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NON-COMMUTATIVE CREPANT RESOLUTIONS FOR SOME TORIC SINGULARITIES I

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Abstract. We give a criterion for the existence of non-commutative crepant resolutions (NCCR's) for certain toric singularities. In particular we recover Broomhead's result that a 3-dimensional toric Gorenstein singularity has an NCCR. Our result also yields the existence of an NCCR for a 4-dimensional toric Gorenstein singularity which is known to have no toric NCCR.

1. Introduction

In this note we discuss the existence of non-commutative crepant resolutions (NCCRs) for some toric singularities. Let us first recall the definition. For the rationale behind the definition of an NCCR see [VdB04a, Leu12]. Throughout k is an algebraically closed field of characteristic zero.

Definition 1.1. [DITV15, Leu12, ŠVdB17a, VdB04a, Wem16] Let R be a normal Gorenstein domain. A *non-commutative crepant resolution* (NCCR) of R is an R-algebra of finite global dimension of the form $\Lambda = \operatorname{End}_R(M)$ which in addition is Cohen-Macaulay as R-module and where M is a non-zero finitely generated reflexive R-module.

The following proposition is a combination of our main results. For a representation X of a reductive group G we denote by $X^u := \{x \in X \mid 0 \in \overline{Gx}\}$ the unstable locus.

Proposition 1.2 (§6)**.** *Let W be a generic unimodular representation of an abelian reductive group G over k, and let $X := \operatorname{Spec}\operatorname{Sym}(W) = W^\vee$. If $\dim X^u - \dim G \leq 1$ then $\operatorname{Sym}(W)^G$ has an NCCR.*

For the definition of a generic, unimodular representation see Definitions 3.1, 3.2, respectively. Recall that an abelian reductive group over k is a product of a torus and a finite abelian group.

Proposition 1.2 gives a relatively easy proof that three-dimensional toric Gorenstein singularities have an NCCR (see Corollary 6.2), a fact first proved by Broomhead [Bro12]. Actually Broomhead establishes the existence of a "toric" [Boc12] NCCR (M is a sum of reflexive ideals) which is much more difficult and relies on the theory of dimer models. In [ŠVdB17c] we give an alternative proof of Broomhead's result which is however still not easy.

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In [ŠVdB17a, §10.1] we constructed toric NCCRs for toric rings coming from quasi-symmetric representations W (e.g. self-dual), and showed that in general toric NCCRs do not always exist. In other words, Broomhead's result does not extend to higher dimension. In fact in [ŠVdB17a, §10.1] we gave an example of a 4-dimensional toric Gorenstein singularity which does not have a toric NCCR. Using Proposition 1.2 we can now show that it nevertheless has a non-toric NCCR. See Example 6.3 below. On the other hand, Higashitani and Nakajima [HN17] recently constructed toric NCCRs for some natural examples of toric rings not coming from quasi-symmetric representations.

2. Acknowledgment

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3. Notation and conventions

All objects are defined over k. If $\mathcal{X}$ is a stack then we write $D(\mathcal{X})$ for the unbounded derived category $D_{\mathrm{Qch}}(\mathrm{Mod}(\mathcal{O}_{\mathcal{X}}))$ of $\mathcal{O}_{\mathcal{X}}$-modules with quasi-coherent cohomology.

For a reductive group G we denote by $X(G)$(resp. $Y(G)$) the character group (resp. the group of one-parameter subgroups) of G. There is a natural pairing $Y(G) \times X(G) \to \mathbb{Z}$, we denote it by $\langle\, , \rangle$.

If a reductive group G acts on an affine variety X and $\chi \in X(G)$ is a character then we write $X^{ss,\chi}$ for the open subset of X consisting of the χ-semi-stable points in X. In other words (following [Kin94]), $X^{ss,\chi}$ consists of the points $x \in X$ such that for $\lambda \in Y(G)$ of G with the property that $\lim_{t\to 0} \lambda(t)x$ exists then $\langle \lambda, \chi\rangle \geq 0$. We say that $x \in X$ is stable if it has a closed orbit and a finite stabilizer. We denote the locus of stable points in X by X^s. We have $X^s \subset X^{ss,\chi}$ for any χ. Moreover, we write $X^{\mathbf{s}} \subset X$ for the set of points with closed orbit and trivial stabilizer.

We write $X^u = \{x \mid 0 \in \overline{Gx}\}$ for the G-unstable locus or nullcone.

The inclusions between the open subschemes of X that were introduced are summarized in the following diagram

$$\begin{array}{ccccc} X^{\mathbf{s}} & \hookrightarrow & X^{s} & \hookrightarrow & X^{ss,\chi} \\ & & \downarrow & & \downarrow \\ & & X \setminus X^u & \hookrightarrow & X \end{array}$$

Definition 3.1. We say that a reductive group G acts *generically* on a smooth affine variety X if $\mathrm{codim}(X - X^{\mathbf{s}}, X) \geq 2$. If W is a G-representation then we say that W is *generic* if G acts generically on $\operatorname{Spec}\operatorname{Sym}(W) \cong W^\vee$.

Definition 3.2. Let W be d-dimensional representation of an algebraic group G. We say that W is a *unimodular* if $\wedge^d W \cong k$, where k is the trivial representation.

A variety is an integral separated scheme of finite type over k. If X is a variety with an action of a reductive group G then we consider a G-equivariant sheaf on X as a sheaf on the stack X/G.

4. Main result

The next theorem extends [VdB04a, Thm. 5.1] to certain Deligne-Mumford stacks [DM69, LMB00, Ols16]. As a consequence we obtain NCCRs (see Corollary 4.4). The proof of the theorem is given in §5.

Let us recall that if X is an affine variety then $X/\!\!/G = \operatorname{Spec} k[X]^G$, $X^{ss,\chi}/\!\!/G = \operatorname{Proj} \Gamma_*(X)^G$ where $\Gamma_*(X) = \bigoplus_n \Gamma(X, \chi^{-n} \otimes \mathcal{O}_X)$. Note that $\Gamma_0(X) = k[X]^G$ and the inclusion $\Gamma_0(X) \hookrightarrow \Gamma_*(X)$ defines a natural projective map $\theta : X^{ss,\chi}/\!\!/G \to X/\!\!/G$.

