Electronic Journal of Differential Equations, Vol. 2020 (2020), No. 90, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

FINITE CYCLICITY OF THE CONTACT POINT IN SLOW-FAST INTEGRABLE SYSTEMS OF DARBOUX TYPE

RENATO HUZAK

Abstract. Using singular perturbation theory and family blow-up we prove that nilpotent contact points in deformations of slow-fast Darboux integrable systems have finite cyclicity. The deformations are smooth or analytic depending on the region in the parameter space. This article is a natural continuation of [1, 3], where one studies limit cycles in polynomial deformations of slow-fast Darboux integrable systems, around the "integrable" direction in the parameter space. We extend the existing finite cyclicity result of the contact point to analytic deformations, and under some assumptions we prove that the contact point has finite cyclicity around the "slow-fast" direction in the parameter space.

1. Introduction

A typical example of *slow-fast Darboux integrable systems* studied in [1, 3] is given by

$$X_\epsilon : \begin{cases} \dot{x} = y - x^2 - \epsilon + \epsilon y \\ \dot{y} = -2\epsilon x + 2\epsilon xy, \end{cases} \tag{1.1}$$

where $\epsilon \geq 0$, $\epsilon \sim 0$, is the singular perturbation parameter. System X_ϵ has the first integral (of Darboux type)

$$H(x, y) = (y - x^2)^\epsilon (1 - y).$$

When $\epsilon > 0$, X_ϵ has a family of periodic orbits $\mathcal{O}_\epsilon$ bounded by $\{y - x^2 = 0\}$ and $\{1 - y = 0\}$ (see Figure 1). The *fast subsystem* X_0 of X_ϵ, has the curve of singularities $\{y - x^2 = 0\}$, often called the *critical* or *slow curve*, and (fast) regular horizontal orbits (see Figure 2).

The goal in [1, 3] was to prove ϵ-uniform finiteness of the number of limit cycles bifurcating from the compact set $\bar{\mathcal{O}} := \overline{\mathcal{O}_\epsilon}$ in a polynomial deformation $X_{\epsilon,\delta}$ of X_ϵ, with $\delta = (\delta_1, \dots, \delta_m) \in \mathbb{R}^m$, $m \in \mathbb{N} \setminus \{0\}$, and $\delta \sim 0$. (Note that $\bar{\mathcal{O}}$ is the ϵ-independent region bounded by $\{y - x^2 = 0\}$ and $\{1 - y = 0\}$, including the boundary.) More precisely, we consider a vector field $X_{\epsilon,\delta} := X_\epsilon + Q_1(x, y, \delta)\frac{\partial}{\partial x} +$

2010 *Mathematics Subject Classification.* 34C26, 34E15, 34E17.
Key words and phrases. Blow-up; cyclicity; Darboux systems; singular perturbation theory; slow-fast systems.

Submitted December 1, 2019. Published September 6, 2020.

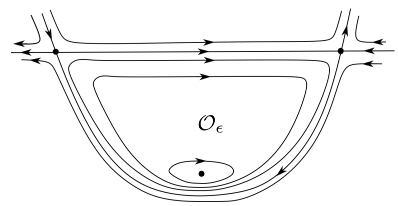

FIGURE 1. Dynamics of X_ϵ, with $\epsilon > 0$.

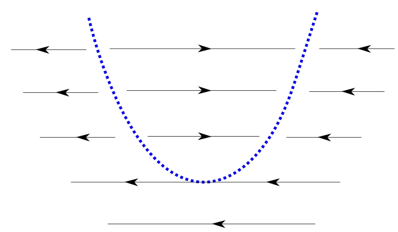

FIGURE 2. Dynamics of X_0.

$Q_2(x, y, \delta)\frac{\partial}{\partial y}$ where Q_i is an analytic δ-family of real polynomials in (x, y) and $Q_i(x, y, 0) \equiv 0$, $i = 1, 2$. Let $\epsilon > 0$ be small and fixed. Let $C(\epsilon) := \text{Cycl}(X_{\epsilon,\delta}, \bar{\mathcal{O}})$ be the maximal number of limit cycles of the system $X_{\epsilon,\delta}$, bifurcating from $\bar{\mathcal{O}}$, for $\delta \sim 0$. *Following* [1, Theorem 2.1], *the cyclicity* $C(\epsilon)$ *is finite and uniformly bounded in* $\epsilon > 0$, *under the assumption that the parameter* (ϵ, δ) *is kept in a narrow region around the* ϵ*-axis, i.e. around the integrable direction (see Figure 3).*

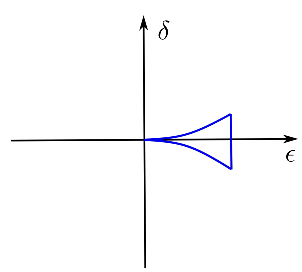

FIGURE 3. Integrable direction in the parameter space (ϵ, δ) studied in [1, 3].

Let us recall that article [1] is a natural continuation of [2, 3] where the ϵ-uniform finiteness property has been obtained in the integrable direction (see Figure 3) for any compact set K contained in $\text{Int}\bar{\mathcal{O}}$ by studying zeros of pseudo-Abelian integrals. Thus, the cyclicity of the polycycle, i.e. the boundary of $\bar{\mathcal{O}}$, has not been studied in [3]. In [1], Dulac maps and a technique based on the so-called "Petrov trick" (see also [14]) have been used to obtain the ϵ-uniform finiteness property for $\bar{\mathcal{O}}$. One of the reasons why the polynomial deformations of X_ϵ have been studied in [1, 3] is Hilbert's 16th problem (see [13, 21]).

The main purpose of this article is to initiate the study of limit cycles of $X_{\epsilon,\delta}$, bifurcating from the compact set $\bar{\mathcal{O}}$ (or any other compact set in the phase space), *around the δ-axis in the (ϵ,δ)-space.* Our focus is on the cyclicity of the *(generic) nilpotent contact point* $(0,0)\in\bar{\mathcal{O}}$, i.e. the nilpotent singularity of X_0 at which the critical curve $\{y-x^2=0\}$ of X_0 has a (quadratic) contact with fast orbits (see Figure 2). We use geometric singular perturbation theory and the family blow-up. All singularities of the critical curve, located away from the origin, are normally hyperbolic (attracting when $x>0$ and repelling when $x<0$). The contact point is called *turning point* if it allows the passage from the attracting part of the critical curve to the repelling part of the critical curve in $X_{\epsilon,\delta}$, with $(\epsilon,\delta)\neq(0,0)$. We can have diverse (nilpotent) contact points, depending on the region in the (ϵ,δ)-space: *jump points* (see [11, 20]), *slow-fast saddle points*, *slow-fast Hopf points* (sometimes called *generic turning points*)(see [11, 12, 16, 19]), *slow-fast Bogdanov-Takens points* (see [8]), *slow-fast codimension 3 saddle or elliptic points* (see [9, 10, 15, 18]), *slow-fast codimension 4 saddle-node points* (see [17]), etc.

To determine the type of our nilpotent contact point, we first blow up the origin in the parameter space (ϵ,δ) by using the following homogeneous rescaling:

$$(\epsilon,\delta)=(\bar\epsilon E,\bar\epsilon D),\quad \bar\epsilon\geq 0,\quad \bar\epsilon\sim 0,\quad (E,D)\in\mathbb{S}^m, \tag{1.2}$$

where $D=(D_1,\dots,D_m)\in\mathbb{R}^m$. The advantage of using the blow-up (1.2) is that we now have only one small parameter $\bar\epsilon$ which is a singular perturbation parameter ($(E,D)\neq(0,0)$ because $(E,D)\in\mathbb{S}^m$). Note that if we vary $\bar\epsilon\geq 0$ and (E,D), kept on the unit sphere, then we cover a complete (small) neighborhood of the origin in the (ϵ,δ)-space. As usual, we use different charts of the sphere:

(1) (*"Integrable" direction $E=\pm 1$*) We have $(\epsilon,\delta)=(\pm\bar\epsilon,\bar\epsilon D)$ where $\bar\epsilon\geq 0$, $\bar\epsilon\sim 0$ and $D\sim 0\in\mathbb{R}^m$. See Figure 4(a). This region is covered by [1, 3] working with polynomial deformations $X_{\epsilon,\delta}$ of X_ϵ as explained above. When $E=1$ (resp. $E=-1$) we deal with a slow-fast Hopf point (resp. a slow-fast saddle point). In the slow-fast Hopf region we prove ϵ-uniform finiteness of the number of limit cycles Hausdorff close to the contact point $(x,y)=(0,0)$ under *analytic* deformations of X_ϵ (Q_1 and Q_2 are analytic). In the slow-fast saddle region the compact set $\bar{\mathcal{O}}$ produces no limit cycles under *smooth* deformations (Q_1 and Q_2 are smooth). For more details see Theorem 2.1.

