (2.1)-(2.2), i.e.

$$\begin{cases} \langle u_m''(t),\phi\rangle + \langle u_{mx}(t),\phi_x\rangle + \lambda_1\langle\Psi_{r_1}(u_m'(t)),\phi\rangle + \mu_1 u_m'(1,t)\phi(1) \\ \qquad\qquad = K_1\Psi_{p_1}(u_m(1,t))\phi(1) + \langle f_1(u_m,v_m),\phi\rangle + \langle F_{1m}(t),\phi\rangle, \\ \langle v_m''(t),\tilde{\phi}\rangle + \langle v_{mx}(t),\tilde{\phi}_x\rangle + \lambda_2\langle\Psi_{r_2}(v_m'(t)),\tilde{\phi}\rangle + \mu_2 v_m'(0,t)\tilde{\phi}(0) \\ \qquad\qquad = K_2\Psi_{p_2}(v_m(0,t))\tilde{\phi}(0) + \langle f_2(u_m,v_m),\tilde{\phi}\rangle + \langle F_{2m}(t),\tilde{\phi}\rangle, \end{cases} \tag{3.54}$$

for all $(\phi,\tilde{\phi}) \in \mathbb{V}_1\times\mathbb{V}_2$, together with the initial conditions

$$(u_m(0), u_m'(0)) = (u_{0m}, u_{1m}), \quad (v_m(0), v_m'(0)) = (v_{0m}, v_{1m}). \tag{3.55}$$

Moreover, the smoothness of $(u_m, v_m)$ on the interval $[0, T_*]$ is said by (3.53) and we recall below the uniform boundedness (independent of $m$) of $\mathcal{S}_m(t)$ on $[0, T_*]$ in (3.5) due to the same arguments derived above.

$$\mathcal{S}_m(t) = \|u_m'(t)\|^2 + \|v_m'(t)\|^2 + \|u_{mx}(t)\|^2 + \|v_{mx}(t)\|^2 + 2\lambda_1 \int_0^t \|u_m'(s)\|_{L^{r_1}}^{r_1} ds \tag{3.56}$$
$$+ 2\lambda_2 \int_0^t \|v_m'(s)\|_{L^{r_2}}^{r_2} ds + 2\mu_1 \int_0^t |u_m'(1,s)|^2 ds + 2\mu_2 \int_0^t |v_m'(0,s)|^2 ds \le C_T,$$

where $C_T$ denotes a positive constant independent of $m$ and $t$ and $t$ moves along the interval $[0, T_*]$.

Define $U_{m,k} := u_m - u_k$ and $V_{m,k} := v_m - v_k$, then these quantities satisfy

$$\begin{cases} \langle U_{m,k}''(t), \phi\rangle + \langle \nabla U_{m,k}(t), \phi_x\rangle + \lambda_1 \langle \Psi_{r_1}(u_m'(t)) - \Psi_{r_1}(u_k'(t)), \phi\rangle \\ +\mu_1 U_{m,k}'(1,t)\phi(1) = K_1 [\Psi_{p_1}(u_m(1,t)) - \Psi_{p_1}(u_k(1,t))]\phi(1) \\ +\langle f_1(u_m, v_m) - f_1(u_k, v_k), \phi\rangle + \langle F_{1m}(t) - F_{1k}(t), \phi\rangle, \\ \langle V_{m,k}''(t), \tilde{\phi}\rangle + \langle \nabla V_{m,k}(t), \tilde{\phi}_x\rangle + \lambda_2 \langle \Psi_{r_2}(v_m'(t)) - \Psi_{r_2}(v_k'(t)), \tilde{\phi}\rangle \\ +\mu_2 V_{m,k}'(0,t)\tilde{\phi}(0) = K_2 [\Psi_{p_2}(v_m(0,t)) - \Psi_{p_2}(v_k(0,t))]\tilde{\phi}(0) \\ +\langle f_2(u_m, v_m) - f_2(u_k, v_k), \tilde{\phi}\rangle + \langle F_{2m}(t) - F_{2k}(t), \tilde{\phi}\rangle, \end{cases} \tag{3.57}$$

for all $(\phi, \tilde{\phi}) \in \mathbb{V}_1 \times \mathbb{V}_2$ and the initial conditions are

$$\begin{aligned} \left(U_{m,k}(0), U_{m,k}'(0)\right) &= (u_{0m} - u_{0k}, u_{1m} - u_{1k}), \\ \left(V_{m,k}(0), V_{m,k}'(0)\right) &= (v_{0m} - v_{0k}, v_{1m} - v_{1k}). \end{aligned} \tag{3.58}$$

It is obvious to obtain the fact that

$$\begin{aligned} \mathcal{S}_{m,k}(t) = \mathcal{S}_{m,k}(0) &+ 2\int_0^t \big(\langle f_1(u_m, v_m) - f_1(u_k, v_k), U_{m,k}'(s)\rangle \\ &+\langle f_2(u_m, v_m) - f_2(u_k, v_k), V_{m,k}'(s)\rangle\big) ds \\ &+ 2\int_0^t \left[\langle F_{1m}(s) - F_{1k}(s), U_{m,k}'(s)\rangle + \langle F_{2m}(s) - F_{2k}(s), V_{m,k}'(s)\rangle\right] ds \\ &+ 2K_1 \int_0^t [\Psi_{p_1}(u_m(1,s)) - \Psi_{p_1}(u_k(1,s))] U_{m,k}'(1,s) ds \\ &+ 2K_2 \int_0^t [\Psi_{p_2}(v_m(0,s)) - \Psi_{p_2}(v_k(0,s))] V_{m,k}'(0,s) ds \\ &= \mathcal{S}_{m,k}(0) + \mathcal{Z}_1 + \mathcal{Z}_2 + \mathcal{Z}_3 + \mathcal{Z}_4, \end{aligned} \tag{3.59}$$

where we have denoted by

$$\begin{aligned} \mathcal{S}_{m,k}(t) = &\left\|U_{m,k}'(t)\right\|^2 + \left\|V_{m,k}'(t)\right\|^2 + \|\nabla U_{m,k}(t)\|^2 + \|\nabla V_{m,k}(t)\|^2 \\ &+ 2\lambda_1 \int_0^t \langle \Psi_{r_1}(u_m'(s)) - \Psi_{r_1}(u_k'(s)), U_{m,k}'(s)\rangle ds \\ &+ 2\lambda_2 \int_0^t \langle \Psi_{r_2}(v_m'(s)) - \Psi_{r_2}(v_k'(s)), V_{m,k}'(s)\rangle ds \\ &+ 2\mu_1 \int_0^t \left|U_{m,k}'(1,s)\right|^2 ds + 2\mu_2 \int_0^t \left|V_{m,k}'(0,s)\right|^2 ds, \end{aligned} \tag{3.60}$$

$$\mathcal{S}_{m,k}(0) = \|u_{1m} - u_{1k}\|^2 + \|v_{1m} - v_{1k}\|^2 + \|u_{0mx} - u_{0kx}\|^2 + \|v_{0mx} - v_{0kx}\|^2 . \tag{3.61}$$

The above calculations are done by a valid replacement of the test functions $\phi$ and $\tilde{\phi}$ by $U'_{m,k}$ and $V'_{m,k}$, respectively, in (3.57) and then integrating with respect to $t$. By the same strategy and using (3.38), (3.56) and (3.60), we can estimate the terms on the right-hand side of (3.59) and obtain

$$\mathcal{S}_{m,k}(t) \le \mathcal{R}_{m,k} + 2\left(1 + \eta(p_1, p_2) + 8\mathcal{R}_T\right) \int_0^t \mathcal{S}_{m,k}(s)\, ds, \tag{3.62}$$

where the involved terms are appropriately defined as follows:

$$\mathcal{R}_T := \max_{i=1,2} \sup_{|y|,|z| \le \sqrt{C_T}} \left( \left| \frac{\partial f_i}{\partial y}(y,z) \right| + \left| \frac{\partial f_i}{\partial z}(y,z) \right| \right),$$

$$\eta(p_1, p_2) := \frac{K_1^2}{\mu_1}(p_1 - 1)^2 C_T^{p_1 - 2} + \frac{K_2^2}{\mu_2}(p_2 - 1)^2 C_T^{p_2 - 2},$$

$$\mathcal{R}_{m,k} := 2\mathcal{S}_{m,k}(0) + 2\|F_{1m} - F_{1k}\|^2_{L^2(Q_T)} + 2\|F_{2m} - F_{2k}\|^2_{L^2(Q_T)} .$$

Here we remark that $\mathcal{R}_{m,k}$ approaches zero as $m$ and $k$ tend to infinity. By the aid of Gronwall's inequality, it follows from (3.62) that for all $t \in [0, T_*]$

$$\mathcal{S}_{m,k}(t) \le \mathcal{R}_{m,k} \exp\left(2T\left(1 + \eta(p_1, p_2) + 8\mathcal{R}_T\right)\right). \tag{3.63}$$

Thus, one can show that the right-hand side of (3.63) goes to zero as the indexes $m$ and $k$ tend to infinity by the direct argument concerning convergences of $\{(u_{0m}, u_{1m})\}$ and $\{(v_{0m}, v_{1m})\}$. As by-product, it gives us the following results

$$(u_m, v_m) \to (u, v) \text{ strong in } C\left([0, T_*]; \mathbb{V}_1 \times \mathbb{V}_2\right) \cap C^1\left([0, T_*]; L^2 \times L^2\right), \tag{3.64}$$

$$(u'_m, v'_m) \to (u', v') \text{ strong in } L^{r_1}(Q_{T_*}) \times L^{r_2}(Q_{T_*}), \tag{3.65}$$

$$(u_m(1, \cdot), v_m(0, \cdot)) \to (u(1, \cdot), v(0, \cdot)) \text{ strong in } [H^1(0, T_*)]^2. \tag{3.66}$$

An important point should be mentioned here is that by (3.56) we can extract a subsequence of $\{(u_m, v_m)\}$ (still relabel with the old index $m$) which reads as

$$(u_m, v_m) \to (u, v) \text{ weak-* in } L^\infty(0, T_*; \mathbb{V}_1 \times \mathbb{V}_2), \tag{3.67}$$

$$(u'_m, v'_m) \to (u', v') \text{ weak-* in } L^\infty\left(0, T_*; L^2 \times L^2\right), \tag{3.68}$$

and inheriting from (3.64)-(3.66), one deduces

$$(f_1(u_m, v_m), f_2(u_m, v_m)) \to (f_1(u, v), f_2(u, v)) \text{ strong in } [L^2(Q_{T_*})]^2, \tag{3.69}$$

$$(\Psi_{r_1}(u'_m), \Psi_{r_2}(v'_m)) \to (\Psi_{r_1}(u'), \Psi_{r_2}(v')) \text{ strong in } [L^2(Q_{T_*})]^2. \tag{3.70}$$

