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THE DIAGRAM CATEGORY OF FRAMED TANGLES AND INVARIANTS OF QUANTIZED SYMPLECTIC GROUP

ZHANKUI XIAO, YUPING YANG, AND YINHUO ZHANG

ABSTRACT. In this paper we present a categorical version of the first and second fundamental theorems of the invariant theory for the quantized symplectic groups. Our methods depend on the theory of braided strict monoidal categories which are pivotal, more explicitly the diagram category of framed tangles.

1. INTRODUCTION

The fundamental theorems of the classical invariant theory are concerned with describing generators and relations for invariants of the classical group actions [28]. There are several but essentially equivalent ways to formulate these fundamental theorems.

Let G be the general (special) linear group, or the symplectic group, or the orthogonal group. One formulation of the fundamental theorems is in terms of the endomorphism algebra $\text{End}_G(V^{\otimes n})$, where V is the corresponding natural representation. In this case, the first fundamental theorem (FFT) describes the endomorphism algebra as the homomorphic image of some known algebra and the FFT of this formulation is also known as Schur-Weyl duality. For the general (special) linear group, the endomorphism algebra $\text{End}_G(V^{\otimes n})$ is the homomorphic image of the group algebra of the symmetric group following Schur [28]. For the symplectic or the orthogonal group, the endomorphism algebra $\text{End}_G(V^{\otimes n})$ is the homomorphic image of the Brauer algebra with specialized parameters (see [2, 5] for the symplectic case and [2, 7] for the orthogonal case).

On the other hand, except the case of general (special) linear groups, it is an open problem for several decades to find a standard form of the second fundamental theorem (SFT) in the formulation of endomorphism algebras. It needs to describe a suitable ideal in the Brauer algebra using the standard generators of the Brauer algebra. Recently Hu and the first author in [13] proved the SFT for the symplectic group and Lehrer-Zhang in [19] gave the SFT for the orthogonal group, where they took advantage of the different versions of invariant theory. Furthermore, the methods in [19] give rise to the possibility that there should be a unified description for the different formulations of the fundamental theorems and this goal was achieved by Lehrer and Zhang in [20] using the category of Brauer diagrams, a symmetric braided strict monoidal category in the sense of Joyal and Street [16]. The main

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motivation of this paper is to give a quantum analogue of [20, Theorem 4.8] for the quantized symplectic group.

The invariant theory of quantum groups [4, 22] in a broad sense has been studied extensively. An interesting and important aspect of it is the connection with the topological quantum field theory, i.e. the quantum group theoretical construction of the Jones polynomial of knots [15]. We refer the reader to the book [27] for a comprehensive understanding of this topic.

Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra associated to the finite dimensional complex simple Lie algebra \mathfrak{g} [3, 22]. The Schur-Weyl duality formulation of the FFT for quantum invariants describes the endomorphism algebra $\text{End}_{U_q(\mathfrak{g})}(V^{\otimes n})$, with V the natural representation, as the homomorphic image of the Hecke algebra of type A [14, 8] or the Birman-Murakami-Wenzl algebra (BMW algebra for short) with some specialized parameters [11, 12]. However, in this case, we do not know so much for the SFT. When q is an indeterminate and $\mathfrak{g} = \mathfrak{sp}_{2m}$, the symplectic Lie algebra, the SFT can be given from the detailed structure and representations of the BMW algebra [13, 6], where the proofs involve a somewhat technical computation. Therefore, it is desirable to provide a standard and explicit formulae for the SFT of the quantized symplectic group and the orthogonal group which has been suggested in [20, Section 8].

This note can be seen as our first attempt, following the idea of Lehrer and Zhang, to build an explicit connection between different formulations of the fundamental theorems for the quantum invariant theory [3, 21, 22]. We would like to remark that the proof of [20, Theorem 4.8] depends on the linear formulation of the fundamental theorems of the symplectic or the orthogonal group. However, in quantum case, the linear formulation of the fundamental theorems do not exist as far as we know. Our categorical approach here relies highly on the pivotal structure of braided strict monoidal categories. More precisely, we introduce the diagram category of framed tangles [10], which is also known as the category of non-directed ribbon graphs [25, 27]. The fact is that the Reshetikhin-Turaev functor (RT-functor for short, see Section 3) is pivotal, a full monoidal functor from the diagram category of framed tangles to the category of tensor representations of the quantized symplectic group. This fact has been used in the classical or non-quantum case in [20, 26].

The contents of this note is organized as follows. In Section 2 we first introduce some definitions and basic properties related to the diagram category of framed tangles, as a strict braided pivotal monoidal category. We then present explicitly, in Section 3, the RT-functor between the diagram category of framed tangles and the category of tensor representations of $U_q(\mathfrak{sp}_{2m})$. In Section 4, we study the fundamental theorems of invariants for the quantized symplectic group over the rational function field, and prove a quantum analogue of [20, Theorem 4.8].

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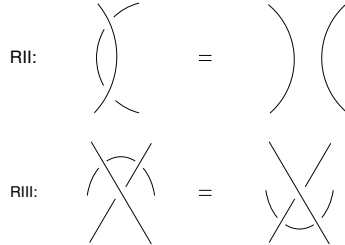
2. DIAGRAM CATEGORY OF FRAMED TANGLES

In the present section, we introduce some basic definitions and facts about the diagram category of framed tangles, the BMW algebra. All the tangles in this paper are assumed to be non-directed. For the terminologies about monoidal and tensor categories, we refer the reader to [9] for their precise meanings.

