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# On Lie Algebras Having a Primitive Universal Enveloping Algebra

ALFONS I. OOMS

*Department of Mathematics, University of Rochester, Rochester, New York 14627*

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## 1. INTRODUCTION

In his book "Structure of Rings" [7, p. 23] Professor Jacobson raised the following open question: "What are the conditions on a finite dimensional Lie algebra $L$ over a field $k$ that insure that its universal enveloping algebra $U(L)$ is primitive?" [Since $U(L)$ has an anti-automorphism the notions left and right primitive are the same for $U(L)$.]

If $k$ is of characteristic $p \neq 0$, then $U(L)$ cannot be primitive unless $L = 0$ [7, p. 255]. Therefore we may assume from now on that $L$ is a nonzero finite dimensional Lie algebra over a field $k$ of characteristic zero. For each linear functional $f \in L^*$ we denote by $L[f]$ the set of all $x \in L$ such that $f(Ex) = 0$ for all $E$ in the algebraic hull of $\operatorname{ad} L \subset \operatorname{End} L$. Clearly $L[f]$ is a Lie subalgebra of $L$ containing the center $Z(L)$ of $L$.

The aim of this paper is to prove the following.

THEOREM. *If $U(L)$ is primitive then $L[f] = 0$ for some $f \in L^*$. Moreover, the converse holds if $L$ is solvable and $k$ is algebraically closed.*

If we denote by $D(L)$ the division ring of quotients of $U(L)$, $Z(D(L))$ its center, we shall prove that the condition that $L[f] = 0$ for some $f \in L^*$ is equivalent with $Z(D(L)) = k$ (which forces the centers of both $L$ and $U(L)$ to be trivial). In particular, $U(L)$ cannot be primitive if $L$ is either nilpotent or semi-simple. Finally, we shall give some examples of Lie algebras (of which one is not solvable) that do have a primitive universal enveloping algebra.

## 2. SOME RESULTS ON PRIMITIVE RINGS

PROPOSITION 1. *Let $U$ be a primitive ring with 1, contained as a subring in a ring $Q$. Suppose $C$ is a subring of $Q$ such that $[C, U] = 0$ and such that for* *each nonzero element $c$ of $C$ there exist nonzero elements $x, y$ in $U$ such that $cx = y$. If $V$ is a faithful, irreducible $U$-module then $C$ is isomorphic to a subring of the center of the division ring $\Delta = \mathrm{End}_U V$.*

*Proof.* Professor Martindale has shown this result in case $Q$ is the complete ring of right quotients of $U$ and $C$ is the center of $Q$. [8, p. 453]. However, exactly the same proof works also in the situation above.

PROPOSITION 2. *Let $Q$ be an associative $k$-algebra with 1 and $U \subset Q$ a primitive subalgebra, $1 \in U$. Suppose $C \subset Q$ is a subalgebra of $Q$ such that $[C, U] = 0$ and such that for each nonzero element $c$ of $C$ there exist nonzero elements $x, y$ of $U$ such that $cx = y$. Then $C$ is algebraic over $k$ if one of the following two conditions is satisfied.*

(1) $\dim_k U < \mathrm{card}\, k$,

(2) *$U$ is the union of an increasing filtration $U_0 \subset U_1 \subset \cdots$ of subspaces such that $1 \in U_0$, $U_p U_q \subset U_{p+q}$ and such that the associated graded algebra $\mathrm{gr}(U) = \bigoplus_p U_{p/U_{p-1}}$ is a finitely generated commutative $k$-algebra.*

*Proof.* We may regard $U$ as an irreducible ring of endomorphisms of a vector space $V$ over $k$. Since $k$ can be considered as a subfield of the division ring $\Delta = \mathrm{End}_U V$, it is easy to check that the isomorphism we have established in the preceding proposition between $C$ and a subring of the center $Z(\Delta)$ is in fact a $k$-isomorphism. The result then follows immediately, since each one of the conditions 1, 2 implies that $\Delta$ is algebraic over $k$. (See [10].)

LEMMA 1. *Let $k$ be a commutative integral domain, $Q$ an associative $k$-algebra and $U$ a subalgebra endowed with an increasing filtration of $k$-submodules $U_0 = k \cdot 1 \subset U_1 \subset U_2 \subset \cdots$ with $U$ as their union, $U_p U_q \subset U_{p+q}$ and such that the associated graded algebra $\mathrm{gr}(U)$ is a unique factorization domain (U.F.D.). Suppose $c \in Q$ is an element for which there exist nonzero elements $x, y$ in $U$ such that $cx = y$ and $[c, x] = 0$. If $c$ is algebraic over $k$ then it follows that $(ac - b)x = 0$ for some nonzero $a, b \in k$. (So, in case $x$ is regular in $Q$, then we may consider $c$ as being an element of the quotient field of $k$.)*