Theorem 4.1. *Let G be an abelian reductive group over k and let X be a smooth affine G-variety containing a G-stable point. Let $\chi \in X(G)$ be a character such that every point in $X^{ss,\chi}$ has finite stabilizer (i.e. $X^{ss,\chi}/G$ is a Deligne-Mumford stack) and assume in addition that $\theta : X^{ss,\chi}/\!\!/G \to X/\!\!/G$ has fibers of dimension ≤ 1. Then $\operatorname{coh}(X^{ss,\chi}/G)$ contains an object $\mathcal{T}$ with the following properties.*

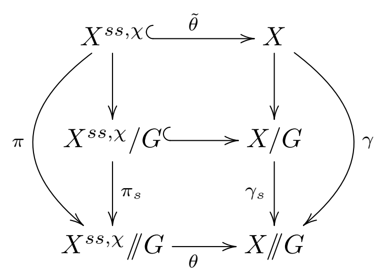
1. *$\mathcal{T}$ is a vector bundle on $X^{ss,\chi}$.*
2. *$\operatorname{Ext}^i_{X^{ss,\chi}/G}(\mathcal{T}, \mathcal{T}) = 0$ for $i > 0$.*
3. *$\mathcal{T}$ is a generator[1] for $D(X^{ss,\chi}/G)$.*

An object satisfying (1)(2)(3) is sometimes called a *tilting bundle*.

Remark 4.2. If G is an abelian reductive group acting linearly on an affine variety then $X^{ss,\chi}/G$ will be a Deligne-Mumford stack if $\chi \in X(G)$ is chosen generically. Indeed we may choose a closed embedding of X in a G-representation and for a representation the claim follows from [CLS11, Theorem 14.3.14] (see also [HLS16, Proposition 2.1]).

Whenever we are in the setting of Theorem 4.1 we will use the following diagram

(1)



where $\tilde\theta$ is an inclusion, θ the induced map on the quotients (coming from the definition of the quotients), π, γ are quotient maps, and π_s, γ_s are stack morphisms. More precisely, the morphism $\gamma : X \to X/\!\!/G$ is G-equivariant and hence it factors through X/G which yields γ_s. A similar statement holds for π_s.

Under some genericity conditions (in the sense of Definition 3.1) one may obtain an NCCR from Theorem 4.1. We denote by $R = k[X]^G$ the coordinate ring of $X/\!\!/G$.

Corollary 4.3. *Let $X, G, \chi, \mathcal{T}$ be as in Theorem 4.1. Then $D(X^{ss,\chi}/G) \cong D(\Lambda)$ where $\Lambda = \operatorname{End}_{X^{ss,\chi}/G}(\mathcal{T})$. One has $\operatorname{gl\,dim} \Lambda < \infty$. Moreover, if G acts generically on X then $\Lambda = \operatorname{End}_R(T)$ where $T = \Gamma(X^{ss,\chi}/G, \mathcal{T}) = \Gamma(X^{ss,\chi}, \mathcal{T})^G$ which is a reflexive R-module.*

[1]See Definition 5.1 for the definition of the generation.

Proof. The derived equivalence claim follows from [Kel94, Theorem 4.3]. The derived equivalence implies $\operatorname{gl\,dim} \Lambda \leq \infty$ since $X^{ss,\chi}/G$ is smooth (see [HVdB07, Theorem 7.6]). We now refer to [ŠVdB17a, §3,4] for some generalities concerning reflexive sheaves we use below. Recall in particular that reflexive sheaves $\mathcal{F}$, $\mathcal{G}$ on a normal variety Z form a rigid monoidal category with tensor product $\mathcal{F} \otimes \mathcal{G} := (\mathcal{F} \otimes_Z \mathcal{G})^{\vee\vee}$. Assume that $\operatorname{codim}(X - X^{\mathbf{s}}, X) \geq 2$. Then also $\operatorname{codim}(X - X^{ss,\chi}, X) \geq 2$ and hence $\tilde{\theta}_*$ defines a monoidal equivalence between the categories of reflexive sheaves on $X^{ss,\chi}$ and X. Using again the condition $\operatorname{codim}(X - X^{\mathbf{s}}, X) \geq 2$, taking G-invariants defines a monoidal equivalence between G-equivariant reflexive sheaves on X and reflexive sheaves on $X/\!\!/G$ by [ŠVdB17c, Lemma 4.1.3]. Therefore $T = \Gamma(X^{ss,\chi}, \mathcal{T})^G = \Gamma(X, \tilde{\theta}_*\mathcal{T})^G = \Gamma(X/\!\!/G, (\gamma_*\tilde{\theta}_*\mathcal{T})^G)$, and in particular T is a reflexive R-module. Again using the mentioned monoidal equivalences we obtain $\Lambda = \operatorname{End}_{X^{ss,\chi}/G}(\mathcal{T}) = \operatorname{End}_{X/\!\!/G}((\gamma_*\tilde{\theta}_*\mathcal{T})^G) = \operatorname{End}_R(T)$. □

Corollary 4.4. *Let X, G, χ be as in Theorem 4.1. Assume in addition that $X = W^\vee$ where W is generic unimodular G-representation. Then $R = \operatorname{Sym}(W)^G$ has an NCCR.*

Proof. Let $\mathcal{A} = \mathcal{E}nd_{X^{ss,\chi}/G}(\mathcal{T})$ where $\mathcal{T}$ is as in Theorem 4.1. Then $\mathcal{A}$ is a sheaf of algebras on $X^{ss,\chi}/G$. By Corollary 4.3 we have to show that $\Lambda = Rf_{s,*}\mathcal{A}$ is Cohen-Macaulay. Using Lemma 4.5 below we have by the same argument as [VdB04b, Lemma 3.2.9] noting that $f_s = \theta \circ \pi_s$ is proper (since π_s, θ are proper by [Ols16, Exercise 11.E], [CLS11, Proposition 14.1.12], resp.) and referring to [Nir09, Corollary 2.10] for the first equality

$$\begin{aligned}
\operatorname{RHom}_{X/\!\!/G}(Rf_{s,*}\mathcal{A}, \omega_{X/\!\!/G}) &= \operatorname{RHom}_{X^{ss,\chi}/G}(\mathcal{A}, f_s^!\omega_{X/\!\!/G}) \\
&= \operatorname{RHom}_{X^{ss,\chi}/G}(\mathcal{A}, \omega_{X^{ss,\chi}/G}) \\
&= \operatorname{RHom}_{X^{ss,\chi}/G}(\mathcal{A}, \mathcal{O}_{X^{ss,\chi}/G}) \\
&= Rf_{s,*}\mathcal{A}^\vee \\
&= Rf_{s,*}\mathcal{A} \\
&= f_{s,*}\mathcal{A}
\end{aligned}$$

This finishes the proof. □

We have used the following lemma.