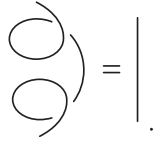
2.1. **Framed tangle category.** Let us first recall some definitions from [10, 17].

Definition 2.1. A *tangle* is a knot diagram inside a rectangle consisting of a finite number of vertices in the top and the bottom row of the rectangle (not necessarily the same number) and a finite number of arcs inside the rectangle such that each vertex is connected to another vertex by exactly one arc, and arcs either connect two vertices or are closed curves.

Definition 2.2. Two tangles are *regularly isotopic* if they are equivalent by a sequence of the following Reidemeister Moves II and III



and the isotopies fixing the boundary of the rectangle (or any rotation of them in a local portion of the rectangle). By $T(s, t)$ we denote the set of all tangles with s vertices in the top row and t vertices in the bottom row subject to the relations of regular isotopy and one more relation, named the modified Reidemeister Move I, as follows:



There are two operations on tangles:

$$(2.1) \quad \circ : T(s, t) \times T(t, l) \longrightarrow T(s, l),$$

$$(2.2) \quad \otimes : T(s, t) \times T(l, m) \longrightarrow T(s + l, t + m).$$

The *composition* \circ is defined by concatenation of tangles, reading from up to down for our late convenience, and the *tensor product* \otimes is given by juxtaposition of tangles. More explicitly, $D \otimes D'$ means placing D' on the right of D without overlapping. In order to obtain the explicit definition of the framed tangle category, we need to add an element id_0 in $T(0, 0)$ and define the corresponding composition and tensor product as follows: for any $f \in T(0, t)$, $g \in T(s, 0)$ and $h \in T(s, t)$,

$$(2.3) \quad id_0 \circ f = f, \quad g \circ id_0 = g,$$

$$(2.4) \quad h \otimes id_0 = id_0 \otimes h = h.$$

With these notations, we can give the definition of the framed tangle category following [10, Definition 3.8].

Definition 2.3. The *framed tangle category*, denoted \mathcal{FT} , is a strict monoidal category with objects $\mathbb{N} = \{0, 1, 2, \dots\}$ and morphisms given by $\text{Hom}_{\mathcal{FT}}(s, t) = T(s, t)$. The tensor product on objects is defined by $m \otimes n := m + n$, and the composition and tensor product on morphisms are defined by (2.1-2.4).

The framed tangle category \mathcal{FT} is a free pivotal strict monoidal category with a self-dual generator, see [10, Theorem 3.6]. The duality functor $*$ is a contravariant functor taking each object to itself, and acting on morphisms by rotating the rectangle containing a tangle through π about an axis perpendicular to the rectangle. It is clear that $**$ is just the identity functor $\text{Id}_{\mathcal{FT}}$, see [10] for more details. Here we remark that $T(s, t) = \emptyset$ by definition if $s + t$ is an odd number.

2.2. Diagram category of framed tangles. To meet our need, we give the linear version of the framed tangle category. Let \mathbb{K} be the rational function field $\mathbb{Q}(r, q)$ with r, q being indeterminates. The linear framed tangle category, denoted $\mathbb{K}(\mathcal{FT})$, consists of the same objects of \mathcal{FT} , but $\text{Hom}_{\mathbb{K}(\mathcal{FT})}(s, t)$ is the \mathbb{K} -linear span of $T(s, t)$. The composition and the tensor product of morphisms are given by bilinear extensions of those in \mathcal{FT} . The diagram category of framed tangles (see below Definition 2.5) can be obtained as a quotient of $\mathbb{K}(\mathcal{FT})$ subject to certain relations on morphisms.

Definition 2.4. For $s, t \in \mathbb{N}$ we denote by $\mathcal{D}(s, t)$ the \mathbb{K} -linear span of $T(s, t)$ subject to the following relations:

$$\text{Q1: } \begin{array}{c} \diagup \\ \diagdown \end{array} = \begin{array}{c} \diagdown \\ \diagup \end{array} + (q - q^{-1}) \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \right) \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \right) - (q - q^{-1}) \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

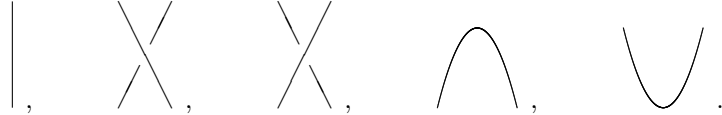
$$\text{Q2: } \begin{array}{c} \text{---} \\ \text{---} \end{array} = r^{-1} \begin{array}{c} | \\ | \end{array}$$

$$\text{Q3: } \begin{array}{c} \text{---} \\ \text{---} \end{array} = r \begin{array}{c} | \\ | \end{array}$$

$$\text{Q4: } \bigcirc = x$$

where $x = 1 + \frac{r-r^{-1}}{q-q^{-1}}$.

We shall explain the composition defined by (2.1) can be extended to $\mathcal{D}(s, t)$. According to [10, Theorem 3.5], any tangle in $T(s, t)$ can be generated through compositions and tensor products by the following five tangles:



We shall refer to these generators as the *elementary* tangles and denote them by I, X, X^{op}, A, U respectively. In order to show that the composition can be extended to $\mathcal{D}(s, t)$, We only need to show that when the terms in both handsides of the relations Q1-Q4 compose with an elementary tangle, the results are still identities. This follows from a direct verification. It is obvious that there is a natural tensor operation over $\mathcal{D}(s, t)$. Therefore we can now give the following new definition, which is a little stronger than the one in [10, Definition 4.1.2].

