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ON THE NONCOMMUTATIVE BONDAL-ORLOV CONJECTURE FOR SOME TORIC VARIETIES

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WITH AN APPENDIX BY JASON P. BELL

Abstract. We show that all toric noncommutative crepant resolutions (NCCRs) of affine GIT quotients of "weakly symmetric" unimodular torus representations are derived equivalent. This yields evidence for a non-commutative extension of a well known conjecture by Bondal and Orlov stating that all crepant resolutions of a Gorenstein singularity are derived equivalent. We prove our result by showing that *all* toric NCCRs of the affine GIT quotient are derived equivalent to a fixed Deligne-Mumford GIT quotient stack associated to a generic character of the torus. This extends a result by Halpern-Leistner and Sam which showed that such GIT quotient stacks are a geometric incarnation of a family of *specific* toric NCCRs constructed earlier by the authors.

1. Introduction

The Bondal-Orlov conjecture [BO02] asserts that all crepant resolutions of Gorenstein singularities are derived equivalent. Later the conjecture was further generalized to a noncommutative setting.

Definition 1.1. Let S be a normal noetherian Gorenstein domain. A *noncommutative crepant resolution* (NCCR) of S is an S-algebra of a finite global dimension of the form $\operatorname{End}_S(M)$, which is Cohen-Macaulay as an S-module, where M is a nonzero finitely generated reflexive S-module.

In [VdB04a, Conjecture 4.6] it is then conjectured that crepant resolutions should also be equivalent to noncommutative crepant resolutions. In [IW13, Conjecture 1.4] the noncommutative part is singled out as "noncommutative Bondal-Orlov".

Conjecture 1.2. [VdB04a, IW13] All noncommutative crepant resolutions are derived equivalent.

NCCRs do not always exist, however in [ŠVdB17a] they were constructed for a large class of quotient singularities for reductive groups. Notably they exist (under mild genericity condition, see Definition 3.3) for quotient singularities for quasi-symmetric representations W of a torus T. Here "quasi-symmetric" means that the sum of weights of W on each line through the origin is zero (see [Kit17] for a

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geometric interpretation of quasi-symmetric representations). The NCCRs in loc. cit. are given by *modules of covariants* $M = (U \otimes_k \operatorname{Sym}(W))^T$ for a representation U of T, and in this setting (as T is a torus and hence U is a sum of characters) are called *toric*.

In this note we prove Conjecture 1.2 for toric NCCRs in the above context(thus in particular the existence of toric NCCRs is guaranteed). In fact, our assumptions are slightly weaker and apply to "weakly symmetric" (generalizing quasi-symmetric) representations, see Definition 3.2 (for which however toric NCCRs do not always exist, see [ŠVdB17a, Example 10.1]). The following is our main result.

Theorem 1.3. *Let W be a unimodular, generic and weakly symmetric representation W of a torus T. Then all toric NCCRs of $\operatorname{Sym}(W)^T$ are derived equivalent.*

To prove the theorem we follow a strategy of Halpern-Leistner and Sam [HLS16] and embed any toric NCCR of $\operatorname{Sym}(W)^T$ into the derived category of a *fixed* generic GIT stack quotient of $W^* = \operatorname{Sym} W$. Such GIT quotient stacks are Calabi-Yau and hence they do not admit non-trivial semi-orthogonal decompositions. Therefore the constructed embedding is in fact an equivalence, proving the theorem.

We note that Halpern-Leistner and Sam already showed that generic GIT stack quotients of W^* for quasi-symmetric W are derived equivalent to the specific toric NCCRs constructed in [ŠVdB17a] (already mentioned above). But since we start with an arbitrary toric NCCR we are dealing with a more general class of Cohen-Macaulay modules than in [HLS16] and therefore the construction of the embedding is more intricate and requires the use of the Cohen-Macaulayness criterion for modules of covariants from [VdB93]. Luckily this criterion turns out to considerably simplify in the weakly symmetric case.

In §7 we present an alternative proof of Theorem 1.3 in the case $T = G_m$. It is a based on a crucial combinatorial lemma provided by Jason Bell in Appendix A, which is used to describe "maximal Cohen-Macaulay cliques" of $X(T)$ (see (1)). Moreover, we introduce the notion of *toric maximal modification algebras* (toric MMAs) (see the paragraph after Definition 7.1) and show that in the case $T = G_m$ they coincide with NCCRs (see Proposition 7.2).

We note that Iyama and Wemyss [IW13] provide a sufficient criterion under which Conjecture 1.2 holds, which in particular covers dimension ≤ 3. However, this criterion does not seem to be easily applicable to our setting.

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3. Notation and conventions

Throughout k is an algebraically closed field of characteristic 0. Let W be a d-dimensional T-representation, $\dim T = s$, and let $R = \operatorname{Sym} W$, $X = \operatorname{Spec} R = W^*$. Let $(\alpha_i)_{i=1}^d$ be the weights of W and let $(w_i)_i$ be the corresponding weight vectors.

We denote by $X(T)$ (resp. $Y(T)$) the character group (resp. the group of one-parameter subgroups) of T. There is a natural pairing $Y(T) \times X(T) \to \mathbb{Z}$, which extends to $Y(T)_\mathbb{R} \times X(T)_\mathbb{R} \to \mathbb{R}$, we denote it by $\langle\ ,\ \rangle$. On $Y(T)_\mathbb{R}$ we choose a positive definite quadratic form, and we denote the corresponding norm by $\|\ \|$.

If U is a 1-dimensional representation of T given by $\mu \in X(T)$ then we write $M(\mu)$ instead of $M(U)$. In our main reference [VdB93] the results are written in terms of semi-invariants R^T_μ, defined as the sum of all irreducible representations of T in R with character μ; i.e. $R^T_\mu \cong M(-\mu)$. We will use both notations, depending on the context. More generally, if M is any T-module, we write M^T_μ for the sum of all irreducible representations of T in M with character μ.

Let us denote

$$X^{\lambda,+} = \{x \in X \mid \lim_{t\to 0} \lambda(t)x \text{ exists}\},$$
$$X^{\lambda,>0} = \{x \in X \mid \lim_{t\to 0} \lambda(t)x = 0\},$$

and let $W^{\lambda,>}$, $W^{\lambda,+}$ be the subspace of W generating the defining ideal $I^{\lambda,>}$, $I^{\lambda,+}$ of $X^{\lambda,>0}$ and $X^{\lambda,+}$. Note that these subspaces are spanned by the weight vectors w_i such that $\langle\lambda,\alpha_i\rangle \geq 0$, $\langle\lambda,\alpha_i\rangle > 0$, respectively.