*Proof.* We notice that since $\mathrm{gr}(U)$ is an integral domain, so is $U$. [5, p. 7]. Suppose $a_n c^n + a_{n-1} c^{n-1} + \cdots + a_1 c + a_0 = 0$ for some $a_i \in k$, $a_n \neq 0$. Multiplying by $x^n$ and taking into account that $[c, x] = 0$, we obtain that

$$a_n (cx)^n + a_{n-1} (cx)^{n-1} x + \cdots + a_1 (cx)\, x^{n-1} + a_0 x^n = 0.$$

Hence

$$a_n y^n + a_{n-1} y^{n-1} x + \cdots + a_1 y x^{n-1} + a_0 x^n = 0. \tag{1}$$

We may assume that $a_0 \neq 0$. (Indeed, since $y$ is not 0, at least one of the $a_i$ other than $a_n$ is $\neq 0$. Let $a_i$ be the last nonzero coefficient,

$$a_n y^n + a_{n-1} y^{n-1} x + \cdots + a_i y^i x^{n-i} = 0,$$

and by cancelling out the common factor $y^i$ we obtain a relation similar to (1).) We recall that if $u \in U_n \backslash U_{n-1}$ we define $\deg u = n$ and $[u] = u \bmod U_{n-1}$. It is well known that $\deg(uv) = \deg u + \deg v$ and $[uv] = [u][v]$ for $u, v \in U$. (1) implies that $\deg x = \deg y$. Indeed, if $\deg y > \deg x$ the left-hand side of

$$a_n y^n = -(a_{n-1} y^{n-1} x + \cdots + a_0 x^n)$$

is clearly of degree $n \cdot \deg y$, while each term of the opposite side would be of a lower degree. A similar reasoning shows that $\deg y < \deg x$ cannot occur either. Hence $\deg x = \deg y$ and therefore (1) implies that

$$a_n[y]^n + a_{n-1}[y]^{n-1}[x] + \cdots + a_1[y][x]^{n-1} + a_0[x]^n = 0.$$

Let $g$ be a greatest common divisor of $[x]$ and $[y]$ in $\operatorname{gr}(U)$. We may write that $[x] = gu$ and $[y] = gv$ where $u$ and $v$ are nonzero relatively prime elements of $\operatorname{gr}(U)$. After cancelling the factor $g^n$, we obtain that

$$a_n v^n + a_{n-1} v^{n-1} u + \cdots + a_1 v u^{n-1} + a_0 u^n = 0.$$

Clearly $u$ divides $a_n v^n$ and since $u$, $v$ are relatively prime, $u$ also divides $a_n$. Hence $u \in k$ and similarly $v \in k$. Then the fact that $u[y] = v[x]$ forces $uy$ and $vx$ to have the same leading term. In particular, $\deg(uy - vx) < \deg x$. Finally, we have that $(uc - v)x = uy - vx$. Clearly $uc - v$ is algebraic over $k$ and commutes with $x$. Hence $uy - vx = 0$ (otherwise it would follow as before that $\deg(uy - vx) = \deg x$) and therefore

$$(uc - v)x = 0, \text{ where } u, v \text{ are nonzero elements of } k.$$

Combining this Lemma with Proposition 2, we obtain.

THEOREM 1. *Let $U$ be a primitive associative algebra over the field $k$, endowed with an increasing filtration $U_0 = k \cdot 1 \subset U_1 \subset \cdots$ such that the associated graded algebra* $\operatorname{gr}(U)$ *is a finitely generated commutative $k$-algebra. Then the center $C$ of the ring of quotients of $U$ is a field algebraic over $k$. Moreover, $C = k$ if* $\operatorname{gr}(U)$ *is in addition a unique factorization domain.*

Since $\operatorname{gr}(U)$ is left and right Noetherian, so is $U$ [5, p. 7]. Thus $U$ is a Goldie ring and therefore has a left and right ring of quotients.

## 3. Application of the Preceding Section to $U(L)$

The universal enveloping algebra $U(L)$ of a Lie algebra $L$ has a natural increasing filtration of which the associated graded algebra $\mathrm{gr}(U(L))$ is isomorphic to the (commutative) polynomial algebra $k[X_1,..., X_n]$, $n = \dim L$, by the Poincaré–Birkhoff–Witt theorem. Therefore the following is an immediate consequence of Theorem 1.

Proposition 3. *Let $L$ be a Lie algebra over $k$ and $I$ a primitive ideal of $U(L)$. Then the center $C$ of the ring of quotients of $U(L)/I$ is algebraic over $k$.*

*Remark.* Clearly, $C = k$ if $k$ is algebraically closed, a result already shown by Rais [11] and which is a slight improvement of a theorem due to Dixmier [3] by removing the requirement of the uncountability of $k$. On the other hand, there are cases where $C \neq k$. The following example was pointed out to us by Professor Seligman. Let $L$ be the 1-dimensional real Lie algebra generated by $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, which acts irreducibly on the plane. $U(L)$ can be identified with $\mathbf{R}[X]$ and the kernel $I$ of the representation is the ideal generated by $X^2 + 1$. Therefore $U(L)/I \simeq \mathbf{C}$ and $C = \mathbf{C}$.