Henceforward, it is advantageous to state our limit processing now. In fact, passing to the limit in (3.54) associated with (3.55) and evidenced by (3.64)-(3.70), we obtain a couple of functions $(u, v)$ satisfying the variational problem

$$\begin{cases} \frac{d}{dt}\langle u'(t), \phi\rangle + \langle u_x(t), \phi_x\rangle + \lambda_1 \langle \Psi_{r_1}(u'(t)), \phi\rangle + \mu_1 \Psi_{q_1}(u'(1,t))\phi(1) \\ \qquad = K_1 \Psi_{p_1}(u(1,t))\phi(1) + \langle f_1(u,v), \phi\rangle + \langle F_1(t), \phi\rangle, \\ \frac{d}{dt}\left\langle v'(t), \tilde{\phi}\right\rangle + \left\langle v_x(t), \tilde{\phi}_x\right\rangle + \lambda_2 \left\langle \Psi_{r_2}(v'(t)), \tilde{\phi}\right\rangle + \mu_2 \Psi_{q_2}(v'(0,t))\tilde{\phi}(0) \\ \qquad = K_2 \Psi_{p_2}(v(0,t))\tilde{\phi}(0) + \left\langle f_2(u,v), \tilde{\phi}\right\rangle + \left\langle F_2(t), \tilde{\phi}\right\rangle, \end{cases}$$

for all $(\phi, \tilde{\phi}) \in \mathbb{V}_1 \times \mathbb{V}_2$, and endowed with (2.2). This also indicates the existence of a local solution. The uniqueness of such a weak solution is directly obtained by using the regularization procedure investigated by Lions (see e.g. [13]). Hence, we end up with the proof of the theorem. □

**Remark 3.5.** If one has $N = \frac{1}{2}\max\left\{2;\alpha;\beta;\frac{q_1(p_1-1)}{q_1-1};\frac{q_2(p_2-1)}{q_2-1}\right\} \leq 1$ (cf. Remark 3.3) and considers the assumptions $(\mathbf{B_1})$ and $(\mathbf{B}_2)$, the integral $\mathcal{S}_m(t)$ (which has been bounded by a constant $C_T$; see in (3.10)) can be estimated globally (and uniformly as well) in time, i.e.

$$\mathcal{S}_m(t) \leq C_T,\ \forall m \in \mathbb{N},\ \forall t \in [0,T],\ \forall T > 0.$$

Consequently, all arguments used in the proof of Theorem 3.3 are applicable to prove that there exists a global weak solution $(u,v)$ to the problem $(P)$ and satisfying

$$\begin{aligned}&(u,v) \in L^\infty(0,T;\mathbb{V}_1\times\mathbb{V}_2),(u',v') \in L^\infty\left(0,T;L^2\times L^2\right),\\&u(1,\cdot),v(0,\cdot)\in H^1(0,T).\end{aligned} \tag{3.71}$$

Nevertheless, it should be noticed that the aforementioned case of $N$ does not imply the weak solution belongs to $C([0,T];\mathbb{V}_1\times\mathbb{V}_2)\cap C^1\left([0,T];L^2\times L^2\right)$ and one cannot also say any further statement for the uniqueness in this case.

4. **Finite time blow-up.** In principle, the blow-up phenomenon of a solution to a time-dependent equation is devoted to the study of maximal time domain for which it is defined by a finite length. At the endpoint of that interval, the solution behaves in such a way that either it goes to infinity in some specific senses, or it stops being smooth, and so forth. Our main objective here is to show that if the initial energy is negative, then every weak solution of $(P)$ blows up in finite time. The result here draws from ideas in the treatment of a single wave equation [7, 12, 17] and also, for example, the recent results for systems in [6, 14], but our proofs have to be radically altered. In addition, the technique we use here is an adaptation of an argument postulated in [18] to treat the boundary damping terms.

Let us first make a brief note concerning the so-called total energy. From the mathematical point of view, the energy method plays a vital role in the study of partial differential equations and the energy is usually used to derive the well-posedness of such equations. For a very fundamental scalar wave equation which was originally discovered by d'Alembert, one may easily find the energy integral computed by

$$\frac{1}{2}\int_\Omega |u_t|^2\,dx + \frac{1}{2}\int_\Omega |u_x|^2\,dx,$$

where we basically multiply the equation by $u_t$ and integrate over $\Omega$ and use the divergence theorem.

In the energy equation, the first term is typically called as the kinetic energy, while the second term is referred to as the potential energy. If one imposes mixed boundary conditions, says $au + bu_x = 0$ on the boundary, where $a$ and $b$ have the same sign, the energy can only decrease. If they have opposite sign, the problem is unstable in the sense that the energy will increase. In this section, the total energy are computed in a common way and then it would be a decreasing function along the trajectories, starting from a negative initial value.

From here on, we aim to consider the problem $(P)$ in a specific case: linear damping $r_i = 2$ with $F_i = 0$, and $q_i = 2$, $p_i > 2$, $K_i > 0$, $\lambda_i > 0$, $\mu_i > 0$ for $i = 1, 2$. In this case, the total quadratic-type energy $E(t)$ associated with the solution $(u,v)$

is defined by

$$E(t) := \frac{1}{2}\left(\|u'(t)\|^2 + \|v'(t)\|^2 + \|u_x(t)\|^2 + \|v_x(t)\|^2\right) - \left(\frac{K_1}{p_1}|u(1,t)|^{p_1} + \frac{K_2}{p_2}|v(0,t)|^{p_2}\right) - \int_0^1 \mathcal{F}(u(x,t), v(x,t))\,dx.$$

To prove the solution in this case blows up in finite time, we consider the following assumptions on the interior sources and on the initial energy:

**($\mathbf{A}_3'$)** there exists a $C^2$-function $\mathcal{F} : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{\partial \mathcal{F}}{\partial u}(u,v) = f_1(u,v), \quad \frac{\partial \mathcal{F}}{\partial v}(u,v) = f_2(u,v),$$

and there also exists the constants $\alpha,\ \beta > 2;\ d_1,\ d_2,\ \bar{d}_1,\ \bar{d}_2 > 0$ such that

$$d_1\mathcal{F}(u,v) \le uf_1(u,v) + vf_2(u,v) \le d_2\mathcal{F}(u,v) \quad \text{for all } (u,v) \in \mathbb{R}^2, \tag{4.1}$$

$$\bar{d}_1\left(|u|^\alpha + |v|^\beta\right) \le \mathcal{F}(u,v) \le \bar{d}_2\left(|u|^\alpha + |v|^\beta\right) \quad \text{for all } (u,v) \in \mathbb{R}^2. \tag{4.2}$$

**($\mathbf{A}_4$)** define $H(t) := -E(t)$ and assume that

$$-H(0) = \frac{1}{2}\left(\|\tilde{u}_1\|^2 + \|\tilde{v}_1\|^2 + \|\tilde{u}_{0x}\|^2 + \|\tilde{v}_{0x}\|^2\right) - \left(\frac{K_1}{p_1}|\tilde{u}_0(1)|^{p_1} + \frac{K_2}{p_2}|\tilde{v}_0(0)|^{p_2}\right) - \int_0^1 \mathcal{F}(\tilde{u}_0(x), \tilde{v}_0(x))\,dx < 0. \tag{4.3}$$

It is worth noting that the time derivative of $H$ is non-negative along the trajectories in a local time, namely

$$H'(t) = \lambda_1\|u'(t)\|^2 + \lambda_2\|v'(t)\|^2 + \mu_1|u'(1,t)|^2 + \mu_2|v'(0,t)|^2 \ge 0, \tag{4.4}$$

for all $t \in [0, T_*)$ where we have multiplied (1.1) by $(u'(x,t), v'(x,t))$ and integrated the resulting equation over $\Omega$. Together with the fact that $H(0) > 0$, we have

$$0 < H(0) \le H(t),\ \forall t \in [0, T_*). \tag{4.5}$$

Observe the final term in the total energy $E(t)$, it follows from (4.2) that

$$\bar{d}_1\left(\|u(t)\|_{L^\alpha}^\alpha + \|v(t)\|_{L^\beta}^\beta\right) \le \int_0^1 \mathcal{F}(u(x,t), v(x,t))\,dx \le \bar{d}_2\left(\|u(t)\|_{L^\alpha}^\alpha + \|v(t)\|_{L^\beta}^\beta\right). \tag{4.6}$$

Combining (4.5) and (4.6) gives

$$\begin{aligned} 0 < H(t) &\le \frac{K_1}{p_1}|u(1,t)|^{p_1} + \frac{K_2}{p_2}|v(0,t)|^{p_2} + \bar{d}_2\left(\|u(t)\|_{L^\alpha}^\alpha + \|v(t)\|_{L^\beta}^\beta\right) \\ &\le \bar{D}_2\left(|u(1,t)|^{p_1} + |v(0,t)|^{p_2} + \|u(t)\|_{L^\alpha}^\alpha + \|v(t)\|_{L^\beta}^\beta\right), \end{aligned} \tag{4.7}$$

for all $t \in [0, T_*)$ and $\bar{D}_2 = \max\left\{\frac{K_1}{p_1}, \frac{K_2}{p_2}, \bar{d}_2\right\}$. It also allows us to derive that for all $t \in [0, T_*)$

$$\begin{aligned} &\frac{1}{2}\left(\|u_x(t)\|^2 + \|v_x(t)\|^2\right) \\ &\le \frac{K_1}{p_1}|u(1,t)|^{p_1} + \frac{K_2}{p_2}|v(0,t)|^{p_2} + \int_0^1 \mathcal{F}(u(x,t), v(x,t))\,dx \\ &\le \bar{D}_2\left(|u(1,t)|^{p_1} + |v(0,t)|^{p_2} + \|u(t)\|_{L^\alpha}^\alpha + \|v(t)\|_{L^\beta}^\beta\right). \end{aligned} \tag{4.8}$$

Now we construct the following functional

$$L(t) := H^{1-\xi}(t) + \varepsilon\psi(t), \tag{4.9}$$

where we have defined

$$\psi(t) := \langle u(t), u'(t)\rangle + \langle v(t), v'(t)\rangle + \frac{\lambda_1}{2}\|u(t)\|^2 + \frac{\lambda_2}{2}\|v(t)\|^2 + \frac{\mu_1}{2}u^2(1,t) + \frac{\mu_2}{2}v^2(0,t), \tag{4.10}$$

for $\varepsilon > 0$ sufficiently small and $\xi \in \left(0, \min\left\{\frac{\alpha-2}{2\alpha}, \frac{\beta-2}{2\beta}\right\}\right] \subset \left(0, \frac{1}{2}\right)$.

