

SIMPLICITY OF RINGS OF DIFFERENTIAL OPERATORS IN PRIME CHARACTERISTIC

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ABSTRACT. Let W be a finite dimensional representation of a linearly reductive group G over a field k . Motivated by their work on classical rings of invariants, Levasseur and Stafford asked whether the ring of invariants under G of the symmetric algebra of W has a simple ring of differential operators. In this paper, we show that this is true in prime characteristic. Indeed, if R is a graded subring of a polynomial ring over a perfect field of characteristic $p > 0$ and if the inclusion $R \hookrightarrow S$ splits, then $D_k(R)$ is a simple ring. In the last section of the paper, we discuss how one might try to deduce the characteristic zero case from this result. As yet, however, this is a subtle problem and the answer to the question of Levasseur and Stafford remains open in characteristic zero.

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1. INTRODUCTION

The ring of differential operators on the coordinate ring R of a smooth connected affine algebraic variety over a field k is fairly well understood. This ring, denoted $D_k(R)$, is a simple ring; in the event that k has characteristic zero, $D_k(R)$ is a finitely generated k -algebra. Despite intense study by various authors, however, the structure of $D_k(R)$ for a non-smooth variety remains a mystery. It is known that it need not be simple nor finitely generated in general [3]. A problem remains to identify classes of varieties for which the corresponding ring of differential operators is simple or is finitely generated.

A general expectation persists that rings of differential operators on rings of invariants for linearly reductive groups should have various nice properties [18] [21][25][32]. One such expectation is reflected in the following conjecture.

Conjecture 1.1. Let W be a finite dimensional k -representation of a linearly reductive group G . Let R be the ring of invariants for G acting on the symmetric algebra $S(W)$. Then $D_k(R)$ is a simple ring.

This conjecture was stated as a question by Levasseur and Stafford [18] in the slightly more restrictive setting of G reductive and k of characteristic zero. To date, much research into the structure of rings of differential operators has focused on the case of affine rings over a field of characteristic zero. This may be founded in a (perhaps misguided) feeling that rings of differential operators are better behaved in characteristic zero. In fact, differential operators have an especially interesting form in prime characteristic, and their nice structure enables us to prove a general theorem that implies Conjecture 1.1 in characteristic p .

In a fairly delicate argument using the structure theory of primitive ideals in an enveloping algebra, Levasseur and Stafford were able to prove Conjecture 1.1 for the “classical” rings of invariants in characteristic zero [18]. Earlier, Kantor [14] had proven the conjecture for finite groups, assuming the ground field to be the complex numbers. Van den Bergh has studied some representations of $\mathrm{SL}(2, k)$ when k has characteristic zero [35]. The case of torus invariants in characteristic zero is covered by [22]. Very little had been known about the structure of $D_k(R)$, however, in the case where k has characteristic $p > 0$.

The simplicity of $D_k(R)$ imposes strong restrictions on R . If $D_k(R)$ is a simple ring, then R must be simple as a $D_k(R)$ module (see §4.4). For any reduced ring R , it is easy to see that each minimal prime of R is a $D_k(R)$ submodule of R , so that simplicity of $D_k(R)$ implies that R is a domain. In [32], it is shown that simplicity of $D_k(R)$ implies that R is Cohen-Macaulay. In characteristic $p > 0$, the simplicity of $D_k(R)$ forces the tight closure of an ideal I , usually a very subtle and difficult to compute closure operation, to take an especially simple form [28].

When R is the ring of invariants of a linearly reductive group acting on the symmetric algebra $S = S(W)$ of a finite dimensional k -representation W , the Reynolds operator provides an R module splitting of the inclusion map $R \hookrightarrow S$. The simplicity of $D_k(R)$ may be the result of the elementary algebraic features of R inherited by virtue of its being a direct summand of a regular ring rather than a consequence of some of the more subtle issues arising because of the structure and action of G . We propose the following conjecture.

Conjecture 1.2. Let $R \hookrightarrow S$ be an inclusion of k -algebras, where k is any field. Assume that this inclusion splits as a map of R modules. If $D_k(S)$ is a simple ring, then $D_k(R)$ is a simple ring.

In the question of Levasseur and Stafford, S is a polynomial ring and R is some graded subring. Thus to prove Conjecture 1.1, it would suffice to prove this more general conjecture even just for the case where R is a graded subring of a polynomial ring S . This is exactly what we accomplish in this paper in prime characteristic. The following important corollary follows easily from the main theorem of this paper (Theorem 4.2.1).

Theorem 1.3. *Let R be a graded subring of a polynomial ring S over a perfect field k of characteristic $p > 0$. Assume that the inclusion $R \hookrightarrow S$ splits in the category of graded R modules. Then $D_k(R)$ is a simple ring.*

With the notation of the theorem above, $D_k(R)$ is simply the ring of all additive maps from R to itself linear over some subring R^{p^e} of p^e -th powers of the elements of R (see §2.5). Our work heavily exploits this very interesting description of the ring of differential operators.

The study of differential operators in characteristic $p > 0$ is partially motivated by the connections with the theory of *tight closure*. Tight closure, introduced by M. Hochster and C. Huneke, is a closure operation performed on ideals in a ring of prime characteristic which has led to deep new insight into the structure of commutative rings containing a field. We refer the reader to [11] for more about tight closure. The relationship with differential operators is developed by the first author in [28]. Although this paper is independent of the theory of tight closure, some of the motivation for this work, and many of the ideas within, are inspired by tight closure.

The paper is organized as follows. Section 2 establishes the notation and summarizes the relevant facts to be used throughout the paper. Section 3 contains a study of the behavior of modules under the Frobenius functor. In particular, we introduce the concept of “Finite F-representation type.” This is a fairly strong representation theoretic property of commutative rings of characteristic $p > 0$. Its name is intended to recall the similar (though much stronger) property of finite representation type. In Section 4, the main results about the simplicity of rings of differential operators in characteristic $p > 0$ are developed. In the final section of this paper, we address the question of how one may attempt to deduce the characteristic zero case from this theorem. As yet, however, this problem is a subtle one and the above conjectures remain open in characteristic zero.

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2. GENERALITIES

Throughout this paper, the word “module” always means left module, except where otherwise indicated. The notation M_R indicates that M is a right R -module,

and the notation ${}_R M$ indicates that M is a left R -module. Frequently, an abelian group M may be a module or bi-module over several different rings, in which case the notation M_R (or ${}_R M$) indicates that the structure under consideration is the right (or left) R -module structure.

2.1. Differential Operators: Definitions. Let k be any commutative ring and let R be any commutative k -algebra. For any two R modules M and N , the module $D_{R/k}(M, N)$ (or $D_k(M, N)$ when R is understood) of k linear differential operators from M to N is a certain distinguished submodule of $\text{Hom}_k(M, N)$, defined inductively as follows:

$$D_k(M, N) = \bigcup_{n \in \mathbb{N}} D_k^n(M, N),$$

where

$$D_k^0(M, N) = \text{Hom}_R(M, N) \text{ and}$$

$$D_k^n(M, N) = \{\theta \in \text{Hom}_k(M, N) \mid \text{for all } r \in R, [r, \theta] \in D_k^{n-1}(M, N)\}.$$

The symbol $[r, \theta]$ denotes the commutator operator $(r \circ \theta - \theta \circ r) \in \text{Hom}_k(M, N)$, where the symbol “ \circ ” denotes composition of operators. A differential operator $\theta \in D_k^n(M, N)$ but not in $D_k^{n-1}(M, N)$ is said to be of *order* n . When $M = N$, the differential operators $D_k(M, M)$ form a ring, denoted $D_k(M)$. Note that $D_k(M, N)$ is a $D_k(N) - D_k(M)$ bimodule where the action is given by composition of maps. Of particular interest is the ring of differential operators on R itself, $D_k(R)$. The unadorned symbol $D(R)$ indicates that the base ring k is \mathbb{Z} .

Another, entirely equivalent, point of view is the following. Consider the ring $R \otimes_k R$. Note that $\text{Hom}_k(M, N)$ is a (left) module over $R \otimes_k R$ in an obvious way: an element $r \otimes s$ acts on θ to produce $r \circ \theta \circ s \in \text{Hom}_k(M, N)$. Let $J_{R/k}$ denote the kernel of the multiplication map $R \otimes_k R \rightarrow R$ sending an element $r \otimes s$ to the product rs . Note that $J_{R/k}$ is generated by elements of the form $r \otimes 1 - 1 \otimes r$, and that such an element acts on $\theta \in \text{Hom}_k(M, N)$ to produce the commutator $[r, \theta]$. It is thus clear that an operator $\theta \in \text{Hom}_k(M, N)$ is a differential operator of order less than or equal to n if and only if θ is annihilated by $J_{R/k}^{n+1}$. That is, there are canonical isomorphisms

$$D_k^n(M, N) = \text{Hom}_{R \otimes_k R}((R \otimes_k R)/J_{R/k}^{n+1}, \text{Hom}_k(M, N)) = \text{Hom}_R(P_{R/k}^n \otimes M, N)$$

where $P_{R/k}^n$ is simply $(R \otimes_k R)/J_{R/k}^{n+1}$, regarded as an R -bimodule, and the homomorphisms are as left R modules. By definition, an element m of an S -module \mathcal{M} is in the zero-th local cohomology module $H_J^0(\mathcal{M})$ with support in the ideal $J \subset S$ if m is annihilated by some power of J . Thus, differential operators can be thought of as the elements of the local cohomology module

$$(2.1) \quad D_k(M, N) = H_{J_{R/k}}^0(\text{Hom}_k(M, N)).$$

2.2. Derived functors of differential operators. Of substantial interest to us are the higher derived functors of differential operators. This is because, as we will show in Section 5, they essentially control the behavior of differential operators under reduction to prime characteristic. Denote by $R^i D_k(M, -)$ the right derived functors of the left exact covariant functor $D_k(M, -)$. Clearly

$$R^i D_k(M, N) = \varinjlim_n \text{Ext}_R^i(P_{R/k}^n \otimes_R M, N)$$

Although $D_k(-, N)$ is a left exact contravariant functor (so that it would make sense to compute its derived functors), in general $R^i D_k(M, N)$ is not the derived functor of its first argument; indeed, there is no reason why $R^i D_k(R, M)$ should vanish for $i > 0$ (see example 5.1.1). However, using the “associativity formula” $\text{Hom}_R(M, \text{Hom}_R(P_{R/k}^n, N)) = \text{Hom}_R(P_{R/k}^n \otimes M, N)$, we do see that there are two spectral sequences with the same limit whose E^2 terms are (dropping the subscripts from the notation):

$$\text{Ext}^p(M, \text{Ext}^q(P^n, N)) \quad \text{and} \quad \text{Ext}^q(\text{Tor}_p(P^n, M), N)$$

(see chapter XVI of [4]).

In very good cases, these spectral sequences collapse, as we now describe. Recall that a k -algebra R is *formally smooth* if, for every k -algebra C and every ideal $J \subset C$ such that $J^2 = 0$, the map $\text{Hom}_k(R, C) \rightarrow \text{Hom}_k(R, C/J)$ is surjective ([7], IV.0.19.3.1, or IV.17.1.1). If k is a field, then the coordinate ring R of a smooth affine algebraic variety over k is k -formally smooth, as is any local ring of a smooth k -variety.

Proposition 2.2.1. *Assume that R is formally smooth over k , and that each $P_{R/k}^n$ is a finitely presented (left) R module. Then*

- (1) $R^i D_k(R, N) = 0$ for $i > 0$;
- (2) $R^i D_k(M, N)$ is a derived functor of its first argument;
- (3) Assume in addition that R is noetherian and that M is finitely generated.

Then there are natural identifications

$$\begin{aligned} R^i D_k(M, N) &= \text{Ext}_R^i(M, (N \otimes_R D_k(R))_R) \\ &= \text{Ext}_{D_k(R)}^i(M \otimes_R D_k(R), N \otimes_R D_k(R)) \end{aligned}$$

Proof. The main point is that if R is formally smooth over k , then each $P_{R/k}^n$ is left and right projective as an R module ([7], 16.10.1). It easily follows that

$$\text{Ext}^i(M, \text{Hom}(P^n, N)) \cong \text{Ext}^i(P^n \otimes M, N)$$

for all i (this can be seen also from the collapsing of both spectral sequences at E^2). Taking the direct limit (which commutes with the computation of cohomology), we see that both (1) and (2) hold.