Definition 2.5. The *diagram category of framed tangles* with parameters r, q , denoted $\mathbb{D}(r, q)$, is a strict monoidal category with objects $\mathbb{N} = \{0, 1, 2, \dots\}$ and morphisms $\text{Hom}_{\mathbb{D}(r, q)}(s, t) = \mathcal{D}(s, t)$. The composition \circ and the tensor product \otimes are inherited from those of $\mathbb{K}(\mathcal{FT})$.

From now on we only focus us on the diagram category of framed tangles. The category $\mathbb{D}(r, q)$ has a contravariant functor $*$: $\mathbb{D}(r, q) \rightarrow \mathbb{D}(r, q)$ such that $n^* = n$ for any object $n \in \mathbb{N}$. In order to get a clear picture of morphisms under the functor $*$, we define the following useful tangles with notations compatible with those in the general theory of tensor category [9]. Let $U_n : n \otimes n \rightarrow 0$ and $A_n : 0 \rightarrow n \otimes n$ be tangles defined by

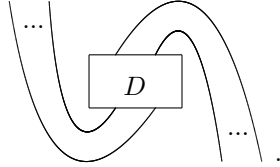
$$(2.5) \quad U_n = (I^{\otimes(n-1)} \otimes U \otimes I^{\otimes(n-1)}) \circ \dots \circ (I \otimes U \otimes I) \circ U,$$

$$(2.6) \quad A_n = A \circ (I \otimes A \otimes I) \circ \dots \circ (I^{\otimes(n-1)} \otimes A \otimes I^{\otimes(n-1)}).$$

These are depicted as follows



Let $I_n := I^{\otimes n}$ be the identity morphism id_n . The linear mapping $*$: $\mathcal{D}(s, t) \rightarrow \mathcal{D}(t, s)$ defined for any tangle $D \in \mathcal{D}(s, t)$ by $D^* := (I_t \otimes A_s) \circ (I_t \otimes D \otimes I_s) \circ (U_t \otimes I_s)$ can be described as follows



By [10, Lemma 3.3], we have $** = \text{Id}$, the identity functor. Moreover, the following compositions

$$n \xrightarrow{A_n \otimes I_n} n \otimes n \otimes n \xrightarrow{I_n \otimes U_n} n,$$

$$n \xrightarrow{I_n \otimes A_n} n \otimes n \otimes n \xrightarrow{U_n \otimes I_n} n$$

are both the identity morphism. Hence the contravariant functor $*$ gives rise to the duality of $\mathbb{D}(r, q)$ with evaluation U and coevaluation A . Hence we have

Proposition 2.6. *The diagram category of framed tangles $\mathbb{D}(r, q)$ is pivotal.*

The following analogue of [20, Theorem 2.6] to some extent is known for experts (see [10, 25, 27]). However, we haven't found it in literatures, and hence sketch the proof here for the completeness.

Theorem 2.7. *Any morphism of $\mathbb{D}(r, q)$ is generated by four elementary tangles I, X, A, U through linear combination, composition and tensor product. A complete set of relations among these four generators is given by the following identities and*

their dualities:

$$(2.7) \quad I \circ I = I, (I \otimes I) \circ X = X, A \circ (I \otimes I) = A,$$

$$(2.8) \quad X \circ X = I \otimes I + (q - q^{-1})X - r^{-1}(q - q^{-1})U \circ A,$$

$$(2.9) \quad (X \otimes I) \circ (I \otimes X) \circ (X \otimes I) = (I \otimes X) \circ (X \otimes I) \circ (I \otimes X),$$

$$(2.10) \quad A \circ X = r^{-1}A,$$

$$(2.11) \quad A \circ U = x,$$

$$(2.12) \quad (A \otimes I) \circ (I \otimes X) = (I \otimes A) \circ ((X + (q^{-1} - q)(I \otimes I - U \circ A)) \otimes I),$$

$$(2.13) \quad (A \otimes I) \circ (I \otimes U) = I.$$

Proof. Since any tangle can be generated by I, X, X^{op}, A, U through the composition and the tensor product, according to [10, Theorem 3.5], any morphism in $\mathcal{D}(s, t)$ can be obtained by I, X, A, U through linear combination, composition and tensor product because of the relation Q1. Thus, we proved the first claim of the theorem.

For the second part of the theorem, we first explain how to obtain the identities (2.7-2.11) and their dualities. The identities (2.7) and their dualities are obvious. The identity (2.8) can be obtained from the relation Q1 by composing with elementary tangle X , using the Reidemeister move RII and the identity Q2. The identities (2.9-2.11) are corresponding to the Reidemeister move RIII, Q2 and Q4 respectively. Finally, (2.12-2.13) are deduced from the singular isotopy, called sliding and straightening respectively, see [10, Definition 2.3]. Now we need to prove that these identities (2.7-2.13) are complete. For any $s, t \in \mathbb{N}$, it follows from the definition of $\mathcal{D}(s, t)$ that all relations among the four generators are determined by the regular isotopy and the relations Q1-Q4. Set $X^{op} := X - (q - q^{-1})I \otimes I - (q - q^{-1})U \circ A$. Then the regular isotopy, the modified Reidemeister move I, and the relations Q1-Q4 can be deduced from the identities (2.7-2.13) and their dualities. \square

For any object $n > 0$, the set of morphisms $\mathcal{D}(n, n)$ forms a unital associative \mathbb{K} -algebra under the composition of tangles. This is the Kauffman's tangle algebra [17] which is isomorphic to a BMW algebra. We express BMW algebra by generators and relations for late use.