(2) (*"Slow-fast" direction $D\in\mathbb{S}^{m-1}$*) We have $(\epsilon,\delta)=(\bar\epsilon E,\bar\epsilon D)$ where $\bar\epsilon\geq 0$, $\bar\epsilon\sim 0$, $D\in\mathbb{S}^{m-1}$ and E is kept in a large compact set in $\mathbb{R}$. See Figure 4(b). This region is covered by our paper. As we will see in Sections 2 and 3, the type of the nilpotent contact point is closely related to the order of vanishing of the *slow dynamics* of $X_{\epsilon,\delta}$ (defined along the slow curve $\{y-x^2=0\}$) at the contact point. The slow dynamics is given by

$$x'=-E+Ex^2+\frac{\langle D,\tilde Q(x,x^2,0)\rangle}{2x},\quad x\neq 0,$$

where we write $Q_2(x,y,\delta)=\langle\delta,\tilde Q(x,y,\delta)\rangle$ ($\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^m$). For more details about the definition of the slow dynamics see Section 2.1. Using $\langle D,\tilde Q(x,x^2,0)\rangle=\alpha_0+\alpha_1x+\alpha_2x^2+\alpha_3x^3+\alpha_4x^4+O(x^5)$, the slow dynamics can be written as

$$x'=\frac{\alpha_0}{2x}+(\frac{\alpha_1}{2}-E)+\frac{\alpha_2}{2}x+(\frac{\alpha_3}{2}+E)x^2+\frac{\alpha_4}{2}x^3+O(x^4),\quad x\neq 0.$$

When $\alpha_0 \neq 0$ for some $D = D^0 \in \mathbb{S}^{m-1}$, we deal with a jump contact point and any compact set in the phase space (x, y) produces no limit cycles under *smooth* deformations, for each $D \sim D^0$ and E kept in a large compact set. See Theorem 2.2.

When $\alpha_0 = 0$ for some $D = D^0 \in \mathbb{S}^{m-1}$, then the contact point (resp. any compact set in the phase space (x, y)) produces a finite number of limit cycles under *analytic* deformations (resp. no limit cycles under *smooth* deformations) uniformly in (E, D), with $D \sim D^0$ and $\frac{\alpha_1}{2} - E < 0$ (resp. $\frac{\alpha_1}{2} - E > 0$). Like in the integrable direction, we deal with a slow-fast Hopf point or a slow-fast saddle point. For more details see Theorem 2.4.

When $\alpha_0 = 0$, $\frac{\alpha_1}{2} - E = 0$ and $\alpha_2 \neq 0$, for some $(E, D) = (E^0, D^0) \in \mathbb{R} \times \mathbb{S}^{m-1}$, then we deal with a slow-fast Bogdanov-Takens point and any compact set in the phase space (x, y) produces at most 1 limit cycle under *smooth* perturbations, with $(E, D) \sim (E^0, D^0)$. See Theorem 2.6.

When $\alpha_0 = \frac{\alpha_1}{2} - E = \alpha_2 = 0$, $\frac{\alpha_3}{2} + E > 0$ (resp. $\frac{\alpha_3}{2} + E < 0$) and $\alpha_4 \neq 0$ for some $(E, D) = (E^0, D^0) \in \mathbb{R} \times \mathbb{S}^{m-1}$, then any compact set in the phase space (x, y) (resp. the contact point) produces at most 2 limit cycles under *smooth* perturbations, for $(E, D) \sim (E^0, D^0)$. We deal with a slow-fast codimension 3 saddle (resp. elliptic) point. See Theorem 2.8.

When $\alpha_0 = \frac{\alpha_1}{2} - E = \alpha_2 = \frac{\alpha_3}{2} + E = 0$ and $\alpha_4 \neq 0$ for some $(E, D) = (E^0, D^0) \in \mathbb{R} \times \mathbb{S}^{m-1}$, then we deal with a slow-fast codimension 4 saddle-node point and any compact set in the phase space (x, y) produces at most 2 limit cycle under *smooth* perturbations, with $(E, D) \sim (E^0, D^0)$. See Theorem 2.9.

The cases "$\alpha_0 = \frac{\alpha_1}{2} - E = \alpha_2 = \alpha_4 = 0$, $\frac{\alpha_3}{2} + E \neq 0$" and "$\alpha_0 = \frac{\alpha_1}{2} - E = \alpha_2 = \frac{\alpha_3}{2} + E = \alpha_4 = 0$" are topics of further study.

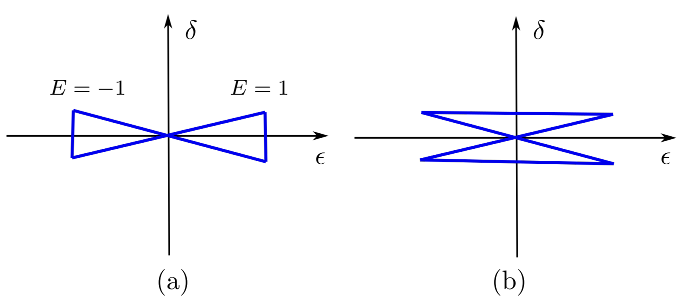

FIGURE 4. (a) The "integrable" direction in the parameter space (ϵ, δ). (b) The "slow-fast" direction in the (ϵ, δ)-space.

Besides the small-amplitude limit cycles of $X_{\epsilon,\delta}$ studied in this paper, we can also have limit cycles in $X_{\epsilon,\delta}$ bifurcating from so-called detectable canard limit periodic sets consisting of a fast orbit of $X_{0,0}$ and the part of the critical curve of $X_{0,0}$ between the α-limit set and the ω-limit set of the fast orbit (see Figure 2). In the "slow-fast" direction, these canard limit cycles are possible only if the slow dynamics points from the attracting part of the critical curve to the repelling part of the critical curve. This happens, for instance, in the slow-fast Hopf case and the slow-fast codimension 3 elliptic case. More precisely, we have to deal with detectable canard limit cycles when the order of the *slow dynamics* at the contact point is $0, 2, 4, 6, \dots$, with a negative coefficient. In the slow-fast Hopf case, i.e. when the order of the slow dynamics is 0, we can study zeros of the so-called slow divergence integral if the slow dynamics is regular (see [4]). If the slow dynamics has isolated singularities (away from the contact point), we can use the results of [5]. When the order of the slow dynamics is ≥ 2, we study zeros of the derivative of the slow divergence integral if the slow dynamics is regular (see [6, 7]). If the slow dynamics has isolated zeros, away from the contact point, we can combine [6, 7] with [5]. The same can be done in the "integrable" direction (the slow-fast Hopf case). Since there are a lot of different possibilities, we prefer to deal with the detectable canard limit cycles in a separate paper. This is a topic of further study.

In the "slow-fast" direction, compact regions in which we have to study detectable canard cycles can be larger than the compact region $\bar{\mathcal{O}}$ in the "integrable" direction. This happens, for example, in the slow-fast Hopf case with regular slow dynamics. In the "integrable" direction, the slow dynamics has two hyperbolic saddles $x = \pm 1$ for $(E, D) = (1, 0)$. See also Figure 1.

In Section 2 we state our main results. We prove the results in Section 3. Although in this paper we are mostly interested in the slow-fast Darboux system (1.1), our methods can be used in a more general framework of slow-fast integrable systems of Darboux type introduced in [3], under assumption that our contact point is of generic nilpotent type. See Section 4.

2. Slow dynamics and statement of results

In Section 2.1 we define the notion of slow dynamics of (2.1) along the critical curve $\{y - x^2 = 0\}$. We state our main results in Section 2.2.

2.1. **The fast subsystem and slow dynamics.**

We consider an $(\epsilon, \delta, \lambda)$-family of 2-dimensional vector fields

$$X_{\epsilon,\delta,\lambda} : \begin{cases} \dot{x} = y - x^2 - \epsilon + \epsilon y + Q_1(x, y, \delta, \lambda) \\ \dot{y} = -2\epsilon x + 2\epsilon xy + Q_2(x, y, \delta, \lambda) \end{cases} \tag{2.1}$$

where $\epsilon \in (\mathbb{R}, 0)$, $\delta \in (\mathbb{R}^m, 0)$, $\lambda \in (\mathbb{R}^n, \lambda_0)$ $(m, n \in \mathbb{N} \setminus \{0\})$ and Q_1 and Q_2 are smooth functions with $Q_1(x, y, 0, \lambda) = Q_2(x, y, 0, \lambda) \equiv 0$. (For the sake of generality, Q_1 and Q_2 may depend on extra parameter λ.) *We focus on system $X_{\bar{\epsilon}E,\bar{\epsilon}D,\lambda}$ where $\bar{\epsilon} \geq 0$, $\bar{\epsilon} \sim 0$ and $(E, D) \in \mathbb{S}^m$* (see (1.2)).