For the first identification in part (3), we have

$$\begin{aligned} R^i D_k(M, N) &= \varinjlim \text{Ext}_R^i(M, \text{Hom}_R(P^n, N)) \\ &\cong \text{Ext}_R^i(M, \varinjlim \text{Hom}_R(P^n, N)) \\ &\cong \text{Ext}_R^i(M, (N \otimes_R D(R))_R). \end{aligned}$$

The first isomorphism above follows from the fact that M is finitely generated and R is noetherian. Under these hypotheses it is easily seen that Ext commutes with direct limits in its second argument.

The last isomorphism uses the assumption that each P^n is finitely presented and projective, in order to verify that the natural map $N \otimes \text{Hom}_R(P^n, R) \rightarrow \text{Hom}_R(P^n, N)$ is an isomorphism. Taking the direct limit, we have the natural isomorphism of right $D(R)$ modules (and hence right R modules) $N \otimes D(R) = D(R, N)$.

For the second identification in (3), we point out that $D(R)$ is a flat R module, since it is a direct limit of projective R modules. Furthermore, for any right $D(R)$ module I , and any right R module M , we have the natural adjointness

$$\mathrm{Hom}_R(M, I_R) = \mathrm{Hom}_{D(R)}(M \otimes_R D(R), I).$$

It follows that any injective right $D(R)$ module is also injective considered as a right R module, since the functor $\mathrm{Hom}_R(-, I_R) = \mathrm{Hom}_{D(R)}(- \otimes_R D(R), I)$ is exact. The computation of $\mathrm{Ext}_R^i(M, (N \otimes_R D(R))_R)$ can therefore be accomplished by resolving $N \otimes D(R)$ by right $D(R)$ injectives, and then viewing this resolution as a resolution by right R modules. In light of the adjointness above, the identification with $\mathrm{Ext}_{D_k(R)}^i(M \otimes_R D_k(R), N \otimes_R D_k(R))$ is immediate. \square

The formula (2.1) suggests a corresponding formula for higher local cohomology. The following proposition shows that this is essentially correct.

Proposition 2.2.2. *Assume that R and M are projective over k and that M is a finitely generated R -module. Assume furthermore that $R \otimes_k R$ is Noetherian¹. Then*

$$R^i D_k(M, N) = H_{J_{R/k}}^i(\mathrm{Hom}_k(M, N))$$

Proof. Because M is a projective k module, the functor $H_{J_{R/k}}^0(\mathrm{Hom}_k(M, -))$ is a left exact covariant functor, and the $\{H_{J_{R/k}}^i(\mathrm{Hom}_k(M, -))\}$ form a δ functor (so that short exact sequences of R modules give rise to long exact sequences involving $H_{J_{R/k}}^i(\mathrm{Hom}_k(M, -))$). By (2.1), $H_{J_{R/k}}^i(\mathrm{Hom}_k(M, -))$ agrees with $R^i D(M, N)$ when $i = 0$. Therefore, to prove the proposition, it will be sufficient to show that $H_{J_{R/k}}^i(\mathrm{Hom}_k(M, -))$ vanishes on injective R modules when $i > 0$. Now

$$H_{J_{R/k}}^i(\mathrm{Hom}_k(M, I)) = \varinjlim_n \mathrm{Ext}_{R \otimes_k R}^i(P^n, \mathrm{Hom}_k(M, I))$$

Let A^\cdot be a projective $R \otimes_k R$ -resolution of P^n . We have

$$\begin{aligned} \mathrm{Ext}_{R \otimes_k R}^i(P^n, \mathrm{Hom}_k(M, I)) &= H^i(\mathrm{Hom}_{R \otimes_k R}(A^\cdot, \mathrm{Hom}_k(M, I))) \\ &= H^i(\mathrm{Hom}_R(A^\cdot \otimes_R M, I)) \\ &= \mathrm{Hom}_R(H_i(A^\cdot \otimes_R M), I) \end{aligned}$$

The assumption that R is k -projective implies that any projective $R \otimes_k R$ module is also a projective R module (on either side). Thus the $R \otimes R$ projective resolution A^\cdot can be viewed as a projective resolution of P^n as a right R -module. So

$$H_i(A^\cdot \otimes_k M) = \mathrm{Tor}_i^R((P^n)_R, M)$$

and we obtain

$$(2.2) \quad H_{J_{R/k}}^i(\mathrm{Hom}_k(M, I)) = \varinjlim_n \mathrm{Hom}_R(\mathrm{Tor}_i^R(P^n, M), I)$$

Now let B^\cdot be a resolution of M by finitely generated projective R -modules. Then

$$\mathrm{Tor}_i^R(P^n, M) = H_i(P^n \otimes B^\cdot) = H_i(\tilde{B}^\cdot / J_{R/k}^n \tilde{B}^\cdot)$$

where for $B \in R\text{-mod}$, \tilde{B} denotes the $R \otimes_k R$ -module $R \otimes_k B$.

¹ R is a quotient of $R \otimes_k R$, so if $R \otimes_k R$ is Noetherian, then so is R . The converse is false; a counter example is given by a rational function field in infinitely many variables.

Lemma 2.2.3 below implies that for $i > 0$ there exists an m such that for all n the maps

$$H_i(\tilde{B}/J_{R/k}^{n+m}\tilde{B}) \rightarrow H_i(\tilde{B}/J_{R/k}^n\tilde{B})$$

are zero. This implies that the right hand side of (2.2) is zero when $i > 0$, and so we are done. \square

The following (well-known) lemma, used above, is essentially a more precise statement of the flatness of completion at an arbitrary ideal in a Noetherian ring.

Lemma 2.2.3. *Let S be a commutative, Noetherian ring and let $J \subset S$ be an ideal. Let*

$$P \xrightarrow{\psi} Q \xrightarrow{\phi} T$$

be an exact sequence of finitely generated S -modules. Denote by ψ_n, ϕ_n the induced maps

$$P/J^n P \xrightarrow{\psi_n} Q/J^n Q \xrightarrow{\phi_n} T/J^n T$$

and let M_n be the middle homology of this sequence. Then there exist $m \geq 0$ such that the induced maps $M_{m+n} \rightarrow M_n$ are zero for all n .

Proof. This is an immediate consequence of the Artin-Rees lemma. The Artin-Rees lemma asserts that given a J -adic filtration on a finitely generated module over a Noetherian ring, the induced filtration on any submodule is “eventually” J -stable (see any text on commutative algebra, e.g. [1]). In our situation, we see that there exists m such that for all n

$$\phi(Q) \cap J^{n+m}T \subset J^n\phi(Q)$$

Now let $q \in Q$ represent, modulo $J^{m+n}Q$, an element of M_{n+m} . Thus

$$\phi(q) \in J^{n+m}T \cap \phi(Q) \subset J^n\phi(Q) = \phi(J^n Q)$$

So there exists $a \in J^n Q$ such that $\phi(q - a) = 0$, so that $q - a = \psi(p)$ for some $p \in P$. This yields $q \bmod J^n Q = \psi_n(p \bmod J^n P)$ and thus the image of q in M_n is zero. \square

Below we indicate some corollaries of Proposition 2.2.2. They seem to suggest that the $R^i D_k(M, N)$ are interesting objects in their own right.

Corollary 2.2.4. *Let $S \rightarrow R$ be a surjective map of k -algebras satisfying the hypotheses of Proposition 2.2.2. Let M and N be R -modules, where M also satisfies the hypotheses of 2.2.2.*

(1) *There is a natural identification*

$$R^i D_{R/k}(M, N) = R^i D_{S/k}(sM, sN)$$

(2) *If R is finitely generated over k , then $R^i D_k(M, N) = 0$ for i exceeding the minimal number of algebra generators for R over k . If R is finitely generated over a field, then $R^i D_k(M, N) = 0$ for i exceeding twice the Krull dimension of R .*

(3) *The functors $\{R^i D(M, N)\}_i$ form a δ -functor in M on the sub-category of finitely generated R modules which are projective as k -modules.*

Proof. The first assertion follows because $J_{R/k} = J_{S/k}R$ and in general for local cohomology $H_{JR}^i(A) = H_J^i({}_S A)$ where $J \subset S$ is an ideal in a Noetherian ring, R is a Noetherian S -algebra, and A is any R module. For (2), note that J is generated by $x \otimes 1 - 1 \otimes x$ where x ranges through the k -algebra generators for R , and that the Krull dimension of $R \otimes_k R$ is twice the dimension of R , for finitely generated algebras over a field. For (3), note that $\text{Hom}_k(-, N)$ is exact, so that $H_J^0(\text{Hom}_k(-, N))$ is left exact, and thus the long exact sequence property follows from that of $H_J^0(-)$. \square

2.3. The Frobenius Functor. We first recall some general facts about rings of prime characteristic $p > 0$. The letter e always denotes a non-negative integer and q always denotes an integer power $q = p^e$ of p .

For each $q = p^e$, the set R^q of q^{th} powers in R forms a subring. We have a descending chain of subrings of R :

$$R \supset R^p \supset R^{p^2} \supset R^{p^3} \supset \dots$$

Every perfect field contained in R is contained in every subring R^q ; in particular, $\mathbb{Z}/p\mathbb{Z} \subset \cap R^q$. Of course, any R module M may be viewed as a module over any one of these subrings R^q .

The process of viewing an R module M as a module over one of these subrings is often described in terms of the ‘‘Frobenius functor’’ on the set of R modules. The Frobenius map is the ring map $F : R \rightarrow R$ sending $r \mapsto r^p$. We often consider not just the Frobenius map, but also its iterates $F^e : R \rightarrow R$ sending $r \mapsto r^q$. Assuming that R is reduced, the Frobenius map F^e is an isomorphism onto its image R^q . For any R module M , we denote by ${}^e M$ the module M with its R module structure pulled back via F^e . As an abelian group, ${}^e M$ is the same as M , but its R module structure is given by $r \cdot m = F^e(r)m = r^q m$.

In this paper, we will frequently be examining the structure of R as an R^q module, or equivalently, the structure of ${}^e R$ as an R module. For reduced R , there is yet a third way to view this algebra extension which is sometimes more convenient. We have an extension $R \subset R^{1/q}$ where $R^{1/q}$ is simply the over-ring of q^{th} roots of elements in R . The Frobenius map F^e affords an isomorphism of $R \hookrightarrow R^{1/q}$ with $R^q \hookrightarrow R$. The descending chain above amounts to an ascending chain of over-rings of R :

$$R \subset R^{1/p} \subset R^{1/p^2} \subset \dots$$

We use the notation ${}^e m$ to indicate that an element m in an R module M is being regarded as an element in ${}^e M$. In particular, for an element x in a reduced ring R , the element ${}^e x$ in the R module ${}^e R$ corresponds to the element $x^{1/q}$ in the R module $R^{1/q}$ under the correspondence discussed above.

Although our formal statements will most often involve the Frobenius functor notation, the reader is encouraged to bear all three interpretations in mind. Depending on the context, one of the following three equivalent notions may make a particular statement the most transparent: the R^q module R , the R module ${}^e R$, or the R module $R^{1/q}$.

2.4. Strong F-Regularity. Throughout this section, we assume that R is finitely generated as a module over its subring R^p of p^{th} powers. This weak assumption (often called ‘‘F-finiteness’’) is satisfied whenever R is a finitely generated algebra over a perfect field k or whenever R is a complete local Noetherian ring with a

perfect residue field k . More generally, k need not even be perfect, as long as $[k : k^p] < \infty$. F-finiteness is preserved under localizations.

Two properties of F-finite rings will be of particular interest to us.

Definition 2.4.1. If the Frobenius map splits, we say that R is F-split. That is, R is F-split if the inclusion $R^p \hookrightarrow R$ splits as a map of R^p modules.

This property is also called F-purity in the literature. Strictly speaking, F-purity, the property that the Frobenius map is pure, is a weaker condition than F-splitting. However, for rings R finitely generated over their subrings R^p , F-splitness is equivalent to F-purity (see [12]).

Note that if $R^p \subset R$ splits as a map of R^p modules, then for all q , $R^q \subset R$ splits over R^q .

A stronger, but closely related property is *strong F-regularity*.

Definition 2.4.2. A reduced F-finite ring is said to be strongly F-regular, if for all $c \in R$ not in any minimal prime, there exists $q = p^e$ such that the map $R \rightarrow R^{1/q}$ sending $1 \mapsto c^{1/q}$ splits as an R -module homomorphism.

The property of strong F-regularity was introduced by Hochster and Huneke in [10]. They have conjectured that all ideals of R are tightly closed² if and only if R is strongly F-regular. This is known to be true for Gorenstein rings [10], for rings of Krull dimension no more than three [36], and for rings with (at worst) isolated singularities [19].

It is not hard to see that strongly F-regular rings are F-split. The relationship of this property to the study of differential operators is apparent from the following fact, proved in [28].

