Perverse schobers and GKZ systems

PERVERSE SCHOBERS AND GKZ SYSTEMS

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Abstract. Perverse schobers are categorifications of perverse sheaves. In prior work we constructed a perverse schober on a partial compactification of the stringy Kähler moduli space (SKMS) associated by Halpern-Leistner and Sam to a quasi-symmetric representation of a reductive group. When the group is a torus the SKMS corresponds to the complement of the GKZ discriminant locus (which is a hyperplane arrangement in the quasi-symmetric case shown by Kite). We show here that a suitable variation of the perverse schober we constructed provides a categorification of the associated GKZ hypergeometric system in the case of non-resonant parameters. As an intermediate result we give a description of the monodromy of such "quasi-symmetric" GKZ hypergeometric systems.

CONTENTS

1. INTRODUCTION

The classical Riemann-Hilbert correspondence yields an equivalence between the triangulated category of (regular holonomic) D-modules and that of constructible

sheaves [Del70, Kas84, Meb84]. Under this equivalence the abelian category of regular holonomic D-modules, more or less the same as systems of linear partial differential equations with regular singularities, corresponds to the abelian category of perverse sheaves [Kas75, BBD82]. Recently, guided by mirror symmetry applications, Kapranov and Schechtman [KS15] have introduced categorifications of perverse sheaves and called these perverse schobers. It is intended that perverse schobers would serve as coefficient data in the construction of Fukaya like categories.

The aim of this paper is to show that a variant of the perverse schober constructed in [ŠVdB19] provides a categorification of the GKZ hypergeometric system in the case of toric data associated to a quasi-symmetric representation (see below).

As an intermediate result we obtain a general formula for the monodromy of such "quasi-symmetric" GKZ systems which we believe is new. The result applies to all classical, one variable, hypergeometric equations (generalizing in particular [BH89]), and also to some of the classical higher dimensional GKZ systems such as Lauricella F_D [Beu16, §9] and Appell F_1 [Bod13, p.94] (for those particular systems we recover the results from [DM86, Pic81, Ter83]).

1.1. **Perverse schobers.** *Perverse schobers* [BKS18, KS15] are categorifications
of perverse sheaves on suitably stratified topological spaces. There is no general definition for perverse schobers but they have been defined for example on complex vector spaces stratified by complexified real hyperplane arrangements [KS15]. This is accomplished by categorifying the work of [KS16]. Work is also ongoing in the case of Riemann surfaces [Don19, DKSS].

We discuss the case of hyperplane arrangements. Let V be a real vector space and let $\mathcal{H}$ be an affine hyperplane arrangement in V. Let $\mathcal{C}$ be the open cell complex on V induced by $\mathcal{H}$, ordered by $C' < C$ iff $C' \subset \bar{C}$ and let $\mathcal{C}^0 \subset \mathcal{C}$ be the set of chambers. To this data one associates a particular kind of perverse schober on $V_\mathbb{C}$ which is called an $\mathcal{H}$-schober in [BKS18] and whose definition is recalled in §8.1.

For the purpose of this introduction we will just mention that the data defining an $\mathcal{H}$-schober consists of a family of triangulated categories $(\mathcal{E}_C)_{C \in \mathcal{C}}$ such that $\mathcal{E}_{C'} \subset \mathcal{E}_C$ if $C < C'$. Extra conditions are imposed on this data which imply in particular that the collection $(\mathcal{E}_C)_{C \in \mathcal{C}^0}$ defines a representation of the fundamental groupoid of $V_\mathbb{C} \setminus \mathcal{H}_\mathbb{C}$ in the category of triangulated categories. Informally: a local system of triangulated categories on $V_\mathbb{C} \setminus \mathcal{H}_\mathbb{C}$.

The *decategorification* $K^0_\mathbb{C}(\mathcal{E})$ of an $\mathcal{H}$-schober $\mathcal{E}$ assigns the vector space $E_C := K^0(\mathcal{E}_C)_\mathbb{C}$ to $C \in \mathcal{C}$. The definition of an $\mathcal{H}$-schober is such that $(E_C)_{C \in \mathcal{C}}$ is precisely the data required by [KS16] to define a perverse sheaf on $V_\mathbb{C}$ which is smooth with respect to $\mathcal{H}_\mathbb{C}$. We will denote this perverse sheaf by $\tilde{K}^0_\mathbb{C}(\mathcal{E})$.

1.2. **A perverse schober using Geometric Invariant Theory.** In the paper
[HLS20] Halpern-Leistner and Sam observed that the Stringy Kähler Moduli Space (SKMS) of suitable GIT quotients is given by the complement of a (toric) hyperplane arrangement and they used this observation to construct a local system of triangulated categories on the SKMS. In [ŠVdB19], reviewed in §9, we were able to extend this local system to an $\mathcal{H}$-schober (denoted by $\mathcal{S}^c$ below).

For the convenience of the reader we briefly give some indications about the construction. We restrict ourselves to the torus case as this is the case we will be

concerned with in this paper. Let T be a torus acting faithfully on a representation W with weights $(b_i)_i \in X(T)$. In the rest of this introduction, and in most of the paper, we also assume that W is *quasi-symmetric* [ŠVdB17a], i.e. $\sum_{b_i \in \ell} b_i = 0$ for each line $\ell \subset X(T)_{\mathbb{R}}$, passing through the origin.