Definition 2.8. ([1, 24]) The generic *BMW algebra* $B_n(r, q)$ is a unital associative \mathbb{K} -algebra generated by the elements $T_i^{\pm 1}$ and E_i for $1 \leq i \leq n - 1$ subject to the relations:

$$(2.14) \quad T_i - T_i^{-1} = (q - q^{-1})(1 - E_i), \quad \text{for } 1 \leq i \leq n - 1,$$

$$(2.15) \quad E_i^2 = xE_i, \quad \text{for } 1 \leq i \leq n - 1,$$

$$(2.16) \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad \text{for } 1 \leq i \leq n - 2,$$

$$(2.17) \quad T_i T_j = T_j T_i, \quad \text{for } |i - j| > 1,$$

$$(2.18) \quad E_i E_{i+1} E_i = E_i, E_{i+1} E_i E_{i+1} = E_{i+1}, \quad \text{for } 1 \leq i \leq n - 2,$$

$$(2.19) \quad T_i T_{i+1} E_i = E_{i+1} E_i, T_{i+1} T_i E_{i+1} = E_i E_{i+1}, \quad \text{for } 1 \leq i \leq n - 2,$$

$$(2.20) \quad E_i T_i = T_i E_i = r^{-1} E_i \quad \text{for } 1 \leq i \leq n - 1,$$

$$(2.21) \quad E_i T_{i+1} E_i = r E_i, E_{i+1} T_i E_{i+1} = r E_{i+1}, \quad \text{for } 1 \leq i \leq n - 2,$$

where $x = 1 + \frac{r - r^{-1}}{q - q^{-1}}$.

Theorem 2.9. *Kauffman's tangle algebra $\mathcal{D}(n, n)$ is isomorphic to the BMW algebra $B_n(r, q)$ with the isomorphism given by*

$$T_i \mapsto I_{i-1} \otimes X \otimes I_{n-1-i}, \quad E_i \mapsto I_{i-1} \otimes (U \circ A) \otimes I_{n-1-i}.$$

3. THE RESHETIKHIN-TURAEV FUNCTOR

In this section, we present explicitly the RT-functor between the diagram category of framed tangles and the category of tensor representations in the symplectic case [25]. Let $m \in \mathbb{N}$, and q an indeterminate. Let $U_q(\mathfrak{sp}_{2m})$ be the quantized enveloping algebra over $\mathbb{Q}(q)$ associated to \mathfrak{sp}_{2m} (see [22]). Let V be the natural representation of $U_q(\mathfrak{sp}_{2m})$. Then $\dim V = 2m$ equipped with a skew symmetric bilinear form $(-, -)$. We refer the reader to [12, Section 2] for the explicit action of $U_q(\mathfrak{sp}_{2m})$ on the tensor space $V^{\otimes n}$.

There is a right action of the BMW algebra $B_n(-q^{2m+1}, q)$, i.e. r specialized to $-q^{2m+1}$, on the tensor space $V^{\otimes n}$, which we now recall. For each integer i with $1 \leq i \leq 2m$, set $i' := 2m + 1 - i$. We fix an ordered basis $\{v_i\}_{i=1}^{2m}$ of V such that

$$(v_i, v_j) = 0 = (v_{i'}, v_{j'}), \quad (v_i, v_{j'}) = \delta_{ij} = -(v_{j'}, v_i), \quad \forall 1 \leq i, j \leq m.$$

We set

$$(\rho_1, \dots, \rho_{2m}) := (m, m-1, \dots, 1, -1, \dots, -m+1, -m),$$

and $\epsilon_i := \text{sign}(\rho_i)$. For any $i, j \in \{1, 2, \dots, 2m\}$, we use $E_{i,j} \in \text{End}_{\mathbb{Q}(q)}(V)$ to denote the elementary matrix whose entries are all zero except 1 for the (i, j) -th entry. Let us define (see [12, Section 3] for the same notations)

$$\begin{aligned} \beta' &:= \sum_{1 \leq i \leq 2m} \left(qE_{i,i} \otimes E_{i,i} + q^{-1}E_{i,i'} \otimes E_{i',i} \right) + \sum_{\substack{1 \leq i, j \leq 2m \\ i \neq j, j'}} E_{i,j} \otimes E_{j,i} + \\ &\quad (q - q^{-1}) \sum_{1 \leq i < j \leq 2m} \left(E_{i,i} \otimes E_{j,j} - q^{\rho_j - \rho_i} \epsilon_i \epsilon_j E_{i,j'} \otimes E_{i',j} \right), \\ \gamma' &:= \sum_{1 \leq i, j \leq 2m} q^{\rho_j - \rho_i} \epsilon_i \epsilon_j E_{i,j'} \otimes E_{i',j}. \end{aligned}$$

Note that the operators β', γ' are related to each other by the equation

$$\beta' - (\beta')^{-1} = (q - q^{-1})(\text{id}_{V^{\otimes 2}} - \gamma').$$

For $i = 1, 2, \dots, n-1$, we set

$$\beta'_i := \text{id}_{V^{\otimes i-1}} \otimes \beta' \otimes \text{id}_{V^{\otimes n-i-1}}, \quad \gamma'_i := \text{id}_{V^{\otimes i-1}} \otimes \gamma' \otimes \text{id}_{V^{\otimes n-i-1}}.$$

By [3, (10.2.5)] and [11, Section 4], the mapping which sends the generator T_i to β'_i and E_i to γ'_i can be naturally extended to a right action of $B_n(-q^{2m+1}, q)$ on the tensor space $V^{\otimes n}$. This right action commutes with the left action of $U_q(\mathfrak{sp}_{2m})$ on $V^{\otimes n}$.