Theorem 2.4.3. *Let R be reduced ring finitely generated over R^p . Then R is strongly F-regular if and only if it is F-split and it is simple as a left module over its ring of \mathbb{Z} linear differential operators.*

Strongly F-regular rings are always Cohen-Macaulay and normal [10]. In particular, any local strongly F-regular ring is a domain.

One more fact we will need about strong F-regularity follows directly from the definitions (see [10]):

Theorem 2.4.4. *If $R \subset S$ splits as a map of R modules and S is strongly F-regular, then R is also strongly F-regular.*

2.5. Rings of Differential Operators in Prime Characteristic. Rings of differential operators are especially interesting in prime characteristic. Let R be a reduced ring finitely generated over its subring R^p of p^{th} powers. It is not difficult to check that

$$(2.3) \quad D_{\mathbb{Z}}(M, N) = \bigcup_{q=p^e} \text{Hom}_{R^q}(M, N)$$

We recall Yekutieli's proof here [37] for convenience of the reader.

²This paper is independent of the theory of tight closure, but see [11] for more about this beautiful subject.

Suppose that θ is a differential operator of order $\leq n$. For each $q > n$, $J_{R/\mathbb{Z}}^q \theta = 0$. In particular $(r \otimes 1 - 1 \otimes r)^q = r^q \otimes 1 - 1 \otimes r^q$ kills θ . Thus for all $r \in R$, $r^q \theta = \theta r^q$, and θ is R^q linear.

For the converse, suppose that R is generated by x_1, x_2, \dots, x_d as an algebra over R^p . The same elements generate R over R^q , and thus for all e , J_{R/R^q} is generated by $(x_1 \otimes 1 - 1 \otimes x_1), \dots, (x_d \otimes 1 - 1 \otimes x_d)$. In particular, $J_{R/R^q}^{dq} \subset ((x_1^q \otimes 1 - 1 \otimes x_1^q), \dots, (x_d^q \otimes 1 - 1 \otimes x_d^q)) = 0$. Therefore, any element $\theta \in \text{Hom}_{R^q}(M, N)$ is killed by J^{dq} and is therefore an R^q linear differential operator. In particular, θ is a differential operator from M to N linear over any perfect field contained in R .

Remark 2.5.1. It follows from the proof that whenever R is finitely generated over its subring R^p of p^{th} powers,

$$D_{\mathbb{Z}}(M, N) = D_{(\mathbb{Z}/p\mathbb{Z})}(M, N) = D_k(M, N) = \bigcup_{q=p^e} \text{Hom}_{R^q}(M, N)$$

where k is any perfect field contained in R .

3. THE BEHAVIOR OF MODULES UNDER FROBENIUS

3.1. Finite F-representation type. Our goal is to understand the structure of $D_{\mathbb{Z}}(R) = \bigcup_{q=p^e} \text{End}_{R^q}(R)$. We are led to analyze the structure of R as an R^q module, or equivalently, of ${}^e R$ as an R module, as $e \rightarrow \infty$. This task will be much easier if there are only finitely many isomorphism classes of R modules that appear as indecomposable summands of ${}^e R$. This observation leads naturally to the introduction of rings with *Finite F-representation type*.

Finite F-representation type (FFRT) is a representation theoretic property of commutative rings that makes sense only in characteristic p . To be able to state it we need a class of rings for which the Krull-Schmidt theorem holds. We also need that R is a finitely generated R^p -module. Therefore we usually restrict ourselves to the following two classes of rings.

- (A) Complete local Noetherian rings with residue field k having the property $[k : k^p] < \infty$ (the “complete” case).
- (B) \mathbb{N} -graded rings of the form $R = k \oplus R_1 \oplus R_2 \oplus \dots$ with k a field such that $[k : k^p] < \infty$ and with R finitely generated over k (the “graded” case).

Throughout this section, R will always denote a ring of either type (A) or type (B).

Let $M(R)$ stand for the category of finitely generated R -modules (resp. finitely generated \mathbb{Q} -graded R modules). For $M \in M(R)$ we let $[M]$ stand for the isomorphism class of M . In the graded case, we do not require the isomorphism to be degree preserving: $[M] = [N]$ if and only if $N \cong M(\alpha)$ via a degree preserving isomorphism where $M(\alpha)$ is the graded R -module defined by $M(\alpha)_i = M_{i+\alpha}$. If $M \in M(R)$ then recall that the notation ${}^e M$ means that one is viewing M as an R -module via restriction of scalars from the Frobenius map. In the graded case we grade ${}^e M$ by putting $({}^e M)_\alpha = M_{p^e \alpha}$.

By the Krull-Schmidt theorem there is a decomposition in $M(R)$

$${}^e R = M_1^{(e)} \oplus \dots \oplus M_{n_e}^{(e)}$$

with $M_i^{(e)}$ indecomposable. When R is reduced, the reader may find it easier to think about the (canonically) isomorphic decomposition of $R^{1/q}$ as an R module where $q = p^e$. See Section 2.3

Definition 3.1.1. We say that R has Finite F-representation type (FFRT) if the set

$$\{[M_i^{(e)}] \mid e \in \mathbb{N}, i = 1, \dots, n_e\}$$

is finite. That is, R has FFRT if there exists a finite set \mathcal{S} of isomorphism classes of R modules such that any indecomposable R module summand of $R^{1/q}$, for any $q = p^e$, is isomorphic to some element of \mathcal{S} .

If R is regular, then it is either a polynomial ring or a power series ring over k and eR is free over R . Thus

Observation 3.1.2. Regular rings have finite F-representation type.

More generally, suppose that R is Cohen-Macaulay. Then eR is a Cohen-Macaulay R module of maximal dimension, and hence so are all $M_i^{(e)}$. Now recall that R is said to be of *finite representation type* if it has only a finite number of indecomposable maximal Cohen-Macaulay modules, up to isomorphism. So we obtain:

Observation 3.1.3. If R is Cohen-Macaulay and has finite representation type, then R has finite F-representation type.

This observation applies, for example, to quadric hypersurfaces in at least three variables that have an isolated singularity [15].

Below we will show that finite representation type is much stronger than FFRT. Our first result in this direction is the following.

Proposition 3.1.4. *Assume that $R \subset S$ is an inclusion of rings of type (A) or (B) such that S is a finite R -module and such that fR is a direct summand of fS as R -modules for some f . Then if S has FFRT, then so does R .*

Proof. We have to show that the indecomposable summands of all eR are finite in number as $e \rightarrow \infty$. It is sufficient to consider the case $e \geq f$. For each $e \geq f$, eR is a direct summand of eS as an R module. Let M be an indecomposable R module summand of eR , $e \geq f$. Because eR is an R module summand of eS , M is also an R module summand of eS . Since the S module decomposition of eS is automatically an R module decomposition as well (though not necessarily into indecomposables), M must be an R module summand of some indecomposable S summand of eS . Since the latter are finite in number by hypothesis, the number of possibilities for M is also finite. \square

This proposition shows that rings of invariants of regular rings under finite groups of order prime to the characteristic will have FFRT. On the other hand, in dimension three or higher, these rings of invariants do not usually have finite representation type. Indeed, every ring with finite representation type must have an isolated singularity [2], whereas rings of invariants need not. Yoshino's book is a good source of information about rings of finite representation type [38].

Now we discuss FFRT in the graded case. The following lemma shows that there is a restriction, independent of e , on the degrees of the generators of eR as R -module.

Lemma 3.1.5. *Let R be a graded ring as in (B). Assume that R is generated as k -algebra by generators in degrees a_1, a_2, \dots, a_n . Then eR is generated as R -module by generators whose degrees are contained in the half open interval $[0, a_1 + \dots + a_n[$.*

Proof. We may assume that R is a polynomial ring $k[X_1, \dots, X_n]$, $\deg X_i > 0$. Then $R^{1/q} \cong k^{1/q}[X_1^{1/q}, \dots, X_n^{1/q}]$, $q = p^e$. Let u_1, \dots, u_q be the generators of $k^{1/q}$ as k -vector space. Then the generators of $R^{1/q}$ as R -module are given by $u_i X_1^{\alpha_1/q} \dots X_n^{\alpha_n/q}$, $0 \leq \alpha_j \leq q - 1$. The degree of such a generator is given by $(\sum a_i \alpha_i)/q$, which lies indeed in $[0, a_1 + \dots + a_n[$. \square

Proposition 3.1.6. *Assume that $R \subset S$ are graded rings satisfying (B). Assume furthermore that $[S_0 : R_0] < \infty$ and that fR is an R -module direct summand of fS for some f . Then if S has FFRT, so does R .*

Proof. We have to describe the indecomposable R module summands of eR . As before, we may assume $e \geq f$.

Let N_1, \dots, N_t be the list of indecomposable S module summands of eS , up to isomorphism and shift of degree. Without loss of generality, we may, and we will, assume that the grading on the N_i is normalized in such a way that $(N_i)_0 \neq 0$ and $(N_i)_\alpha = 0$ for $\alpha < 0$.

For any \mathbb{Q} -graded S -module M we have a decomposition as S -module

$$M = \bigoplus_{\alpha \in [0, 1[\cap \mathbb{Q}} [M]_{\alpha \bmod \mathbb{Z}}$$

where the index $\alpha \bmod \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$ gives rise to $[M]_{\alpha \bmod \mathbb{Z}} = \bigoplus_{n \in \mathbb{Z}} M_{\alpha+n}$. Applying this decomposition to the indecomposable N_i we obtain $N_i = [N_i]_{0 \bmod \mathbb{Z}}$. Thus the N_i are \mathbb{N} -graded.

Now let M be an indecomposable graded summand of eR , $e \geq f$. Note that any minimal generator of M is a minimal generator of eR , so by Lemma 3.1.5, the degrees of the minimal generators of M are bounded above by a , where a is independent of e . Furthermore, M is also an R -module summand of eS and hence an R -module summand of some $N_i(\alpha)$, $\alpha \leq 0$. Thus $M' = M(-\alpha)$ is a R -summand of N_i . Note that the minimal generators of M' have degrees bounded by $\leq a + \alpha \leq a$, where a is independent of e .

So we have shown that any direct R -summand of eR is, after shifting degrees, a direct summand of some N_i and is generated in degree $\leq a$. Now let

$$(3.1) \quad M'_1, M'_2, \dots, M'_n, \dots$$

be a (possibly infinite) list of non-isomorphic direct summands of N_i , generated in degree $\leq a$. By the Krull-Schmidt theorem, $M'_1 \oplus \dots \oplus M'_n$ is a direct summand of N_i for all n . Now since all $[N_i]_k$ are finite dimensional S_0 -vector spaces, they also are finite dimensional R_0 vector spaces. Furthermore all M'_i contain at least one element in degree $\leq a$. So comparing dimensions in degrees $\leq a$, we see that the list (3.1) must be finite.

Hence there are only a finite number of non-isomorphic direct summands generated in degree $\leq a$ of a given N_i . Since the N_i are also finite in number we are done. \square

This proposition applies for example to rings of invariants of *linearly* reductive groups. A linearly reductive group is one for which all finite dimensional representations are completely reducible. In characteristic $p > 0$, the only linearly reductive

groups are extensions of finite groups whose order is prime to p by tori [23]. For groups defined over a field of characteristic zero, linearly reductive is equivalent to reductive [8]. In the next section, we discuss rings of invariants of linearly reductive groups in characteristic p in some detail.

Example 3.1.7. The cubic cone, given by $R = k[X, Y, Z]/(X^3 + Y^3 + Z^3)$, is example of a graded ring which does not have FFRT, at least when k is algebraically closed of characteristic $p = 1 \pmod{3\mathbb{Z}}$. Indeed, in [30], Tango uses Atiyah's classification of vector bundles to give an explicit decomposition of the structure sheaf $F_*\mathcal{O}_X$ of an elliptic curve X viewed as an \mathcal{O}_X module via the Frobenius endomorphism. His method easily generalizes to give a decomposition of $F_*^e\mathcal{O}_X$ over \mathcal{O}_X , and hence of eR over R , where R is the coordinate ring for $X \subset \mathbb{P}^2$ (e.g. the cubic cone above). In the case where X is not super-singular (equivalently, where R is F-split), the decomposition is

$$F_*^e\mathcal{O}_X \cong \bigoplus_{i=0}^{q-1} \mathcal{L}_i,$$

where the \mathcal{L}_i run through the various (non-isomorphic) degree zero line bundles corresponding to the $q = p^e$ distinct q -torsion points³ of X . This decomposition corresponds to an R module decomposition $({}^eR)_{0 \bmod \mathbb{Z}} \cong \bigoplus_{i=0}^{q-1} M_i$, where the M_i are indecomposable graded R modules whose sheafifications produce the \mathcal{L}_i .