We write $X^u = \{x \mid 0 \in \overline{Tx}\}$ for the T-unstable locus (also called "nullcone") of X. A defining ideal for X^u is $R(R^T)^+$ where $(R^T)^+$ is the augmentation ideal of R^T. By the Hilbert-Mumford criterion, $X^u = \cup_{\lambda\in Y(T)} X^{\lambda,>0}$. Let $\chi \in X(T)$. Then by definition $X^{ss,\chi}$ consists of the points $x \in X$ such that if $\lambda \in Y(T)$ is such that $\lim_{t\to 0}\lambda(t)x$ exists then $\langle\lambda,\chi\rangle \geq 0$. We write $X^{u,\chi} := X \setminus X^{ss,\chi} = \bigcup_{\lambda:\langle\lambda,\chi\rangle<0} X^{\lambda,+}$ and call it the χ-unstable locus.

For further reference we introduce some extra notation, mostly consistent with [VdB93]: $T_\lambda = \{i \mid \langle\lambda,\alpha_i\rangle < 0\}$, $d_\lambda = \operatorname{codim}(X^{\lambda,>0}, X) = d - |T_\lambda|$. Moreover, $T^0_\lambda = \{i \mid \langle\lambda,\alpha_i\rangle = 0\}$, $d^0_\lambda = |T^0_\lambda|$. $T^+_\lambda = \{i \mid \langle\lambda,\alpha_i\rangle > 0\}$. On $Y(T)_\mathbb{R}$ we let $\lambda \sim \lambda'$ iff $T_\lambda = T_{\lambda'}$, $B = \{\lambda \in Y(T)_\mathbb{R} \mid \|\lambda\| < 1\}$, $\Lambda = B/\!\sim$, $B_\lambda = \{\mu \in B \mid \mu \sim \lambda\}$, $\Phi_\lambda = \bar{B}_\lambda - B_\lambda$, $H_\lambda = \operatorname{span}\{(\alpha_i)_{i\in T^0_\lambda}\}$, $h^0_\lambda = \dim H_\lambda$.

We introduced some properties

Definition 3.1. A T-representation W is *generic* if the set X^s of points in X with closed orbit and trivial stabilizer is nonempty and satisfies $\operatorname{codim}(X - X^s) \geq 2$ (equivalently for all $\lambda \in Y(T)\setminus\{0\}$ there exist two $1 \leq i \leq d$ such that $\langle\lambda,\alpha_i\rangle > 0$).

Definition 3.2. A T-representation W is *quasi-symmetric* if for every line $\ell \subset X(T)_\mathbb{R}$ through the origin we have $\sum_{\alpha_i\in\ell}\alpha_i = 0$. It is *weakly symmetric* if for every ℓ the cone spanned by $\alpha_i \in \ell$ is either zero or ℓ.

We denote $\Sigma := \{\sum_i a_i\alpha_i \mid a_i \in]-1,0]\}$.

Definition 3.3. We say that $\chi \in X(T)$ is *generic for W* if it is parallel to Σ but not parallel to any face of Σ.

4. Local cohomology in the weakly symmetric case

As alluded to in the introduction we will construct an embedding of the derived category of a toric NCCR $\Lambda = \operatorname{End}_{R^T}(\oplus_\gamma M(\gamma))$, into the derived category of a fixed GIT quotient stack of the form $X^{ss,\chi}/T$. To relate the Cohen-Macaulayness of Λ to the necessary vanishing on $X^{ss,\chi}/T$ (see Proposition 5.1 and its proof) it will turn out that we need to compare (as T-modules) the local cohomology of $\mathcal{O}_X$ supported in X^u and in $X^{u,\chi}$. We have a good understanding of the former by [VdB93] and we employ the HKKN[1] stratification [Kir84] for the latter.

[1]Hesselink-Kempf-Kirwan-Ness

4.1. **Support in the nullcone.** In the first part of this section W is arbitrary; i.e. not necessarily weakly symmetric or generic. The local cohomology modules of $\mathcal{O}_X$ supported in the nullcone are described in [VdB93]. In particular, they provide a criterion for Cohen-Macaulayness of modules of covariants. At the end of this section we show that when W is weakly symmetric and generic the criterion becomes more concrete.

Lemma 4.1. [VdB93, Theorem 3.4.1] *Let $I = R(R^T)^+$ be the defining ideal of X^u (see §3). Then*

$$H^i_{(R^T)^+}(R^T_\mu) \cong H^i_I(R)^T_\mu, \tag{4.1}$$

and hence R^T_μ is Cohen-Macaulay if and only if $H^i_I(R)^T_\mu = 0$ for $0 \le i < \dim R^T$.

Theorem 4.2. [VdB93, Proposition 3.3.1, Theorem 3.4.1] *There is a T-equivariant filtration on $H^i_{X^u}(X, \mathcal{O}_X)$ together with a $\mathbb{Z}^d$-graded isomorphism of R-modules*

$$\operatorname{gr} H^i_{X^u}(X, \mathcal{O}_X) \cong \bigoplus_{\lambda\in\Lambda} \tilde{H}^{i+s-d_\lambda-1}(\Phi_\lambda, k) \otimes H^{d_\lambda}_{X^{\lambda,>0}}(X, \mathcal{O}_X). \tag{4.2}$$

Furthermore there is a $\mathbb{Z}^d$-graded isomorphism

$$\begin{aligned} H^{d_\lambda}_{X^{\lambda,>0}}(X, \mathcal{O}_X) &\cong (\wedge^{d_\lambda} W^{\lambda,>})^* \otimes \bigoplus_t \operatorname{Sym}^t(W^{\lambda,>})^* \otimes R/I^{\lambda,>} \\ &\cong (\wedge^{d_\lambda} W^{\lambda,>})^* \otimes \bigoplus_t \operatorname{Sym}^t(W^{\lambda,>*} \oplus W/W^{\lambda,>0}) \end{aligned} \tag{4.3}$$

In our situation the Φ_λ are easy to describe.