Lemma 4.5. *Let X, G, χ be as in Theorem 4.1. Assume in addition that $X = W^\vee$ where W is a generic unimodular G-representation. The map $f_s = \theta \circ \pi_s$ is crepant and $\omega_{X^{ss,\chi}/G} \cong \mathcal{O}_{X^{ss,\chi}/G}$.*

Proof. The hypothesis imply that $\omega_{X/\!\!/G}$ is invertible and moreover $\omega_{X/\!\!/G} \cong \mathcal{O}_{X/\!\!/G}$ by [Kno89, Satz 2] because of the unimodularity. A Deligne-Mumford stack is étale locally a quotient stack for a finite group and in particular $\omega_{X^{ss,\chi}/G}$ is a reflexive sheaf (it is in fact invertible but already reflexivity suffices our purposes). We claim $f_s^*\omega_{X/\!\!/G} = \omega_{X^{ss,\chi}/G}$ and hence in particular $\omega_{X^{ss,\chi}/G} \cong \mathcal{O}_{X^{ss,\chi}/G}$. This follows from the fact both $\omega_{X/\!\!/G}$ and $\omega_{X^{ss,\chi}/G}$ are reflexive and f_s is the identity on $X^{\mathbf{s}}/G \cong X^{\mathbf{s}}/\!\!/G$, using that the complement of $X^{\mathbf{s}}$ in X is of codimension ≥ 2 by the genericity assumption. □

Remark 4.6. The assumption that W is generic simplifies the proof of the previous lemma but it is in fact superfluous. This is a consequence of the theory of toric DM stacks [BH06]. See e.g. [ŠVdB17c, Lemma A.2].

5. Proof of Theorem 4.1

We refer to [ŠVdB16, Definition 3.3.1] for the definition of a good quotient. Assume Y is such that a good quotient $Y/\!\!/G$ exists (in particular G is reductive). For an open $U \subset Y/\!\!/G$ we write $\tilde{U} = U \times_{Y/\!\!/G} Y \subset Y$.

Definition 5.1. Let $\mathcal{Y}$ be a stack. We say that a derived category $D(\mathcal{Y})$ is generated by an object $E \in D(\mathcal{Y})$ if $E^\perp = 0$. We say that $D(\mathcal{Y})$ for $\mathcal{Y} = Y/G$ such that a good quotient $Y/\!\!/G$ exists is *locally generated* by a perfect object E if $D(\tilde{U}/G)$ is generated by $E|\tilde{U}$ for every affine open $U \subset Y/\!\!/G$, i.e. $(E|\tilde{U})^\perp = 0$.

Remark 5.2. From the fact that G-equivariant complexes on $\tilde{U}$ can be extended to complexes on Y (for example by pushforward), it follows that E being a local generator is equivalent to the statement that $R\mathcal{H}om_{Y/G}(E, \mathcal{F}) = 0$ implies $\mathcal{F} = 0$.

The following is a variant on [ŠVdB16, Lemma 3.5.4]. It can be deduced from a more general result (see [OS03, Lemma 1.3, Theorem 5.7]). However it seems useful to give a direct proof in our simple setting.

Lemma 5.3. *Let G be a reductive group acting on an algebraic variety Y such that a good quotient $\pi : Y \to Y/\!\!/G$ exists and such that Y/G is a Deligne-Mumford stack. Then $D(Y/G)$ is locally generated by $V \otimes \mathcal{O}_Y$ for a single finite dimensional representation V of G.*

Proof. By Remark 5.2, we need to find V such that $\pi_{s,*}R\mathcal{H}om_{Y/G}(V \otimes \mathcal{O}_Y, \mathcal{F}) = 0$ implies $\mathcal{F} = 0$, where $\pi_s : Y/G \to Y/\!\!/G$ is the morphism of stacks associated to π. Since $\pi_{s,*}$ is exact, as π is a good quotient, and $V \otimes \mathcal{O}_Y$ is a vector bundle we have $H^*(\pi_{s,*}R\mathcal{H}om_{Y/G}(V \otimes \mathcal{O}_Y, \mathcal{F})) = \pi_{s,*}\mathcal{H}om_{Y/G}(V \otimes \mathcal{O}_Y, H^*(\mathcal{F}))$. In other words, it is sufficient to prove that $\pi_{s,*}\mathcal{H}om_{Y/G}(V \otimes \mathcal{O}_Y, \mathcal{F}) = 0$ implies $\mathcal{F} = 0$ for $\mathcal{F} \in \operatorname{Qch}(Y/G)$.

If a certain V works then any representation containing V works as well. Hence we claim that the existence of V is a local property for the étale topology on $Y/\!\!/G$. Let $(U_i \to Y/\!\!/G)_i$ be an étale covering of $Y/\!\!/G$. Let $\tilde{U}_i = Y \times_{Y/\!\!/G} U_i$. Since G is reductive (and π is a good quotient) one can see that $U_i = \tilde{U}_i/\!\!/G$.[2] We denote $\pi_s^i : \tilde{U}_i/G \to U_i/\!\!/G$. Let us assume that for every i there exists V_i such that $\pi^i_{s,*}\mathcal{H}om_{\tilde{U}_i/G}(V_i \otimes \mathcal{O}_{\tilde{U}_i}, \mathcal{F}) = 0$ implies $\mathcal{F} = 0$ for $\mathcal{F} \in \operatorname{Qch}(\tilde{U}_i/G)$. As $Y/\!\!/G$ is quasi-compact (and as an étale map is open) we only need a finite number of U_i such that $(U_i \to Y/\!\!/G)_{i=1}^n$ is an étale covering. Let $V = \oplus_{i=1}^n V_i$. Assume that $\mathcal{H} = \pi_{s,*}\mathcal{H}om_{Y/G}(V \otimes \mathcal{O}_Y, \mathcal{F}) = 0$. We need to prove that $\mathcal{F} = 0$. Let us write $\mathcal{G}_i$ for the pullback of $\mathcal{G} \in \operatorname{Qch}(Y/G)$ to $\tilde{U}_i/G$. The restriction $\mathcal{H}_i$ to $\tilde{U}_i/G$ is 0. Moreover, flatness of étale morphisms implies that $\mathcal{H}_i = \pi^i_{s,*}\mathcal{H}om_{\tilde{U}_i/G}(V_i \otimes \mathcal{O}_{\tilde{U}_i}, \mathcal{F}_i) = 0$. Thus, $\mathcal{F}_i = 0$ by our assumption, and hence $\mathcal{F} = 0$.