There are infinitely many non-isomorphic M_i as $e \rightarrow \infty$. To see this, note that if $M_i \cong M_j(n)$ for some integer n , then $\mathcal{L}_i \cong \mathcal{L}_j \otimes \mathcal{O}_X(n)$ as \mathcal{O}_X modules. Because each \mathcal{L}_i has degree zero, this is impossible unless $n = 0$, a contradiction if $i \neq j$. Since $({}^eR)_{0 \bmod \mathbb{Z}}$ is a direct summand of eR , we infer that R does not have FFRT.

3.2. Invariants under linearly reductive groups. In this section we give a more explicit description of the indecomposable summands of eR when R is a ring of invariants for the action of a linearly reductive group on the symmetric algebra $S = S(W)$ of some representation W over a perfect field k . This section is independent of the rest of the paper.

Let U be another representation of G . Then $R(U) = (S \otimes U)^G$ is an R -module, called a *module of covariants*. Clearly $R(U \oplus V) = R(U) \oplus R(V)$. For the basic properties of modules of covariants (in characteristic zero) see [34][31]. Below we show that there is a finite set of irreducible representations $(U_i)_{i=1, \dots, n}$ such that all eR can be written as direct sums of copies of $R(U_i)$. These U_i will be described in terms of *Frobenius twists* of the original representation, so we now digress to review this idea; see [13] for a more detailed discussion.

3.2.1. Frobenius Twisted Representations. Let G be a group and let U be any finite dimensional representation over a perfect field k . As before, we may consider the k module eU defined as the abelian group U but with k structure defined via Frobenius: $\lambda \cdot v = \lambda^{p^e} v$ for $\lambda \in k$ and $v \in {}^eU$. Because $G \rightarrow GL_k(U) \rightarrow GL_k({}^eU)$, it is clear that eU is also a representation of G . This representation is called a *Frobenius Twist* of U . It has traditionally been more standard in representation theory circles to denote this representation by $U^{(-e)}$. In order to be consistent with this tradition, we will adopt this notation here as well.

³The p^e -torsion points on a non-super-singular elliptic curve form a cyclic group of order p^e [26, p 137].

The apparent anomaly of notation is motivated by the following observation. Fix a basis e_1, \dots, e_n for U . Suppose that the representation of G is defined by $g \cdot e_i = \sum_{j=1}^n c_{ij}(g)e_j$ for some functions c_{ij} on G . Because the e_i also form a basis for eU (assuming k is perfect), the action of G on eU is described by $g \cdot e_i = \sum_{j=1}^n (c_{ij}(g))^{p^{-e}} \cdot e_j$. Therefore, a collection of matrices defining the representation $U^{(-e)} (= {}^eU)$ is given by raising each of the entries in a defining collection of matrices for U to the p^{-e} -th power.

This explicit description of the representation $U^{(-e)}$ brings to light an important point. Suppose that $G \rightarrow GL_k(U)$ is an algebraic representation, *i.e.*, that G is an algebraic group defined over k and that the functions c_{ij} above are rational functions on G . The twisted representation $G \rightarrow GL_k({}^eU)$ is not *a priori* an algebraic representation, as the functions $c_{ij}^{p^{-e}}$ need not be in $k[G]$. The representation $U^{(e)}$ is algebraic if $e \geq 0$. For $e < 0$, $U^{(e)}$ is not algebraic in general, though $U^{(e)}$ is easily checked algebraic for all $e \in \mathbb{Z}$ when G is a finite group.

When G is a torus, the fact that the representations eU are not algebraic can be side-stepped by appropriately grading the representation U . Thus, though the ideas are similar, there are some slight differences between our treatments of the finite group case and the torus case. With a little effort, but at the cost of some extra technicality, similar results can be proved for the case where G is an extension of a finite group by a torus. We leave this to the reader.

3.2.2. G finite. Let G be a finite group and let W be a finite dimensional G representation over a perfect field k . Let S be the symmetric algebra $S = S(W)$ and let $S_+ = \bigoplus_{n>0} S_n$. The notation $S_+^{[p^e]}$ denotes the ideal of S generated by the p^e -th powers of the elements in S_+ .

Proposition 3.2.1. *With notation as above, set $R = S^G$, and assume p does not divide $|G|$.*

- (1) *As R -modules we have*

$${}^eR \cong \left((S/S_+^{[p^e]})^{(-e)} \otimes_k S \right)^G$$

In particular, eR is a module of covariants.

- (2) *Let U_1, \dots, U_n be the list of irreducible representations of G occurring in $(S^l W)^{(f)}$ for some l, f . Then eR is a direct sum of modules of covariants $R(U_i)$ and conversely every such module of covariants $R(U_i)$ is a direct summand of some eR .*

Proof. For (1), note that we have a G -equivariant surjective map $S \rightarrow S/S_+^{[p^e]}$. Since G is linearly reductive this map splits G -equivariantly. This yields a G -equivariant map of S^{p^e} -modules

$$(3.2) \quad S/S_+^{[p^e]} \otimes_k S^{p^e} \rightarrow S$$

Since these are both projective S^{p^e} -modules, and (3.2) is clearly an isomorphism after the base change $\otimes_{S^{p^e}} S^{p^e}/S_+^{[p^e]}$, Nakayama's lemma yields that (3.2) is an isomorphism. Hence

$$(3.3) \quad S/S_+^{[p^e]} \otimes_k S^{p^e} \cong {}^eS$$

as (G, S) modules, where S acts on the left hand side of (3.3) through F^e , that is, $s \cdot (\bar{x} \otimes r^{p^e}) = (\bar{x} \otimes s^{p^e} r^{p^e})$. Furthermore, the bijection

$$(3.4) \quad (S/S_+^{[p^e]})^{(-e)} \otimes S \rightarrow S/S_+^{[p^e]} \otimes S^{p^e} : \bar{x} \otimes s \mapsto \bar{x} \otimes s^{p^e}$$

is easily checked to be an isomorphism of (G, S) -modules, where S now acts in the natural way on the left hand side of (3.4). Since $({}^e S)^G = {}^e R$, this proves (1).

For (2), we have to decompose $(S/S_+^{[p^e]})^{(-e)}$ into irreducible G -representations. Now all irreducible representations of G are defined over some finite extension of the prime field. Hence there exists a u such that for every finite dimensional G -representation one has $U^{(u)} \cong U$.

Now every U_i occurs by definition in some $(S^l W)^{(-e)}$. Because of the remark in the previous paragraph we may assume $e \gg 0$. Then U_i also occurs in $(S/S_+^{[p^e]})^{(-e)}$. Hence $R(U_i)$ is a direct summand of ${}^e R$.

Conversely $(S/S_+^{[p^e]})^{(-e)}$ is a quotient of $S^{(-e)}$ and hence is a direct sum of the U_i . So ${}^e R$ is a direct sum of the $R(U_i)$. \square

3.2.3. G a torus. Let G be a torus, split over the perfect ground field k . Let $X(G)$ be the character group of G ; we will use additive notation in working with $X(G)$. The notation $X(G)_{\mathbb{Q}}$ stands for the group $X(G) \otimes_{\mathbb{Z}} \mathbb{Q}$.

For $\chi \in X(G)$ we denote by L_{χ} the corresponding one-dimensional G -representation. Let W be any finite dimensional representation of G over k . Diagonalizing the action of G on W , we have a decomposition $W \cong \bigoplus L_{\alpha}$ where $\alpha \in X(G)$ runs through the weights of W . The symmetric algebra $S = S(W) = S(\bigoplus L_{\alpha})$ is an $X(G)$ graded k -algebra. The graded component S_{χ} corresponding to $\chi \in X(G)$ consists of all elements $s \in S$ for which $g \cdot s = \chi(g)s$. The k -algebra ${}^e S = S({}^e W)$ can be given an $X(G)_{\mathbb{Q}}$ grading. In general, for any $X(G)_{\mathbb{Q}}$ graded object U , let ${}^e U$ denote the same abelian group U , but with the grading shrunk by $[{}^e U]_{\alpha} = [U]_{\alpha p^e}$.

The decomposition of ${}^e R$ will be described in terms of the *strongly critical characters* of W . Recall the definition.

Definition 3.2.2. Let $\alpha_1, \dots, \alpha_d$ be the weights of W . A character $\chi \in X(G)$ is *strongly critical* with respect to W if $\chi = \sum_i u_i \alpha_i$ in $X(G)_{\mathbb{Q}}$ with $u_i \in]-1, 0] \cap \mathbb{Q}$.

The importance of this property is that it is a tractable criterion for $R(L_{-\chi})$ to be Cohen-Macaulay [29][33].

The following result covers the torus case.

Proposition 3.2.3. *Let G, W be as above. Let $\chi_1, \dots, \chi_n \in X(G)$ be those characters that are strongly critical with respect to W and that are weights of some $S^l W$. Then all ${}^e R$ are direct sums of the modules of covariants $R(L_{-\chi_i})$ and conversely every such module of covariants $R(L_{-\chi_i})$ is a direct summand of some ${}^e R$.*

Proof. The proof parallels that of Proposition 3.2.1. First diagonalize the action of G on W so as to assume that $S = k[x_1, \dots, x_d]$ where $g \cdot x_i = \alpha_i(g)x_i$ for $g \in G$. For any character $\chi \in X(G)$, it is easy to check that the module of covariants $R(L_{\chi})$ is R -isomorphic to the graded piece $S_{-\chi}$. Note that this is zero unless $-\chi$ is some weight of some $S^l(W)$. For an $X(G)_{\mathbb{Q}}$ graded object U , we employ the notation $\text{Supp } U$ to indicate the set $\{\chi \in X(G)_{\mathbb{Q}} \mid U_{\chi} \neq 0\}$.

The analog of (3.3)(3.4) is

$${}^e S \cong (S/S_+^{[p^e]}) \otimes_k S$$

as $X(G)_{\mathbb{Q}}$ graded S -modules. If we decompose $(S/S_+^{[p^e]})$ as $\bigoplus L_{\chi}$, then ${}^e(S/S_+^{[p^e]}) \cong \bigoplus {}^e L_{\chi} \cong \bigoplus L_{\chi/p^e}$.

So

$${}^e R = ({}^e S)_0 \cong \left({}^e(S/S_+^{[p^e]}) \otimes_k S \right)_0 \cong \bigoplus_{\chi} R(L_{-\chi/p^e})$$

where in this direct sum $\frac{\chi}{p^e}$ runs through

$$- \text{Supp} \left({}^e(S/S_+^{[p^e]}) \right) \cap \text{Supp} S$$

with appropriate multiplicities.

Now

$$(3.5) \quad - \text{Supp} \left({}^e(S/S_+^{[p^e]}) \right) = \left\{ \sum -\frac{v_i}{p^e} \alpha_i \mid 0 \leq v_i \leq p^e - 1 \right\}$$

and this is contained in the set of strongly critical characters.

Conversely let $\chi = \sum_i u_i \alpha_i$, $u_i \in]-1, 0]$ be a strongly critical weight, contained in $\text{Supp} S$. Then $\chi = \sum_i a_i \alpha_i$, $a_i \in \mathbb{N}$. Assume that n is a common denominator for the $(u_i)_i$. Then we can find for all $\epsilon > 0$, u and e such that

$$1 \leq \frac{un}{p^e} \leq 1 + \epsilon$$

Put $\eta = \frac{un}{p^e} - 1$ and put $v_i = u_i + \eta(u_i - a_i)$. Clearly $\chi = \sum_i v_i \alpha_i$ and furthermore $v_i \in]-1, 0]$ if we choose ϵ small enough. Since the denominators of the v_i divide p^e we find that $\chi \in \text{RHS}(3.5)$. This concludes the proof. \square

3.3. Growth. In this section, we say more about the structure of R as an R^q module for rings of finite F-representation type.

Assume that R has finite F-representation type. Then there is a finite set of indecomposable modules in $M(R)$ such that for each e ,

$${}^e R = M_1^{\oplus a_1} \oplus \dots \oplus M_n^{\oplus a_n}$$

(with appropriate shifts in the graded case).

Obviously the *multiplicities* a_1, \dots, a_n depend on e . In this section we show that under the additional hypothesis of strong F-regularity these multiplicities grow like p^{de} where d is the Krull dimension of R .

For $M, N \in M(R)$ indecomposable, $e \in \mathbb{N}$ let us denote by $m(e, M, N)$ the multiplicity of M in ${}^e N$. Note that

$$(3.6) \quad m(e+f, M, N) = \sum_K m(e, M, K) m(f, K, N)$$

where the sum runs through all isomorphism classes of indecomposable objects in $M(R)$. This formula follows from the observation that ${}^{e+f} N = {}^e({}^f N) = \bigoplus_K {}^e K^{m(f, K, N)} = \bigoplus_K \bigoplus_M M^{m(e, M, K) m(f, K, N)}$ but also ${}^{e+f} N = \bigoplus_M M^{m(e+f, M, N)}$.