To show that one can choose $\varepsilon > 0$ small enough such that $L(t)$ is non-decreasing for all $t \in [0, T_*)$, namely

$$L(t) \geq L(0) > 0, \quad \forall t \in [0, T_*), \tag{4.11}$$

a very clear way is to consider the derivative of such a function. In fact, let us prove the following lemma.

**Lemma 4.1.** *There exists a positive constant $\gamma$ such that*

$$L'(t) \geq \gamma\Big[H(t) + \|u'(t)\|^2 + \|v'(t)\|^2 + \|u_x(t)\|^2 + \|v_x(t)\|^2 + \|u(t)\|_{L^\alpha}^\alpha + \|v(t)\|_{L^\beta}^\beta + |u(1,t)|^{p_1} + |v(0,t)|^{p_2}\Big]. \tag{4.12}$$

*Proof of Lemma 4.1.* Multiplying (1.1) by $(u(x,t), v(x,t))$ and then integrating the resulting equation over $\Omega$, the derivative of $\psi(t)$ can be defined as follows:

$$\psi'(t) = \|u'(t)\|^2 + \|v'(t)\|^2 - \left(\|u_x(t)\|^2 + \|v_x(t)\|^2\right) + K_1|u(1,t)|^{p_1} + K_2|v(0,t)|^{p_2} + \langle f_1(u(t), v(t)), u(t)\rangle + \langle f_2(u(t), v(t)), v(t)\rangle.$$

By using this formulation and the fact computed from (4.9) that

$$\begin{aligned} L'(t) &= (1-\xi)H^{-\xi}(t)H'(t) + \varepsilon\left(\|u'(t)\|^2 + \|v'(t)\|^2\right) \\ &\quad - \varepsilon\left(\|u_x(t)\|^2 + \|v_x(t)\|^2\right) + \varepsilon\left(K_1|u(1,t)|^{p_1} + K_2|v(0,t)|^{p_2}\right) \\ &\quad + \varepsilon\left(\langle f_1(u(t), v(t)), u(t)\rangle + \langle f_2(u(t), v(t)), v(t)\rangle\right), \end{aligned}$$

in combination with (4.4), (4.5), (4.8) and the following inequality

$$\begin{aligned} &\langle f_1(u(t), v(t)), u(t)\rangle + \langle f_2(u(t), v(t)), v(t)\rangle \\ &\geq d_1\int_0^1 \mathcal{F}(u(x,t), v(x,t))\,dx \geq d_1\bar{d}_1\left(\|u(t)\|_{L^\alpha}^\alpha + \|v(t)\|_{L^\beta}^\beta\right), \end{aligned}$$

we get

$$\begin{aligned} L'(t) &\geq \varepsilon\left(\|u'(t)\|^2 + \|v'(t)\|^2\right) - 2\varepsilon\bar{D}_2\left(|u(1,t)|^{p_1} + |v(0,t)|^{p_2} + \|u(t)\|_{L^\alpha}^\alpha + \|v(t)\|_{L^\beta}^\beta\right) \\ &\quad + \varepsilon\left[K_1|u(1,t)|^{p_1} + K_2|v(0,t)|^{p_2} + d_1\bar{d}_1\left(\|u(t)\|_{L^\alpha}^\alpha + \|v(t)\|_{L^\beta}^\beta\right)\right] \\ &\geq \varepsilon\left(\|u'(t)\|^2 + \|v'(t)\|^2\right) \\ &\quad + \varepsilon\left(\bar{D}_1 - 2\bar{D}_2\right)\left(|u(1,t)|^{p_1} + |v(0,t)|^{p_2} + \|u(t)\|_{L^\alpha}^\alpha + \|v(t)\|_{L^\beta}^\beta\right), \end{aligned} \tag{4.13}$$

where we have put $\bar{D}_1 = \min\left\{K_1, K_2, d_1\bar{d}_1\right\}$.

If one assumes that $2\max\left\{\frac{K_1}{p_1},\frac{K_2}{p_2},\bar{d}_2\right\}<\min\left\{K_1,K_2,d_1\bar{d}_1\right\}$, we deduce

$$0<\bar{D}_3=\bar{D}_1-2\bar{D}_2=\min\left\{K_1,K_2,d_1\bar{d}_1\right\}-2\max\left\{\frac{K_1}{p_1},\frac{K_2}{p_2},\bar{d}_2\right\} \tag{4.14}$$
$$<\min\left\{K_1,K_2,d_1\bar{d}_1\right\}.$$

Recall from (4.7) that

$$\bar{D}_2\left(|u(1,t)|^{p_1}+|v(0,t)|^{p_2}+\|u(t)\|_{L^\alpha}^{\alpha}+\|v(t)\|_{L^\beta}^{\beta}\right)\geq H(t),$$

and also thanks to (4.8) and (4.13) with the assumption to get (4.14), it is sufficient to show that there exists $\gamma>0$ such that (4.12) holds. Hence, we complete the proof of the lemma. □

From Lemma 4.1, we obtain (4.11). The assumption $2\max\left\{\frac{K_1}{p_1},\frac{K_2}{p_2},\bar{d}_2\right\}<\min\left\{K_1,K_2,d_1\bar{d}_1\right\}$ is additionally necessary to approach the blow-up result. Before going to the main result, let us consider the following supplementary inequalities, whose proof can be found in Appendix A.

**Lemma 4.2.** *Let $\xi>0$ such that*

$$\begin{cases}2\leq 2/(1-2\xi)\leq\min\{\alpha,\beta\},\\ 2\leq 2/(1-\xi)\leq\min\{\alpha,\beta,p_1,p_2\},\end{cases}$$

*then the following inequalities hold*

$$\|u\|_{L^\alpha}^{2/(1-2\xi)}+\|u\|^{2/(1-\xi)}+|u(1)|^{2/(1-\xi)}\leq 3\left(\|u_x\|^2+\|u\|_{L^\alpha}^{\alpha}+|u(1)|^{p_1}\right),\ \forall u\in\mathbb{V}_1, \tag{4.15}$$

$$\|v\|_{L^\beta}^{2/(1-2\xi)}+\|v\|^{2/(1-\xi)}+|v(0)|^{2/(1-\xi)}\leq 3\left(\|v_x\|^2+\|v\|_{L^\beta}^{\beta}+|v(0)|^{p_2}\right),\ \forall v\in\mathbb{V}_2. \tag{4.16}$$

**Theorem 4.3.** *Consider $r_i=2$ with $F_i=0$ and $q_i=2,p_i>2$ for $i=1,2$ in the problem $(P)$. Suppose that* $(\mathbf{A}_1)$, $(\mathbf{A}_3')$ *and* $(\mathbf{A}_4)$ *hold. If $2\max\left\{\frac{K_1}{p_1},\frac{K_2}{p_2},\bar{d}_2\right\}<\min\left\{K_1,K_2,d_1\bar{d}_1\right\}$ holds in* $(\mathbf{A}_3')$, *then any weak solution $(u,v)$ of $(P)$ blows up in a finite time in the sense that there exists $T_*>0$ such that*

$$\lim_{t\to T_*^-}\left(\|u'(t)\|^2+\|v'(t)\|^2+\|u_x(t)\|^2+\|v_x(t)\|^2\right)=+\infty.$$

*Proof of Theorem 4.3.* By using an elementary inequality

$$\left(\sum_{i=1}^{7}z_i\right)^r\leq 7^{r-1}\sum_{i=1}^{7}z_i^r,\ \forall r>1,\ x_i\geq 0,\ i=\overline{1,7},$$

and thanks to (4.9), (4.10), one can show that

$$L^{1/(1-\xi)}(t)\leq C\left(H(t)+|\langle u(t),u'(t)\rangle|^{1/(1-\xi)}+|\langle v(t),v'(t)\rangle|^{1/(1-\xi)}\right. \tag{4.17}$$
$$\left.+\|u(t)\|^{2/(1-\xi)}+\|v(t)\|^{2/(1-\xi)}+u^{2/(1-\xi)}(1,t)+v^{2/(1-\xi)}(0,t)\right),$$

where

$$C = 2^{-1/(1-\xi)} \max \Big\{ 2^{1/(1-\xi)}, 2^{1/(1-\xi)} \varepsilon^{1/(1-\xi)}, (\lambda_1 \varepsilon)^{1/(1-\xi)}, \\ (\lambda_2 \varepsilon)^{1/(1-\xi)}, (\mu_1 \varepsilon)^{1/(1-\xi)}, (\mu_2 \varepsilon)^{1/(1-\xi)} \Big\} > 0.$$

Moreover, we find

$$|\langle u(t), u'(t) \rangle|^{1/(1-\xi)} \le \|u(t)\|^{1/(1-\xi)} \|u'(t)\|^{1/(1-\xi)} \le \|u(t)\|_{L^\alpha}^{1/(1-\xi)} \|u'(t)\|^{1/(1-\xi)}, \tag{4.18}$$

where we have used the Cauchy-Schwartz inequality and the Hölder inequality.

