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LOW-MACH CONSISTENCY OF A CLASS OF LINEARLY IMPLICIT SCHEMES FOR THE COMPRESSIBLE EULER EQUATIONS

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Abstract: In this note, we give an overview of the authors' paper [6] which deals with asymptotic consistency of a class of linearly implicit schemes for the compressible Euler equations. This class is based on a linearization of the nonlinear fluxes at a reference state and includes the scheme of Feistauer and Kučera [3] as well as the class of RS-IMEX schemes [8, 5, 1] as special cases. We prove that the linearization gives an asymptotically consistent solution in the low-Mach limit under the assumption of a discrete Hilbert expansion. The existence of the Hilbert expansion is shown under simplifying assumptions.

Keywords: asymptotic preserving schemes, compressible Euler equations, low-Mach limit, Hilbert expansion

MSC: 76N10, 76M45, 76B03, 65M12

1. Introduction

We consider the compressible Euler equations of gas dynamics written in the form of a first order nonlinear hyperbolic system of partial differential equations

$$\partial_t \boldsymbol{w} + \nabla\cdot \boldsymbol{f}(\boldsymbol{w}) = 0, \tag{1}$$

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where the non-dimensional state and flux vectors are defined as

$$\boldsymbol{w} := \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix}, \qquad \boldsymbol{f} := \begin{pmatrix} \rho \boldsymbol{u} \\ \rho \boldsymbol{u} \otimes \boldsymbol{u} + \frac{p}{\varepsilon^2} Id \\ \boldsymbol{u}(E+p) \end{pmatrix}. \tag{2}$$

By ε we denote a reference Mach number. Here $\boldsymbol{u}$ is the velocity vector $\boldsymbol{u} := (u, v)$; the equations come along with the dimensionless equation of state:

$$E = \frac{p}{\gamma - 1} + \frac{\varepsilon^2}{2} \rho |\boldsymbol{u}|^2. \tag{3}$$

It is known that for $\varepsilon \to 0$, the solution $\boldsymbol{w}$ converges towards the solution of the incompressible equations if initial and boundary data are so-called *well-prepared*, see Def. 2 and also [7]. We note that the non-dimensional form in (2) and (3) differs from the standard non-dimensional form by the factors ε^2. This results from a different choice of the reference variables and is more suitable for the analysis of the low-Mach case. For ease of presentation we will deal with the 2D case.

The system (1) is hyperbolic in the sense that for any unit vector $\boldsymbol{n} \in \mathbb{R}^2$ and any $\boldsymbol{w} \in \mathbb{R}^4$ such that the corresponding density and pressure are positive, the Jacobian matrix $\boldsymbol{f}'(\boldsymbol{w}) \cdot \boldsymbol{n}$ is real diagonalizable with eigenvalues $\lambda_1(\boldsymbol{w}, \boldsymbol{n}), \ldots, \lambda_4(\boldsymbol{w}, \boldsymbol{n})$. It can be shown that the minimal and maximal eigenvalues are $\mathcal{O}(\varepsilon^0)$ and $\mathcal{O}(\varepsilon^{-1})$, respectively, as $\varepsilon \to 0$. This stiff behavior leads to problems with the time discretization, e.g. for explicit schemes the CFL condition imposes a small time step of order $\mathcal{O}(\varepsilon \Delta x)$, which becomes infeasible for very small ε. Fully implicit schemes, on the other hand, necessitate solving large systems of nonlinear equations, whose condition number deteriorates as $\varepsilon \to 0$. Our focus here is on IMEX (implicit-explicit) schemes, which attempt to split the system into a fast part (treated implicitly) and a slow part (treated explicitly).

Our goal is to compare asymptotic properties, as $\varepsilon \to 0$ of the scheme [3], which we call Dolejší-Feistauer-Kučera in this work, with the RS-IMEX scheme presented in [9]. Although different in type, numerically, both schemes perform very well in the $\varepsilon \to 0$ limit. For the RS-IMEX scheme, a formal *asymptotic consistency* analysis has been given in [4]; no such analysis has been presented for the Dolejší-Feistauer-Kučera scheme. To this end, we unify the two schemes by constructing a generalized framework consisting of a class of linearly implicit schemes based on a reference state. The goal is to prove that the numerical schemes from this class give the correct solution as $\varepsilon \to 0$, converging to the incompressible limit.