In order to define the RT-functor explicitly in our case, we need the following $\mathbb{Q}(q)$ -linear maps:

$$\begin{aligned} \check{R} : V \otimes V &\longrightarrow V \otimes V, & v \otimes w &\mapsto \beta'(v \otimes w), \\ C : \mathbb{Q}(q) &\longrightarrow V \otimes V, & 1 &\mapsto \alpha := \sum_{1 \leq k \leq 2m} q^{-\rho_k} \epsilon_k v_k \otimes v_{k'}, \\ E : V \otimes V &\longrightarrow \mathbb{Q}(q), & v_i \otimes v_j &\mapsto q^{-\rho_i} \epsilon_j (v_i, v_j). \end{aligned}$$

Clearly $\gamma' = E \circ C$, where the composition reads from left to right, because of the *right* action of $B_n(-q^{2m+1}, q)$ on $V^{\otimes n}$. By the above statements, the maps \check{R}, C and E are all $U_q(\mathfrak{sp}_{2m})$ -homomorphisms. They have the following properties.

Lemma 3.1. *Denote the identity map on V by id . Then the maps \check{R}, C and E satisfy the relations:*

$$(3.1) \quad \check{R}^2 = \text{id}^{\otimes 2} + (q - q^{-1})(\check{R} - r^{-1}E \circ C),$$

$$(3.2) \quad (\check{R} \otimes \text{id}) \circ (\text{id} \otimes \check{R}) \circ (\check{R} \otimes \text{id}) = (\text{id} \otimes \check{R}) \circ (\check{R} \otimes \text{id}) \circ (\text{id} \otimes \check{R}),$$

$$(3.3) \quad C \circ \check{R} = r^{-1}C,$$

$$(3.4) \quad C \circ E = x,$$

$$(3.5) \quad (C \otimes \text{id}) \circ (\text{id} \otimes \check{R}) = (\text{id} \otimes C) \circ (\check{R}^{-1} \otimes \text{id}),$$

$$(3.6) \quad (C \otimes \text{id}) \circ (\text{id} \otimes E) = \text{id},$$

where $r = -q^{2m+1}$ and the composition reads from left to right.

Proof. Note that the mapping which sends each T_i to β'_i and each E_i to γ'_i can be naturally extended to a right action of $B_n(-q^{2m+1}, q)$ on the tensor space $V^{\otimes n}$. Then \check{R} is invertible and $\check{R}^{-1} = \check{R} + (q^{-1} - q)(\text{id}^{\otimes 2} - E \circ C)$. The identities (3.1-3.2) now follow from Definition 2.8 and the remaining relations can be verified directly. \square

Definition 3.2. We denote by $\mathcal{T}(V)$ the full subcategory of $U_q(\mathfrak{sp}_{2m})$ -modules with objects $V^{\otimes n}$, where $n \in \mathbb{N}$ and $V^{\otimes 0} = \mathbb{Q}(q)$ by convention. The usual tensor product of $U_q(\mathfrak{sp}_{2m})$ -modules and of $U_q(\mathfrak{sp}_{2m})$ -homomorphisms is a bi-functor $\mathcal{T}(V) \times \mathcal{T}(V) \rightarrow \mathcal{T}(V)$, which will be called the tensor product of this category. We call $\mathcal{T}(V)$ the *category of tensor representations* of $U_q(\mathfrak{sp}_{2m})$.

Since $V \cong V^*$ as $U_q(\mathfrak{sp}_{2m})$ -modules, the category $\mathcal{T}(V)$ is also a pivotal strict monoidal category with the evaluation E and the coevaluation C . Under the duality decided by E and C , V and V^* are the same object in the category $\mathcal{T}(V)$. Furthermore, $\mathcal{T}(V)$ has a (non-symmetric) braiding structure given by an R -matrix of $U_q(\mathfrak{sp}_{2m})$, see [18, 25].

Let $r = -q^{2m+1}$ and $\mathbb{D}(\mathfrak{sp}_{2m}) := \mathbb{D}(r, q)$. We have the following results.

Theorem 3.3. *There exists a unique additive covariant functor $F : \mathbb{D}(\mathfrak{sp}_{2m}) \rightarrow \mathcal{T}(V)$ satisfying the following properties:*

- (i) F sends the object n to $V^{\otimes n}$ and the morphism $D : s \rightarrow t$ to $F(D) : V^{\otimes s} \rightarrow V^{\otimes t}$, where $F(D)$ is defined on the generators of tangles by

$$F \left(\begin{array}{c} | \\ | \end{array} \right) = \text{id}_V, \quad F \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \right) = \check{R},$$

$$F \left(\begin{array}{c} \cap \end{array} \right) = C, \quad F \left(\begin{array}{c} \cup \end{array} \right) = E.$$

- (ii) F is a pivotal monoidal functor, i.e. F preserves the tensor products and the dualities.

Proof. Since the linear maps \check{R}, C and E are $U_q(\mathfrak{sp}_{2m})$ -homomorphisms, by Theorem 2.7 it is clear that F preserves tensor products and dualities of objects, i.e.

$F(s \otimes t) = F(s) \otimes F(t)$ and $F(n^*) = F(n) = V^{\otimes n} = F(n)^*$. As a covariant functor, F preserves the composition of tangles, and by (ii) F preserves the tensor products and the dualities of morphisms.

Thanks to [25, Theorem 5.1] we just need to check that F is well-defined. In fact, it is enough to show that the images of the generators of tangles satisfy the relations in Theorem 2.7. This is clear from Lemma 3.1. \square

Following [20, §8.3], the functor F in Theorem 3.3 is called the *Reshetikhin-Turaev functor* (RT-functor for short as seen in the introduction). Now let us end this section with a technical lemma.