According to Young's inequality introduced in (3.6), by choosing $\delta = 1$, $q = \frac{2(1-\xi)}{1-2\xi}$, $q' = 2(1-\xi)$ and letting $a = \|u(t)\|_{L^\alpha}^{1/(1-\xi)}$, $b = \|u'(t)\|^{1/(1-\xi)}$ one easily obtains from (4.18) that

$$|\langle u(t), u'(t) \rangle|^{1/(1-\xi)} \le c_1 \left( \|u(t)\|_{L^\alpha}^{2/(1-2\xi)} + \|u'(t)\|^2 \right), \tag{4.19}$$

where $c_1 = \max \left\{ \frac{1-2\xi}{2(1-\xi)}, \frac{1}{2(1-\xi)} \right\} < 1$. Similarly, we have for a constant $c_2 \in (0,1)$

$$|\langle v(t), v'(t) \rangle|^{1/(1-\xi)} \le c_2 \left( \|v(t)\|_{L^\beta}^{2/(1-\xi)} + \|v'(t)\|^2 \right). \tag{4.20}$$

Therefore, there always exists a positive constant (here we reuse the notation $C$ for simplicity) such that

$$L^{1/(1-\xi)}(t) \le C \Big( H(t) + \|u'(t)\|^2 + \|v'(t)\|^2 + \|u_x(t)\|^2 + \|v_x(t)\|^2 \\ + \|u(t)\|_{L^\alpha}^\alpha + \|v(t)\|_{L^\beta}^\beta + |u(1,t)|^{p_1} + |v(0,t)|^{p_2} \Big), \ \forall t \in [0, T_*). \tag{4.21}$$

In addition, the estimate (4.12) together with (4.21) allows us to take a positive constant $\bar{C}$ such that

$$L'(t) \ge \bar{C} L^{1/(1-\xi)}(t), \quad \forall t \in [0, T_*). \tag{4.22}$$

Now, integrating (4.22) over $(0,t)$ one has

$$L^{\xi/(1-\xi)}(t) \ge \frac{1}{L^{-\xi/(1-\xi)}(0) - \frac{\bar{C}\xi}{1-\xi} t}, \ t \in \left[ 0, \frac{1-\xi}{\bar{C}\xi} L^{-\xi/(1-\xi)}(0) \right),$$

which yields that $L(t)$ definitely blows up in a finite time $T_* = \frac{1-\xi}{\bar{C}\xi} L^{-\xi/(1-\xi)}(0)$. Hence, we complete the proof of the theorem. □

5. **Exponential decay.** While the previous section employs the total energy to find the finite time blow-up result, the exponential decay is proved in this section. In particular, we study the global solution $(u,v)$ of $(P)$ satisfying (3.71) with $r_1 = r_2 = q_1 = q_2 = 2$ and $p_1, p_2 > 2$. Like the blow-up phenomenon, this sort of results can be seen as an extension of many previous works from the single wave equation in, for example, [8, 19] to the system of equations.

Our result here relies on the construction of a Lyapunov functional by performing a suitable modification of the energy. To this end, for $\delta > 0$ being chosen later, we define

$$\mathcal{L}(t) := E(t) + \delta \psi(t), \tag{5.1}$$

where we have recalled the function $\psi(t)$ in (4.10).

Under some additional conditions, it is sufficient to see that $\mathcal{L}(t)$ and $E(t)$ are equivalent in the sense that there exist two positive constants $\beta_1$ and $\beta_2$ depending on $\delta$ such that for $t \geq 0$,

$$\beta_1 E(t) \leq \mathcal{L}(t) \leq \beta_2 E(t). \tag{5.2}$$

Before explicitly providing the statement of this equivalence, let us first consider the time derivative of the total energy, which can be defined in the same way as (4.4), by the following lemma.

**Lemma 5.1.** *The time derivative of the total energy satisfies*

$$\begin{aligned} E'(t) \leq &-\lambda_* \left( \|u'(t)\|^2 + \|v'(t)\|^2 \right) - \mu_* \left( |u'(1,t)|^2 + |v'(0,t)|^2 \right) \\ &+ \frac{1}{2} \left( \|F_1(t)\| + \|F_2(t)\| \right) + \frac{1}{2} \left( \|F_1(t)\| + \|F_2(t)\| \right) \left( \|u'(t)\|^2 + \|v'(t)\|^2 \right), \end{aligned} \tag{5.3}$$

$$\begin{aligned} E'(t) \leq &-\left( \lambda_* - \frac{\varepsilon_1}{2} \right) \left( \|u'(t)\|^2 + \|v'(t)\|^2 \right) - \mu_* \left( |u'(1,t)|^2 + |v'(0,t)|^2 \right) \\ &+ \frac{1}{2\varepsilon_1} \left( \|F_1(t)\|^2 + \|F_2(t)\|^2 \right), \end{aligned} \tag{5.4}$$

*Proof of Lemma 5.1.* Multiplying (1.1) by $(u'(x,t), v'(x,t))$, integrating over $\Omega$ and using the integration by parts, we obtain

$$\begin{aligned} E'(t) = &-\lambda_1 \|u'(t)\|^2 - \lambda_2 \|v'(t)\|^2 - \mu_1 |u'(1,t)|^2 \\ &- \mu_2 |v'(0,t)|^2 + \langle F_1(t), u'(t) \rangle + \langle F_2(t), v'(t) \rangle. \end{aligned} \tag{5.5}$$

To obtain (5.3), we only need to estimate the last two terms. It is straightforward that by using the standard inequalities which read as

$$\begin{aligned} \langle F_1(t), u'(t) \rangle &\leq \frac{1}{2} \|F_1(t)\| + \frac{1}{2} \|F_1(t)\| \|u'(t)\|^2, \\ \langle F_2(t), v'(t) \rangle &\leq \frac{1}{2} \|F_2(t)\| + \frac{1}{2} \|F_2(t)\| \|v'(t)\|^2, \end{aligned}$$

it yields

$$\begin{aligned} &\langle F_1(t), u'(t) \rangle + \langle F_2(t), v'(t) \rangle \\ &\leq \frac{1}{2} \left( \|F_1(t)\| + \|F_2(t)\| \right) + \frac{1}{2} \left( \|F_1(t)\| + \|F_2(t)\| \right) \left( \|u'(t)\|^2 + \|v'(t)\|^2 \right). \end{aligned}$$

For the second estimate (5.4), we also use the same approach to prove. Indeed, by Young's inequality one deduces the following estimate

$$\begin{aligned} &\langle F_1(t), u'(t) \rangle + \langle F_2(t), v'(t) \rangle \\ &\leq \frac{1}{2\varepsilon_1} \left( \|F_1(t)\|^2 + \|F_2(t)\|^2 \right) + \frac{\varepsilon_1}{2} \left( \|u'(t)\|^2 + \|v'(t)\|^2 \right). \end{aligned}$$

Hence, the proof of the lemma is complete. □

Let us next define the following functions $I_i(t) := I_i(u(t))$ for $i = 1, 2$ and $J(t) := J(u(t))$ by rewriting the total energy $E(t)$:

$$E(t) = \frac{1}{2} \left( \|u'(t)\|^2 + \|v'(t)\|^2 \right) + J(t), \tag{5.6}$$

$$J(t) = \frac{1}{2} \left( 1 - \frac{1}{p_1} - \frac{1}{p_2} \right) \left( \|u_x(t)\|^2 + \|v_x(t)\|^2 \right) + \frac{I_1(t)}{p_1} + \frac{I_2(t)}{p_2}, \tag{5.7}$$

$$I_1(t) = \frac{1}{2}\left(\|u_x(t)\|^2 + \|v_x(t)\|^2\right) - K_1 |u(1,t)|^{p_1} - \frac{p_1}{2}\int_0^1 \mathcal{F}(u(x,t), v(x,t))\, dx, \tag{5.8}$$

$$I_2(t) = \frac{1}{2}\left(\|u_x(t)\|^2 + \|v_x(t)\|^2\right) - K_2 |v(0,t)|^{p_2} - \frac{p_2}{2}\int_0^1 \mathcal{F}(u(x,t), v(x,t))\, dx. \tag{5.9}$$

Furthermore, let us provide the following assumption:

$(\mathbf{B}_2')$ $F_1,\, F_2 \in L^\infty(\mathbb{R}_+; L^2) \cap L^1(\mathbb{R}_+; L^2)$.

From now on, our main result in this section is established where the proof of the equivalence between $\mathcal{L}(t)$ and $E(t)$ is also included. It says that if there is an exponential rate of energy decay for the external functions $F_1$ and $F_2$ and influenced by such functions, the initial-related energy function, denoted by $E_*$, is properly suited in a particular set, then the quadratic-type total energy decays exponentially.

**Theorem 5.2.** *Consider $(u, v)$ of the problem $(P)$ satisfying (3.71) with $r_1 = r_2 = q_1 = q_2 = 2$ and $p_1,\, p_2 > 2$. Suppose that $(\mathbf{A_1})$, $(\mathbf{A}_3')$ and $(\mathbf{B}_2')$ hold, together with $d_2 < \min\{p_1, p_2\}$ in $(\mathbf{A}_3')$. Assume that $I_1(0), I_2(0) > 0$ and the initial energy $E(0)$ satisfies*

$$\eta_* = \frac{p_1 + p_2}{2}\bar{d}_2\left[(p_* E_*)^{\frac{\alpha}{2}-1} + (p_* E_*)^{\frac{\beta}{2}-1}\right] + K_1 (p_* E_*)^{\frac{p_1}{2}-1} + K_2 (p_* E_*)^{\frac{p_2}{2}-1} < 1, \tag{5.10}$$

*where $p_* = \frac{2p_1p_2}{p_1p_2 - p_1 - p_2}$, $E_* = [E(0) + \rho]\, exp(2\rho)$, $\rho = \frac{1}{2}\int_0^\infty (\|F_1(s)\| + \|F_2(s)\|)\, ds$. Moreover, assume the external functions decays exponentially in the sense that*

$$\|F_1(t)\|^2 + \|F_2(t)\|^2 \le \eta_1 exp(-\eta_2 t), \quad \forall t \ge 0, \tag{5.11}$$

*where $\eta_1$ and $\eta_2$ are two positive constants. Then there exist positive constants $C$ and $\gamma$ such that*

$$E(t) \le C exp(-\gamma t), \quad \forall t \ge 0. \tag{5.12}$$

*Proof of Theorem 5.2.* First, we claim that $I_i(t) \ge 0$ for $i = 1, 2$ and for all $t \ge 0$. In fact, since $I_i(t)$, $i = 1, 2$ is continuous and their initial values are positive, thus there exist two positive constants $T_1$ and $T_2$ such that

$$I_1(t) \ge 0, \quad \forall t \in [0, T_1], \text{ and } I_2(t) \ge 0, \quad \forall t \in [0, T_2]$$

which also leads to the fact that

$$J(t) \ge \frac{1}{2}\left(1 - \frac{1}{p_1} - \frac{1}{p_2}\right)\left(\|u_x(t)\|^2 + \|v_x(t)\|^2\right), \ \forall t \in \left[0, \bar{T}\right],$$

where $\bar{T} = \min\{T_1, T_2\} > 0$. In other words, we can say that

$$\|u_x(t)\|^2 + \|v_x(t)\|^2 \le \frac{2p_1p_2}{p_1p_2 - p_1 - p_2} J(t) \le \frac{2p_1p_2}{p_1p_2 - p_1 - p_2} E(t), \ \forall t \in \left[0, \bar{T}\right]. \tag{5.13}$$