The dynamics of the fast subsystem $X_{0,0,\lambda}$ is given in Figure 2, and the dynamics of $X_{\bar{\epsilon}E,\bar{\epsilon}D,\lambda}$, with $\bar{\epsilon} \sim 0$ and $\bar{\epsilon} > 0$, away from the critical curve, is governed by the dynamics of $X_{0,0,\lambda}$. The dynamics of $X_{\bar{\epsilon}E,\bar{\epsilon}D,\lambda}$ near the critical curve, away from the contact point $(x, y) = (0, 0)$, can be studied using the slow dynamics. Let us define the slow dynamics. Center manifolds at the normally hyperbolic (or semi-hyperbolic) singularity $(x, y, \bar{\epsilon}) = (x, x^2, 0)$, $x \neq 0$, of $X_{\bar{\epsilon}E,\bar{\epsilon}D,\lambda} + 0\frac{\partial}{\partial \bar{\epsilon}}$ can be written as

$$y = x^2 + \bar{\epsilon}\Big(- \langle D, \bar{Q}(x, x^2, 0, \lambda)\rangle + \frac{\langle D, \tilde{Q}(x, x^2, 0, \lambda)\rangle}{2x} + O(\bar{\epsilon})\Big),$$

with $Q_1(x, y, \delta, \lambda) = \langle \delta, \bar{Q}(x, y, \delta, \lambda)\rangle$ and $Q_2(x, y, \delta, \lambda) = \langle \delta, \tilde{Q}(x, y, \delta, \lambda)\rangle$. The dynamics inside these center manifolds can be obtained from the first equation in

(2.1):

$$\dot{x} = \bar{\epsilon}\Big(-E + Ex^2 + \frac{\langle D, \tilde{Q}(x, x^2, 0, \lambda)\rangle}{2x} + O(\bar{\epsilon})\Big).$$

If we divide this equation by $\bar{\epsilon}$ and if we let $\bar{\epsilon} \to 0$, we find the slow dynamics

$$x' = f(x, E, D, \lambda) := -E + Ex^2 + \frac{\langle D, \tilde{Q}(x, x^2, 0, \lambda)\rangle}{2x}, \quad x \neq 0. \tag{2.2}$$

Using (2.2) we can write

$$f(x, E, D, \lambda) = \sum_{k=-1}^{3} \beta_k x^k + O(x^4),$$

where

$$\begin{gathered}
\beta_{-1}(D, \lambda) = \frac{\langle D, \tilde{Q}(0, 0, 0, \lambda)\rangle}{2};\\
\beta_0(E, D, \lambda) = \frac{\langle D, \frac{\partial \tilde{Q}}{\partial x}(0, 0, 0, \lambda)\rangle}{2} - E;\\
\beta_1(D, \lambda) = \frac{\langle D, \big(\frac{\partial^2 \tilde{Q}}{\partial x^2} + 2\frac{\partial \tilde{Q}}{\partial y}\big)(0, 0, 0, \lambda)\rangle}{4};\\
\beta_2(E, D, \lambda) = \frac{\langle D, \big(\frac{\partial^3 \tilde{Q}}{\partial x^3} + 6\frac{\partial^2 \tilde{Q}}{\partial x \partial y}\big)(0, 0, 0, \lambda)\rangle}{12} + E;\\
\beta_3(D, \lambda) = \frac{\langle D, \big(\frac{\partial^4 \tilde{Q}}{\partial x^4} + 12\frac{\partial^3 \tilde{Q}}{\partial x^2 \partial y} + 12\frac{\partial^2 \tilde{Q}}{\partial y^2}\big)(0, 0, 0, \lambda)\rangle}{48}.
\end{gathered}$$

Clearly, the order of vanishing of the slow dynamics f, with $\lambda = \lambda_0$, at $x = 0$ depends on $(E, D) \in \mathbb{S}^m$, the function $\tilde{Q}$ and its partial derivatives at $(x, y, \delta, \lambda) = (0, 0, 0, \lambda_0)$. As explained in Section 1, limit cycles of (2.1) cannot be studied uniformly. We have to use different techniques depending on the order of vanishing of f at $x = 0$.

2.2. **Statement of results.** In the parameter space (ϵ, δ), we distinguish between the integrable direction $\{E = \pm 1\}$, with $D \sim 0 \in \mathbb{R}^m$, and the slow-fast direction $\{D \in \mathbb{S}^{m-1}\}$, with E kept in a large compact set in $\mathbb{R}$.

2.2.1. *Integrable direction.* In this section we study limit cycles of system $X_{\pm\bar{\epsilon}, \bar{\epsilon}D, \lambda}$ where $(\bar{\epsilon}, D, \lambda) \sim (0, 0, \lambda_0) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n$ and $\bar{\epsilon} \geq 0$. We have $\beta_{-1} = 0$ and $\beta_0 \neq 0$ for $(E, D, \lambda) = (\pm 1, 0, \lambda_0)$. We obtain the following result.

Theorem 2.1. *Let the family $X_{\pm\bar{\epsilon}, \bar{\epsilon}D, \lambda}$ be as defined above. The following statements are true:*

1. *(Slow-fast Hopf case $E = 1$) Suppose that Q_1 and Q_2 are analytic. Then there exists $\bar{\epsilon}_0 > 0$, a neighborhood $\mathcal{W}$ of $(0, \lambda_0)$ in the (D, λ)-space, a neighborhood $\mathcal{V}$ of $(x, y) = (0, 0)$ and $N \in \mathbb{N}$ such that system $X_{\bar{\epsilon}, \bar{\epsilon}D, \lambda}$ has at most N limit cycles in $\mathcal{V}$, for each value $(\bar{\epsilon}, D, \lambda) \in [0, \bar{\epsilon}_0] \times \mathcal{W}$.*
2. *(Slow-fast saddle case $E = -1$) Let Q_1 and Q_2 be smooth and let K be an arbitrary compact set in the phase space (x, y). Then there exists $\bar{\epsilon}_0 > 0$ and a neighborhood $\mathcal{W}$ of $(0, \lambda_0)$ in the (D, λ)-space such that system $X_{-\bar{\epsilon}, \bar{\epsilon}D, \lambda}$ has no limit cycles in K, for each value $(\bar{\epsilon}, D, \lambda) \in [0, \bar{\epsilon}_0] \times \mathcal{W}$.*

To prove Theorem 2.1 in the region $E = 1$, first we bring the analytic family $X_{\bar\epsilon,\bar\epsilon D,\lambda}$ near $(x, y) = (0, 0)$ to a normal form of Liénard type using [16]. Then, we use a finite cyclicity result for analytic slow-fast Hopf points of Liénard type obtained in [12]. See Section 3.1.

2.2.2. *Slow-fast direction.* In this section we deal with limit cycles of system $X_{\bar\epsilon E,\bar\epsilon D,\lambda}$ where $\bar\epsilon \sim 0$, $\bar\epsilon > 0$, E is kept in a large compact set in $\mathbb{R}$, $D \in \mathbb{S}^{m-1}$ and $\lambda \sim \lambda_0$. We start with the simplest case when $\beta_{-1} \neq 0$, i.e. with the jump case.

Theorem 2.2 (Jump case)**.** *Let Q_1 and Q_2 be smooth and $\beta_{-1}(D^0, \lambda_0) \neq 0$ for some $D^0 \in \mathbb{S}^{m-1}$. For each compact set K in the phase space (x, y) and for each compact set C in the parameter space E there exists $\bar\epsilon_0 > 0$ and a neighborhood $\mathcal{W}$ of (D^0, λ_0) in the (D, λ)-space such that $X_{\bar\epsilon E,\bar\epsilon D,\lambda}$ has no limit cycles in K for each value $(\bar\epsilon, E, D, \lambda) \in [0, \bar\epsilon_0] \times C \times \mathcal{W}$.*

The above theorem will be proved in Section 3.2.2. When $m = 1$, we have $D = \pm 1$ and the condition $\beta_{-1} \neq 0$ is equivalent to $\tilde{Q}(0, 0, 0, \lambda_0) \neq 0$. Now, as a direct consequence of Theorem 2.1 and Theorem 2.2, we obtain the following finite cyclicity result of the contact point in analytic families (2.1), in a *full neighborhood of* $(\epsilon, \delta) = (0, 0)$.