We may therefore assume that Y is affine, and furthermore it suffices to show that $\mathcal{F}$ is zero in a neighbourhood of any closed orbit by [ŠVdB16, Lemma 4.4.3]. Invoking the Luna slice theorem we may assume that π is of the form $G \times^H S \to$

[2]It is easy to see that a good quotient (in the sense of [ŠVdB17b, Definition 3.3.1]) is compatible with arbitrary base extension; i.e. if $Y \to X$ is a good quotient and $Z \to X$ is arbitrary, then $Y \times_X Z \to Z$ is also a good quotient. To see this note that $Y \to X$ is built by gluing morphisms $\operatorname{Spec} A \to \operatorname{Spec} A^G$ and this allows us to reduce to the affine case. Let $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} A^G$, $Z = \operatorname{Spec} B$. Then the dual statement $B = (A \otimes_{A^G} B)^G$ holds since the inclusion $A^G \hookrightarrow A$ is split by the Reynolds operator.

$(G \times^H S)/\!\!/G \cong S/\!\!/H$ where S is an étale slice at $y \in Y$ with closed orbit and $H = \operatorname{Stab}(y)$. Since Y/G is a Deligne-Mumford stack, H is finite. Let kH be the regular H-representation. Then $\mathcal{H}om_{S/H}(kH \otimes \mathcal{O}_S, \mathcal{F}) = 0$ implies $\mathcal{F} = 0$. Since $S/H \cong (G \times^H S)/G$, $kH \otimes_k \mathcal{O}_S$ corresponds to a G-equivariant vector bundle $\mathcal{E}$ on $G \times^H S$. It now suffices, using the reduction to $Y = G \times^H S$ and $Y/G \cong S/H$, to write $\mathcal{E}$ as a quotient of $V \otimes \mathcal{O}_{G \times^H S}$ for some finite dimensional G-representation V. □

Lemma 5.4. *Let G be a reductive group acting on an algebraic variety Y which is projective over an affine variety and let $\mathcal{M}$ be an ample G-equivariant line bundle on Y. Let $Y^{ss} \subset Y$ be the semi-stable locus corresponding to the linearization given by $\mathcal{M}$ and let $\pi : Y^{ss} \to Y^{ss}/\!\!/G$ be the (good)[3] quotient map. Then up to replacing $\mathcal{M}$ by a strictly positive multiple we may assume that $(\mathcal{M}|_{Y^{ss}})^G$ is an ample line bundle on $Y^{ss}/\!\!/G$ generated by global sections such that moreover $\pi^*((\mathcal{M}|_{Y^{ss}})^G) \cong \mathcal{M}|_{Y^{ss}}$.*

Proof. Put
$$\Gamma_*(Y) = \bigoplus_{n \geq 0} \Gamma(Y, \mathcal{M}^{\otimes n}).$$
Then $Y^{ss}/\!\!/G = \operatorname{Proj} \Gamma_*(Y)^G$. Since $\Gamma_*(Y)$ and $\Gamma_*(Y)^G$ are finitely generated, there is an N such that the N'th Veronese subalgebras of $\Gamma_*(Y)$ and $\Gamma_*(Y)^G$ are both generated in degree one. We then replace $\mathcal{M}$ by $\mathcal{M}^{\otimes N}$. □

Lemma 5.5. *Let G be a reductive group acting on an algebraic variety Y which is projective over an affine variety and let $\mathcal{M}$ be an ample G-equivariant line bundle on Y. Let $Y^{ss} \subset Y$ be the semi-stable locus corresponding to the linearization given by $\mathcal{M}$ and let $\pi : Y^{ss} \to Y^{ss}/\!\!/G$, $\pi_s : Y^{ss}/G \to Y^{ss}/\!\!/G$ be the associated quotient maps. We assume that Y^{ss}/G is a Deligne-Mumford stack.*

In addition we assume that we have replaced $\mathcal{M}$ by a strictly positive multiple such that $\mathcal{L} := (\mathcal{M} \mid Y^{ss})^G \in \operatorname{Pic}(Y^{ss}/\!\!/G)$ has the properties exhibited in Lemma 5.4. Put $Z = \operatorname{Spec} \Gamma(Y, \mathcal{O}_Y)$. Let d be the maximum of the dimension of the fibers of $Y^{ss}/\!\!/G \to Z/\!\!/G$. Let V be a finite dimensional representation of G such that $V \otimes \mathcal{O}_{Y^{ss}}$ is a local generator for $D(Y^{ss}/G)$ as in Lemma 5.3. Put $V = \bigoplus_{i=1}^n V_i$ with V_i irreducible and fix $m_i \in \mathbb{Z}$ for $i = 1, \ldots, n$ and $l \geq 1$. Then
$$\bigoplus_{j=0}^{d} \bigoplus_{i=1}^{n} V_i \otimes \pi_s^*(\mathcal{L})^{\otimes lj + m_i} = \bigoplus_{j=0}^{d} \bigoplus_{i=1}^{n} V_i \otimes \mathcal{M}^{\otimes lj + m_i}\Big|_{Y^{ss}}$$
is a compact generator for $D(Y^{ss}/G)$.

Proof. Replacing $\mathcal{M}$ by $\mathcal{M}^{\otimes l}$ we may assume $l = 1$. Put $\mathcal{E} = \bigoplus_{i=1}^n V_i \otimes \pi_s^*(\mathcal{L})^{\otimes m_i}$. Then since $\pi_s^*(\mathcal{L})$ is locally free on Y^{ss}/G, $\mathcal{E}$ is a local generator for $D(Y^{ss}/G)$. We must prove that $\bigoplus_{j=0}^d \mathcal{E} \otimes \pi_s^*(\mathcal{L})^{\otimes j}$ is a generator for $D(Y^{ss}/G)$.

Assume $\mathcal{F} \in D(Y^{ss}/G)$ is such that $\operatorname{RHom}_{Y^{ss}/G}(\bigoplus_{j=0}^d \mathcal{E} \otimes \pi_s^*(\mathcal{L})^{\otimes j}, \mathcal{F}) = 0$. Then $\operatorname{RHom}_{Y^{ss}/\!\!/G}(\bigoplus_{j=0}^d \mathcal{L}^{\otimes j}, \pi_{s,*} \operatorname{R}\mathcal{H}om_{Y^{ss}/G}(\mathcal{E}, \mathcal{F})) = 0$. By [VdB04b, Lemma 3.2.2] this implies $\pi_{s,*} \operatorname{R}\mathcal{H}om_{Y^{ss}/G}(\mathcal{E}, \mathcal{F}) = 0$. Since $\mathcal{E}$ is a local generator this implies $\mathcal{F} = 0$. □

Lemma 5.6. *Let X, G, χ be as in Theorem 4.1. Then θ is birational and it is true that $R\theta_* \mathcal{O}_{X^{ss,\chi}/\!\!/G} = \mathcal{O}_{X/\!\!/G}$. Finally $R^i\theta_* = 0$ for $i > 1$.*

[3] See e.g. [ŠVdB16, §3.4].

