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A classification of Roter type spacetimes

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Abstract

An algebraic classification of the Roter type spacetimes is given. It follows that the Roter-type curvature condition is essentially equivalent with the pseudosymmetry condition on 4-dimensional Lorentzian manifolds.

1 Introduction

In the search and classification of solutions of the Einstein field equations, curvature conditions have always played a central role. Typically, various types of conditions on the Ricci, Weyl or Einstein tensor are imposed with geometrical and/or physical interpretations. Besides the aforementioned tensors, combinations of these tensors have been extensively studied in the Riemannian and Lorentzian case. In [5, 13, 15, 19, 20, 24, 29] spacetimes which satisfy a condition on the tensor $R \cdot R$, with R the Riemann curvature tensor, have been studied and classified.

In [6, 7] a particular decomposition of the Riemann curvature tensor was explicitely introduced and manifolds satisfying this condition were called Roter type manifolds. In the following we give a complete classification of the Roter-type spacetimes and show that this condition is essentially equivalent with the pseudosymmetry condition for spacetimes.

2 Preliminaries

Throughout this paper, all manifolds are assumed to be connected paracompact manifolds of class C^∞. Let (M, g) be a semi-Riemannian manifold of dimension $n \geq 2$, let ∇ be its Levi-Civita

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connection and $\Xi(M)$ the Lie algebra of vector fields on M. We define on M the endomorphisms $X \wedge_A Y$ and $\mathcal{R}(X,Y)$ of $\Xi(M)$, respectively, by

$$\begin{aligned}(X \wedge_A Y)Z &= A(Y,Z)X - A(X,Z)Y, \\ \mathcal{R}(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,\end{aligned}$$

where A is a symmetric $(0,2)$-tensor on M and $X,Y,Z \in \Xi(M)$. The Ricci tensor S, the Ricci operator $\mathcal{S}$, the tensor S^2 and the scalar curvature κ of a manifold (M,g) are defined by

$$S(X,Y) \;=\; \operatorname{tr}\{Z \rightarrow \mathcal{R}(Z,X)Y\}, \quad g(\mathcal{S}X,Y) \;=\; S(X,Y), \quad S^2(X,Y) \;=\; S(\mathcal{S}X,Y),$$

and $\kappa = \operatorname{tr} \mathcal{S}$, respectively.

The endomorphism $\mathcal{C}(X,Y)$ of a manifold (M,g), $n \geq 3$, is defined by

$$\mathcal{C}(X,Y)Z \;=\; \mathcal{R}(X,Y)Z - \frac{1}{n-2}\left(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y\right) Z.$$

The Riemann-Christoffel curvature tensor R and the Weyl conformal curvature tensor C of a manifold (M,g) are defined by

$$\begin{aligned}R(X_1,X_2,X_3,X_4) &= g(\mathcal{R}(X_1,X_2)X_3,X_4), \\ C(X_1,X_2,X_3,X_4) &= g(\mathcal{C}(X_1,X_2)X_3,X_4),\end{aligned}$$

respectively, where $X_1,X_2,X_3,X_4 \in \Xi(M)$.

Let $\mathcal{B}$ be a tensor field sending any $X,Y \in \Xi(M)$ to a skew-symmetric endomorphism $\mathcal{B}(X,Y)$ and let B be a $(0,4)$-tensor associated with $\mathcal{B}$ by

$$B(X_1,X_2,X_3,X_4) \;=\; g(\mathcal{B}(X_1,X_2)X_3,X_4)\,. \tag{1}$$

The tensor B is said to be a *generalized curvature tensor* if the following conditions are satisfied

$$\begin{aligned}&B(X_1,X_2,X_3,X_4) \;=\; B(X_3,X_4,X_1,X_2)\,, \\ &B(X_1,X_2,X_3,X_4) + B(X_3,X_1,X_2,X_4) + B(X_2,X_3,X_1,X_4) \;=\; 0\,.\end{aligned}$$

For $\mathcal{B}$ as above, let B be again defined by (1). We extend the endomorphism $\mathcal{B}(X,Y)$ to a derivation $\mathcal{B}(X,Y)\cdot$ of the algebra of tensor fields on M, assuming that it commutes with contractions and $\mathcal{B}(X,Y)\cdot f = 0$, for any smooth function f on M.