By (5.3) and thanks to $I_i(0) > 0$ for $i = 1, 2$ associated with (5.6), we deduce

$$E(t) \le E(0) + \frac{1}{2}\int_0^\infty (\|F_1(s)\| + \|F_2(s)\|)\, ds$$
$$+ \int_0^t (\|F_1(s)\| + \|F_2(s)\|)\, E(s)\, ds, \ \forall t \in \left[0, \bar{T}\right],$$

then using Gronwall's inequality leads to the following

$$E(t) \leq \left[E(0) + \frac{1}{2}\int_0^\infty \left(\|F_1(s)\| + \|F_2(s)\|\right) ds\right] \exp\left(\int_0^t \left(\|F_1(s)\| + \|F_2(s)\|\right) ds\right) \tag{5.14}$$
$$\leq [E(0) + \rho] \exp(2\rho).$$

Therefore, with $E_* = [E(0) + \rho]\exp(2\rho)$, $p_* = \frac{2p_1p_2}{p_1p_2 - p_1 - p_2}$ we combine (5.14) with (5.13) to obtain

$$\|u_x(t)\|^2 + \|v_x(t)\|^2 \leq p_* E_*, \quad \forall t \in \left[0, \bar{T}\right]. \tag{5.15}$$

By the assumption (4.2) in $(\mathbf{A}_3')$ and also (5.15), it yields

$$K_1 |u(1,t)|^{p_1} + K_2 |v(0,t)|^{p_2} \leq K_1 \|u_x(t)\|^{p_1} + K_2 \|v_x(t)\|^{p_2} \tag{5.16}$$
$$\leq K_1 (p_* E_*)^{\frac{p_1}{2}-1} \|u_x(t)\|^2 + K_2 (p_* E_*)^{\frac{p_2}{2}-1} \|v_x(t)\|^2,$$

and

$$(p_1 + p_2)\int_0^1 \mathcal{F}(u(x,t), v(x,t))\, dx \leq (p_1 + p_2)\bar{d}_2 \left(\|u(t)\|_{L^\alpha}^\alpha + \|v(t)\|_{L^\beta}^\beta\right) \tag{5.17}$$
$$\leq (p_1 + p_2)\bar{d}_2 \left[(p_* E_*)^{\frac{\alpha}{2}-1} \|u_x(t)\|^2 + (p_* E_*)^{\frac{\beta}{2}-1} \|v_x(t)\|^2\right].$$

Thus, it follows from (5.16) and (5.17) that

$$K_1 |u(1,t)|^{p_1} + K_2 |v(0,t)|^{p_2} + \frac{p_1 + p_2}{2}\int_0^1 \mathcal{F}(u(x,t), v(x,t))\, dx$$
$$\leq \eta_* \left(\|u_x(t)\|^2 + \|v_x(t)\|^2\right) < \|u_x(t)\|^2 + \|v_x(t)\|^2, \ \forall t \in \left[0, \bar{T}\right],$$

where $\eta_*$ is given by (5.10).

So we now claim that $I_1(t)$ and $I_2(t)$ are positive for all $t \in \left[0, \bar{T}\right]$. If we put

$$T_* = \sup\left\{T > 0 : I_1(t) \text{ and } I_2(t) \text{ are positive } \forall t \in [0, T]\right\},$$

and if $T_* < \infty$, then (by the continuity of $I_1(t)$ and $I_2(t)$) we have $I_1(T_*)$ and $I_2(T_*)$ are non-negative. Therefore, by the same arguments it is possible to show that there exists $\bar{T}_* > T_*$ such that $I_i(t) > 0$ for $i = 1, 2$ for all $t \in \left[0, \bar{T}_*\right]$. Hence, we can conclude that $I_i(t) \geq 0$ $(i = 1, 2)$ for all $t \geq 0$.

Next, we end up with the equivalence of $\mathcal{L}(t)$ and $E(t)$. Let us recall that by (5.1), (5.3), (5.4), (5.6), (5.7) and (4.10) the function $\mathcal{L}(t)$ can be rewritten as

$$\mathcal{L}(t) = \frac{1}{2}\left(\|u'(t)\|^2 + \|v'(t)\|^2\right) + \frac{1}{2}\left(1 - \frac{1}{p_1} - \frac{1}{p_2}\right)\left(\|u_x(t)\|^2 + \|v_x(t)\|^2\right)$$
$$+ \frac{I_1(t)}{p_1} + \frac{I_2(t)}{p_2} + \delta\langle u(t), u'(t)\rangle + \delta\langle v(t), v'(t)\rangle$$
$$+ \frac{\delta\lambda_1}{2}\|u(t)\|^2 + \frac{\delta\lambda_2}{2}\|v(t)\|^2 + \frac{\delta\mu_1}{2}u^2(1,t) + \frac{\delta\mu_2}{2}v^2(0,t).$$

It is straightforward to see that

$$\mathcal{L}(t) \leq \frac{1}{2}(1+\delta)\left(\|u'(t)\|^2 + \|v'(t)\|^2\right) + \frac{I_1(t)}{p_1} + \frac{I_2(t)}{p_2} \tag{5.18}$$
$$+ \frac{1}{2}\left(1 - \frac{1}{p_1} - \frac{1}{p_2} + \delta(\lambda_1 + \lambda_2 + \mu_1 + \mu_2)\right)\left(\|u_x(t)\|^2 + \|v_x(t)\|^2\right),$$

where we have used an elementary inequality

$$\langle u(t), u'(t)\rangle + \langle v(t), v'(t)\rangle \le \frac{1}{2}\left(\|u_x(t)\|^2 + \|u'_x(t)\|^2\right) + \frac{1}{2}\left(\|v_x(t)\|^2 + \|v'_x(t)\|^2\right).$$

Therefore, one can choose from (5.18) that

$$\begin{aligned}\beta_2 &= \max\left\{1+\delta, \frac{1-\frac{1}{p_1}-\frac{1}{p_2}+\delta(\lambda_1+\lambda_2+\mu_1+\mu_2)}{1-\frac{1}{p_1}-\frac{1}{p_2}}\right\}\\ &= \max\left\{1+\delta, 1+\frac{\delta(1+\lambda_1+\lambda_2+\mu_1+\mu_2)}{1-\frac{1}{p_1}-\frac{1}{p_2}}\right\},\end{aligned}$$

to obtain $\mathcal{L}(t) \le \beta_2 E(t)$.

Furthermore, one easily finds that

$$\begin{aligned}\mathcal{L}(t) \ge{}& \frac{1}{2}(1-\delta)\left(\|u'(t)\|^2 + \|v'(t)\|^2\right)\\ &+ \frac{I_1}{2p_1} + \frac{I_2}{2p_2} + \frac{1}{2}\left(1-\frac{1}{p_1}-\frac{1}{p_2}-\delta\right)\left(\|u_x(t)\|^2 + \|v_x(t)\|^2\right).\end{aligned}$$

Thus, choosing $\delta > 0$ small enough for which $\beta_1 = \min\left\{1-\delta, 1-\frac{\delta}{1-\frac{1}{p_1}-\frac{1}{p_2}}\right\} > 0$, most likely we choose $\delta \in \left(0, 1-\frac{\delta}{1-\frac{1}{p_1}-\frac{1}{p_2}}\right)$ to obtain $\mathcal{L}(t) \le \beta_2 E(t)$. Hereby, our Lyapunov functional $\mathcal{L}(t)$ is definitely equivalent to the total energy $E(t)$ for all $t \ge 0$.

It remains to consider the functional $\psi(t)$ defined in (4.10). The time derivative of $\psi(t)$ can be found by multiplying (1.1) by $(u(x,t), v(x,t))$ and then integrating the resulting equation over $\Omega$. It therefore has the following expression

$$\begin{aligned}\psi'(t) ={}& \|u'(t)\|^2 + \|v'(t)\|^2 - \|u_x(t)\|^2 - \|v_x(t)\|^2 + K_1|u(1,t)|^{p_1}\\ &+ K_2|v(0,t)|^{p_2} + \langle f_1(u(t), v(t)), u(t)\rangle + \langle f_2(u(t), v(t)), v(t)\rangle\\ &+ \langle F_1(t), u(t)\rangle + \langle F_2(t), v(t)\rangle.\end{aligned} \tag{5.19}$$

On the one hand, we have

$$\langle f_1(u(t), v(t)), u(t)\rangle + \langle f_2(u(t), v(t)), v(t)\rangle \le d_2\int_0^1 \mathcal{F}(u(x,t), v(x,t))\,dx,$$

and by (5.8) and (5.9), we continue to estimate the above inequality by

$$\begin{aligned}&\langle f_1(u(t), v(t)), u(t)\rangle + \langle f_2(u(t), v(t)), v(t)\rangle\\ &\le d_2\left[\frac{1}{2}\left(\frac{1}{p_1}+\frac{1}{p_2}\right)\left(\|u_x(t)\|^2 + \|v_x(t)\|^2\right)\right.\\ &\left.-\left(\frac{K_1}{p_1}|u(1,t)|^{p_1} + \frac{K_2}{p_2}|v(0,t)|^{p_2}\right) - \left(\frac{I_1(t)}{p_1}+\frac{I_2(t)}{p_2}\right)\right].\end{aligned} \tag{5.20}$$

On the other hand, one easily has

$$\begin{aligned}\langle F_1(t), u(t)\rangle + \langle F_2(t), v(t)\rangle \le{}& \frac{\varepsilon_2}{2}\left(\|u_x(t)\|^2 + \|v_x(t)\|^2\right)\\ &+ \frac{1}{2\varepsilon_2}\left(\|F_1(t)\|^2 + \|F_2(t)\|^2\right),\ \forall\varepsilon_2 > 0.\end{aligned} \tag{5.21}$$