Lemma 4.3. *Assume that T acts faithfully on W and that W is weakly symmetric. Let $\lambda \in Y(T)_\mathbb{R}$. Then Φ_λ is homeomorphic with a sphere (of maximal dimension) in $H^\perp_\lambda \subset Y(T)_\mathbb{R}$.*

Proof. We claim that

$$B_\lambda = \{\mu \in B \cap H^\perp_\lambda \mid \forall i \in T_\lambda : \langle \mu, \alpha_i\rangle < 0\}.$$

Indeed, if $\mu \in B \cap H^\perp_\lambda$ then $T^0_\lambda \subset T^c_\mu$, and if $T_\lambda \subset T_\mu$ then $T_{-\lambda} \subset T^c_\mu$ by the weak symmetricity, respectively. As $\{i \mid 1 \le i \le d\} = T_\lambda \sqcup T^0_\lambda \sqcup T_{-\lambda}$ we have $T_\mu = T_\lambda$ and thus $\mu \in B_\lambda$. For the converse it is enough to observe that $\mu \in B_\lambda$ implies $\mu \in H^\perp_\lambda$. Let $\mu \in B_\lambda$ and write $\mu = \mu_0 + \mu_\perp$ for $\mu_0 \in H_\lambda$, $\mu_\perp \in H^\perp_\lambda$. If $\mu_0 \neq 0$ then $\langle \mu_0, \alpha_i\rangle < 0$ for some $i \in T^0_\lambda$ since W is weakly symmetric. Thus, $\mu \in H^\perp_\lambda$.

Let $\Gamma \subset X(T)_\mathbb{R}/H_\lambda$ be the cone spanned by the images of the weights $(-\alpha_i)_{i\in T_\lambda}$ in $X(T)_\mathbb{R}/H_\lambda$. Note that by faithfulness and weak symmetry Γ is of maximal dimension. $\langle -, -\rangle$ descends to a non-degenerate pairing between $H^\perp_\lambda$ and $X(T)_\mathbb{R}/H_\lambda$. Let $\Gamma^\vee \subset H^\perp_\lambda$ be the dual cone of Γ. Then we have

$$\begin{aligned} B_\lambda &= \{\mu \in B \cap H^\perp_\lambda \mid \mu|(\Gamma - \{0\}) > 0\} \\ &= B \cap \operatorname{relint} \Gamma^\vee. \end{aligned}$$

The conclusion now easily follows. □

Proposition 4.4. *Assume T acts faithfully and W is weakly symmetric. Then*

$$\operatorname{gr} H^i_{X^u}(X, \mathcal{O}_X) \cong \bigoplus_{\lambda\in\Lambda, i=d_\lambda-h^0_\lambda} H^{d_\lambda}_{X^{\lambda,>0}}(X, \mathcal{O}_X).$$

Proof. By Lemma 4.3 Φ_λ is a sphere of dimension $s-1-h^0_\lambda$. Hence the non-zero terms in (4.1) correspond to $i+s-d_\lambda-1=s-1-h^0_\lambda$ or $i=d_\lambda-h^0_\lambda$. □

Corollary 4.5. *Assume W is generic and weakly symmetric. Then R^T_μ is Cohen-Macaulay if and only if for all $0\neq\lambda\in Y(T)$ we have $H^{d_\lambda}_{X^{\lambda,>0}}(X,\mathcal{O}_X)_\mu=0$.*

Proof. Since T acts generically we have $\dim R^T=d-s$.

According to Proposition 4.4 and Lemma 4.1 R^T_μ is Cohen-Macaulay if and only if for all $\lambda\in Y(T)$ such that $d_\lambda-h^0_\lambda<d-s$ we have $H^{d_\lambda}_{X^{\lambda,>0}}(X,\mathcal{O}_X)_\mu=0$.

If $\lambda=0$ then $d_\lambda-h^0_\lambda=d-s$ and hence the condition $d_\lambda-h^0_\lambda<d-s$ does not hold. Assume now $\lambda\neq 0$. We will show that $d_\lambda-h^0_\lambda<d-s$ now always holds. This proves the lemma.

Assume on the contrary that $d_\lambda-h^0_\lambda\geq d-s$. Since $d_\lambda=d-|T_\lambda|$ this is equivalent to $|T_\lambda|+h^0_\lambda\leq s$. Since W is generic $|T_\lambda|\geq 2$ and in particular $T_\lambda\neq\emptyset$. Moreover since T acts in particular faithfully we have $h^0_\lambda<s$. Since W is weakly symmetric all weights of W not in H_λ are in $\cup_{i\in T_\lambda}\mathbb{R}\alpha_i$. Fix $f\in T_\lambda$ and let $\langle\gamma,-\rangle=0$ be a hyperplane in $X(T)_{\mathbb{R}}$ containing H_λ and $\{\alpha_i\mid i\in T_\lambda, i\neq f\}$ such that $\langle\gamma,\alpha_f\rangle\geq 0$ (this is possible by the hypothesis $|T_\lambda|+h^0_\lambda\leq s$, $h^0_\lambda<s$). Then it easy to see that there is *at most* one weight such that $\langle\gamma,\alpha_i\rangle>0$ (namely α_f). This contradicts the hypothesis that W is generic. □

Remark 4.6. Corollary 4.5 does not hold true if W is not weakly-symmetric, see [VdB93, §4.5]. See also Remark 4.12 for an example of a unimodular W.

Corollary 4.7. *Assume W is generic and weakly symmetric. Then the weights of $H^{d_\lambda}_{X^{\lambda,>0}}(X,\mathcal{O}_X)_\mu$ are of the form*

$$-\sum_{i\in T^+_\lambda}\alpha_i-\sum_{i\in T^+_\lambda}a_i\alpha_i+\sum_{i\in T_\lambda}b_i\alpha_i+\sum_{i\in T^0_\lambda}c_i\alpha_i \tag{4.4}$$

$a_i,b_i,c_i\in\mathbb{Z}$, $a_i>0$, $b_i>0$. Thus R^T_μ is a Cohen-Macaulay R^T-module if and only if μ is not of the form (4.4) *for all $\lambda\in Y(T)\setminus\{0\}$.*

Proof. This is a consequence of (4.3) and Corollary 4.5. We use the fact that since W is weakly generic the positive integral linear combinations of $(\alpha_i)_{i\in T^0_\lambda}$ form a lattice. □

4.2. **Support in $X^{\lambda,+}$.** We will study the local cohomology of $\mathcal{O}_X$ supported in $X^{u,\chi}$ inductively using the HKKN stratification. In this section we prove the relevant vanishing theorem. More precisely it will follow from Corollary 4.9 below that if R^T_μ is Cohen-Macaulay then for all HKKN-strata S and all $i\geq 0$ we have $H^i_S(X,\mathcal{O}_X)^T_\mu=0$ where we follow the convention that if S is locally closed in X then $H^*_S(X,-):=H^*_S(U,-)$ where U is an open subset of X such that S is closed in U. By excision this definition does not depend on the choice of U.