Proof. Both $X^{ss,\chi}/\!\!/G$ and $X/\!\!/G$ contain $X^s/\!\!/G$ as an nonempty hence dense subscheme. So they are birational. Both $X^{ss,\chi}/\!\!/G$ and $X/\!\!/G$ are quotients by reductive groups and hence they have rational singularities (see [Bou87, Corollaire]). This proves the claim about $R\theta_*\mathcal{O}_{X^{ss,\chi}/\!\!/G}$. The last claim follows from the hypothesis that the fibers of θ have dimension ≤ 1. □

Lemma 5.7. *Let G, X, χ be as in Theorem 4.1. Then there exist characters $(\chi_u)_{u=1,\dots,N}$ such that $\mathcal{L}_u = \chi_u \otimes \mathcal{O}_{X^{ss,\chi}}$ for $i = 1, \dots, N$ generate $D(X^{ss,\chi}/G)$ and such that moreover we have*

$$\operatorname{Ext}^i_{X^{ss,\chi}/G}(\mathcal{L}_u, \mathcal{L}_u) = 0 \qquad \text{for } i > 0 \tag{2}$$

$$\operatorname{Ext}^i_{X^{ss,\chi}/G}(\mathcal{L}_u, \mathcal{L}_v) = 0 \qquad \text{for } i > 1 \tag{3}$$

$$\operatorname{Ext}^1_{X^{ss,\chi}/G}(\mathcal{L}_u, \mathcal{L}_v) = 0 \qquad \text{for } u < v \tag{4}$$

Proof. According to Lemma 5.5 (with $Y = X$, $\mathcal{M} = \chi \otimes \mathcal{O}_{X^{ss,\chi}}$) after replacing χ by some strict positive multiple there exist $\mu_i \in X(G)$ (corresponding to the character of $V_i = \mu_i \otimes \mathcal{O}_{X^{ss,\chi}}$ in Lemma 5.5) such that for any collection of $m_i \in \mathbb{Z}$ and for any $\ell \geq 1$ the object

$$\bigoplus_{j=0}^{1} \bigoplus_{i=1}^{n} \mu_i \otimes \chi^{lj+m_i} \otimes \mathcal{O}_{X^{ss,\chi}}$$

is a compact generator of $D(X^{ss,\chi}/G)$. We put $\chi_1 = \mu_1 \otimes \chi^{m_1}$, $\chi_2 = \mu_1 \otimes \chi^{l+m_1}$, $\chi_3 = \mu_2 \otimes \chi^{m_2}$, ..., $\chi_{2n} = \mu_n \otimes \chi^{l+m_n}$, $N = 2n$. Then (2), (3) follow directly from Lemma 5.6 (because $\pi_{s,*}$ is exact since π is a good quotient (see Lemma 5.4) and $X/\!\!/G$ is affine).

To make (4) true we choose $l, (m_i)_i$ in such a way that

$$m_1 \ll l + m_1 \ll m_2 \ll m_2 + l \ll m_3 \ll \cdots$$

Then in (4) we have $\mathcal{L}_u = \mu_i \otimes \chi^a \otimes \mathcal{O}_{X^{ss}}$, $\mathcal{L}_v = \mu_j \otimes \chi^b \otimes \mathcal{O}_{X^{ss,\chi}}$ with $a \ll b$. Put $\mathcal{L} = \pi_{s,*}(\chi \otimes \mathcal{O}_{X^{ss,\chi}})$. By our choice of χ, $\mathcal{L}$ is ample on $X^{ss,\chi}/\!\!/G$ and $\pi^{s,*}\mathcal{L} = \chi \otimes \mathcal{O}_{X^{ss,\chi}}$ by Lemma 5.4. Using the projection formula we have

$$\begin{aligned} \operatorname{RHom}_{X^{ss,\chi}/G}(\mathcal{L}_u, \mathcal{L}_v) &= R\Gamma(X^{ss,\chi}/\!\!/G, \mathcal{L}^{\otimes b-a} \otimes \pi_{s,*}(\mu_2 \otimes \mu_1^{-1} \otimes \mathcal{O}_{X^{ss,\chi}})) \\ &= \Gamma(X^{ss,\chi}/\!\!/G, \mathcal{L}^{\otimes b-a} \otimes \pi_{s,*}(\mu_2 \otimes \mu_1^{-1} \otimes \mathcal{O}_{X^{ss,\chi}})) \end{aligned}$$

where in the second line we use that $\mathcal{L}$ is ample and $b - a \gg 0$. □

Proof of Theorem 4.1. If E, F are objects in an abelian category $\mathcal{A}$ such that the Yoneda extension $\operatorname{Ext}^1_{\mathcal{A}}(E, F)$ is a finitely generated right $\operatorname{Hom}_{\mathcal{A}}(E, E)$-module with generators $c_1, \dots, c_n$ then we define the corresponding semi-universal extension of E and F to be the middle term of the extension

$$0 \to F \to \bar{F} \to E^{\oplus n} \to 0$$

corresponding to $(c_i)_i$.

Let $(\mathcal{L}_u)_{u=1,\dots,N}$ be as in Lemma 5.7. Using the conditions (2,3,4) as in Lemma 5.7 we may construct the object $\mathcal{T}$ by taking successive semi-universal extensions among the $(\mathcal{L}_u)_u$. See [HP14, Lemma 3.1] for details. In loc. cit. universal extensions are considered but the argument also works with semi-universal extensions. □

6. Combinatorial interpretation

We let X, G, χ be as in Theorem 4.1, without a priori assuming that the fibers of $\theta : X^{ss,\chi}/\!\!/G \to X/\!\!/G$ have dimension ≤ 1.

Proposition 6.1. *Assume $X = W^\vee$ for a G-representation W. Then if*

$$\dim X^u - \dim G \leq 1 \tag{5}$$

the fibers of θ have dimension ≤ 1.