Thus, we obtain from (5.19)-(5.21) that

$$\psi'(t) \le \|u'(t)\|^2 + \|v'(t)\|^2 - \left(1 - \frac{\varepsilon_2}{2} - \frac{d_2}{2}\left(\frac{1}{p_1} + \frac{1}{p_2}\right)\right)\left(\|u_x(t)\|^2 + \|v_x(t)\|^2\right) \tag{5.22}$$
$$+ \left(1 - \frac{d_2}{p_1}\right) K_1 |u(1,t)|^{p_1} + \left(1 - \frac{d_2}{p_2}\right) K_2 |v(0,t)|^{p_2}$$
$$- d_2\left(\frac{I_1(t)}{p_1} + \frac{I_2(t)}{p_2}\right) + \frac{1}{2\varepsilon_2}\left(\|F_1(t)\|^2 + \|F_2(t)\|^2\right).$$

By (5.4), (5.22) and thanks to (5.10) and (5.16), the time derivative of our Lyapunov functional $\mathcal{L}(t)$ can be estimated as follows:

$$\mathcal{L}'(t) \le -\left(\lambda_* - \delta - \frac{\varepsilon_1}{2}\right)\left(\|u'(t)\|^2 + \|v'(t)\|^2\right) - \delta d_2\left(\frac{I_1(t)}{p_1} + \frac{I_2(t)}{p_2}\right) \tag{5.23}$$
$$+ \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\left(\|F_1(t)\|^2 + \|F_2(t)\|^2\right)$$
$$- \delta\left[(1-\eta_*)\left(1 - \frac{d_2}{\max\{p_1,p_2\}}\right) - \frac{\varepsilon_2}{2}\right]\left(\|u_x(t)\|^2 + \|v_x(t)\|^2\right)$$

for all $\delta$, $\varepsilon_1$, $\varepsilon_2 > 0$. Here we imply

$$d_2 < \min\{p_1,p_2\}, \quad 0 < \varepsilon_2 < 2(1-\eta_*)\left(1 - \frac{d_2}{\max\{p_1,p_2\}}\right).$$

Then for $\delta$ sufficiently small, satisfying $0 < \delta < \lambda_*$, and let $\varepsilon_1 > 0$ such that $0 < \varepsilon_1 < 2(\lambda_* - \delta)$, the equivalence of $\mathcal{L}(t)$ and $E(t)$ together with (5.23) and (5.11) says that there exists a constant $\gamma \in (0, \eta_2)$ such that

$$\mathcal{L}'(t) \le -\gamma\mathcal{L}(t) + \bar{\eta}_1 \exp(-\eta_2 t), \quad \forall t \ge 0,$$

and by Gronwall's inequality, we have

$$\mathcal{L}(t) \le \left(\mathcal{L}(0) + \tfrac{\bar{\eta}_1}{\eta_2}\right)\exp(-\gamma t), \ \forall t \ge 0,$$

which leads to (5.12) by the equivalence of $\mathcal{L}(t)$ and $E(t)$. Hence, we complete the proof of the theorem. □

6. **A numerical example.** The one-dimensional linear damped system of nonlinear wave equations has been qualitatively investigated in the sense of exponential decays. Therefore, in this section the emphasis is put on the illustrative framework. Particularly, our problem here reduces to the following form:

$$\begin{cases} u_{tt} - u_{xx} + \lambda_1 u_t = f_1(u,v) + F_1(x,t), \\ v_{tt} - v_{xx} + \lambda_2 v_t = f_2(u,v) + F_2(x,t), \end{cases}$$

along with the boundary conditions (1.3) and initial conditions (1.4). Notice that the interior sources $f_1$ and $f_2$ have been introduced in Remark 3.1. In fact, the functional $\mathcal{F} \in C^2(\mathbb{R}^2;\mathbb{R})$ (cf. (2.5) as an example) can be established by

$$f_1(u,v) = \alpha\left(\gamma_1 |u|^{\alpha-2} + \frac{\gamma_2}{2}|u|^{\frac{\alpha}{2}-2}|v|^{\frac{\beta}{2}}\right)u,$$
$$f_2(u,v) = \beta\left(\gamma_1 |v|^{\beta-2} + \frac{\gamma_2}{2}|u|^{\frac{\alpha}{2}}|v|^{\frac{\beta}{2}-2}\right)v.$$

By virtue of these explicit expressions, (2.3) obviously holds. Moreover, we deduce the fact that for all $(u,v) \in \mathbb{R}^2$

$$\left(\gamma_1 - \frac{\gamma_2}{2}\right)\left(|u|^\alpha + |v|^\beta\right) \le \mathcal{F}(u,v) \le \left(\gamma_1 + \frac{\gamma_2}{2}\right)\left(|u|^\alpha + |v|^\beta\right),$$

$$\min\{\alpha,\beta\}\,\mathcal{F}(u,v) \le u f_1(u,v) + v f_2(u,v) \le \max\{\alpha,\beta\}\,\mathcal{F}(u,v),$$

which imply $d_1 = \min\{\alpha,\beta\}$, $d_2 = \max\{\alpha,\beta\}$, $\bar{d}_1 = \gamma_1 - \frac{\gamma_2}{2}$, and $\bar{d}_2 = \gamma_1 + \frac{\gamma_2}{2}$ in (4.1), (4.2). If one can choose an appropriate set of constants $\alpha$, $\beta$, $\gamma_1$, and $\gamma_2$ such that $\gamma_2 < 2\gamma_1$ and $d_2 < \min\{p_1, p_2\}$, $(\mathbf{A}_3')$ is clearly valid.

In this example, we take $\alpha = \beta = 4$, $p_1 = p_2 = 6$, $q_1 = q_2 = 2$ and $K_i = \mu_i = \lambda_i = 1$ for $i = 1,2$ with $\gamma_1 = \frac{3}{4}$, $\gamma_2 = \frac{1}{2}$ (so, $d_2 = 4$ and $\bar{d}_2 = 1$). Next, the initial conditions are given by

$$\begin{aligned}
&\tilde{u}_0(x) = x\left(e^9+1\right)^{\frac{-1}{4}}, \ \ \tilde{u}_1(x) = -xe^9\left(e^9+1\right)^{\frac{-5}{4}},\\
&\tilde{v}_0(x) = (1-x)\left(e^9+1\right)^{\frac{-1}{4}}, \ \ \tilde{v}_1(x) = (x-1)\,e^9\left(e^9+1\right)^{\frac{-5}{4}}.
\end{aligned}$$

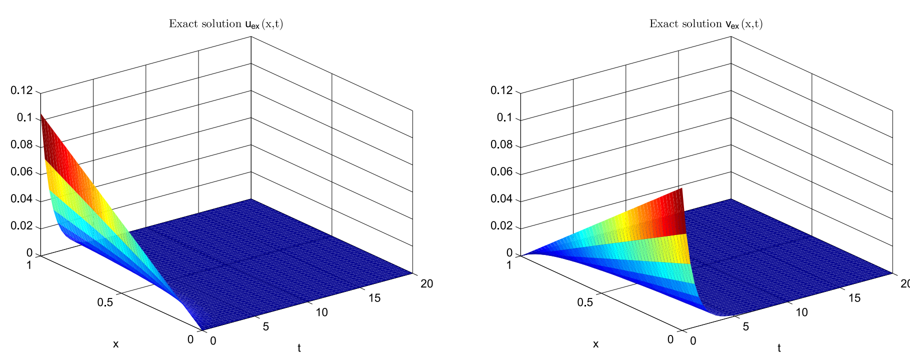
Then our external functions are

$$\begin{aligned}
F_1(x,t) &= -\frac{4x^3 - 2x^2 + x}{\left(e^{9+4t}+1\right)^{3/4}} - \frac{5e^{9+4t}x}{\left(e^{9+4t}+1\right)^{9/4}},\\
F_2(x,t) &= \frac{(x-1)\left(4x^2 - 6x + 3\right)}{\left(e^{9+4t}+1\right)^{3/4}} - \frac{5e^{9+4t}(1-x)}{\left(e^{9+4t}+1\right)^{9/4}}.
\end{aligned}$$

This way we can find the exact solutions in the following form:

$$u_{ex}(x,t) = \frac{x}{\sqrt[4]{e^{9+4t}+1}}, \ \ v_{ex}(x,t) = \frac{1-x}{\sqrt[4]{e^{9+4t}+1}}, \tag{6.1}$$

but they shall be neglected since we want to consider the illustrative approximation of solutions.




(A) Exact solution $u_{ex}(x,t)$ (B) Exact solution $v_{ex}(x,t)$

FIGURE 1. Exact solutions.

In order to fulfill all assumptions of Theorem 5.2, it remains to check (5.10) and (5.11). Obviously, one has $p_* = 3$ and $\rho$ is computed by the following

$$\rho = \frac{1}{2}\int_0^\infty \left(\|F_1(s)\| + \|F_2(s)\|\right) ds \le \frac{1}{2}\int_0^\infty \left(\frac{12}{\left(e^{9+4s}+1\right)^{3/4}} + \frac{18}{\left(e^{9+4s}+1\right)^{3/4}}\right) ds \tag{6.2}$$
$$= 15\int_0^\infty \frac{1}{\left(e^{9+4s}+1\right)^{3/4}} ds \le 15\int_0^\infty \frac{1}{e^{\frac{27}{4}+3s}} ds = 15e^{\frac{-27}{4}}\int_0^\infty e^{-3s} ds$$
$$= 5e^{\frac{-27}{4}} \approx 0.58\times 10^{-4} \approx 0,$$

and the initial energy can be estimated by

$$E(0) = \frac{1}{2}\left(\|\tilde{u}_1\|^2 + \|\tilde{v}_1\|^2 + \|\tilde{u}_{0x}\|^2 + \|\tilde{v}_{0x}\|^2\right) - \left(|\tilde{u}_0(1)|^6 + |\tilde{v}_0(0)|^6\right) \tag{6.3}$$
$$- \frac{3}{4}\int_0^1 \left(|\tilde{u}_0(x)|^4 + |\tilde{v}_0(x)|^4\right) dx - \frac{1}{2}\int_0^1 |\tilde{u}_0(x)|^2 |\tilde{v}_0(x)|^2 dx$$
$$= \frac{1}{3}e^{18}\left(e^9+1\right)^{-\frac{5}{2}} + \left(e^9+1\right)^{\frac{-1}{2}} - 2\left(e^9+1\right)^{-\frac{3}{2}}$$
$$- \frac{3}{10}\left(e^9+1\right)^{-1} - \frac{1}{60}\left(e^9+1\right)^{-1} < 0.015.$$

Combining (6.2) and (6.3), the checking argument for (5.11) is made by (6.2) and we obtain that

$$E_* = [E(0) + \rho]\exp(2\rho) \approx E(0) < 0.015,$$

which leads to $\eta_* = 12p_*E_* + 2\left(p_*E_*\right)^2 < 1$.

