K-theory of locally compact modules over orders

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K-THEORY OF LOCALLY COMPACT MODULES OVER ORDERS

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ABSTRACT. We present a quick approach to computing the K-theory of the category of locally compact modules over any order in a semisimple $\mathbb{Q}$-algebra. We obtain the K-theory by first quotienting out the compact modules and subsequently the vector modules. Our proof exploits the fact that the pair (vector modules plus compact modules, discrete modules) becomes a torsion theory after we quotient out the finite modules. Treating these quotients as exact categories is possible due to a recent localization formalism.

CONTENTS

1. INTRODUCTION

Suppose A is a finite-dimensional semisimple $\mathbb{Q}$-algebra and $\mathfrak{A} \subset A$ is any $\mathbb{Z}$-order. We write $\operatorname{mod}(-)$ for the category of finitely generated right modules, $A_{\mathbb{R}}$ for $A \otimes_{\mathbb{Q}} \mathbb{R}$, and $\mathsf{LCA}_{\mathfrak{A}}$ for the exact category of locally compact topological $\mathfrak{A}$-modules [8]. We give a new proof for the following theorem.

Theorem 1.1. *For every localizing invariant $K\colon \operatorname{Cat}_{\infty}^{\operatorname{Ex}} \to \mathcal{D}$ (where $\mathcal{D}$ is any stable ∞-category), there is a canonical fiber sequence:*

$$K(\operatorname{mod}(\mathfrak{A})) \to K(\operatorname{mod}(A_{\mathbb{R}})) \to K(\mathsf{LCA}_{\mathfrak{A}}),$$

where the first map is induced by the natural embedding $- \otimes_{\mathbb{Z}} \mathbb{R}\colon \mathfrak{A} \to \mathfrak{A}_{\mathbb{R}} = A_{\mathbb{R}}$.

For localizing invariants and $\operatorname{Cat}_{\infty}^{\operatorname{Ex}}$ we refer to the framework and notation of [2]. The principal example is non-connective K-theory taking values in spectra (and we will indicatively always denote the invariant by K throughout the paper). When convenient, and for example in the above statement, we write $K(\mathsf{C})$ even if C is an exact (or one-sided exact) category, for $K(\mathsf{D}_{\infty}^{\mathsf{b}}(\mathsf{C}))$, where $\mathsf{D}_{\infty}^{\mathsf{b}}(\mathsf{C})$ is the stable ∞-category of bounded complexes attached to C.

The first theorem of the above kind is due to Clausen [5, Theorem 3.4], who proved it in the special case $A = \mathbb{Q}$ and $\mathfrak{A} = \mathbb{Z}$ (with an additional, but ultimately inconsequential, restriction to second-countable topologies) with an eye to applications in class field theory. To this end, he set up a cone construction on the level of stable ∞-categories. The above version stems from [3, Theorem 11.4]. It was based on Schlichting's Localization Theorem from [13]. In our new approach, we use the recent Localization Theorem of [6, 7], which uses the additional flexibility of one-sided exact categories. These devices

can be thought of as convenient tools to avoid having to handle the underlying stable ∞-categories (or triangulated categories) manually.

The proof of the main theorem is given in §4. We start by considering the quotient of $\mathsf{LCA}_{\mathfrak{A}}$ by the subcategory of compact modules $\mathsf{LCA}_{\mathfrak{A},\mathsf{C}}$. While this subcategory does not satisfy the s-filtering conditions of Schlichting's Localization Theorem, it does satisfy the conditions of the recent Localization Theorem of [6, 7]. The latter theory endows the quotient $\mathcal{E} \coloneqq \mathsf{LCA}_{\mathfrak{A}} / \mathsf{LCA}_{\mathfrak{A},\mathsf{C}}$ with the structure of a one-sided exact category (in the sense of [1, 12]), which can then canonically be embedded in its exact hull $\mathcal{E}^{\mathrm{ex}}$. It is from this exact hull that we take a further quotient, this time by the subcategory $\mathcal{V}$ of vector modules of $\mathsf{LCA}_{\mathfrak{A}}$. Finally, we show that the resulting category $\mathcal{F} \coloneqq \mathcal{E}^{\mathrm{ex}}/\mathcal{V}$ is equivalent to $\mathrm{Mod}\,\mathfrak{A}/\mathrm{mod}\,\mathfrak{A}$; it is from this equivalence that we obtain the sequence in Theorem 1.1.

The equivalence $\mathcal{F} \simeq \mathrm{Mod}\,\mathfrak{A}/\mathrm{mod}\,\mathfrak{A}$ is based on the universal properties of the aforementioned quotients and the exact hull, as well as on the following observation (see Theorem 3.8): after quotienting the finite modules out, the Structure Theorem of $\mathsf{LCA}_{\mathfrak{A}}$ (Theorem 3.4) implies that the pair $(\mathsf{LCA}_{\mathfrak{A},\mathsf{C}\mathbb{R}}, \mathsf{LCA}_{\mathfrak{A},\mathsf{D}})$ becomes a torsion pair. This means that the sequences "$M_{\mathrm{compact}} \oplus M_{\mathrm{vector}} \rightarrowtail M \twoheadrightarrow M_{\mathrm{discrete}}$" given by the Structure Theorem become essentially unique in the quotient.

Using these new tools, our proof of Theorem 1.1 is considerably shorter and technically less involved. We never seriously leave the world of 1-categories, do not use the ∞-categorical cone construction of [5], nor do we need the tedious verifications of Schlichting's left/right s-filtering conditions done in [3].

2. Localizations of exact categories

This section is preliminary in nature. We summarize the results of [6, 7] about localizations of exact categories. One salient feature of the localizations we consider is that the resulting category need not be exact, but will be one-sided exact (in the sense of [1, 12]).

2.1. **One-sided exact categories.**

Definition 2.1. A *conflation category* is an additive category $\mathcal{C}$ together with a chosen class of kernel-cokernel pairs (closed under isomorphisms), called *conflations*. The kernel part of a conflation is called an *inflation* and the cokernel part of a conflation is called a *deflation*. We depict inflations by $\rightarrowtail$ and deflations by $\twoheadrightarrow$.

An additive functor $F\colon \mathcal{C} \to \mathcal{D}$ between conflation categories is called *conflation-exact* if conflations are mapped to conflations.

Definition 2.2. A conflation category $\mathcal{E}$ is called an *inflation-exact* or *left exact category* if $\mathcal{E}$ satisfies the following axioms:

L0 For each $X \in \mathcal{E}$, the map $0 \to X$ is an inflation.
L1 The composition of two inflations is again an inflation.
L2 The pushout of any morphism along an inflation exists, moreover, inflations are stable under pushouts.

Dualizing the above axioms yields the notion of a *deflation-exact* category. A *Quillen exact* category is simply a conflation category which is both inflation-exact and deflation-exact by [9, Appendix A].