For a $(0,k)$-tensor field T, $k \geq 1$, we can define the $(0,k+2)$-tensor $B \cdot T$ by

$$\begin{aligned}&(B \cdot T)(X_1,\ldots,X_k,X,Y) \;=\; (\mathcal{B}(X,Y)\cdot T)(X_1,\ldots,X_k) \\ =\; &-T(\mathcal{B}(X,Y)X_1,X_2,\ldots,X_k) - \cdots - T(X_1,\ldots,X_{k-1},\mathcal{B}(X,Y)X_k)\,.\end{aligned}$$

If A is a symmetric $(0,2)$-tensor then we define the $(0,k+2)$-tensor $Q(A,T)$ by

$$\begin{aligned}&Q(A,T)(X_1,\ldots,X_k,X,Y) \;=\; (X \wedge_A Y \cdot T)(X_1,\ldots,X_k) \\ =\; &-T((X \wedge_A Y)X_1,X_2,\ldots,X_k) - \cdots - T(X_1,\ldots,X_{k-1},(X \wedge_A Y)X_k)\,.\end{aligned}$$

The tensor $Q(A,T)$ is called the *Tachibana tensor of the tensors* A and T, or shortly the Tachibana tensor (see, e.g., [8, 12, 13, 16]). Thus we can construct the $(0,6)$-tensors $R \cdot R$, $R \cdot C$, $C \cdot R$, $C \cdot C$, $Q(g,R)$, $Q(S,R)$, $Q(g,C)$ and $Q(S,C)$, as well as the $(0,4)$-tensors $R \cdot S$ and $Q(g,S)$.

The Kulkarni-Nomizu product $A \wedge B$ of $(0,2)$-tensors A and B is defined by (see, e.g., [8])

$$\begin{aligned}(A \wedge B)(U,V;X,Y) \;=\; &A(U,Y)B(V,X) + A(V,X)B(U,Y) \\ &-A(U,X)B(V,Y) - A(V,Y)B(U,X),\end{aligned} \tag{2}$$

where $X, Y, U, V \in \Xi(M)$. Now we can present the Weyl tensor C of (M, g), $n \geq 3$, in the form (see, e.g., [8])

$$C = R - \frac{1}{n-2}\, g \wedge S + \frac{\kappa}{2(n-2)(n-1)}\, g \wedge g. \tag{3}$$

We define the set $\mathcal{U}_R = \left\{ x \in M \,|\, R - \frac{\kappa}{2(n-1)n}\, g \wedge g \neq 0 \text{ at } x \right\}$. Further, we define by $\mathcal{U}_S$ the set of all points of (M, g) at which S is not proportional to g, i.e., $\mathcal{U}_S = \left\{ x \in M \,|\, S - \frac{\kappa}{n}\, g \neq 0 \text{ at } x \right\}$ and by $\mathcal{U}_C$ the set of all points of M at which $C \neq 0$. We note that $\mathcal{U}_S \cup \mathcal{U}_C = \mathcal{U}_R$ (see, e.g. [8]).

The manifold (M, g), $n \geq 2$, is said to be an *Einstein manifold* if at every point of M its Ricci tensor S is proportional to the metric tensor g, i.e., $S = \frac{\kappa}{n}\, g$ on M, assumed that κ is constant when $n = 2$.

It is known that if a semi-Riemannian manifold (M, g), $n \geq 3$, is locally symmetric then

$$\nabla R = 0 \tag{4}$$

on M. Locally indecomposable locally symmetric manifolds are Einstein manifolds. Equation (4) implies the integrability condition $\mathcal{R}(X, Y) \cdot R = 0$ or, briefly,

$$R \cdot R = 0. \tag{5}$$

Semi-Riemannian manifolds satisfying (5) are called *semisymmetric*. Semisymmetric manifolds form a subclass of the class of pseudosymmetric manifolds.

A semi-Riemannian manifold (M, g), $n \geq 3$, is said to be *pseudosymmetric* if the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent at every point of M (see, e.g., [1, Chapter 8.5.3], [2, Section 15] [30, Chapter 6] [34, Chapter 12.4] and [40]. [8, 10, 16, 24] and references therein). This is equivalent to

$$R \cdot R = L_R\, Q(g, R) \tag{6}$$

on the set $\mathcal{U}_R$, where L_R is some function on this set. Every semisymmetric manifold is pseudosymmetric. The converse statement is not true (see, e.g., [24]). We note that (6) implies

$$R \cdot S = L_R\, Q(g, S) \tag{7}$$

and

$$R \cdot C = L_R\, Q(g, C). \tag{8}$$

The conditions (6), (7) and (8) are equivalent on the set $\mathcal{U}_S \cap \mathcal{U}_C$ of any warped product manifold $M_1 \times_F M_2$, with $\dim M_1 = \dim M_2 = 2$ (see, e.g., [13] and references therein).

