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Implicit and hybrid methods for attribute weighting in multi-attribute decision-making: a review study

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Abstract Attribute weighting is a task of paramount relevance in multi-attribute decision-making (MADM). Over the years, different approaches have been developed to face this problem. Despite the effort of the community, there is a lack of consensus on which method is the most suitable one for a given problem instance. This paper is the second part of a two-part survey on attribute weighting methods in MADM scenarios. The first part introduced a categorization in five classes while focusing on explicit weighting methods. The current paper addresses implicit and hybrid approaches. A total of 20 methods are analyzed in order to identify their strengths and limitations. Toward the end, we discuss possible alternatives to address the detected drawbacks, thus paving the road for further research directions. The implicit weighting with additional information category resulted in the most coherent approach to give effective solutions. Consequently, we encourage the development of future methods with additional preference information.

Keywords attribute weighting · multiple attribute decision making · implicit weighting methods · hybrid weighting methods

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1 Introduction

Decision-making is probably the most frequently performed activity of human beings. Hence, it makes sense that academics have put a great deal of effort into mathematically modeling these processes, especially in complex scenarios where intuition may fail. The best decision is always linked to the best interests of decision-makers, which are often associated with several variables (attributes) simultaneously. However, not all attributes have the same importance. For this reason, decision-making processes often benefit from weighting the attributes, which is a decisive step toward generating solutions for the problem at hand. The application of attribute weighting approaches has also been extended to other areas such as data mining [38,7,86,52].

One of the first attribute weighting methods reported in the literature was based on direct ranking where the decision-makers are requested to assign a weight to each attribute given a fixed scale. ELECTRE I [65] and II [66] are examples of this approach. Another early introduced method was the qualitative approach [51], which only requires ordering the attributes by their relevance. However, this type of method is not recommended when dealing with more complex decision-making problems having high uncertainty or dependency among the attributes. Later research efforts (e.g., [34] [76] [41] [10]) allowed handling problems with these characteristics.

Currently, most research efforts focus on developing methods based on optimization models where the information given by decision-makers is modeled as constraints while the objective functions match with the weighting strategy. Some of these methods can be found in [77,25,40,80,54,9,75,74,35,15, 23,88,11,10,56,87,72,30]. Recent papers reviewing and comparing weighting methods can be found in [50,8] and [29], respectively. This paper complements these pieces of research by focusing on the calculation procedures, interpretation and situations where these methods may produce misleading results. Whenever possible, we discuss some ideas that can be used by the community as starting points towards correcting the detected issues.

In the literature, attribute weighting methods are often gathered into three categories: subjective, objective, and hybrid [81,11]. However, in a recently published paper [57], we proposed a different taxonomy (see Figure 1) that facilitates the study of such methods. The first two classes gather explicit weighting methods, which include strong explicit methods (e.g., [17,62,6,39, 18,23,70,69,88]) and weak implicit methods (e.g., [71,77,25,40,80,54,9,75, 74,15,35,21,37]). Overall, 25 methods from both classes were categorized and analyzed (please refer to [57] for more details).

Explicit methods operate on the information provided by decision-makers on the importance of each attribute. However, sometimes decision-makers are not able to provide such information. In addition, if the main objective is attribute weighting, then the availability of weights should not be assumed. In both cases, the viable alternative would be the implicit approaches, where weights are derived indirectly. Methods grouped in Class III (methods without additional information) and Class IV (methods with additional preference

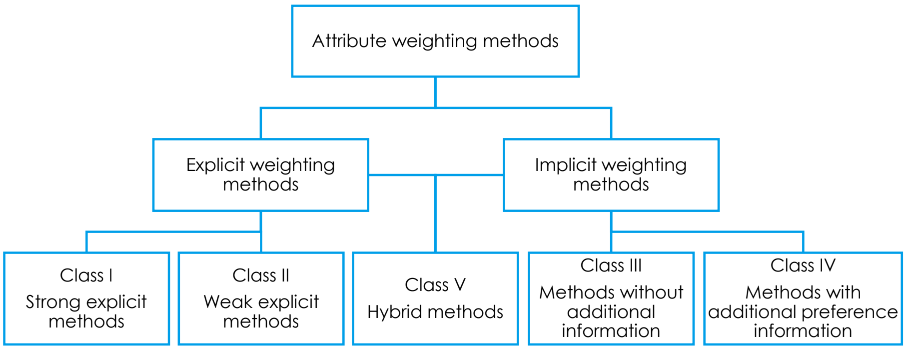

Fig. 1 Categorization of attributes weighting methods [57]. In this paper, we cover the methods belonging to the last three classes.

information) perform implicit weighting. Class V (hybrid weighting methods) combines both explicit and implicit methods. In this paper, we cover the methods belonging to the last three classes (see Table 1), thus providing closure to our previously published review.

Table 1 Distribution of methods to be analyzed in groups within Classes III, IV and V.

Class	Group	Methods
III	1	The entropy method [14,67,73,89], the standard deviation method [14,16], importance of criteria through inter-criteria correlation [53, 33], correlation coefficient and standard deviation [81]
	2	Subjective and objective integrated approach [46,47], attribute weighting method by incompatibility among attributes [11]
	3	Objective method based on intuitionistic fuzzy sets entropy [10], continuous entropy method based on interval-valued intuitionistic fuzzy sets with unknown weights information [35]
	4	Multi-objective programming model that takes one objective without weight information [79], rational model with dissonance minimization based on correlation measures [56], TOPSIS method based on single-valued neutrosophic sets [4]
IV	1	Swing weighting [19], the TRADE-OFFS method [36], MacBeth's weighting method [2]
	2	Linear programming for attributes' weighting on the performance evaluation process [31], multiple attribute decision making based on fuzzy preference information on alternatives [20], attribute weighting using preference comparisons [32]
V	1	Approach to integrate subjective preferences and objective information [82], integrated weighting method of attributes "1" [44], integrated weighting method of attributes "2" [64]

Before moving forward, it seems convenient to describe the review methodology. Firstly, a comprehensive search of articles proposing new weighting methods published up to 2019 was conducted. We gave priority to recent journal papers while avoiding journals requesting publication fees. More than 70 approaches/methods were identified and grouped according to their weighting approach, type of information requested from decision-makers, and the calculation procedures used to compute the weights. Secondly, a representative method of each group was selected and carefully revised.

Overall, this paper has three main objectives. The first one is concerned with revising representative weighting methods, which include both implicit and hybrid approaches. These methods belong to Classes III, IV and V. The second objective focuses on conducting a critical analysis to identify the principal shortcomings of these methods. The third goal is aimed at discussing potential alternatives to overcome the detected issues, thus serving as a guide for future algorithmic and theoretical developments.

The rest of this paper is organized as follows. Section 2, 3 and 4 revise representative methods belonging to Classes III, IV and V, respectively. Section 5 summarizes the main drawbacks of these methods and elaborates on possible solutions. Finally, Section 6 concludes the paper.

2 Implicit weighting methods without additional information

In this section, we will introduce the implicit weighting methods without additional information. These methods, categorized as Class III, only use the decision matrix [57]. This group can be further divided into four subgroups, each described in a separate subsection.

We will assume the following notation (unless otherwise stated). The set of alternatives is $O = \{O_1, O_2, \ldots, O_m\}$ with $m \geq 2$, the set of attributes is $A = \{A_1, A_2, \ldots, A_n\}$ with $n \geq 2$, the weight vector is $w = \{w_1, w_2, \ldots, w_n\}^\top$, where $\sum_{j=1}^{n} w_j = 1, w_j \geq 0$ and w_j denotes the weight of the attribute A_j, the set of decision-makers is $E = \{E_1, E_2, \ldots, E_K\}$ with $K \geq 1$, and the evaluations of the alternatives in each attribute or elements of the decision matrix is $x_{ij}, i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n$.

2.1 Description of methods in Group III.1

The methods described next compute the weights based on the dispersion of the evaluations associated with the available alternatives.

2.1.1 The entropy method

The method uses an entropy index [14, 67, 73, 89], which establishes a measure of the variation among the evaluations of an attribute. Attributes with fairly distributed evaluations are deemed to be important.

The first step of this weighting method consists in normalizing the evaluations x_{ij} associated with each attribute in the decision matrix. The normalized values a_{ij} are computed as follows:

$$a_{ij} = \begin{cases} \frac{x_{ij}}{\sum_{i=1}^{m} x_{ij}} & \text{if } A_j \text{ is for benefit} \\ \\ \frac{1/x_{ij}}{\sum_{i=1}^{m}(1/x_{ij})} & \text{if } A_j \text{ is for cost} \end{cases} . \quad (1)$$

After that, the entropy index E_j associated with attribute A_j is calculated according to Equation (2),

$$E_j = -k \sum_{i=1}^{m} a_{ij} log(a_{ij}); \;\; (k = \frac{1}{log(m)}). \quad (2)$$

Finally, considering the diversity coefficient $D_j = 1 - E_j$, we can compute the weight associated with D_j as $w_j = \frac{D_j}{\sum_{j=1}^{n} D_j}$.

A recent version of this method based on the Evidential Reasoning (ER) [24–28,11,12,49,63,3,83,85] is developed in [11] (view subsection 2.2.2 for a summary of ER). The main difference with respect to the original formalism relies on the information format used by the decision-maker, which is based on the ER instead of being real numbers.