Theorem 2. *Let $L$ be a Lie algebra over $k$. If $U(L)$ is primitive then $Z(D(L)) = k$. Moreover, the converse holds if $L$ is solvable and $k$ is algebraically closed.*

*Proof.* The first part follows easily from Theorem 1. For the converse we observe that since $U(L)$ is semi-simple [7, p. 22] the intersection of all primitive ideals of $U(L)$ is 0. For this reason, Dixmier's argument used in the proof of his well-known criterion for the primitive ideals of $U(L)$, works in this situation without requiring the uncountability of $k$. Indeed, if $Z(D(L)) = k$ then Lemma 3.4 of [3, p. 28] guarantees that the intersection of all nonzero prime ideals of $U(L)$ is not 0. In particular the same is true for all nonzero primitive ideals. Hence (0) is necessarily a primitive ideal.

## Condition in Terms of the Algebra Structure on $L$, Equivalent with $Z(D(L)) = k$

Let $L$ be a Lie algebra over $k$ and let $H$ be the algebraic hull of $\mathrm{ad}\, L$ in $\mathrm{End}\, L$. For each linear functional $f \in L^*$, we have defined $L[f]$ to be the collection of elements $x \in L$ such that $f(Ex) = 0$ for all $E \in H$. We have that $L[f]$ is an ideal of $L(f)$, $L(f)$ being the radical of the alternating bilinear form $(x, y) \to f([x, y])$ on $L$. Since $L(f)$ is abelian for $f$ lying in some open dense subset $0$ of $L^*$, the same is true for $L[f]$. ($0$ is the set of all $f \in L^*$ for which

$L(f)$ has minimum dimension [1, p. 17]. It is obvious that $L[f] = L(f)$ if $L$ is ad-algebraic (i.e., $\operatorname{ad} L = H$).

We recall that each endomorphism $E \in \operatorname{End} L$ can uniquely be extended to a derivation of the quotient field $K(L)$ of the symmetric algebra $S(L)$ of $L$. We are interested in the subfield $K(L)^I$ of the invariants of $K(L)$ with respect to $\operatorname{ad} L$ (i.e., $K(L)^I$ is the collection of elements of $K(L)$ annihilated by all $E \in \operatorname{ad} L$). It can be shown that $\operatorname{ad} L$ and its algebraic hull $H$ have the same invariants in $K(L)$ [9, p. 25]. Because of this and Dixmier's formula for the transcendency degree of the invariants of an algebraic Lie algebra of endomorphisms [2, p. 336] we obtain that:

$$\operatorname{tr} \deg_k(K(L)^I) = \dim L - \operatorname{rank}_{K(L)}((E_i x_j)_{ij}),$$

whenever $\{x_1, \ldots, x_n\}$ is a basis for $L$ and $\{E_1, \ldots, E_r\}$ a basis for $H$. Since for each $f \in L^*$

$$\dim L[f] = \dim L - \operatorname{rank}(f(E_i x_j)_{ij})$$

we may conclude that

$$\operatorname{tr} \deg_k(K(L)^I) = \min_{f \in L^*} \dim L[f].$$

(In fact, it can be shown that this number is also equal to the transcendency degree of $Z(D(L))$ over $k$ in case $L$ is either solvable or ad-algebraic [9].)

The following is the main tool of this section.

PROPOSITION 4. *Let $L$ be a Lie algebra over $k$, then the following are equivalent*:

(1) $Z(D(L)) = D(Z(L))$,

(2) $K(L)^I = K(Z(L))$,

(3) $L[f] = Z(L)$ *for some* $f \in L^*$.

Before we can go over to the proof of this, we need to introduce an increasing filtration in $U(L)$, other than the usual one. We denote by $s$ the canonical linear isomorphism of $S(L)$ onto $U(L)$, which for every system $y_1, \ldots, y_m$ of $L$ maps the product $y_1 \cdots y_m$ into $(1/m!) \sum_p y_{p(1)} \cdots y_{p(m)}$, where the sum is taken over all permutations $p$ of $\{1, \ldots, m\}$. Let $\{e_1, \ldots, e_c; x_1, \ldots, x_n\}$ be a basis for $L$ such that $\{e_1, \ldots, e_c\}$ is a basis for the center $Z(L)$. Put $R = S(Z(L))$. Obviously, $R$ can be identified with $U(Z(L))$ since $Z(L)$ is commutative. In particular $R \subset Z(U(L))$. Each element of $S(L)$ can be considered as a polynomial in the $x_i$'s with coefficients in $R$ (i.e., $S(L) \simeq R[X_1, \ldots, X_n]$). Clearly $S(L)$ is the direct sum of the subspaces $S^m$ of polynomials homogeneous of degree $m$ in the $x_i$'s. We have that $S^m S^t \subset S^{m+t}$ for all positive integers $m$, $t$.