A semi-Riemannian manifold (M, g), $n \geq 3$, is called *Ricci-pseudosymmetric* if the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent at every point of M (see, e.g., [1, Chapter 8.5.3], [10]). This is equivalent on $\mathcal{U}_S$ to

$$R \cdot S = L_S\, Q(g, S), \tag{9}$$

where L_S is some function on this set. Every warped product manifold $\overline{M} \times_F \widetilde{N}$ with an 1-dimensional manifold $(\overline{M}, \overline{g})$ and an $(n-1)$-dimensional Einstein semi-Riemannian manifold $(\widetilde{N}, \widetilde{g})$, $n \geq 3$, and a warping function F, is a Ricci-pseudosymmetric manifold, see, e.g., [3, Section 1] and [13, Example 4.1].

A semi-Riemannian manifold (M, g), $n \geq 4$, is said to be *Weyl-pseudosymmetric* if the tensors $R \cdot C$ and $Q(g, C)$ are linearly dependent at every point of M [8, 10]. This is equivalent on $\mathcal{U}_C$ to

$$R \cdot C = L_1\, Q(g, C), \tag{10}$$

where L_1 is some function on this set. For a presentation of results on the problem of the equivalence of pseudosymmetry, Ricci-pseudosymmetry and Weyl-pseudosymmetry we refer to [10, Section 4].

A semi-Riemannian manifold (M, g), $n \geq 4$, is said to have *pseudosymmetric Weyl tensor* if the tensors $C \cdot C$ and $Q(g, C)$ are linearly dependent at every point of M (see, e.g., [8, 10, 13]). This is equivalent on $\mathcal{U}_C$ to

$$C \cdot C \;=\; L_C\, Q(g, C), \tag{11}$$

where L_C is some function on this set.

Warped products manifolds $\overline{M} \times_F \widetilde{N}$, of dimension ≥ 4, satisfying on $\mathcal{U}_C \subset \overline{M} \times_F \widetilde{N}$ the condition (11) were studied among others in [13]. In that paper it was proved that the warped products manifold $\overline{M} \times_F \widetilde{N}$, with a 2-dimensional base $(\overline{M}, \overline{g})$ and an $(n-2)$-dimensional space of constant curvature $(\widetilde{N}, \widetilde{g})$, $n \geq 4$, satisfies (11) [13, Theorem 7.1 (i)].

Investigations on semi-Riemannian manifolds (M, g), $n \geq 4$, satisfying (6) and (11) on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ lead to the following condition (see, e.g., [9, Section 6], [13, Section 1], [18, Section 2])

$$R \;=\; \frac{\phi}{2}\, S \wedge S + \mu\, g \wedge S + \frac{\eta}{2}\, g \wedge g, \tag{12}$$

where ϕ, μ and η are some functions on $\mathcal{U}_S \cap \mathcal{U}_C$. We note that if (12) is satisfied at a point of $\mathcal{U}_S \cap \mathcal{U}_C$ then at this point we have $\mathrm{rank}(S - \alpha\, g) > 1$ for any $\alpha \in \mathbb{R}$. A semi-Riemannian manifold (M, g), $n \geqslant 4$, satisfying (12) on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ is called a *Roter type manifold*, or a *Roter type space*, or a *Roter space* (see, e.g., [6, 13, 18]) We have