2.1.2 The standard deviation method

The idea behind this method is similar to the previous one [14,16]. The main difference is that the index used to express the attribute' distribution is the standard deviation (SD). In this method, the x_{ij} values are normalized first such that we obtain the matrix $M_{m\times n} = (a_{ij})_{m\times n}$. The standard deviation of the j-th attribute σ_j is determined as $\sigma_j = \frac{1}{m}\sqrt{\sum_{i=1}^{m}(a_{ij} - \bar{a_j})^2}$, where $\bar{a_j} = \frac{\sum_{i=1}^{m} a_{ij}}{m}$. The weight of each attribute is given as $w_j = \frac{\sigma_j}{\sum_{k=1}^{m} \sigma_k}$. A new version based on the ER approach can be found in [11].

2.1.3 Importance of criteria through inter-criteria correlation

This method is based on the importance of criteria through inter-criteria correlation (CRITIC) [53,33]. It determines the weights based on the correlation coefficients r_{jk} between the evaluations on the pairs of attributes (A_j, A_k). Equation (3) formalizes this idea,

$$r_{jk} = \frac{\sum_{i=1}^{m}(a_{ij} - \bar{a_j})(a_{ik} - \bar{a_k})}{\sqrt{\sum_{i=1}^{m}(a_{ij} - \bar{a_j})^2 \sum_{i=1}^{m}(a_{ik} - \bar{a_k})^2}}; \;\; j,k = 1,2,\ldots,n \quad (3)$$

where $\bar{a_j} = \frac{\sum_{i=1}^{m} a_{ij}}{m}$ and $\bar{a_k} = \frac{\sum_{i=1}^{m} a_{ik}}{m}$. Equation (4) shows how to compute the importance coefficient C_j for each attribute A_j,

$$C_j = \sigma_j \sum_{k=1}^{n}(1 - r_{jk}); \;\; j = 1,2,\ldots,n. \quad (4)$$

The final weights are determined after normalizing the coefficients C_j as $w_j = \frac{C_j}{\sum_{k=1}^{n} C_k}$. As in the previous cases, the reader is referred to [11] for a version of this method based on the ER approach.

2.1.4 Correlation coefficient and standard deviation

This method uses the Correlation Coefficient and Standard Deviation (CCSD) [81]. It integrates both the Pearson's correlation coefficient and the SD. This method starts by considering the Simple Additive Weighting (SAW) on each alternative O_i, removing one of the attributes $d_{ij} = \sum_{k=1,k\neq j}^{n} w_k a_{ik}$. Consequently, both the correlation among the evaluations and the SAW weights of all the alternatives are determined without considering the A_j attribute. Equation (5) shows how to compute this,

$$R_j = \frac{\sum_{i=1}^{m}(a_{ij} - \bar{a_j})(d_{ik} - \bar{d_k})}{\sqrt{\sum_{i=1}^{m}(d_{ij} - \bar{d_j})^2 \sum_{i=1}^{m}(a_{ik} - \bar{a_k})^2}} \tag{5}$$

where $\bar{a_j} = \frac{\sum_{i=1}^{m} a_{ij}}{m}$ and $\bar{d_k} = \frac{\sum_{i=1}^{m} d_{ik}}{m}$.

Assuming that σ_j is the SD of the evaluations associated with A_j, the weight of A_j is determined as $w_j = \frac{\sigma_j\sqrt{1-R_j}}{\sum_{k=1}^{n}\sigma_k\sqrt{1-R_k}}$. The weight w_j requires knowledge about R_j that is subject to d_{ij}, which in turn depends on the weights w_j. This generates a dependency cycle. The optimization function presented in Equation (6) avoids this inconvenience,

$$minimize\ J = \sum_{j=1}^{n}(w_j - \frac{\sigma_j\sqrt{1-R_j}}{\sum_{k=1}^{n}\sigma_k\sqrt{1-R_k}})^2\ ;\ \ j = 1, 2, \ldots, n. \tag{6}$$

2.1.5 Advantages and disadvantages of methods in Group III.1

Advantages

a) These methods are suitable for finding the attributes with the highest impact on the alternatives.
   When facing decision-making problems, we often rely on those attributes with the highest influence on the evaluations. By analyzing the attributes that influence such a variability allows adopting policies that lead to an optimal evaluation distribution. For example, let us consider a company selling cars is not doing great when compared to previous years. After an attribute analysis, we find out that there is little variability on the values of the attribute *speed*. This suggests that customers do not take this attribute into consideration when buying the cars of that company, thus the speed is not the reason for the drops in the sales.

Disadvantages

a) It is assumed that a larger dispersion (to the center) in the alternatives' evaluations on the same attribute influences its importance.
An attribute with the alternatives' evaluations widely dispersed discriminates better among them. Therefore, it would make sense to assume that such attributes contain more information to distinguish one alternative from another. However, we might question to what extent this criterion properly reflects the importance of an attribute when it comes to the opinion/objectives of the decision-maker. The reader can fairly argue that such information is already contained in the data records.
b) The option of decision-makers regarding the attributes' relevance is not taken into consideration.
The opinion of decision-makers is a fundamental pillar to solve decision-making problems, even when the information on their preferences is limited. The relevance of each attribute should include human perception, which may vary depending on many factors.

Case study. Let us consider a couple who wants to buy an apartment. The couple has a budget of \$ 100,000 and focuses on three attributes: price, size and location. They would like to save money for furniture although they prefer spacious apartments. The key elements that influence the location are distance to the workplace and access to supermarkets and medical services. Also, it would be desirable to analyze options in areas with good access to public transport. Table 2 shows the different options, where MS and SC represent the closest distances to medical services and a shopping center, respectively, while WP is the distance to the workplace.

Table 2 Summary of the three purchase options.

	Price \$	Size m^2	Location km(MS/SC/WP)
O_1	70,000	120	2.0 /3.0 / 4.0
O_2	90,000	84	0.3 /0.4 / 0.5
O_3	100,000	196	2.0 /2.0 / 3.0

After a qualitative evaluation of the information, the most important attribute should be price, because the amounts are relatively close to the available capital. Location and size should not be so significant due to favorable public transportation and the fact that only two people are moving to the new house. Under such conditions, O_1 arises as the best option. However, implicit weighting methods neglect the information regarding the buyer's opinion, thus very likely generating sub-optimal weights.

Aiming at further elaborating on this issue, let us analyze Table the normalized values per option (see Table 3). The criteria associated with the attribute location were averaged before normalization.

Based on the values in Table 3, the attribute weights and the alternatives' evaluations by simple additive weighting (SAW, $E(O_i)$, $i = 1, 2, 3$) were calculated for the four methods in Group III.1. Table 4 shows the results obtained using Mathematica 12.0 software package [48].

Table 3 Normalized values using the functions in Section 2.1.1.

	Price	Size	Location
O_1	0.40	0.10	0.30
O_2	0.31	0.77	0.21
O_3	0.28	0.13	0.49

Table 4 Attributes' weights and SAW of the alternatives' evaluation according to the four methods in Group III.1.

Method	w (Price)	w (Location)	w (Size)	Evaluation of alternatives by SAW
Entropy	0.025	0.133	0.842	$E(O_2) = 0.68 > E(O_3) = 0.185 > E(O_1) = 0.135$
SD	0.108	0.250	0.642	$E(O_2) = 0.578 > E(O_3) = 0.239 > E(O_1) = 0.183$
CRITIC	0.136	0.214	0.650	$E(O_2) = 0.585 > E(O_3) = 0.231 > E(O_1) = 0.184$
CCSD	0.094	0.257	0.649	$E(O_2) = 0.580 > E(O_3) = 0.240 > E(O_1) = 0.180$

The results show that the attribute with the highest importance is the location because its values have a larger central dispersion. Additionally, the SAW method indicates that O_1 is the worst alternative. Both results are inconsistent with the real situation of the problem.

2.2 Description of methods in Group III.2

The methods in this group are closely related to the ones described in the previous subsection since they assume a dependency between the dispersion of an attribute's evaluations and its importance.

2.2.1 Subjective and objective integrated approach

In this method, the weights are derived from the vector $w^* = \frac{H^{-1}e}{e^T H^{-1} e}$, where $e = [1, \ldots, 1]_{n\times 1}$ and H is a diagonal matrix of $m \times n$ [46,47]. The elements of H are $h_{jj} = \sum_{i=1}^{m}(b_j^* - b_{ij})^2$, where $b_j^* = \max_{1\leq i\leq m}\{b_{ij}\}\forall j = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$ (b_{ij} are the normalized values of x_{ij}).

The values of the main diagonal of H quantify the distance between the maximum evaluation b_j^* and the rest of the evaluations for the same attribute. Note that H^{-1} is the diagonal matrix whose diagonal elements are the reciprocals of H. Therefore, $H^{-1}e$ is a vector that attributes a larger value to each attribute in the measure of how small is the distance between its evaluations set and its maximum evaluation. The w^* vector is the normalization of $H^{-1}e$, given that $e^\top H^{-1}e$ is the sum of the latter vector's elements. In conclusion, this weighting method assigns large weights to the attributes whose evaluation distributions are closer to their maximum evaluation.

