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Inequalities for the Casorati Curvature of Statistical Manifolds in Holomorphic Statistical Manifolds of Constant Holomorphic Curvature

Simona Decu [1,2,*,†], **Stefan Haesen** [3,4,†] and **Leopold Verstraelen** [5,†]

1 Department of Applied Mathematics, Bucharest University of Economic Studies, 6 Piaţa Romană, 010374 Bucharest, Romania
2 Romanian Academy, *Costin C. Kiriţescu* National Institute of Economic Research—Centre of Mountain Economy (CE-MONT), 13 Calea 13 Septembrie, 030508 Bucharest, Romania
3 Department of Mathematics, University of Hasselt, BE 3590 Diepenbeek, Belgium; stefan.haesen@uhasselt.be
4 Department of Teacher Education, Thomas More University College, 2290 Vorselaar, Belgium
5 Department of Mathematics, Katholieke Universiteit Leuven, Celestijnenlaan 200B, Box 2400, BE-3001 Leuven, Belgium; leopold.verstraelen@wis.kuleuven.be
* Correspondence: simona.decu@gmail.com or simona.marinescu@csie.ase.ro
† These authors contributed equally to this work.

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Abstract: In this paper, we prove some inequalities in terms of the normalized δ-Casorati curvatures (extrinsic invariants) and the scalar curvature (intrinsic invariant) of statistical submanifolds in holomorphic statistical manifolds with constant holomorphic sectional curvature. Moreover, we study the equality cases of such inequalities. An example on these submanifolds is presented.

Keywords: Casorati curvature; statistical submanifold; holomorphic statistical manifold

1. Introduction

The problem of discovering simple relationships between the main *intrinsic invariants* and the main *extrinsic invariants* of submanifolds is a basic problem in submanifold theory [1]. In this respect, beautiful results focus on certain types of geometric inequalities. Moreover, another basic problem in this field is to study the *ideal submanifolds* in a space form, namely to investigate the submanifolds which satisfy the equality case of such inequalities [2].

The method of looking for *Chen invariants* answers the problems posed above. First, Chen demonstrated in [3] an optimal inequality for a submanifold on a real space form between the intrinsically defined δ-curvature and the extrinsically defined squared mean curvature. This approach initiated a new line of research and was extended to various types of submanifolds in several types of ambient spaces, e.g., submanifolds in complex space forms of constant holomorphic sectional curvature (see [4–7]). The submanifolds attaining the equality of these inequalities (called Chen ideal submanifolds) were also investigated. Recently, Chen et al. classified $\delta(2, n-2)$-ideal Lagrangian submanifolds in complex space forms in [8].

Moreover, new solutions to the above problems are given by the inequalities involving *δ-Casorati curvatures*, initiated in [9,10]. In the search for a true measure of curvature, Casorati in 1890 proposed the curvature which nowadays bears his name because it better corresponds with our *common intuition of curvature* than Gauss and mean curvature [11]. However, this notion of curvature was soon forgotten and was rediscovered by Koenderink working in the field of computer vision [12]. Verstraelen developed some geometrical models for early vision, presenting perception via the Casorati curvature of sensation [13]. A geometrical interpretation of this type of curvature for submanifolds in

Riemannian spaces was given in [14]. In [15], the isotropical Casorati curvature of production surfaces was studied. The Casorati curvature was used to obtain optimal inequalities between intrinsic and extrinsic curvatures of submanifolds in real space forms in [9,10]. Later, this knowledge was extended (e.g., see [16–21]). Submanifolds which satisfy these equalities are named *Casorati ideal submanifolds*. Recently, Vîlcu established an optimal inequality for Lagrangian submanifolds in complex space forms involving Casorati curvature [22]. Aquib et al. obtained a classification of Casorati ideal Lagrangian submanifolds in complex space forms [23]. Very recently, Suceavă and Vajiac studied inequalities involving some Chen invariants, mean curvature, and Casorati curvature for strictly convex Euclidean hypersurfaces [24]. Brubaker and Suceavă investigated a geometric interpretation of Cauchy–Schwarz inequality in terms of Casorati curvature [25].

The concept of *statistical manifold* was defined by Amari in 1985, in the basic study on information geometry [26]. Currently, interest in the field of statistical manifolds is increasing, being focused on applications in differential geometry, information geometry, statistics, machine learning, etc. (see, e.g., [27–29]). Cuingnet et al. introduced a continuous framework to spatially regularize support vector machines (SVM) for brain image analysis, considering the images as elements of a statistical manifold, in order to classify patients with Alzheimer's disease [30]. The study of curvature invariants of submanifolds in statistical manifolds gives other solutions to the above research problems. Aydin et al. established some inequalities (Chen–Ricci and Wintgen) for submanifolds in statistical manifolds of constant curvature in [31,32]. Lee et al. obtained inequalities on Sasakian statistical manifolds in terms of Casorati curvatures [33]. Aquib and Shahid [34] proved some inequalities involving Casorati curvatures on statistical submanifolds in quaternion Kähler-like statistical space forms. The quaternionic theory of statistical manifolds is investigated in [35]. Very recently, new results have been published. Aytimur et al. established some Chen inequalities for submanifolds in Kähler-like statistical manifolds [36]. Aquib et al. achieved generalized Wintgen-type inequalities for submanifolds in generalized space forms [37]. Chen et al. established a Chen first inequality for statistical submanifolds in Hessian manifolds of constant Hessian curvature [38]. Moreover, Siddiqui et al. studied a Chen inequality for statistical warped products statistically immersed in a statistical manifold of constant curvature [39].

Recently, Furuhata et al. [40] defined the notion of a *holomorphic statistical manifold*, which can be considered as a generalization of a special Kähler manifold. The authors establish the basics for statistical submanifolds in holomorphic statistical manifolds.

In order to find out new solutions for the problems under debate, we obtain inequalities for statistical submanifolds in holomorphic statistical manifolds. The invariants involved in such inequalities are the extrinsic normalized δ-Casorati curvatures and the intrinsic scalar curvature. The method is focused on a constrained extremum problem. Moreover, the equality cases are investigated. This study revealed that the equality at all points characterizes submanifolds that are totally geodesic with respect to the Levi–Civita connection.

2. Preliminaries

Let $(\tilde{M}, \tilde{g})$ be a $2n$-dimensional manifold, $\tilde{\nabla}$ an affine connection on $\tilde{M}$, and $\tilde{g}$ a Riemannian metric on $\tilde{M}$. Consider $\tilde{T} \in \Gamma(T\tilde{M}^{(1,2)})$ the torsion tensor field of $\tilde{\nabla}$.

A pair $(\tilde{\nabla}, \tilde{g})$ is called a *statistical structure* on $\tilde{M}$ if the torsion tensor field $\tilde{T}$ vanishes and $\tilde{\nabla}\tilde{g} \in \Gamma(T\tilde{M}^{(0,3)})$ is symmetric.