Proof. We refer to the diagram (1). By semi-continuity it is sufficient to bound the dimension of $\theta^{-1}(\bar{0})$, where $\bar{0} = \pi(0)$. Now $\theta^{-1}(\bar{0}) = \pi(\tilde{\theta}^{-1}(\gamma^{-1}(\bar{0})))$. Since the fibers of π have constant dimension $\dim G$ we deduce

$$\begin{aligned}\dim \theta^{-1}(\bar{0}) &= \dim(\gamma^{-1}(\bar{0}) \cap X^{ss,\chi}) - \dim G \\ &= \dim(X^u \cap X^{ss,\chi}) - \dim G \leq \dim X^u - \dim G \quad \square.\end{aligned}$$

Proof of Proposition 1.2. We claim that we can apply Corollary 4.4 to obtain an NCCR. We need to verify that there exists $\chi \in X(G)$ such $X = W^\vee$, G χ satisfy assumptions of Theorem 4.1. By Remark 4.2, we may choose $\chi \in X(G)$ such that $X^{ss,\chi}/G$ is a DM stack. Moreover, the fibers of θ have dimension ≤ 1 by Proposition 6.1. $\square$

Corollary 6.2. *Assume $X = W^\vee$ for a G-representation W. If W is generic and $\dim X/\!\!/G = \dim X - \dim G \leq 3$ then the fibers of θ have dimension ≤ 1. In particular we recover the result by Broomhead that "affine Gorenstein toric singularities of dimension 3 have an NCCR".*

Proof. Let $\beta_1, \ldots, \beta_d$ be the weights of W. The fact that W is generic implies that for every $0 \neq \lambda \in Y(G)$ we have that there are at least two i such that $\langle \lambda, \beta_i \rangle > 0$ (as otherwise $X - X^{\mathbf{s}}$ contains a codimension 1 variety given by the vanishing of the coordinate x_i corresponding to the only i for which $\langle \lambda, \beta_i \rangle > 0$). Hence $\dim X^u \leq \dim X - 2$. Thus $\dim X^u - \dim G \leq \dim X - \dim G - 2 \leq 3 - 2 = 1$. In other words (5) holds.

For the last statement we employ Proposition 1.2 after noting that every affine toric variety X can be written in the form $W^\vee/\!\!/G$ for a generic G-representation W. This is explained e.g. in [ŠVdB17a, §11.6.1]. If X is Gorenstein then of course so is $W^\vee/\!\!/G$ and this is equivalent to W being unimodular by [Kno89]. $\square$

Note that (5) may hold for higher dimensional $X/\!\!/G$. Below we recall an example from [ŠVdB17a] of a 4-dimensional variety $X/\!\!/G$ which does not have a toric NCCR. For this variety (5) is satisfied, and it therefore has a (non-toric) NCCR by Proposition 1.2 which we explicitly construct.

Example 6.3. Consider the example [ŠVdB17a, §10.1]. Then we have that $G = G_m^2$ is a two dimensional torus and (after the identifying $X(G) \cong \mathbb{Z}^2$) the weights $(\beta_i)_i$ of W are given by $(3,0), (1,1), (0,3), (-1,0), (-3,-3), (0,-1)$ (see Figure 1). We have $X^u = \bigcup_{1 \leq i \leq 6} \{x_i = x_{i+1} = x_{i+2} = 0\}$ with cyclic indices, hence $\dim X^u = 3$ and moreover W is generic and unimodular so that by Proposition 1.2 $R = \operatorname{Sym}(W)^G = k[x_2x_4x_6, x_1x_3x_5, x_1x_4^3, x_3x_6^3, x_2^3x_5] \cong k[a,b,c,d,e]/(a^3b - cde)$ has an NCCR. However this NCCR is not toric which is the same as saying that it is not given by a module of covariants (a module of the form $M(U) = (U \otimes SW)^G$ for a finite dimensional G-representation U). In fact an NCCR given by a module of covariants does not exist in this case as is shown in loc. cit.

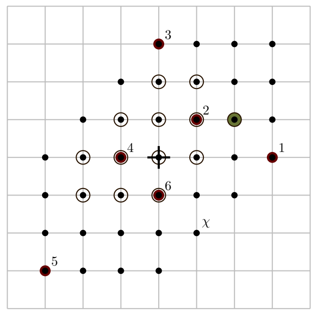


FIGURE 1. $\bullet^i$ weights , $\bullet$ CM weights, ○● $\mathcal{L}$

We will now describe the construction of an explicit NCCR for this example. We have not literally followed the proof of Proposition 1.2 which appeared computationally too expensive. Instead we obtain an NCCR using a similar but more adhoc procedure.

First we give some heuristic motivation for the construction. Assuming an appropriately strengthened version of the Bondal-Orlov conjecture asserting that all (stacky) commutative and non-commutative crepant resolutions are derived equivalent [BO02, IW13, VdB04a] the number of indecomposable summands of the module defining a non-commutative crepant resolution that we need is given by the rank of K_0 of a (stacky) crepant commutative resolution of $\operatorname{Spec} R$ (since K_0 is invariant under derived equivalence).

It is easy to verify that $\operatorname{Spec} R$ as a (singular) toric variety corresponds to the fan given by the cone over a 3-dimensional polytope P shown in Figure 2. The volume of this polytope equals 13/6, therefore the rank of K_0 of the stacky crepant resolution of $\operatorname{Spec} R$, corresponding to a triangulation of P, is 13 (see Theorem A.1).

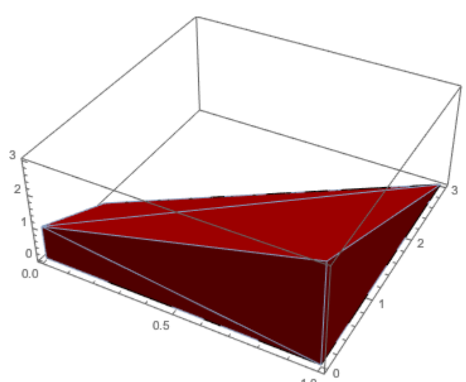


FIGURE 2

After these heuristics we describe the actual construction. Let $\mathcal{L}$[4] be given by weights corresponding to encircled dots in Figure 6.3 and let $\mathcal{L}' = \mathcal{L} \setminus \{(2,1)\}$. We write $M(\mu) = (\mu \otimes SW)^G$. The endomorphism ring $\operatorname{End}_R(\oplus_{\mu\in\mathcal{L}'} M(\mu))$ is Cohen-Macaulay (see [ŠVdB17a, Example 10.1]). Since $|\mathcal{L}'| = 12$ we expect to need a single additional indecomposable R-module K such that $\Lambda = \operatorname{End}_R(\oplus_{\mu\in\mathcal{L}'} M(\mu) \oplus K)$ is an NCCR. By loc. cit. K cannot be a module of covariants.