Lemma 3.4. *Let $H(s, t) := \text{Hom}_{U_q(\mathfrak{sp}_{2m})}(V^{\otimes s}, V^{\otimes t})$ for all $s, t \in \mathbb{N}$. Define the linear maps*

$$\mathbb{U}_s^t := (- \otimes I_t) \circ (I_s \otimes U_t) : \mathcal{D}(n, s+t) \longrightarrow \mathcal{D}(n+t, s),$$

$$\mathbb{A}_t^n := (I_n \otimes A_t) \circ (- \otimes I_t) : \mathcal{D}(n+t, s) \longrightarrow \mathcal{D}(n, s+t).$$

Then we have:

(i) *the $\mathbb{Q}(q)$ -linear maps*

$$F\mathbb{U}_s^t := (- \otimes \text{id}_V^{\otimes t})(\text{id}_V^{\otimes s} \otimes F(U_t)) : H(n, s+t) \longrightarrow H(n+t, s),$$

$$F\mathbb{A}_t^n := (\text{id}_V^{\otimes n} \otimes F(A_t))(- \otimes \text{id}_V^{\otimes t}) : H(n+t, s) \longrightarrow H(n, s+t)$$

are well-defined and are mutually inverses of each other;

(ii) *the functor F induces a linear map*

$$F_t^s : \mathcal{D}(s, t) \longrightarrow H(s, t) = \text{Hom}_{U_q(\mathfrak{sp}_{2m})}(V^{\otimes s}, V^{\otimes t}), \quad D \mapsto F(D),$$

and the following diagrams are commutative:

$$\begin{array}{ccc} \mathcal{D}(n+t, s) & \xrightarrow{\mathbb{A}_t^n} & \mathcal{D}(n, s+t) & & \mathcal{D}(n, s+t) & \xrightarrow{\mathbb{U}_s^t} & \mathcal{D}(n+t, s) \\ F_s^{n+t} \downarrow & & \downarrow F_{s+t}^n & & F_{s+t}^n \downarrow & & \downarrow F_s^{n+t} \\ H(n+t, s) & \xrightarrow{F\mathbb{A}_t^n} & H(n, s+t) & & H(n, s+t) & \xrightarrow{F\mathbb{U}_s^t} & H(n+t, s). \end{array}$$

Proof. First of all, we show that \mathbb{A}_t^n and \mathbb{U}_s^t are mutually inverses of each other. Let $D \in \mathcal{D}(n, s+t)$. Then we have

$$\begin{aligned} \mathbb{A}_t^n \mathbb{U}_s^t(D) &= (I_n \otimes A_t) \circ (D \otimes I_t \otimes I_t) \circ (I_s \otimes U_t \otimes I_t) \\ &= D \circ (I_s \otimes I_t \otimes A_t) \circ (I_s \otimes U_t \otimes I_t) \\ &= D \circ (I_s \otimes I_t) \\ &= D. \end{aligned}$$

Hence $\mathbb{A}_t^n \mathbb{U}_s^t$ is the identity morphism of $\mathcal{D}(n, s+t)$. Similarly one can show that $\mathbb{U}_s^t \mathbb{A}_t^n$ is the identity morphism of $\mathcal{D}(n+t, s)$ for each $n \in \mathbb{N}$. Now the Claim (i) follows from Theorem 3.3.

Since the RT-functor F preserves both the composition and the tensor product of tangles, it is obvious that $F(\mathbb{A}_t^n(D)) = F\mathbb{A}_t^n(F(D))$ for any tangle $D \in \mathcal{D}(n+t, s)$. Hence we obtain the commutativity of the first diagram. Similarly, we have the second one. \square

4. FUNDAMENTAL THEOREMS FOR QUANTIZED SYMPLECTIC GROUP

In this section we present a categorical version of the fundamental theorems of invariants for the quantized symplectic group. For each positive integer n we introduce the quantum integer

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}}, \quad \text{and} \quad [n]! := [n][n-1] \cdots [1],$$

with convention $[0] := 1$.

Definition 4.1. For $k \in \mathbb{N}$ and $1 \leq i < n$ define $Y_i(k) \in \mathcal{D}(n, n) = B_n(r, q)$ by

$$Y_i(k) := \frac{-1}{[k+1]} \left([k]T_i - q^k + \frac{q^k - q^{-k}}{1 + rq^{-2k+1}} E_i \right).$$

Proposition 4.2. *The elements satisfy the following relations:*

$$\begin{aligned} Y_i(k)Y_{i+1}(k+h)Y_i(h) &= Y_{i+1}(h)Y_i(k+h)Y_{i+1}(k), \\ Y_i(k)Y_j(h) &= Y_j(h)Y_i(k) \quad \text{for } |i-j| > 1. \end{aligned}$$

Proof. The second identity is clear. For the first relation, we can assume $i = 1$ without loss of generality. Using the relations of BMW algebra and Theorem 2.9, it can be verified via a direct but complicated calculation in the 15-dimensional algebra $B_3(r, q)$ \square