Proposition 3.3.1. *Let R be complete or graded as in (A) or (B). Assume in addition that R is strongly F-regular and has FFRT. Let d be the Krull dimension of R and let M_1, \dots, M_n be the list of indecomposable summands of ${}^e R$ as e ranges over all natural numbers. Then*

$$\lim_{e \rightarrow \infty} \frac{m(e, M_i, M_j)}{p^{de}}$$

exists and is strictly positive.

Proof. Put $e_{ij} = m(1, M_i, M_j)$ and let E be the $n \times n$ matrix whose $(ij)^{th}$ entry is (e_{ij}) . From (3.6) it follows that

$$(3.7) \quad m(e, M_i, M_j) = (E^e)_{ij},$$

the $(ij)^{th}$ entry of the matrix E^e obtained by multiplying E by itself e times. We claim that some power E^u of E has strictly positive entries. This means that $m(u, M_i, M_j) > 0$ for all i, j .

Since $m(u+v, M_i, M_j) \geq m(u, M_i, R)m(v, R, M_j)$, it is sufficient to show that there exists u_0 such that for all $u \geq u_0$, and all i , we have $m(u, M_i, R) > 0$ and $m(u, R, M_i) > 0$.

Because R is strongly F-regular, it is F-split, and thus each ${}^e R$ is a direct summand of ${}^{e+1}R$. In particular, once an R module M_i appears as a direct summand of some ${}^e R$, it appears as a direct summand also of each ${}^f R$ for $f \geq e$. Hence, for all $u \gg 0$, $m(u, M_i, R) > 0$.

Now we consider $m(u, R, M_i)$. As above, M_i is a direct summand of ${}^{e_i}R$. Pick an arbitrary $0 \neq c \in M_i$. By the definition of strongly F-regular the map $R \rightarrow {}^{f_i}M_i \hookrightarrow {}^{e_i+f_i}R$ given by sending 1 to c will split for some f_i . So R is a direct summand of some ${}^{f_i}M_i$. Consider any $u \geq \max\{f_i\}$. Since R is F-split, R is a direct summand of 1R ; hence if R is a direct summand of ${}^f M$ for some f , then R is also a direct summand of ${}^{f+1}M$. So R is a direct summand of all ${}^u M_i$ and consequently $m(u, R, M_i) > 0$ for all i .

Because strongly F-regular rings are normal, the fact that R is (graded) local implies that R is a domain. So objects in $M(R)$ have a well defined rank. Since

$$\mathrm{rk}_R {}^e M = \mathrm{rk}_R {}^e R \cdot \mathrm{rk}_R M = p^{de} \mathrm{rk}_R M$$

we obtain the formula

$$(3.8) \quad \sum_i m(1, M_i, M_j) \mathrm{rk} M_i = p^d \mathrm{rk} M_j.$$

Let w be the row vector in \mathbb{Q}^n whose i^{th} component is the integer $\mathrm{rk} M_i$. With this notation, formula (3.8) becomes $wE = p^d w$. Hence w is a row eigenvector for E with strictly positive entries. The standard linear algebra lemma below (3.3.2) guarantees that p^d is the eigenvalue of E with largest absolute value. Furthermore, Lemma 3.3.2 also ensures that $\lim_{e \rightarrow \infty} E^e / p^{de}$ exists and has strictly positive entries. But by (3.7), this means that

$$\lim_{e \rightarrow \infty} \frac{m(e, M_i, M_j)}{p^{de}}$$

exists and is positive, so the multiplicities grow like p^{de} and the proof is complete. \square

Lemma 3.3.2. *Assume that E is a matrix with non-negative real entries such that some power has strictly positive entries. Then E has a unique eigenvalue λ of largest absolute value. Furthermore λ is real and strictly positive. If v is a (row or column) eigenvector corresponding to λ then v can be chosen to have strictly positive entries and v is the only eigenvector with this property (up to scalar multiples). Furthermore for every vector w with positive entries one has*

$$(3.9) \quad \lim_{n \rightarrow \infty} \frac{E^n w}{\lambda^n} = a v$$

for some $a > 0$.

Proof. When E itself has strictly positive entries this is not difficult to prove and in any case is the well-known Perron-Frobenius theorem [6]. Assume $E^u > 0$. The eigenvalues of E^u are of the form λ^u with λ an eigenvalue of E and the corresponding eigenvectors are equal. Let λ be an eigenvalue of E of largest absolute value and let v be a corresponding column eigenvector (the case of a row eigenvector is similar). Then λ^u has largest absolute values among the eigenvalues of E^u and hence λ^u is uniquely determined by E^u and is strictly positive. Furthermore v is an eigenvector of E^u corresponding to λ^u , so, changing signs if necessary, v may be assumed to have strictly positive entries. Hence v is unique (up to scalar multiple) among the eigenvectors of E^u (or E) with this property. Then the equation $E^r v = \lambda^r v$ together with the fact that E has non-negative entries implies that λ is real and non-negative. Using $\lambda = (\lambda^u)^{1/u}$ we deduce that λ is unique and strictly positive.

We also have for any vector w with non-negative components

$$\lim_{n \rightarrow \infty} \frac{E^{nu+r} w}{\lambda^{nu+r}} = \frac{E^r}{\lambda^r} \left(\lim_{n \rightarrow \infty} \frac{E^{nu} w}{\lambda^{nu}} \right) = \frac{a}{\lambda^r} E^r v = av$$

for some $a > 0$. Since this a is independent of r , (3.9) follows. \square

4. RINGS OF DIFFERENTIAL OPERATORS

4.1. Endomorphism rings. In this section, we summarize a few basic properties of endomorphism rings needed in the next section. These properties follow easily from the definitions, and can be found in a standard text, such as [20]. We assume throughout that R is a ring and that $M(R)$ is a category of finitely generated R modules for which the Krull-Schmidt theorem applies. For example, R could be a ring of type (A) or of type (B) as defined in Section 3. In the graded case, the adjectives ‘‘graded’’ and ‘‘homogeneous’’ implicitly modify all modules and maps; see [24] for basic facts about graded rings.

Suppose that $M \in M(R)$. Let Λ denote the ring $\text{End}_R M$. Recall that if M is indecomposable, then Λ is a local ring, that is, the set of non-units forms an ideal, which is necessarily maximal. (Readers who usually work with commutative rings are reminded that a local non-commutative ring has a unique maximal (two-sided) ideal but the converse is false in general.) If M is a direct sum of n copies of the indecomposable R module N , then Λ is isomorphic to an $n \times n$ matrix ring with entries in $\text{End}_R N$, that is $\Lambda \cong M(n, \text{End}_R N)$. This ring also has a unique maximal ideal: the set of all matrices in $M(n, \text{End}_R N)$ with non-units in each entry.

More generally, suppose that

$$M = M_1^{\oplus a_1} \oplus \cdots \oplus M_n^{\oplus a_n}$$

where each M_i is indecomposable and $M_i \not\cong M_j$ for $i \neq j$. Then $\phi \in \Lambda$ may be written in block matrix⁴ form $(\phi_{ij})_{i,j=1,\dots,n} \in \text{Hom}_R(M_i^{a_i}, M_j^{a_j})$; in particular, ϕ_{ij} is now itself an $a_i \times a_j$ -matrix with entries in $\text{Hom}_R(M_i, M_j)$. The maximal ideals of this ring are in one-to-one correspondence with the indecomposable modules M_i : the i^{th} maximal ideal is the set of elements with no units in the i^{th} diagonal block $\text{End}_R M_i^{a_i}$.

⁴We adopt the convention that matrices act on the right, so that elements of a direct sum are represented by row matrices.

In particular, an element ϕ is in the intersection of all maximal ideals (that is, $\phi \in \text{rad } \Lambda$) if and only if each of the $a_i \times a_i$ matrices $(\phi_{ii})_{i=1, \dots, n}$ consists of only non-invertible entries.

In other words,

$$(4.1) \quad \Lambda / \text{rad } \Lambda \cong \prod_{i=1}^n M(a_i, D_i).$$

where $D_i = \text{End}_R M_i / \text{rad } \text{End}_R M_i$. If we let k denote the residue field of R , then it is clear that $[D_i : k] < \infty$ and from the Krull-Schmidt theorem it follows that D_i is a division algebra. The preceding remarks are summarized by the following lemma.

Lemma 4.1.1. *Let (R, m) be a complete local or graded ring with residue field k . Let M be an arbitrary finitely generated R module and set $\Lambda = \text{End}_R(M)$. If $M \cong M_1^{\oplus a_1} \oplus \dots \oplus M_n^{\oplus a_n}$ is a decomposition of the R module M into indecomposable R modules, then both the simple Λ modules and the maximal two-sided ideals of Λ are indexed by the M_i . In particular,*

- (1) *Using the isomorphism (4.1) the simple Λ -modules can be identified with $D_i^{\oplus a_i}$.*
- (2) *The maximal two-sided ideals of Λ are of the form*

$$P_i = \{\phi \in \Lambda \mid \phi_{ii} \text{ contains only non-invertible entries}\}$$

where Λ is identified with the ring of n by n matrices whose $(ij)^{\text{th}}$ entry is an a_i by a_j matrix ϕ_{ij} with entries in $\text{Hom}_R(M_i, M_j)$.

4.2. Simplicity of Rings of Differential Operators. The following theorem is the main result of this paper.

Theorem 4.2.1. *Let R be a commutative Noetherian ring of characteristic p , either complete (type (A)) or graded (type (B)) as defined in Section 3. If R is strongly F -regular and has finite F -representation type, then the ring of differential operators $D(R)$ is a simple ring.*

Proof. We use the formula (2.3).

$$D(R) = \varinjlim_q \text{End}_{R^q}(R) = \varinjlim_e \text{End}_R({}^e R)$$

Let us define $\Lambda_e = \text{End}_R({}^e R)$. We have $\Lambda_0 = R$. In particular, all Λ_e -modules are automatically R -modules. Given an element ϕ in the direct limit defining $D(R)$, we will denote by ${}^e \phi$ the corresponding representative in Λ_e . Thus elements ${}^e \phi \in \Lambda_e$ and ${}^f \phi \in \Lambda_f$ determine the same element $\phi \in D(R)$.

Step 1. Let $0 \neq {}^e \phi \in \Lambda_e$. Then there exist $f \geq e$ such that ${}^f \phi \notin \text{rad } \Lambda_f$.

Proof. Assume that ${}^f \phi \in \text{rad } \Lambda_f$ for all $f \geq e$. This means that the left ideal of Λ_f generated by ${}^f \phi$ is contained in $\text{rad } \Lambda_f$. Therefore, by Nakayama's Lemma, we have an inclusion of proper left Λ_f submodules of R ,

$$(\Lambda_f {}^f \phi)R \subset (\text{rad } \Lambda_f)R \subsetneq R.$$

In particular, for every $f \geq e$, $(\Lambda_f {}^f \phi)R$ is a proper Λ_f (and hence R)-submodule of R .

Now, if ϕ is not the zero operator, then $\phi \cdot r \neq 0$ for some element $r \in R$. Because R is F-regular, it is simple as a $D(R)$ module (*c.f.* Theorem 2.4.3). Thus there exists some $\psi \in D(R)$ such that $(\psi \circ \phi) \cdot r = 1$. In particular, $({}^f\psi \circ {}^f\phi) \cdot r = 1$ for all $f \gg 0$, so that $(\Lambda_f {}^f\phi)R = R$, a contradiction. \square

Step 2. Let ${}^e\phi \in \Lambda_e$. Then there exists an $f \geq e$ such that ${}^f\phi$ is not contained in any maximal two-sided ideal of Λ_f .

Proof. Let M_1, \dots, M_n be the non-isomorphic indecomposable modules occurring as direct summands of $({}^eR)_{e \geq 0}$. By Proposition 3.3.1 (or its proof) there exists a $u > 0$ such that $m(u, M_i, M_j) > 0$ for all i, j .

By Step 1 we may assume that ${}^e\phi \notin \text{rad } \Lambda_e$. So in the block matrix representation $({}^e\phi_{ij})_{ij}$, some ${}^e\phi_{ii}$ contains an entry ${}^e\psi \in \text{End}_R(M_i)$ that is an automorphism (see the notation and discussion in Section 4.1).