2. Linearly implicit schemes based on a reference state

First, we formulate a unified framework containing both the Dolejší-Feistauer-Kučera and the RS-IMEX scheme. The scheme consists of a first order difference approximation of the time derivative in (1) along with a linearization of the nonlinear fluxes using a reference state. Let $\Omega \subset \mathbb{R}^2$ be a a bounded domain with Lipschitz boundary. We define the time partition $0 = t_0 < t_1 < \ldots$ and choose a reference state $\boldsymbol{w}_R^n$ for each $n = 0, 1, \ldots$.

Definition 1 (Linearly implicit scheme based on a reference state)**.** *We seek* $\boldsymbol{w}^n : \Omega \to \mathbb{R}^4$ *for* $n = 0, 1, \ldots$ *satisfying*

$$\frac{\boldsymbol{w}^{n+1} - \boldsymbol{w}^n}{\Delta t} = -\nabla\cdot \Big(\boldsymbol{f}(\boldsymbol{w}^n) + \boldsymbol{f}'(\overline{\boldsymbol{w}}_R^n)(\boldsymbol{w}^{n+1} - \boldsymbol{w}^n) \Big). \tag{4}$$

Taking the reference state to be the discretization at time level n, i.e., $\overline{\boldsymbol{w}}_R^n = \boldsymbol{w}^n$, then (4) reduces to

$$\frac{\boldsymbol{w}^{n+1} - \boldsymbol{w}^n}{\Delta t} = -\nabla\cdot \Big(\boldsymbol{f}'(\boldsymbol{w}^n)\boldsymbol{w}^{n+1} \Big), \tag{5}$$

since the Euler flux is homogeneous of degree one, i.e. $\boldsymbol{f}(\boldsymbol{w}) = \boldsymbol{f}'(\boldsymbol{w})\boldsymbol{w}$. This is the basis of the Dolejší-Feistauer-Kučera scheme, proposed in [2, 3].

On the other hand, the scheme (4) can be rewritten in the RS-IMEX form

$$\frac{\boldsymbol{w}^{n+1} - \boldsymbol{w}^n}{\Delta t} = -\nabla\cdot \Big(\widetilde{\boldsymbol{f}}(\boldsymbol{w}^{n+1}; \boldsymbol{w}_R^{n+1}) + \widehat{\boldsymbol{f}}(\boldsymbol{w}^n; \boldsymbol{w}_R^n) \Big), \tag{6}$$

where the stiff and non-stiff fluxes are defined as

$$\widetilde{\boldsymbol{f}}(\boldsymbol{w}; \boldsymbol{w}_R) := \boldsymbol{f}(\boldsymbol{w}_R) + \boldsymbol{f}'(\boldsymbol{w}_R)(\boldsymbol{w} - \boldsymbol{w}_R), \quad \widehat{\boldsymbol{f}}(\boldsymbol{w}; \boldsymbol{w}_R) \; := \boldsymbol{f}(\boldsymbol{w}) - \widetilde{\boldsymbol{f}}(\boldsymbol{w}, \boldsymbol{w}_R). \tag{7}$$

This is the basis of the scheme used in [9], where $\boldsymbol{w}_R$ is taken as the solution of the incompressible Euler equations. The motivation is that the Jacobian matrix $\widetilde{\boldsymbol{f}}'$ contains all eigenvalues of order ε^{-1}, hence $\widetilde{\boldsymbol{f}}$ is discretized implicitly. The Jacobian $\widehat{\boldsymbol{f}}'$ contains eigenvalues of order ε^0, and $\widehat{f}$ will hence be discretized explicitly.

3. Formal asymptotic expansion

In this section, we analyze the asymptotic preserving properties of the scheme (4) for the Euler equations. Due to lack of space, the analysis cannot be performed in its entirety, cf. [6] for details. The purpose here is to demonstrate the basic principles of performing such an analysis.