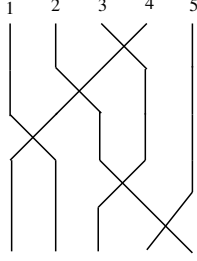
Let \mathfrak{S}_n be the symmetric group on n words with the standard Coxeter generators s_1, s_2, \dots, s_{n-1} . To a reduced expression $s_{i_1}s_{i_2} \cdots s_{i_l}$, we associate an element $Y(s_{i_1}s_{i_2} \cdots s_{i_l}) \in B_n(r, q)$ of the form:

$$Y(s_{i_1}s_{i_2} \cdots s_{i_l}) := Y_{i_1}(k_1)Y_{i_2}(k_2) \cdots Y_{i_l}(k_l),$$

where the integers k_1, \dots, k_l are determined as follows. Firstly we identify each element of \mathfrak{S}_n with an n -string diagram as usual and label the vertices of an n -string diagram in the top and bottom row by the indices $1, 2, \dots, n$ from left to right. Under this identification, we draw the n -string diagram of the reduced expression $s_{i_1}s_{i_2} \cdots s_{i_l}$ (from up to down) such that each factor s_{i_j} , $1 \leq j \leq l$, corresponds to an individual n -string diagram and to a crossing of two adjacent strings. Since the word is reduced, any two strings can cross at most once. We label the crossings by their corresponding position in the reduced expression, and label each string with the number of its starting vertex on the most top of the diagram. Then for the j -th crossing, if the numbers of the two strings are a_j and b_j with $a_j < b_j$, we set $k_j = b_j - a_j$.

The element $Y(s_{i_1}s_{i_2} \cdots s_{i_l})$ will be called the *Yang-Baxter element* associated to the reduced expression $s_{i_1}s_{i_2} \cdots s_{i_l}$.

Example 4.3. The 5-string diagram corresponding to the reduced expression $s_3s_2s_1s_3s_4$ is



and hence the Yang-Baxter element associated to it is

$$Y(s_3 s_2 s_1 s_3 s_4) = Y_3(1)Y_2(2)Y_1(3)Y_3(1)Y_4(3).$$

Lemma 4.4. *If two reduced expressions represent the same permutation, then the associated Yang-Baxter elements in $B_n(r, q)$ are the same.*

Proof. It follows from the Matsumoto's theorem [23] that two reduced expressions represent the same permutation if and only if they are related by a finite sequence of braid moves of the form $s_i s_{i+1} s_i \rightarrow s_{i+1} s_i s_{i+1}$, $s_{i+1} s_i s_{i+1} \rightarrow s_i s_{i+1} s_i$, and $s_i s_j = s_j s_i$ for $|i-j| > 1$. Let us consider the braid move $s_i s_{i+1} s_i \rightarrow s_{i+1} s_i s_{i+1}$ and the other two types of braid moves can be proved similarly. Let $w = w_1 s_i s_{i+1} s_i w_2$ be a reduced expression with subword $s_i s_{i+1} s_i$ for some $1 \leq i < n$. Assume $(i)w_1^{-1} = a$, $(i+1)w_1^{-1} = b$ and $(i+2)w_1^{-1} = c$. Then $a < b < c$ since w and w_1 are reduced. Hence the Yang-Baxter element associated to $w_1 s_i s_{i+1} s_i w_2$ contains subword $Y_i(b-a)Y_{i+1}(c-a)Y_i(c-b)$ corresponding to the subword $s_i s_{i+1} s_i$. On the other hand, the Yang-Baxter element associated to $w_1 s_{i+1} s_i s_{i+1} w_2$ contains subword $Y_{i+1}(c-b)Y_i(c-a)Y_{i+1}(b-a)$ corresponding to the subword $s_{i+1} s_i s_{i+1}$. Now Proposition 4.2 completes the proof of this lemma. \square

Lemma 4.4 tells us that the Yang-Baxter element can be defined for each permutation, independent of its explicit reduced expressions. Let $Y_n \in B_n(r, q)$ be the Yang-Baxter element associated to the longest length permutation in \mathfrak{S}_n .

We will show that the element Y_n provides the one-dimensional sign representation ρ of the BMW algebra $B_n(r, q)$ [13, Section 3], which is defined on generators by

$$\rho(T_i) = -q^{-1}, \quad \rho(E_i) = 0.$$

Lemma 4.5. *The Yang-Baxter element Y_n satisfies*

$$bY_n = \rho(b)Y_n = Y_n b,$$

for any $b \in B_n(r, q)$. Furthermore Y_n is a central idempotent.

Proof. For each $1 \leq j < n$ there exists a reduced expression for the longest length permutation such that the reduced expression begins with s_j . Then Y_n has left factor $Y_j(1)$. We have

$$\begin{aligned} -[2]T_j Y_j(1) &= T_j \left(T_j - q + \frac{q - q^{-1}}{1 + r q^{-1}} E_j \right) \\ &= 1 + (q - q^{-1})(T_j - E_j T_j) - q T_j + \frac{q - q^{-1}}{1 + r q^{-1}} T_j E_j \\ &= 1 - q^{-1} T_j + (q - q^{-1}) \frac{-q^{-1}}{1 + r q^{-1}} E_j \\ &= (-q)^{-1} (-[2]Y_j(1)). \end{aligned}$$

A similar calculation shows $E_j Y_j(1) = 0$, and hence $bY_n = \rho(b)Y_n$. The equation $\rho(b)Y_n = Y_n b$ for any $b \in B_n(r, q)$ can be proved similarly. Thus $Y_j(k)Y_n = Y_n$ for any $k \in \mathbb{N}$ and this yields Y_n is a central idempotent. \square

Proposition 4.6. *Let $r = -q^{2m+1}$. We have $F(Y_{m+1}) = 0$.*

Proof. It follows from Theorem 3.3 and Lemma 4.5 that $F(Y_{m+1})$ is an idempotent in $\text{End}_{U_q(\mathfrak{sp}_{2m})}(V^{\otimes(m+1)})$. Since the RT-functor F is pivotal, it preserves the traces of morphisms (see [10, P.160]). The rank of an idempotent is equal to its trace and hence it is sufficient to show that the trace of Y_{m+1} equals zero.