On the other hand $s(ax) = as(x)$ for all $a \in R$, $x \in S(L)$. (This is clear if $a \in Z(L)$, hence also if $a$ is of the form $y_1 \cdots y_m$, $y_i \in Z(L)$; the general case follows by linearity of $s$.) As a result $s$ may be considered as an isomorphism of $R$-modules. $U(L)$ is the direct sum of the subspaces $U^m$, $U^m$ being the image of $S^m$ under $s$. Next put $U_q = \bigoplus_{m \leqslant q} U^m$. It is easy to verify that the monomials $x_{i_1} \cdots x_{i_p}$ with $i_1 \leqslant \cdots \leqslant i_p$ and $p \leqslant q$ form a basis of $U_q$ over $R$ and $U_q U_t \subset U_{q+t}$. Therefore the subspaces $U_q$ form an increasing filtration in $U(L)$ and the associated graded algebra $\mathrm{gr}(U(L))$ is isomorphic to $R[X_1, ..., X_n] \simeq S(L)$. The elements $u \in U_q \backslash U_{q-1}$ are said to be of degree $q$ and $[u] = u \bmod U_{q-1}$ is called the leading term of $u$. For all nonzero $u, v \in U(L)$: $[uv] = [u][v]$ and $\deg(uv) = \deg(u) + \deg(v)$. Furthermore, if $y = y_m + \cdots + y_0$, $y_m \neq 0$ is the decomposition of $y \in S(L)$ into homogeneous components ($y_i \in S^i$) then it follows immediately from the definition of $s$ that $[s(y)] = y_m$.

Finally, we recall that each derivation $E$ of $L$ can uniquely be extended to a derivation of $S(L)$ (and $K(L)$) on the one hand and to a derivation of $U(L)$ (and $D(L)$) on the other hand. If we denote both extensions by $E$ again then the diagram

$$\begin{array}{ccc} S(L) & \xrightarrow{E} & S(L) \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} \\ U(L) & \xrightarrow{E} & U(L) \end{array}$$

is commutative. This implies in particular that $s\colon S(L)^I \to Z(U(L))$ is a linear bijection. Moreover each $E \in \mathrm{ad}\, L$ maps $S^m$ into itself and the same is true for $U^m$ and $U_m$.

*Proof of the Proposition*

We note first that $Z(D(L)) \supset D(Z(L))$, $K(L)^I \supset K(Z(L))$ and $L[f] \supset Z(L)$ for all $f \in L^*$.

$1 \Rightarrow 2$.

Take $u \in K(L)^I$. We may assume that $u = xy^{-1}$, $y \neq 0$, such that $x, y$ are relatively prime in $S(L)$. $Eu = 0$ for all $E \in \mathrm{ad}\, L$ implies that $yEx = xEy$. Since $x$ and $y$ are relatively prime and since $\deg(Ex) \leqslant \deg x$ we obtain that $Ex = \lambda(E)x$ and $Ey = \lambda(E)y$ for some $\lambda \in (\mathrm{ad}\, L)^*$. It follows that $Es(x) = s(Ex) = \lambda(E)\, s(x)$ and similarly $Es(y) = \lambda(E)\, s(y)$ for all $E \in \mathrm{ad}\, L$. Next put $z = s(x)\, s(y)^{-1} \in D(L)$. For each $E \in \mathrm{ad}\, L$:

$$Ez = (Es(x) - s(x)\, s(y)^{-1} Es(y))\, s(y)^{-1} = 0.$$

Hence $z \in Z(D(L)) = D(Z(L))$. Consequently $z = b^{-1}a$ for some $a, b \in R$,

$b \neq 0$. ($R = U(Z(L))$). But $s(x)\, s(y)^{-1} = b^{-1}a$ implies that $s(ay) = as(y) = bs(x) = s(bx)$. Therefore $ay = bx$ and $u = xy^{-1} = ab^{-1} \in K(Z(L))$. Hence $K(L)^I = K(Z(L))$.

$2 \Rightarrow 1$.

We remark that $K(L)^I = K(Z(L))$ implies that $S(L)^I = S(Z(L))$. (Indeed let $u \in S(L)^I$; $u = ab^{-1}$ for some $a, b \in S(Z(L)) = R$, $b \neq 0$; hence $a = bu$ which forces the degree of $u$ to be 0 and $u \in R$.) By taking the image under $s$ we see that also $Z(U(L)) = U(Z(L)) = R$. Now let $z$ be a nonzero element of $Z(D(L))$. We define $d(z) = \min\{\deg u \mid z = uv^{-1} u, v \in U(L), v \neq 0\}$. We shall prove by induction on $d(z)$ that $z \in D(Z(L))$. Let $u, v$ be nonzero elements of $U(L)$ such that $z = uv^{-1}$ and $\deg u = d(z)$.