Theorem 1 *[10, 26] If (M, g), $n \geqslant 4$, is a semi-Riemannian Roter space satisfying (12) on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ then, on this set, we have*

$$S^2 \;=\; \alpha_1\, S + \alpha_2\, g, \quad \alpha_1 \;=\; \kappa + \frac{(n-2)\mu - 1}{\phi}, \quad \alpha_2 \;=\; \frac{\mu\kappa + (n-1)\eta}{\phi}, \tag{13}$$

$$R \cdot R \;=\; L_R\, Q(g, R), \quad L_R \;=\; \frac{1}{\phi}\left((n-2)(\mu^2 - \phi\eta) - \mu\right), \tag{14}$$

$$R \cdot C \;=\; L_R\, Q(g, C), \quad R \cdot S \;=\; L_R\, Q(g, S),$$

$$R \cdot R \;=\; Q(S, R) + L\, Q(g, C), \quad L \;=\; L_R + \frac{\mu}{\phi} \;=\; \frac{n-2}{\phi}(\mu^2 - \phi\eta), \tag{15}$$

$$C \cdot C \;=\; L_C\, Q(g, C), \quad L_C \;=\; L_R + \frac{1}{n-2}\left(\frac{\kappa}{n-1} - \alpha_1\right), \tag{16}$$

$$C \cdot R \;=\; L_C\, Q(g, R), \quad C \cdot S \;=\; L_C\, Q(g, S),$$
$$R \cdot C - C \cdot R \;=\; \left(\frac{1}{\phi}(\mu - \frac{1}{n-2}) + \frac{\kappa}{n-1}\right) Q(g, R) + \left(\frac{\mu}{\phi}(\mu - \frac{1}{n-2}) - \eta\right) Q(S, G),$$

$$C \cdot R - R \cdot C \;=\; Q(S, C) - \frac{\kappa}{n-1}\, Q(g, C).$$

Remark 2 *(i) In the standard Schwarzschild coordinates $(t; r; \theta; \phi)$, and the physical units $(c = G = 1)$, the Reissner-Nordström-de Sitter $(\Lambda > 0)$, and the Reissner-Nordström-anti-de Sitter $(\Lambda < 0)$ metrics are given by the line element (see, e.g., [38]) $ds^2 = -h(r)\, dt^2 + h(r)^{-1}\, dr^2 + r^2\, (d\theta^2 + \sin^2\theta\, d\phi^2)$,*
$h(r) \;=\; 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}$, where M, Q and Λ are non-zero constants.

(ii) [9, Section 6] (see also [14, Remark 2.5] and [18, Section 2]) The metric (2) satisfies (12) with

$$\phi = \frac{3}{2}(Q^2 - Mr)r^4Q^{-4}, \quad \mu = \frac{1}{2}(Q^4 + 3Q^2\Lambda r^4 - 3\Lambda M r^5)Q^{-4},$$
$$\eta = \frac{1}{12}(3Q^6 + 4Q^4\Lambda r^4 - 3Q^4 Mr + 9Q^2\Lambda^2 r^8 - 9\Lambda^2 M r^9)r^{-4}Q^{-4}.$$

If we set $\Lambda = 0$ in (2) then we obtain the line element of the Reissner-Nordström spacetime, see, e.g., [28, Section 9.2] and references therein. It seems that the Reissner-Nordström spacetime is the oldest known example of a Roter type warped product manifold.

(iii) Some comments on pseudosymmetric manifolds (also called Deszcz symmetric spaces), as well as Roter spaces, are given in [4, Section 1]: "From a geometric point of view, the Deszcz symmetric spaces may well be considered to be the simplest Riemannian manifolds next to the real space forms." and "From an algebraic point of view, Roter spaces may well be considered to be the simplest Riemannian manifolds next to the real space forms." For further comments we refer to [39].

(iv) Roter type manifolds and in particular Roter type hypersurfaces in space forms were studied among other in: [6, 8, 12, 17, 21, 22, 23, 26, 32, 33].

(v) According to Theorem 3.1 of ([25]) (p. 338), it follows that if a semi-Riemannian manifold (M, g), $n \geq 4$, is a pseudosymmetric manifold (i.e., we have (6)) satisfying (11) then on the set $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ we have

$$Q\left(S - \left(L_C - L_R + \frac{\kappa}{n-1}\right) g, C - \frac{\mu}{(n-2)n}\,\frac{1}{2}\, g \wedge g\right) = 0, \qquad (17)$$

where μ is some function on $\mathcal{U}_S \cap \mathcal{U}_C$. From (17) we can conclude that (12) holds on the set V of all points of $\mathcal{U}_S \cap \mathcal{U}_C$ at which rank $\left(S - \left(L_C - L_R + \frac{\kappa}{n-1}\right)\right) > 1$.