Lemma 4.8. *We have as T-representations:*

$$H^i_{X^{\lambda,+}}(X,\mathcal{O}_X)=(\wedge^d W)^*\otimes_k H^{d-i}_{X^{-\lambda,>0}}(X,\mathcal{O}_X)^*, \tag{4.5}$$

$$H^i_{X^{\lambda,+}}(X,\mathcal{O}_X)=0 \quad \textit{for } i\neq d-d_{-\lambda}. \tag{4.6}$$

Proof. This follows from the fact that the defining ideals of $X^{\lambda,+}$ and $X^{-\lambda,>0}$ are generated by complementary subspaces of W together with [VdB93, Proposition 3.3.1]. □

For $f \in R$ we write $X_f = \{f \neq 0\} \subset X$,

Corollary 4.9. *Assume W is generic and weakly symmetric. Assume $\lambda \neq 0$ and let $f = \prod_{j\in J} x_j$ for a subset $J \subset T_\lambda^0$. If R_μ^T is Cohen-Macaulay R^T-module then $R\Gamma_{X_f^{\lambda,+}}(X, \mathcal{O}_X)_\mu^T = 0$.*

Proof. Let $\delta = \sum_i \alpha_i$ be the character of $\wedge^d W$. As

$$H^i_{X_f^{\lambda,+}}(X, \mathcal{O}_X) \cong H^i_{X_f^{\lambda,+}}(X_f, \mathcal{O}_{X_f}) \cong R_f \otimes_R H^i_{X^{\lambda,+}}(X, \mathcal{O}_X),$$

we see that $H^i_{X_f^{\lambda,+}}(X, \mathcal{O}_X) = 0$ for $i \neq d - d_{-\lambda}$ by (4.6) and for $i = d - d_{-\lambda}$ the weights μ' of $R_f \otimes_R H^{d-d_{-\lambda}}_{X_f^{\lambda,+}}(X, \mathcal{O}_X)$ are of the form

$$\mu' = -\delta + \sum_{i\in T^+_{-\lambda}} \alpha_i + \sum_{i\in T^+_{-\lambda}} a_i\alpha_i - \sum_{i\in T_{-\lambda}} b_i\alpha_i + \sum_{i\in T^0_\lambda} c_i\alpha_i \tag{4.7}$$

$a_i, b_i, c_i \in \mathbb{Z}$, $a_i > 0$, $b_i > 0$ by (4.4).

On the other hand since W is generic we have by [Kno86] $\omega_{R^T} = R^T_{-\delta}$ and since $\operatorname{Hom}_{R^T}(R^T_\mu, \omega_{R^T}) = R^T_{-\mu-\delta}$ by [ŠVdB17a, Lemma 4.1.3] as W is generic, $R^T_{-\mu-\delta}$ is also a Cohen-Macaulay R^T-module. Hence by applying (4.4) for $-\mu-\delta, -\lambda$ we find

$$-\mu - \delta \neq -\sum_{i\in T^+_{-\lambda}} \alpha_i - \sum_{i\in T^+_{-\lambda}} a_i\alpha_i + \sum_{i\in T_{-\lambda}} b_i\alpha_i + \sum_{i\in T^0_\lambda} c_i\alpha_i \tag{4.8}$$

for all $a_i, b_i, c_i \in \mathbb{Z}$, $a_i > 0$, $b_i > 0$. Clearly (4.8) implies that (4.7) cannot be satisfied for $\mu' = \mu$. □

4.3. **Support in the χ-unstable locus.** In this section we proceed to compute the local cohomology supported in the χ-unstable locus $X^{u,\chi}$ using the HKKN stratification and applying the vanishing results from the previous section. We show that $H^*_{X^{u,\chi}}(X, \mathcal{O}_X)^T_\mu$ vanishes if R^T_μ is Cohen-Macaulay.

We first recall some properties of the HKKN stratifications that we will use.

Let $(S_i)_{i=1}^N$ be the *HKKN stratification* of $X^{u,\chi}$ (see [Kir84] and [BFK] for its application in a similar context). It satisfies the following properties:

1. The closure of S_i is contained in $\bigcup_{j\geq i} S_j$.
2. If S is an HKKN stratum then S is an open subset of $X^{\lambda,+}$ for some $\lambda \neq 0$ by the proof of [Kir84, Corollary 13.2]. Moreover, S is the intersection of $X^{\lambda,+}$ with a union U of open sets of the form X_{f_J} where $f_J = \prod_{j\in J} x_j$ for some $J \subset T^0_\lambda$ (see [Kir84, Definition 12.20]).

Lemma 4.10. *Assume W is generic and weakly symmetric. Let S be an HKKN stratum in $X^{u,\chi}$. If R^T_μ is a Cohen-Macaulay R^T-module then $R\Gamma_S(X, \mathcal{O}_X)_\mu = 0$.*

Proof. Let the notations be as in (2) above. As $X^{\lambda,+}$ is closed in X, S is closed in U. Hence by definition $H^i_S(X, \mathcal{O}_X) \cong H^i_S(U, \mathcal{O}_U)$. We have

$$R\Gamma_S(U, \mathcal{O}_U) = R\Gamma(U, \mathcal{H}^{c_\lambda}_{X^{\lambda,+}}(X, \mathcal{O}_X))[-c_\lambda],$$

where $c_\lambda = d - d_{-\lambda}$ and hence

$$H^i_S(U, \mathcal{O}_U) = H^{i-c_\lambda}(U, \mathcal{H}^{c_\lambda}_{X^{\lambda,+}}(X, \mathcal{O}_X)), \tag{4.9}$$

Recall that U has a covering consisting of open sets of the form X_{f_J}, $J \subset T^0_\lambda$. We can compute the right-hand side of (4.9) using the Čech complex with respect to this covering noting that $X_{f_{J_1}} \cap X_{f_{J_2}} = X_{f_{J_1 \cup J_2}}$. To prove the lemma it is then sufficient to prove that $H^{i-c_\lambda}(X_{f_J}, \mathcal{H}^{c_\lambda}_{X^{\lambda,+}}(X, \mathcal{O}_X))_\mu = 0$. This follows from Corollary 4.9. □

Proposition 4.11. *Assume W is generic and weakly symmetric. If R^T_μ is a Cohen-Macaulay R^T-module then the natural map*

$$R^T_\mu \to R\Gamma(X^{ss,\chi}, \mathcal{O}_X)^T_\mu$$

is an isomorphism.