A Riemannian manifold $(\tilde{M}, \tilde{g})$ is called a *statistical manifold* if it is endowed with a pair of torsion-free affine connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$ satisfying

$$Z\,\tilde{g}(X, Y) = \tilde{g}(\tilde{\nabla}_Z X, Y) + \tilde{g}(X, \tilde{\nabla}^*_Z Y),$$

for any $X, Y, Z \in \Gamma(T\tilde{M})$. Denote $(\tilde{M}, \tilde{g}, \tilde{\nabla})$ as the statistical manifold. The connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$ are named *dual connections* or *conjugate connections*.

Remark 1. *If* $(\tilde{M}, \tilde{g}, \tilde{\nabla})$ *is a statistical manifold, then we remark that*

1. $(\tilde{\nabla}^*)^* = \tilde{\nabla}$;
2. $(\tilde{M}, \tilde{g}, \tilde{\nabla}^*)$ *is also a statistical manifold;*
3. $\tilde{\nabla}$ *always has a dual connection* $\tilde{\nabla}^*$ *satisfying*

$$\tilde{\nabla} + \tilde{\nabla}^* = 2\tilde{\nabla}^0, \tag{1}$$

where $\tilde{\nabla}^0$ *is the Levi–Civita connection on* $\tilde{M}$.

Let M be an m-dimensional submanifold of a $2n$-dimensional statistical manifold $(\tilde{M}, \tilde{g})$ and g the induced metric on M. The Gauss formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$
$$\tilde{\nabla}^*_X Y = \nabla^*_X Y + h^*(X, Y),$$

for any $X, Y \in \Gamma(TM)$, where h and h^* are symmetric and bilinear $(0, 2)$-tensors, called the *imbedding curvature tensor* of M in $\tilde{M}$ for $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively.

Denote the curvature tensor fields of ∇ and $\tilde{\nabla}$ by R and $\tilde{R}$, respectively. Then, the *Gauss equation* concerning the connection $\tilde{\nabla}$ is ([41])

$$\tilde{g}(\tilde{R}(X, Y)Z, W) = g(R(X, Y)Z, W) + \tilde{g}(h(X, Z), h^*(Y, W)) - \tilde{g}(h^*(X, W), h(Y, Z)), \tag{2}$$

for any $X, Y, Z, W \in \Gamma(TM)$.

In addition, denote the curvature tensor fields of the connections ∇^* and $\tilde{\nabla}^*$ by R^* and $\tilde{R}^*$, respectively. Then the *Gauss equation* concerning the connection $\tilde{\nabla}^*$ is ([41])

$$\tilde{g}(\tilde{R}^*(X, Y)Z, W) = g(R^*(X, Y)Z, W) + \tilde{g}(h^*(X, Z), h(Y, W)) - \tilde{g}(h(X, W), h^*(Y, Z)), \tag{3}$$

for any $X, Y, Z, W \in \Gamma(TM)$.

If M is a submanifold of a statistical manifold $(\tilde{M}, \tilde{g}, \tilde{\nabla})$, then (M, g, ∇) is also a statistical manifold with the induced metric g and the induced connection ∇.

Let S be the *statistical curvature tensor field of a statistical manifold* (M, g, ∇), where $S \in \Gamma(TM^{(1,3)})$ is defined by [40]

$$S(X, Y)Z = \frac{1}{2}\{R(X, Y)Z + R^*(X, Y)Z\}, \tag{4}$$

for $X, Y, Z \in \Gamma(TM)$.

If $\pi = span_{\mathbb{R}}\{u_1, u_2\}$ is a 2-dimensional subspace of T_pM, for $p \in M$, then the *sectional curvature* of M is defined by [40]:

$$\sigma(\pi) = \frac{g(S(u_1, u_2)u_2, u_1)}{g(u_1, u_1)g(u_2, u_2) - g^2(u_1, u_2)}. \tag{5}$$

Let $\{e_1, ..., e_m\}$ be an orthonormal basis of the tangent space T_pM, for $p \in M$, and let $\{e_{m+1}, ..., e_{2n}\}$ be an orthonormal basis of the normal space $T_p^\perp M$. The *scalar curvature* τ at p is given by

$$\tau(p) = \sum_{1 \le i < j \le m} \sigma(e_i \wedge e_j) = \sum_{1 \le i < j \le m} g(S(e_i, e_j)e_j, e_i), \tag{6}$$

and the *normalized scalar curvature* ρ of M is defined as

$$\rho = \frac{2\tau}{m(m-1)}. \tag{7}$$

The *mean curvature* vector fields of M, denoted by H and H^*, are given by

$$H = \frac{1}{m}\sum_{i=1}^{m} h(e_i, e_i), \quad H^* = \frac{1}{m}\sum_{i=1}^{m} h^*(e_i, e_i).$$

From Equation (1), we get $2h^0 = h + h^*$ and $2H^0 = H + H^*$, where h^0 and H^0 are the second fundamental form and the mean curvature field of M, respectively, with respect to the Levi–Civita connection ∇^0 on M.

The *squared mean curvatures* of the submanifold M in $\tilde{M}$ have the expressions

$$\|H\|^2 = \frac{1}{m^2}\sum_{\alpha=m+1}^{2n}\left(\sum_{i=1}^{m} h_{ii}^{\alpha}\right)^2, \quad \|H^*\|^2 = \frac{1}{m^2}\sum_{\alpha=m+1}^{2n}\left(\sum_{i=1}^{m} h_{ii}^{*\alpha}\right)^2,$$

where $h_{ij}^{\alpha} = \tilde{g}(h(e_i, e_j), e_\alpha)$ and $h_{ij}^{*\alpha} = \tilde{g}(h^*(e_i, e_j), e_\alpha)$, for $i, j \in \{1, ..., m\}$, $\alpha \in \{m+1, ..., 2n\}$.

Denote by $\mathcal{C}$ and $\mathcal{C}^*$ the *Casorati curvatures* of the submanifold M, defined by the squared norms of h and h^*, respectively, over the dimension m, as follows:

$$\mathcal{C} = \frac{1}{m}\|h\|^2 = \frac{1}{m}\sum_{\alpha=m+1}^{2n}\sum_{i,j=1}^{m}\left(h_{ij}^{\alpha}\right)^2,$$

$$\mathcal{C}^* = \frac{1}{m}\|h^*\|^2 = \frac{1}{m}\sum_{\alpha=m+1}^{2n}\sum_{i,j=1}^{m}\left(h_{ij}^{*\alpha}\right)^2.$$

Let L be an s-dimensional subspace of T_pM, $s \geq 2$ and let $\{e_1, \ldots, e_s\}$ be an orthonormal basis of L. Hence, the Casorati curvatures $\mathcal{C}(L)$ and $\mathcal{C}^*(L)$ of L are given by

$$\mathcal{C}(L) = \frac{1}{s}\sum_{\alpha=m+1}^{2n}\sum_{i,j=1}^{s}\left(h_{ij}^{\alpha}\right)^2, \quad \mathcal{C}^*(L) = \frac{1}{s}\sum_{\alpha=m+1}^{2n}\sum_{i,j=1}^{s}\left(h_{ij}^{*\alpha}\right)^2.$$