Let $\mathcal{E}$ be one-sided exact category. Analogous to exact categories, one can define the bounded derived category $\mathsf{D}^{\mathsf{b}}(\mathcal{E})$ as the Verdier localization $\mathsf{K}^{\mathsf{b}}(\mathcal{E})/\langle \mathbf{Ac}(\mathcal{E})\rangle_{\mathrm{thick}}$ of the bounded homotopy category by the thick closure of the triangulated subcategory of acyclic complexes (see [1, Corollary 7.3]). The canonical embedding $i\colon \mathcal{E} \to \mathsf{D}^{\mathsf{b}}(\mathcal{E})$, mapping objects to stalk complexes in degree zero, is a fully faithful embedding mapping conflations to triangles. We write $\mathsf{D}^{\mathsf{b}}_{\infty}(\mathcal{E})$ for the corresponding stable ∞-category; in particular, the homotopy category of $\mathsf{D}^{\mathsf{b}}_{\infty}(\mathcal{E})$ recovers $\mathsf{D}^{\mathsf{b}}(\mathcal{E})$.

The derived category allows a construction of the *exact hull* $\mathcal{E}^{\mathrm{ex}}$ of $\mathcal{E}$ (see [6]): the exact hull is given by the extension closure of $\mathcal{E}$ in the (bounded) derived category $\mathsf{D}^{\mathsf{b}}(\mathcal{E})$. A sequence $X \to Y \to Z$ in $\mathcal{E}^{\mathrm{ex}}$ is a conflation if and only if it fits in a triangle $X \to Y \to Z \to \Sigma X$. The following proposition is shown in [6].

Proposition 2.3. *Let* $j\colon \mathcal{E} \to \mathcal{E}^{\mathrm{ex}}$ *be the embedding of an inflation-exact category in its exact hull.*

(1) *The embedding* $j\colon \mathcal{E} \to \mathcal{E}^{\mathrm{ex}}$ *is 2-universal among conflation-exact functors to exact categories.*
(2) *The embedding* j *lifts to an equivalence* $\mathsf{D}^{\mathsf{b}}_{\infty}(\mathcal{E}) \overset{\simeq}{\to} \mathsf{D}^{\mathsf{b}}_{\infty}(\mathcal{E}^{\mathrm{ex}})$ *of stable ∞-categories.*

2.2. Strictly percolating subcategories.

Definition 2.4. Let $\mathcal{E}$ be an exact category. A full subcategory $\mathcal{A} \subseteq \mathcal{E}$ is called a *strictly inflation-percolating subcategory* if the following properties are satisfied:

A1 The category $\mathcal{A}$ is a *Serre subcategory* of $\mathcal{E}$, i.e. for any conflation $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{E}$, we have $Y \in \mathcal{A} \Leftrightarrow X, Z \in \mathcal{A}$.

A2 Every morphism $f \colon A \to X$ with $A \in \mathcal{A}$ is *strict*, i.e. factors as $A \twoheadrightarrow \mathrm{im}(f) \rightarrowtail X$, and $\mathrm{im}(f) \in \mathcal{A}$.

Remark 2.5. If $\mathcal{A} \subseteq \mathcal{E}$ is a strictly inflation-percolating subcategory, then $\mathcal{A}$ is a fully exact abelian subcategory of $\mathcal{E}$.

The following observation will be of use later.

Proposition 2.6. *Let $\mathcal{E}$ be an inflation-exact category and let $\mathcal{V} \subseteq \mathcal{E}$ be a full additive subcategory satisfying axioms **A1** and **A2**. If every object of $\mathcal{V}$ is injective in $\mathcal{E}$, then $\mathcal{V}$ is a strictly inflation-percolating subcategory of the exact hull $\mathcal{E}^{\mathrm{ex}}$.*

Proof. As each $V \in \mathcal{V} \subseteq \mathcal{E}$ is injective, $\mathrm{Hom}_{\mathcal{E}}(-, V)$ is exact and hence $\mathrm{Ext}^1_{\mathcal{E}}(-, V) = 0$. As $\mathcal{E}^{\mathrm{ex}}$ is the extension-closure of $\mathcal{E}$ in $\mathcal{E}^{\mathrm{ex}}$, it follows that V is injective in $\mathcal{E}^{\mathrm{ex}}$ as well.

Note that $\mathcal{E}^{\mathrm{ex}} = \bigcup_{n \geq 0} \mathcal{E}_n$ where $\mathcal{E}_0 = \mathcal{E}$ and $\mathcal{E}_n$ for $n \geq 1$ is defined recursively as extensions of objects in $\mathcal{E}_{n-1}$. As $\mathcal{V} \subseteq \mathcal{E}^{\mathrm{ex}}$ consists of injective objects, it follows that $\mathcal{V} \subseteq \mathcal{E}^{\mathrm{ex}}$ is an extension-closed subcategory.

We now show axiom **A2**. Let $f \colon V \to Y$ be a map in $\mathcal{E}^{\mathrm{ex}}$ with $V \in \mathcal{V}$. By definition, there is an n such that $Y \in \mathcal{E}_n$. We proceed by induction on $n \geq 0$. If $n = 0$, the result follows as $\mathcal{V} \subseteq \mathcal{E}$ satisfies axiom **A2**. If $n \geq 1$, then there is a conflation $X \overset{\iota}{\rightarrowtail} Y \overset{\rho}{\twoheadrightarrow} Z$ in $\mathcal{E}^{\mathrm{ex}}$ with $X, Z \in \mathcal{E}_{n-1}$. By the induction hypothesis, the composition $\rho \circ f$ factors as $V \overset{p}{\twoheadrightarrow} V'' \overset{h}{\rightarrowtail} Z$ with $V'' \in \mathcal{V}$. As $\mathcal{V} \subseteq \mathcal{E}^{\mathrm{ex}}$ is an abelian subcategory, $(V' :=) \ker(p) \in \mathcal{V}$. Note that there is an induced map $g \colon V' \to X$ such that $fi = \iota g$. Again the induction hypothesis yields that g factors as $V' \overset{g'}{\twoheadrightarrow} U \overset{g''}{\rightarrowtail} X$ with $U \in \mathcal{V}$. Taking the pushout of g' along i in $\mathcal{E}^{\mathrm{ex}}$ yields the following commutative diagram (where the rows are conflations):

$$
\begin{array}{ccccc}
V' & \overset{i}{\rightarrowtail} & V & \overset{p}{\twoheadrightarrow} & V'' \\
\downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle f'} & & \| \\
U & \overset{i'}{\rightarrowtail} & W & \overset{p'}{\twoheadrightarrow} & V'' \\
\downarrow{\scriptstyle g''} & & \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle h} \\
X & \overset{\iota}{\rightarrowtail} & Y & \overset{\rho}{\twoheadrightarrow} & Z
\end{array}
$$

Here the upper-left square is bicartesian, $f''f' = f$ and $W \in \mathcal{V}$ as $\mathcal{V} \subseteq \mathcal{E}^{\mathrm{ex}}$ is extension-closed. It follows from [4, Corollary 3.2] that f'' is an inflation in $\mathcal{E}^{\mathrm{ex}}$. This shows axiom **A2**.