3 Spacetimes

Let (M, g) be a 4-dimensional, connected, time-oriented, Lorentzian manifold, or in short a spacetime.
Let $p \in M$ and $\{e_0, e_1, e_2, e_3\}$ be an orthonormal basis of T_pM, with $g(e_0, e_0) = -1$. From this basis a null frame $\{k, l, m, \overline{m}\}$ can be constructed as follows

$$k = \frac{1}{\sqrt{2}}(e_0 + e_1)\ , \quad l = \frac{1}{\sqrt{2}}(e_0 - e_1) \ \text{ and } \ m = \frac{1}{\sqrt{2}}(e_2 + i\, e_3). \qquad (18)$$

Once such a null frame is given, one still has the freedom to perform Lorentz transformations. They can be divided into three classes:

null rotations which leave k fixed,

$$k' = k\ , \quad l' = l + E\,\overline{m} + \overline{E}\,m + E\,\overline{E}\,k\ , \quad m' = m + E\,k,$$

null rotations which leave l fixed,

$$l' = l\ , \quad k' = k + B\,\overline{m} + \overline{B}\,m + B\,\overline{B}\,l\ , \quad m' = m + B\,l, \qquad (19)$$

boosts in the $k - l$-plane and spatial rotations in the $m - \overline{m}$-plane,

$$k' = A\,k\ , \quad l' = A^{-1}\,l\ , \quad m' = e^{i\,\Theta},$$

with B, E complex and $\Theta, A (> 0)$ real.

The Weyl tensor C is completely determined by the five complex coefficients Ψ_i, $i = 0, \ldots, 4$, defined by

$$\begin{array}{lll} \Psi_0 = C(k,m,k,m) \;, & & \Psi_3 = C(k,l,\overline{m},l), \\ \Psi_1 = C(k,l,k,m) \;, & & \Psi_4 = C(\overline{m},l,\overline{m},l), \\ \Psi_2 = C(k,m,\overline{m},l) & = & \frac{1}{2}\Big(C(k,l,k,l) - C(k,l,m,\overline{m})\Big). \end{array}$$

The Weyl tensor can be characterized by the number of principal null directions k with the property that $\Psi_0 = C(k,m,k,m) = 0$ [35]. There are at most four such vectors. These can be obtained by starting from an arbitrary null frame $\{k', l', m', \overline{m}'\}$ and applying the Lorentz transformation (19). Under this transformation k' can be brought into any null direction except l', while l' is kept fixed. The complex Weyl scalars change under this transformation. In particular, the Weyl scalar Ψ_0 of the new tetrad reads,

$$\Psi_0 = \Psi_0' - 4E\Psi_1' + 6E^2\Psi_2' - 4E^3\Psi_3' + E^4\Psi_4' \;.$$

Hence, assuming that the new null direction k is a principal direction, i.e., $\Psi_0 = 0$, gives an algebraic equation of at most fourth order for the complex number E. If there are four distinct roots E, or equivalently, four distinct principal null directions, the Weyl tensor is said to be of Petrov type I or algebraically general. If there is at least one multiple principal null direction the Weyl tensor is said to be algebraically general. Several cases can be distinguised. If there is only one multiple principal direction with multiplicity 2, the Weyl tensor is said to be of Petrov type II, if it is of multiplicity 3 the Petrov type is III and has multiplicity 4 the Petrov type is N. If the Weyl tensor has two multiple principal null directions, both with multiplicity 2, the Petrov type is D. Corresponding with each Petrov type there exists a canonical null basis with respect to which the Weyl scalars have characteristic forms. For example, the Weyl tensor is of Petrov type D if and only if with respect to the null frame of principal directions $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$ and $\Psi_2 \neq 0$.

The components of the Ricci tensor S with respect to the null basis (18) are denoted by

$$\begin{array}{ll} 2\Phi_{00} = S(k,k) \;, & 4\Phi_{11} = S(k,l) + S(m,\overline{m}), \\ 2\Phi_{01} = S(k,m) \;, & 2\Phi_{12} = S(l,m), \\ 2\Phi_{02} = S(m,m) \;, & 2\Phi_{22} = S(l,l). \end{array}$$

Just like the Weyl tensor, also the Ricci tensor of a spacetime can be classified into algebraic types according to its principal directions and eigenvalues, see e.g. [31, 37] for a detailed account. It turns out that the Ricci tensor always can be transformed into one of the following four canonical forms:

A1:

$$S = -\rho_0\omega^0 \otimes \omega^0 + \rho_1\omega^1 \otimes \omega^1 + \rho_2\omega^2 \otimes \omega^2 + \rho_3\omega^3 \otimes \omega^3,$$

