Polar coordinates and generalized h-type indices

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Highlights

- the study of generalized h- and g-indices is continued
- a new definition is proposed
- this definition applies to functions which are not necessary decreasing
- these generalize indices are essentially polar coordinates

Polar coordinates and generalized h-type indices

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Abstract

This article highlights the importance of using polar coordinates when studying functions, in particular in relation to generalized h-type indices. Concretely, generalized h-type indices are essentially polar coordinates. This observation ties informetric ideas to standard mathematics. This article is essentially meant to provide tools for further studies.

Keywords: polar coordinates; generalized h-index; generalized g-index

1. Introduction

A. Functions

Let X and Y be sets. A relation between X and Y is any subset of the Cartesian product $X_x Y = \{(x,y), x \in X, y \in Y\}$. A relation is said to be a function, denoted as f, if with any element $x \in X$ (or a subset A of X) corresponds exactly one element $y \in Y$. This is denoted as

$$f: A \subset X \rightarrow Y: x \rightarrow y = f(x) \tag{1}$$

The set A is called the domain of the function f, denoted as $\boldsymbol{D}(f)$ and the set $f(A) = \{y \in Y: \exists x \in A \; such \; that \; f(x) = y\}$ is called the range of f, denoted as $\boldsymbol{R}(f)$. If each $y \in \boldsymbol{R}(f)$ is the image of exactly one x in A, then f is said to be injective.

In expression (1) we have defined a function f in an explicit manner: we describe to which element y each element x is mapped. When working with real-valued functions one may consider the set $\{(x,f(x)) \subset \mathbf{R}^2\}$ which is called the graph of the function f. Besides in an explicit manner it often happens that functions are described implicitly. This means that an expression $F(x,y) = 0$ is given, from which it is theoretically possible to find y, given x. Each real-valued function given in explicit form, $y = f(x)$, can be rewritten in an implicit form, namely as $y - f(x) = 0$. The point is that the reverse is often not possible in practice, because $F(x,y) = 0$ may only define a relation and not a function. The circle provides a simple example. Indeed, $x^2 + y^2 - 1 = 0$ leads to a graph which is a circle with center in the origin of a Cartesian coordinate system and with radius 1. Yet, this relation does not correspond to a unique function. If we wish so we may derive two functions from it, namely

$$f_1 : [-1, +1] \rightarrow \mathbf{R} : x \rightarrow y = +\sqrt{1 - x^2}$$

which corresponds to the upper half of the circle, and

$$f_2 :]-1, +1[\rightarrow \mathbf{R} : x \rightarrow y = -\sqrt{1 - x^2}$$

which corresponds to its lower half. If we define f_2 also on [-1, +1] then the two graphs have two points in common.

Points in the two-dimensional plane can uniquely be described by their Cartesian coordinates x and y. Yet such a point can also be described by polar coordinates, among others. These are described in the next section.

B. Classical polar coordinates

For simplicity we will work in this article in the first quadrant of the Euclidean plane, hence we restrict our explanation to this case. Working in the first quadrant means that in Cartesian coordinates (x,y), $x \geq 0$ and $y \geq 0$. Points in this quadrant can also be described using polar coordinates: $\varphi \in [0, \pi/2]$ and $\rho \geq 0$, with ρ a function of φ. Let P be a point with Cartesian coordinates (x,y). Then $\rho(\varphi)$ denotes the distance between the origin and the point P and is called the radial distance, while φ denotes the angle, expressed in radians, between the x-axis and the line segment connecting the origin and the point P. It is called the polar

angle (actually 'a' polar angle as φ and φ+2kπ, for k any integer, lead to the same Cartesian point).

In the first quadrant, the relation between Cartesian coordinates (x,y) and polar coordinates (φ, ρ) is given by

$$x = \rho \cos(\varphi) \text{ and } y = \rho \sin(\varphi) \text{ (φ expressed in radians)}$$

and, vice versa: $\varphi = arctan\left(\frac{y}{x}\right)$ and $\rho = \sqrt{x^2 + y^2}$, if (x,y) ≠ (0,0); the origin corresponds to ρ = 0 and its polar angle can have any value.

If y=Z(x) in Cartesian coordinates, then ρ_Z.sin(φ) = Z(ρ_Z.cos(φ)) is the implicit equation of the graph of Z in polar coordinates.