To show axiom **A1**, it remains to show that given a conflation $X \overset{\iota}{\rightarrowtail} V \overset{\rho}{\twoheadrightarrow} Z$ in $\mathcal{E}^{\mathrm{ex}}$ with $V \in \mathcal{V}$, X, Z belong to $\mathcal{V}$ as well. By axiom **A2**, ρ is admissible with image in $\mathcal{V}$. It follows that $Z \in \mathcal{V}$. As X is the kernel of a morphism in $\mathcal{V}$ and $\mathcal{V}$ is an abelian subcategory of $\mathcal{E}$, we know that X belongs to $\mathcal{V}$ as well. This concludes the proof. $\square$

2.3. Quotients by strictly percolating subcategories.
The next definition is based on [13, Definition 1.12].

Definition 2.7. Let $\mathcal{E}$ be an inflation-exact category and let $\mathcal{A} \subseteq \mathcal{E}$ be a strictly inflation-percolating subcategory. A morphism $f \colon X \to Y$ in $\mathcal{E}$ is called a *weak $\mathcal{A}$-isomorphism* (or simply a *weak isomorphism*) if f is strict and $\ker(f), \mathrm{coker}(f) \in \mathcal{A}$. The set of weak $\mathcal{A}$-isomorphisms is denoted by $S_{\mathcal{A}}$.

The following theorem summarizes the main results of [6, 7].

Theorem 2.8. *Let $\mathcal{A}$ be a strictly inflation-percolating subcategory of an exact category $\mathcal{E}$.*

(1) The set $S_{\mathcal{A}}$ of weak $\mathcal{A}$-isomorphisms is a left multiplicative system.

(2) The natural localization functor $Q \colon \mathcal{E} \to \mathcal{E}[S_{\mathcal{A}}^{-1}]$ endows the localization $\mathcal{E}[S_{\mathcal{A}}^{-1}]$ with an inflation-exact structure such that Q preserves and reflects conflations.

(3) The localization functor Q is also a quotient in the category of conflation categories, i.e. it satisfies the following universal property: if $F \colon \mathcal{E} \to \mathcal{F}$ is a conflation-exact functor between conflation categories such that $F(\mathcal{A}) = 0$, then F factors uniquely through Q via a conflation-exact functor $F' \colon \mathcal{E}/\mathcal{A} = \mathcal{E}[S_{\mathcal{A}}^{-1}] \to \mathcal{F}$.

Moreover, the localization sequence $\mathcal{A} \hookrightarrow \mathcal{E} \overset{Q}{\to} \mathcal{E}/\mathcal{A}$ induces a Verdier localization sequence on the bounded derived categories

$$\mathsf{D}^{\mathsf{b}}_{\mathcal{A}}(\mathcal{E}) \to \mathsf{D}^{\mathsf{b}}(\mathcal{E}) \to \mathsf{D}^{\mathsf{b}}(\mathcal{E}/\mathcal{A})$$

where $\mathsf{D}^{\mathsf{b}}_{\mathcal{A}}(\mathcal{E})$ is the thick subcategory of $\mathsf{D}^{\mathsf{b}}(\mathcal{E})$ generated by $\mathcal{A}$ under the canonical embedding $\mathcal{E} \hookrightarrow \mathsf{D}^{\mathsf{b}}(\mathcal{E})$.

If $\mathcal{A}$ has enough $\mathcal{E}$-injectives, then the natural embedding $\mathsf{D}^{\mathsf{b}}(\mathcal{A}) \hookrightarrow \mathsf{D}^{\mathsf{b}}_{\mathcal{A}}(\mathcal{E})$ is a triangle equivalence, and there is an exact sequence in $\mathrm{Cat}^{\mathrm{Ex}}_{\infty}$:

$$\mathsf{D}^{\mathsf{b}}_{\infty}(\mathcal{A}) \to \mathsf{D}^{\mathsf{b}}_{\infty}(\mathcal{E}) \to \mathsf{D}^{\mathsf{b}}_{\infty}(\mathcal{E}/\mathcal{A}).$$

3. STRUCTURE THEORY OF LOCALLY COMPACT MODULES OVER AN ORDER

Let LCA be the exact category of locally compact abelian groups, cf. [8]. Let A denote a finite-dimensional semisimple $\mathbb{Q}$-algebra and $\mathfrak{A} \subset A$ is a $\mathbb{Z}$-order, i.e. a subring of A which is a finitely generated $\mathbb{Z}$-module such that $\mathbb{Q} \cdot \mathfrak{A} = A$. In this section, we have a closer look at the category $\mathsf{LCA}_{\mathfrak{A}}$ of locally compact right $\mathfrak{A}$-modules.

Definition 3.1. We define the category $\mathsf{LCA}_{\mathfrak{A}}$ of locally compact right modules over $\mathfrak{A}$ as follows:

(1) An object $M \in \mathsf{LCA}_{\mathfrak{A}}$ is a right $\mathfrak{A}$-module such that the additive group $(M, +) \in \mathsf{LCA}$ is a locally compact group and such that right multiplication by any $\alpha \in M$ is a continuous endomorphism of $(M, +)$.
(2) A morphism $f \colon M \to N$ is a continuous right $\mathfrak{A}$-module map.

The following theorem is standard, see [3, Proposition 3.4 and Lemma 3.6] (based on the earlier [8]).

Theorem 3.2. *The category $\mathsf{LCA}_{\mathfrak{A}}$ is a quasi-abelian category; the inflations are given by closed injections and deflations are given by open surjections.*

Definition 3.3. We consider the following subcategories of $\mathsf{LCA}_{\mathfrak{A}}$:

(1) $\mathsf{LCA}_{\mathfrak{A},\mathsf{C}}$ denotes the full subcategory of compact $\mathfrak{A}$-modules.
(2) $\mathsf{LCA}_{\mathfrak{A},\mathsf{D}}$ denotes the full subcategory of discrete $\mathfrak{A}$-modules.
(3) $\mathsf{LCA}_{\mathfrak{A},\mathbb{R}}$ denotes the full subcategory of vector $\mathfrak{A}$-modules, i.e. those $\mathfrak{A}$-modules whose underlying locally compact abelian group is isomorphic to $\mathbb{R}^n$ for some finite $n \geq 0$.
(4) $\mathsf{LCA}_{\mathfrak{A},\mathsf{C}\mathbb{R}}$ denotes the full subcategory of $\mathfrak{A}$-modules which are a direct sum of a compact and a vector $\mathfrak{A}$-module.
(5) $\mathsf{LCA}_{\mathfrak{A},\mathsf{f}}$ denotes the full subcategory of finite $\mathfrak{A}$-modules.

The Structure Theorem for locally compact abelian groups extends to $\mathfrak{A}$-modules in the following sense (see [3, Lemma 6.5]).