We define K by the exact sequence

$$0 \to K \to M(0,-1) \oplus M(1,1) \oplus M(-1,1) \xrightarrow{\psi} M(2,1) \to 0, \tag{6}$$

where $\psi(r_1,r_2,r_3) = r_1 d + r_2 a + r_3 ab/c$. (Note that $M(0,-1) \cong (a,e)$, $M(1,1) \cong (a,d)$, $M(-1,1) \cong (a^2, ac, cd)$, $M(2,1) \cong (a^2, ad, de)$.) Considering $M(\mu)$ as subsets of $\operatorname{Sym}(W)$ we can write $\psi(r_1,r_2,r_3) = r_1 x_2 x_5 + r_2 x_4 + r_3 x_3 x_5$.

It is easy to check that Λ is a Cohen-Macaulay R-module (using e.g. Macaulay2), suggesting that it might be an NCCR of R. Below we verify this fact by constructing an appropriate tilting bundle on a particular stacky resolution $X^{ss,\chi}/G$ of $\operatorname{Spec} R$.

Let $\chi = (1,-2)$. We claim that $\mathcal{E} = \bigoplus_{\mu\in\mathcal{L}} \mu \otimes \mathcal{O}_{X^{ss,\chi}}$ generates $D(X^{ss,\chi}/G)$. One can use a similar algorithm as in the proof of [ŠVdB17a, Theorem 1.5.1]. We refer to [ŠVdB17a, §11.1-3] for some unexplained notation. In loc. cit. the complexes $C_{\lambda,\mu}$ with cohomology supported on $X^{\lambda,\geq 0}$ relate projectives P_ν, $\nu \in X(T)$, in $D(X/G)$. Thus, if $\langle \chi, \lambda\rangle < 0$ then $C_{\lambda,\mu}$ is exact when restricted to $X^{ss,\chi}/G$ (recall that $X^{ss,\chi}$ consists of $x \in X$ such that if $\lambda \in Y(T)$ is such that $\lim_{t\to 0} \lambda(t)x$ exists then $\langle \lambda, \chi\rangle \geq 0$ which is equivalent to saying that $-\chi$ is in the cone generated by $(\beta_i)_{x_i\neq 0}$). Assume that $\tilde{\mathcal{L}} \subset X(T)$ is such that $\nu \otimes \mathcal{O}_{X^{ss,\chi}}$, $\nu \in \tilde{\mathcal{L}}$, belong to the subcategory of $D(X^{ss,\chi}/G)$ generated by $\mathcal{E}$ (e.g. $\tilde{\mathcal{L}} = \mathcal{L}$). Then we may enlarge $\tilde{\mathcal{L}}$ by $\nu \in X(T)$ if for some $\langle \lambda, \chi\rangle < 0$ all components except for $\nu \otimes \mathcal{O}_{X^{ss,\chi}}$ of either of the complexes $C_{\lambda,\nu}$, $C_{\lambda,\nu-\sum_{\langle\lambda,\beta_i\rangle>0}\beta_i}$ are of the form $\mu \otimes \mathcal{O}_{X^{ss,\chi}/G}$ for $\mu \in \tilde{\mathcal{L}}$. Note that if $\tilde{\mathcal{L}}$ contains $X(T) \cap \Sigma$, then we may enlarge $\tilde{\mathcal{L}}$ to $X(T)$. (See also the proof of [HLS16, Theorem 3.2].)

In our example we may easily verify by hand (or by computer, cf. [ŠVdB17a, Remark 11.3.2]) that we can indeed enlarge $\mathcal{L}$ to $\Sigma \cap X(T)$ (where in this case $\Sigma \cap X(T)$ is given by weights corresponding to black dots in the above picture), and therefore $\mathcal{E}$ generates $D(X^{ss,\chi}/G)$.

Since the endomorphism ring $\operatorname{End}_R(\oplus_{\mu\in\mathcal{L}'} M(\mu))$ is Cohen-Macaulay, we have $\operatorname{Ext}^1_{X^{ss,\chi}/G}(\mathcal{E}', \mathcal{E}') = 0$ for $\mathcal{E}' = \bigoplus_{\mu\in\mathcal{L}'} \mu \otimes \mathcal{O}_{X^{ss,\chi}}$ (see [VdB93, Corollary 3.3.2]).

Denote $M = \{(0,-1),(1,1),(-1,1)\} \subset \mathcal{L}'$. Let $\tilde{\psi} : \bigoplus_{\mu\in M} \mu \otimes \mathcal{O}_{X^{ss,\chi}} \to \mu_{(2,1)} \otimes \mathcal{O}_{X^{ss,\chi}}$ be the lift of ψ to $X^{ss,\chi}/G$, and let $\mathcal{K}$ be the lift of K (see the proof of Corollary 4.3). We claim that (6) induces an exact sequence

$$0 \to \mathcal{K} \to \bigoplus_{\mu\in M} \mu \otimes \mathcal{O}_{X^{ss,\chi}} \xrightarrow{\tilde{\psi}} \mu_{(2,1)} \otimes \mathcal{O}_{X^{ss,\chi}} \to 0. \tag{7}$$

Since $\tilde{\psi}$ is a restriction of the map $\Psi : \oplus_{\mu\in M} \mu \otimes \mathcal{O}_X \to \mu_{(2,1)} \otimes \mathcal{O}_X$, induced from ψ, we need to check that the cokernel $\mathcal{N}$ of this map has support in the complement of $X^{ss,\chi}$. The support of the cokernel is defined by the ideal (x_2x_5, x_4, x_3x_5). Let $x = (x_1, \ldots, x_6)$ belong to the support. Then either $x_2 = x_3 = x_4 = 0$ or

[4] This notation is in accordance with notation in [ŠVdB17c, §11] which we will refer to in the sequel. It should not be confused with the notation for line bundles used in the previous sections.

$x_4 = x_5 = 0$. Since $-\chi$ does not lie in the cone generated by neither $\beta_1, \beta_5, \beta_6$ nor $\beta_1, \beta_2, \beta_3, \beta_6$, x does not belong to $X^{ss,\chi}$.