2.2.2 Attribute weighting by incompatibility among attributes

The weighting by incompatibility among attributes uses an information modeling from an ER perspective [11]. Let us assume a decision-making problem with M alternatives $a_l, l = 1, 2, \ldots, M$ and L attributes $e_i, i = 1, 2, \ldots, L$. The weights of the L attributes are denoted by $w = (w_1, w_2, \ldots, w_L)$, where $0 \leq w_i \leq 1, \sum_{i=1}^{L} w_i = 1$. The values $\Omega = \{H_1, H_2, \ldots, H_N\}$ are used to evaluate the M alternatives for the L attributes. The elements of the decision matrix can be seen as an evaluation vector distributed across e_i attribute for an alternative $a_l : B(e_i(a_l)) = \{H_n, \beta_{n,i}(a_l), n = 1, 2, \ldots, N\}$. In this formalization, $\beta_{n,i}(a_l)$ is the degree of belief to which the appropriate level for the evaluation e_i regarding the alternative a_l is H_N, such that $\beta_{n,i}(a_l) \geq 0; \sum_{n=1}^{N} \beta_{n,i}(a_l) \leq 1$, and $\sum_{n=1}^{N} \beta_{n,i}(a_l) + \beta_{\Omega,i}(a_l) = 1$. Likewise, $\beta_{\Omega,i}(a_l)$ denotes the degree of global ignorance. If $\beta_{\Omega,i}(a_l) = 0$, then the evaluation is considered to be complete, otherwise it is regarded as incomplete.

To generate a solution in the ER approach, the $B(e_i(a_l))$ evaluations weighted by w are added to form the $B(y(a_l)) = \{H_n, \beta_n(a_l), n = 1, 2, \ldots, N;$ $(\Omega, \beta_\Omega(a_l))$ evaluations, using the analytical algorithm [83]. In this context, $\beta_\Omega(a_l)$ represents the degree of aggregate global ignorance.

Fu Yang et al. in [26] defined a compatibility measure (cm) between two belief structures (BS) based on the distance between the betting commitments of two BS developed by [45]. In this context, in [26] the authors quantify the deviation incompatibility C_σ^i on the attribute e_i as follows:

$$C_\sigma^i = \frac{\sum_{l=1}^{M-1} \sum_{j=l+1}^{M} [1 - cm[B(e_i(a_l)), B(e_i(a_j))]]}{M(M-1)/2}, i = 1, 2, \ldots, L. \quad (7)$$

Consequently, they proposed a decision incompatibility measure C_d^i on the attribute e_i as follows:

$$C_d^i = \frac{\sum_{l=1}^{M} [1 - cm[B(e_i(a_l)), B(y(a_l))]]}{M}, i = 1, 2, \ldots, L \quad (8)$$

such that the aggregated assessments $B(y(a_l))$ were determined using the analytical algorithm presented in [83].

They also proposed a relaxation coefficient $0 \leq \theta \leq 1$ to combine C_σ^i with C_d^i in order to form a complete incompatibility coefficient denoted by $C_s^i = \theta C_\sigma^i + (1-\theta) C_d^i$. The weights are computed as $w_i = \frac{C_s^i}{\sum_{k=1}^{L} C_s^k}$. Therefore, an attribute is deemed important as long as it presents more incompatibilities between the assessments of its different alternatives. This formula is poorly defined. Let us consider the notation $y \leftarrow x$ where the variable y depends on the variable x, so the dependency relation $w_i \leftarrow C_s^i \leftarrow C_d^i \leftarrow B(y(a_l))$ is true. According to the analytical algorithm in [83], $B(y(a_l)) \leftarrow w_i$ meaning that $w_i \leftarrow w_i$, which is not consistent. To avoid this problem, the authors in [11] proposed the following optimization problem:

$$minimize\ W = \sum_{i=1}^{L}(w_i - \frac{C_s^i}{\sum_{k=1}^{L} C_s^k})^2 \tag{9}$$

where $\sum_{i=1}^{L} w_i = 1$, and $w_i \geq 0$, $i = 1, 2, \ldots, L$. This optimization problem allows approximating the values generated by $\frac{C_s^i}{\sum_{k=1}^{L} C_s^k}$ and the w_i values used to obtain $B(y(a_l))$, since they do not match in general.

2.2.3 Advantages and disadvantages of methods in Group III.2

Advantages

a) They are suitable for finding the attributes with the highest impact on the alternatives' differences.
These methods assume that the importance of each attribute is linked to the distribution of the evaluations. The key difference between the methods belonging to Groups III.1 and III.2 relies on the criteria used to indicate how the distributions of the evaluations affect the weights.

Disadvantages

a) It is assumed that the distribution of the attribute evaluations by all the alternatives determines its importance.
Although all the methods in this group show this limitation, we focus on the one described in subsection 2.2.1. Let us the case where the attributes' evaluations are all the same. In such a situation, determining the weights by using the expression $w^* = \frac{H^{-1}e}{e^\top H^{-1}e}$ loses meaning since H^{-1} would not be defined. It could happen that all evaluations on the same attribute are almost equal (A_j). If so, even though w^* is well defined, the weight of A_j would be very large because the coefficient associated with its value in the matrix H^{-1} is $\frac{1}{\sum_{i=1}^{m}(b_j^* - b_{ij})^2}$. This coefficient would reach large values as the denominator would be very close to 0. If all the evaluations of A_j are small, and a new alternative is added with a larger evaluation, then the weight of A_j decreases radically. This means that only one component would primarily determine the attribute weight.
b) The option of decision-makers regarding the attributes' relevance is not taken into consideration.
If we neglect the subjective opinion of decision-makers, then for two decision-making problems with the same number of alternatives, attributes, and decision matrix confirmation, the model defined in Equation (9) will generate identical solutions. However, the decision contexts and the attributes' nature could be entirely different.

2.3 Description of methods in Group III.3

The methods described in this subsection minimize the fuzzy entropy of the decision matrix to generate the weights.

2.3.1 Objective method based on intuitionist fuzzy set entropy

This method uses intuitionist fuzzy sets (IFS) to generate the decision matrix's values [10]. An IFS A in a set X is defined as $A = \{\langle x, \mu_A(x), v_A(x) | x \in X\rangle\}$, where $\mu_A : X \longrightarrow [0,1]$ and $v_A : X \longrightarrow [0,1]$. The values μ_A and v_A represent the degrees of membership and non-membership of x to A, respectively, such that $0 \leq \mu_A(x) + v_A(x) \leq 1$. In addition, the degree of ignorance of A is defined by $\pi_A(x) = 1 - (\mu_A(x) + v_A(x))$. Within this mathematical formalism, the concept of fuzzy entropy concept emerged.

In [10] several approaches to determine the fuzzy entropy are discussed. As not all of them could be discussed in this paper, we will adopt one of these approaches to contextualize the concept of fuzzy entropy.

Let us assume that the entropy of a set X with IFS values is $E(X) = \sum_{i=1}^{n} \pi_i$, where $X = \{x_1, \ldots, x_n\}, x_i = (\mu_i, v_i, \pi_i)$. The first step to calculate the weights consists in computing the fuzzy entropy matrix such that each element $x_{ij} = (\mu_{ij}, v_{ij}, \pi_{ij})$ of the decision matrix is replaced by its singular entropy value E_{ij} (as $E_{ij} = \pi_{ij}$). Therefore, for each attribute A_j we have a unique entropy of the alternatives' evaluation $E_{ij}, i = \{1, 2, \ldots, m\}$. The values of E_{ij} are normalized as $h_{ij} = \frac{E_{ij}}{\max\{E_{ij}, E_{2j}, \ldots, E_{mj}\}}$. If we consider that $a_j = \sum_{i=1}^{m} h_{ij}$ and $T = \sum_{i=1}^{n} a_j$, then the weights can be computed as $w_j = \frac{1-a_j}{n-T}$. According to this expression, larger weights are assigned to attributes whose fuzzy entropy values are small.

2.3.2 Continuous entropy method based on interval-valued IFS with unknown weight information

Continuous entropy method based on interval-valued IFS with unknown weight information is a group of decision-making methods where each decision-maker builds the decision matrix with interval-valued IFS (IVIFS) [35, 78, 77, 90, 84, 61]. A set $\bar{A}$ of IVIFS associated with the universe $X = \{x_1, x_2, \ldots, x_k\}$ of fixed real values is defined as $\bar{A} = \{\langle x, \bar{\mu}_{\bar{A}}(x), \bar{v}_{\bar{A}}(x) | x \in X\rangle\}$, where $\bar{\mu}_{\bar{A}}(x) : X \longrightarrow L([0,1]), \bar{v}_{\bar{A}}(x) : X \longrightarrow L([0,1])$, while $L([0,1])$ denotes the set of all sub-intervals of $[0,1]$. Likewise, $\bar{\mu}_{\bar{A}}(x)$ and $\bar{v}_{\bar{A}}(x)$ represent the degree of membership and non-membership of $\bar{A}$ to X, respectively.

Similarly, $\bar{A}$ can be expressed as $\{\langle x, [\bar{\mu}^L_{\bar{A}}(x), \bar{\mu}^R_{\bar{A}}(x)], [\bar{v}^L_{\bar{A}}(x), \bar{v}^R_{\bar{A}}(x)] | x \in X\rangle\}$, where $\bar{\mu}^L_{\bar{A}}(x)$ and $\bar{v}^L_{\bar{A}}(x)$ are the lower limits of the intervals and $\bar{\mu}^R_{\bar{A}}(x)$ and $\bar{v}^R_{\bar{A}}(x)$ are the upper values, so that $0 \leq \bar{\mu}^L_{\bar{A}}(x) \leq \bar{\mu}^R_{\bar{A}}(x) \leq 1$, $0 \leq \bar{v}^L_{\bar{A}}(x) \leq \bar{v}^R_{\bar{A}}(x) \leq 1$ and $\bar{\mu}^R_{\bar{A}}(x), \bar{v}^R_{\bar{A}}(x) \leq 1, \forall x \in X$.