Theorem 3.4 (Structure Theorem for locally compact modules). *For each $M \in \mathsf{LCA}_{\mathfrak{A}}$, there exists a (non-canonical) conflation*

$$C_M \oplus V_M \overset{i_M}{\rightarrowtail} M \overset{p_M}{\twoheadrightarrow} D_M$$

with $C_M \in \mathsf{LCA}_{\mathfrak{A},\mathsf{C}}, D_M \in \mathsf{LCA}_{\mathfrak{A},\mathsf{D}}$ and $V_M \in \mathsf{LCA}_{\mathfrak{A},\mathbb{R}}$.

It is well known that vector groups are both injective and projective in LCA (see for example [11, Corollary 3 to Theorem 3.3]). This result extends to $\mathfrak{A}$-modules (see [3, Theorem 5.13]).

Theorem 3.5. *The vector $\mathfrak{A}$-modules are both injective and projective in $\mathsf{LCA}_{\mathfrak{A}}$.*

The next lemma will be useful later.

Lemma 3.6. *Let $X \in \mathsf{LCA}_{\mathfrak{A},\mathsf{C}\mathbb{R}}$, thus $X \cong C \oplus V$ with $C \in \mathsf{LCA}_{\mathfrak{A},\mathsf{C}}$ and $V \in \mathsf{LCA}_{\mathfrak{A},\mathbb{R}}$.*

(1) If $X \rightarrowtail Y \twoheadrightarrow Z$ is a conflation with $Z \in \mathsf{LCA}_{\mathfrak{A},\mathsf{C}}$, then $Y \in \mathsf{LCA}_{\mathfrak{A},\mathsf{C}\mathbb{R}}$.
(2) If $f \colon X \twoheadrightarrow Y$ is a deflation, then $Y \in \mathsf{LCA}_{\mathfrak{A},\mathsf{C}\mathbb{R}}$.
(3) If $g \colon Y \rightarrowtail X$ is an inflation such that $Y \in \mathsf{LCA}_{\mathfrak{A},\mathsf{D}}$, then Y is a finitely generated $\mathfrak{A}$-module.

Proof. (1) As V is injective, the conflation $X \rightarrowtail Y \twoheadrightarrow Z$ is a direct sum of conflations $V \rightarrowtail V \twoheadrightarrow 0$ and $C \rightarrowtail C' \twoheadrightarrow Z$. Since $\mathsf{LCA}_{\mathfrak{A},\mathsf{C}}$ is closed under extensions, we find that $C' \in \mathsf{LCA}_{\mathfrak{A},\mathsf{C}}$, as required.

(2) Applying the Structure Theorem to Y yields a conflation $C' \oplus V' \rightarrowtail Y \twoheadrightarrow D$, with $D \in \mathsf{LCA}_{\mathfrak{A},\mathsf{D}}$. From the previous statement, we see that it suffices to show that $D \in \mathsf{LCA}_{\mathfrak{A},\mathsf{C}}$ (thus, D is finite). Write h for the composition $X \twoheadrightarrow Y \twoheadrightarrow D$ and consider the conflation $\ker h \rightarrowtail X \twoheadrightarrow D$. As $X \cong C \oplus V$ and $\mathrm{Hom}(V, D) = 0$, we see that this conflation is the direct sum of conflations $V \rightarrowtail V \twoheadrightarrow 0$ and $K \rightarrowtail C \twoheadrightarrow D$. As C is compact, we find that D is compact as well.

(3) It suffices to show that Y is finitely generated as an abelian group. As Y is a closed subgroup of X, the Pontryagin dual of [10, Chapter 2, Corollary 2 to Theorem 7] implies that Y is of the form $\mathbb{R}^m \oplus \mathbb{Z}^l \oplus D$ with D a finite (discrete) group. As Y is discrete by assumption, we see that Y is a finitely generated group. $\qquad\square$

We now interpret these results in the quotient category $\overline{\mathsf{LCA}}_{\mathfrak{A}} := \mathsf{LCA}_{\mathfrak{A}} / \mathsf{LCA}_{\mathfrak{A},\mathrm{f}}$. We write $\overline{\mathsf{LCA}}_{\mathfrak{A},\mathsf{D}}$ for the full subcategory of $\overline{\mathsf{LCA}}_{\mathfrak{A}}$ consisting of those objects which are discrete groups; the subcategory $\overline{\mathsf{LCA}}_{\mathfrak{A},\mathsf{C}\mathbb{R}}$ of $\overline{\mathsf{LCA}}_{\mathfrak{A}}$ is defined similarly.

Proposition 3.7. *The category $\mathsf{LCA}_{\mathfrak{A},\mathrm{f}}$ is a strictly two-sided percolating subcategory of $\mathsf{LCA}_{\mathfrak{A}}$. Moreover, the following hold:*

(1) the quotient $\mathsf{LCA}_{\mathfrak{A}} / \mathsf{LCA}_{\mathfrak{A},\mathrm{f}}$ is quasi-abelian,
(2) the localization $Q_f \colon \mathsf{LCA}_{\mathfrak{A}} \to \mathsf{LCA}_{\mathfrak{A}} / \mathsf{LCA}_{\mathfrak{A},\mathrm{f}}$ commutes with finite limits and colimits,
(3) the subcategories $\overline{\mathsf{LCA}}_{\mathfrak{A},\mathsf{D}}$ and $\overline{\mathsf{LCA}}_{\mathfrak{A},\mathsf{C}\mathbb{R}}$ of $\overline{\mathsf{LCA}}_{\mathfrak{A}}$ are closed under isomorphisms,
(4) $\overline{\mathsf{LCA}}_{\mathfrak{A},\mathsf{D}} = \mathsf{LCA}_{\mathfrak{A},\mathsf{D}} / \mathsf{LCA}_{\mathfrak{A},\mathrm{f}}$ and $\overline{\mathsf{LCA}}_{\mathfrak{A},\mathsf{C}\mathbb{R}} = \mathsf{LCA}_{\mathfrak{A},\mathsf{C}\mathbb{R}} / \mathsf{LCA}_{\mathfrak{A},\mathrm{f}}$,
(5) any morphism $X \to Y$ in $\overline{\mathsf{LCA}}_{\mathfrak{A},\mathsf{D}}$ is strict.

Proof. It is easy to verify that $\mathsf{LCA}_{\mathfrak{A},\mathrm{f}}$ is a strictly inflation- and deflation-percolating in $\mathsf{LCA}_{\mathfrak{A}}$. It follows from [7] that $\mathsf{LCA}_{\mathfrak{A}} / \mathsf{LCA}_{\mathfrak{A},\mathrm{f}}$ is quasi-abelian. As the set $S_{\mathsf{LCA}_{\mathfrak{A},\mathrm{f}}}$ is a left and a right multiplicative set, the localization Q_f commutes with finite limits and colimits. For (3), recall from Theorem 2.8 that $S_{\mathsf{LCA}_{\mathfrak{A},\mathrm{f}}}$ is saturated. So, we can reduce to showing that for any weak isomorphism $s \colon X \xrightarrow{\sim} Y$, we have that $X \in \mathsf{LCA}_{\mathfrak{A},\mathsf{D}}$ (or in $\mathsf{LCA}_{\mathfrak{A},\mathsf{C}\mathbb{R}}$) if and only if $Y \in \mathsf{LCA}_{\mathfrak{A},\mathsf{D}}$ (or in $\mathsf{LCA}_{\mathfrak{A},\mathsf{C}\mathbb{R}}$). As every weak isomorphism is a composition of inflations and deflations in (with cokernel and kernel in $\mathsf{LCA}_{\mathfrak{A},\mathrm{f}}$), we can furthermore assume that s is of this form. These cases are then easily handled separately.