Theorem 2.3. *Suppose that Q_1 and Q_2 are analytic, $m = 1$ and $\frac{\partial Q_2}{\partial \delta}(0, 0, 0, \lambda_0) \neq 0$. There exists a neighborhood $\mathcal{W}$ of $(\epsilon, \delta, \lambda) = (0, 0, \lambda_0)$, a neighborhood $\mathcal{V}$ of $(x, y) = (0, 0)$ and $N \in \mathbb{N}$ such that system $X_{\epsilon,\delta,\lambda}$ has at most N limit cycles in $\mathcal{V}$, for each $(\epsilon, \delta, \lambda) \in \mathcal{W}$.*

As in Section 2.2.1, in the slow-fast direction we can also encounter slow-fast Hopf and saddle cases ($\beta_{-1} = 0$ and $\beta_0 \neq 0$).

Theorem 2.4. *Suppose that $\beta_{-1}(D^0, \lambda_0) = 0$ and $\beta_0(E^0, D^0, \lambda_0) \neq 0$ for some $(E^0, D^0) \in \mathbb{R} \times \mathbb{S}^{m-1}$. The following statements are true:*

(1) *(Slow-fast Hopf case $\beta_0 < 0$) Suppose that Q_1 and Q_2 are analytic. If $\beta_0(E^0, D^0, \lambda_0) < 0$, then there exists $\bar\epsilon_0 > 0$, a neighborhood $\mathcal{W}$ of (E^0, D^0, λ_0) in the (E, D, λ)-space, a neighborhood $\mathcal{V}$ of $(x, y) = (0, 0)$ and $N \in \mathbb{N}$ such that system $X_{\bar\epsilon E,\bar\epsilon D,\lambda}$ has at most N limit cycles in $\mathcal{V}$, for each value $(\bar\epsilon, E, D, \lambda) \in [0, \bar\epsilon_0] \times \mathcal{W}$.*
(2) *(Slow-fast saddle case $\beta_0 > 0$) Let Q_1 and Q_2 be smooth functions, let $\beta_0(E^0, D^0, \lambda_0) > 0$ and let K be an arbitrary compact set in the phase space (x, y). Then there exists $\bar\epsilon_0 > 0$ and a neighborhood $\mathcal{W}$ of (E^0, D^0, λ_0) in the (E, D, λ)-space such that system $X_{\bar\epsilon E,\bar\epsilon D,\lambda}$ has no limit cycles in K, for each value $(\bar\epsilon, E, D, \lambda) \in [0, \bar\epsilon_0] \times \mathcal{W}$.*

The proof of the above theorem is similar to the proof of Theorem 2.1 (see Section 3.1). When $m = 1$, Theorem 2.1, Theorem 2.4 and the definition of β_{-1}, β_0 imply

Theorem 2.5. *Let Q_1 and Q_2 be analytic functions, $m = 1$, $\frac{\partial Q_2}{\partial \delta}(0, 0, 0, \lambda_0) = 0$ and let*

$$S_\pm(\rho) := \Big\{(\bar\epsilon E, \pm\bar\epsilon) \mid \bar\epsilon \in]0, 1[, E \in B\Big(\pm \frac{\frac{\partial^2 Q_2}{\partial x \partial \delta}(0, 0, 0, \lambda_0)}{2}, \rho\Big)\Big\}, \quad \rho > 0, \tag{2.3}$$

where B is an open ball in $\mathbb{R}$. For each (small) $\rho > 0$ there exists a neighborhood $\mathcal{W}_1$ of $(\epsilon, \delta) = (0, 0)$, a neighborhood $\mathcal{W}_2$ of $\lambda = \lambda_0$, a neighborhood $\mathcal{V}$ of $(x, y) = (0, 0)$ and $N \in \mathbb{N}$ such that system $X_{\epsilon,\delta,\lambda}$ has at most N limit cycles in $\mathcal{V}$, for each $(\epsilon, \delta, \lambda) \in (\mathcal{W}_1 \setminus (S_+(\rho) \cup S_-(\rho))) \times \mathcal{W}_2$.

When $\beta_{-1}=\beta_0=0$ and $\beta_1\neq 0$, we deal with the slow-fast Bogdanov-Takens case.

Theorem 2.6 (Slow-fast Bogdanov-Takens case)**.** *Let Q_1 and Q_2 be smooth functions, $\beta_{-1}(D^0,\lambda_0)=\beta_0(E^0,D^0,\lambda_0)=0$ and $\beta_1(D^0,\lambda_0)\neq 0$ for some $(E^0,D^0)\in\mathbb{R}\times\mathbb{S}^{m-1}$. For each compact set K in the (x,y)-space there exists $\bar\epsilon_0>0$ and a neighborhood $\mathcal{W}$ of (E^0,D^0,λ_0) in the (E,D,λ)-space such that system $X_{\bar\epsilon E,\bar\epsilon D,\lambda}$ has at most 1 limit cycle in K, for each value $(\bar\epsilon,E,D,\lambda)\in[0,\bar\epsilon_0]\times\mathcal{W}$.*

We prove Theorem 2.6 in Section 3.2.3. When $m=1$, Theorem 2.6 gives us better understanding of limit cycles in the narrow parameter region $S_+(\rho)\cup S_-(\rho)$ defined in (2.3).

Theorem 2.7. *Let Q_1 and Q_2 be smooth functions, $m=1$, $\frac{\partial Q_2}{\partial\delta}(0,0,0,\lambda_0)=0$ and $\big(\frac{\partial^3 Q_2}{\partial x^2\partial\delta}+2\frac{\partial^2 Q_2}{\partial y\partial\delta}\big)(0,0,0,\lambda_0)\neq 0$. For each compact set K in the (x,y)-space there exists $\rho>0$, a neighborhood $\mathcal{W}_1$ of $(\epsilon,\delta)=(0,0)$ and a neighborhood $\mathcal{W}_2$ of $\lambda=\lambda_0$ such that system $X_{\epsilon,\delta,\lambda}$ has at most 1 limit cycle in K, for each $(\epsilon,\delta,\lambda)\in(\mathcal{W}_1\cap(S_+(\rho)\cup S_-(\rho)))\times\mathcal{W}_2$.*

When $\beta_{-1}=\beta_0=\beta_1=0$ and $\beta_2>0$ (resp. $\beta_2<0$), we deal with the slow-fast codimension 3 saddle (resp. elliptic) case. Moreover, when $\beta_3\neq 0$, we have the following result.

Theorem 2.8. *Let Q_1 and Q_2 be smooth functions, $\beta_{-1}=\beta_0=\beta_1=0$, $\beta_2\neq 0$ and $\beta_3\neq 0$ for some $(E^0,D^0)\in\mathbb{R}\times\mathbb{S}^{m-1}$. The following statements are true:*

(1) *(Slow-fast codimension 3 saddle case) Let $\beta_2(E^0,D^0,\lambda_0)>0$ and let K be an arbitrary compact set in the phase space (x,y). Then there exists $\bar\epsilon_0>0$ and a neighborhood $\mathcal{W}$ of (E^0,D^0,λ_0) in the (E,D,λ)-space such that system $X_{\bar\epsilon E,\bar\epsilon D,\lambda}$ has at most 2 limit cycles in K, for each value $(\bar\epsilon,E,D,\lambda)\in[0,\bar\epsilon_0]\times\mathcal{W}$.*

(2) *(Slow-fast codimension 3 elliptic case) Let $\beta_2(E^0,D^0,\lambda_0)<0$. Then there exists $\bar\epsilon_0>0$, a neighborhood $\mathcal{W}$ of (E^0,D^0,λ_0) in the (E,D,λ)-space and a neighborhood $\mathcal{V}$ of $(x,y)=(0,0)$ such that system $X_{\bar\epsilon E,\bar\epsilon D,\lambda}$ has at most 2 limit cycles in $\mathcal{V}$, for each value $(\bar\epsilon,E,D,\lambda)\in[0,\bar\epsilon_0]\times\mathcal{W}$.*

We prove the above theorem in Section 3.2.4.

When $\beta_{-1}=\beta_0=\beta_1=\beta_2=0$ and $\beta_3\neq 0$, we deal with the slow-fast codimension 4 saddle-node case.