Sometimes it is easy to go from the Cartesian to the polar equation such as in the case $y = \sqrt{R^2 - x^2}$ (R > 0), which leads to ρ = R, the part of the circle with origin (0,0) and radius R, situated in the first quadrant. Yet, in many cases it is not possible to describe explicitly a Cartesian equation in polar coordinates. An example is y(x) = x^a + b, a < 0 for which no explicit expression in polar coordinates exists.

C. Generalized continuous h-type indices

Before continuing our discussion we like to point out that although this article can be placed within a series of studies on generalized h-type indices (Egghe & Rousseau, 2019a,b,c,d) no prior knowledge of these articles is necessary to understand what is presented in the following sections. Occasionally, though, we will refer to these earlier articles to point out that a result has already been obtained.

Definition

Given a positive function Z and a real number θ ∈]0, +∞[, we define h_θ(Z) as the x-coordinate of the unique intersection, if it exists, of the line y = θx with the graph of y = Z(x).

Hence h_θ(Z) is characterized by:

$$Z\big(h_\theta(Z)\big) = \theta\, h_\theta(Z) \tag{2}$$

We further consider the function $h(Z): \theta \rightarrow h_\theta(Z)$, defined on its domain, $\boldsymbol{D}$(h(Z)). Even if this domain exists, it may consist of a single point.

Consider e.g. the function $Z(x) = x^2+1$. Then it is easy to check that $\boldsymbol{D}(h(Z))$ = {2} as for other values of θ the line $y = \theta x$ has either no or two intersection points with $Z(x)$. For $\theta = 2$, we find $h_2(x^2+1) = 1$. This example has been provided by a reviewer for which we thank him/her.

Note that here $\theta = \tan(\varphi)$ denotes a slope, not an angle, and that h_θ is a function from a set of functions to the real numbers. The function h_θ is called the generalized continuous h-index. In our previous work (Egghe & Rousseau, 2019c) we used this definition only for decreasing functions, defined on an interval of the form [0,T]. Here, however, this is not required anymore.

If Z is strictly decreasing, $\theta = 1$, $h_\theta = h_1$ is the continuous h-index as introduced in (Egghe & Rousseau, 2006). In the discrete case h_1 has been introduced by Hirsch (2005), while the general h_θ has been introduced in (van Eck & Waltman, 2008) under the name of h_α-index.

2. A new existence result for h_θ

In previous work we proved that h_θ exists for positive, non-constant, decreasing, differentiable functions defined on an interval [0,T] (Egghe & Rousseau, 2019c; Proposition 1). Now we will show the (unique) existence of h_θ for a larger family of functions.

In what follows we will only use positive functions $Z(x)$ such that for every $x \in \boldsymbol{D}(Z)$, there exists a unique $\theta > 0$ such that $Z(x) = \theta x$.

This condition implies that a function such as $Z(x) = x^2+1$ is not covered by our theory. The theory does apply to positive, decreasing, continuous functions defined on an interval [0,T], except for the case $Z(x) = 0$.

Theorem 1

Given the positive function $Z(x)$, defined on $A \subset]0, +\infty[$ then the complete graph of $Z(x)$ can be described in polar coordinates $\rho_Z : \boldsymbol{D}(\rho_Z) \subset [0,\pi/2] : \varphi \rightarrow \rho_Z(\varphi)$, with $\theta = \tan(\varphi)$, if and only if $h_\theta(Z)$ is defined for every $\theta \in \boldsymbol{D}(h(Z)) \subset]0, +\infty[$ with $\boldsymbol{D}(h(Z)) = \boldsymbol{R}(Z(x)/x)$.

Proof. If $(x, Z(x))$ is a point on the graph of Z, then $(\varphi, \rho_Z(\varphi))$ is its polar form if ρ_Z is equal to the distance between $(0,0)$ and $(x, Z(x))$ and $\varphi = \arctan(Z(x)/x)$. We have to check if $h(Z)$ is a function of θ and if for every

$x \in \boldsymbol{D}(Z)$ there exists a unique θ such that $Z(x)/x = \theta$. By the definition of h, uniqueness is the same as stating that $\tan(\boldsymbol{D}(\rho_Z)) \subset \boldsymbol{D}(h(Z))$. Moreover, we know that $\boldsymbol{D}(h(Z)) \subset \boldsymbol{R}(Z(x)/x)$. Now, as we work with functions $Z(x)$ such that for every $x \in \boldsymbol{D}(Z)$, there exists a unique $\theta > 0$ such that $Z(x) = \theta x$, this implies that $\boldsymbol{R}(Z(x)/x) \subset \boldsymbol{D}(h(Z))$. Hence we conclude that $\boldsymbol{R}(Z(x)/x) = \boldsymbol{D}(h(Z))$.