If $d(z) = 0$ then clearly $u \in R = U(Z(L))$ and therefore

$$v = uz^{-1} \in Z(U(L)) = U(Z(L)).$$

Consequently $z = uv^{-1} \in D(Z(L))$. So, we may assume that $d(z) = n > 0$. Since $z$ commutes in particular with $v$, we see that $z = v^{-1}u$. Take $E \in \operatorname{ad} L$. Since $Ez = 0$, we obtain from $zv = u$ that $zEv = Eu$. Hence $uEv = vEu$. Choose $x, y$ in $S(L)$ such that $s(x) = u$ and $s(y) = v$. Let $x = x_n + \cdots + x_0$, $x_n \neq 0$ and $y = y_m + \cdots + y_0$, $y_m \neq 0$ be their decomposition into homogeneous components ($x_i \in S^i$, $y_j \in S^j$). Since each $E \in \operatorname{ad} L$ maps each $S^i$ into itself we see that $Ex = Ex_n + \cdots + Ex_0$ and $Ey = Ey_m + \cdots + Ey_0$ are the decompositions into homogeneous components of $Ex$ and $Ey$. Therefore if $Ex = 0$ then each $Ex_i = 0$, similarly for $Ey$. Next we observe that $s(x)\, s(Ey) = s(x)\, Es(y) = uEv = vEu = s(y)\, Es(x) = s(y)\, s(Ex)$. From this we see in particular that $Ex = 0$ if and only if $Ey = 0$. Denote by $Ex_q$ and $Ey_p$ the leading (nonzero) terms of $Ex$ and $Ey$ in case $Ex \neq 0$. Then $[s(x)][s(Ey)] = [s(y)][s(Ex)]$ implies that $x_n Ey_p = y_m Ex_q$ and $n + p = m + q$ (by taking degrees of both sides). We see that $Ex_n = 0$ if and only if $Ey_m = 0$. In any case we have that $x_n Ey_m = y_m Ex_n$ for all $E \in \operatorname{ad} L$. This forces the element $x_n y_m^{-1} \in K(L)$ to be annihilated by all $E \in \operatorname{ad} L$, i.e., $x_n y_m^{-1} \in K(L)^I = K(Z(L))$. Hence $x_n y_m^{-1} = ab^{-1}$ for some nonzero $a, b \in R$. Considering that $[bu] = b[s(x)] = bx_n = ay_m = a[s(y)] = [av]$ we conclude that $\deg(bu - av) < \deg(bu) = \deg u = d(z)$. Next put $z_1 = (bu - av)(av)^{-1} = ba^{-1}z - 1 \in Z(D(L))$. By induction $z_1 \in D(Z(L))$ since $d(z_1) < d(z)$. Consequently

$$z = ab^{-1}(z_1 + 1) \in D(Z(L)) \qquad \text{and} \qquad Z(D(L)) = D(Z(L)).$$

$2 \Leftrightarrow 3$.

We know that

$$\operatorname{tr}\deg_k(K(L)^I) = \min_{f \in L^*} \dim L[f] \qquad \text{and} \qquad K(Z(L)) \simeq k(X_1, \ldots, X_c)$$

with $c = \dim Z(L)$. Hence $2 \Rightarrow 3$ is clear. Conversely, if $L[f] = Z(L)$ for some $f \in L^*$, then $c$ is the degree of transcendence over $k$ of both the fields $K(L)^I$ and $K(Z(L))$. Hence $K(L)^I$ is algebraic over $K(Z(L))$. Therefore each element $u$ of $K(L)^I$ satisfies a nontrivial equation of the form

$$a_m X^m + \cdots + a_0 = 0,$$

$a_i \in K(Z(L))$. By multiplying this with a common denominator of the $a_i$, we see that we may assume that all $a_i \in S(Z(L)) = R$. Consequently $u$ is algebraic over $R$. By Lemma 1 it follows that $u$ is in the quotient field of $R$, which is $K(Z(L))$. Hence $K(L)^I = K(Z(L))$.

COROLLARY 1. *Let $L$ be a Lie algebra over $k$, for which there exists a linear functional $f \in L^*$ such that $L[f] = Z(L)$. Then $Z(U(L)) = U(Z(L))$ and is therefore isomorphic to $k[X_1, \ldots, X_c]$ with $c = \dim Z(L)$; its quotient field is $Z(D(L))$, which is in fact equal to $K(L)^I$.*

COROLLARY 2. *Let $L$ be a Lie algebra over $k$. Then the following are equivalent*:

(1) $Z(D(L)) = k$,

(2) $K(L)^I = k$,

(3) $L[f] = 0$ *for some* $f \in L^*$.

This corollary combined with Theorem 2 yields the main result, announced in the Introduction.

THEOREM 3. *Let $L$ be a Lie algebra over $k$. If $U(L)$ is primitive then $L[f] = 0$ for some $f \in L^*$. Moreover, the converse holds if $L$ is solvable and $k$ is algebraically closed.*

## EXAMPLES

1. Let $L$ be the Lie algebra over an algebraically closed field $k$ with basis $\{x_1, \ldots, x_n; x_{n+1}\}$ and with the following nonvanishing brackets: $[x_{n+1}, x_i] = a_i x_i$, $a_i \in k$ $i$: $1, \ldots, n$. Then $U(L)$ is primitive if and only if $a_1, \ldots, a_n$ are linearly independent over $\mathbf{Q}$.