The trace tr_n of an endomorphism $D \in \mathcal{D}(n, n)$ for any $n \in \mathbb{N}$ in the pivotal category $\mathbb{D}(\mathfrak{sp}_{2m})$ is defined by the evolution and the coevolution as follows

$$\text{tr}_n(D) := A_{2n} \circ (D \otimes I_n) \circ U_{2n}.$$

The following properties can be verified directly

$$\begin{aligned} \text{tr}_{n+1}(D \otimes I) &= x \text{tr}_n(D), \\ \text{tr}_{n+1}((D \otimes I) \circ T_n \circ (D' \otimes I)) &= r \text{tr}_n(D \circ D'), \\ \text{tr}_{n+1}((D \otimes I) \circ E_n \circ (D' \otimes I)) &= \text{tr}_n(D \circ D'), \end{aligned}$$

for any tangles $D, D' \in \mathcal{D}(n, n)$, where T_n, E_n are identified with tangles in $\mathcal{D}(n+1, n+1)$ by Theorem 2.9.

A particular choice of the reduced expression for the longest length permutation gives

$$Y_{m+1} = Y_1(1)Y_2(2) \cdots Y_m(m)Y_m.$$

Hence, we have

$$\begin{aligned} &\text{tr}_{m+1}(Y_{m+1}) \\ &= \frac{-1}{[m+1]} \text{tr}_{m+1} \left(Y_1(1)Y_2(2) \cdots Y_{m-1}(m-1) \left([m]T_m - q^m + \frac{q^m - q^{-m}}{1 + rq^{-2m+1}} E_m \right) Y_m \right) \\ &= \frac{-1}{[m+1]} \left([m]r - q^m x + \frac{q^m - q^{-m}}{1 + rq^{-2m+1}} \right) \text{tr}_m(Y_1(1)Y_2(2) \cdots Y_{m-1}(m-1)Y_m). \end{aligned}$$

When $r = -q^{2m+1}$, $x = 1 - [2m+1]$ and then $\text{tr}_{m+1}(Y_{m+1}) = 0$. \square

Definition 4.7. Let $\langle Y_{m+1} \rangle$ be the subspace of $\bigoplus_{s,t} \mathcal{D}(s, t)$ spanned by the morphisms in $\mathbb{D}(\mathfrak{sp}_{2m})$ obtained from Y_{m+1} by composition and tensor product. Set $\langle Y_{m+1} \rangle_t^s := \langle Y_{m+1} \rangle \cap \mathcal{D}(s, t)$.

The FFT and SFT of invariants for the quantized symplectic group can be respectively interpreted as Parts (i) and (ii) of the following theorem, which is the quantum version of [20, Theorem 4.8] in the symplectic case.

Theorem 4.8. *With notations as above and $r = -q^{2m+1}$, we have*

- (i) *the RT-functor $F : \mathbb{D}(\mathfrak{sp}_{2m}) \rightarrow \mathcal{T}(V)$ is full, i.e. F is surjective on Hom spaces;*
- (ii) *the map $F_t^s : \mathcal{D}(s, t) \rightarrow H(s, t)$ is injective if $s + t \leq 2m$, and $\text{Ker}(F_t^s) = \langle Y_{m+1} \rangle_t^s$ if $s + t > 2m$.*

Proof. It is clear that the theorem is true when $s + t$ is odd. Now assume that $s + t = 2n$ for some $n \in \mathbb{N}$. It follows from Lemma 3.4 that we have a canonical isomorphism $\mathcal{D}(s, t) \cong \mathcal{D}(n, n)$, and the study of F_t^s is equivalent to that of F_n^n .

(i). The surjectivity of F_n^n follows from the Schur-Weyl duality of type C , see [3, 10.2], or [11, 12] for example.

(ii). When $2n \leq 2m$, i.e. $n \leq m$, the injectivity of F_n^n also follows from the Schur-Weyl duality of type C , see [12, Theorem 5.5] for example. When $m < n$, we set $Y_{m+1}^{(n)} := Y_{m+1} \otimes I_{n-m-1} \in B_n(-q^{2m+1}, q)$. Hence by Proposition 4.6 it is sufficient to show

$$\dim B_n(-q^{2m+1}, q)/(Y_{m+1}^{(n)}) \leq \dim H(n, n) = \dim \text{End}_{U_q(\mathfrak{sp}_{2m})}(V^{\otimes n}),$$

where $(Y_{m+1}^{(n)})$ means the ideal of $B_n(-q^{2m+1}, q)$ generated by $Y_{m+1}^{(n)}$. In fact

$$\begin{aligned} \dim(Y_{m+1}^{(n)}) &\geq \dim_{\mathbb{Q}}(Y_{m+1}^{(n)} \downarrow_{q=1}) \\ &= \dim B_n(-q^{2m+1}, q) - \dim \text{End}_{U_q(\mathfrak{sp}_{2m})}(V^{\otimes n}), \end{aligned}$$

where the first inequality follows from the specialization $\lim_{q \rightarrow 1}$, see the Part (i) of [20, Theorem 8.2]. This completes the proof of the theorem. \square

Remark 4.9. Theorem 4.8 with $s = 2n, t = 0$ yields the linear formulation of fundamental theorems for the quantized symplectic group, while the endomorphism algebra formulation arises from the case $s = t = n$. The equivalence of these two versions is a consequence of Lemma 3.4.

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