Theorem 2.9 (Slow-fast codimension 4 saddle-node case)**.** *Let Q_1 and Q_2 be smooth functions, $\beta_{-1}=\beta_0=\beta_1=\beta_2=0$ and $\beta_3\neq 0$ for some $(E^0,D^0)\in\mathbb{R}\times\mathbb{S}^{m-1}$. For each compact set K in the (x,y)-space there exists $\bar\epsilon_0>0$ and a neighborhood $\mathcal{W}$ of (E^0,D^0,λ_0) in the (E,D,λ)-space such that system $X_{\bar\epsilon E,\bar\epsilon D,\lambda}$ has at most 2 limit cycles in K, for each value $(\bar\epsilon,E,D,\lambda)\in[0,\bar\epsilon_0]\times\mathcal{W}$.*

The above theorem will be proved in Section 3.2.5.

3. Proof of Theorems 2.1–2.9

There are essentially two types of normal forms of (2.1) near the origin $(x,y)=(0,0)$ which turn out to be useful for proving the results stated in Section 2.2. One type is analytic Liénard normal form (3.1), used in the slow-fast Hopf region in the parameter space (see Section 3.1). The other type of normal form is smooth and given in (3.11). We use it to prove the results in the other regions in the parameter space (see Section 3.2).

3.1. **Proof of Theorems 2.1 and 2.4.**

3.1.1. *Slow-fast Hopf case.* We consider slow-fast systems

$$X_{\bar\epsilon,\mu}:\begin{cases}\dot x = y - F(x,\mu)\\ \dot y = \bar\epsilon G(x,\mu),\end{cases} \tag{3.1}$$

where $\bar\epsilon\ge 0$, $\bar\epsilon\sim 0$, $\mu\sim\mu_0\in\mathbb{R}^p$ and F and G are analytic. Following [12], we say that system $X_{\bar\epsilon,\mu}$ for $(\bar\epsilon,\mu)=(0,\mu_0)$ has a slow-fast Hopf point (at $(x,y)=(0,0)$) if $F(0,\mu_0)=\frac{\partial F}{\partial x}(0,\mu_0)=G(0,\mu_0)=0$, $\frac{\partial^2 F}{\partial x^2}(0,\mu_0)\neq 0$ and $\frac{\partial G}{\partial x}(0,\mu_0)<0$. Suppose that system $X_{\bar\epsilon,\mu}$ has a slow-fast Hopf point for $(\bar\epsilon,\mu)=(0,\mu_0)$. Then Theorem 1.2 in [12] implies that the slow-fast Hopf point in the analytic Liénard family $X_{\bar\epsilon,\mu}$ has a finite cyclicity. More precisely, there exists $\bar\epsilon_0>0$, a neighborhood $\mathcal{V}$ of $(x,y)=(0,0)$, a neighborhood $\mathcal{W}$ of μ_0 in the μ-space and some $N\in\mathbb{N}$ such that $X_{\bar\epsilon,\mu}$ has at most N limit cycles in $\mathcal{V}$, for all $(\bar\epsilon,\mu)\in[0,\bar\epsilon_0]\times\mathcal{W}$.

To prove Theorem 2.1.1 (resp. Theorem 2.4.1), it suffices to bring (non-Liénard analytic) system $X_{\bar\epsilon E,\bar\epsilon D,\lambda}$, near $(x,y)=(0,0)$, into an analytic Liénard normal form of type (3.1) and to show that the obtained normal form has a slow-fast Hopf point at $(x,y)=(0,0)$ for $(\bar\epsilon,E,D,\lambda)=(0,E^0,D^0,\lambda_0)$ where $(E^0,D^0)=(1,0)$ (resp. (E^0,D^0) is introduced in Theorem 2.4.1). When $(\bar\epsilon,E,D,\lambda)=(0,E^0,D^0,\lambda_0)$, the linearized vector field of $X_{0,0,\lambda_0}=(y-x^2)\partial_x+0\partial_y$ at $(x,y)=(0,0)$ is of nilpotent type. This implies that the linear part of $X_{0,0,\lambda_0}$ at the origin is not radial, i.e. not of the form $\alpha x\partial_x+\alpha y\partial_y$. Using this and the fact that Q_1 and Q_2 are analytic we can find an analytic $(\bar\epsilon,E,D,\lambda)$-family of coordinate changes

$$(\bar x,\bar y)=\Phi^{\bar\epsilon,E,D,\lambda}(x,y)=\big(\Phi_1^{\bar\epsilon,E,D,\lambda}(x,y),\Phi_2^{\bar\epsilon,E,D,\lambda}(x,y)\big),$$

with $(x,y)\sim(0,0)$, $(\bar\epsilon,E,D,\lambda)\sim(0,E^0,D^0,\lambda_0)$ and $\Phi^{0,E^0,D^0,\lambda_0}(0,0)=(0,0)$, and a nowhere zero analytic function $\psi^{\bar\epsilon,E,D,\lambda}(\bar x,\bar y)$ such that

$$\psi^{\bar\epsilon,E,D,\lambda}(\bar x,\bar y)\cdot\begin{pmatrix}\bar y-F(\bar x,\bar\epsilon,E,D,\lambda)\\ \bar G(\bar x,\bar\epsilon,E,D,\lambda)\end{pmatrix}=D\Phi^{\bar\epsilon,E,D,\lambda}(x,y)X_{\bar\epsilon E,\bar\epsilon D,\lambda}(x,y), \tag{3.2}$$

for some analytic functions F and $\bar G$, $F(0,0,E^0,D^0,\lambda_0)=\bar G(0,0,E^0,D^0,\lambda_0)=0$. Thus, there exists a local analytic $(\bar\epsilon,E,D,\lambda)$-family of coordinate changes transforming $X_{\bar\epsilon E,\bar\epsilon D,\lambda}$ to an analytic $(\bar\epsilon,E,D,\lambda)$-family of Liénard equations, up to multiplication by a nowhere zero analytic function. See [16, Theorem 1].

First, let us show that $\bar G=O(\bar\epsilon)$. When $\bar\epsilon=0$, system $X_{\bar\epsilon E,\bar\epsilon D,\lambda}$ has the critical curve $\{y=x^2\}$. Combining this fact with (3.2) we obtain

$$\bar G\big(\Phi_1^{0,E,D,\lambda}(x,x^2),0,E,D,\lambda\big)=0 \tag{3.3}$$

and

$$\Phi_2^{0,E,D,\lambda}(x,x^2)-F(\Phi_1^{0,E,D,\lambda}(x,x^2),0,E,D,\lambda)=0. \tag{3.4}$$

If we differentiate (3.4) with respect to x, we obtain

$$\frac{\partial\Phi_2^{0,E^0,D^0,\lambda_0}}{\partial x}(0,0)-\frac{\partial F}{\partial x}(0,0,E^0,D^0,\lambda_0)\frac{\partial\Phi_1^{0,E^0,D^0,\lambda_0}}{\partial x}(0,0)=0. \tag{3.5}$$

It follows from (3.5) that $\frac{\partial\Phi_1^{0,E^0,D^0,\lambda_0}}{\partial x}(0,0)\neq 0$. Using this and (3.3) we obtain $\bar G(\bar x,\bar\epsilon,E,D,\lambda)=\bar\epsilon G(\bar x,\bar\epsilon,E,D,\lambda)$ where G is analytic and $\bar x\sim 0$.