The *normalized δ-Casorati curvatures* $\delta_{\mathcal{C}}(m-1)$ and $\hat{\delta}_{\mathcal{C}}(m-1)$ of the submanifold M^n are given by

$$\delta_{\mathcal{C}}(m-1)|_p = \frac{1}{2}\mathcal{C}\,|_p + \frac{m+1}{2m}\inf\{\mathcal{C}(L)|L \text{ a hyperplane of } T_pM\}$$

and

$$\hat{\delta}_{\mathcal{C}}(m-1)|_p = 2\mathcal{C}\,|_p - \frac{2m-1}{2m}\sup\{\mathcal{C}(L)|L \text{ a hyperplane of } T_pM\}.$$

Moreover, the *dual normalized δ^*-Casorati curvatures* $\delta^*_{\mathcal{C}}(m-1)$ and $\widehat{\delta}^*_{\mathcal{C}}(m-1)$ of the submanifold M in $\tilde{M}$ are defined as

$$\delta^*_{\mathcal{C}}(m-1)|_p = \frac{1}{2}\mathcal{C}^*\,|_p + \frac{m+1}{2m}\inf\{\mathcal{C}^*(L)|L \text{ a hyperplane of } T_pM\}$$

and

$$\hat{\delta}^*_{\mathcal{C}}(m-1)|_p = 2\mathcal{C}^*\,|_p - \frac{2m-1}{2m}\sup\{\mathcal{C}^*(L)|L \text{ a hyperplane of } T_pM\}.$$

Denote by $\delta_{\mathcal{C}}(r; m-1)$ and $\hat{\delta}_{\mathcal{C}}(r; m-1)$, the *generalized normalized δ-Casorati curvatures* of M, defined in [10] as

$$\delta_{\mathcal{C}}(r; m-1)|_p = r\,\mathcal{C}\,|_p + a(r)\,\inf\{\mathcal{C}(L) \mid L \text{ a hyperplane of } T_pM\},$$

if $0 < r < m(m-1)$, and

$$\hat{\delta}_{\mathcal{C}}(r; m-1)|_p = r\,\mathcal{C}\,|_p + a(r)\sup\{\mathcal{C}(L) \mid L \text{ a hyperplane of } T_pM\},$$

if $r > m(m-1)$, for $a(r)$ set as

$$a(r) = \frac{(m-1)(r+m)(m^2-m-r)}{mr},$$

where $r \in \mathbb{R}_+$ and $r \neq m(m-1)$.

Furthermore, denote by $\delta^*_{\mathcal{C}}(r; m-1)$ and $\hat{\delta}^*{}_{\mathcal{C}}(r; m-1)$ the *dual generalized normalized δ^*-Casorati curvatures* of the submanifold M, defined as follows:

$$\delta^*_{\mathcal{C}}(r; m-1)|_p = r\,\mathcal{C}^* \;|_p + a(r) \inf\{\mathcal{C}^*(L) \mid L \text{ a hyperplane of } T_pM\},$$

if $0 < r < m(m-1)$, and

$$\hat{\delta}^*_{\mathcal{C}}(r; m-1)|_p = r\,\mathcal{C}^* \;|_p + a(r) \sup\{\mathcal{C}^*(L) \mid L \text{ a hyperplane of } T_pM\},$$

if $r > m(m-1)$, for $a(r)$ set above.

A statistical submanifold (M, g, ∇) of $(\tilde{M}, \tilde{g}, \tilde{\nabla})$ is called *totally geodesic* with respect to the connection $\tilde{\nabla}$ if the second fundamental form h of M for $\tilde{\nabla}$ vanishes identically [40].

Let $\tilde{M}$ be an almost complex manifold with almost complex structure $J \in \Gamma(T\tilde{M}^{(1,1)})$. A quadruplet $(\tilde{M}, \tilde{\nabla}, \tilde{g}, J)$ is called a *holomorphic statistical manifold* if

1. $(\tilde{\nabla}, \tilde{g})$ is a statistical structure on $\tilde{M}$; and
2. ω is a $\tilde{\nabla}$-parallel 2-form on $\tilde{M}$,

where ω is defined by $\omega(X, Y) = \tilde{g}(X, JY)$, for any $X, Y \in \Gamma(T\tilde{M})$.

For a holomorphic statistical manifold, the following formula holds:

$$\tilde{g}(\tilde{S}(Z, W)JY, JX) = \tilde{g}(\tilde{S}(JZ, JW)Y, X) = \tilde{g}(\tilde{S}(Z, W)Y, X), \tag{8}$$

for any $X, Y, Z, W \in \Gamma(T\tilde{M})$.

A holomorphic statistical manifold $(\tilde{M}, \tilde{\nabla}, \tilde{g}, J)$ is said to be of *constant holomorphic sectional curvature* $c \in \mathbb{R}$ if the following formula holds [42]:

$$\tilde{S}(X, Y)Z \;=\; \frac{c}{4}\{\tilde{g}(Y, Z)X - \tilde{g}(X, Z)Y + \tilde{g}(JY, Z)JX - \tilde{g}(JX, Z)JY + 2\tilde{g}(X, JY)JZ\}, \tag{9}$$

for any $X, Y, Z \in \Gamma(T\tilde{M})$, where $\tilde{S}$ is the statistical curvature tensor field of $\tilde{M}$.

Remark 2 ([43])**.** *Let $(\tilde{M}, \tilde{g}, J)$ be a Kähler manifold. If we define a connection $\tilde{\nabla}$ as $\tilde{\nabla} = \nabla^{\tilde{g}} + K$, where $K \in \Gamma(T\tilde{M}^{(1,2)})$ satisfying the conditions*

$$K(X, Y) = K(Y, X), \tag{10}$$

$$\tilde{g}(K(X, Y), Z) = \tilde{g}(Y, K(X, Z)), \tag{11}$$

$$K(X, JY) = -JK(X, Y), \tag{12}$$

for any $X, Y, Z \in \Gamma(T\tilde{M})$, then $(\tilde{M}, \tilde{\nabla}, \tilde{g}, J)$ is a holomorphic statistical manifold.

Let M be an m-dimensional statistical submanifold of a holomorphic statistical manifold $(\tilde{M}, \tilde{\nabla}, \tilde{g}, J)$. For any vector field X tangent to M we can decompose

$$JX = PX + FX, \tag{13}$$

where PX and FX are the tangent component and the normal component, respectively, of JX. Given a local orthonormal frame $\{e_1, e_2, \cdots, e_m\}$ of M, then the squared norm of P is expressed by

$$\|P\|^2 = \sum_{i,j=1}^{m} g^2(Pe_i, e_j).$$

Next, we consider the constrained extremum problem

$$\min_{x \in M} f(x), \tag{14}$$

where M is a Riemannian submanifold of a Riemannian manifold $(\tilde{M}, \tilde{g})$, and $f : \tilde{M} \to \mathbb{R}$ is a function of differentiability class C^2.

Theorem 1 ([44]). *If M is complete and connected, $(\text{grad} f)(p) \in T_p^{\perp} M$ for a point $p \in M$, and the bilinear form $\mathcal{A} : T_p M \times T_p M \to \mathbb{R}$ defined by*

$$\mathcal{A}(X, Y) = \text{Hess}(f)(X, Y) + \tilde{g}(h^0(X, Y), \text{grad} f), \tag{15}$$

is positive definite in p, then p is the optimal solution of the Problem (14).