Moreover, any map from $\bigoplus_{\mu \in \mathcal{L}'} \mu \otimes \mathcal{O}_X$ to $\mu_{(2,1)} \otimes \mathcal{O}_X$ factors through Ψ, since its image is zero in $\mathcal{N}$ which easily follows from the fact that $\mathcal{L}'$ does not intersect the semigroups generated by $\beta_1, \beta_5, \beta_6$ and $\beta_1, \beta_2, \beta_3, \beta_6$, resp., shifted by $\mu_{(2,1)}$. Therefore, employing again the proof of Corollary 4.3, the map $\mathrm{Hom}_{X^{ss,\chi}/G}(\mu_i \otimes \mathcal{O}_{X^{ss,\chi}}, \bigoplus_{\mu \in M} \mu \otimes \mathcal{O}_{X^{ss,\chi}}) \to \mathrm{Hom}_{X^{ss,\chi}/G}(\mu_i \otimes \mathcal{O}_{X^{ss,\chi}}, \mu_{(2,1)} \otimes \mathcal{O}_{X^{ss,\chi}})$ induced from (7) is surjective. Thus, $\mathrm{Ext}^1_{X^{ss,\chi}/G}(\mu_i \otimes \mathcal{O}_{X^{ss,\chi}}, \mathcal{K}) = 0$.

Applying $\mathrm{Hom}_{X^{ss,\chi}/G}(-, \mu_i \otimes \mathcal{O}_{X^{ss,\chi}})$ and $\mathrm{Hom}_{X^{ss,\chi}/G}(-, \mathcal{K})$ to (7) further implies that $\mathrm{Ext}^1_{X^{ss,\chi}/G}(\mathcal{K}, \mu_i \otimes \mathcal{O}_{X^{ss,\chi}}) = 0$ and $\mathrm{Ext}^1_{X^{ss,\chi}/G}(\mathcal{K}, \mathcal{K}) = 0$.

Since $\mathcal{E}$ generates $D(X^{ss,\chi}/G)$, the same holds for $\mathcal{F} = \bigoplus_{\mu \in \mathcal{L}'} \mu \otimes \mathcal{O}_{X^{ss,\chi}} \bigoplus \mathcal{K}$ by (7), and we moreover have $\mathrm{Ext}^1_{X^{ss,\chi}/G}(\mathcal{F}, \mathcal{F}) = 0$. Thus, $\mathrm{End}_R(\bigoplus_{\mu \in \mathcal{L}'} M(\mu) \bigoplus K)$ is an NCCR of R by Corollary 4.4.

Remark 6.4. The discussion on the "universality" of Ψ in fact implies that ψ in (6) is the minimal $\mathrm{add}(\oplus_{\mu \in \mathcal{L}'} M(\mu))$-approximation of $M(2,1)$ in the sense that every map $M(\mu) \to M(2,1)$ for $\mu \in \mathcal{L}'$ factors through ψ.

Remark 6.5. Let $S = k[a,b,c,d,e]$. The module K introduced in the above example may also be described by a matrix factorization (d_0, d_1) of $f = a^3 b - cde$:

$$d_0 = \begin{pmatrix} ab & 0 & ce & 0 \\ 0 & ab & -ac & -cd \\ -d & 0 & -a^2 & 0 \\ -a & -e & 0 & a^2 \end{pmatrix}, \qquad d_1 = \begin{pmatrix} a^2 & 0 & ce & 0 \\ 0 & a^2 & -ac & cd \\ -d & 0 & -ab & 0 \\ a & e & 0 & ab \end{pmatrix},$$

where $d_0, d_1 : S^4 \to S^4$ and $K = \mathrm{coker}(d_0)$.

Appendix A. Grothendieck group of a toric DM stack

Here we recall some results about the Grothendieck group of a toric DM stack. We mainly follow [BH06].

Let Σ be a fan, refining a cone over an $n-1$-dimensional convex lattice polyhedron $P \times \{1\}$. Let $\mathbf{\Sigma}$ be a stacky fan $(\Sigma, (v_i)_{i=1}^l)$, where $v_i \in \mathbb{Z}^{n-1} \times \{1\}$ define 1-dimensional cones in Σ. We denote by $P_{\mathbf{\Sigma}}$ (resp. P_Σ) the corresponding toric DM stack (resp. toric variety). Note that $P_{\mathbf{\Sigma}}$ (resp. P_Σ) equals $Y^{ss,\chi}/G$ (resp. $Y^{ss,\chi}/\!\!/G$) for an action of $G \subset k^{*l}$ on $Y = k^n$ via characters determined by the images of the generators $e_i \in \mathbb{Z}^l$ (e_i denotes the i-th generator) in $\mathbb{Z}^l/\rho(M) \cong X(G)$ ($\rho : m \mapsto (\langle m, v_i \rangle)_i$) and a generic $\chi \in X(G)$ (see [BH06, Section 2], [CLS11, Theorem 15.1.10]).

Let $\mu_i = \bar{e_i} \in X(G)$. We denote by R_i the class of the invertible sheaf $\mu_i \otimes \mathcal{O}_{Y^{ss,\chi}}$ in $K_0(P_{\mathbf{\Sigma}})$.

Theorem A.1. [BH06] *Let $P_{\mathbf{\Sigma}}$ be a toric DM stack. Let B be the quotient of the Laurent polynomial ring $\mathbb{Z}[x_1, x_1^{-1}, \dots, x_l, x_l^{-1}]$ by the ideal generated by the relations*

- $\prod_{i=1}^l x_i^{\langle m, v_i \rangle} = 1$ *for all $m \in M$,*
- $\prod_{i \in I}(1 - x_i) = 0$ *for any set $I \subseteq \{1, \dots, l\}$ such that v_i, $i \in I$, are not contained in any cone of Σ.*

Then the map $\phi : B \to K_0(P_{\mathbf{\Sigma}})$ which sends x_i to R_i is an isomorphism. If P_Σ is a triangulation of a cone over a polyhedron P, then $\mathrm{rk} K_0(P_{\mathbf{\Sigma}}) = (n-1)!\mathrm{Vol}(P)$.

Proof. First part follows by [BH06, Theorem 4.10], while the last statement follows from [BH06, Remark 3.11, Theorem 5.3]. Indeed, we only need to show that $(1-t)^l \sum_{n\in N\cap\sigma} t^{\deg(n)}$ evaluated at 1 equals $(n-1)!\operatorname{Vol}(P)$.

Note that $N\cap\sigma = \bigcup_{d\in\mathbb{N}} dP\times\{d\}$, and $\deg(n)=d$ for $n\in dP\times\{d\}$. Moreover, the number of lattice points in $dP\times\{d\}$ equals $\operatorname{Ehr}_P(d)$, where Ehr denotes the Ehrhart polynomial (see e.g. [CLS11, Theorem 9.4.2]). Since the degree of Ehr is $n-1$ (as P is $n-1$-dimensional) and its leading coefficient equals $(n-1)!\operatorname{Vol}(P)$ (see e.g. [CLS11, Exercise 9.4.7]) we obtain that the above sum evaluated at 1 equals $(n-1)!\operatorname{Vol}(P)$. □

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