A2:

$$S = \frac{\rho_0}{2}(k^\flat \otimes l^\flat + l^\flat \otimes k^\flat) + \rho_1(k^\flat \otimes k^\flat - l^\flat \otimes l^\flat) + \rho_2\omega^2 \otimes \omega^2 + \rho_3\omega^3 \otimes \omega^3,$$

A3:

$$S = -\rho_1(k^\flat \otimes l^\flat + l^\flat \otimes k^\flat) + \rho_2\omega^2 \otimes \omega^2 + \rho_3\omega^3 \otimes \omega^3 \pm k^\flat \otimes k^\flat,$$

B:

$$S = \frac{\rho_1}{2}(k^\flat \otimes l^\flat + l^\flat \otimes k^\flat) + \frac{1}{2}(k^\flat \otimes \omega^2 + \omega^2 \otimes k^\flat) + \rho_2\omega^2 \otimes \omega^2 + \rho_3\omega^3 \otimes \omega^3 \;,$$

with $\{\omega^0, \omega^1, \omega^2, \omega^3\}$ the one-forms metrically equivalent with the orthonormal basis $\{e_0, e_1, e_2, e_3\}$ and $\{k^\flat, l^\flat\}$ the one-forms metrically equivalent with the null vectors $\{k, l\}$.

4 Classification of the Roter type spacetimes

A spacetime (M, g) is said to be of *Roter type* if there exist functions ϕ, μ, η on $\mathcal{U} \subset M$, such that the Riemann-Christoffel curvature tensor has the form

$$R = \frac{\phi}{2} S \wedge S + \mu\, g \wedge S + \frac{\eta}{2} g \wedge g. \tag{20}$$

We can verify that if the condition (20) is satisfied on a semi-Riemannian manifold (M, g), $\dim M \geq 4$, and its Ricci tensor S is of the form

$$S = \Lambda\, g + \lambda\, \omega \otimes \omega, \tag{21}$$

with ω a one-form and Λ and λ some functions on M, then the Weyl tensor C of (M, g) vanishes. If the Ricci tensor S of a semi-Riemannian manifold (M, g), $\dim M \geq 3$, satisfies (21) then the manifold (M, g) is said to be quasi-Einstein.

Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations and the investigation on quasi-umbilical hypersurfaces of conformally flat spaces, see, e.g., [10, 13] and references therein. Quasi-Einstein manifolds satisfying some pseudosymmetry type curvature conditions were investigated among others in [3, 8, 12]. Quasi-Einstein hypersurfaces in semi-Riemannian space forms were studied among others in [11, 17, 27], see also [10] and references therein.

We note that the Roter type condition (20) will only be considered on the subset $\mathcal{U} \subset M$ of a semi-Riemannian manifold (M, g), $\dim M \geq 4$, where the Weyl tensor does not vanish and the Ricci tensor does not satisfy the quasi-Einstein condition. We then have the following result.

Theorem 3 *A spacetime (M, g) is of Roter type if and only if on $\mathcal{U} \subset M$ the Weyl tensor is of Petrov type D and the Ricci tensor is of Segré type A1, with two distinct eigenvalues with multiplicity 2.*

proof The condition (20) written with respect to a null basis (18) gives rise to the following set of 13 algebraic relations between the Ricci and Weyl scalars:

$$\Psi_2 + \overline{\Psi}_2 - 2\Phi_{11} + \frac{\kappa}{12} = 4\phi \left\{ \left(\Phi_{11} - \frac{\kappa}{8}\right)^2 - \Phi_{00}\Phi_{22} \right\} - 4\mu \left(\Phi_{11} - \frac{\kappa}{8}\right) + \eta,$$

$$\Psi_1 - \Phi_{01} = 4\phi \left\{ \Phi_{01} \left(\Phi_{11} - \frac{\kappa}{8}\right) - \Phi_{00}\Phi_{12} \right\} - 2\mu\Phi_{01},$$

$$-\overline{\Psi}_3 + \Phi_{12} = 4\phi \left\{ \Phi_{01}\Phi_{22} - \Phi_{12} \left(\Phi_{11} - \frac{\kappa}{8}\right) \right\} + 2\mu\Phi_{12},$$