The last two statements follow easily from (3). $\qquad\square$

Theorem 3.8 (Structure Theorem for $\overline{\mathsf{LCA}}_{\mathfrak{A}}$). *The pair $(\overline{\mathsf{LCA}}_{\mathfrak{A},\mathsf{C}\mathbb{R}}, \overline{\mathsf{LCA}}_{\mathfrak{A},\mathsf{D}})$ is a torsion pair in $\overline{\mathsf{LCA}}_{\mathfrak{A}}$, meaning that $\mathrm{Hom}(\overline{\mathsf{LCA}}_{\mathfrak{A},\mathsf{C}\mathbb{R}}, \overline{\mathsf{LCA}}_{\mathfrak{A},\mathsf{D}}) = 0$ and every $M \in \overline{\mathsf{LCA}}_{\mathfrak{A}}$ fits in a conflation*

$$C_M \overset{i_M}{\rightarrowtail} M \overset{p_M}{\twoheadrightarrow} D_M$$

with $C_M \in \overline{\mathsf{LCA}}_{\mathfrak{A},\mathsf{C}\mathbb{R}}$ and $D_M \in \overline{\mathsf{LCA}}_{\mathfrak{A},\mathsf{D}}$. Such a conflation is then unique up to unique isomorphism.

Proof. Directly from Theorem 3.4. $\qquad\square$

Corollary 3.9. *Let $X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z$ be a conflation in $\overline{\mathsf{LCA}}_{\mathfrak{A}}$.*

(1) If $Y \in \overline{\mathsf{LCA}}_{\mathfrak{A},\mathsf{C}\mathbb{R}}$, then $Z \in \overline{\mathsf{LCA}}_{\mathfrak{A},\mathsf{C}\mathbb{R}}$.
(2) If $Y \in \overline{\mathsf{LCA}}_{\mathfrak{A},\mathsf{D}}$, then $X, Z \in \overline{\mathsf{LCA}}_{\mathfrak{A},\mathsf{D}}$.
(3) If $Y \in \overline{\mathsf{LCA}}_{\mathfrak{A},\mathsf{C}\mathbb{R}}$ and $X \in \overline{\mathsf{LCA}}_{\mathfrak{A},\mathsf{D}}$, then X is finitely generated.

Proof. We only prove the first statement. The other statements can be proven in an analogous way. Let $X \overset{\sim}{\leftarrow} X' \overset{f'}{\to} Y$ be a roof in $\mathsf{LCA}_{\mathfrak{A}}$ representing f; in $\overline{\mathsf{LCA}}_{\mathfrak{A}}$, we have $\mathrm{coker}\, f \cong \mathrm{coker}\, f'$. It follows from Lemma 3.6 that $\mathrm{coker}\, f' \in \mathsf{LCA}_{\mathfrak{A},\mathsf{C}\mathbb{R}}$ and hence $\mathrm{coker}\, f \in \mathsf{LCA}_{\mathfrak{A},\mathsf{C}\mathbb{R}}$ by Proposition 3.7.(3). $\qquad\square$

4. *K*-THEORY OF LOCALLY COMPACT MODULES OVER AN ORDER

Throughout this section, A denotes a finite-dimensional semisimple $\mathbb{Q}$-algebra and $\mathfrak{A} \subset A$ is a $\mathbb{Z}$-order. The aim of this section is to show Theorem 1.1 from the introduction. We proceed in four steps.

4.1. **The localization $Q_\mathsf{C} \colon \mathsf{LCA}_{\mathfrak{A}} \to \mathsf{LCA}_{\mathfrak{A}} / \mathsf{LCA}_{\mathfrak{A},\mathsf{C}}$.** The following proposition (see [6]) reduces the study of localizing invariants of $\mathsf{LCA}_{\mathfrak{A}}$ (such as non-connective K-theory) to that of the quotient category $\mathsf{LCA}_{\mathfrak{A}} / \mathsf{LCA}_{\mathfrak{A},\mathsf{C}}$, which we shall call $\mathcal{E}$.

Proposition 4.1. *The subcategory $\mathsf{LCA}_{\mathfrak{A},\mathsf{C}} \subseteq \mathsf{LCA}_{\mathfrak{A}}$ is a strictly inflation-percolating subcategory. The quotient $Q_\mathsf{C} \colon \mathsf{LCA}_{\mathfrak{A}} \to \mathcal{E}(= \mathsf{LCA}_{\mathfrak{A}} / \mathsf{LCA}_{\mathfrak{A},\mathsf{C}})$ induces an exact sequence of stable ∞-categories*

$$\mathsf{D}^{\mathsf{b}}_{\infty}(\mathsf{LCA}_{\mathfrak{A},\mathsf{C}}) \to \mathsf{D}^{\mathsf{b}}_{\infty}(\mathsf{LCA}_{\mathfrak{A}}) \to \mathsf{D}^{\mathsf{b}}_{\infty}(\mathcal{E}).$$

As every object in $\mathsf{D}^{\mathsf{b}}_{\infty}(\mathsf{LCA}_{\mathfrak{A},\mathsf{C}})$ can be trivialized using an Eilenberg swindle with infinite products, for any localizing invariant K, there is an equivalence $K(\mathsf{LCA}_{\mathfrak{A}}) \simeq K(\mathcal{E})$.

4.2. **The functor** $Q_{\mathbb{R}} \colon \mathsf{LCA}_{\mathfrak{A}} / \mathsf{LCA}_{\mathfrak{A},\mathsf{C}} \to \mathcal{F}$. We now write $\mathcal{V}$ for the full additive subcategory of $\mathcal{E}$ generated by the vector $\mathfrak{A}$-modules. Our first goal is to show that $\mathcal{V}$ is a strictly inflation-percolating subcategory of $\mathcal{E}^{\mathrm{ex}}$, the exact hull of $\mathcal{E}$, so that we can consider the quotient $\mathcal{F} := \mathcal{E}^{\mathrm{ex}}/\mathcal{V}$. We start with the following lemma.

Lemma 4.2.

(1) *For any vector $\mathfrak{A}$-module V, the localization functor Q_{C} induces a natural equivalence*

$$Q_{\mathsf{C}} \colon \operatorname{Hom}_{\mathsf{LCA}_{\mathfrak{A}}}(-, V) \to \operatorname{Hom}_{\mathcal{E}}(Q_{\mathsf{C}}(-), Q_{\mathsf{C}}(V)).$$

In particular, it follows that V is injective in $\mathcal{E}$.