Remark 3 ([44]). *If the bilinear form $\mathcal{A}$ defined by Equation (15) is positive semi-definite on the submanifold M, then the critical points of $f|M$ are global optimal solutions of the Problem (14).*

3. Main Inequalities

Theorem 2. *Let M be an m-dimensional statistical submanifold of a $2n$-dimensional holomorphic statistical manifold $(\tilde{M}, \tilde{\nabla}, \tilde{g}, J)$ of constant holomorphic sectional curvature c. Then we have*

(i)

$$\begin{aligned} 2\tau &\le \delta^0_{\mathcal{C}}(r; m-1) + m\mathcal{C}^0 - 2m^2\|H^0\|^2 \\ &+ m^2\tilde{g}(H, H^*) + \frac{3c}{4}\|P\|^2 + \frac{c}{4}m(m-1), \end{aligned} \tag{16}$$

for any real number r such that $0 < r < m(m-1)$, where $\delta^0_{\mathcal{C}}(r; m-1) = \frac{\delta_{\mathcal{C}}(r;m-1) + \delta^*_{\mathcal{C}}(r;m-1)}{2}$ and $\mathcal{C}^0 = \frac{\mathcal{C} + \mathcal{C}^*}{2}$; and

(ii)

$$\begin{aligned} 2\tau &\le \hat{\delta}^0_{\mathcal{C}}(r; m-1) + m\mathcal{C}^0 - 2m^2\|H^0\|^2 \\ &+ m^2\tilde{g}(H, H^*) + \frac{3c}{4}\|P\|^2 + \frac{c}{4}m(m-1), \end{aligned} \tag{17}$$

for any real number r such that $r > m(m-1)$, where $\hat{\delta}^0_{\mathcal{C}}(r; m-1) = \frac{\hat{\delta}_{\mathcal{C}}(r;m-1) + \hat{\delta}^*_{\mathcal{C}}(r;m-1)}{2}$.

Moreover, the equality cases of Inequalities (16) and (17) hold identically at all points $p \in M$ if and only if the following condition is satisfied:

$$h + h^* = 0, \tag{18}$$

where h and h^* are the imbedding curvature tensors of the submanifold associated to the dual connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively.

Proof. The relations (Equations (2)–(4)) imply

$$\begin{aligned} 2\tilde{g}(\tilde{S}(X, Y)Z, W) &= 2g(S(X, Y)Z, W) - \tilde{g}(h(Y, Z), h^*(X, W)) + \tilde{g}(h(X, Z), h^*(Y, W)) \\ &\quad - \tilde{g}(h^*(Y, Z), h(X, W)) + \tilde{g}(h^*(X, Z), h(Y, W)), \end{aligned} \tag{19}$$

where $X, Y, Z, W \in \Gamma(TM)$.

For $p \in M$, we choose $\{e_1, ..., e_m\}$ and $\{e_{m+1}, ..., e_{2n}\}$ orthonormal bases of T_pM and $T_p^{\perp}M$, respectively. For $X = Z = e_i$ and $Y = W = e_j$ with $i, j \in \{1, ..., m\}$, from the Equation (19), it follows that

$$\begin{aligned} 2\tau(p) &= m^2\tilde{g}(H, H^*) - \sum_{1\leq i,j\leq m} \tilde{g}(h^*(e_i, e_j), h(e_i, e_j)) \\ &\quad + \frac{c}{4}(m^2 - m + 3\|P\|^2). \end{aligned} \tag{20}$$

Denoting $2H^0 = H + H^*$ and $2\mathcal{C}^0 = \mathcal{C} + \mathcal{C}^*$, Equation (20) becomes

$$\begin{aligned} 2\tau(p) &= 2m^2\|H^0\|^2 - \frac{m^2}{2}\|H\|^2 - \frac{m^2}{2}\|H^*\|^2 \\ &\quad -2m\mathcal{C}^0 + \frac{m}{2}(\mathcal{C} + \mathcal{C}^*) + \frac{c}{4}(m^2 - m + 3\|P\|^2). \end{aligned} \tag{21}$$

Let $\mathcal{P}$ be the quadratic polynomial defined by

$$\begin{aligned} \mathcal{P} &= r\mathcal{C}^0 + a(r)\,\mathcal{C}^0(L) + \frac{m}{2}(\mathcal{C} + \mathcal{C}^*) - \frac{m^2}{2}(\|H\|^2 + \|H^*\|^2) \\ &\quad -2\tau(p) + \frac{c}{4}(m^2 - m + 3\|P\|^2), \end{aligned} \tag{22}$$

where L is a hyperplane of T_pM.

We consider that the hyperplane L is spanned by the tangent vectors $e_1, ..., e_{m-1}$, without loss of generality. Therefore, we get

$$\mathcal{P} = \sum_{\alpha=m+1}^{2n}\left[\frac{2m+r}{m}\sum_{i,j=1}^{m}(h_{ij}^{0\alpha})^2 + a(r)\frac{1}{m-1}\sum_{i,j=1}^{m-1}(h_{ij}^{0\alpha})^2 - 2\left(\sum_{i=1}^{m}h_{ii}^{0\alpha}\right)^2\right]. \tag{23}$$

Then, Equation (23) yields

$$\begin{aligned} \mathcal{P} &= \sum_{\alpha=m+1}^{2n}\Bigg\{\left[\frac{2(2m+r)}{m} + \frac{2a(r)}{m-1}\right]\sum_{1\leq i<j\leq m-1}(h_{ij}^{0\alpha})^2 + \left[\frac{2(2m+r)}{m} + \frac{2a(r)}{m-1}\right]\sum_{i=1}^{m-1}(h_{i\,m}^{0\alpha})^2 \\ &\quad + \left(\frac{2m+r}{m} + \frac{a(r)}{m-1} - 2\right)\sum_{i=1}^{m-1}(h_{ii}^{0\alpha})^2 \\ &\quad -4\sum_{1\leq i<j\leq m}h_{ii}^{0\alpha}h_{jj}^{0\alpha} + \left(\frac{2m+r}{m} - 2\right)(h_{mm}^{0\alpha})^2\Bigg\} \\ &\geq \sum_{\alpha=m+1}^{2n}\left[\frac{r(m-1)+a(r)m}{m(m-1)}\sum_{i=1}^{m-1}(h_{ii}^{0\alpha})^2 + \left(\frac{r}{m}\right)(h_{mm}^{0\alpha})^2 - 4\sum_{1\leq i<j\leq m}h_{ii}^{0\alpha}h_{jj}^{0\alpha}\right]. \end{aligned}$$

Let f_α be a quadratic form defined by $f_\alpha : \mathbb{R}^m \to \mathbb{R}$ for any $\alpha \in \{m+1, ..., 2n\}$,

$$\begin{aligned} f_\alpha(h_{11}^{0\alpha}, h_{22}^{0\alpha}, ..., h_{mm}^{0\alpha}) &= \sum_{i=1}^{m-1}\frac{r(m-1)+a(r)m}{m(m-1)}(h_{ii}^{0\alpha})^2 \\ &\quad + \frac{r}{m}(h_{mm}^{0\alpha})^2 - 4\sum_{1\leq i<j\leq m}h_{ii}^{0\alpha}h_{jj}^{0\alpha}. \end{aligned}$$

We investigate the constrained extremum problem

$$\min f_\alpha$$

with the constraint

$$Q : h_{11}^{0\alpha} + h_{22}^{0\alpha} + ... + h_{mm}^{0\alpha} = k^{\alpha},$$

where k^{α} is a real constant.