In the rest of this section we prove that the analytic family

$$\begin{aligned} \dot{\bar{x}} &= \bar{y} - F(\bar{x}, \bar{\epsilon}, E, D, \lambda) \\ \dot{\bar{y}} &= \bar{\epsilon} G(\bar{x}, \bar{\epsilon}, E, D, \lambda) \end{aligned} \tag{3.6}$$

has a slow-fast Hopf point at $(\bar{x}, \bar{y}) = (0,0)$ for $(\bar{\epsilon}, E, D, \lambda) = (0, E^0, D^0, \lambda_0)$. We have $\Phi_2^{0,E^0,D^0,\lambda_0}(x,0) \equiv 0$ because $\Phi^{0,E^0,D^0,\lambda_0}$ preserves the line $\{y = 0\}$. Using this and the first component of (3.2) we obtain

$$\begin{aligned} &\psi^{0,E^0,D^0,\lambda_0}\big(\Phi^{0,E^0,D^0,\lambda_0}(x,0)\big) F(\Phi_1^{0,E^0,D^0,\lambda_0}(x,0), 0, E^0, D^0, \lambda_0) \\ &= \frac{\partial \Phi_1^{0,E^0,D^0,\lambda_0}}{\partial x}(x,0) x^2. \end{aligned} \tag{3.7}$$

Since $\Phi^{0,E^0,D^0,\lambda_0}(0,0) = (0,0)$, $\frac{\partial \Phi_1^{0,E^0,D^0,\lambda_0}}{\partial x}(0,0) \neq 0$ and $\psi^{0,E^0,D^0,\lambda_0}(0,0) \neq 0$, (3.7) implies that $F(0,0,E^0,D^0,\lambda_0) = \frac{\partial F}{\partial x}(0,0,E^0,D^0,\lambda_0) = 0$ and

$$\frac{\partial^2 F}{\partial x^2}(0,0,E^0,D^0,\lambda_0) = \frac{2}{\psi^{0,E^0,D^0,\lambda_0}(0,0) \frac{\partial \Phi_1^{0,E^0,D^0,\lambda_0}}{\partial x}(0,0)} \neq 0. \tag{3.8}$$

On the other hand, as a consequence of the Implicit Function Theorem, the x-nullcline of $X_{\bar{\epsilon}E,\bar{\epsilon}D,\lambda}$ near the origin, defined in (2.1), is given by $y = \eta(x, \bar{\epsilon}, E, D, \lambda)$ where η is analytic and $\eta = x^2 + O(\bar{\epsilon})$. If we substitute the function $\eta(x, \bar{\epsilon}, E, D, \lambda)$ for y in (3.2), divide the second component of (3.2) by $\bar{\epsilon}$ and let $(\bar{\epsilon}, E, D, \lambda) \to (0, E^0, D^0, \lambda_0)$, we obtain

$$\begin{aligned} &\psi^{0,E^0,D^0,\lambda_0}\big(\Phi^{0,E^0,D^0,\lambda_0}(x,x^2)\big).G(\Phi_1^{0,E^0,D^0,\lambda_0}(x,x^2), 0, E^0, D^0, \lambda_0) \\ &= \frac{\partial \Phi_2^{0,E^0,D^0,\lambda_0}}{\partial y}(x,x^2) 2x f(x, E^0, D^0, \lambda_0) \end{aligned} \tag{3.9}$$

where f is the slow dynamics defined in (2.2). Note that $\beta_{-1}(D^0, \lambda_0) = 0$ and $\beta_0(E^0, D^0, \lambda_0) < 0$ for $(E^0, D^0) = (1, 0)$ or for a parameter (E^0, D^0) satisfying the assumptions of Theorem 2.4.1 where β_{-1} and β_0 are defined after (2.2). Since $\frac{\partial \Phi_2^{0,E^0,D^0,\lambda_0}}{\partial x}(0,0) = 0$ ($\Phi_2^{0,E^0,D^0,\lambda_0}(x,0) \equiv 0$), we have $\frac{\partial \Phi_2^{0,E^0,D^0,\lambda_0}}{\partial y}(0,0) \neq 0$ and (3.9) implies $G(0,0,E^0,D^0,\lambda_0) = 0$ and

$$\frac{\partial G}{\partial x}(0,0,E^0,D^0,\lambda_0) = \frac{2 \frac{\partial \Phi_2^{0,E^0,D^0,\lambda_0}}{\partial y}(0,0) \beta_0(E^0, D^0, \lambda_0)}{\psi^{0,E^0,D^0,\lambda_0}(0,0) \frac{\partial \Phi_1^{0,E^0,D^0,\lambda_0}}{\partial x}(0,0)} \neq 0. \tag{3.10}$$

Since the critical curve $\{y = x^2\}$ of $X_{0,0,\lambda_0}$ is concave up, we conclude that $\frac{\partial \Phi_2^{0,E^0,D^0,\lambda_0}}{\partial y}(0,0) > 0$ (resp. $\frac{\partial \Phi_2^{0,E^0,D^0,\lambda_0}}{\partial y}(0,0) < 0$) if the critical curve of (3.6), for $(\bar{\epsilon}, E, D, \lambda) = (0, E^0, D^0, \lambda_0)$, is concave up, i.e. (3.8) is positive (resp. concave down, i.e. (3.8) is negative). Now comparing expressions (3.8) and (3.10), we see that $\frac{\partial G}{\partial x}(0,0,E^0,D^0,\lambda_0) < 0$. This completes the proof of Theorem 2.1.1 and Theorem 2.4.1.

3.1.2. *Slow-fast saddle case.* In this section we prove Theorems 2.1.2 and 2.4.2. Let us recall that $\beta_{-1}(D^0, \lambda_0) = 0$ and $\beta_0(E^0, D^0, \lambda_0) > 0$ for $(E^0, D^0) = (-1, 0)$ or for a parameter (E^0, D^0) satisfying the assumptions of Theorem 2.4.2. It can be easily seen that system $X_{\bar{\epsilon}E,\bar{\epsilon}D,\lambda}$ has a hyperbolic saddle near $(x, y) = (0, 0)$, for

$(\bar\epsilon, E, D, \lambda) \sim (0, E^0, D^0, \lambda_0)$ and $\bar\epsilon > 0$. Thus we have no limit cycles near the origin in the phase space. On the other hand, there are no detectable canard limit cycles for $(\bar\epsilon, E, D, \lambda) \sim (0, E^0, D^0, \lambda_0)$, $\bar\epsilon \geq 0$, because the slow dynamics (2.2) points from the repelling part of the critical curve to the attracting part of the critical curve near $x = 0$ ($x' = \beta_0 + O(x) > 0$ for $(E, D, \lambda) = (E^0, D^0, \lambda_0)$).

3.2. **Proof of Theorems 2.2, 2.6, 2.8 and 2.9.**

3.2.1. *Bringing $X_{\bar\epsilon E,\bar\epsilon D,\lambda}$ to a smooth normal form for generic nilpotent contact points.* We consider slow-fast systems

$$\begin{aligned} \dot x &= y \\ \dot y &= -xy + \bar\epsilon g(x, \bar\epsilon, \mu) + \bar\epsilon y^2 H(x, y, \bar\epsilon, \mu), \end{aligned} \tag{3.11}$$

where $\bar\epsilon \sim 0$, $\bar\epsilon \geq 0$, $\mu \sim \mu_0 \in \mathbb{R}^p$ and g and H are smooth functions. System (3.11) has a generic nilpotent contact point at the origin. The cyclicity of this nilpotent contact point has been studied in [8, 9, 17], depending on the singularity order at the contact point, i.e. the order of vanishing of $g(x, 0, \mu_0)$ at $x = 0$. Suppose that Q_1 and Q_2 in (2.1) are smooth. We transform system $X_{\bar\epsilon E,\bar\epsilon D,\lambda}$, near $(x, y) = (0, 0)$, to a slow-fast system of type (3.11).

Lemma 3.1. *Consider a smooth slow-fast system $X_{\bar\epsilon E,\bar\epsilon D,\lambda}$, defined in (2.1), with $(E, D, \lambda) \sim (E^0, D^0, \lambda_0)$, $\bar\epsilon \sim 0$ and $\bar\epsilon \geq 0$. There exists a local smooth $(\bar\epsilon, E, D, \lambda)$-family of coordinate changes*

$$(x, y) \mapsto \Phi^{\bar\epsilon,E,D,\lambda}(x, y),$$

with $\Phi^{0,E,D,\lambda}(0, 0) = (0, 0)$, bringing $X_{\bar\epsilon E,\bar\epsilon D,\lambda}$ in a smooth $(\bar\epsilon, E, D, \lambda)$-family (up to multiplication by a smooth strictly positive function)

$$\begin{aligned} \dot x &= y \\ \dot y &= -xy + \bar\epsilon g(x, \bar\epsilon, E, D, \lambda) + \bar\epsilon y^2 H(x, y, \bar\epsilon, E, D, \lambda), \end{aligned} \tag{3.12}$$

with smooth functions g and H, and

$$g(x, 0, E, D, \lambda) = x f(\frac{x}{\sqrt{2}}, E, D, \lambda), \tag{3.13}$$

where f is the slow dynamics of $X_{\bar\epsilon E,\bar\epsilon D,\lambda}$.