Now we wish to follow what becomes of ${}^e\psi$ when we map further along in the direct limit system, say to Λ_u where $u \geq e$. Then ${}^u\psi$ can again be written in block matrix form $({}^u\psi_{kl})_{kl}$. Since ${}^u\psi$ is invertible it is not contained in any two-sided maximal ideal of $\text{End}_R({}^eM_i)$. Hence by §4.1 every block of the form ${}^u\psi_{kk}$ for $k = 1, 2, \dots, n$ must contain invertible entries. Now since ${}^u\psi_{kk}$ can be identified with a sub-matrix of ${}^u\phi_{kk}$, we can conclude that each of the block matrices ${}^u\phi_{kk}$ along the diagonal of ${}^u\phi$ will contain invertible entries. So again by §4.1, ${}^u\phi$ is not contained in any maximal two-sided ideal of Λ_u . \square

Step 3. Now we conclude the proof of Theorem 4.2.1. Assume that $0 \neq I \subset D(R)$ is a non-trivial two-sided ideal. Then there exists an e such that $I \cap \Lambda_e \neq 0$ (actually since R is a domain, $e = 0$). Pick $0 \neq {}^e\phi \in I \cap \Lambda_e$. By the previous step there exists an $f \geq e$ such that ${}^f\phi$ is not contained in any maximal two-sided ideal of Λ_f . Since ${}^f\phi \in I \cap \Lambda_f$ we deduce $I \cap \Lambda_f = \Lambda_f$. Therefore $I = D(R)$, and $D(R)$ is simple. \square

Corollary 4.2.2. *Let R be a graded ring satisfying (B). Assume that $R \subset S$ is a split inclusion of graded rings where S is a (weighted homogeneous) polynomial ring, with the extension of residue fields $R_0 \subset S_0$ being finite. Then $D(R)$ is simple.*

Proof. Polynomial rings always have finite representation type, so from Proposition 3.1.6, we know that R has finite F-representation type. On the other hand, R is strongly F-regular by Theorem 2.4.4. The corollary is thus an immediate consequence of Theorem 4.2.1. \square

A result of this generality is not known in characteristic zero. Indeed, a characteristic zero analog of this result would give an affirmative answer to the question of Levasseur and Stafford (Conjecture 1.1). However it is known in the case that $R = S^G$ where G is either finite [14] or a torus [22].

Theorem 4.2.1 also applies to quadric hypersurfaces.

Corollary 4.2.3. *Suppose that $R = k[X_1, X_2, \dots, X_N]/(Q)$ where k is a field of characteristic $p > 2$ and Q is a quadratic form in the X_i 's of rank ≥ 3 (so in particular $N \geq 3$). Then the ring of differential operators $D_k(R)$ is simple.*

Proof. Since rings of differential operators are compatible with flat base change (see §5.1) we may assume that k is algebraically closed. We may then change coordinates so as to assume that $Q = \sum_{i=1}^m X_i^2$ with $\text{rk } Q \geq 3$. It is an immediate application of Theorem 4.1 of [27] that R is strongly F-regular in all characteristics $p > 2$. (For $p \gg 0$, this was first proven by Fedder [5].)

It is easy to see that the FFRT property is preserved under taking polynomial extensions, whence we may assume that Q has maximal rank. In this case, R has finite representation type [15], so it certainly has FFRT. We can therefore use Theorem 4.2.1 to conclude that $D(R)$ is simple. \square

The fact that $D(R)$ is simple in the quadric hypersurface case when k is characteristic zero was proved in [17] (see also [16]). The characteristic zero proof uses the structure of primitive ideals in enveloping algebras and is much more intricate than our proof of the characteristic p -case.

4.3. Finite dimensional representations. Let R be either graded or complete as in (A) or (B). In both case R contains a copy of its residue field k . A $D(R)$ -module is said to be a *finite dimensional representation* if it has finite length as an R module, *ie*, if it is finite dimensional as a vector space over k . As is to be expected, the finite dimensional representations of $D(R)$ are controlled by the direct summands of eR . In fact, in this section we give a necessary and sufficient, representation theoretic, criterion for $D(R)$ to have no finite dimensional representations.

Let W be a $D(R)$ module. The annihilator in $D(R)$ is clearly a two-sided ideal of $D(R)$. If W is a finite dimensional $D(R)$ module, then because W is in fact killed by some power of the maximal ideal of R , this annihilator must be non-trivial. Therefore, if $D(R)$ is simple, then it has no finite dimensional representations (at least if $\dim R > 0$). Unfortunately there seems to be, a priori, no reason why the converse should hold.

For an indecomposable finitely generated R module M , let us define $d(M)$ to be the dimension of the division algebra $\text{End}(M)/\text{rad End}(M)$. Of course if k is algebraically closed then $d(M)=1$.

Proposition 4.3.1. *Let R be complete or graded as in (A) or (B). Then the minimal dimension of a finite dimensional $D(R)$ -representation is given by*

$$u = \sup_e \left\{ \min_{M|{}^eR} d(M)m(e, M, R) \right\}$$

where the M range through all the indecomposable R modules appearing as direct summands in eR with non-zero multiplicity. If particular $D(R)$ has no finite dimensional representations if and only if $u = \infty$. The “sup” here means supremum.

Proof. This follows from the formula $D(R) = \varinjlim \text{End}_R({}^eR)$. By Lemma 4.1.1, the simple Λ_e modules are all of the form $D^{\oplus m(e, M, R)}$ where M is one of the indecomposable summands appearing in an R module decomposition of eR , D is the division algebra $\text{End } M/\text{rad End } M$, and $m(e, M, R)$ is the multiplicity of M in eR . Given a finite dimensional $D(R)$ module, we can, of course, view it as a finite dimensional Λ_e module. Note that

$$\min_{M; m(e, M, R) \neq 0} d(M)m(e, M, R)$$

is the minimal dimension of a finite dimensional $\text{End}_R({}^eR)$ -module. We can then invoke Lemma 4.3.2. \square

Lemma 4.3.2. *Assume that $(\Lambda_e)_{e \in \mathbb{N}}$ is a directed system of rings containing a field k each having only a finite number of non-isomorphic simple representations. Let u_e be the minimal dimension of a finite dimensional (over k) Λ_e -representation and let $\Lambda = \varinjlim \Lambda_e$. Then $u = \sup_e u_e$ is the minimal dimension of a finite dimensional Λ -representation.*

Proof. We first observe that $(u_e)_e$ is a non-decreasing sequence of integers, since restriction of scalars defines a dimension preserving functor from $\Lambda_e\text{-mod}$ to $\Lambda_{e-1}\text{-mod}$.

The case $u = \infty$ If W is a finite dimensional Λ -module then W is also a Λ_e -module for every e . So $\dim W \geq u_e$. Since this holds for all e , $\dim W \geq u = \infty$. Hence in this case one certainly has $\dim W = u$.

The case $u < \infty$ In this case there exists an f such that for $e \geq f$ one has $u_e = u$. Define for all $e \geq f$:

$$\mathcal{F}_e = \{\text{isomorphism classes of simple } \Lambda_e\text{-modules of dimension } u\}$$

By hypothesis \mathcal{F}_e is a non-empty finite set. Furthermore the restriction functor from $\Lambda_e\text{-mod}$ to $\Lambda_{e-1}\text{-mod}$ defines a map $\mathcal{F}_e \rightarrow \mathcal{F}_{e-1}$. Define $\mathcal{F} = \varprojlim \mathcal{F}_e$. But an inverse limit of finite sets is always non-empty, so $\mathcal{F} \neq \emptyset$.

Let $(W_e)_e \in \mathcal{F}$. We may assume that W_e is a simple Λ_e module of dimension u representing an isomorphism class in \mathcal{F}_e . Restriction of scalars (possibly composed with another isomorphism) gives an isomorphism of Λ_{e-1} modules $W_e \rightarrow W_{e-1}$. Therefore, we have isomorphisms $W_e \rightarrow W_{e+1}$ of Λ_e modules for all $e \in \mathbb{N}$. The direct limit $W = \varinjlim W_i$ is a finite dimensional Λ -module of dimension u .

It is clear that there can be no Λ module of smaller dimension. For if W is a Λ module of dimension $< u$, then by restriction of scalars, W is a Λ_e module of dimension $< u$ for each e , contrary to the definition of u . \square

4.4. D-simplicity. Closely related to the simplicity of the ring $D_k(R)$ is the simplicity of R as a module over $D_k(R)$. If $D_k(R)$ is a simple ring, then R is simple as a $D_k(R)$ module, as the annihilator of any $D_k(R)$ module of the form R/I is a non-zero two-sided ideal of $D_k(R)$. The converse, however, is false; see the example of Chamarie, Levasseur and Stafford in the introduction of [18].

How does one verify that a given k -algebra R is simple as a $D_k(R)$ module? Clearly R is D -simple if and only if each non-zero element $c \in R$ generates all of R as a $D_k(R)$ module; that is, if and only if for each non-zero $c \in R$, there exists some differential operator $\theta \in D_k(R)$ sending c to $1 \in R$. In practice we would like to be able to check this condition, not for all $c \in R$, but for a *single element* $c \in R$. We prove in this section that this is indeed the case for a large class of rings of characteristic p , describing a specific c that governs the D -simplicity of R .

Note that if k is not a field, but merely a commutative ring, then R is virtually never a simple $D_k(R)$ module, because any ideal of k expands to an ideal of R that is stable under $D_k(R)$. This can be remedied by introducing a concept of *relative D-simplicity*. A k -algebra R is said to be relatively D_k simple if every non-zero $D_k(R)$ submodule of R has non-zero intersection with the set of non-zero-divisors in $\text{im } k \subset R$. If k is a field, this is equivalent to R being a simple $D_k(R)$ module.

The next proposition lets us reduce the problem of checking relative D_k simplicity to checking that certain elements can be sent to k . We abuse notation throughout by speaking of elements of k as if k were a subring of R , though of course the structure map $k \rightarrow R$ need not be injective.

Proposition 4.4.1. *Let R be a k -algebra, where k is an arbitrary (commutative) ground ring. Let c be any non-zero-divisor of R such that R_c is a relatively simple $D_k(R_c)$ module. Then R is relatively simple as a $D_k(R)$ module if and only if for each integer n , there is some element $\theta_n \in D_k(R)$ such that $\theta_n \cdot c^n$ is a non-zero-divisor in k .*

Proof. To prove that R is relatively simple as a $D_k(R)$ module, we need only show that for each non-zero $x \in R$, there is an operator θ sending x to a non-zero-divisor in k . Let $\frac{x}{1}$ denote the image of x in R_c . Because c is a non-zero-divisor, $\frac{x}{1}$ is non-zero provided that x is non-zero. Because R_c is relatively simple as a $D_k(R_c)$ module, there is a differential operator θ' in $D(R_c)$ taking $\frac{x}{1}$ to $\lambda \in k$, where λ is not a zero-divisor. Since $D_k(R_c) \cong D_k(R) \otimes_R R_c$, we may assume that $\theta' = \frac{\theta}{c^n}$ for some $\theta \in D_k(R)$. Therefore $\theta \cdot x = \lambda c^n$ in R .

We have assumed that there is an operator $\theta_n \in D_k(R)$ such that $\theta_n \cdot c^n$ is sent to a non-zero-divisor in k . Therefore, the composition $\theta_n \circ \theta \in D_k(R)$ is a differential operator sending x to a non-zero-divisor in k . The proof is complete. \square

We would like to be able to check that R is simple (or relatively simple) over $D_k(R)$ by checking just one condition, not infinitely many. We pose the question:

Question 4.4.2. Does there exist an element $c \in R$ such that R is simple as a $D_k(R)$ module if and only if $\theta \cdot c = 1$ for some $\theta \in D(R)$?

Amazingly, this turns out to be true in characteristic $p > 0$.

Theorem 4.4.3. *Assume that R is a domain of characteristic $p > 0$ finitely generated over its subring R^p . Suppose that R is module finite over some F -finite regular domain, T , such that the corresponding extension of fraction fields is separable. Let c be a discriminant of R over T , i.e. $c = \det(\text{trace}(r_i r_j))$, where $r_1, r_2, \dots, r_n \in R$ is a basis for $R \otimes_T K$ over K , the fraction field of T . Then R is simple as a $D(R)$ module if and only if there is a differential operator sending c^2 to 1.*

More generally, R need not be a domain if it is reduced and if it is module finite, torsion-free, and generically smooth over the regular subring T .

Note that the restrictions on R above are quite weak in general. By a variant of Noether normalization, any domain finitely generated over a perfect field satisfies the hypothesis. Similarly, any complete local domain with a perfect residue field satisfies the hypothesis.

Proof. Localizing at the discriminant c , the map $T_c \hookrightarrow R_c$ becomes étale, so R_c is regular. Because regular rings are $D_{\mathbb{Z}/p\mathbb{Z}}$ simple, R_c is simple as a $D(R_c)$ module. By Proposition 4.4.1, it therefore suffices to check that there are differential operators taking each power c^n to 1. Because multiplication by c^{q-n} is a differential operator on R , there is no harm in assuming that n is a large power of p .