To define the fuzzy entropy coefficient, in [35] the authors convert IVIFS values into IFS ones. This conversion is carried out by transforming an interval into a real value using the function $F_Q(a) = F_Q([a^-, a^+]) = \lambda a^+ + (1-\lambda)a^-$, where $\lambda \in [0,1]$ determines the preference of the decision-maker towards a^- or a^+, respectively. In addition, the ignorance degree associated with each of these IFS values formed is $\pi_{F_Q(\tilde{\alpha}_{ij})} = 1 - (F_Q(\mu_{\tilde{\alpha}_{ij}}) + F_Q(v_{\tilde{\alpha}_{ij}}))$. The fuzzy entropy of $\tilde{\alpha}_{ij}$ is defined as follows,

$$\varepsilon(\tilde{\alpha}_{ij}) = \frac{1 - |F_Q(\mu_{\tilde{\alpha}_{ij}}) - F_Q(v_{\tilde{\alpha}_{ij}})| + \pi_{F_Q(\tilde{\alpha}_{ij})}}{1 + |F_Q(\mu_{\tilde{\alpha}_{ij}}) - F_Q(v_{\tilde{\alpha}_{ij}})| + \pi_{F_Q(\tilde{\alpha}_{ij})}}. \tag{10}$$

When considering the opinion of multiple decision-makers $(e_1, e_2, \ldots, e_K)$, the entropy of the attribute A_j is calculated as $E(A_j) = \sum_{k=1}^{K} l_k \sum_{i=1}^{m} \varepsilon(\tilde{\alpha}_{ij})$, where l_k denotes the weight of the k-th decision. The weights based on the entropy of the IFS evaluations are calculated as follows:

$$w_j = \frac{\sum_{k=1}^{K} l_k \sum_{i=1}^{m}(1 - \varepsilon(\tilde{\alpha}_{ij}^k))}{\sum_{j=1}^{n}\sum_{k=1}^{K} l_k \sum_{i=1}^{m}(1 - \varepsilon(\tilde{\alpha}_{ij}^k))}; \quad j = 1, 2, \ldots, L. \tag{11}$$

2.3.3 Advantages and disadvantages and of methods in Group III.3

Advantages

a) These methods are suitable when the quantity and quality of information available on some attributes are poor.
If there is uncertainty in the evaluation of a given attribute, it is reasonable not to rely too heavily on that information. One way to handle such uncertainty is to reduce the relevance of that attribute when making the decisions. This is the idea behind these methods.

Disadvantages

a) Inconsistency in the objective method based on IFS entropy.
The methods based on fuzzy entropy are intended to assign larger weights to those attributes whose evaluations have lower entropy values (as explained in [10]) since this is a desirable characteristic of the decisions matrix. However, the formula for calculating weights it is not consistent with this line of thought.
In the expression $w_j = \frac{1-a_j}{n-T}$ (where $a_j = \sum_{i=1}^{m} h_{ij}$ and $T = \sum_{i=1}^{n} a_j$), a_j is subtracted from 1. This means that larger fuzzy entropy values correspond with lower weights. It should be noticed that $a_j \geq 1 \forall j = 1, 2, \ldots, n$, so $1 - a_j \leq 0$. Likewise, $T = \sum_{i=1}^{n} a_j \geq \sum_{i=1}^{n} 1 = n$. The case where $a_j = 1$ can be ruled out since is unlikely in practice, which means that $1 - a_j < 0$ and $T > n$. Then, $w_j = \frac{1-a_j}{n-T}$ takes a positive value because the numerator and denominator are negative. However, larger values of a_j increase the modular value of $1 - a_j$ so the expression $\frac{1-a_j}{n-T}$ increases as well. In other words, large fuzzy entropy values result in large weights, which contradicts the intuition behind these methods.
As an alternative to overcome this drawback could change the expression $w_j = \frac{1-a_j}{n-T}$ to $w_j = \frac{m-a_j}{mn-T}$. Given that $h_{ij} \leq 1 \longrightarrow a_j = \sum_{i=1}^{m} h_{ij} \leq \sum_{i=1}^{m} 1 = m$ and $T = \sum_{i=1}^{n} a_j \leq \sum_{i=1}^{n} m = nm$, then the assumption that both the numerator and denominator are always positive is fulfilled. In addition, larger values of a_j would decrease the numerator such that the expression would behave as desired.

The expression $w_j = \frac{m-a_j}{mn-T}$ can be generalized to $w_j = \frac{M-a_j}{Mn-T}$ where M is a constant such that $M \in R, M \geq m$. Note that $\sum_{j=1}^{n} w_j = \sum_{j=1}^{n} \frac{(M-a_j)}{Mn-T} = \frac{\sum_{j=1}^{n}(M-a_j)}{Mn-T} = \frac{nM-\sum_{j=1}^{n} a_j}{Mn-T} = \frac{nM-T}{Mn-T} = 1$.

b) A small fuzzy entropy in the evaluations of the same attribute does not objectively contribute to its importance.

Case study. A company wants to invest based on two attributes, the investment cost (A_1) and the expected increase earnings per year (A_2). The company gives priority to the second attribute.

According to the method described in Section 2.3.2. Let us assume that A_2 is more important than A_1. Moreover, since the values of A_1 and A_2 depend on factors generating uncertainty, it would make sense to express the evaluations with IVIFS values. In this example, we will model the problem with IFS for the sake of simplicity. Table 5 shows the values of $F_Q(\mu_{\tilde{\alpha}_{ij}})$ and $F_Q(v_{\tilde{\alpha}_{ij}})$, and the evaluations of the investment options O_1 and O_2.

Table 5 Evaluations of O_1 and O_2 for attributes A_1 and A_2, respectively.

	Investment cost A_1	Expected increase in annual earnings A_2
O_1	$F_Q(\mu_{\tilde{\alpha}_{11}}) = 0.50, F_Q(v_{\tilde{\alpha}_{11}}) = 0.40$	$F_Q(\mu_{\tilde{\alpha}_{12}}) = 0.60, F_Q(v_{\tilde{\alpha}_{12}}) = 0.20$
O_2	$F_Q(\mu_{\tilde{\alpha}_{21}}) = 0.70, F_Q(v_{\tilde{\alpha}_{21}}) = 0.20$	$F_Q(\mu_{\tilde{\alpha}_{22}}) = 0.40, F_Q(v_{\tilde{\alpha}_{22}}) = 0.40$

In this example, the alternative O_1 is less valued than O_2 with respect to A_1. In addition, A_1 has a small degree of ignorance 0.1 for both alternatives $\pi_{F_Q(\tilde{\alpha}_{ij})} = 0.1 = 1-(0.5+0.4) = 1-(0.7+0.2)$. However, O_1 is better valued than O_2 with respect to the most important attribute A_2. Notice that A_2 is the more uncertain attribute, so its degree of ignorance 0.2, is greater than the one associated with $A_1, \pi_{F_Q(\tilde{\alpha}_{ij})} = 0.2 = 1-(0.6+0.2) = 1-(0.4+0.4)$. Table 6 shows the entropy values for the decision matrix evaluations, and the attributes' weights according to Equations (10) and (11).

Table 6 Entropy of the evaluations according to Equation (10) and weights according to Equation (11).

	A_1	A_2
O_1	$\varepsilon(\tilde{\alpha}_{11}) = 0.833$	$\varepsilon(\tilde{\alpha}_{12}) = 0.500$
O_2	$\varepsilon(\tilde{\alpha}_{21}) = 0.375$	$\varepsilon(\tilde{\alpha}_{22}) = 0.100$
w_j	$w_1 = 0.612$	$w_2 = 0.387$

The weights generated by this method indicate that A_1 is more important than A_2, which does not correspond with the company's interest.

According to the method described in Section 2.3.1. The expression $w_j = (1 - a_j)/(n - T)$ fails in assigning larger weights to attributes having smaller entropy values. Hence, the expression $w_j = (M - a_j)/(nM - T)$ will be used instead. Table 6 shows the entropy values obtained according to $E_{ij} = \pi_{ij}$ (see Table 7) while Table 8 shows the normalized values.

Table 7 Entropy of the evaluations according to $E_{ij} = \pi_{ij}$.

	A_1	A_2
O_1	$E_{11} = 0.1$	$E_{11} = 0.2$
O_2	$E_{21} = 0.1$	$E_{22} = 0.2$

Table 8 Normalized entropy values according to the proposed format.

	A_1	A_2
O_1	$E_{11} = 1$	$E_{12} = 1$
O_2	$E_{21} = 1$	$E_{22} = 1$

Given the fact that all values associated with each attribute match with each other, then we can conclude that $w_1 = w_2$. However, this conclusion would conflict with the problem statement.

2.4 Description of methods in Group III.4

As mentioned in Section 1, the weighting methods were grouped into different classes according to their similarity to facilitate their analysis.

2.4.1 Multi-objective programming model that takes one objective without weight information

The idea behind this model is that the most appropriate weights are the ones that maximize the evaluations [79]. Assuming that z_{ij} represents the normalized values of each alternative, the authors in [79] proposed the optimization model described below,

$$maximizing\ J(W) = \frac{1}{m}\sum_{i=1}^{m} D_i(W) = \frac{1}{n}\sum_{i=1}^{n} z_{ij}w_j. \quad (12)$$

Overall, the objective of this optimization model is to find the set of weights that maximizes the average of the simple additive evaluations.