Proof. We can write the slow-fast system $X_{\bar\epsilon E,\bar\epsilon D,\lambda}$ as

$$\begin{aligned} \dot x &= y - x^2 + O(\bar\epsilon) \\ \dot y &= \bar\epsilon\big(- 2Ex + 2Exy + \langle D, \tilde Q(x, y, \bar\epsilon D, \lambda)\rangle\big) \end{aligned} \tag{3.14}$$

where $O(\bar\epsilon)$ is a smooth function in $(x, y, \bar\epsilon, E, D, \lambda)$ and $\tilde Q$ is a smooth function defined in Section 2.1. Using the (smooth) coordinate change $Y = y - x^2 + O(\bar\epsilon)$ near $(x, y) = (0, 0)$, the vector field (3.14) changes into

$$\begin{aligned} \dot x &= y \\ \dot y &= \bar\epsilon\big(- 2Ex + 2Ex(y + x^2) + \langle D, \tilde Q(x, y + x^2, 0, \lambda)\rangle + O(\bar\epsilon)\big) \\ &\quad + (-2x + O(\bar\epsilon))y \end{aligned} \tag{3.15}$$

where we denote Y again by y and where the $O(\bar{\epsilon})$-terms are smooth functions in $(x, y, \bar{\epsilon}, E, D, \lambda)$. Using the first-order Taylor expansion of the y-component in (3.15) w.r.t. y about $y = 0$, the vector field (3.15) can be written as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= (-2x + O_1(\bar{\epsilon}))y + \bar{\epsilon}\big(2xf(x, E, D, \lambda) + O_2(\bar{\epsilon})\big) + O(\bar{\epsilon}y^2) \end{aligned} \tag{3.16}$$

where f is defined in (2.2), $O_1(\bar{\epsilon})$, $O_2(\bar{\epsilon})$ and $O(\bar{\epsilon}y^2)$ are smooth functions and $O_1(\bar{\epsilon})$ and $O_2(\bar{\epsilon})$ are independent of y. After a translation $x \to x + \bar{\epsilon}\alpha$, with α smooth in $(\bar{\epsilon}, E, D, \lambda)$, the vector field (3.16) becomes

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -xl(x, \bar{\epsilon}, E, D, \lambda)y + \bar{\epsilon}\big(2xf(x, E, D, \lambda) + O(\bar{\epsilon})\big) + O(\bar{\epsilon}y^2) \end{aligned} \tag{3.17}$$

where l is a smooth function, $l(x, 0, E, D, \lambda) \equiv 2$, $O(\bar{\epsilon})$ and $O(\bar{\epsilon}y^2)$ are smooth and $O(\bar{\epsilon})$ is independent of y. Since l is strictly positive near $x = 0$, there exists a strictly positive smooth function $L(x, \bar{\epsilon}, E, D, \lambda)$ $(L(x, 0, E, D, \lambda) \equiv 2)$ such that

$$\frac{x^2}{2}L(x) = \int_0^x sl(s)ds.$$

After differentiating this with respect to x and after division by x we obtain

$$L(x) + \frac{x}{2}L'(x) = l(x). \tag{3.18}$$

Using the coordinate change $X = x\sqrt{L(x)}$ (we write $x = X\tilde{L}(X)$ where $\tilde{L}$ is a smooth function and $\tilde{L}(x, 0, E, D, \lambda) \equiv \frac{1}{\sqrt{2}}$), the expression (3.18) and multiplication by $\frac{\sqrt{L(x)}}{l(x)} > 0$, the vector field (3.17) changes into

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -xy + \bar{\epsilon}\Big(\frac{2x}{l(x\tilde{L}(x))}f(x\tilde{L}(x), E, D, \lambda) + O(\bar{\epsilon})\Big) + O(\bar{\epsilon}y^2) \end{aligned} \tag{3.19}$$

where we denote X again by x and $O(\bar{\epsilon})$ and $O(\bar{\epsilon}y^2)$ are new smooth functions. It is clear now that the vector field (3.19) is of type (3.12) with

$$g(x, \bar{\epsilon}, E, D, \lambda) = \frac{2x}{l(x\tilde{L}(x))}f(x\tilde{L}(x), E, D, \lambda) + O(\bar{\epsilon}).$$

This implies that $g(x, 0, E, D, \lambda) = xf(\frac{x}{\sqrt{2}}, E, D, \lambda)$. □

We use the normal form (3.12) when we study limit cycles of $X_{\bar{\epsilon}E,\bar{\epsilon}D,\lambda}$ near $(x, y) = (0, 0)$ (see Sections 3.2.2–3.2.5).

3.2.2. *Proof of Theorem 2.2.* Suppose that Q_1, Q_2 are smooth and $\beta_{-1}(D^0, \lambda_0) \neq 0$ for some $D^0 \in \mathbb{S}^{m-1}$. Since $\beta_{-1}(D^0, \lambda_0) \neq 0$, (3.13) implies that the order of vanishing of $g(x, 0, E, D^0, \lambda_0)$ at $x = 0$ is 0 for each E kept in a compact subset of $\mathbb{R}$. Thus, the contact point of (3.12) (or $X_{\bar{\epsilon}E,\bar{\epsilon}D,\lambda}$) is of jump type and there are no limit cycles of $X_{\bar{\epsilon}E,\bar{\epsilon}D,\lambda}$. See e.g. [9, Section 3.4] for a detailed study of the jump point. (Note that system (3.12) (or $X_{\bar{\epsilon}E,\bar{\epsilon}D,\lambda}$) has no singularities in a fixed neighborhood of $(x, y) = (0, 0)$ for $\bar{\epsilon} \sim 0$, $\bar{\epsilon} > 0$, $(D, \lambda) \sim (D^0, \lambda_0)$ and E kept in the compact subset of $\mathbb{R}$.)

3.2.3. *Proof of Theorem 2.6.* Suppose that Q_1 and Q_2 are smooth, $\beta_{-1}(D^0,\lambda_0)=\beta_0(E^0,D^0,\lambda_0)=0$ and $\beta_1(D^0,\lambda_0)\neq 0$ for some $(E^0,D^0)\in\mathbb{R}\times\mathbb{S}^{m-1}$. Using (3.13) we have that the order of vanishing of $g(x,0,E^0,D^0,\lambda_0)$ at $x=0$ is 2. This implies that the contact point of (3.12) (or $X_{\bar\epsilon E,\bar\epsilon D,\lambda}$) is of slow-fast Bogdanov-Takens type and we can apply the results of [8]. Following [8], system (3.12) has at most 1 (hyperbolic) limit cycle in an $(\bar\epsilon,E,D,\lambda)$-uniform neighborhood of $(x,y)=(0,0)$ for $(\bar\epsilon,E,D,\lambda)\sim(0,E^0,D^0,\lambda_0)$ and $\bar\epsilon\geq 0$. On the other hand, there are no detectable canard limit cycles in $X_{\bar\epsilon E,\bar\epsilon D,\lambda}$ for $(\bar\epsilon,E,D,\lambda)\sim(0,E^0,D^0,\lambda_0)$ and $\bar\epsilon\geq 0$ because the passage from the attracting part to the repelling part of the critical curve is not possible (for $(E,D,\lambda)=(E^0,D^0,\lambda_0)$, $x\sim 0$ and $x\neq 0$, the slow dynamics (2.2) is given by $x'=x(\beta_1(D^0,\lambda_0)+O(x))$). This completes the proof of Theorem 2.6.

3.2.4. *Proof of Theorem 2.8.* Suppose that Q_1 and Q_2 are smooth, $\beta_{-1}=\beta_0=\beta_1=0$, $\beta_2\neq 0$ and $\beta_3\neq 0$ for some $(E^0,D^0)\in\mathbb{R}\times\mathbb{S}^{m-1}$. Since $\beta_2\neq 0$, (3.13) implies that the order of vanishing of $g(x,0,E^0,D^0,\lambda_0)$ at $x=0$ is 3, and the contact point of (3.12) is of slow-fast codimension 3 saddle ($\beta_2>0$) or elliptic ($\beta_2<0$) type studied in [9]. Following [9, 10, 15, 18], the number of limit cycles near $(x,y)=(0,0)$ depends on the higher order terms in $g(x,0,E^0,D^0,\lambda_0)$ and when the (symmetry breaking) coefficient in front of the quartic term in $g(x,0,E^0,D^0,\lambda_0)$ is nonzero (i.e. $\beta_3\neq 0$), system (3.12) (or $X_{\bar\epsilon E,\bar\epsilon D,\lambda}$) has at most 2 limit cycles in an $(\bar\epsilon,E,D,\lambda)$-uniform neighborhood of $(x,y)=(0,0)$ for $(\bar\epsilon,E,D,\lambda)\sim(0,E^0,D^0,\lambda_0)$ and $\bar\epsilon\geq 0$. Moreover, in the saddle case detectable canard limit cycles of $X_{\bar\epsilon E,\bar\epsilon D,\lambda}$ are not possible because the slow dynamics of $X_{\bar\epsilon E,\bar\epsilon D,\lambda}$ points from the repelling part of the critical curve to the attracting part of the critical curve ($x'=x^2(\beta_2(E^0,D^0,\lambda_0)+O(x))>0$, for $(E,D,\lambda)=(E^0,D^0,\lambda_0)$, $x\sim 0$ and $x\neq 0$). This completes the proof of Theorem 2.8.