An important property of the discriminant, observed by Hochster and Huneke, is that for each power q of p , $cR^{1/q} \subset T^{1/q} \otimes_T R$ (Lemma 6.5 [11]; see also first sentence of proof of Lemma 6.4 in [11]). Note also that because $T^{1/q}$ is free over T , the ring $T^{1/q} \otimes_T R$ is free over R . Thus there is an R linear map $T^{1/q} \otimes_T R \rightarrow R$ which sends $1 \otimes 1$ to 1. Pre-composing with multiplication by c , we have an R linear map $R^{1/q} \xrightarrow{\text{mult by } c} T^{1/q} \otimes_T R \rightarrow R$ which sends 1 to c . Raising everything to the q^{th} power, there is an R^q linear map π from R to R^q sending 1 to c^q .

We wish to find a differential operator in $D(R)$ sending c^q to 1. Suppose that $\theta \in \text{End}_{R^q} R \subset D(R)$ sends c^2 to 1. Let $\theta^{[q]} \in \text{End}_{R^{q^2}} R^q$ be the operator

$$R^q \rightarrow R^q \text{ sending } x^q \mapsto (\theta \cdot x)^q.$$

Consider the composition

$$R \xrightarrow{\pi} R^q \xrightarrow{\theta^{[q]}} R.$$

The element c^q is sent to $\theta^{[q]} \cdot c^{2q} = (\theta \cdot c^2)^q = 1$. Because this map is R^{q^2} linear, it is a differential operator, and the proof is complete. \square

Let R be a finitely generated algebra over a perfect field k of characteristic $p > 0$. The above theorem tells us that there is a *single* element, namely the square of any discriminant of a (separable) Noether normalization, which governs whether or not R is simple as an $D_k(R)$ module. We do not know whether or not a similar result holds in characteristic zero or in the relative setting. An interesting question: Why does the discriminant play this special role with respect to differential operators?

5. CHARACTERISTIC ZERO

Ultimately, we would like to be able to prove Conjecture 1.1 in characteristic zero as well. We would like to accomplish this by a reduction mod p argument and then invoking Theorem 4.2.1. In order to do this, three major issues need to be addressed:

- (1) An understanding of how differential operators behave mod p , at least for invariant rings;
- (2) An understanding of the FFRT property for finitely generated algebras over a field of characteristic zero;
- (3) An understanding of strong F-regularity in characteristic zero.

The idea is as follows. Suppose we are given a finitely generated k -algebra $R \cong k[X_1, \dots, X_n]/(F_1, \dots, F_r)$ where k is a field of characteristic zero. We build a finitely generated \mathbb{Z} -algebra A that contains all of the elements of k necessary to define R ; *i.e.* A should contain all the coefficients of the polynomials F_1, \dots, F_r defining the ideal of relations on the generators for R . We thus have an A -algebra $R_A = A[X_1, \dots, X_n]/(F_1, \dots, F_r)$ such that the natural map $R_A \otimes_A k \rightarrow R$ is an isomorphism. By the lemma of generic freeness, we may invert a single element of A so as to assume that R_A is A free. Each of the closed fibers of the map $A \rightarrow R_A$ is a finitely generated algebra over a perfect (finite!) field. We think of the family of closed fibers as being a model for the original k -algebra R .

Several natural questions come to mind: How does $D_A(R_A) \otimes L$ compare to $D_L(R_A \otimes_A L)$ where L is an A -algebra? If R is simple as a $D_k(R)$ -algebra, does this mean that the closed fibers $R_A \otimes_A A/\mu$ are simple as $D_{A/\mu}(R_A \otimes_A A/\mu)$ modules? If R is a graded direct summand of a polynomial ring, is it true that $R_A \otimes_A L$ has finite F-representation type where L is a perfect field of characteristic p to which A maps?

5.1. Reduction to characteristic p and rings of differential operators. We discuss item (1) in somewhat more detail. Unfortunately, differential operators are not well behaved “mod p ” in general. In [28], an example is given to show that in some sense there are “more” differential operators in characteristic $p > 0$. It is the cubic cone $k[X, Y, Z]/(X^3 + Y^3 + Z^3)$. Nevertheless, reduction *does* work for some nice classes of rings such as regular rings and rings of invariants for finite groups. So it is not unreasonable to expect that reduction should work for some other good rings, such as rings of invariants under reductive groups. Unfortunately, we have not been able to prove this.

What we can do, however, is describe the major obstruction against reduction mod p . It is based upon the first derived functor of the left exact functor $D(R, -)$ (see §2.2). We write $R^1D(R)$ for $R^1D(R, R)$.

Let R_A be flat and finitely generated over A . For simplicity we assume that A is a Dedekind domain. Recall that

$$(5.1) \quad D_A(R_A) = \varinjlim_n \operatorname{Hom}_{R_A}(P_{R_A/A}^n, R_A)$$

where $P_{R_A/A}^n = (R_A \otimes_A R_A)/J_A^{n+1}$ with J_A is the kernel of the multiplication map $R_A \otimes_A R_A \rightarrow R_A$; see 2.1. Let L be any A -algebra. Because R_A is A -flat, it is clear that $J_A \otimes_A L$ is the kernel of the corresponding multiplication map for $R_A \otimes_A L$. Furthermore, by inverting a single element of A , the modules $P_{R_A/A}^n$ for all n may be assumed to be all A -flat. To see this, consider the short exact sequence $0 \rightarrow J^n/J^{n+1} \rightarrow P^{n+1} \rightarrow P^n \rightarrow 0$. The A modules J^n/J^{n+1} can be assumed A free because they are the graded pieces of the finitely generated A -algebra $\operatorname{Gr}_J S = S/J \oplus J/J^2 \oplus J^2/J^3 \oplus \dots$, with $S = R_A \otimes_A R_A$, and this graded ring may be assumed A -flat after inverting a single element of A , by the lemma of generic freeness. That all the P^n can be assumed A -flat now follows immediately by induction on n . We conclude that $P_{R_A/A}^n \otimes_A L = P_{R_A \otimes_A L/L}^n$.

Because tensor product commutes with direct limits, we see that we therefore have a natural map for any A -algebra L

$$D_A(R_A) \otimes_A L \rightarrow D_L(R_A \otimes_A L).$$

The question is to determine when this map is an isomorphism. It always is when L is a flat A -algebra, for instance, when L is the fraction field of A . But our main concern is when $L = A/\mu$, for some $\mu \in \max \operatorname{Spec} A$. On a Zariski dense open subset of $\max \operatorname{Spec} A$ the R -modules $P_{R_A/A}^n$ are A -flat. One can then easily show using (5.1) and the Universal Coefficient Theorem that for the same open set of μ 's in A , there is a short exact sequence

$$0 \rightarrow D_A(R_A) \otimes_A A/\mu \rightarrow D_{A/\mu}(R_A \otimes_A A/\mu) \rightarrow \operatorname{Tor}_1^A(A/\mu, R^1 D_A(R_A)) \rightarrow 0$$

So for reduction to work we should have $\operatorname{Tor}_1^A(A/\mu, R^1 D_A(R_A)) = 0$ on a dense set of maximal ideals μ in A . Unfortunately $R^1 D(R)$ seems to be very hard to compute explicitly. One could naively hope that $R^1 D(R)$ is always zero but this is contradicted by the quadric hypersurface, which we discuss below.

Example 5.1.1. Assume that $R = S/I$ where S is a graded polynomial ring over A . Using the results in Section 2.2 we find that

$$\begin{aligned} R^i D(R) &= R^i D({}_S R, {}_S R) = \operatorname{Ext}_{D(S)}^i(R \otimes_S D(S), R \otimes D(S)) \\ &= \operatorname{Ext}_{D(S)}^i(D(S)/ID(S), D(S)/ID(S)) \end{aligned}$$

Let us now assume that $R = A[x_1, \dots, x_n]/(f)$ where $f = \sum_i x_i^2$. Then

$$R^1 D(R) = \operatorname{Ext}_{D(S)}^1(D(S)/fD(S), D(S)/fD(S)) = \frac{D(S)}{fD(S) + D(S)f}$$

Assume now that A is of characteristic zero and let k be the algebraic closure of the fraction field of A . Put $\bar{S} = k \otimes_A S$. We will show that

$$k \otimes_A R^1 D(R) = \frac{D(\bar{S})}{fD(\bar{S}) + D(\bar{S})f}$$

is not zero.

Since we are in characteristic zero

$$D(\bar{S}) = k[x_1, \dots, x_n, \partial_1, \dots, \partial_n]$$

where $\partial_i = \frac{\partial}{\partial x_i}$.

Let $O(n)$ be the orthogonal group over k , acting in the standard way on \bar{S} . It is easy to see that

$$D(\bar{S})^{O(n)} = k[f, g, h]$$

where $f = \sum_i x_i^2$, $g = \sum_i x_i \partial_i$ and $h = \sum_i \partial_i^2$. f, g, h satisfy the relations

$$\begin{aligned} [g, f] &= 2f \\ [g, h] &= -2h \\ [h, f] &= 4g + 2n \end{aligned}$$

So we see that $\mathfrak{g} = kf + k(g + \frac{n}{2}) + kh$ is a Lie algebra (isomorphic to \mathfrak{sl}_2). Hence we have a surjective map $U(\mathfrak{g}) \rightarrow D(\bar{S})^{O(n)}$. By comparing the dimensions of the associated graded rings on both sides one sees that there can be no extra relations among f, g, h . So $D(\bar{S})^{O(n)}$ is actually isomorphic to the enveloping algebra of \mathfrak{g} . Using this isomorphism we obtain

$$(5.2) \quad (k \otimes_A R^1 D(R))^{O(n)} = \frac{U(\mathfrak{g})}{fU(\mathfrak{g}) + U(\mathfrak{g})f}$$

Since f is in the augmentation ideal of $U(\mathfrak{g})$, the right hand side of (5.2) cannot be zero.

We pose the following problem.

Question 5.1.2. Let A be a domain finitely generated as an algebra over \mathbb{Z} . Suppose that R_A is a finitely generated A -algebra such that $R_A \otimes_A K$ is a direct summand of a regular ring for K the fraction field of A . Is the A -module $\text{Tor}_1^A(A/\mu, R^1 D_A(R_A))$ zero for all maximal $\mu \in \text{Spec } A$ except on some Zariski closed subset? What if $R_A \otimes_A K$ is a ring of invariants for a linear action of a reductive group on a polynomial ring?

An affirmative answer would tell us that the ring of differential operators on a ring of invariants can be studied by reduction to characteristic $p > 0$.

5.2. Reduction mod p and strong F-regularity. Strong F-regularity, though defined in §2.4 as a characteristic p notion, can be made meaningful for finitely generated algebras over a field of characteristic zero.

Let $R = k[X_1, X_2, \dots, X_n]/(F_1, F_2, \dots, F_r)$ be a finitely generated algebra over a field k of characteristic zero. Choose a finitely generated \mathbb{Z} subalgebra of k over which R is defined and set $R_A = A[X_1, X_2, \dots, X_n]/(F_1, F_2, \dots, F_r)$.

We say that R is strongly F-regular⁵ if there exists such a choice of A for which all the closed fibers $R_{\bar{A}}$ are strongly F-regular on some non-empty Zariski open set of $\text{Spec } A$. Similarly, we say that R has finite F-representation type or that R is F-split if either of these properties holds on a non-empty Zariski open set of $\text{Spec } A$.

We investigate what sort of k -algebras will be strongly F-regular. We first note that D-simplicity in characteristic zero implies it on a Zariski open set of closed fibers.

Theorem 5.2.1. *Let R_A be a reduced finitely generated A -algebra where A is a domain finitely generated as a \mathbb{Z} -algebra. If the K -algebra $R_A \otimes_A K$ (where K is any field containing A) is simple as a $D_K(R)$ module, then the same is true for*

⁵In the tight closure literature, this would be called strongly F-regular type.

almost all closed fibers: for all maximal $\mu \in \text{Spec } A$ in a Zariski dense open set, the ring \bar{R} is simple as a $D_{\bar{A}}(\bar{R})$ module, where the “bar” indicates reduction mod μ .