We obtain the system of first-order partial derivatives:

$$\begin{cases} \dfrac{\partial f_{\alpha}}{\partial h_{ii}^{0\alpha}} = 2\dfrac{r(m-1)+a(r)m}{m(m-1)}h_{ii}^{0\alpha} - 4\left(\displaystyle\sum_{k=1}^{m} h_{kk}^{0\alpha} - h_{ii}^{0\alpha}\right) = 0 \\ \dfrac{\partial f_{\alpha}}{\partial h_{mm}^{0\alpha}} = \dfrac{2r}{m}h_{mm}^{0\alpha} - 4\displaystyle\sum_{k=1}^{m-1} h_{kk}^{0\alpha} = 0, \end{cases}$$

for every $i \in \{1, ..., m-1\}$, $\alpha \in \{m+1, ..., 2n\}$.

It follows that the constrained critical point is

$$h_{ii}^{0\alpha} = \frac{2m(m-1)}{(m-1)(2m+r)+ma(r)}k^{\alpha}$$

$$h_{mm}^{0\alpha} = \frac{2m}{2m+r}k^{\alpha},$$

for any $i \in \{1, ..., m-1\}$, $\alpha \in \{m+1, ..., 2n\}$.

For $p \in$ Q, let $\mathcal{A}$ be a 2-form, $\mathcal{A} : T_pQ \times T_pQ \to \mathbb{R}$ defined by

$$\mathcal{A}(X,Y) = \mathrm{Hess}(f_{\alpha})(X,Y) + \langle h^{'}(X,Y), (\mathrm{grad} f_{\alpha})(p)\rangle,$$

where h' is the second fundamental form of Q in $\mathbb{R}^{m+1}$ and $\langle \cdot,\cdot \rangle$ is the standard inner product on $\mathbb{R}^m$.

The Hessian matrix of f_{α} is given by

$$\mathrm{Hess}(f_{\alpha}) = \begin{pmatrix} \lambda & -4 & \ldots & -4 & -4 \\ -4 & \lambda & \ldots & -4 & -4 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -4 & -4 & \ldots & \lambda & -4 \\ -4 & -4 & \ldots & -4 & \frac{2r}{m} \end{pmatrix},$$

where $\lambda = 2\frac{(m-1)(r+2m)+ma(r)}{m(m-1)}$ is a real constant.

The condition $\sum_{i=1}^{m} X_i = 0$ is satisfied, for a vector field $X \in T_pQ$, as the hyperplane Q is totally geodesic in $\mathbb{R}^m$. Then, we achieve

$$\begin{aligned} \mathcal{A}(X,X) &= \lambda \sum_{i=1}^{m-1} X_i^2 + \frac{2r}{m}X_m^2 - 8\sum_{i,j=1(i\neq j)}^{m} X_iX_j \\ &= \lambda \sum_{i=1}^{m-1} X_i^2 + \frac{2r}{m}X_m^2 + 4\left(\sum_{i=1}^{m} X_i\right)^2 - 8\sum_{i,j=1(i\neq j)}^{m} X_iX_j \\ &= \lambda \sum_{i=1}^{m-1} X_i^2 + \frac{2r}{m}X_m^2 + 4\sum_{i=1}^{m} X_i^2 \\ &\geq 0. \end{aligned}$$

Applying Remark 3, the critical point $(h_{11}^{0\alpha}, ..., h_{mm}^{0\alpha})$ of f_{α} is the global minimum point of the problem. Since $f_{\alpha}(h_{11}^{0\alpha}, ..., h_{mm}^{0\alpha}) = 0$, we get $\mathcal{P} \geq 0$.

We have then proved Inequalities (16) and (17), considering infimum and supremum, respectively, over all tangent hyperplanes L of T_pM.

In addition, we study the equality cases of Inequalities (16) and (17). First, we find out the critical points of $\mathcal{P}$

$$h^c = (h_{11}^{0\,m+1}, h_{12}^{0\,m+1}, \ldots, h_{m\,m}^{0\,m+1}, \ldots, h_{11}^{0\,2n}, \ldots, h_{m\,m}^{0\,2n})$$

as the solutions of following system of linear homogeneous equations:

$$\begin{cases} \dfrac{\partial \mathcal{P}}{\partial h_{ii}^{0\alpha}} = 2\left[\dfrac{2m+r}{m} + \dfrac{a(r)}{m-1} - 2\right] h_{ii}^{0\alpha} - 4\displaystyle\sum_{k\neq i, k=1}^{m} h_{kk}^{0\alpha} = 0, \\ \dfrac{\partial \mathcal{P}}{\partial h_{mm}^{0\alpha}} = 2\dfrac{r}{m} h_{mm}^{0\alpha} - 4\displaystyle\sum_{k=1}^{m-1} h_{kk}^{0\alpha} = 0, \\ \dfrac{\partial \mathcal{P}}{\partial h_{ij}^{0\alpha}} = 4\left[\dfrac{2m+r}{m} + \dfrac{a(r)}{m-1}\right] h_{ij}^{0\alpha} = 0, \;\; i \neq j, \\ \dfrac{\partial \mathcal{P}}{\partial h_{im}^{0\alpha}} = 4\left[\dfrac{2m+r}{m} + \dfrac{a(r)}{m-1}\right] h_{im}^{0\alpha} = 0. \end{cases}$$

The critical points satisfy $h_{ij}^{0\alpha} = 0$, with $i, j \in \{1, ..., m\}$ and $\alpha \in \{m+1, ..., 2n\}$. On the other hand, we know that $\mathcal{P} \geq 0$ and $\mathcal{P}(h^c) = 0$, then the critical point h^c is a minimum point of $\mathcal{P}$. Consequently, the cases of equality hold in both Inequalities (16) and (17) if and only if $h_{ij}^{\alpha} = -h_{ij}^{*\alpha}$, for $i, j \in \{1, ..., m\}$, $\alpha \in \{m+1, ..., 2n\}$. $\square$

Remark 4. *Under Equation (18), the submanifold M is totally geodesic with respect to the Levi–Civita connection $\tilde{\nabla}^0$. Then, the equality cases of Inequalities (16) and (17) hold for all unit tangent vectors at p if and only if p is a totally geodesic point with respect to the Levi–Civita connection.*

By virtue of Theorem 2, the generalized normalized δ-Casorati curvatures satisfy Inequalities (16) and (17). If the normalized δ-Casorati curvatures $\delta_{\mathcal{C}}(m-1)$ and $\delta_{\mathcal{C}}^*(m-1)$, respectively, $\hat{\delta}_{\mathcal{C}}(m-1)$ and $\hat{\delta}_{\mathcal{C}}^*(m-1)$ are involved, then we can state the following result.