(2) *The category $\mathcal{V}$ is equivalent to the category $\operatorname{mod}(A_{\mathbb{R}})$ where $A_{\mathbb{R}} := \mathfrak{A} \otimes_{\mathbb{Z}} \mathbb{R}$. In particular, $\mathcal{V}$ is a fully exact abelian subcategory of $\mathcal{E}$.*

Proof.   (1) Let $f \in \operatorname{Hom}_{\mathsf{LCA}_{\mathfrak{A}}}(X, V)$ such that $Q_{\mathsf{C}}(f) = 0$. There exists a weak isomorphism $t \colon V \xrightarrow{\sim} Y$ in $S_{\mathsf{LCA}_{\mathfrak{A},\mathsf{C}}}$ (thus, with $\ker f \in \mathsf{LCA}_{\mathfrak{A},\mathsf{C}}$) such that $t \circ f = 0$ in $\mathsf{LCA}_{\mathfrak{A}}$. As the only compact submodule of V is trivial, t is a monomorphism. It follows that $f = 0$ in $\mathsf{LCA}_{\mathfrak{A}}$. This shows $\operatorname{Hom}_{\mathsf{LCA}_{\mathfrak{A}}}(X, V) \to \operatorname{Hom}_{\mathcal{E}}(X, V)$ is an injection.

To show that it is a surjection, let $g \in \operatorname{Hom}_{\mathcal{E}}(X, V)$ be represented by a roof $X \xrightarrow{f} Y \xleftarrow{s} V$ with $s \in S_{\mathsf{LCA}_{\mathfrak{A},\mathsf{C}}}$. Note that s is an inflation (as s is strict and V only has the trivial compact submodule). As V is injective in $\mathsf{LCA}_{\mathfrak{A}}$, the inflation $V \xrightarrow{s} Y$ is a coretraction. Let $t \colon Y \twoheadrightarrow V$ be the corresponding retraction, i.e. $ts = 1_V$. It follows that $Q_{\mathsf{C}}(t \circ f) = g$. This shows the desired bijection.
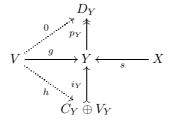
As V is injective in $\mathsf{LCA}_{\mathfrak{A}}$, $\operatorname{Hom}_{\mathsf{LCA}_{\mathfrak{A}}}(-, V)$ is an exact functor. Hence $\operatorname{Hom}_{\mathcal{E}}(-, V)$ is exact as well and thus V is injective in $\mathcal{E}$ (this characterization of injective objects remains valid for inflation-exact categories, see [6, Proposition 3.22]).

(2) This follows immediately from the above equivalence. $\square$

Proposition 4.3. *The category $\mathcal{V}$ is a strictly inflation-percolating subcategory of the exact hull $\mathcal{E}^{\mathrm{ex}}$ of $\mathcal{E}$.*

Proof. By Proposition 2.6 and Lemma 4.2.(1), it suffices to show that $\mathcal{V} \subseteq \mathcal{E}$ satisfies axioms **A1** and **A2**.

We first show axiom **A2**. Let $f \in \operatorname{Hom}_{\mathcal{E}}(V, X)$ be represented by a roof $V \xrightarrow{g} Y \xleftarrow{s} X$ with $s \in S_{\mathsf{LCA}_{\mathfrak{A},\mathsf{C}}}$. As the image of a connected space is connected, the Structure Theorem yields the following commutative diagram in $\mathsf{LCA}_{\mathfrak{A}}$:

Here V_Y is a vector $\mathfrak{A}$-module. As the projection $C_Y \oplus V_Y \twoheadrightarrow V_Y$ is an isomorphism in $\mathcal{E}$ and the composition $V \to V_Y$ is strict, we know that $V \to C_Y \oplus V_Y$ is strict in $\mathcal{E}$. As $\mathcal{E}$ is an inflation-exact category, axiom **L1** yields that the composition of inflations is an inflation. It follows that the composition $i_Y \circ h$ is strict in $\mathcal{E}$. This shows axiom **A2**.

We now show axiom **A1**, i.e. that $\mathcal{V}$ is a Serre subcategory of $\mathcal{E}$. Let $X \xrightarrow{\iota} Y \xrightarrow{\rho} Z$ be a conflation in $\mathcal{E}$. Assume that $Y \in \mathcal{V}$. By axiom **A2**, ρ is strict with image in $\mathcal{V}$, thus $Z \in \mathcal{V}$. As $X \cong \ker(\rho)$ is the kernel of a morphism in $\mathcal{V}$ and $\mathcal{V} \subseteq \mathcal{E}$ is an abelian subcategory by Lemma 4.2.(2), $X \in \mathcal{V}$.

Conversely, assume that $X, Z \in \mathcal{V}$. By Lemma 4.2.(1), X is injective in $\mathcal{E}$. It follows that the conflation (ι, ρ) splits and thus $Y \cong X \oplus Z$ belongs to $\mathcal{V}$. $\square$

Corollary 4.4. *The quotient functor $Q_{\mathbb{R}} \colon \mathcal{E}^{\mathrm{ex}} \to \mathcal{F}(:= \mathcal{E}^{\mathrm{ex}}/\mathcal{V})$ induces a fibre sequence*

$$K(\mathcal{V}) \to K(\mathcal{E}^{\mathrm{ex}}) \to K(\mathcal{F}),$$

where K is any localizing invariant.

Proof. By Lemma 4.2, we know that $\mathcal{V}$ contains enough $\mathcal{E}^{\mathrm{ex}}$-injective objects. The statement then follows from Theorem 2.8. $\square$
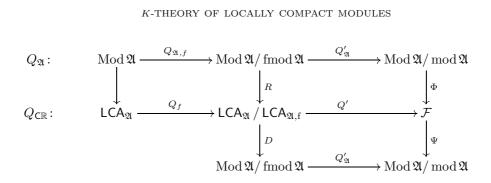
$$
\begin{array}{ccccc}
Q_{\mathfrak{A}}: & \operatorname{Mod}\mathfrak{A} \xrightarrow{\;Q_{\mathfrak{A},f}\;} \operatorname{Mod}\mathfrak{A}/\operatorname{fmod}\mathfrak{A} \xrightarrow{\;Q'_{\mathfrak{A}}\;} \operatorname{Mod}\mathfrak{A}/\operatorname{mod}\mathfrak{A} \\
& \downarrow \qquad\qquad \downarrow{\scriptstyle R} \qquad\qquad \downarrow{\scriptstyle \Phi} \\
Q_{\mathbb{C}\mathbb{R}}: & \operatorname{LCA}_{\mathfrak{A}} \xrightarrow{\;Q_f\;} \operatorname{LCA}_{\mathfrak{A}}/\operatorname{LCA}_{\mathfrak{A},f} \xrightarrow{\;Q'\;} \mathcal{F} \\
& \qquad\qquad \downarrow{\scriptstyle D} \qquad\qquad \downarrow{\scriptstyle \Psi} \\
& \operatorname{Mod}\mathfrak{A}/\operatorname{fmod}\mathfrak{A} \xrightarrow{\;Q'_{\mathfrak{A}}\;} \operatorname{Mod}\mathfrak{A}/\operatorname{mod}\mathfrak{A}
\end{array}
$$

FIGURE 1. Overview of the functors from Construction 4.8.