3.2.5. *Proof of Theorem 2.9.* Suppose that Q_1 and Q_2 are smooth, $\beta_{-1}=\beta_0=\beta_1=\beta_2=0$ and $\beta_3\neq 0$ for some $(E^0,D^0)\in\mathbb{R}\times\mathbb{S}^{m-1}$. Then the order of $g(x,0,E^0,D^0,\lambda_0)$ at $x=0$ is 4 and the contact point of (3.12) is of slow-fast codimension 4 saddle-node type studied in [17]. Following [17], system (3.12) (i.e. $X_{\bar\epsilon E,\bar\epsilon D,\lambda}$) has at most 2 limit cycles in an $(\bar\epsilon,E,D,\lambda)$-uniform neighborhood of $(x,y)=(0,0)$ for $(\bar\epsilon,E,D,\lambda)\sim(0,E^0,D^0,\lambda_0)$ and $\bar\epsilon\geq 0$. Large (canard) limit cycles of $X_{\bar\epsilon E,\bar\epsilon D,\lambda}$ are not possible because the passage from the attracting part of the critical curve to the repelling part of the critical curve is not possible. We have $x'=x^3(\beta_3(D^0,\lambda_0)+O(x))$ for $(E,D,\lambda)=(E^0,D^0,\lambda_0)$, $x\sim 0$ and $x\neq 0$. This completes the proof of Theorem 2.9.

4. Generalization of the slow-fast Darboux integrable system X_ϵ

We consider a slow-fast Darboux integrable system

$$Y_\epsilon:\begin{cases}\dot x=-P_0(x,y)\dfrac{\partial P_1}{\partial y}(x,y)-\epsilon P_1(x,y)\dfrac{\partial P_0}{\partial y}(x,y)\\[2mm] \dot y=P_0(x,y)\dfrac{\partial P_1}{\partial x}(x,y)+\epsilon P_1(x,y)\dfrac{\partial P_0}{\partial x}(x,y),\end{cases}$$

where $\epsilon\sim 0$ and P_0 and P_1 are smooth or analytic functions. System Y_ϵ has the first integral (of Darboux type) $H=P_0^\epsilon P_1$. (When $P_0=y-x^2$ and $P_1=1-y$, Y_ϵ becomes X_ϵ defined in (1.1).) The fast subsystem of Y_ϵ is given by Y_0. We assume that the vector field Y_ϵ satisfies the following conditions (see [3] or [1]).

- There is a compact region $\bar{\mathcal{O}}$ bounded by $\{P_0 = 0\}$ and $\{P_1 = 0\}$ and the fast subsystem Y_0 has no singularities in the interior of $\bar{\mathcal{O}}$ (i.e. $\nabla P_1 \neq (0,0)$ in the interior of $\bar{\mathcal{O}}$).
- The curve $\{P_0 = 0\}$ is transverse to Y_0, i.e. $\langle \nabla P_0, (-\frac{\partial P_1}{\partial y}, \frac{\partial P_1}{\partial x})\rangle \neq 0$ (normally hyperbolic singularities), except for one point where we deal with a nilpotent singularity, i.e. $\langle \nabla P_0, (-\frac{\partial P_1}{\partial y}, \frac{\partial P_1}{\partial x})\rangle = 0$, $\nabla P_0 \neq (0,0)$ and $\nabla P_1 \neq (0,0)$. Moreover, we assume that the contact at the nilpotent singularity is quadratic.

Like in Sections 1 and 2, we can try to find a suitable blow-up at $(\epsilon, \delta) = (0,0)$ and to study the cyclicity of the nilpotent contact point in smooth or analytic deformations of Y_ϵ, in different directions in the parameter space (ϵ, δ). In the integrable direction, polynomial deformations of Y_ϵ have been studied in [3].

References

[1] M. Bobieński, L. Gavrilov; *Finite cyclicity of slow-fast Darboux systems with a two-saddle loop*, Proc. Amer. Math. Soc., **144** (2016), no. 10, 4205–4219.

[2] M. Bobieński, P. Mardesic, D. Novikov; *Pseudo-Abelian integrals: unfolding generic exponential*, J. Differential Equations, **247** (2009), no. 12, 3357–3376.

[3] M. Bobieński, P. Mardesic, D. Novikov; *Pseudo-abelian integrals on slow-fast Darboux systems*, Ann. Inst. Fourier (Grenoble), **63** (2013), no. 2, 417–430.

[4] P. De Maesschalck, F. Dumortier; *Time analysis and entry-exit relation near planar turning points*, J. Differential Equations, **215** (2005), no. 2, 225–267.

[5] P. De Maesschalck, F. Dumortier; *Canard cycles in the presence of slow dynamics with singularities*, Proc. Roy. Soc. Edinburgh Sect. A , **138** (2008), no. 2, 265–299.

[6] P. De Maesschalck, F. Dumortier; *Singular perturbations and vanishing passage through a turning point*, J. Differential Equations, **248** (2010), no. 9, 2294–2328.

[7] P. De Maesschalck, F. Dumortier; *Detectable canard cycles with singular slow dynamics of any order at the turning point*, Discrete Contin. Dyn. Syst., **29** (2011), no. 1, 109–140.

[8] P. De Maesschalck, F. Dumortier; *Slow-fast Bogdanov-Takens bifurcations*, J. Differential Equations, **250** (2011), no. 2, 1000–1025.

[9] P. De Maesschalck, F. Dumortier, R. Huzak; *Limit cycles in slow-fast codimension 3 saddle and elliptic bifurcations*, J. Differential Equations, **255** (2013), no. 11, 4012–4051.

[10] P. De Maesschalck, F. Dumortier, R. Huzak; *Primary birth of canard cycles in slow-fast codimension 3 elliptic bifurcations*, Communications on Pure and Applied Analysis, **13** (2014), no. 6, 2641–2673.

[11] F. Dumortier, R. Roussarie; *Canard cycles and center manifolds*, Mem. Amer. Math. Soc., **121** (1996), no. 577, x+100, With an appendix by Li Chengzhi.

[12] F. Dumortier, R. Roussarie; *Birth of canard cycles*, Discrete Contin. Dyn. Syst. Ser. S, **2** (2009), no. 4, 723–781.

[13] F. Dumortier, R. Roussarie, C. Rousseau; *Hilbert's 16th problem for quadratic vector fields*, J. Differential Equations,**110** (1994), no. 1, 86–133.

[14] L. Gavrilov; *On the number of limit cycles which appear by perturbation of Hamiltonian two-saddle cycles of planar vector fields*, Bull. Braz. Math. Soc. (N.S.), **42** (2011), no. 1, 1–23.

[15] R. Huzak; *Cyclicity of the origin in slow-fast codimension 3 saddle and elliptic bifurcations*, Discrete Contin. Dyn. Syst. ,**36** (2016), no. 1, 171–215.

[16] R. Huzak; *Normal forms of Liénard type for analytic unfoldings of nilpotent singularities*, Proc. Amer. Math. Soc., **145** (2017), no. 10, 4325–4336.

[17] R. Huzak; *Regular and slow-fast codimension 4 saddle-node bifurcations*, J. Differential Equations, **262** (2017), no. 2, 1119–1154.

[18] R. Huzak; *Canard explosion near non-Liénard type slow-fast Hopf point*, J. Dynam. Differential Equations, **31** (2019), no. 2, 683–709.

[19] M. Krupa, P. Szmolyan; *Relaxation oscillation and canard explosion*, J. Differential Equations, **174** (2001), no. 2, 312–368.

[20] E. F. Mishchenko, Yu. S. Kolesov, A. Yu. Kolesov, N. Kh. Rozov; *Asymptotic methods in singularly perturbed systems*, Monographs in Contemporary Mathematics, Consultants Bureau, New York, 1994, Translated from the Russian by Irene Aleksanova.

[21] S. Smale; *Mathematical problems for the next century*, Math. Intelligencer, **20** (1998), no. 2, 7–15.

Renato Huzak
Hasselt University, Campus Diepenbeek, Agoralaan Gebouw D, 3590 Diepenbeek, Belgium
Email address: renato.huzak@uhasselt.be