Corollary 1. *Let M be an m-dimensional statistical submanifold of a $2n$-dimensional holomorphic statistical manifold $(\tilde{M}, \tilde{\nabla}, \tilde{g}, J)$ of constant holomorphic sectional curvature c. Then, we have*

(i)

$$\begin{aligned} \rho \;\; &\leq \;\; \delta_{\mathcal{C}}^0(m-1) + \frac{1}{m-1}\mathcal{C}^0 - \frac{2m}{m-1}\|H^0\|^2 \\ &+ \;\; \frac{m}{m-1}\tilde{g}(H, H^*) + \frac{3c}{4m(m-1)}\|P\|^2 + \frac{c}{4}, \end{aligned} \tag{24}$$

where $2\delta_{\mathcal{C}}^0(m-1) = \delta_{\mathcal{C}}(m-1) + \delta_{\mathcal{C}}^*(m-1)$ and $2\mathcal{C}^0 = \mathcal{C} + \mathcal{C}^*$, and

(ii)

$$\begin{aligned} \rho \;\; &\leq \;\; \hat{\delta}_{\mathcal{C}}^0(m-1) + \frac{1}{m-1}\mathcal{C}^0 - \frac{2m}{m-1}\|H^0\|^2 \\ &+ \;\; \frac{m}{m-1}\tilde{g}(H, H^*) + \frac{3c}{4m(m-1)}\|P\|^2 + \frac{c}{4}, \end{aligned} \tag{25}$$

where $2\hat{\delta}_{\mathcal{C}}^0(m-1) = \hat{\delta}_{\mathcal{C}}(m-1) + \hat{\delta}_{\mathcal{C}}^*(m-1)$.

Moreover, the equality cases of Inequalities (24) and (25) hold identically at all points if and only if h and h^* satisfy the condition in Equation (18), which implies that M is a totally geodesic submanifold with respect to the Levi–Civita connection.

4. An Example

Example 1. *Let (x_1, x_2, y_1, y_2) be a standard system on $\mathbb{R}^4$, g the Euclidean metric. Define $t = (y_1^2 + y_2^2)/2$ $(t \geq 0)$ and the functions u, v on $\mathbb{R}^4$ as*

$$u(x_1, x_2, y_1, y_2) = a(t), \;\; v(x_1, x_2, y_1, y_2) = b(t),$$

where a is a function $a : [0, \infty) \to (0, \infty)$, and $b(t) = -a(t)a'(t)(2ta'(t) - a(t))^{-1}$, assuming that $a(t) + 2tb(t) > 0$ for $t \geq 0$.

Let G be a g-natural metric on $\mathbb{R}^4$ and J a complex structure defined by Oproiu ([45]) such that $\mathbb{R}^4$ is Kählerian, as follows:

$$\begin{aligned} G \;\; = \;\; & (u + vy_1^2)dx_1dx_1 + 2vy_1y_2dx_1dx_2 + (u + vy_2^2)dx_2dx_2 + \frac{u + vy_2^2}{u(u + 2tv)}dy_1dy_1 \\ - \;\; & 2\frac{vy_1y_2}{u(u + 2tv)}dy_1dy_2 + \frac{u + vy_1^2}{u(u + 2tv)}dy_2dy_2, \end{aligned} \tag{26}$$

$$\begin{cases} J\dfrac{\partial}{\partial x_1} = (u + vy_1^2)\dfrac{\partial}{\partial y_1} + vy_1y_2\dfrac{\partial}{\partial y_2}, \\ J\dfrac{\partial}{\partial x_2} = vy_1y_2\dfrac{\partial}{\partial y_1} + (u + vy_2^2)\dfrac{\partial}{\partial y_2}, \\ J\dfrac{\partial}{\partial y_1} = -\dfrac{u + vy_2^2}{u(u + 2tv)}\dfrac{\partial}{\partial x_1} + \dfrac{vy_1y_2}{u(u + 2tv)}\dfrac{\partial}{\partial x_2}, \\ J\dfrac{\partial}{\partial y_2} = \dfrac{vy_1y_2}{u(u + 2tv)}\dfrac{\partial}{\partial x_1} - \dfrac{u + vy_1^2}{u(u + 2tv)}\dfrac{\partial}{\partial x_2}. \end{cases} \tag{27}$$

Let the function u be defined as $u(x_1, x_2, y_1, y_2) = \frac{1+\sqrt{1+4t}}{2}$. Therefore, the function v becomes $v(x_1, x_2, y_1, y_2) = 1$. Then, for the metric G and the complex structure J, there exists a tensor field K such that $(\mathbb{R}^4, \tilde{\nabla} := \nabla^G + K, \tilde{g} := G, J)$ is a special Kähler manifold [46]. Notice that a holomorphic statistical structure of holomorphic curvature 0 is nothing but a special Kähler manifold [43].

In this respect, define a $(1,2)$-tensor field K on $\mathbb{R}^4$:

$$K = \sum_{i,j,l=1}^{4} k_{ij}^l \frac{\partial}{\partial x^l} \otimes dx^i \otimes dx^j. \tag{28}$$

Let $\alpha_1, \ldots, \alpha_7$ be functions on $\mathbb{R}^4$ and denote $p := u + vy_1^2$, $q := u + vy_2^2$, $r := u + 2tv$, $s := vy_1y_2$. Suppose that α_2 has the expression

$$\alpha_2 = \frac{1}{2}s(u_{y_1} + 2y_1) + \frac{1}{2}qu_{y_2}. \tag{29}$$

Moreover, α_1 and α_3 satisfy the equation

$$\left(\alpha_2\frac{q}{sur} - \alpha_3\frac{1}{ur} - \alpha_1\frac{q}{s}\right)\frac{pur}{q} + \alpha_1\frac{sur}{q} + \alpha_2\frac{s}{q} = \frac{1}{2}p(u_{y_1} + 2y_1) + \frac{1}{2}su_{y_2}, \tag{30}$$

where $u_{y_1} := \frac{\partial u}{\partial y_1}$ and $u_{y_2} := \frac{\partial u}{\partial y_2}$.