*Proof.* Let $N$ be the commutative ideal of $L$ with basis $\{x_1, \ldots, x_n\}$. Clearly $\operatorname{ad} N \subset \operatorname{End} L$ is an algebraic Lie algebra, consisting of nilpotent endomorphisms. Put $E_i = \operatorname{ad} x_i$, $i$: $1, \ldots, n+1$ and denote by $H_1$ the collection of replicas in $\operatorname{End} L$ of $E_{n+1} = \operatorname{ad} x_{n+1}$. Since $E_{n+1}$ is diagonal with respect to the given basis, so is each element of $H_1$. The dimension of $H_1$ is

$\dim_{\mathbf{Q}} \sum_{i=1}^{n} a_i\mathbf{Q}$. (See for example [12]). Let $\{E_{n+1},\ldots, E_{n+p}\}$ be a basis for $H_1$. Since $[H_1, \operatorname{ad} N] \subset \operatorname{ad} N$, $H = H_1 \oplus \operatorname{ad} N$ is the algebraic hull of $\operatorname{ad} L$ in $\operatorname{End} L$ and $\{E_1,\ldots, E_{n+p}\}$ is a basis for $H$. On the other hand, $\min_{f\in L^*} \dim L[f] = \dim L - \operatorname{rank}_{K(L)}((E_i x_j)_{ij})$. Since $E_{n+i}x_j = a_{ij}x_j$ for some $a_{ij} \in k$ ($a_{1j} = a_j$) we have that

$$(E_i x_j)_{ij} = \begin{bmatrix} 0 & \cdots & 0 & -a_1x_1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & -a_nx_n \\ a_{11}x_1 & \cdots & a_{1n}x_n & 0 \\ \vdots & & \vdots & \vdots \\ a_{p1}x_1 & \cdots & a_{pn}x_n & 0 \end{bmatrix}$$

its rank is $p + 1$ since the last $p$ rows are $K(L)$-linearly independent ($E_{n+1},\ldots, E_{n+p}$ are $k$-linearly independent). Hence,

$$\min_{f\in L^*} \dim L[f] = (n + 1) - (p + 1) = n - p = n - \dim_{\mathbf{Q}} \sum_{i=1}^{n} a_i\mathbf{Q}$$

and this is 0 if and only if $a_1,\ldots, a_n$ are linearly independent over $\mathbf{Q}$. $L$ being solvable, the result follows at once from Theorem 3.

Definition. By similarity with the associative case, we shall call a Lie algebra $L$ over $k$ a Frobenius Lie algebra if there exists a linear functional $f \in L^*$ such that the alternating bilinear form on $L$, $(x, y) \to f([x, y])$ is nondegenerate (i.e., $L(f) = 0$). Such a Lie algebra $L$ is clearly even dimensional and $Z(D(L)) = k$ (since $L(f) = 0$ implies $L[f] = 0$). We also notice that ad-algebraic Lie algebras having a primitive universal enveloping algebra are necessarily Frobenius. (Theorem 3.)

*Remark.* Let $L$ be a Lie algebra over $k$, with basis $\{x_1,\ldots, x_n\}$. Then the following conditions are equivalent:

(1) $L$ is Frobenius,

(2) $\det(f([x_i, x_j])) \neq 0$ for a suitable $f \in L^*$,

(3) $\det([x_i, x_j]) \neq 0$ (the entries $[x_i, x_j]$ are considered as elements of the symmetric algebra $S(L)$).

This result follows easily from

$$\dim L(f) = \dim L - \operatorname{rank}(f([x_i, x_j]))$$

and

$$\operatorname{rank}_{K(L)}([x_i, x_j]) = \max_{f\in L^*} \operatorname{rank}(f([x_i, x_j])).$$

2. Assume now that $k$ is algebraically closed. Then each solvable Frobenius Lie algebra over $k$ has a primitive universal enveloping algebra. (Theorem 3.)

*Examples.*

(a) The Lie algebra $N$ with basis $\{x, y\}$ and $[x, y] = y$ is obviously Frobenius.

(b) In the four-dimensional case we have three different types of Frobenius Lie algebras:

(i) $N \oplus N$ (Direct product),

(ii) The Lie algebras of the form $L(a)$, $a \in k$, with basis $\{x_1, x_2, x_3, x_4\}$ and relations $[x_1, x_2] = ax_2$, $[x_1, x_3] = (1 - a)x_3$, $[x_1, x_4] = x_4$, $[x_2, x_3] = x_4$. We have that $L(a) \simeq L(b)$ if and only if $a = b$ or $a + b = 1$.

(iii) The Lie algebra $L$ with basis $\{x_1, x_2, x_3, x_4\}$ and relations $[x_1, x_2] = \frac{1}{2}x_2 + x_3$, $[x_1, x_3] = \frac{1}{2}x_3$, $[x_1, x_4] = x_4$, $[x_2, x_3] = x_4$.

3. Finally, we shall give an example of a nonsolvable Frobenius Lie algebra over the complex numbers having a primitive universal enveloping algebra.