Let $\boldsymbol{w} := (w_1, w_2, w_3, w_4)^\top$, then we can write the Euler fluxes in terms of $\boldsymbol{w}$ as

$$\boldsymbol{f}_1(\boldsymbol{w}) = \begin{pmatrix} w_2 \\ \frac{3-\gamma}{2}\frac{w_2^2}{w_1} + \frac{1-\gamma}{2}\frac{w_3^2}{w_1} + \frac{\gamma-1}{\varepsilon^2}w_4 \\ \frac{w_2 w_3}{w_1} \\ \frac{\gamma w_2 w_4}{w_1} - \frac{\varepsilon^2(\gamma-1)}{2}\frac{w_2^3 + w_2 w_3^2}{w_1^2} \end{pmatrix}, \; \boldsymbol{f}_2(\boldsymbol{w}) = \begin{pmatrix} w_3 \\ \frac{w_2 w_3}{w_1} \\ \frac{1-\gamma}{2}\frac{w_2^2}{w_1} + \frac{3-\gamma}{2}\frac{w_3^2}{w_1} + \frac{\gamma-1}{\varepsilon^2}w_4 \\ \frac{\gamma w_3 w_4}{w_1} - \frac{\varepsilon^2(\gamma-1)}{2}\frac{w_2^2 w_3 + w_3^3}{w_1^2} \end{pmatrix}. \tag{8}$$

Using this notation, (1) is simply

$$\partial_t \boldsymbol{w} + \partial_x \boldsymbol{f}_1(\boldsymbol{w}) + \partial_y \boldsymbol{f}_2(\boldsymbol{w}) = 0. \tag{9}$$

The Jacobi matrices of $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ with respect to $\boldsymbol{w}$ (written in terms of the physical variables density ρ, momentum $\rho\boldsymbol{u}$ and energy E) are given by

$$\boldsymbol{f}_1'(\boldsymbol{w}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{\gamma-3}{2}u^2 + \frac{\gamma-1}{2}v^2 & (3-\gamma)u & (1-\gamma)v & \frac{\gamma-1}{\varepsilon^2} \\ -uv & v & u & 0 \\ -\frac{\gamma E u}{\rho} + \varepsilon^2(\gamma-1)u(u^2+v^2), & \frac{\gamma E}{\rho} - \varepsilon^2\frac{\gamma-1}{2}(3u^2+v^2), & \varepsilon^2(1-\gamma)uv, & \gamma u \end{pmatrix}, \tag{10}$$

$$\boldsymbol{f}_2'(\boldsymbol{w}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -uv & v & u & 0 \\ \frac{\gamma-1}{2}u^2 + \frac{\gamma-3}{2}v^2 & (1-\gamma)u & (3-\gamma)v & \frac{\gamma-1}{\varepsilon^2} \\ -\frac{\gamma E v}{\rho} + \varepsilon^2(\gamma-1)v(u^2+v^2), & \varepsilon^2(1-\gamma)uv, & \frac{\gamma E}{\rho} - \varepsilon^2\frac{\gamma-1}{2}(u^2+3v^2), & \gamma v \end{pmatrix}. \tag{11}$$

We assume that the physical quantities $\rho, \boldsymbol{u}, E$ and p on each time level have a formal Hilbert expansion w.r.t. ε of the form (written e.g. for ρ^n)

$$\rho^n(x) = \rho^n_{(0)}(x) + \varepsilon\rho^n_{(1)}(x) + \varepsilon^2\rho^n_{(2)}(x) + O(\varepsilon^3), \tag{12}$$

similarly, this is assumed for the reference state $\boldsymbol{w}_R$. The existence of the Hilbert expansion is proved under simplifying assumptions in [6].

Substituting the Hilbert expansions into (10) and (11) and gathering terms according to powers of ε gives the expansion

$$\boldsymbol{f}_s'(\boldsymbol{w}) = \varepsilon^{-2}\boldsymbol{f}'_{s,(-2)}(\boldsymbol{w}) + \varepsilon^{-1}\boldsymbol{f}'_{s,(-1)}(\boldsymbol{w}) + \varepsilon^0\boldsymbol{f}'_{s,(0)}(\boldsymbol{w}) + O(\varepsilon), \tag{13}$$

for $s = 1, 2$, where

$$\boldsymbol{f}'_{1,(-2)}(\boldsymbol{w}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\gamma-1) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{f}'_{2,(-2)}(\boldsymbol{w}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\gamma-1) \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{14}$$

and $\boldsymbol{f}'_{s,(-1)}(\boldsymbol{w}) = 0$ for $s = 1, 2$. Finally, since

$$\frac{1}{\rho} = \frac{1}{\rho_{(0)}} - \frac{\rho_{(1)}}{\rho^2_{(0)}}\varepsilon + O(\varepsilon^2) \tag{15}$$