Proof. We first note that it is sufficient to prove the theorem when K is the fraction field of A . Let L be the fraction field of A . Because $L \subset K$ splits over L , the inclusion $D_L(R_A \otimes L) \cong D_A(R_A) \otimes L \subset D_K(R_A \otimes K) \cong D_A(R_A) \otimes K$ splits over $D_L(R_A \otimes L)$. Assume that $R_A \otimes K$ is $D_K(R)$ simple. Then for each element $c \otimes 1 \in R_A \otimes L \subset R_A \otimes K$, we can find an operator in $D_A(R_A) \otimes K$ sending $c \otimes 1$ to 1. This operator is of the form $\sum \theta_i \otimes \kappa_i$ where $\theta_i \in D_A(R_A)$ and $\kappa_i \in K$. Applying the splitting above, there is a corresponding operator in $D_L(R_A \otimes L)$, $\sum \theta_i \otimes \lambda_i$ where $\lambda_i \in L$. Using the splitting we see that $\sum \kappa_i \theta_i \cdot c = \sum \lambda_i \theta_i \cdot c = 1$, so that $R_A \otimes L$ is simple over $D_L(R_A \otimes L)$. Thus we may assume that K is the fraction field of A .

The point is that (after possibly inverting a single element of A) we can find a Noether normalization for R_A , *i.e.*, a polynomial subring $T_A \hookrightarrow R_A$ over which R_A is module finite and generically smooth. The discriminant of this algebra extension is an element c_A of T_A whose image in almost all the fibers is the corresponding discriminant for the algebra extension in the fibers (see [9], Discussion 2.4.5). Thus, by Theorem 4.4.3, in order to check that \bar{R} is simple over $D(\bar{R})$, we need only find a differential operator in $D(R)$ taking \bar{c}_A^2 to 1.

Because the generic fiber is D-simple, there is a differential operator $\theta \in D(R \otimes K)$ sending c_A to 1. Without loss of generality, we may assume that $\theta \in D_A(R_A)$. But because for almost all fibers we have $D_A(R_A) \otimes \bar{A} \subset D_{\bar{A}}(\bar{R})$, this means that $\bar{\theta}$ is differential operator on \bar{R} sending \bar{c}^2 to 1, as needed. \square

Strong F-regularity in characteristic zero is decidedly more subtle. It is no longer clear that a finitely generated k -algebra that is a direct summand of a regular ring in characteristic zero will be strongly F-regular. Although the preceding result shows that the D-simplicity descends to characteristic p , the issue of F-splitting is harder to deal with. Using results of Hochster and Roberts [12], one can deduce the F-splitting for graded Gorenstein direct summands of a polynomial ring, at least on a dense set of closed fibers. Using the more recent theory of tight closure in characteristic zero due to Hochster and Huneke [9], one can prove strong F-regularity for a Zariski open set of fibers. We record this proof below.

Lemma 5.2.2. *Let R be a graded Gorenstein ring finitely generated as a k -algebra, where k is a field of characteristic zero. Assume that R is a direct summand (as a graded R module) of a graded regular ring. Then R is strongly F-regular on a Zariski open set of fibers of any finitely generated \mathbb{Z} -algebra A over which R is defined.*

Proof. By Proposition 3.1 of [28], R will be simple as a $D_k(R)$ module. By Proposition 5.2.1, it follows that for any choice of A , the closed fibers $R_A \otimes_A A/\mu$ are simple as $D_{A/\mu}(R_A \otimes_A A/\mu)$ modules on a Zariski dense open set of $\max \text{Spec } A$. Therefore, by Theorem 2.4.3, it is enough to show that a non-empty Zariski open set of the fibers is F-split.

We choose to do this using the theory of tight closure in characteristic zero (see [9]). The point is that because R is pure in a regular ring, all ideals of R are tightly closed (in the characteristic zero theory). Unfortunately, this is not known to imply in general that a Zariski open set of fibers of some $A \rightarrow R_A$ are all strongly F-regular. However, this is true when R is Gorenstein.

The point is that for a graded Gorenstein ring R , we may choose a homogeneous system of parameters x_1, x_2, \dots, x_d for R and a socle generator z for $R/(x_1, x_2, \dots, x_d)R$. We now choose A so that the x_1, x_2, \dots, x_d and the socle generator z are all defined over A . In almost all of the closed fibers $R_{\bar{A}}$, the images of the x_1, x_2, \dots, x_d are a system of parameters for the graded Gorenstein ring $R_{\bar{A}}$ and the image of z is a socle generator for the quotient.

The fact that all ideals of R are tightly closed implies that \bar{z} cannot be in the tight closure $(\bar{x}_1, \dots, \bar{x}_d)^*$ in the fiber $R_{\bar{A}}$, for almost all fibers. Because R is Gorenstein, this implies that all ideals of $R_{\bar{A}}$ are tightly closed, so that in particular $R_{\bar{A}}$ is F-split. We conclude that $R_{\bar{A}}$ is strongly F-regular on a dense open set of $\max\text{Spec } A$.

Even if we had begun with a \mathbb{Z} -algebra A over which the given system of parameters x_1, x_2, \dots, x_d and the socle generator z were not defined, we can still conclude that $R_{\bar{A}}$ is strongly F-regular on a non-empty Zariski open set of $\text{Spec } A$. Simply enlarge A to A' over which they are defined. Note that $R_{A/\mu \cap A} \subset R_{A'/\mu}$ is faithfully flat for all $\mu \in \max\text{Spec } A'$. Therefore, because all ideals of $R_{\bar{A}'}$ are tightly closed (on a dense open set), the same is true of $R_{\bar{A}}$. In particular, $R_{\bar{A}}$ is F-split. Thus R is D-simple and F-split, and hence strongly F-regular, on a dense open set of any finitely generated \mathbb{Z} -algebra over which R is defined. This concludes the proof⁶. \square

Theorem 5.2.3. *Let S^G be the invariant ring for the action of a reductive group G on the symmetric algebra S for a finitely dimensional representation of G of characteristic zero. Then R is strongly F-regular.*

Proof. Because G is reductive, there is a subgroup H of G which is semi-simple and such that the quotient G/H is an extension of a finite group by a torus. Note that the quotient group G/H acts on the ring of invariants S^H for the semi-simple group: $\bar{g} \in G/H$ acts on $f \in S^H$ by $g \cdot h$ where g is any lifting to G of \bar{g} . It is easy to verify that $S^G = (S^H)^{G/H}$. Because H is semi-simple, the ring S^H is Gorenstein. Thus by the preceding lemma, it is strongly F-regular. On the other hand, G/H is linearly reductive and thus the inclusion $(S^H)^{G/H} \hookrightarrow S^H$ is split by the Reynolds operator. This splitting descends to characteristic p for all $p > 0$. Therefore, because S^H is strongly F-regular in almost all fibers, so is its direct summand $S^G = (S^H)^{G/H}$. \square

The issue of how to keep track of the property of finite F-representation type in descending to characteristic $p > 0$ is wide open.

Question 5.2.4. If R is a graded direct summand of a polynomial ring of characteristic zero, does R have Finite F-representation type? How about if R is a ring of invariants for a semi-simple group acting linearly on a polynomial ring?

REFERENCES

- [1] M. Artin and D. Mumford, *Some elementary examples of unirational varieties which are not rational*, Proc. London Math. Soc. (3) **25** (1972), 75–95.

⁶An alternate way to finish of the proof is to invoke Theorem 2.3.15 of [9] to conclude that each of the $R_{\bar{A}}$ are Gorenstein, so that strong F-regularity is equivalent to all ideals being tightly closed [10]. We expect that these technical points will eventually be worked out and appear in the developing manuscript [9].

- [2] M. Auslander, *Isolated singularities and existence of almost split sequences*, Proc. ICRA IV, Lecture Notes in Mathematics, vol. 1178, Springer Verlag, 1986, pp. 194–241.
- [3] Bernstein, Gelfand, and Gelfand, *Differential operators on the cubic cone*, Russian Math. Surveys **27** (1972), 169–174.
- [4] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton University Press, Princeton, NJ, 1956.
- [5] R. Fedder, *A Frobenius characterization of rational singularities in two dimensional graded rings*, Trans. Amer. Math. Soc., vol. 2, 1993, pp. 655–668.
- [6] F. Gantmacher, *The theory of matrices*, vol. 2, Chelsea Publishing Co., New York, 1959.
- [7] A. Grothendieck and J. Dieudonné, *Elements de géométrie algébrique*, Inst. Hautes Études Sci. Publ. Math., 1960–1967.
- [8] W. Haboush, *Reductive groups are geometrically reductive*, Ann. of Math. (2) **102** (1975), no. 1, 67–83.
- [9] M. Hochster and C. Huneke, *Tight closure in equal characteristic zero*, University of Michigan preprint, 1995.
- [10] ———, *Tight closure and strong F -regularity*, Mém. Soc. Math. France (N.S.) **38** (1989), 119–133.
- [11] ———, *Tight closure, invariant theory and the Briançon-Skoda theorem*, J. Amer. Math. Soc. **3** (1990), no. 1, 31–116.
- [12] M. Hochster and J. Roberts, *The purity of Frobenius and local cohomology*, Adv. in Math. **21** (1976), 117–172.
- [13] J. C. Jantzen, *Representations of algebraic groups*, Academic Press, Orlando, FA, 1987.
- [14] J. Kantor, *Formes et opérateurs différentiels sur les espaces analytiques complexes*, Bull. Soc. Math. France **53** (1977), 5–80.
- [15] H. Knörrer, *Cohen-Macaulay modules on hypersurface singularities I*, Invent. Math. **116** (1985), 147–164.
- [16] T. Levasseur, *La dimension de Krull de $U(\mathfrak{sl}(3))$* , J. Algebra **102** (1986), 39–59.
- [17] T. Levasseur, S. P. Smith, and J. T. Stafford, *The minimal nilpotent orbit, the Joseph ideal and differential operators*, J. Algebra **116** (1986), 480–501.
- [18] T. Levasseur and J. T. Stafford, *Rings of differential operators on classical rings of invariants*, Memoirs of the AMS, vol. 412, Amer. Math. Soc., 1989.
- [19] B. MacCrimmon, *Strong and weak f -regularity for isolated singularities*, Ph.D. thesis, University of Michigan, 1995.
- [20] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*, John Wiley & Sons, New York, 1987.
- [21] I. M. Musson, *Rings of differential operators on invariants of tori*, Trans. Amer. Math. Soc. **303** (1987), 805–827.
- [22] I. M. Musson and M. Van den Bergh, *Invariants of rings of differential operators under tori*, preprint of the University of Limburg, available as URL “<http://www.luc.ac.be/Research/Algebra/Publications/musson.html>”, 1996.
- [23] M. Nagata, *Complete reducibility of rational representations of a matrix group*, J. Math. Kyoto Univ. **1** (1961), 87.
- [24] C. Nastasescu and F. Van Oystaeyen, *Graded ring theory*, North-Holland, 1982.
- [25] G. Schwarz, *Lifting differential operators from orbitspaces*, Ann. Sci. École Norm. Sup. (4) **28** (1995), 253–306.
- [26] J. H. Silverman, *The arithmetic of elliptic curves*, Graduate Texts in Mathematics, vol. 106, Springer Verlag, New York, 1986.
- [27] K. E. Smith, *Tight closure in graded rings*, MIT preprint, 1996.
- [28] ———, *The D -module structure of F -split rings*, Research Letters in Mathematics (1995).
- [29] R. P. Stanley, *Linear diophantine equations and local cohomology*, Invent. Math. **68** (1982), 175–193.
- [30] H. Tango, *On the behavior of extensions of vector bundles*, Nagoya Math. J. **48** (1972), 73–89.
- [31] M. Van den Bergh, *Cohen-Macaulayness of modules of covariants*, Invent. Math. **106** (1991), 389–409.
- [32] ———, *Differential operators on semi-invariants for tori and weighted projective spaces*, Topics in invariant theory (M. P. Malliavin, ed.), Lecture Notes in Mathematics, vol. 1478, Springer Verlag, 1991, pp. 255–272.

- [33] ———, *Cohen-Macaulayness of semi-invariants for tori*, Trans. Amer. Math. Soc. **336** (1993), no. 2, 557–580.
- [34] ———, *Local cohomology of modules of covariants*, Proceedings of the ICM 94, vol. 1, Birkäuser, 1995, pp. 352–362.
- [35] ———, *Some rings of differential operators for SL_2 -invariants are simple*, J. Pure Appl. Algebra **107** (1996), 309–335.
- [36] L. Williams, *Uniform stability of kernels of Koszul cohomology indexed by the Frobenius endomorphism*, J. Algebra **172** (1995), 721–743.
- [37] A. Yekutieli, *An explicit construction of the Grothendieck residue complex*, Asterisque, vol. 208, Soc. Math. France, 1992.
- [38] Yoshino, *Cohen-Macaulay modules over Cohen-Macaulay rings*, London Mathematical Society Lecture Note Series, vol. 146, Cambridge University Press, New York, 1990.

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