Therefore, the assumptions needed to check are all satisfied.

At the discretization level for this problem, a uniform grid of mesh-points $(x_k, t_n)$ is used. Here $x_k = k\Delta x$ and $t_n = n\Delta t$ where $k$ and $n$ are integers and $\Delta x = \frac{1}{K}$, $\Delta t = \frac{T}{N}$ the equivalent mesh-widths in space $x$ and time $t$, respectively. We first consider the following differential system for the unknowns $(U_k(t), V_k(t)) \equiv (u(x_k,t), v(x_k,t))$ for $k = \overline{0,K}$:

$$\frac{d\bar{U}_1}{dt}(t) = K^2\left(U_2(t) - 2U_1(t)\right) - \bar{U}_1(t) \tag{6.4}$$
$$+ 3U_1^3(t) + V_1^2(t)U_1(t) + F_1(x_1,t),$$
$$\frac{d\bar{U}_k}{dt}(t) = K^2\left(U_{k-1}(t) - 2U_k(t) + U_{k+1}(t)\right) - \bar{U}_k(t)$$
$$+ 3U_k^3(t) + V_k^2(t)U_k(t) + F_1(x_k,t),\ k = \overline{2,K-1},$$
$$\frac{d\bar{U}_K}{dt}(t) = K\left(U_K^5(t) - \bar{U}_K(t)\right) - K^2\left(U_K(t) - U_{K-1}(t)\right)$$
$$- \bar{U}_K(t) + 3U_K^3(t) + F_1(x_K,t),$$

and

$$\begin{aligned}
\frac{d\bar{V}_0}{dt}(t) &= K\left(V_0^5(t) - \bar{V}_0(t)\right) + K^2\left(V_1(t) - V_0(t)\right) \\
&\quad - \bar{V}_0(t) + 3V_0^3(t) + F_2(x_0, t), \\
\frac{d\bar{V}_k}{dt}(t) &= K^2\left(V_{k-1}(t) - 2V_k(t) + V_{k+1}(t)\right) - \bar{V}_k(t) \\
&\quad + 3V_k^3(t) + U_k^2(t)\,V_k(t) + F_2(x_k, t),\ k = \overline{1, K-2}, \\
\frac{d\bar{V}_{K-1}}{dt}(t) &= K^2\left(V_{K-2}(t) - 2V_{K-1}(t)\right) - \bar{V}_{K-1}(t) \\
&\quad + 3V_{K-1}^3(t) + V_{K-1}^2(t)\,U_{K-1}(t) + F_2(x_{K-1}, t),
\end{aligned} \tag{6.5}$$

where $\left(\bar{U}_k(t), \bar{V}_k(t)\right)$ identically stands for $\left(\frac{dU_k}{dt}(t), \frac{dV_k}{dt}(t)\right)$ and the initial conditions are

$$\left(U_k(0), V_k(0)\right) = \left(\tilde{u}_0(x_k), \tilde{v}_0(x_k)\right), \left(\bar{U}_k(0), \bar{V}_k(0)\right) = \left(\tilde{u}_1(x_k), \tilde{v}_1(x_k)\right),\ k = \overline{0, K}. \tag{6.6}$$

Here we also notice that the values of $U_0(t)$ and $V_K(t)$ are known, so those cases are not considered.

Below a basic numerical approach is achieved by using the linear recursive method where the nonlinear terms are linearized. Accordingly, after some rearrangements we rewrite the linearized differential system to be a differential equation where the solution includes all discrete solutions of the linearized system. Such a unifying way also guarantees that the numerical solution uniquely exists. When doing this, at the $m$-th iterative stage $(m \geq 1)$ the linearized differential system of (6.4)-(6.6) one by one becomes

$$\begin{aligned}
\frac{d\bar{U}_1^{(m)}}{dt}(t) &= K^2\left(U_2^{(m)}(t) - 2U_1^{(m)}(t)\right) - \bar{U}_1^{(m)}(t) \\
&\quad + 3\left(U_1^{(m-1)}(t)\right)^3 + \left(V_1^{(m-1)}(t)\right)^2 U_1^{(m)}(t) + F_1(x_1, t), \\
\frac{d\bar{U}_k^{(m)}}{dt}(t) &= K^2\left(U_{k-1}^{(m)}(t) - 2U_k^{(m)}(t) + U_{k+1}^{(m)}(t)\right) - \bar{U}_k^{(m)}(t) \\
&\quad + 3\left(U_k^{(m-1)}(t)\right)^3 + \left(V_k^{(m-1)}(t)\right)^2 U_k^{(m)}(t) + F_1(x_k, t),\ k = \overline{2, K-1}, \\
\frac{d\bar{U}_K^{(m)}}{dt}(t) &= K\left[\left(U_K^{(m-1)}(t)\right)^5 - \bar{U}_K^{(m)}(t)\right] - K^2\left(U_K^{(m)}(t) - U_{K-1}^{(m)}(t)\right) \\
&\quad - \bar{U}_K^{(m)}(t) + 3\left(U_K^{(m-1)}(t)\right)^3 + F_1(x_K, t),
\end{aligned} \tag{6.7}$$

and

$$\frac{d\bar{V}_0^{(m)}}{dt}(t) = K\left[\left(V_0^{(m-1)}(t)\right)^5 - \bar{V}_0^{(m)}(t)\right] + K^2\left(V_1^{(m)}(t) - V_0^{(m)}(t)\right) \tag{6.8}$$
$$- \bar{V}_0^{(m)}(t) + 3\left(V_0^{(m-1)}(t)\right)^3 + F_2(x_0, t),$$
$$\frac{d\bar{V}_k^{(m)}}{dt}(t) = K^2\left(V_{k-1}^{(m)}(t) - 2V_k^{(m)}(t) + V_{k+1}^{(m)}(t)\right) - \bar{V}_k^{(m)}(t)$$
$$+ 3\left(V_k^{(m-1)}(t)\right)^3 + \left(U_k^{(m-1)}(t)\right)^2 V_k^{(m)}(t) + F_2(x_k, t),\ k = \overline{1, K-2},$$
$$\frac{d\bar{V}_{K-1}^{(m)}}{dt}(t) = K^2\left(V_{K-2}^{(m)}(t) - 2V_{K-1}^{(m)}(t)\right) - \bar{V}_{K-1}^{(m)}(t)$$
$$+ 3\left(V_{K-1}^{(m-1)}(t)\right)^3 + \left(U_{K-1}^{(m-1)}(t)\right)^2 V_{K-1}^{(m)}(t) + F_2(x_{K-1}, t),$$

$$\left(U_k^{(m)}(0), V_k^{(m)}(0)\right) = \left(\tilde{u}_0(x_k), \tilde{v}_0(x_k)\right), \tag{6.9}$$
$$\left(\bar{U}_k^{(m)}(0), \bar{V}_k^{(m)}(0)\right) = \left(\tilde{u}_1(x_k), \tilde{v}_1(x_k)\right), k = \overline{0, K}.$$

Then, a matrix-type differential system derived from (6.7)-(6.9) can be expressed as follows:

$$\frac{d\mathbb{U}^{(m)}}{dt}(t) = \mathbb{A}^{(m)}(t)\,\mathbb{U}^{(m)}(t) + \mathbb{F}_1^{(m)}(t),\ \frac{d\mathbb{V}^{(m)}}{dt}(t) = \mathbb{B}^{(m)}(t)\,\mathbb{V}^{(m)}(t) + \mathbb{F}_2^{(m)}(t), \tag{6.10}$$

where the solutions $\mathbb{U}^{(m)}, \mathbb{V}^{(m)} \in \mathbb{R}^{2K}$, and the block matrices $\mathbb{A}^{(m)}, \mathbb{B}^{(m)} \in \mathbb{R}^{2K} \times \mathbb{R}^{2K}$, and the functions $\mathbb{F}_1^{(m)}, \mathbb{F}_2^{(m)} \in \mathbb{R}^{2K}$ are defined by

$$\mathbb{U}^{(m)} = \begin{bmatrix} U_1^{(m)} \\ \vdots \\ U_K^{(m)} \\ \bar{U}_1^{(m)} \\ \vdots \\ \bar{U}_{K-1}^{(m)} \\ \bar{U}_K^{(m)} \end{bmatrix},\quad \mathbb{F}_1^{(m)}(t) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \alpha_1^{(m)}(t) \\ \vdots \\ \alpha_{K-1}^{(m)}(t) \\ \alpha_K^{(m)}(t) + K\left(U_K^{(m-1)}(t)\right)^5 \end{bmatrix},$$

$$\mathbb{V}^{(m)} = \begin{bmatrix} V_0^{(m)} \\ V_1^{(m)} \\ \vdots \\ V_{K-1}^{(m)} \\ \bar{V}_0^{(m)} \\ \bar{V}_1^{(m)} \\ \vdots \\ \bar{V}_{K-1}^{(m)} \end{bmatrix},\quad \mathbb{F}_2^{(m)}(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \beta_0^{(m)}(t) + K\left(V_0^{(m-1)}(t)\right)^5 \\ \beta_1^{(m)}(t) \\ \vdots \\ \beta_{K-1}^{(m)}(t) \end{bmatrix},$$

$$\alpha_k^{(m)}(t) = 3\left(U_k^{(m-1)}(t)\right)^3 + F_1(x_k, t),\ k = \overline{1, K},$$

$$\beta_k^{(m)}(t) = 3\left(V_k^{(m-1)}(t)\right)^3 + F_2(x_k,t), \ k = \overline{0, K-1},$$

$$\mathbb{A}^{(m)} = \begin{bmatrix} & & O & & & \begin{matrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{matrix} \\ a_1^{(m)} & K^2 & \cdots & \cdots & 0 & \begin{matrix} -1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & -1 & 0 \\ 0 & \cdots & \cdots & 0 & -1-K \end{matrix} \\ K^2 & a_2^{(m)} & K^2 & & \vdots & \\ \vdots & \ddots & \ddots & \ddots & \vdots & \\ \vdots & & K^2 & a_{K-1}^{(m)} & K^2 & \\ 0 & \cdots & \cdots & K^2 & -K^2 & \end{bmatrix},$$

$$\mathbb{B}^{(m)} = \begin{bmatrix} & & O & & & \begin{matrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{matrix} \\ -K^2 & K^2 & \cdots & \cdots & 0 & \begin{matrix} -1-K & 0 & \cdots & \cdots & 0 \\ 0 & -1 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & -1 & 0 \\ 0 & \cdots & \cdots & 0 & -1 \end{matrix} \\ K^2 & b_1^{(m)} & K^2 & & \vdots & \\ \vdots & \ddots & \ddots & \ddots & \vdots & \\ \vdots & & K^2 & b_{K-2}^{(m)} & K^2 & \\ 0 & \cdots & \cdots & K^2 & b_{K-1}^{(m)} & \end{bmatrix},$$

$$a_k^{(m)} := a_k^{(m)}(t) = -2K^2 + \left(V_k^{(m-1)}(t)\right)^2,$$

$$b_k^{(m)} := b_k^{(m)}(t) = -2K^2 + \left(U_k^{(m-1)}(t)\right)^2, \ k = \overline{1, K-1},$$

associated with the initial data

$$\mathbb{U}^{(m)}(0) = \begin{bmatrix} \tilde{u}_0(x_1) \\ \vdots \\ \tilde{u}_0(x_K) \\ \tilde{u}_1(x_1) \\ \vdots \\ \tilde{u}_1(x_K) \end{bmatrix}, \quad \mathbb{V}^{(m)}(0) = \begin{bmatrix} \tilde{v}_0(x_0) \\ \vdots \\ \tilde{v}_0(x_{K-1}) \\ \tilde{v}_1(x_0) \\ \vdots \\ \tilde{v}_1(x_{K-1}) \end{bmatrix}. \tag{6.11}$$