due to Taylor expansion, we have

$$\boldsymbol{f}'_{1,(0)}(\boldsymbol{w}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{\gamma-3}{2}u^2_{(0)} + \frac{\gamma-1}{2}v^2_{(0)}, & (3-\gamma)u_{(0)}, & (1-\gamma)v_{(0)}, & 0 \\ -u_{(0)}v_{(0)} & v_{(0)} & u_{(0)} & 0 \\ -\frac{\gamma E_{(0)}u_{(0)}}{\rho_{(0)}} & \frac{\gamma E_{(0)}}{\rho_{(0)}} & 0 & \gamma u_{(0)} \end{pmatrix}, \tag{16}$$

$$\boldsymbol{f}'_{2,(0)}(\boldsymbol{w}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -u_{(0)}v_{(0)} & v_{(0)} & u_{(0)} & 0 \\ \frac{\gamma-1}{2}u_{(0)}^2 + \frac{\gamma-3}{2}v_{(0)}^2, & (1-\gamma)u_{(0)}, & (3-\gamma)v_{(0)}, & 0 \\ -\frac{\gamma E_{(0)}v_{(0)}}{\rho_{(0)}} & 0 & \frac{\gamma E_{(0)}}{\rho_{(0)}} & \gamma v_{(0)} \end{pmatrix}. \tag{17}$$

4. Asymptotic consistency analysis

Taking all the expansions (12)–(17) and substituting into the linearized problem (4), we gather terms according to the powers of ε. One can then proceed to derive properties of the individual terms in the expansion of the unknowns. From the ε^{-2} and ε^{-1} terms, we get the following lemma.

Lemma 1. *The functions $E^n_{(0)}$, $E^n_{(1)}$, $p^n_{(0)}$ and $p^n_{(1)}$ are constant in space for every n.*

Proof. By gathering the terms of order ε^{-2} and ε^{-1} from (4), we obtain

$$\nabla\big(p^n_{(0)} + (\gamma-1)(E^{n+1}_{(0)} - E^n_{(0)})\big) = 0, \tag{18}$$

$$\nabla\big(p^n_{(1)} + (\gamma-1)(E^{n+1}_{(1)} - E^n_{(1)})\big) = 0. \tag{19}$$

Taking the ε^0 and ε^1 terms from the equation of state (3) at time level n gives

$$E^n_{(0)} = \frac{p^n_{(0)}}{\gamma-1}, \quad E^n_{(1)} = \frac{p^n_{(1)}}{\gamma-1}. \tag{20}$$

Substituting into (18) and (19) gives $\nabla E^{n+1}_{(0)} = \nabla E^{n+1}_{(1)} = 0$, hence $E^{n+1}_{(0)}$ and $E^{n+1}_{(1)}$ are constant in space for every n. Equation (20) implies the same for $p^{n+1}_{(0)}$ and $p^{n+1}_{(1)}$. □

Lemma 2. *Assuming either slip boundary conditions for $\boldsymbol{u}_R$ and $\boldsymbol{u}^n$ for all n or periodic boundary conditions, the functions $E^n_{(0)}$ and $p^n_{(0)}$ are constant in space and independent of n.*

Proof. Collecting the ε^0 terms of the energy equation, we get

$$\frac{E^{n+1}_{(0)} - E^n_{(0)}}{\Delta t} + \nabla\cdot\Bigg(\big(E^n_{(0)} + p^n_{(0)}\big)\boldsymbol{u}^n_{(0)} - \gamma\frac{E_{R,(0)}\boldsymbol{u}_{R,(0)}}{\rho_{R,(0)}}(\rho^{n+1}_{(0)} - \rho^n_{(0)}) \\ + \gamma\frac{E_{R,(0)}}{\rho_{R,(0)}}(\rho^{n+1}_{(0)}\boldsymbol{u}^{n+1}_{(0)} - \rho^n_{(0)}\boldsymbol{u}^n_{(0)}) + \gamma\boldsymbol{u}_{R,(0)}(E^{n+1}_{(0)} - E^n_{(0)})\Bigg) = 0. \tag{21}$$