Proof. Put $S_0 = X^{ss,\chi}$ and $X_i := \bigcup_{j \leq i} S_j$. $X^{ss,\chi} = X_0 \subset X_1 \subset \cdots \subset X_N = X$ is a filtration of X by open subsets such that $S_l = X_l - X_{l-1}$ is closed in X_l. We thus have a distinguished triangle

$$R\Gamma_{S_l}(X_l, \mathcal{O}_{X_l}) \to R\Gamma(X_l, \mathcal{O}_{X_l}) \to R\Gamma(X_{l-1}, \mathcal{O}_{X_{l-1}}) \to$$

Since $R\Gamma_{S_l}(X_l, \mathcal{O}_{X_l})^T_\mu = R\Gamma_{S_l}(X, \mathcal{O}_X)^T_\mu = 0$ by Lemma 4.10 we have $R\Gamma(X_l, \mathcal{O}_{X_l})^T_\mu \cong R\Gamma(X_{l-1}, \mathcal{O}_{X_{l-1}})^T_\mu$. Thus, $R\Gamma(X^{ss,\chi}, \mathcal{O}_{X^{ss,\chi}})^T_\mu \cong R\Gamma(X, \mathcal{O}_X)^T_\mu = R^T_\mu$. □

Remark 4.12. Proposition 4.11 does not hold without the weak symmetry assumption. The reason goes back to Corollary 4.5, see Remark 4.6. It fails for instance in the example [VdB93, §4.5] (for $\mu = (-3, -3)$, $\chi = (-2, -1)$). Slightly tweaking the example in loc. cit. we can also get a unimodular W, which we mention here but omit the details. Let $T = G_m^2$ and let W be with weights $(1, 0)$, $(2, 0)$, $(0, 1)$, $(-1, 1)$, $(-1, -1)^{\oplus 2}$, $\mu = (-3, -1)$, $\chi = (1, -2)$. By [VdB93] (c.f. §4.1), R_μ is Cohen-Macaulay, however it is easy to verify that $H^3_{X^{(2,1),>0}}(X, \mathcal{O}_X)_\mu \neq 0$ (c.f. 4.3) and thus also $H^3_{X^{(2,1),+}}(X, \mathcal{O}_X)_\mu \neq 0$ by Lemma 4.8. One can check (using the Mayer-Vietoris sequence) that $H^3_{X^{(2,1),+}}(X, \mathcal{O}_X)$ occurs in $H^3_{X^{u,\chi}}(X, \mathcal{O}_X)$. Thus, Proposition 4.11 does not hold in this case.

5. GIT quotient stacks vs NCCRs

In this section we show that any NCCR is derived equivalent to $X^{ss,\chi}/T$, which also proves Theorem 1.3. For a family of specific NCCRs in the quasi-symmetric case, that were constructed in [ŠVdB17a, Theorem 1.6.2.], this result had been established in [HLS16, Corollary 4.2, Remark 4.3].

Proposition 5.1. *Assume that W is unimodular, generic and weakly symmetric. Let Λ be a toric NCCR of R^T, and let χ be a generic character of T. Then* $\mathrm{D}(\Lambda) \cong \mathrm{D}(X^{ss,\chi}/T)$.

Proof. Let $\Lambda = \mathrm{End}_{R^T}(M(U))$ for $U = \bigoplus_{i \in I} \chi_i$. Put $\mathcal{E} = \bigoplus_{i \in I} \chi_i \otimes_k \mathcal{O}_{X^{ss,\chi}}$. Note that $\Lambda \cong M(\mathrm{End}(U)) \cong \bigoplus_{i,j \in I} M(\chi_i - \chi_j)$ as W is generic (see [ŠVdB17a, Lemma 4.1.3]). Proposition 4.11 shows that the functor

$$- \overset{L}{\otimes}_\Lambda \mathcal{E} : \mathrm{D}(\Lambda) \to \mathrm{D}(X^{ss,\chi}/T)$$

is fully faithful. As $\operatorname{gl\,dim} \Lambda < \infty$, $\mathrm{D}(\Lambda)$ is an admissible subcategory in $\mathrm{D}(X^{ss,\chi}/T)$. Since χ is generic $X^{ss,\chi}/T$ is a Deligne-Mumford stack (see e.g. [HLS16, Proposition

2.1], the proof goes through under the weak symmetry assumption). We now use [ŠVdB17c, Corollary A.5] which asserts that $\mathrm{D}(X^{ss,\chi}/T)$ has no nontrivial semi-orthogonal decomposition. Thus, $\mathrm{D}(\Lambda) \cong \mathrm{D}(X^{ss,\chi}/T)$. □

Proof of Theorem 1.3. The result follows immediately from Proposition 5.1. □

6. Example

If $\Delta \subset \mathbb{R}^n$ is a bounded closed convex polygon and $\varepsilon \in \mathbb{R}^n$ then $\Delta_\varepsilon = \bigcup_{r>0} \Delta \cap (r\varepsilon + \Delta)$. Assume W is quasi-symmetric and generic. The NCCRs that were constructed in [ŠVdB17a, Theorem 1.6.2.] are given by modules of covariants $M(U)$ where U is the sum of the characters contained in $1/2\bar{\Sigma}_\varepsilon \cap X(T)$ for a generic (in the sense of Definition 3.3) $\varepsilon \in X(T)$. One can check that the proofs hold true also if we replace $1/2\bar{\Sigma}_\varepsilon$ by $\nu + 1/2\overline{\Sigma}$ for $\nu \in X(T)_\mathbb{R}$ such that $(\nu + (1/2)\partial\overline{\Sigma}) \cap X(T) = \emptyset$.

In this section we give an example which shows that not all toric NCCRs come from $(\nu + 1/2\overline{\Sigma}) \cap X(T)$ (for ν as above), so Proposition 5.1 does not follow from [HLS16, Corollary 4.2, Remark 4.3].

We take $T = G_m$ and let $-3, -2, -2, 2, 2, 3$ be the weights of W. The set of Cohen-Macaulay modules of covariants is given by $\{i \mid -7 < i < 7\} \cup \{-8, 8\}$, which can be deduced from [VdB93] (c.f. §4.1). Note that $\Sigma = (-7, 7)$. Let U have weights $-4, -2, -1, 0, 1, 2, 4$. This set of weights is not an interval and hence it is not of the form $(\nu + 1/2\bar{\Sigma}) \cap X(T)$. However, it is a maximal Cohen-Macaulay clique (see (1) below). Thus Proposition 7.2 below implies that $\Lambda = \mathrm{End}_{R^T}(M(U))$ is an NCCR of R^T.

