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Counterexamples to a conjecture of Las Vergnas

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Abstract

We present counterexamples to a 30-year-old conjecture of Las Vergnas [J. Combin. Theory Ser. B, 1988] regarding the Tutte polynomial of binary matroids.

Based on an evaluation established for the Tutte polynomial of plane graphs on $(3, 3)$, Michel Las Vergnas made three conjectures in [LV88], in increasing strength, regarding the Tutte polynomial of binary matroids. The first and weakest of these [LV88, Conjecture 4.1] was proved in [Jae89] and in a more general setting in [Bou91].

Theorem 1 ([Jae89, Bou91])**.** *For every binary matroid M, the value $T_M(3,3)/T_M(-1,-1)$ is an odd integer.*

We remark that, for a binary matroid M, $T_M(-1,-1) = (-1)^{|E(M)|}(-2)^{b(M)}$, where $b(M)$ is the dimension of the bicycle space of M [RR78, Theorem 9.1]. The other two conjectures remained open for a long time (and were recalled again in 2004 in [ELV04]). It was shown by Gordon Royle [Roy13] that $M(K_8)$ is a counterexample to the third and strongest of the three conjectures [LV88, Conjecture 4.3]. In fact, an exhaustive search using the dataset of binary matroids with at most 15 elements of [FW11] reveals several more counterexamples.

We now state the second conjecture [LV88, Conjecture 4.2], which is stronger than the first and weaker than the third conjecture.

Conjecture 2 ([LV88])**.** *For every binary matroid M and every integer z, the value $T_M(-1+4z,-1+4z)/T_M(-1,-1)$ is an odd integer.*

Thus Theorem 1 corresponds to the value $z = 1$ in Conjecture 2. It turns out that $M(K_8)$ is not a counterexample to this conjecture. Also, an exhaustive search using the above-mentioned dataset of [FW11] reveals no counterexample to this conjecture. Consequently, any counterexample has at least 16 elements. In fact, for each binary matroid M with less than 16 elements, $Q_M(z) := T_N(-1+4z,-1+4z)/T_N(-1,-1)$ turns out to have only integer coefficients. This, together with the fact that both $Q_M(0) = 1$ and $Q_M(1)$ are odd (the latter by Theorem 1), implies that $Q_M(z)$ is an odd integer for all integers z.

Using SageMath [Sage] we found that the binary matroid G with 24 elements corresponding to the extended binary Golay code (see, e.g., the appendix of [Oxl11] for a definition) is a counterexample to Conjecture 2. Moreover, the rank-6 minor N of G with 18 elements having the following reduced representation over GF(2)

$$\begin{pmatrix}
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1
\end{pmatrix}$$

is another counterexample. Indeed, N has the following Tutte polynomial $T_N(x, y)$

$$y^{12} + 6y^{11} + 21y^{10} + 56y^9 + 126y^8 + 252y^7 + x^6 + 45xy^5 + 462y^6 + 12x^5 + 6x^4y + 225xy^4 + 747y^5 + 72x^4 + 111x^3y + 240x^2y^2 + 675xy^3 + 1017y^4 + 247x^3 + 591x^2y + 1095xy^2 + 1057y^3 + 417x^2 + 909xy + 723y^2 + 231x + 231y.$$

We have $T_N(-1, -1) = 2^6$ and $Q_N(z) = T_N(-1 + 4z, -1 + 4z)/T_N(-1, -1)$ is equal to

$$262144z^{12} - 393216z^{11} + 344064z^{10} - 180224z^9 + 73728z^8 - 18432z^7 + 8320z^6 - 1248z^5 + 2616z^4 - 1012z^3 + \frac{195}{2}z^2 - \frac{15}{2}z + 1.$$

Consequently, $Q_N(z)$ is even for $z \in \{-2, -1, 2\}$, contradicting Conjecture 2.

Finally, the self-dual (but not identically self-dual) rank-9 minor N' of G having the following reduced representation over GF(2)

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

is yet another counterexample with 18 elements.

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References

[Bou91] A. Bouchet. Tutte-Martin polynomials and orienting vectors of isotropic systems. *Graphs and Combinatorics*, 7(3):235–252, 1991.

[ELV04] G. Etienne and M. Las Vergnas. The Tutte polynomial of a morphism of matroids III. Vectorial matroids. *Advances in Applied Mathematics*, 32:198–211, 2004.

[FW11] H. Fripertinger and M. Wild. *A catalogue of small regular matroids and their Tutte polynomials*, 2011. arXiv:1107.1403. Dataset at `imsc.uni-graz.at/fripertinger/html/matroids/matroide_neu.html`. Retrieved: July 24, 2018.

[Jae89] F. Jaeger. On Tutte polynomials of matroids representable over GF(q). *European Journal of Combinatorics*, 10:247–255, 1989.

[LV88] M. Las Vergnas. On the evaluation at $(3, 3)$ of the Tutte polynomial of a graph. *Journal of Combinatorial Theory, Series B*, 45(3):367–372, 1988.

[Oxl11] J. Oxley. *Matroid theory, Second Edition.* Oxford University Press, 2011.

[Roy13] G. Royle. A Las Vergnas conjecture, 2013. URL `symomega.wordpress.com/2013/06/30/a-las-vergnas-conjecture/`.

[RR78] P. Rosenstiehl and R. Read. On the principal edge tripartition of a graph. In B. Bollobás, editor, *Advances in Graph Theory*, volume 3 of *Annals of Discrete Mathematics*, pages 195–226. Elsevier, 1978.

[Sage] *SageMath, the Sage Mathematics Software System.* URL `sagemath.org`.