If K performs the conditions in Equations (10)–(12) and also the conditions in Equations (29), (30), then we get $(\mathbb{R}^4, \tilde{\nabla} := \nabla^G + K, \tilde{g} := G, J)$ a special Kähler manifold [46] with K constructed as follows:

$$k_{14}^1 = k_{41}^1 = k_{13}^2 = k_{31}^2 = -k_{34}^3 = -k_{43}^3 = \alpha_1,\; k_{11}^4 = k_{12}^3 = k_{21}^3 = \alpha_2,\; k_{12}^4 = k_{21}^4 = k_{22}^3 = \alpha_3,$$

$$k_{24}^1 = k_{42}^1 = k_{23}^2 = k_{32}^2 = -k_{34}^4 = -k_{43}^4 = \alpha_4,\; k_{22}^2 = -k_{24}^4 = -k_{42}^4 = \alpha_5,$$

$$k_{11}^1 = k_{12}^1 = k_{21}^1 = -k_{23}^3 = -k_{32}^3 = 0,\ k_{12}^2 = k_{21}^2 = -k_{24}^3 = -k_{42}^3 = \alpha_7 \frac{q}{s},$$

$$k_{33}^1 = \alpha_6,\ k_{11}^2 = k_{14}^3 = k_{41}^3 = 0,\ k_{14}^2 = k_{41}^2 = -\alpha_2 \frac{s}{urq} + \alpha_3 \frac{p}{urq} - \alpha_1 \frac{s}{q},$$

$$k_{23}^1 = k_{32}^1 = \alpha_2 \frac{q}{urp} - \alpha_4 \frac{s}{p} - \alpha_3 \frac{s}{urp},\ k_{11}^3 = \alpha_1 \frac{2s^4 - u^2r^2}{sq} + \alpha_2 \frac{ur + 2s^2}{sq} - \alpha_3 \frac{p}{q},$$

$$k_{22}^4 = -\alpha_2 \frac{q}{p} - \alpha_4 \frac{u^2r^2}{sp} + \alpha_3 \frac{ur + 2s^2}{sp},\ k_{13}^1 = k_{31}^1 = \alpha_2 \frac{q}{sur} - \alpha_3 \frac{1}{ur} - \alpha_1 \frac{q}{s},$$

$$k_{33}^1 = -\alpha_2 \frac{q}{urp} + \alpha_3 \frac{s}{urp} + \alpha_4 \frac{s}{p},\ k_{44}^1 = -k_{24}^2 = -k_{42}^2 = \alpha_2 \frac{1}{ur} - \alpha_3 \frac{p}{sur} + \alpha_4 \frac{p}{s},$$

$$k_{44}^3 = \alpha_2 \frac{s}{qur} - \alpha_3 \frac{p}{qur} + \alpha_1 \frac{s}{q},\ k_{33}^3 = -\alpha_2 \frac{q}{sur} + \alpha_3 \frac{1}{ur} + \alpha_1 \frac{q}{s},$$

$$k_{22}^1 = -k_{23}^4 = -k_{32}^4 = -\alpha_5 \frac{s}{p},$$

$$k_{34}^1 = k_{43}^1 = -\alpha_6 \frac{s}{q} - \alpha_5 \frac{s^2}{urpq},$$

$$k_{44}^1 = \alpha_6 \frac{s^2}{q^2} + \alpha_5 \frac{s(2s^2 + ur)}{urpq^2},$$

$$k_{13}^3 = k_{31}^3 = k_{14}^4 = k_{41}^4 = 0,$$

$$k_{13}^4 = k_{31}^4 = 0,\ k_{33}^2 = -\alpha_6 \frac{p}{s},$$

$$k_{34}^2 = k_{43}^2 = \alpha_6 \frac{p}{q} + \alpha_5 \frac{s}{urq},$$

$$k_{44}^2 = -\alpha_5 \frac{pq + s^2}{urq^2} - \alpha_6 \frac{sp}{q^2}.$$

Then, $\tilde{M} = (\mathbb{R}^4, \tilde{\nabla} := \nabla^G + K, G, J)$ *is a holomorphic statistical manifold of holomorphic curvature* 0.

Next, let M *be any* m*-dimensional submanifold* ($m < 4$) *of* $\tilde{M}$*. Then, Inequalities* (16) *and* (17) *are satisfied. Moreover, the statistical submanifold* M *of* $\tilde{M}$ *attains equality in both these inequalities, provided that* M *is totally geodesic.*

5. Conclusions

In this research study, we provided new solutions to the fundamental problem of finding simple relationships between various invariants (intrinsic and extrinsic) of the submanifolds. In this respect, we obtained inequalities involving the normalized δ-Casorati curvatures (extrinsic invariants) and the scalar curvature (intrinsic invariant) of statistical submanifolds in holomorphic statistical manifolds with constant holomorphic sectional curvature. In addition, we characterized the equality cases. These results may stimulate new research aimed at obtaining similar relationships in terms of various invariants, for statistical submanifolds in other ambient spaces.

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References

1. Chen, B.Y. Riemannian submanifolds. In *Handbook of Differential Geometry*; Dillen, F., Verstraelen, L., Eds.; Elsevier North-Holland: Amsterdam, The Netherlands, 2000; Volume 1, pp. 187–418.
2. Chen, B.Y. Strings of Riemannian invariants, inequalities, ideal immersions and their applications. In Proceedings of the Third Pacific Rim Geometry Conference, Seoul, South Korea, 4–7 December 1996; International Press: Cambridge, UK, 1998; pp. 7–60.
3. Chen, B.Y. Some pinching and classification theorems for minimal submanifolds. *Arch. Math.* **1993**, *60*, 569–578. [CrossRef]
4. Chen, B.Y. A general inequality for submanifolds in complex-space-forms and its applications. *Arch. Math.* **1996**, *67*, 519–528. [CrossRef]
5. Oiagă, A.; Mihai, I. B.Y. Chen inequalities for slant submanifolds in complex space forms. *Demonstr. Math.* **1999**, *32*, 835–846. [CrossRef]
6. Oprea, T. Chen's inequality in Lagrangian case. *Colloq. Math.* **2007**, *108*, 163–169. [CrossRef]
7. Chen, B.Y. *Pseudo-Riemannian Geometry, δ-Invariants and Applications*; World Scientific: Singapore, 2011.
8. Chen, B.Y.; Dillen, F.; der Veken, J.V.; Vrancken, L. Classification of $\delta(2, n-2)$-ideal Lagrangian submanifolds in n-dimensional complex space forms. *J. Math. Anal. Appl.* **2018**, *458*, 1456–1485. [CrossRef]
9. Decu, S.; Haesen, S.; Verstraelen, L. Optimal inequalities involving Casorati curvatures. *Bull. Transilv. Univ. Braşov Ser. B Suppl.* **2007**, *14*, 85–93.
10. Decu, S.; Haesen, S.; Verstraelen, L. Optimal inequalities characterising quasi-umbilical submanifolds. *J. Inequal. Pure Appl. Math.* **2008**, *9*.
11. Casorati, F. Mesure de la courbure des surfaces suivant l'idée commune. *Acta Math.* **1890**, *14*, 95–110. [CrossRef]
12. Koenderink, J.J. *Shadows of Shapes*; De Clootcrans Press: Utrecht, The Netherlands, 2012.
13. Verstraelen, P. A geometrical description of visual perception-The Leuven Café Erasmus model and Bristol Café Wall illusion. *Kragujevac J. Math.* **2005**, *28*, 7–17.
14. Haesen, S.; Kowalczyk, D.; Verstraelen, L. On the extrinsic principal directions of Riemannian submanifolds. *Note Mat.* **2009**, *29*, 41–53.
15. Decu, S.; Verstraelen, L. A note of the isotropical geometry of production surfaces. *Kragujevac J. Math.* **2013**, *37*, 217–220.
16. He, G.; Liu, H.; Zhang, L. Optimal inequalities for the Casorati curvatures of submanifolds in generalized space forms endowed with semi-symmetric non-metric connections. *Symmetry* **2016**, *8*, 113. [CrossRef]
17. Lee, C.W.; Yoon, D.W.; Lee, J.W. Optimal inequalities for the Casorati curvatures of submanifolds of real space forms endowed with semi-symmetric metric connections. *J. Inequal. Appl.* **2014**, *2014*, 327. [CrossRef]
18. Park, K. Inequalities for the Casorati curvatures of real hypersurfaces in some Grassmannians. *Taiwan. J. Math.* **2018**, *22*, 63–77. [CrossRef]
19. Zhang, P.; Zhang, L. Casorati Inequalities for Submanifolds in a Riemannian Manifold of Quasi-Constant Curvature with a Semi-Symmetric Metric Connection. *J. Inequal. Appl.* **2016**, *8*, 19. [CrossRef]
20. Zhang, P.; Zhang, L. Inequalities for Casorati curvatures of submanifolds in real space forms. *Adv. Geom.* **2016**, *16*, 329–335. [CrossRef]
21. Lee, C.; Lee, J.; Vîlcu, G.E. Optimal inequalities for the normalized δ-Casorati curvatures of submanifolds in Kenmotsu space forms. *Adv. Geom.* **2017**, *17*, 355–362. [CrossRef]
22. Vîlcu, G. An optimal inequality for Lagrangian submanifolds in complex space forms involving Casorati curvature. *J. Math. Anal. Appl.* **2018**, *465*, 1209–1222. [CrossRef]
23. Aquib, M.; Lee, J.; Vîlcu, G.E.; Yoon, D. Classification of Casorati ideal Lagrangian submanifolds in complex space forms. *Differ. Geom. Appl.* **2019**, *63*, 30–49. [CrossRef]
24. Suceavă, B.; Vajiac, M. Estimates of B.-Y. Chen's $\hat{\delta}$-Invariant in Terms of Casorati Curvature and Mean Curvature for Strictly Convex Euclidean Hypersurfaces. *Int. Electron. J. Geom.* **2019**, *12*, 26–31.
25. Brubaker, N.; Suceavă, B. A Geometric Interpretation of Cauchy-Schwarz inequality in terms of Casorati Curvature. *Int. Electron. J. Geom.* **2018**, *11*, 48–51.
26. Amari, S. *Differential-Geometrical Methods in Statistics*; Lecture Notes in Statistics; Berger, J., Fienberg, S., Gani, J., Krickeberg, K., Olkin, I., Singer, B., Eds.; Springer: Berlin, Germany, 1985; Volume 28.