$$\overline{\Psi}_2 - \Psi_2 = 4\phi \Big(\overline{\Phi}_{01}\Phi_{12} - \Phi_{01}\overline{\Phi}_{12} \Big),$$

$$\Psi_0 = 4\phi \Big(\Phi_{01}^2 - \Phi_{00}\Phi_{02} \Big),$$

$$\Phi_{00} = 4\phi \left\{ \Phi_{00} \left(\Phi_{11} + \frac{\kappa}{8}\right) - \Phi_{01}\overline{\Phi}_{01} \right\} + 2\mu\Phi_{00},$$

$$\Phi_{02} = 4\phi \left\{ \Phi_{01}\Phi_{12} - \Phi_{02} \left(\Phi_{11} - \frac{\kappa}{8}\right) \right\} + 2\mu\Phi_{02},$$

$$-\Psi_2 + \frac{\kappa}{12} = 4\phi \left\{ \overline{\Phi}_{01}\Phi_{12} - \left(\Phi_{11} - \frac{\kappa}{8}\right) \left(\Phi_{11} + \frac{\kappa}{8}\right) \right\} + \frac{\kappa}{2}\mu + \eta,$$

$$\Psi_1 + \Phi_{01} = 4\phi\left\{\Phi_{01}\left(\Phi_{11} + \frac{\kappa}{8}\right) - \overline{\Phi}_{01}\Phi_{02}\right\} + 2\mu\Phi_{01},$$

$$\overline{\Psi}_4 = 4\phi\Big(\Phi_{12}^2 - \Phi_{22}\Phi_{02}\Big),$$

$$\Phi_{22} = 4\phi\left\{\Phi_{22}\left(\Phi_{11} + \frac{\kappa}{8}\right) - \Phi_{12}\overline{\Phi}_{12}\right\} + 2\mu\Phi_{22},$$

$$\overline{\Psi}_3 + \Phi_{12} = 4\phi\left\{\Phi_{12}\left(\Phi_{11} + \frac{\kappa}{8}\right) - \Phi_{02}\overline{\Phi}_{12}\right\} + 2\mu\Phi_{12},$$

$$\Psi_2 + \overline{\Psi}_2 + 2\Phi_{11} + \frac{\kappa}{12} = 4\phi\left\{\left(\Phi_{11} + \frac{\kappa}{8}\right)^2 - \Phi_{02}\overline{\Phi}_{02}\right\} + 4\mu\left(\Phi_{11} + \frac{\kappa}{8}\right) + \eta.$$

Inserting each canonical form of the Ricci tensor in the above relations, shows that only a Ricci tensor of Segré type A1, with two distinct eigenvalues with multiplicity 2, is compatible. If we denote the principal values with ρ_1 and ρ_2, it further follows that

$$2\mu = 1 - \phi(\rho_1 + \rho_2)\ , \quad 3\eta = \phi(\rho_1^2 + \rho_2^2 + \rho_1\rho_2) - (\rho_1 + \rho_2),$$

and $6\Psi_2 = \phi(\rho_1 - \rho_2)^2 \neq 0$, $\Psi_i = 0$, with $i = 0, 1, 3, 4$. $\diamondsuit$

In the particular case that the two distinct eigenvalues of the Ricci tensor have equal magnitude and opposite sign, these types of spacetimes can be physically interpreted as being filled with a *non-null electromagnetic field* if the corresponding energy-momentum tensor satisfies the Maxwell equations.

Corollary 4 *Every Petrov type D spacetime which is filled with a non-null electromagnetic field is of Roter type.*

In [37, p.322] the general metric of this type is given.

In [29] an algebraic classification of the pseudosymmetric spacetimes was given. It was shown that the pseudosymmetric spacetimes are either of Petrov type D, N or conformally flat, with further extra conditions on the Ricci tensor. Examples of conformally flat spacetimes are the Robertson-Walker spaces, while examples of Petrov type D pseudosymmetric spacetimes are the Schwarzschild and Reissner-Nordström spaces. On comparing this classification of pseudosymmetric spacetimes with the above classification of the Roter type spacetimes, we find the following result.

Corollary 5 *On the subset $\mathcal{U}$, a spacetime is of the Roter type if and only if it is pseudosymmetric.*

Remark 6 *That this equivalence does not hold for 6-dimensional semi-Riemannian manifolds follows from example 4.4 in [36] of a pseudosymmetric manifold which is of non-Roter type.*

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