7. Toric NCCRs in the case $T = G_m$

In this section we give an explicit combinatorial criterion (based on Appendix A written by Jason Bell) for recognizing toric NCCRs in the case $T = G_m$, and prove that they are all related by a "mutation" procedure, which in particular gives a new proof of Theorem 1.3 in this case. Meanwhile we obtain some relations between NCCRs and "maximal modification algebras" which we recall first.

Definition 7.1. [IW14a] Let S be a noetherian Cohen-Macaulay ring. A reflexive S-module M is *modifying* if $\mathrm{End}_S(M)$ is Cohen-Macaulay. It is *maximal modifying* (MM) if it is modifying and if $M \oplus M'$ is modifying for a reflexive module M' then $M' \in \mathrm{addM}$ (i.e. M' is a direct summand of direct sums of M). If M is an MM module, then $\mathrm{End}_S(M)$ is a *maximal modification algebra* (MMA).

We consider a toric variant of this definition. We say that $\mathrm{End}_{R^T}(M)$ is a *toric MMA* if M is a module of covariants and maximal modifying with respect to modifying modules of covariants. Hence an MMA is automatically a toric MMA but the converse might not hold. See [ŠVdB17a, Example 10.1], [ŠVdB17b, Example 3.3] for a (counter)example.

An NCCR is always an MMA by [IW14a, Proposition 4.5] and the converse fails in general (see [VdB04b, Example A.1] and [IW14b, Theorem 4.16, Remark 4.17]). Surprisingly, for toric MMAs and NCCRs in the case $T = G_m$ there is no distinction.

Proposition 7.2. *(see §7.2.2 below) If $T = G_m$ and W is a generic and unimodular T-representations then toric MMAs and toric NCCRs of* $\mathrm{Sym}(W)^T$ *coincide.*

We next describe our strategy for an alternative proof of Theorem 1.3. Note that we may and we will assume that W has no zero weights since extra zero weights do not affect the NCCR property.

(1) We say that $S \subset X(T) \cong \mathbb{Z}$ is a *Cohen-Macaulay clique* [Boc12] if for every $i, j \in S$ the module of covariants $M(i-j)$ is Cohen-Macaulay. Note that toric MMAs correspond precisely to maximal Cohen-Macaulay cliques. A combinatorial argument provided by Jason Bell (see Appendix A) shows that each maximal Cohen-Macaulay clique contains exactly one element congruent to i for $0 \le i < N := \sum_{\alpha_i > 0} \alpha_i$.
(2) Using the complexes connecting projective (T, R)-modules [ŠVdB17c, §11] and (1) we show that for a toric MMA $\Lambda = \operatorname{End}_{R^T}(M)$ and for $\Lambda' = \operatorname{End}_{R^T}(M')$, where $M' = M(U')$ is obtained from $M = M(U)$ by replacing the highest weight $\mu_{\max}$ of U by $\mu_{\max} - N$, the (Λ, Λ')-bimodule $\operatorname{Hom}_{R^T}(M', M)$ defines a derived equivalence between Λ' and Λ.
(3) By induction on the maximum difference between the weights of U we can therefore, using (1), construct a derived equivalence between the "standard" NCCR $\operatorname{End}_{R^T}(\oplus_{0 \le \mu < N} M(\mu))$ [VdB04a, Theorem 8.9] and a toric MMA $\operatorname{End}_{R^T}(M(U))$.

7.1. **Toric MMAs.** In this section we deduce from Appendix A that all maximal Cohen-Macaulay cliques have the same size, and moreover that they have a particular form which will be of vital importance in §7.2.

We write $T^+ = T_1^+ = \{\alpha_i > 0\}$, $T^- = T_1^- = \{\alpha_i < 0\}$. Setting $\{a_i\}_i = \{\alpha_i\}_{i \in T^+} \cup \{-\alpha_i\}_{i \in T^-}$ and $N = \sum_{i \in T^+} \alpha_i = -\sum_{i \in T^-} \alpha_i$ we deduce from Lemma 4.1 and Theorem 4.2 that $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-$ in Appendix A corresponds to the set of non-Cohen-Macaulay weights (i.e. weights μ such that $M(\mu)$ is not Cohen-Macaulay).

Remark 7.3. Note that $\mathcal{S} = -\mathcal{S}$ so $\mathcal{M}$ is a Cohen-Macaulay clique if for all $i, j \in \mathcal{M}$ we have $M(i-j)$ or $M(j-i)$ is Cohen-Macaulay.

The following corollary is an immediate consequence of Corollary A.2.

Corollary 7.4. *If $\mathcal{M}$ is a maximal Cohen-Macaulay clique, then for every $0 \le i < N$ there exists a unique $m \in \mathcal{M}$ such that $m \equiv i\,(N)$.*

For the construction of derived equivalence via "mutation" we will also need the following easy corollary.

Corollary 7.5. *Let $\mathcal{M}$ be a maximal Cohen-Macaulay clique, let $m_{\max} = \max \mathcal{M}$ and let $\emptyset \ne S \subseteq T^-$. Then $m_{\max} + \sum_{i \in S} \alpha_i \in \mathcal{M}$ and $(\mathcal{M} \setminus \{m_{\max}\}) \cup \{m_{\max} - N\}$ is a maximal Cohen-Macaulay clique.*

Proof. By Corollary 7.4 and the maximality assumption we have that $m := m_{\max} + \sum_{i \in S} \alpha_i - kN \in \mathcal{M}$ for some $k \ge 0$. Since $m_{\max} - m = kN - \sum_{i \in S} \alpha_i \notin \mathcal{S}^+$ we must have $k = 0$.

For the second claim we need to show that $-N < m_{\max} - N - m \notin \mathcal{S}_+$ for $m \in \mathcal{M}$ by Remark 7.3. If $m_{\max} - N - m \in \mathcal{S}_+$ then also $m_{\max} - m = (m_{\max} - N - m) + N \in \mathcal{S}_+$, a contradiction. □

7.2. **Derived equivalence.** Let $\Lambda = \operatorname{End}_{R^T}(M(U))$ be a toric MMA. Up to Morita equivalence we may, and will, assume that every weight in U occurs with multiplicity 1. Let μ_{max} be the maximal weight of U and let U' be a representation given by replacing $\mu_{\max}$ in U by $\mu_{\max} - N$. We will refer to $\Lambda' = \operatorname{End}_{R^T}(M(U'))$ as a *mutation* of Λ.