2.4.2 Rational model with dissonance minimization based on correlation measures

In [56] the authors proposed a rational model with dissonance minimization, which is based on the cognitive crisis theory [22]. This method introduces several indicators to correlate the evaluations for pairs of alternatives. These indicators increase with the similarity between the paired evaluations of alternatives. The optimization model in [56] gives more relevance to the weights that maximize the correlation between all pairs of alternatives, and the similarity between each alternative and the alternative having an *optimal evaluation*. In the context of decision making, the alternative with optimal evaluation is the one with the best attribute rating.

2.4.3 TOPSIS method based on single-valued neutrosophic sets

Neutrosophic sets (NS) can be seen as a generalization of fuzzy sets and IFS. These sets are defined as $NS = \langle T_N(x), I_N(x), F_N(x)\rangle$, where $T_N(x)$ is the truth membership function, $I_N(x)$ is the indeterminacy-membership function and $F_N(x)$ is the falsity-membership function. These independent functions represent uncertainty but also vagueness, imprecision, error, contradiction and redundancy [4]. Unlike IFS, in NS-based methods, the sum of the three functions satisfies the condition $0 \leq T_N(x) + I_N(x) + F_N(x) \leq 3$.

In [4] the ratings provided by decision-makers about each attribute's alternative take the form of single-valued NS, thus generating the following decision matrix $X_k = \langle T_{ij}, I_{ij}, F_{ij}\rangle_{m\times n}$ where T_{ij}, I_{ij}, F_{ij} denote the degree of truth-membership, indeterminacy-membership and falsity-membership, respectively, of alternative Oi with respect to attribute A_j. Equation (13) displays the weighting averaging operator used to aggregate the decision matrix associated to the k-th decision maker,

$$X_k \oplus W = X^w = \langle x_{ij}^{w_j}\rangle_{m\times n} = \langle T_{ij}^{w_j}, I_{ij}^{w_j}, F_{ij}^{w_j}\rangle_{m\times n}. \tag{13}$$

Afterwards, the relative positive $RP^* = [x_1^{w+}, x_2^{w+}, \ldots, x_n^{w+}]$ and negative RN^* ideal solutions are defined from the aggregated weighted decision matrix. The Euclidean distance is used to determine the gap from each alternative O_j to the positive and negative ideal solutions. Thus, the relative closeness coefficient (C_i^*) of each alternative to the neutrosophic positive ideal solution is calculated by Equation (14). This coefficient is used as a baseline such that values above the coefficient indicate better alternatives.

$$C_i^* = \frac{D_{Euc}^{i-}(x_{ij}^{w_j}, x_j^{w-})}{D_{Euc}^{i+}(x_{ij}^{w_j}, x_j^{w+}) + D_{Euc}^{i-}(x_{ij}^{w_j}, x_j^{w-})}, \text{ for } i = 1, 2, ..., m. \tag{14}$$

The similarity measure is an important tool to compute the degree of similarity between objects. In [58] the authors proposed other vector similarity measures of single-valued and interval NS by hybridizing the concepts of Dice and Cosine similarity measures.

MADM problems using single-valued NS have the drawback of asymmetrical behavior and undefined phenomena. In [59] the authors proposed a method able to deal with unknown weights of attributes and unknown weights of decision-makers. Recently published papers [60, 13, 5] expand the single-valued representation of ratings on each attribute's alternative to interval-valued, trapezoidal, among others types of NS.

2.4.4 Advantages and disadvantages of methods in Group III.4

Next, we analyze the shortcomings of methods in Group III.4. The advantages and disadvantages listed in a) refer to the method described in Section 2.4.1, the ones in b) refer to the method in Section 2.4.2 while the ones in c) refer to the method in Section 2.4.3.

Advantages

a) These methods are suitable for low-risk environments.
   By assuming that the evaluations are maximal, decision-makers take an optimistic position. If the decision environment is high-risk, this position can lead to significant losses. This might result in a considerable deviation of $J(w)$ from its real value. However, if the environment is low-risk, then incorrect results would have little consequences.
b) They are suitable in situations where the characteristics of the available alternatives are different from each other.
c) These methods are capable of handling indeterminate and inconsistent information.

Disadvantages

a) The maximization of the evaluations is not an objective criterion to determine the attributes' weights.
   Large evaluation values do not always express the intrinsic attractiveness of attributes as this also depends on human perception. It seems opportune to highlight that these methods do not take into account the decision-makers' preferences. Consequently, it would difficult to guarantee the compatibility between both pieces of information.
b) The global correlation maximization between the pairs of alternatives (as an indicator of low cognitive dissonance) does not seem to be an objective criterion to determine the weight of each attribute.
   Human beings prefer to avoid uncertainty when facing decision-making situations. According to the method described in Section 2.4.2, one of the main reasons for decision-makers to be uncertain is that evaluations' distributions might differ considerably. If two alternatives have very different characteristics and present weaknesses and strengths, then the comparison between becomes difficult. Moreover, the fact that a set of weights implies more similarities between the alternatives does not imply a change in the real characteristics for them to be more similar. Hence, the premise that

the cognitive dissonance phenomenon can be limited without changing the alternatives' characteristics seems to lack objectivity.

3 Implicit weighting methods with additional information

In this section, we will discuss weighting methods that do not require direct information about the attributes. Nevertheless, they need additional information about the decision matrix.

3.1 Description of methods in Group IV.1

The methods in this subsection use ideal reference alternatives, which are used by decision-makers to establish their preferences.

3.1.1 Swing weighting

This method uses an initial ideal alternative where all attributes have minimum evaluation. Then, starting with the most important attribute, it increases its level to be the maximal one while inspecting the new alternatives [19]. The change in the attractiveness of the new alternative for the reference is attached with a numerical reference value (commonly 100), which is the value assigned to the most important attribute. The same process is performed for each one of the other attributes by increasing the level from minimum to maximum. The changes produced in the alternatives are weighted with values lower than the one that was initially granted to the most relevant attribute. These values are given to the remaining attributes. Finally, the assigned values are normalized, thus generating the definitive weights.

3.1.2 The TRADE-OFFS method

The TRADE-OFFS method starts with the maximum and minimum attributes' evaluations $A_1^k, A_2^k, \ldots, A_n^k$ and $A_1^1, A_2^1, \ldots, A_n^1$, respectively, where k is the maximum evaluation level that an attribute can reach while 1 is the minimum [36]. For each attribute A_i, decision-makers are asked for the value of p of the lottery alternative $(A_1^k, A_2^k, \ldots, A_n^k, p; A_1^1, A_2^1, \ldots A_n^1, 1-p))$ to be preferentially equivalent to the alternative $(A_1^1, A_2^1, \ldots, A_i^k, \ldots, A_n^1)$. The larger the value of p, the most important the attribute. The following step consists in sorting the attributes according to their importance.

To determine the weights, the authors in [36] used the most important attribute (A_j) as a reference. For each attribute A_i, $i = 1, 2, \ldots, n$, $i \neq j$ they ask decision-makers to set the levels for $A_j^{k'}$, A_j and $A_i^{k''}$, A_i, $k', k'' = 1, 2, \ldots, k$. Any level associated with the remaining attributes does not change for an alternative with $A_j^{k'}$, A_i^1 and another with $A_j^1, A_i^{k''}$. Then, considering that an attribute evaluated with 0 does not report any utility, the equality between these two alternatives is $r_j u_j(A_j^{k'}) = r_i u_i(A_i^{k''})$, where $u_j(A_j^{k'})$ and $u_i(A_i^{k''})$ are the utilities (established by decision-makers) that report the levels $A_j^{k'}$ and $A_i^{k''}$, respectively. This leads to $r_i = r_j \frac{u_j(A_j^{k'})}{u_i(A_j^{k''})}$. After setting a value for r_j, it is possible to derive all possible values from r_i, which can be considered a relevance indicator for each attribute. Therefore, it would only be necessary to normalize these values to obtain the final weights.

3.1.3 MacBeth's weighting method

The main difference between MacBeth's method [2] and the others reported in the literature relies on the scale is that the scale used to evaluate the alternatives. Instead of fixing this scale, MacBeth's method uses a generated scale that takes into account the set of impact levels that different evaluations have on each attribute. For example, the reached attribute's level A_i can have a high impact (S_k), a huge impact (S_{k-1}), ..., a little impact (S_1) such that the set $S = (S_1, S_2, \ldots, S_k)$ determines an ordinal scale, where S_k is greater than S_{k-1} and so on, S is called the "attribute descriptor". Within this ordinal scale, two particular levels must be selected, one neutral (S_1 will be assumed) and another good ($S_{k'}$ will be assumed).

To express the difference in attractiveness between the elements of S, we should consider another ordinal scale. This new scale is characterized by the following elements: an element without a difference in attractiveness (C_0), an element with a small difference (C_1), ..., and so on until reaching a huge attractive difference (C_d), where $d+1$ is the total possible levels for an attractive difference. The set $C = (C_0, C_1, \ldots, C_d)$ must be generated.

For each attribute, the decision-makers have to assign a value $C_i, i = 1, 2, \ldots, d$ to differentiate between each pair of elements $S_j, S'_j \in S$ as shown in Table 9. These values will be denoted as $(S_j, S_{j'}) \in C_i$, where S_j is preferred over $S_{j'}$. The reader can notice that more than a couple of elements of S may be associated with the same value of C.

Table 9 Assignments of the attractiveness differences C_i to each pair of impact levels.