We integrate (21) over Ω and apply Green's theorem. Since $E^n_{(0)}$ and $E^{n+1}_{(0)}$ are constant by Lemma 1, we get

$$|\Omega|\frac{E^{n+1}_{(0)} - E^n_{(0)}}{\Delta t} + \int_{\partial\Omega} \mathcal{E}\cdot\boldsymbol{n}\,\mathrm{d}\sigma = 0, \tag{22}$$

where $\mathcal{E}$ corresponds to the terms under the divergence symbol in (21). Since each of these terms contains either $\boldsymbol{u}_{R,(0)}, \boldsymbol{u}_{(0)}^n$ or $\boldsymbol{u}_{(0)}^{n+1}$, all of which have zero normal component on $\partial\Omega$, the whole boundary integral in (22) vanishes. This is the case of slip-boundary conditions, for periodic boundary conditions, the boundary integral vanishes due to spatial periodicity of all the terms. Altogether, (22) then implies $E_{(0)}^{n+1} = E_{(0)}^n$ and (20) implies $p_{(0)}^{n+1} = p_{(0)}^n$.

$\square$

We wish to prove that the zero order variables from the Hilbert expansion satisfy the incompressible Euler equations. First, we prove the incompressibility.

Lemma 3. *Assume either slip boundary conditions for $\boldsymbol{u}_R$ and $\boldsymbol{u}^n$ for all n or periodic boundary conditions. Let $\rho_{(0)}^n$ and $\rho_{R,(0)}$ be constant in space and let $\nabla\cdot \boldsymbol{u}_{(0)}^n = \nabla\cdot \boldsymbol{u}_{R,(0)} = 0$. Then $\rho_{(0)}^{n+1} = \rho_{(0)}^n$, i.e., $\rho_{(0)}^{n+1}$ is also constant in space, and $\nabla\cdot \boldsymbol{u}_{(0)}^{n+1} = 0$.*

Proof. Collecting the ε^0 terms of the mass equation from (4) gives:

$$\frac{\rho_{(0)}^{n+1} - \rho_{(0)}^n}{\Delta t} + \nabla\cdot (\rho_{(0)}^{n+1}\boldsymbol{u}_{(0)}^{n+1}) = 0. \tag{23}$$

Next, we can simplify the energy equation (21) using Lemma 2 and the assumptions $\nabla\rho_{(0)}^n = 0$ and $\nabla\cdot \boldsymbol{u}_{(0)}^n = \nabla\cdot \boldsymbol{u}_{R,(0)} = 0$ to obtain

$$-\boldsymbol{u}_{R,(0)}\cdot \nabla(\rho_{(0)}^{n+1} - \rho_{(0)}^n) + \nabla\cdot (\rho_{(0)}^{n+1}\boldsymbol{u}_{(0)}^{n+1}) = 0. \tag{24}$$

Substituting this equality into the mass equation (23) gives us

$$\frac{\rho_{(0)}^{n+1} - \rho_{(0)}^n}{\Delta t} + \boldsymbol{u}_{R,(0)}\cdot \nabla\big(\rho_{(0)}^{n+1} - \rho_{(0)}^n\big) = 0. \tag{25}$$

Denoting for simplicity $\varrho := \rho_{(0)}^{n+1} - \rho_{(0)}^n$, we write (25) as

$$\tfrac{1}{\Delta t}\varrho + \boldsymbol{u}_{R,(0)}\cdot \nabla\varrho = 0. \tag{26}$$

We wish to prove that $\varrho = 0$, i.e., that $\rho_{(0)}^{n+1} = \rho_{(0)}^n$. To this end, we multiply (26) by ϱ and integrate over Ω:

$$\frac{1}{\Delta t}\int_\Omega \varrho^2 \,\mathrm{d}x + \int_\Omega \boldsymbol{u}_{R,(0)}\cdot \nabla\varrho\,\varrho \,\mathrm{d}x = 0. \tag{27}$$