Let $V$ be an $n$-dimensional vector space over $k$ and let $L$ be the Lie algebra of endomorphisms of $V$ mapping $V$ into a given $(n - 1)$-dimensional subspace. By choosing a suitable basis in $V$ we see that $L$ is the Lie algebra of $n \times n$ matrices with last row equal to zero. Clearly $L$ is not solvable if $n \geqslant 3$ (its Levi factor is $sl(n - 1)$). Moreover, $L$ is an algebraic Lie algebra and satisfies the Gelfand–Kirillov conjecture [5, p. 14], in fact $D(L)$ is isomorphic to $D_{n(n-1)/2,0}$. The second index being 0 indicates that $Z(D(L)) = k$, which implies that $L$ is Frobenius. (Corollary 2.)

We shall now prove that $U(L)$ is primitive in case $n = 3$ and $k$ is the field of complex numbers. Under these circumstances it is easy to verify that $L$ is six-dimensional with basis $\{h, x, y; e_0, e_1, e_2\}$ and nonvanishing brackets: $[h, x] = 2x$, $[h, y] = -2y$, $[x, y] = h$, $[e_0, e_1] = e_1$, $[e_0, e_2] = e_2$, $[h, e_1] = e_1$, $[h, e_2] = -e_2$, $[x, e_2] = e_1$, $[y, e_1] = e_2$. Obviously, $\{e_0, e_1, e_2\}$ is a basis for the radical of $L$, while $\{h, x, y\}$ is a basis for a Levi factor $S$ of $L$. Next we take $f$ in $L^*$ such that $f(x) = f(e_2) = 1$ and $f(h) = f(y) = f(e_0) = f(e_1) = 0$ (it turns out that $L(f) = 0$, showing again that $L$ is Frobenius).

Denote by $SP(f)$ the set of the solvable polarizations of $f$, i.e., the collection of the solvable Lie subalgebras $H$ of $L$ such that $f([H, H]) = 0$ and with $\dim H = \frac{1}{2}(\dim L + \dim L(f)) = 3$. Let $H$ be the Lie subalgebra with basis $\{h + e_0, e_1, e_2\}$. Clearly $[h + e_0, e_1] = 2e_1$, $[h + e_0, e_2] = 0$, $[e_1, e_2] = 0$. Hence $H \in SP(f)$.

Following Dixmier, we define for each $x \in H$

$$\mathcal{O}(x) = \tfrac{1}{2}(\operatorname{tr}(\operatorname{ad}_H x) - \operatorname{tr}(\operatorname{ad}_L x)).$$

$U(L)$ becomes a right $H$-module (and hence a right $U(H)$-module) by defining for each $u \in U(L)$ and $x \in H$:

$$u \cdot_{L,H} x = ux + \mathcal{O}(x)u.$$

However, in this case $\mathcal{O}(x) = 0$ for all $x \in H$. Indeed, $\operatorname{tr}(\operatorname{ad}_H e_i) = 0 = \operatorname{tr}(\operatorname{ad}_L e_i)$ $i$: 1, 2 and $\operatorname{tr}(\operatorname{ad}_H(h + e_0)) = 2 = \operatorname{tr}(\operatorname{ad}_L(h + e_0))$.

Because $f([H, H]) = 0$, $f$ defines a one-dimensional representation of $H$ and hence of $U(H)$. Denote by $J \subset U(H)$ its kernel. This representation induces a representation of $U(L)$, usually denoted by $\operatorname{Ind}(f \mid H, L)$. Its kernel $I(f)$ is a primitive ideal of $U(L)$ [4, Théorème 1] and is the largest ideal of $U(L)$ contained in $U(L) \cdot_{L,H} J$ [3, Lemma 4.15, p. 36]. We shall prove that in this case $I(f) = 0$ (and hence $U(L)$ is primitive).

By the Poincaré–Birkhoff–Witt Theorem each element of $U(H)$ can uniquely be written in the form $\sum a_{pqr}(h + e_0)^p e_1^q e_2^r$, $a_{pqr} \in \mathbf{C}$ and by induction on the degree in $e_2$ also in the form $\sum b_{pqr}(h + e_0)^p e_1^q(e_2 - 1)^r$, $b_{pqr} \in \mathbf{C}$. Clearly, the latter element is in $J$ if and only if $b_{000} = 0$. Consequently, the monomials $(h + e_0)^p e_1^q(e_2 - 1)^r$, $p + q + r \neq 0$ form a basis for $J$ over $\mathbf{C}$ and $U(H) = J \oplus \mathbf{C} \cdot 1$. Moreover, the same monomials form a basis of

$$K = U(L) \cdot_{L,H} J = U(L)J$$

over $U(S)$, i.e., $K$ consists of all elements of the form $\sum a_{pqr}(h + e_0)^p e_1^q(e_2 - 1)^r$ with $a_{000} = 0$ and $a_{pqr} \in U(S)$. Clearly $K$ is the left ideal of $U(L)$ generated by $h + e_0$, $e_1$, $e_2 - 1$ and $U(L) = U(S) \oplus K$. Furthermore, put $K_1 = U(L) e_1 + U(L)(e_2 - 1)$ and $K_2 = U(L)(e_2 - 1)$; then obviously $K \supset K_1 \supset K_2$ and $K_1 x \subset K_1$, $K_2 y \subset K_2$ (indeed,

$$(ue_1 + v(e_2 - 1))x = (ux) e_1 + (vx)(e_2 - 1) - ve_1 \in K_1 .$$

Hence $K_1 x \subset K_1$; $K_2 y \subset K_2$ since $[y, e_2 - 1] = 0$).