	S_k	S_{k-1}	S_{k-2}	$\ldots$	S_1
S_k	C_0	C_1	C_1	$\ldots$	C_d
S_{k-1}		C_0	C_2	$\ldots$	C_{d-2}
S_{k-2}			C_0	$\ldots$	C_{d-2}
$\ldots$				$\ldots$	$\ldots$
S_1					C_0

The numerical scale $\Phi(S_i)$ associated with the ordinal scale S for a given attribute must be consistent with the elements in the assignment table of C_i. Therefore, the authors in [2] proposed the following two conditions.

Condition 1 (ordinal condition). $\forall S_i, S_j \in S : \Phi(S_i) > \Phi(S_j) \iff S_i$ is more attractive than S_j.

Condition 2 (semantic condition). $\forall d', d'' \in \{0, 1, \ldots, d\}, \forall S_i, S_j, S_{i'}, S_{j'} \in \{S\}$ with $(S_i, S_j) \in C_{d'}$ and $(S_{i'}, S_{j'}) \in C_{d''} : d' \geq d''+1 \implies \Phi(S_i) - \Phi(S_j) > \Phi(S_{i'}) - \Phi(S_{j'})$.

We can build and optimization model using all feasible scales that fulfill these two conditions. Such a model allows generating a scale (μ), which is called the basic MacBeth's scale, as depicted below:

$$\begin{aligned}
&\textit{minimize } \Phi(S_k) \\
&\text{s.t: } \Phi(S_1) = 0;\ \forall S_i, S_j \in S:\ \text{with } (S_i, S_j) \in C_0 \implies \Phi(S_i) = \Phi(S_j) \\
&\forall d', d'' \in \{0, 1, \ldots, d\} \text{ with } d' > d'', \forall (S_i, S_j) \in C_{d'} \text{ and} \\
&\forall (S_{i'}, S_{j'}) \in C_{d''} \implies \Phi(S_i) - \Phi(S_j) \geq \Phi(S_{i'}) - \Phi(S_{j'}) + d' + d''.
\end{aligned} \tag{15}$$

However, the scale $\mu(S)$ generated by this linear optimization problem is not definitive. Its main goal is to serve as a reference, so it does not have to be rebuilt from the beginning because the values were established according to Conditions 1 and 2. We can start from $\mu(S)$ and then ask the decision-makers whether or not the proportion $\frac{\mu(S_i)-\mu(S_j)}{\mu(S_{i'})-\mu(S_{j'})}$ correctly reflects the difference of S_i and S_j with respect to $S_{i'}$ and $S_{j'}$, respectively. In this way, decision-makers will be able to suggest changes in the values of $\mu(S_i), \forall i = 1, 2, \ldots, k$ to generate values fitting their opinion. Of course, the constraints to which the optimization model is subject to must be fulfilled.

Once the scales associated with each descriptor have been established, we use a set of $n + 1$ dummy alternatives to produce the attributes' weights $a_0 = (S_1^1, S_1^2, \ldots, S_1^n)$, $a_1 = (S_k^1, S_1^2, \ldots, S_1^n)$, $a_2 = (S_1^1, S_k^2, \ldots, S_1^n)$, …, $a_n = (S_1^1, S_1^2, \ldots, S_{k'}^n)$. In this formulation, the superscripts indicate the attribute reference. These alternatives are those achieving the neutral level (S_1) for all the attributes but one, where the good level is reached ($S_{k'}$). With the exception that A reached the neutral level in all attributes. Considering that the neutral and good evaluation values are $v = 0$ and $v = 100$, respectively, and assuming that the overall alternative's evaluation O_i can be calculated as $V(O_i) = \sum_{j=1}^{n} w_j v(x_{ij})$, we would have $V(a_i) = \sum_{j=1, j\neq i}^{n} w_j v(S_1^j) + w_i v(S_{k'}^i) = w_i v(S_{k'}^i) = 100 w_i; \forall i = 1, 2, \ldots, n$ and $V(a_0) = 0$.

Since the global alternatives' values $V(a_i)$ are directly related to the weights of attributes, we could ask decision-makers to rank these reference alternatives to determine an order among the weights. Later on, it would be sufficient to define the set of semantic expressions to characterize the attractive differences for each pair of attributes. Then, a matrix of paired comparisons can be formed between these reference alternatives. The same process is applied to determine the numerical scale of the evaluations. This scale will associate each reference alternatives with a value as preferred by the expert. We can obtain the final weights after normalizing these values.

3.1.4 Advantages and disadvantages of methods in Group IV.1

Advantages

a) The features of the reference alternatives defined in these methods allow taking into account the opinion of decision-makers.
   Many weighting methods require decision-makers to establish a hierarchical order among them. However, it is unlikely to obtain this information directly from decision-makers, mainly when the alternatives' characteristics differ significantly from each other. The reference alternatives used by these methods have characteristics that allow for this feature. Since the assessments of most attributes are similar, the possible distortion caused by the variability of the information is limited. At the same time, there are differences in the evaluations of the remaining attributes, thus allowing discriminating among the alternatives.

Disadvantages

a) The concept of weight still does not have an objectively formalized interpretation.
   In the Swing method [19], for example, this disadvantage becomes more evident since the weighting is based on a direct assignment by decision-makers, which naturally implies a very high degree of subjectivity. In the case of other methods (i.e., MacBeth, TRADE-OFFS), the weights are derived from mathematical rules but the concept of weight is not objectively formalized, so its interpretation is rather intuitive. In the TRADE-OFFS method [36], the weights are determined based on their relationship with the most important attribute. This is a piece of minimal information. On the other hand, in MacBeth's method [2] decision-makers have the possibility of changing the values of the basic scale μ to adapt the ratio $\frac{\mu(S_i)-\mu(S_j)}{\mu(S_{i'})-\mu(S_{j'})}$ to a desired value. However, the meaning of the previous relationship is very subjective, which makes it difficult for decision-makers to establish an optimal relationship. Since the concept of weight or the important relationship is only an intuitive idea, it cannot be guaranteed that decision-makers can optimally correct the relations $\frac{\mu(S_i)-\mu(S_j)}{\mu(S_{i'})-\mu(S_{j'})}$.

3.2 Description of methods in Group IV.2

The methods described in this subsection consider two forms of evaluation: functional and subjective. The former approach is based on mathematical expressions describing how to weight the alternatives. The latter approach is based on a specific type of subjective information related to preferences of decision-makers. These methods try to find out which set of weights provides the best trade-off between these criteria.

3.2.1 Linear programming for attributes' weighting on the performance evaluation process

When using linear programming, once all the alternatives have been considered, they need to be sorted according to decision-makers' preferences [31].

Let $(O_{1'}, O_{2'}, \ldots, O_{m'})$ be a permutation of $(O_1, O_2, \ldots, O_m)$ such that $O_{i'}$ is at least as preferred as O'_{i+1}. That is, an m-tuple where the alternatives appear in descending order from left to right in relation to its attractiveness. Then the global evaluation of $O_{i'}$ is carried out as $V_i = \sum_{j=1}^{n} w_j x_{ij}$, where x_{ij} is the normalized value of the evaluated alternative $O_{i'}$ with respect to the attribute A_j in the decision matrix. Based on such assumptions, it is fulfilled that $V_i - V_{i+1} = \sum_{j=1}^{n} w_j(x_{ij} - x_{(i+1)j}) \geq 0,\ \forall i = 1, 2, \ldots, m-1$. Likewise, let $V_i - V_{i+1} - S_i = \sum_{j=1}^{n} w_j(x_{ij} - x_{(i+1)j}) - S_i = 0$ with $S_i \geq 0$. Then, considering that $Z = \min\{S_1, S_2, \ldots, S_{m-1}\}$, the authors of [31] proposed the following linear optimization model to weigh the attributes:

$$\begin{aligned} &maximize\ Z, \\ &\text{s.t:} \sum_{j=1}^{n} w_j(x_{ij} - x_{(i+1)j}) \geq Z; i = 1, ..., m-1; Z \geq 0. \end{aligned} \tag{16}$$

3.2.2 Multiple attribute decision making based on fuzzy preference information on alternatives

In this method, decision-makers must provide the fuzzy preference matrix among the alternatives [20]. This matrix is represented by $P = [p_{ij}]_{m \times m}$, and its elements comprise the fuzzy preference relationships p_{ij} that indicate the preference of O_i over the O_j. The values of p_{ij} satisfy that: $p_{ij} \geq 0 : p_{ij} + p_{ji} = 1,\ \forall i, j = 1, 2, \ldots, m,\ i \neq j$, and $p_{ii} = -$ ($-$ means that the decision-makers are not requested to provide this information).