We apply Green's theorem to the second integral to obtain

$$\int_\Omega \boldsymbol{u}_{R,(0)}\cdot \nabla\varrho\,\varrho\,\mathrm{d}x = \underbrace{\int_{\partial\Omega} \boldsymbol{u}_{R,(0)}\cdot \boldsymbol{n}\varrho^2 \,\mathrm{d}\sigma}_{=0} - \underbrace{\int_\Omega \nabla\cdot \boldsymbol{u}_{R,(0)}\varrho^2 \,\mathrm{d}x}_{=0} - \int_\Omega \boldsymbol{u}_{R,(0)}\cdot \nabla\varrho\,\varrho\,\mathrm{d}x, \tag{28}$$

where the first and second right-hand side terms are zero due to the boundary conditions and the divergence-free assumption on $\boldsymbol{u}_{R,(0)}$, respectively, while the last term equals the left-hand side. Therefore, (28) gives us $\int_\Omega \boldsymbol{u}_{R,(0)} \cdot \nabla \varrho \, \varrho \, \mathrm{d}x = 0$, which together with (27) implies

$$\frac{1}{\Delta t} \int_\Omega \varrho^2 \, \mathrm{d}x = 0 \quad \Longrightarrow \quad \varrho = 0 \text{ a.e. in } \Omega \quad \Longrightarrow \quad \rho_{(0)}^{n+1} = \rho_{(0)}^n. \tag{29}$$

This gives the first statement of the Lemma. Since we now know that $\nabla \rho_{(0)}^{n+1} = \nabla \rho_{(0)}^n = 0$, equation (24) simplifies to $\nabla \cdot \boldsymbol{u}_{(0)}^{n+1} = 0$, which completes the proof. $\square$

The main result of the analysis is that the lowest order terms in the Hilbert expansion satisfy the incompressible Euler equations, implicitly discretized in time.

Theorem 4. *Let the initial condition satisfy $\nabla \cdot \boldsymbol{u}_{(0)}^0 = 0$ and $\rho_{(0)}^0$ being constant in space. Let the reference solution satisfy $\nabla \cdot \boldsymbol{u}_{R,(0)}^n = 0$ and $\rho_{R,(0)}^n$ being constant in space for all n. Assume either slip boundary conditions for $\boldsymbol{u}_R^n$ and $\boldsymbol{u}^n$ for all n or periodic boundary conditions. Then for each n, the pair $\left(\boldsymbol{u}_{(0)}^{n+1}, p_{(2)}^{n+1}/\rho_{(0)}^{n+1}\right)$ solves the implicit semi-discrete incompressible Euler equations*

$$\begin{aligned} \frac{\boldsymbol{u}_{(0)}^{n+1} - \boldsymbol{u}_{(0)}^n}{\Delta t} + \nabla \cdot \left(\boldsymbol{u}_{(0)}^{n+1} \otimes \boldsymbol{u}_{(0)}^{n+1}\right) + \nabla \frac{p_{(2)}^{n+1}}{\rho_{(0)}^{n+1}} &= \mathcal{E}^{n+1}, \\ \nabla \cdot \boldsymbol{u}_{(0)}^{n+1} &= 0, \end{aligned} \tag{30}$$

where $\mathcal{E}^{n+1}$ is a consistency error term satisfying

$$|\mathcal{E}^{n+1}| \le C \|\boldsymbol{u}_{(0)}^{n+1} - \boldsymbol{u}_{(0)}^n\|_{W^{1,\infty}} \left(\|\boldsymbol{u}_{(0)}^{n+1} - \boldsymbol{u}_{(0)}^n\|_{W^{1,\infty}} + \|\boldsymbol{u}_{(0)}^n - \boldsymbol{u}_{R,(0)}^n\|_{W^{1,\infty}} \right), \tag{31}$$

where C depends only on γ.

Proof. We have proved that $u_{(0)}^{n+1}$ is divergence-free in Lemma 3. The proof of the first equality in (30) is lengthy and thus we shall omit it. Nevertheless, the proof follows the same principles as demonstrated in the proofs of the previous lemmas. One collects the ε^0 terms form the x and y momentum equations from (4). These equations are rather lengthy. One then proceeds to simplify these equations using results from Lemmas 1 – 3 and the $O(\varepsilon^2)$ terms from the equation of state (3). The resulting equations can then be rearranged to the from (30) with right-hand side terms that can be rather straightforwardly estimated as in (31). $\square$