27. Boyom, N. Foliations-Webs-Hessian Geometry-Information Geometry-Entropy and Cohomology. *Entropy* **2016**, *18*, 433. [CrossRef]
28. Cheng, Y.; Wang, X.; Moran, B. Optimal Nonlinear Estimation in Statistical Manifolds with Application to Sensor Network Localization. *Entropy* **2017**, *19*, 308. [CrossRef]
29. Zhang, J. Nonparametric Information Geometry: From Divergence Function to Referential-Representational Biduality on Statistical Manifolds. *Entropy* **2013**, *15*, 5384–5418. [CrossRef]
30. Cuingnet, R.; Glaunès, J.; Chupin, M.; Benali, H.; Colliot, O. Anatomical regularization on Statistical Manifolds for the Classification of Patients with Alzheimer's Disease. In *Machine Learning in Medical Imaging. MLMI 2011*; Suzuki, K., Wang, F., Shen, D., Yan, P., Eds.; Lecture Notes in Comput. Sci.; Springer: Berlin/Heidelber, Germany, 2011; Volume 7009.
31. Aydin, M.; Mihai, A.; Mihai, I. Some inequalities on submanifolds in statistical manifolds of constant curvature. *Filomat* **2015**, *29*, 465–477. [CrossRef]
32. Aydin, M.; Mihai, A.; Mihai, I. Generalized Wintgen inequality for statistical submanifolds in statistical manifolds of constant curvature. *Bull. Math. Sci.* **2017**, *7*, 155–166. [CrossRef]
33. Lee, C.; Lee, J. Inequalities on Sasakian Statistical Manifolds in Terms of Casorati Curvatures. *Mathematics* **2018**, *6*, 259. [CrossRef]
34. Aquib, M.; Shahid, M. Generalized normalized δ-Casorati curvature for statistical submanifolds in quaternion Kaehler-like statistical space forms. *J. Geom.* **2018**, *109*, 13. [CrossRef]
35. Vîlcu, A.; Vîlcu, G. Statistical manifolds with almost quaternionic structures and quaternionic Kaehler-like statistical submersions. *Entropy* **2015**, *17*, 6213–6228. [CrossRef]
36. Aytimur, H.; Kon, M.; Mihai, A.; Ozgur, C.; Takano, K. Chen Inequalities for Statistical Submanifolds of Kaehler-Like Statistical Manifolds. *Mathematics* **2019**, *7*, 1202. [CrossRef]
37. Aquib, M.; Boyom, M.; Shahid, M.; Vîlcu, G. The First Fundamental Equation and Generalized Wintgen-Type Inequalities for Submanifolds in Generalized Space Forms. *Mathematics* **2019**, *7*, 1151. [CrossRef]
38. Chen, B.Y.; Mihai, A.; Mihai, I. A Chen First Inequality for Statistical Submanifolds in Hessian Manifolds of Constant Hessian Curvatures. *Results Math.* **2019**, *74*, 165. [CrossRef]
39. Siddiqui, A.; Chen, B.Y.; Bahadir, O. Statistical Solitons and Inequalities for Statistical Warped Product Submanifolds. *Mathematics* **2019**, *7*, 797. [CrossRef]
40. Furuhata, H.; Hasegawa, I. Submanifold theory in holomorphic statistical manifolds. In *Geometry of Cauchy-Riemann Submanifolds*; Dragomir, S., Shahid, M.H., Al-Solamy, F.R., Eds.; Springer Science+Business Media: Singapore, 2016; pp. 179–214.
41. Vos, P. Fundamental equations for statistical submanifolds with applications to the Barlett correction. *Ann. Inst. Statist. Math.* **1989**, *41*, 429–450. [CrossRef]
42. Furuhata, H.; Hasegawa, I.; Okuyama, Y.; Sato, K. Kenmotsu statistical manifolds and warped product. *J. Geom.* **2017**, *108*, 1175–1191. [CrossRef]
43. Furuhata, H. Hypersurfaces in statistical manifolds. *Differ. Geom. Appl.* **2009**, *27*, 420–429. [CrossRef]
44. Oprea, T. *Constrained Extremum Problems in Riemannian Geometry*; University of Bucharest Publishing House: Bucharest, Romania, 2006.
45. Oproiu, V. Some new geometric structures on the tangent bundle. *Publ. Math. Debrecen* **1999**, *55*, 261–281.
46. Milijevic, M. Totally real statistical submanifolds. *Interdiscip. Inform. Sci.* **2015**, *21*, 87–96. [CrossRef]