In this section we show that a toric MMA and its mutation are derived equivalent. Repeating the mutation we obtain that a toric MMA is derived equivalent to the "standard" NCCR, providing an alternative proof of Theorem 1.3.

7.2.1. *Derived equivalence of a toric MMA and its mutation.* Let Λ be a noetherian ring. A finitely generated Λ-module X is *tilting* if

(1) $\operatorname{pdim}_\Lambda X < \infty$,
(2) $\operatorname{Ext}^i_\Lambda(X,X) = 0$ for $i > 0$,
(3) X is a generator of $D(\Lambda)$.

It is a classical result that Λ and $\operatorname{End}_\Lambda(X)$ are derived equivalent [Hap88].

Proposition 7.6. *Let notation be as above and denote $M = M(U)$, $M' = M(U')$. The (Λ,Λ')-bimodule $X = \operatorname{Hom}_{R^T}(M', M)$ is a* tilting *Λ'-module. In particular, Λ' and $\Lambda = \operatorname{End}_{\Lambda'}(X)$ are derived equivalent.*

Proof. We will verify the tilting conditions (1)-(3) by employing complexes used in [ŠVdB17a, §11.2] for constructing NCCRs (and also known as a part of Weyman's geometric method [Wey03]). Let $K_- = \operatorname{span}\{w_i \mid \alpha_i < 0\}$, $K_+ = \operatorname{span}\{w_i \mid \alpha_i > 0\}$, $d^\pm = \dim K_\pm = |T^\pm|$. The complexes are obtained from the Koszul resolutions $\mathcal{K}_\pm$ of $R_\pm = \operatorname{Sym}(W/K_\mp)$, which remain exact after tensoring with $\chi \in X(T)$:

$$0 \to \chi \otimes_k \wedge^{d^\mp} K_{\mp 1} \otimes_k R \to \cdots \to \chi \otimes_k K_{\mp 1} \otimes_k R \to \chi \otimes_k R \to \chi \otimes_k R_\pm \to 0. \tag{7.1}$$

To show (1) let us recall that the indecomposable projective right Λ'-modules are of the form $\operatorname{Hom}_{R^T}(M', M(\mu))$ where μ is a weight of U'. Let $\mu_1 < \cdots < \mu_\ell = \mu_{\max}$ be the weights of U. Since $X = \oplus_{i<\ell} \operatorname{Hom}_{R^T}(M', M(\mu_i)) \oplus \operatorname{Hom}_{R^T}(M', M(\mu_\ell))$, it is enough to show that $\operatorname{pdim}_{\Lambda'} \operatorname{Hom}_{R^T}(M', M(\mu_\ell))$ is finite.

We use (7.1) with $\mathcal{K}_+$ and $\chi = \mu_\ell$. Note that $\operatorname{Hom}_R(U' \otimes_k R, \mu_\ell \otimes_k R_+)^T = 0$ since otherwise $-\mu + \mu_\ell + \sum_{i \in T^+} a_i \alpha_i = 0$ for some weight μ of U' and $a_i \in \mathbb{N}$, and consequently $\mu - \mu_\ell + N = N + \sum_{i \in T^+} a_i \alpha_i \in \mathcal{S}_+$, which contradicts the fact that the set of weights of U' is a Cohen-Macaulay clique by Corollary 7.5. Thus, applying $\operatorname{Hom}_R(U' \otimes_k R, -)^T$ to this complex we obtain, using Corollary 7.5, a Λ'-projective resolution $Q^\bullet$ of $\operatorname{Hom}_{R^T}(M', M(\mu_\ell))$. Hence (1) follows. Moreover, from $Q^\bullet$ we also obtain (3).

For (2) we need to show that $\operatorname{RHom}_{\Lambda'}(X,X)$ has cohomology only in degree 0. Since $\operatorname{Hom}_{R^T}(M', M(\mu_i))$, $i < \ell$, are projective Λ'-modules, it suffices to show that $\operatorname{RHom}_{\Lambda'}(\operatorname{Hom}_{R^T}(M', M(\mu_\ell)), \operatorname{Hom}_{R^T}(M', M(\mu_i)))$ has cohomology only in degree 0. We can replace $\operatorname{Hom}_{R^T}(M', M(\mu_\ell))$ by $Q^\bullet = \operatorname{Hom}_{R^T}(M', \sigma_{<0}(\mathcal{K}_+ \otimes_k \mu_\ell[-1])^T)$. We have

$$\begin{aligned}
&\operatorname{RHom}_{\Lambda'}(\operatorname{Hom}_{R^T}(M', M(\mu_\ell)), \operatorname{Hom}_{R^T}(M', M(\mu_i)))\\
&\qquad = \operatorname{Hom}_{\Lambda'}(Q^\bullet, \operatorname{Hom}_{R^T}(M', M(\mu_i)))\\
&\qquad = \operatorname{Hom}_{\Lambda'}(\operatorname{Hom}_{R^T}(M', \sigma_{<0}(\mathcal{K}_+ \otimes_k \mu_\ell[-1])^T), \operatorname{Hom}_{R^T}(M', M(\mu_i)))\\
&\qquad = \operatorname{Hom}_{R^T}(\sigma_{<0}(\mathcal{K}_+ \otimes_k \mu_\ell[-1])^T, M(\mu_i)))\\
&\qquad = \sigma_{>0}(\operatorname{Hom}_{R^T}((\mathcal{K}_+ \otimes_k \mu_\ell)^T, M(\mu_i))[1])\\
&\qquad = \sigma_{>0} \operatorname{Hom}_R(\mathcal{K}_+, R \otimes_k (-\mu_\ell + \mu_i))^T[1]\\
&\qquad = \sigma_{>0}(\wedge^{d_-} K_-^* \otimes_k \mathcal{K}_+[-d] \otimes_k (-\mu_\ell + \mu_i))^T[1]
\end{aligned}$$

which is exact in degrees > 0 since R_+ is the cohomology of $\mathcal{K}_+$ and

$$((\wedge^{d_-} K_-)^* \otimes_k R_+ \otimes_k (-\mu_\ell + \mu_i))^T = (R_+ \otimes_k (N - \mu_\ell + \mu_i))^T = 0$$

as otherwise $\sum_{i\in T^+} a_i\alpha_i + N - \mu_\ell + \mu_i = 0$ for some $a_i \in \mathbb{N}$ which contradicts the fact that $\mu_\ell - \mu_i \notin \mathcal{S}_+$ (as the set of weights of U is a Cohen-Macaulay clique). □

7.2.2. *Proof of Theorem 1.3 and Proposition 7.2.* Let $\Lambda = \operatorname{End}_{R^T}(M(U))$ be a toric MMA. Recall that every (toric) NCCR is a (toric) MMA. By Morita equivalence we can assume that every weight of U appears with multiplicity 1. Let $\mathcal{L}$ be the set of weights of U. By translation we can assume $0 \in \mathcal{L} \subset \mathbb{N}$. We argue by induction on $\max \mathcal{L}$ that Λ is derived equivalent to $\Lambda_0 = \operatorname{End}_{R^T}(M(U_0))$ where the weights of U_0 are given by $\mathcal{L}_0 = [0, N-1]$.