Proposition 4.5. *The functor $Q_{\mathbb{C}\mathbb{R}}\colon \operatorname{LCA}_{\mathfrak{A}} \to \mathcal{E} \to \mathcal{E}^{\mathrm{ex}} \to \mathcal{F}$ is 2-universal with respect to the conflation-exact functors $F\colon \operatorname{LCA}_{\mathfrak{A}} \to \mathcal{C}$ with $\mathcal{C}$ exact and $F(\operatorname{LCA}_{\mathfrak{A},\mathbb{C}\mathbb{R}}) = 0$, thus the functor $- \circ Q_{\mathbb{C}\mathbb{R}}\colon \operatorname{Fun}(\mathcal{F},\mathcal{C}) \to \operatorname{Fun}(\operatorname{LCA}_{\mathfrak{A}},\mathcal{C})$ is a fully faithful functor whose essential image consists of those $F\colon \operatorname{LCA}_{\mathfrak{A}} \to \mathcal{C}$ for which $F(\operatorname{LCA}_{\mathfrak{A},\mathbb{C}\mathbb{R}}) = 0$.*

Proof. From combining each of the universal properties of $\operatorname{LCA}_{\mathfrak{A}} \to \mathcal{E} \to \mathcal{E}^{\mathrm{ex}} \to \mathcal{F}$. $\qquad\square$

Remark 4.6. Note that in $\mathcal{E}$, we have $\mathfrak{A} \cong \mathbb{R} \otimes_{\mathbb{Z}} \mathfrak{A} \in \mathcal{V}$. In particular, $Q_{\mathbb{C}\mathbb{R}}(\mathfrak{A}) = 0$. Hence, $Q_{\mathbb{C}\mathbb{R}}$ sends every finitely generated discrete $\mathfrak{A}$-module to zero.

4.3. **The equivalence $\mathcal{F} \simeq \operatorname{Mod}\mathfrak{A}/\operatorname{mod}\mathfrak{A}$.** In order to complete the proof of Theorem 1.1, we show that $\mathcal{F} \simeq \operatorname{Mod}\mathfrak{A}/\operatorname{mod}\mathfrak{A}$ (see Proposition 4.9). For this, consider the localization functors $Q_{\mathfrak{A}}\colon \operatorname{Mod}\mathfrak{A} \to \operatorname{Mod}\mathfrak{A}/\operatorname{mod}\mathfrak{A}$ and $Q_{\mathfrak{A},f}\colon \operatorname{Mod}\mathfrak{A} \to \operatorname{Mod}\mathfrak{A}/\operatorname{fmod}\mathfrak{A}$, where $\operatorname{fmod}\mathfrak{A}$ is the full subcategory of $\operatorname{Mod}\mathfrak{A}$ consisting of finite $\mathfrak{A}$-modules. It follows from Proposition 3.7.(4) that $Q_f(\operatorname{LCA}_{\mathfrak{A},\mathsf{D}}) \simeq \operatorname{Mod}\mathfrak{A}/\operatorname{fmod}\mathfrak{A}$. Moreover, the universal property of $Q_{\mathfrak{A},f}$ shows that there is a unique functor $Q'_{\mathfrak{A}}\colon \operatorname{Mod}\mathfrak{A}/\operatorname{fmod}\mathfrak{A} \to \operatorname{Mod}\mathfrak{A}/\operatorname{mod}\mathfrak{A}$ such that $Q_{\mathfrak{A}} = Q'_{\mathfrak{A}} \circ Q_{\mathfrak{A},f}$.

The torsion-free part functor $D\colon \overline{\operatorname{LCA}}_{\mathfrak{A}} \to \overline{\operatorname{LCA}}_{\mathfrak{A},\mathsf{D}}\colon M \mapsto D_M$ from Theorem 3.8 need not be conflation-exact. This can be seen by setting $\mathfrak{A} = \mathbb{Z}$ and starting from the conflation $\mathbb{Z} \rightarrowtail \mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z}$. However, we need not change much to obtain a conflation-exact functor.

Proposition 4.7. *The functor $Q'_{\mathfrak{A}} \circ D\colon \operatorname{LCA}_{\mathfrak{A}}/\operatorname{LCA}_{\mathfrak{A},f} \to \operatorname{Mod}\mathfrak{A}/\operatorname{mod}\mathfrak{A}$ is conflation-exact.*

Proof. Let $X \rightarrowtail Y \twoheadrightarrow Z$ be a conflation in $\operatorname{LCA}_{\mathfrak{A}}/\operatorname{LCA}_{\mathfrak{A},f}$. The Structure Theorem of $\overline{\operatorname{LCA}}_{\mathfrak{A}}$ gives the following commutative diagram

$$
\begin{array}{ccccc}
C_X & \rightarrowtail & X & \twoheadrightarrow & D_X \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle g} \\
C_Y & \rightarrowtail & Y & \twoheadrightarrow & D_Y
\end{array}
$$

where the left vertical arrow is an inflation by the dual of [4, Proposition 7.6] and the rightmost vertical arrow is strict by Proposition 3.7.(5). Applying the Short Snake Lemma ([4, Corollary 8.13]), we find exact sequences $\ker g \xrightarrow{\;i\;} C_Y/C_X \to Z \twoheadrightarrow \operatorname{coker} g$ and $\ker g \rightarrowtail D_X \xrightarrow{\;g\;} D_Y \twoheadrightarrow \operatorname{coker} g$.

It follows from Corollary 3.9 that $C_Y/C_X \in \operatorname{LCA}_{\mathfrak{A},\mathbb{C}\mathbb{R}}$ and hence $\operatorname{coker} i \in \operatorname{LCA}_{\mathfrak{A},\mathbb{C}\mathbb{R}}$. Likewise, we find that $\operatorname{coker} g \in \operatorname{LCA}_{\mathfrak{A},\mathsf{D}}$. This shows that the conflation $\operatorname{coker} i \rightarrowtail Z \twoheadrightarrow \operatorname{coker} g$ is the torsion / torsion-free conflation of Z from Theorem 3.8, hence $\operatorname{coker} g \cong D_Z$.