Conversely, if $\boldsymbol{D}(h(Z)) = \boldsymbol{R}(Z(x)/x)$, then for every point $(x, Z(x))$ on the graph of Z, $\varphi = \arctan(\theta)$ is well-defined and ρ is the distance from $(0,0)$ to $(x, Z(x))$.

Corollary. If $Z(x)$ is continuous and decreasing on $]0, +\infty[$ then $h_\theta(Z)$ is uniquely defined.

Proof. If $Z(x)$ is decreasing, then $Y(x) = Z(x)/x$ is strictly decreasing and hence a bijection to $\boldsymbol{R}(Z(x)/x)$. Therefore $h_\theta(Z)$ is defined for $\theta = Z(x)/x$ for every $x \in \boldsymbol{D}(Z)$, i.e., $\boldsymbol{D}(h(Z)) = \boldsymbol{R}(Z(x)/x)$. Now the corollary follows from the previous theorem.

Remark

A constant function $y = C > 0$ is an example of a function which is not strictly decreasing and for which $h_\theta(Z) = C/\theta$ exists for every $\theta \in]0, +\infty[$.

3. Generalized h-indices lead to a description of the Cartesian equation in polar coordinates

Theorem 2. Let Z be a strictly positive function with $\boldsymbol{D}(Z) \subset \mathbb{R}_0^+$. Suppose further that for every $x \in \boldsymbol{D}(Z)$, there exists a unique $\theta > 0$ such that $Z(x) = \theta x$. Then, $y = Z(x)$, for $x \in \boldsymbol{D}(Z)$, has a polar form:

$$\rho_Z(\varphi) = h_{tan(\varphi)}(Z).\sqrt{1 + tan^2(\varphi)} = h_\theta(Z).\sqrt{1 + \theta^2}, \text{ with } \theta = \tan(\varphi).$$

Proof. For x in the domain of Z, and $y = Z(x)$ we have by definition: $(\rho(x))^2 = x^2 + (Z(x))^2$. As we only deal with positive functions $Z(x)$ such that for every $x \in \boldsymbol{D}(Z)$, there exists a unique $\theta > 0$ such that $Z(x) = \theta x$, we can

write: $(\rho(h_\theta(Z)))^2 = (h_\theta(Z))^2 + (Z(h_\theta(Z)))^2 = (h_\theta(Z))^2 + \theta^2(h_\theta(Z))^2 = (h_\theta(Z))^2(1 + \theta^2)$.

Consequently: $\rho_Z(\varphi) = h_\theta(Z).\sqrt{1+\theta^2}$. If we put $\theta = \tan(\varphi)$ (as tan is a bijection between [0, +∞[and [0, π/2[ ), this leads to:

$$\rho_Z(\varphi) = h_{\tan(\varphi)}(Z).\sqrt{1+\tan^2(\varphi)} \qquad (3)$$

Remark. Equation (3) can be considered as an explicit expression of Z in the polar coordinates (φ, ρ), thanks to the generalized form of the h-index. Note though that equation (3) only holds for φ for which $h_{\tan(\varphi)}(Z)$ is defined. Moreover, $\rho_Z(\varphi)$ is an explicit form for the polar equation and an explicit functional form if $h_{\tan(\varphi)}(Z)$ is given as an explicit expression in φ.

Corollary. If the polar form of Z is known then equation (3) gives an analytic formula for the calculation of $h_\theta(Z)$. Indeed, for all θ > 0, such that $h_\theta(Z) = h_{\tan(\varphi)}(Z)$ is defined, we have:

$$h_\theta(Z) = \frac{\rho_Z(\varphi)}{\sqrt{1+\theta^2}} = \frac{\rho_Z(\arctan(\theta))}{\sqrt{1+\theta^2}} \qquad (4)$$

4. Cartesian polar coordinates in the first quadrant: Another set of coordinates

Besides coordinates (x,y) and (φ, ρ), as recalled in sections 1A and 1B, we may also introduce coordinates (θ, h) which we refer to as Cartesian polar coordinates. The relations between these three coordinate systems are:

$$x = h, \ y = \theta h; \quad \varphi = \arctan(\theta), \ \rho = h\sqrt{1+\theta^2}\ .$$

As an illustration we check what it means that a basic coordinate is constant (C > 0).