By Corollary 7.4, $\max \mathcal{L} \geq N-1$, and if $\max \mathcal{L} = N-1$ then $\Lambda = \Lambda_0$. Thus, we can assume that $\max \mathcal{L} \geq N$. Let $\Lambda' = \operatorname{End}_{R^T}(U')$ be a mutation of Λ, and let $\mathcal{L}'$ be set of weights of U'. Then Λ' is a toric MMA by Corollary 7.5, with $0 \in \mathcal{L}' \subset \mathbb{N}$ and $\max \mathcal{L}' < \max \mathcal{L}$. By induction, Λ' is derived equivalent to Λ_0. We can use Proposition 7.6 to conclude that Λ is derived equivalent to Λ' and thus to Λ_0. In particular, Λ is an NCCR, proving also Proposition 7.2.

Appendix A. Appendix by Jason P. Bell

By §4.1, understanding the maximal Cohen-Macaulay cliques reduces to a purely combinatorial problem which is a subject of this section. Let $a_1, \ldots, a_d \in \mathbb{N}_{>0}$. In addition, we assume that $\gcd(a_i) = 1$. As a consequence every sufficiently large natural number can be expressed as a linear combination of the a_i with nonnegative integer coefficients. Let $N \in \mathbb{N}$. We define

$$\mathcal{S}_+ := \left\{ N + \sum_i c_i a_i \colon c_i \geq 0 \right\}, \quad \mathcal{S}_- = -\mathcal{S}_+, \quad \mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-.$$

Then $\mathcal{S}$ contains all but finitely many integers.

Let

$$\mathfrak{M} = \{\mathcal{M} \subset \mathbb{Z} \mid m, m' \in \mathcal{M} \implies m - m' \notin \mathcal{S}\}.$$

Given $i \in \{0, \ldots, N-1\}$, we define $p(i)$ to be the smallest positive integer p such that $i + pN$ is in $\mathcal{S}_+$. Similarly, $q(i)$ is the largest negative integer q such that $i + qN$ is in $\mathcal{S}_-$. Let $\mathcal{M} \in \mathfrak{M}$ and let $j \in \mathbb{Z} \setminus \mathcal{M}$. We say that $m \in \mathcal{M}$ *blocks* j if $j - m \in \mathcal{S}$. Notice that if $\mathcal{M}$ is a maximal element of $\mathfrak{M}$ then for each element in the complement of $\mathcal{M}$ there is necessarily some element of $\mathcal{M}$ that blocks it.

Lemma A.1. *Let $\mathcal{M} \in \mathfrak{M}$ be a maximal element of $\mathfrak{M}$ containing 0 and let $i \in \{0, \ldots, N-1\}$. Then $\mathcal{M}$ contains an element that is congruent to i modulo N.*

Proof. Suppose not. Then for each $j \in \{q(i), \ldots, p(i)\}$ we can choose some integer $m_j \in \mathcal{M}$ such that m_j blocks $i + jN$. Since both $i + p(i)N$ and $i + q(i)N$ are in $\mathcal{S}$, we can take $m_{p(i)} = m_{q(i)} = 0$. Now let

$$X_\pm = \{j \in \{q(i), \ldots, p(i)\} \colon m_j - (i + jN) \in \mathcal{S}_\pm\}.$$

Then X_+ and X_- are disjoint and their union is all of $\{q(i), \ldots, p(i)\}$. Moreover, since $m_{p(i)} = m_{q(i)} = 0$ and $q(i) < 0 < p(i)$, we have $q(i) \in X_+$ and $p(i) \in X_-$. In particular, there must exist some $j \in \{q(i), \ldots, p(i)-1\}$ such that $j \in X_+$ and $j+1 \in X_-$. Given such a j, we then have

$$m_j - (i + jN) \in \mathcal{S}_+ \qquad \text{and} \qquad m_{j+1} - (i + (j+1)N) \in \mathcal{S}_-.$$

So we may write $m_j-(i+jN)=N+k_1$ and $m_{j+1}-(i+(j+1)N)=-N-k_2$, where k_1 and k_2 are $\mathbb{N}$-linear combinations of the a_i. Subtracting these two equalities, we see

$$m_j-m_{j+1}+N=(m_j-(i+jN))-(m_{j+1}-(i+(j+1)N))=2N+k_1+k_2.$$

In particular, $m_j-m_{j+1}=N+k_1+k_2\in\mathcal{S}_+$. But this contradicts the fact that $m_j,m_{j+1}\in\mathcal{M}\in\mathfrak{M}$. The result follows. □

Corollary A.2. *Let N be a $\mathbb{N}$-linear combination of a_i. Let $\mathcal{M}$ be a maximal element in $\mathfrak{M}$. For every $0\le i<N$ there exists exactly one element $m\in\mathcal{M}$ such that $m\equiv i\,(N)$. In particular, all maximal elements of $\mathfrak{M}$ have size N.*

Proof. Let $\mathcal{M}$ be a maximal element of $\mathfrak{M}$. By translation we can assume that $0\in\mathcal{M}$. By Lemma A.1 we have that for each $i\in\{0,\dots,N-1\}$ there is some element of $\mathcal{M}$ that is congruent to i mod N. Thus $|\mathcal{M}|\ge N$. On the other hand, by definition of $\mathcal{S}_+$ and the assumption on N we have $\{N,2N,3N,\dots\}\subseteq\mathcal{S}$. If $\mathcal{M}$ had size strictly larger than N then there would exist $m,m'\in\mathcal{M}$ with $m>m'$ and m and m' congruent to 0 mod N. But then $m-m'$ is a positive multiple of N and hence in $\mathcal{S}$. □

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