The values x_{ij} of the decision matrix are normalized as shown below, thus leading to the normalized decision matrix $B = [b_{ij}]_{m \times n}$:

$$b_{ij} = \begin{cases} \frac{x_{ij} - x_j^{min}}{x_j^{max} - x_j^{min}}, \text{if } A_j \text{ is of benefit} \\ \frac{x_j^{max} - x_{ij}}{x_j^{max} - x_j^{min}}, \text{if } A_j \text{ is cost-effective.} \end{cases} \tag{17}$$

Considering that the alternative's attractiveness O_i can be measured as $d_i = \sum_{j=1}^{n} b_{ij} w_j$, the authors in [20] suggested generating a type of fuzzy relationship among the alternatives subject to the weights:

$$\bar{p}_{ik} = \frac{d_i}{d_i + d_k} = \frac{\sum_{j=1}^{n} b_{ij} w_j}{\sum_{j=1}^{n} (b_{ij} + b_{kj}) w_j}$$

where the expression

$$g_{ik} = p_{ik} - \bar{p}_{ik} = p_{ik} - \frac{\sum_{j=1}^{n} b_{ij} w_j}{\sum_{j=1}^{n} (b_{ij} + b_{kj}) w_j}$$

suggests that we want to minimize its value, which would imply a high compatibility between the decision-makers' opinion on the alternatives and their evaluations. The denominator of the second term of g_{ik} is multiplied by $h_{ij}(w) =$

$p_{ij} \sum_{j=1}^{n}(b_{ij}+b_{kj})w_j - \sum_{j=1}^{n} b_{ij}w_j, \forall i \neq j$, thus leading to the following linear optimization model to generate the weights:

$$minimize\ H(w) = \sum_{i=1}^{m}\sum_{j=1}^{m}[p_{ik}\sum_{j=1}^{n}(b_{ij}-b_{kj})w_j - \sum_{j=1}^{n} b_{ij}w_j]^2. \tag{18}$$

3.2.3 Attribute weighting using preference comparisons

The attribute weighting using preference comparisons starts by calculating the distance between alternative O_i and the optimal ideal for $D_i = K + \sum_{j=1}^{n} w_j d_{ij} + \epsilon_i,\ i = 1, 2, \ldots, m$, where K is a constant, d_{ij} is the distance between the evaluation of criterion A_j for the alternative $O_i(y_{ij})$ and the evaluation of A_j in the ideal (x_j), while ϵ_i is an error term [32]. The value d_{ij} can be determined as $d_{ij} = f_j(y_{ij} - x_j)$, where f is a function that represents the dependency of d_{ij} with respect to $y_{ij} - x_j$. In addition, the distance between two alternatives O_q and O_p can be measured as $\triangle D_{pj} = D_q - D_p = \sum_{j=1}^{n} w_j(d_{qj} - d_{pj}) + \epsilon_{pq} \geq 0, q, p \in S$. S denotes the set of index pairs q, p such that O_p is preferred to O_q. The distance (D_q) between O_q and the ideal alternative is greater than the distance (D_q) between O_p and the ideal alternative $(D_q \geq D_p)$. In addition $\epsilon_{qp} = \epsilon_q - \epsilon_p$.

It should be noticed that the set S does not discriminate between *small* or *large* evaluations. Consequently, decision-makers are requested to define the categories in order to establish such a distinction: $\triangle D_{pq} \longrightarrow$ any(S_1), small(S_2), moderate(S_3), large(S_4). Equation (19) computes the differences between pairs of alternatives,

$$\triangle D_{sr} - \triangle D_{tm} = \sum_{j=1}^{n} w_j(d_{sj} - d_{rj} - d_{tj} - d_{uj}) + (\epsilon_{sr} - \epsilon_{tu}) \geq 0 \tag{19}$$

where $s, r \in S_k; t, u \in S_k; h = 1, 2, 3$ and $k \in \{h+1, 4\}$. The authors in [32] proposed a model for attribute weighting, which is given below:

$$\begin{aligned}
&minimize \sum_{q,p\in S} z_{qp} + \sum_{h=1;s,r\in S_k;u,t\in S_h;h<k\leq 4}^{3} v_{srtu}\ , \\
&\text{s.t: } \{\sum_{j=1}^{n} w_j(d_{qj} - d_{pj})\} + z_{qp} \geq 0, q, p \in \{S\} \qquad (20)\\
&\{\sum_{j=1}^{n} w_j(d_{sj} - d_{rj} - d_{tj} - d_{uj})\} + v_{rstu} \geq 0;\ s, r, t, u \in S_k;\ h = 1, 2, 3
\end{aligned}$$

where $k \in \{h+1, \ldots, 4\}$ and $w_j, z_{qp}, v_{srtu} \geq 0$. The first term of the objective function is aimed at minimizing the negative deviations related to the differences between the evaluations. In other words, the first term tries to minimize the differences between the evaluations of the alternatives whose sign does not match with the expected one and have the lowest possible value. The intuition behind the second term of the sum is analogous.

3.2.4 Advantages and disadvantages of methods in Group IV.2

Advantages

a) These methods are suitable when the subjective information provided by decision-makers is reliable.
   If the subjective information provided by decision-makers is accurate, these methods are the best options, as functional and subjective evaluations will match, thus leading to accurate weights.

Disadvantages

a) The models do not insufficiently represent decision-makers' preferences.
   Aiming at explaining this disadvantage, we will divide the analysis into three parts, one for each method discussed in Section 3.2.

Linear programming [31]. In this case, the objective function ensures that the weights match with the order given by decision-makers. However, maximizing the minimum possible difference between the evaluations might lead to contradictions. Although decision-makers provide their preferences, some alternatives might be equally appealing. In such cases, maximizing the minimum difference would not fully correspond with their preferences. The maximization model depicted below corrects this issue,

$$\begin{aligned} &maximize\ K = \sum_{i=1}^{m-1} k_i, \\ &\text{s.t: } k_i = \begin{cases} 1, \text{ if } V_i - V_{i+1} \geq 0 \\ 0, \quad \text{other cases} \end{cases}. \end{aligned} \tag{21}$$

The possible maximum for K is $m-1$ while the weight sets where this reaches its absolute maximum are those where $V_i - V_{i+1} \geq 0$, $\forall i = 1, 2, \ldots, m-1$, or equivalently when the generated alternatives sorted by their simple additive weighting values match with the decision-makers' preferences.

This model is a generalization of the one presented in [31] since any solution generated with in the first model is contained into the ones generated by our proposal. However, when difference among alternatives is small, equations of the type $\varepsilon \geq V_i - V_{i+1}$ can be added to the constraints, such that the weights are in concordance with decision-makers' preferences.

Fuzzy preference information on alternatives [20]. After multiplying $g_{ik} = p_{ik} - \bar{p}_{ik}$ by the factor $\sum_{j=1}^{n}(b_{ij} + b_{kj})w_j$ we obtain $h_{ij}(w) = p_{ij}\sum_{j=1}^{n}(b_{ij} + b_{kj})w_j - \sum_{j=1}^{n} b_{ij}w_j$. The values of $h_{ij}(w)$ are the intended variables to be minimized in the model's objective function. In this process, we might lose the compatibility with the decisions-makers' preferences. Individually, the terms to be minimized have the form $h_{ij}(w) = (p_{ik} - \bar{p}_{ik}) \sum_{j=1}^{n}(b_{ij} + b_{kj})w_j$. Hence, the model not just produces weights achieving higher correspondence between the evaluations and the fuzzy preference matrix, but also maximizes $\sum_{j=1}^{n}(b_{ij} + b_{kj})w_j$. It is also desired to minimize the evaluation of each pair of alternatives individually, which eventually minimizes all alternatives' evaluations. The reader can notice however that the term $\sum_{j=1}^{n}(b_{ij} + b_{kj})w_j$ can produce deviations from the desired set of weights. In practice, this can be achieved by minimizing the terms $g_{ik} = p_{ik} - \bar{p}_{ik}$.

Attribute weighting using preference comparisons [32]. Minimizing the negative deviations (z_{qp}) is a suitable criterion to establish a correspondence between the preferences given by the decision-maker (represented by S) and the evaluations given by their distances D_i to the ideal solution. However, this criterion does not allow correcting all possible incompatibilities. Let suppose that the decision-maker determines that the alternative O_1 is much more attractive than O_2 and O_3, while O_2 is slightly more attractive than O_3. Moreover, let us consider a set of weights that generates a value of D_1, such that $D_1 - D_2$ is slightly lower than 0. In this case, no constraint is violated since it is expected that $D_1 - D_2 < 0$, so the model does not try to make corrections due to the difference generated between D_1 and D_2. However, the decision-maker established that O_1 was much more attractive than O_2 and not slightly attractive. It would be expected a larger value for $D_2 - D_1$, so the fact that it has a small value could be regarded as an implicit constraint violation.

4 Hybrid weighting methods

The methods described in this section combine (to some extent) explicit and implicit attribute weighting approaches.

4.1 Description of methods in Group V.1

The number of hybrid weighting methods reported in the literature is small when compared with implicit and explicit approaches. These methods are gathered together within the same group.

4.1.1 Approach to integrate subjective preferences and objective information

This method uses a normalized decision matrix $Z = (z_{ij})_{m\times n}$ such that that the alternative's evaluation O_i is given by $d_i = \sum_{j=1}^{n} z_{ij}w_j$ [82]. This can be rewritten as $D = ZW$ with W being the weight vector. Aiming at deriving the weights, the implicit approach used an Analytic Hierarchy Process [68, 69, 55, 1] that employs a matrix A of paired comparisons between attributes $A = (a_{ij})_{n\times n}$. In this matrix, $a_{ij} = \frac{1}{a_{ji}}$ is the value of the Saaty's scale corresponding to the attribute A_i compared to A_j, and W is the auto-vector associated with its higher-value $A : AW = \lambda_{max}W$. In addition, when a matrix

A is perfectly consistent it is true that $AW = nW$. Then, decision-makers are requested for another type of information consisting on the fuzzy preferences relationships matrix among the alternatives $P = (p_{ij})_{m\times m}$, where $p_{ij} + p_{ji} = 1, p_{ii} = 0.5$ and $p_{ij} \geq 0\ \forall i, j = 1, 2, \ldots, m$. P is a subjective estimate of the matrix $\bar{P}$ whose elements $\bar{p}_{ij}$ satisfy $\bar{p}_{ij} = \frac{d_i}{d_i+d_j}$.