a) Cartesian coordinates: x = C; y = C;

y = C is a horizontal straight line; x = C is not a function of x (it is not of the form y(x)) as defined in section 1A. Yet, it may be considered as a

function x(y), and we know that x = C represents a vertical straight line, which has meaning if we consider the (y,x)-plane instead of the (x,y)-plane

b) polar coordinates: ρ = C; ϕ = C ;

The equation ρ = C, in polar coordinates represents a circle with center in the origin of the plane and radius with length equal to R;

ϕ = C; also here there is a logical problem as in polar coordinates we want ρ as a function of ϕ. Yet it is clear that ϕ = C represents a straight line through the origin and with slope equal to tan(C).

c) Cartesian polar coordinates: h = C , θ = C

h = C represents a vertical straight line

θ = C represents a straight line through the origin with slope θ

We would also like to mention that in the classical triangle with Cartesian coordinates (0,0), (x,0), (x, Z(x)), the lengths of the three sides can be considered the second coordinates in each of these three coordinate systems. Indeed, the vertical side corresponds to the ordinate in Cartesian coordinates; the hypotenuse corresponds to the ordinate in polar coordinates, and the horizontal side corresponds to the ordinate in Cartesian polar coordinates.

5. Examples

1) The quarter circle ρ_Z = C, constant for ϕ = [0, π/2]. Let $Z(x) = \sqrt{C^2 - x^2}$ for x ∈ [0, C]. This graph is completely described by the polar function $\rho(\phi)$ = C. By Theorem 1, $h_\theta(Z)$ is defined and from eq. (4) we derive that

$h_\theta(Z) = \frac{C}{\sqrt{1+\theta^2}}$; in particular: h = h_1 = $\frac{C}{\sqrt{2}}$. We next show that this result can also be derived from the definition (in Cartesian coordinates).

$$Z(x) = y = \sqrt{C^2 - x^2}, x > 0.$$

Now x = h_θ(Z) iff $\sqrt{C^2 - x^2} = \theta x$

$$\Leftrightarrow C^2 - x^2 = \theta^2 x^2, x > 0 \Leftrightarrow (1 + \theta^2)x^2 = C^2 \underset{x>0}{\Longleftrightarrow} x = \frac{C}{\sqrt{1 + \theta^2}}$$

This confirms the previous result derived from eq. (4). The major benefit of equation (4) is that it can be applied for any function Z(x), even in cases where $h_\theta(Z)$ is very difficult or even impossible to be calculated from the definition, i.e. from eq. (2). This is further illustrated in the following examples.

2) The Archimedean spiral $\rho_Z(\varphi) = a + b\varphi;\ a, b > 0;\ \varphi \in [0, \frac{\pi}{2}]$. In general, this is not a function Z(x). Yet there exist α > a and $\psi \in \left]0, \frac{\pi}{2}\right[$ such that Z(x) is a function defined on [0, α], with polar graph ρ(φ) = a + bφ, for $\varphi \in [\psi, \pi/2]$. Then applying eq. (4) yields:

$$h_\theta(Z) = \frac{a + b\arctan(\theta)}{\sqrt{1 + \theta^2}}$$

In particular: if $\frac{\pi}{4} > \psi$ then h = $h_1 = \frac{a + \frac{\pi b}{4}}{\sqrt{2}}$.

3) Conic sections. The following equation $\rho(\varphi) = \frac{a}{1 - e.\cos(\varphi)}$ with a > 0 and e.cos(φ) < 1 represents a conic section in polar coordinates. The parameter e is called the eccentricity of the conic section. If e > 1 the conic section is a hyperbola, if e = 1 it is a parabola and if 0 ≤ e < 1 it is an ellipse. In the special case that e = 0 we have a circle with radius a. Applying formula (4) yields:

$$h_\theta(Z) = \frac{\rho_Z(\varphi)}{\sqrt{1 + \theta^2}} = \frac{\rho_Z(\arctan(\theta))}{\sqrt{1 + \theta^2}} = \frac{a}{(1 - e.\cos(\arctan(\theta)))\sqrt{1 + \theta^2}}$$

$$= \frac{a}{\left(1 - e.\frac{1}{\sqrt{1+\theta^2}}\right)\sqrt{1+\theta^2}} = \frac{a}{\sqrt{1+\theta^2} - e}$$

In this result, y= Z(x) is the graph of the portion of the conic section in the first quadrant and results in polar coordinates ρ(φ) with the following restrictions. For e ≤ 1, $\varphi \in \left]0, \frac{\pi}{2}\right]$ and θ > 0, but for e > 1, $\varphi \in \left]\arccos\left(\frac{1}{e}\right), \frac{\pi}{2}\right]$ and θ > $\sqrt{e^2 - 1}$. We note that if e = 0, h_θ(Z) always exists and is equal to $\frac{a}{\sqrt{1+\theta^2}}$, as already shown in example 1). If θ = 1, h_1 = h = $\frac{a}{\sqrt{2} - e}$.