If we denote $\delta^n := \|\boldsymbol{u}_{(0)}^n - \boldsymbol{u}_{R,(0)}^n\|_{W^{1,\infty}}$, the consistency error (31) is of the order

$$|\mathcal{E}^{n+1}| \le C \Delta t (\Delta t + \delta^n). \tag{32}$$

The Dolejší-Feistauer-Kučera scheme is based on the choice $\boldsymbol{u}_{R,(0)}^n = \boldsymbol{u}_{(0)}^n$, hence $\delta^n = 0$ and the consistency error satisfies

$$\mathcal{E}^{n+1} = O(\Delta t^2). \tag{33}$$

For the RS-IMEX scheme we take $\boldsymbol{u}^n_{R,(0)} = \boldsymbol{u}_{\mathrm{ref}}(t_n)$, hence $\delta^n = O(\Delta t)$ and again $\mathcal{E}^{n+1} = O(\Delta t^2)$. We note that in both cases the consistency error is of the second order which is one order higher than the error of approximating the time derivative in (30). We call this property *superconsistency* of the flux approximation. This phenomenon might explain the excellent performance of the Dolejší-Feistauer-Kučera scheme for computing steady state solutions, where the time derivative is close to zero and the consistency error is of second order due to (33).

4.1. Well prepared initial data

An important property of scheme (4) is that it does not generate acoustics if they are not present in the initial conditions (so-called *well-prepared* initial conditions). Since acoustics are $O(\varepsilon)$ perturbations of density, pressure and divergence of velocity, this assumption amounts to having only $O(\varepsilon^2)$ perturbations in these quantities.

Definition 2. *We say that the initial data $\rho^0, p^0, \boldsymbol{u}^0$ are* well prepared *if*

$$\rho^0 = \text{const} + O(\varepsilon^2), \qquad p^0 = \text{const} + O(\varepsilon^2), \qquad \nabla\cdot \boldsymbol{u}^0 = O(\varepsilon^2). \tag{34}$$

If the mentioned quantities possess Hilbert expansions, Definition 2 amounts to $\rho^0_{(1)} = p^0_{(1)} = \nabla\cdot \boldsymbol{u}^0_{(1)} = 0$. Now we prove that if the initial data are well prepared then also $\rho^n = \text{const} + \mathrm{O}(\varepsilon^2)$, $p^n = \text{const} + \mathrm{O}(\varepsilon^2)$ and $\nabla\cdot \boldsymbol{u}^n = O(\varepsilon^2)$ for all n.

Theorem 5. *Let the assumptions of Theorem 4 hold. Assume also that the initial data are well prepared in the sense of Definition 2 and that $\rho^n_{R,(1)} = 0$ for all n. Then $\rho^n_{(1)} = p^n_{(1)} = \nabla\cdot \boldsymbol{u}^n_{(1)} = 0$ for all n.*

Proof. We collect the ε^1 terms of the mass equation from scheme (4):

$$\frac{\rho^{n+1}_{(1)} - \rho^n_{(1)}}{\Delta t} + \nabla\cdot \left(\rho^{n+1}_{(0)} \boldsymbol{u}^{n+1}_{(1)} + \rho^{n+1}_{(1)} \boldsymbol{u}^{n+1}_{(0)}\right) = 0. \tag{35}$$

Similarly, we collect the ε^1 terms of the energy equation from (4), along with (15):

$$\begin{aligned}
\frac{E^{n+1}_{(1)} - E^n_{(1)}}{\Delta t} + \nabla\cdot \Bigg(&\big(E^n_{(0)} + p^n_{(0)}\big)\boldsymbol{u}^n_{(1)} + \big(E^n_{(1)} + p^n_{(1)}\big)\boldsymbol{u}^n_{(0)} - \gamma \frac{E_{R,(0)}\boldsymbol{u}_{R,(0)}}{\rho_{R,(0)}}(\rho^{n+1}_{(1)} - \rho^n_{(1)}) \\
&- \gamma\Big(\frac{E_{R,(0)}\boldsymbol{u}_{R,(1)} + E_{R,(1)}\boldsymbol{u}_{R,(0)}}{\rho_{R,(0)}} - \frac{E_{R,(0)}\boldsymbol{u}_{R,(0)}(\rho_{R,(1)})^2}{\rho_{R,(0)}}\Big)(\rho^{n+1}_{(0)} - \rho^n_{(0)}) \\
&+ \gamma \frac{E_{R,(0)}}{\rho_{R,(0)}}\big(\rho^{n+1}_{(0)}\boldsymbol{u}^{n+1}_{(1)} + \rho^{n+1}_{(1)}\boldsymbol{u}^{n+1}_{(0)} - \rho^n_{(0)}\boldsymbol{u}^n_{(1)} - \rho^n_{(1)}\boldsymbol{u}^n_{(0)}\big) \\
&+ \gamma\Big(\frac{E_{R,(1)}}{\rho_{R,(0)}} - \frac{E_{R,(0)}\rho_{R,(1)}}{(\rho_{R,(0)})^2}\Big)\big(\rho^{n+1}_{(0)}\boldsymbol{u}^{n+1}_{(0)} - \rho^n_{(0)}\boldsymbol{u}^n_{(0)}\big) \\
&+ \gamma \boldsymbol{u}_{R,(0)}(E^{n+1}_{(1)} - E^n_{(1)}) + \gamma \boldsymbol{u}_{R,(1)}(E^{n+1}_{(0)} - E^n_{(0)})\Bigg) = 0.
\end{aligned} \tag{36}$$