LEMMA. *Suppose* $u \in U(L)$.

(a) *If* $u \in K$ *and* $ux \in K$ *then* $u \in K_1$,

(b) *If* $uy^n \in K_1$ *for all* $n \in \mathbf{N}$ *then* $u \in K_2$,

(c) *If* $uh^n \in K_2$ *for all* $n \in \mathbf{N}$ *then* $u = 0$.

*Proof.* (a) Since $u \in K$ we may write $u = a(h + e_0)^t + v$ with $a \in U(S)$, $v \in K_1$ and $t \geqslant 1$. We observe that $a(h + e_0)^t x = ux - vx \in K$ (since

$ux \in K$, $vx \in K_1$) but $a(h + e_0)^t x = a(h + e_0)^{t-1} x(h + e_0) + 2a(h + e_0)^{t-1} x$ (since $[h + e_0, x] = 2x$). This implies that $a(h + e_0)^{t-1} x \in K$. Repeating the same argument a number of times, we arrive at $ax \in K$. But this implies that $a = 0$ (since $ax \in K \cap U(S) = 0$) and therefore $u = v \in K_1$. (b) Since $u \in K_1$ (take $n = 0$) we have that $u = ve_1{}^m + w(e_2 - 1)$ $v, w \in U(L)$, $m \geqslant 1$ and we may assume that $v, w$ are chosen such that $m$ is maximal (use Poincaré–Birkhoff–Witt Theorem). Clearly

$$ve_1{}^m y^n = uy^n - wy^n(e_2 - 1) \in K_1$$

(since $[y, e_2 - 1] = 0$) for all $n \in \mathbf{N}$. Since

$$[y, e_1{}^m] = \sum_{q=1}^{m} e_1^{m-q}[y, e_1]\, e_1^{q-1} = me_1^{m-1} e_2 = me_1^{m-1} + me_1^{m-1}(e_2 - 1),$$

it follows that

$$ve_1{}^m y = (vy)\, e_1{}^m - v[y, e_1{}^m] = (vy)\, e_1{}^m - mve_1^{m-1} - mve_1^{m-1}(e_2 - 1).$$

By repetition of the same argument, we obtain $ve_1{}^m y^m = (-1)^m m!\, v + z$ for some $z \in K_1$. This implies that $v \in K_1$. Hence $v = ae_1 + b(e_2 - 1)$ for some $a, b \in U(L)$. Consequently, $u = ae_1^{m+1} + (be_1{}^m + w)(e_2 - 1)$. But this contradicts the maximality of $m$, unless $a = 0$. Hence $u \in K_2$. (c) Suppose that $uh^n \in K_2$ for all $n \in \mathbf{N}$. In particular $u \in K_2$. Therefore we may write that $u = v(e_2 - 1)^m$, $m \geqslant 1$, $v \in U(L)$. Again we may assume that $v \in U(L)$ is chosen such that $m$ is maximal, which means that $v \notin K_2$ unless $v = 0$. Consider

$$[h, (e_2 - 1)^m] = \sum_{q=1}^{m} (e_2 - 1)^{m-q}\, [h, e_2 - 1](e_2 - 1)^{q-1}.$$

Hence, since $[h, e_2 - 1] = -e_2$: $[h, (e_2 - 1)^m] = -m(e_2 - 1)^{m-1} e_2 = -m(e_2 - 1)^m - m(e_2 - 1)^{m-1}$. Therefore,

$$\begin{aligned} uh &= v(e_2 - 1)^m h = vh(e_2 - 1)^m - v[h, (e_2 - 1)^m] \\ &= vh(e_2 - 1)^m + mv(e_2 - 1)^m + mv(e_2 - 1)^{m-1} \\ &= v(h + m)(e_2 - 1)^m + mv(e_2 - 1)^{m-1}. \end{aligned}$$

By repetition of the same argument we arrive at:

$$uh^m = v(e_2 - 1)^m h^m = m!\, v + \omega \qquad \text{for some} \quad \omega \in K_2 .$$

It follows that $v \in K_2$. Hence $v = 0$ and $u = 0$.

We can now prove that $I(f) = 0$. Take $u \in I(f)$. Hence $uU(L) \subset I(f) \subset K$ (since $I(f)$ is an ideal of $U(L)$ contained in $K$). In particular, $uh^m y^n x^p \in K$ for all $m, n, p \in \mathbf{N}$. Using (a) of the Lemma we see that $uh^m y^n \in K_1$ for all $m, n \in \mathbf{N}$. However, using (b) we obtain that $uh^m \in K_2$ for all $m \in \mathbf{N}$. Finally, this implies that $u = 0$ (by (c)).

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