4) A general power function

We consider the function $Z(x) = C\,x^a$, $x > 0$. If $a > 0$, this function is increasing; if $a < 0$ it is decreasing and if $a = 0$ we have a constant function. The equation of this general power function in polar coordinates is:

$$\rho_Z(\varphi)sin(\varphi) = C\big(\rho_Z(\varphi)cos(\varphi)\big)^a$$

With $\theta = tan(\varphi)$ we obtain:

$$\rho_Z(\arctan(\theta))sin(\varphi) = C\big(\rho_Z(\arctan(\theta))cos(\varphi)\big)^a$$

Or:

$$\rho_Z(\varphi) = \left(\frac{sin(\varphi)}{C(cos(\varphi))^a}\right)^{1/(a-1)}$$

Therefore, Theorem 1 holds, and we may apply equation (4) leading to:

$$h_\theta(Z) = \frac{1}{\sqrt{1+\theta^2}}\left(\frac{sin(\arctan(\theta))}{C(cos(\arctan(\theta)))^a}\right)^{\frac{1}{a-1}}$$

$$= \frac{1}{\sqrt{1+\theta^2}}\left(\frac{\dfrac{\theta}{\sqrt{1+\theta^2}}}{C\left(\dfrac{1}{\sqrt{1+\theta^2}}\right)^a}\right)^{\frac{1}{a-1}} = \left(\frac{\theta}{C}\right)^{\frac{1}{a-1}}$$

We see that if $a = 0$,

$$h_\theta(Z) = \frac{C}{\theta}$$

a result already obtained in (Egghe & Rousseau, 2019c; Example 1).

If $a < 0$ we obtain a Zipf function; if it is moreover defined on [0, T] then we denoted it, e.g. in (Egghe & Rousseau, 2019c, d) as:

$$Z(x) = \frac{B}{x^\beta}, \quad x \in \,]0, T], \; with\; B = T^\beta$$

Now the parameters a and C in this example, become $-\beta$ and T^β, leading to:

$$h_\theta(Z) = \left(\frac{\theta}{C}\right)^{\frac{1}{a-1}} = \left(\frac{\theta}{T^\beta}\right)^{\frac{1}{-(\beta+1)}} = \frac{T^{\left(\frac{\beta}{\beta+1}\right)}}{\theta^{\left(\frac{1}{\beta+1}\right)}}$$

This result was already obtained in (Egghe & Rousseau, 2019c, example 1). Transforming the Zipf (rank-frequency) form to the Lotka (size-frequency) form i.e. a function of the form $L(n) = \frac{K}{n^\alpha}$, K > 0, α > 1, where, in the original version (Lotka, 1926) n refers to a number of publications and L(n) refers to the number of authors with n publications, see e.g. (Egghe, 2005; Egghe & Rousseau, 2019d) leads to: $\beta = \frac{1}{\alpha-1}$ and hence $\frac{\beta}{\beta+1} = \frac{1}{\alpha}$. Using these parameters gives:

$$h_\theta(Z) = \theta^{\left(\frac{1-\alpha}{\alpha}\right)}T^{\frac{1}{\alpha}}$$

A result already obtained in (Egghe & Rousseau, 2019c).

6. The generalized g-index

Given a function Z(x), defined on [0, +∞[and a real number θ > 0, set $\Gamma(x) = \frac{1}{x}\int_0^x Z(s)ds$, for x > 0. Clearly we have to require that this integral exists. This is always the case if Z is a piecewise continuous function or if Z(x) can be written as a difference of two decreasing (or increasing) functions (Apostol, 1957). Then we define $g_\theta(Z)$ as the x-coordinate of the unique intersection, if it exists, of the line y = θx with the graph of Γ(x).