Moreover, it follows from Corollary 3.9 that $\ker g$ is finitely generated and discrete. Hence, we find a conflation $Q'_{\mathfrak{A}}(D_X) \rightarrowtail Q'_{\mathfrak{A}}(D_Y) \twoheadrightarrow Q'_{\mathfrak{A}}(D_Z)$ in $\operatorname{Mod}\mathfrak{A}/\operatorname{mod}\mathfrak{A}$, as required. $\qquad\square$

Construction 4.8. We now construct the diagram given in Figure 1. We start with the rows. The functor $Q'_{\mathfrak{A}}$ is the unique functor such that $Q_{\mathfrak{A}} = Q'_{\mathfrak{A}} \circ Q_{\mathfrak{A},f}$, and Q' is the unique functor such that $Q_{\mathbb{C}\mathbb{R}} = Q' \circ Q_f$; these are induced by the universal properties of $Q_{\mathfrak{A},f}$ and Q_f, respectively.

For the columns, the functor $\operatorname{Mod}\mathfrak{A} \to \operatorname{LCA}_{\mathfrak{A}}$ is the functor mapping an $\mathfrak{A}$-module to the corresponding discrete $\mathfrak{A}$-module. The functor R is the unique functor making the top-left square commute (it exists by the universal property of $Q_{\mathfrak{A},f}$). The (essential) image of R corresponds to the torsion-free part of the torsion pair in Theorem 3.8, hence R has a left adjoint $D\colon \operatorname{LCA}_{\mathfrak{A}}/\operatorname{LCA}_{\mathfrak{A},f} \to \operatorname{Mod}\mathfrak{A}/\operatorname{fmod}\mathfrak{A}$, given by mapping any object $X \in \operatorname{LCA}_{\mathfrak{A}}/\operatorname{LCA}_{\mathfrak{A},f}$ to its torsion-free part D_X. By construction, $D \circ R \cong 1$.

In the last column, the functor $\Phi\colon \operatorname{Mod}\mathfrak{A}/\operatorname{mod}\mathfrak{A} \to \mathcal{F}$ is the unique functor making the top rectangle commute; it exists by the universal property of $Q_{\mathfrak{A}}\colon \operatorname{Mod}\mathfrak{A} \to \operatorname{Mod}\mathfrak{A}/\operatorname{mod}\mathfrak{A}$ (see Remark 4.6). Note that Φ is also the unique functor such that $\Phi \circ Q'_{\mathfrak{A}} = Q' \circ R$.

The functor $\Psi \colon \mathcal{F} \to \operatorname{Mod}\mathfrak{A}/\operatorname{mod}\mathfrak{A}$ is a functor such that $\Psi \circ Q_{\mathbb{C}\mathbb{R}} \cong Q'_{\mathfrak{A}} \circ D \circ Q_f$; it exists by the universal property of $Q_{\mathbb{C}\mathbb{R}}$ (see Proposition 4.5, note that $Q'_{\mathfrak{A}} \circ D \circ Q_f$ is conflation-exact by Proposition 4.7).

Proposition 4.9. *The functors Ψ and Φ are quasi-inverses.*

Proof. For each $M \in \mathsf{LCA}$, the map $p_M \colon M \mapsto R(D_M)$ corresponds to the unit of the adjunction $D \dashv R$. As $Q'(p_M)$ is an isomorphism, we find that $\Phi \circ \Psi \circ Q_{\mathbb{C}\mathbb{R}} \cong Q' \circ R \circ D \circ Q_f$ is isomorphic to $Q_{\mathbb{C}\mathbb{R}} = Q' \circ Q_f$. It follows from the universal property of $Q_{\mathbb{C}\mathbb{R}}$ that $1_{\mathcal{F}} \to \Phi \circ \Psi$ is a natural equivalence.

For the other direction, we start from $Q'_{\mathfrak{A}} \cong Q'_{\mathfrak{A}} \circ D \circ R = \Psi \circ \Phi \circ Q'_{\mathfrak{A}}$, so that the universal property of $Q'_{\mathfrak{A}}$ yields that $\Psi \circ \Phi \cong 1$, as required. $\qquad\square$

4.4. **Proof of Theorem 1.1.** We are now in a position to prove the main theorem.

Proof of Theorem 1.1. Consider the essentially commutative diagram

$$
\begin{array}{ccccc}
\operatorname{mod}(\mathfrak{A}) & \longrightarrow & \operatorname{Mod}(\mathfrak{A}) & \xrightarrow{\ Q_{\mathfrak{A}}\ } & \operatorname{Mod}(\mathfrak{A})/\operatorname{mod}(\mathfrak{A}) \\
\downarrow & & \downarrow & & {\scriptstyle \Phi}\downarrow{\scriptstyle \wr} \\
\mathcal{V} & \longrightarrow & \mathcal{E}^{\mathrm{ex}} & \longrightarrow & \mathcal{F}
\end{array}
$$

of functors, lifting to an essentially commutative diagram of the bounded derived ∞-categories (where the rows are exact sequences). It was shown in Lemma 4.2.(2) that $\mathcal{V} \simeq \operatorname{mod}\mathfrak{A}_{\mathbb{R}}$; the leftmost downwards arrow is given by $M \mapsto \mathbb{R} \otimes_{\mathbb{Z}} M$. This induces a bicartesian square of stable ∞-categories

$$
\begin{array}{ccc}
\mathsf{D}^{\mathsf{b}}_{\infty}(\operatorname{mod}\mathfrak{A}) & \longrightarrow & \mathsf{D}^{\mathsf{b}}_{\infty}(\operatorname{Mod}\mathfrak{A}) \\
\downarrow & & \downarrow \\
\mathsf{D}^{\mathsf{b}}_{\infty}(\operatorname{mod}\mathfrak{A}_{\mathbb{R}}) & \longrightarrow & \mathsf{D}^{\mathsf{b}}_{\infty}(\mathcal{E}^{\mathrm{ex}}).
\end{array}
$$

Using the Eilenberg swindle with direct sums, shows that every object in $\mathsf{D}^{\mathsf{b}}_{\infty}(\operatorname{Mod}\mathfrak{A})$ gets trivialized under a localizing invariant $K \colon \mathsf{Cat}^{\mathrm{Ex}}_{\infty} \to \mathcal{A}$. Hence, for each such K, there is a fiber sequence $K(\mathsf{D}^{\mathsf{b}}_{\infty}(\operatorname{mod}\mathfrak{A})) \to K(\mathsf{D}^{\mathsf{b}}_{\infty}(\operatorname{mod}\mathfrak{A}_{\mathbb{R}})) \to K(\mathsf{D}^{\mathsf{b}}_{\infty}(\mathcal{E}^{\mathrm{ex}}))$. Combining Theorem 2.3 and Proposition 4.1, we find that $K(\mathsf{D}^{\mathsf{b}}_{\infty}(\mathsf{LCA}_{\mathfrak{A}})) \simeq K(\mathsf{D}^{\mathsf{b}}_{\infty}(\mathcal{E})) \simeq K(\mathsf{D}^{\mathsf{b}}_{\infty}(\mathcal{E}^{\mathrm{ex}}))$. Using our convention to suppress $\mathsf{D}^{\mathsf{b}}_{\infty}$ in the notation whenever convenient, this yields the required fiber sequence as formulated in the introduction. $\quad\square$

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