Now we proceed similarly as in the proofs of Lemmas 2 and 3. We integrate (36) over Ω and apply Green's theorem. Similarly as in (22), the resulting boundary terms are equal to zero due to boundary conditions. This gives us $E_{(1)}^{n+1} = E_{(1)}^n$ for all n. Consequently also $p_{(1)}^{n+1} = p_{(1)}^n$ for all n, by taking the ε^1 terms in (3). This implies that $p_{(1)}^n = p_{(1)}^0 = 0$ for all n.

We proceed by induction and assume that the assumptions of the theorem hold on time level t_n. Gathering the assumptions and all previous results, we have that $E_{(0)}^n$, $E_{(1)}^n$, $p_{(0)}^n$ and $p_{(1)}^n$ are independent of x and n, $\nabla\cdot \boldsymbol{u}_{(0)}^n = \nabla\cdot \boldsymbol{u}_{(0)}^{n+1} = \nabla\cdot \boldsymbol{u}_{(1)}^n = \nabla\cdot \boldsymbol{u}_{R,(0)} = 0$ and $\rho_{(0)}^{n+1} = \rho_{(0)}^n$. These results allow us to simplify (36) to

$$-\boldsymbol{u}_{R,(0)}\nabla\cdot\left(\rho_{(1)}^{n+1} - \rho_{(1)}^n\right) + \nabla\cdot\left(\rho_{(0)}^{n+1}\boldsymbol{u}_{(1)}^{n+1} + \rho_{(1)}^{n+1}\boldsymbol{u}_{(0)}^{n+1}\right) = 0. \tag{37}$$

The second term can be substituted into the mass equation (35) to obtain

$$\frac{\rho_{(1)}^{n+1} - \rho_{(1)}^n}{\Delta t} + \boldsymbol{u}_{R,(0)}\nabla\cdot\left(\rho_{(1)}^{n+1} - \rho_{(1)}^n\right) = 0. \tag{38}$$

Now we proceed similarly as in the proof of Lemma 3 – we multiply (38) by $\rho_{(1)}^{n+1} - \rho_{(1)}^n$ and apply Green's theorem. All resulting integral terms vanish either due to boundary conditions or since $\nabla\cdot \boldsymbol{u}_{R,(0)} = 0$. This implies that $\rho_{(1)}^{n+1} - \rho_{(1)}^n = 0$, hence, by induction $\rho_{(1)}^{n+1} = \rho_{(1)}^0 = 0$. Using this fact in (35) implies $\nabla\cdot \boldsymbol{u}_{(1)}^{n+1} = 0$. □

5. Conclusions and outlook

We have analyzed the asymptotic consistency of a class of linearly implicit schemes for the compressible Euler equations in the low-Mach case. We have shown that the obtained solution tends to the solution of the incompressible Euler equations as the reference Mach number ε tends to zero. Furthermore, we have shown that the scheme treats acoustics correctly in the sense that it does not generate acoustics for well-prepared initial conditions (not containing acoustics). As special cases of our class of schemes, one obtains the Dolejší-Feistauer-Kučera and RS-IMEX schemes.

Several details of the analysis were left out due to limited space, including the proof of Theorem 4. These are contained in the authors' paper [6]. One important topic that was left out of this brief overview was the question of existence of the Hilbert expansion with respect to ε of the form (12). In [6] the existence of the Hilbert expansion is proved in 1D under the simplifying assumption that the reference state $\boldsymbol{w}_R$ is constant in space. Even so the proof is lengthy and technical. Proving the existence of the Hilbert expansion in the general case is left for future work.

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