Hence $g_\theta(Z)$ is characterized by:

$$\Gamma\big(g_\theta(Z)\big) = \theta g_\theta(Z) \tag{5}$$

If Z is decreasing and defined on an interval [0,T] then, if θ = 1, $g_\theta = g_1$ is the continuous version of the g-index as introduced in (Egghe, 2006). Again, for discrete data the general g_θ-index has been introduced in (van Eck & Waltman, 2008), for all θ > 0, such that $h_\theta(Z) = h_{tan(\varphi)}(Z)$ is defined,

Theorem 3

For all θ > 0, such that $h_\theta(\Gamma) = h_{tan(\varphi)}(\Gamma)$ is defined, the polar equation of Γ(x) is:

$$\rho_\Gamma(\varphi) = g_{tan(\varphi)}(Z)\sqrt{1 + (tan(\varphi))^2} = g_\theta(Z)\sqrt{1 + \theta^2} \qquad (6)$$

with tan(φ) = θ.

Proof. We have by definition that, for all θ, where $h_\theta(\Gamma)$ is defined, $h_\theta(\Gamma)$ = $g_\theta(Z)$, a remark already made in (Egghe & Rousseau, 2019c), be it in the context of decreasing continuous functions. This restriction is, however, not necessary. Consequently, equations (6) follow immediately from Theorem 2.

Corollary.

For all θ such that $g_\theta(Z)$ is defined, $g_\theta(Z) = \frac{\rho_\Gamma(\arctan(\theta))}{\sqrt{1+\theta^2}}$, where $\Gamma(x) = \frac{1}{x}\int_0^x Z(s)ds$, for x > 0.

This follows immediately from the previous Theorem 3.

An example.

Again we consider the constant function Z(x) = a, a > 0. Then $\Gamma(x) = \frac{1}{x}\int_0^x a\, ds = a = Z(x)$. Consequently, $g_\theta(Z) = h_\theta(\Gamma) = h_\theta(Z) = a/\theta$. This result also follows by the definition of $g_\theta(Z)$. Indeed: the requirement $\Gamma(g_\theta(Z)) = \theta g_\theta(Z)$ leads to the equation: a = $\theta g_\theta(Z)$ and hence $g_\theta(Z) = \frac{a}{\theta}$. We further note from equation (6) that $\rho_\Gamma(\varphi) = \frac{a}{\theta}\sqrt{1 + \theta^2} = \frac{a}{tan(\varphi)}\sqrt{1 + (tan(\varphi))^2}$.

7. Discussion

This article highlights the importance of using polar coordinates when studying functions, in particular in relation to generalized h-type functions. As such it is essentially meant to provide tools for further studies we intend to finish soon.

For simplicity we have restricted ourselves to a theory of functions Z(x) in the first quadrant. Yet, we think that, with some adaptations, our results

also apply without this restriction. Moreover, if Z(0) is finite and different from zero then all our results apply. Finally, just like in Cartesian coordinates a circle is the graph of two functions, we can calculate generalized h-type indices for graphs that can be described through two or more functions in polar coordinates.

References

Apostol, T.M. (1957). *Mathematical Analysis. A Modern Approach to Advanced Calculus.* Reading (MA): Addison-Wesley.

Egghe, L. (2005). *Power laws in the information production process: Lotkaian informetrics.* Amsterdam: Elsevier.

Egghe, L. (2006). Theory and practise of the g-index. *Scientometrics*, 69(1), 131–152.

Egghe, L., & Rousseau, R. (2006). An informetric model for the Hirsch-index. *Scientometrics, 69*(1), 121-129.

Egghe, L., & Rousseau, R. (2019a). Infinite sequences and their h-type indices. *Journal of Informetrics*, 13(1), 291–298.

Egghe, L., & Rousseau, R. (2019b). A geometric relation between the h-index and the Lorenz curve. *Scientometrics*, 119(2), 1281–1284.

Egghe, L., & Rousseau, R. (2019c). Solution by step functions of a minimum problem in $L^2[0,T]$, using generalized h- and g-indices. *Journal of Informetrics*, 13(3), 785–792.

Egghe, L., & Rousseau, R. (2019d). Measures of linear type lead to a characterization of Zipf functions. *Scientometrics,* 121(3), 1707-1715.

Hirsch, J. E. (2005). An index to quantify an individual's scientific research output. *Proceedings of the National Academy of Sciences of the United States of America,* 102(46), 16569-16572.

Lotka, A.J. (1926). The frequency distribution of scientific productivity. *Journal of the Washington Academy of Sciences, 16*(12), 317-323.

van Eck, N.J., & Waltman, L. (2008). Generalizing the h-and g-indices. *Journal of Informetrics, 2*(4), 263-271.