Let us assume the following identity system

$$\sum_{j=1,j\neq i}^{m} \frac{d_i}{d_i + d_j}(d_i + d_j) = (m-1)d_i,\ \forall i = 1, 2, \ldots, m. \tag{22}$$

Given that P is an accurate estimate of $\bar{P}$, we can substitute $\frac{d_i}{d_i+d_j}$ by p_{ij}. This leads to the following matrix:

$$B = \begin{pmatrix} \sum_{j=2}^{m} p_{1j} & p_{12} & \cdots & p_{1m} \\ p_{21} & \sum_{j=1,j\neq 2}^{m} p_{2j} & \cdots & p_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ p_{m1} & p_{m2} & \cdots & \sum_{j=1}^{m-1} p_{mj} \end{pmatrix}. \tag{23}$$

The system in (22) can rewritten as $BD = (m-1)D$. If we make $D = ZW$ in the previous expression, then we obtain $BZW = (m-1)ZW$. Both $AW = nW$ and $BZW = (m-1)ZW$ are achieved only in an ideal case. Therefore, it would be necessary to consider the next deviation vectors $E = AW - nW = [A - nI]W$ and $\mathit{\Gamma} = [BZ - (M-1)]W$. The components of E and $\mathit{\Gamma}$ are given by $E = (\varepsilon_1, \ldots, \varepsilon_n)^\top$ and $\mathit{\Gamma} = (\gamma_1, \ldots, \gamma_m)^\top$, respectively. The authors in [82] proposed the hybrid optimization model given below,

$$\begin{aligned} &\mathit{minimize}\ J = (\alpha \sum_{i=1}^{m} |\gamma_i|^p + \beta \sum_{j=1}^{n} |\varepsilon_j|^p)^{1/p}, \\ &\text{s.t: } [BZ - (m-1)]W - \mathit{\Gamma} = 0;\ (A - nI)W - E = 0. \end{aligned} \tag{24}$$

Wang and Parkan [82] proposed alternative models based on the same idea (interesting readers are referred to [82] for more details).

4.1.2 Integrated weighting method of attributes "1"

The integrated weighting method of attributes "1" starts with the decision matrix X_k of each decision-maker e_k, such that the evaluation of each attribute is given by x_{ij}^k, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots N$, $k = 1, 2, \ldots, K$. Moreover, the x_{ij}^k values are expressed as linguistic variables [64, 74, 47, 75, 42, 44, 43] and later on are replaced by their normalized values r_{ij}^k (see details in [44]). The weight of each attribute A_i is given by $w_j^o = V_j / \sum_{i=1}^{n} V_i$, where V_i is the variance among all evaluations of that attribute.

Each decision-maker e_k is requested to provide the attributes' weights as linguistic variables. This information is used to calculate the aggregate weight of attribute A_j as $w_j^s = w_{j1}^s \oplus w_{j2}^s \oplus, \ldots, \oplus w_{jK}^s$, where $\oplus$ is a definite sum for linguistic terms and w^s_{jk} the weight corresponding to e_k. Finally, subjective weights are calculated by normalizing the values w^s_{jk} with the expression $W^s_j = w^s_j / \sum_{i=1}^n w^s_i$ The definitive attributes' weights are computed as $w_j = uw^o_j + vw^s_j$, where u and v denote the weights attributed to the objective and subjective approaches, respectively.

4.1.3 Integrated weighting method of attributes "2"

The integrated weighting method of attributes "2" is very similar to the previous one. It starts considering the evaluations for each attribute (x_{ij}) and their normalized values (x^*_{ij}) [64]. The variance of the evaluations for each attribute A_j is calculated as $V_j = \frac{1}{n}\sum_{i=1}^m (x^*_{ij} - (x^*_{ij})_{average})^2$. The objective weighting of each attribute A_j is given by $w^o_j = \frac{V_j}{\sum_{i=1}^n V_i}$.

In [64] the authors proposed a subjective weighting approach similar to the Analytic Hierarchy Process (see [64] for details). It includes the subjective weights w^s_j and the weighting model $w^i_j = W^o w^o_j + W^s w^s_j$, where W^o and W^s are the weights attributed to objective and subjective components, respectively, while $W^0, W^S \in [0, 1]$.

4.1.4 Disadvantages and advantages of methods in Group V.1

Advantages

a) Hybrid approaches are a mathematical generalization of the implicit and explicit approaches in its definition.
   When the coefficient associated with an approximation (implicit or explicit) approaches 1 it means that the results generated by the hybrid method will be similar to the ones obtained with the SIMPLE method. This means that if one method is appropriate for a situation (a) while the other method is appropriate for a situation (b), then the hybrid approach that includes them may be suitable for both situations.

Disadvantages

a) Uncertainty in determining adequate relaxation coefficients.
   The weighting process of a hybrid model (implicit or explicit) is very subjective. Overall, the criteria used to guide decision-makers in this process can vary and are often based on intuition.
b) Lack of interpretability.
   Sometimes, implicit and explicit methods lead to entirely different results. The intuition behind these approaches differs significantly from a method to another, so there is no guarantee of compatibility among them.

5 Summary of the main limitations of the revised methods

In this section, we summarize the main drawbacks of the analyzed methods and suggest guidelines to overcome them.

- **Issue 1**. The concept of weight is rather intuitive.
  Observation. Both the calculation procedures and the information requested from decision-makers can take various nuances. This is often a source of inconsistency, thus leading to different results.
  Suggestion. The concept of weight should be mathematically formalized. Thereby, the decision on which methodology to follow will be subject to the concept of weight itself.
- **Issue 2**. It is often assumed that attribute weights can be determined from the dispersion of alternatives' evaluations.
  Observation. There are a few methods where the weights depend on the dispersion of the evaluations. This brings up two main drawbacks. Firstly, the fact that some alternatives are added or removed in the decision process affects the importance of the attributes. Second, the interest of the decision-maker is marginalized. Note that beyond the evaluations of each attribute, the particular attention of each decision-maker may vary. Therefore this should be taken into account when determining the weights.
  Suggestion. Avoid using criteria associated with the dispersion of evaluations when developing new weighting methods.
- **Issue 3**. The preference information of decision-makers regarding the relevance of attributes is often neglected.
  Observations. This limitation was mentioned as part of the previous issue. However, it is important to mention it in isolation, as it is a problem observed in most weighting methods. In practice, these methods are usually adopted when this information is not available.
  Suggestion. The formats used for the decision-makers to provide their preferences must be as flexible as possible and directly associated with the remaining pieces of information.
- **Issue 4**. There is a lack of objectivity in several objective functions attached to weighting optimization models.
  Observation. Such models aim to optimize objective functions that insufficiently represent reality. They are based on abstract concepts that describe poorly the components of the decision-making problem.
  Suggestion. The objective functions must fulfill mathematical relationships ensuring the compatibility with the information obtained from decision-makers. Additionally, the objective functions should be as general as possible, otherwise, we might overlook relevant solutions when trying to satisfy specific relationships or constraints.

Table 10 shows the issues found in the 20 methods examined in this paper. Issue 1 is observed in all methods. Due to the absence of a formal definition of weight, each researcher provides an interpretation according to the needs of the problem they face.

In general, Class III concentrates methods with a higher number of issues. This class gathers implicit weighting methods without additional information. At the other extreme are Class V methods that combine explicit and implicit (hybrid) attribute weighting approaches.

Table 10 Issues found in each method revised in this study.

No.	Method	Issues			
		1	2	3	4
1	The entropy method [14,67,73,89]	✓	✓	✓	
2	The standard deviation method [14,16]	✓	✓	✓	
3	Importance of criteria through inter-criteria correlation [53,33]	✓	✓	✓	
4	Correlation coefficient and standard deviation [81]	✓	✓	✓	
5	Subjective and objective integrated approach [46]	✓	✓	✓	
6	Attribute weighting method by incompatibility among attributes [11]	✓	✓	✓	✓
7	Objective method based on IFS entropy [10]	✓		✓	
8	Continuous entropy method based on IVIFS with unknown weights information [35]	✓		✓	
9	Multi-objective programming model that takes one objective without weight information [79]	✓		✓	✓
10	Rational model with dissonance minimization based on correlation measures [56]	✓	✓		
11	TOPSIS method based on single-valued neutrosophic sets [4]	✓			
12	Swing weighting [19]	✓			
13	The TRADE-OFFS method [36]	✓			
14	MacBeth's weighting method [2]	✓			
15	Linear programming for attributes' weighting on the performance evaluation process [31]	✓			✓
16	Multiple attribute decision making based on fuzzy preference information on alternatives [20]	✓			✓
17	Attribute weighting using preference comparisons [32]	✓			
18	Approach to integrate subjective preferences and objective information [82]	✓			
19	Integrated weighting method of attributes "1" [44]	✓			
20	Integrated weighting method of attributes "2" [64]	✓			

6 Conclusions

In this paper, we have presented a review of implicit and hybrid methods for weighting attributes in the context of multiple attribute decision-making problems, which provides closure to the study presented in [57]. Perhaps the most important finding of our study is that most methods do not align well with the subjective information of decision-makers on their preferences. Although there are cases where this information is limited, it is necessary to establish alternative ways to extract decision-makers' expertise. Without this information, no weighting method could ensure producing coherent results. In that regard, the weighting methods in Class IV (implicit weighting methods with additional information) were the ones potentially prone to give coherent solutions. Therefore, we encourage the community to further improve the implicit weighting methods with additional information.

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