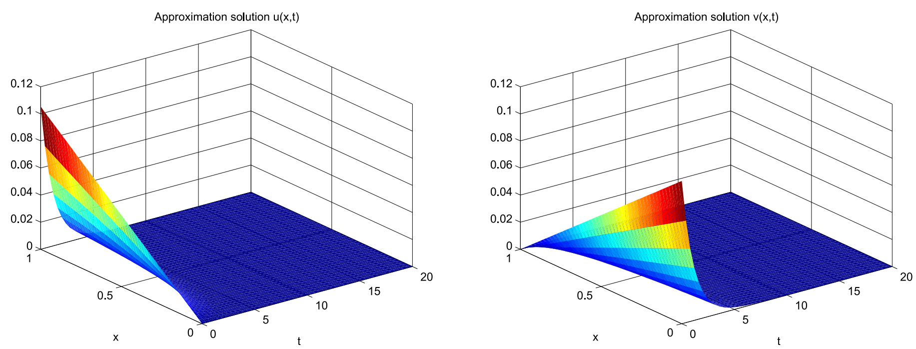
Here the initial guess for our linearization method is simply the initial values, i.e. $\left(\mathbb{U}^{(0)}(t_n), \mathbb{V}^{(0)}(t_n)\right) = \left(\mathbb{U}^{(m)}(0), \mathbb{V}^{(m)}(0)\right)$ for all $n = \overline{0, N}$. In general, it is remarkable that the system (6.10)-(6.11) might be stiff, then it is not good for our implementation since the objective is considering the system in a large time. Understanding the basic instability coming from stiff systems, we therefore apply the well-known implicit Euler method which reads as

$$\mathbb{U}_{n+1}^{(m)} = \left(\mathbb{I} - \frac{T}{N}\mathbb{A}^{(m)}\left(t_{n+1}\right)\right)^{-1}\left(\mathbb{U}_{n}^{(m)} + \frac{T}{N}\mathbb{F}_{1}^{(m)}\left(t_{n+1}\right)\right),$$

$$\mathbb{V}_{n+1}^{(m)} = \left(\mathbb{I} - \frac{T}{N}\mathbb{B}^{(m)}\left(t_{n+1}\right)\right)^{-1}\left(\mathbb{V}_{n}^{(m)} + \frac{T}{N}\mathbb{F}_{2}^{(m)}\left(t_{n+1}\right)\right),$$

for $n = \overline{0, N-1}$, where $\mathbb{I}$ stands for the $2K$-by-$2K$ identity matrix. This discrete system is endowed with the conditions (6.11).

Choosing $m = 5$, $T = 20$, and $K = N = 50$, we have plotted in Figure 2 the approximation function $(u(x,t), v(x,t))$ solutions to our problem $(P)$ considered in this section. Compared to Figure 1, we observe that such functions not only behave like the exact solutions (6.1) (decay exponentially in time), but also completely have the same shapes corresponding to each exact solution.




(A) Approximate solution $u(x,t)$ (B) Approximate solution $v(x,t)$

FIGURE 2. Approximate solutions.

TABLE 1. Numerical results at nodes $\left(\frac{4}{5}, t_n\right)$ for $n \in \{10, 20, 30\}$.

| $n$ | $u_{ex}\left(\frac{4}{5}, t_n\right)$ | $u\left(\frac{4}{5}, t_n\right)$ | $\left\lvert u_{ex}\left(\frac{4}{5}, t_n\right) - u\left(\frac{4}{5}, t_n\right)\right\rvert$ |
|---|---|---|---|
| 10 | $1.54436330E-03$ | $2.91855517E-03$ | $1.37419186E-03$ |
| 20 | $2.82860006E-05$ | $7.20712002E-05$ | $4.37851996E-05$ |
| 30 | $5.18076174E-07$ | $1.77972692E-06$ | $1.26165074E-06$ |

| $n$ | $v_{ex}\left(\frac{4}{5}, t_n\right)$ | $v\left(\frac{4}{5}, t_n\right)$ | $\left\lvert v_{ex}\left(\frac{4}{5}, t_n\right) - v\left(\frac{4}{5}, t_n\right)\right\rvert$ |
|---|---|---|---|
| 10 | $3.86090827E-04$ | $7.29514168E-04$ | $3.43423340E-04$ |
| 20 | $7.07150017E-06$ | $1.80147701E-05$ | $1.09432699E-05$ |
| 30 | $1.29519043E-07$ | $6.22799676E-06$ | $4.93280633E-07$ |

Furthermore, numerical results of the solutions $(u, v)$ together with the exact solutions $(u_{ex}, v_{ex})$ at nodes $\left(\frac{4}{5}, t_n\right)$ for $n \in \{10, 20, 30\}$ and various values of error in the entry-wise norm

$$\mathcal{E}_{N,K}(u) = \max_{1 \le k \le K} \max_{1 \le n \le N} \left| u_{ex}(x_k, t_n) - u(x_k, t_n) \right|,$$

$$\mathcal{E}_{N,K}(v) = \max_{1 \le k \le K} \max_{0 \le n \le N-1} \left| v_{ex}(x_k, t_n) - v(x_k, t_n) \right|,$$

TABLE 2. Numerical results for the $l_\infty$ norm error $\mathcal{E}_{N,K}$.

| $K$ | $N$ | $\mathcal{E}_{N,K}(u)$ | $\mathcal{E}_{N,K}(v)$ |
|---|---|---|---|
| 50 | 50 | $6.68545424E-03$ | $6.68150701E-03$ |
| 100 | 100 | $3.59475057E-03$ | $3.59201931E-03$ |
| 150 | 150 | $2.45841870E-03$ | $2.45632948E-03$ |
| 200 | 200 | $1.86793338E-03$ | $1.86628504E-03$ |

are all given in Table 1-Table 2, respectively. To demonstrate the fact that the $l_\infty$ norm error $\mathcal{E}_{N,K}$ decreases (and obviously tends to zero) as $K, N$ increase without any attention to the discretization level, we show in Table 2 the error values when such constants go from 50 to 200. As expected from the analysis, our numerical method used here is reasonable and efficient.

## Appendix A. Proofs of auxiliary results.

*Proof of Lemma 4.2.* We only need to prove the first inequality since (4.15) and (4.16) are obviously the same. Put $s_1 = 2/(1-2\xi)$ and $s_2 = 2/(1-\xi)$. Our strategy is to independently consider each terms on the left-hand side of (4.15) and furthermore, to estimate those quantities we mainly divide into two cases. For clarity, let us first consider two cases for $\|u\|_{L^\alpha}$.

$(a_1)$ $\|u\|_{L^\alpha} \le 1$: One easily deduces from $2 \le s_1 \le \alpha$ that

$$\|u\|_{L^\alpha}^{s_1} \le \|u\|_{L^\alpha}^{2} \le \|u_x\|^2 + \|u\|_{L^\alpha}^{\alpha} + |u(1)|^{p_1}.$$

$(a_2)$ $\|u\|_{L^\alpha} \ge 1$: Similarly, we get

$$\|u\|_{L^\alpha}^{s_1} \le \|u\|_{L^\alpha}^{\alpha} \le \|u_x\|^2 + \|u\|_{L^\alpha}^{\alpha} + |u(1)|^{p_1}.$$

Next, we consider two cases for $\|u\|$:

$(b_1)$ $\|u\| \le 1$: Since $2 \le s_2 \le \alpha$,

$$\|u\|^{s_2} \le \|u\|^2 \le \|u_x\|^2 + \|u\|_{L^\alpha}^{\alpha} + |u(1)|^{p_1}.$$

$(b_2)$ $\|u\| \ge 1$: In the same manner, we have

$$\|u\|^{s_2} \le \|u\|^\alpha \le \|u\|_{L^\alpha}^{\alpha} \le \|u_x\|^2 + \|u\|_{L^\alpha}^{\alpha} + |u(1)|^{p_1}.$$

Finally, we go through the last term $|u(1)|$:

$(c_1)$ $|u(1)| \le 1$:

$$|u(1)|^{s_2} \le |u(1)|^2 \le \|u\|_{C(\overline{\Omega})}^2 \le \|u_x\|^2 + \|u\|_{L^\alpha}^{\alpha} + |u(1)|^{p_1}.$$

$(c_2)$ $|u(1)| \ge 1$: It is natural to say $|u(1)|^{s_2} \le |u(1)|^{p_1}$.

Combining all of the above inequalities completes the proof of the lemma. □

**Acknowledgments.** The authors would like to thank the anonymous referees for fruitful comments through the improvement of this paper. V.A.K thanks Prof. Pierangelo Marcati (L'Aquila, Italy) for the trust and mathematical training that the Gran Sasso Science Institute has invested in V.A.K through the PhD time. V.A.K gratefully acknowledges Dr. Nguyen Thai Ngoc (Göttingen, Germany) for his unconditional helps during the time V.A.K shortly occupied a research associate position at the institute.

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Received xxxx 20xx; revised xxxx 20xx.

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