To the data (T, W) [HLS20] associates a hyperplane arrangement $\mathcal{H}$ in the real vector space $X(T)_{\mathbb{R}}$ consisting of the translates by elements of $X(T)$ of the hyperplanes in $X(T)_{\mathbb{R}}$ spanned by the faces of the zonotope $\Delta := \sum_i [-1/2, 0]b_i \subset X(T)_{\mathbb{R}}$. The schober $\mathcal{S}^c$ constructed in [ŠVdB19] is such that for $C \in \mathcal{C}$, $\mathcal{S}_C^c$ is the thick subcategory of $D(W/T)$ generated by $\chi \otimes \mathcal{O}_W$ for $\chi \in (\nu + \Delta) \cap X(T)$ with $\nu \in -C$ chosen arbitrarily.[1] We note that the $\mathcal{S}_C^c$ are derived categories of suitable non-commutatve resolutions of $W/\!\!/T$ originally constructed in [ŠVdB17a]. For $C \in \mathcal{C}^0$ they are derived equivalent to certain GIT quotient stacks and these were used in [HLS20] in their description of the local system on $X(T)_{\mathbb{C}} \setminus \mathcal{H}_{\mathbb{C}}$.

Example 1.1. Let $T = G_m$. We identify $X(T)$ with $\mathbb{Z}$ and we write (n) for the one-dimensional T-representation with weight $n \in \mathbb{Z}$. Let $W = (1) \oplus (1) \oplus (-1) \oplus (-1)$ and put $P_n = (n) \otimes \mathcal{O}_W \in D(W/T)$. Then $\mathcal{H} = \mathbb{Z}$. If $C =]a, a+1[\in \mathcal{C}^0$ then $\mathcal{S}_C^c$ is the full thick subcategory of $D(W/T)$ generated P_{-a-1}, P_{-a}. If $C = \{a\} \in \mathcal{C}$ then $\mathcal{S}_C^c$ is generated by P_{-a-1}, P_{-a}, P_{-a+1}.

The $\mathcal{H}$-schober $\mathcal{S}^c$ is in fact invariant under translation by $X(T) \subset X(T)_{\mathbb{R}}$. Hence we can think of it as a perverse schober on the quotient $X(T)_{\mathbb{C}}/X(T)$ and the latter may be canonically identified with the dual torus T^* via the map

$$X(T)_{\mathbb{C}} = Y(T^*)_{\mathbb{C}} = \mathrm{Lie}(T^*) \xrightarrow{\exp(2\pi i -)} T^*$$

where $Y(T^*) = X(T^*)^*$ is the group of cocharacters of T^*. In particular $\tilde{K}^0_{\mathbb{C}}(\mathcal{S}^c)$ descends to a perverse sheaf on T^* which we denote by S^c.

1.3. **The mirror picture.** Since we are in a toric setting the "B-data" given by $(b_i)_{i=1,\dots,d}$ has associated "A-data", referred to as the *Gale dual* of B. Let $\mathbb{T}^d = (\mathbb{C}^*)^d$ be the torus which acts coordinate wise on W. Then by faithfulness we may assume that T acts on W via an inclusion $T \subset \mathbb{T}$. Put $H = \mathbb{T}/T$. We obtain an exact sequence

$$0 \to X(H) \to X(\mathbb{T}) \xrightarrow{B} X(T) \to 0$$

where B sends the standard basis $(e_i)_{i=1,\dots,d}$ of $X(\mathbb{T})$ to $(b_i)_{i=1,\dots,d}$. Then we obtain a morphism A by dualizing this sequence

$$0 \to Y(T) \to Y(\mathbb{T}) \xrightarrow{A} Y(H) \to 0.$$

Usually A is identified with the sequence of H-cocharacters given by $a_i = A(e_i^*)$.

1.4. **The GKZ system.** Gelfand, Kapranov and Zelevinsky discovered a captivating common generalisation of many classical hypergeometric differential equations introduced by Euler, Gauss, Horn, Appell,..., which is now known as the *GKZ (hypergeometric) system*. The GKZ system lives on $\mathbb{T}^*$ and depends on A and in addition on a continuous parameter $\alpha \in Y(H)_{\mathbb{C}}$. It is weakly H^*-invariant (see §A.1.1) and hence with some care it may be regarded as living on $\mathbb{T}^*/H^* = T^*$

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[1]In [ŠVdB19] our signs were slightly different. ν was an element of C (instead of $-C$) and the weights of W were $(-b_i)_i$ (instead of $(b_i)_i$).

(see §4.3 below for the precise procedure). In our setting the GKZ system has regular singularities [Ado94] and so its solution complex $\bar{P}(\alpha)$ is a family of perverse sheaves on T^* parametrized by $\alpha \in Y(H)_{\mathbb{C}}$.

The GKZ system is most easily understood for parameters which are sufficiently generic. Let $\mathcal{I} \subset Y(H)_{\mathbb{R}}$ be the real hyperplane arrangement consisting of the translates by elements of $Y(H)$ of the hyperplanes in $Y(H)_{\mathbb{R}}$ spanned by the facets of the cone $\mathbb{R}_{\geq 0} A$. We say that $\alpha \in Y(H)_{\mathbb{C}}$ is *non-resonant* if $\alpha \notin \mathcal{I}_{\mathbb{C}}$. The following is a basic result in the theory of GKZ systems (see §4.6)

Proposition 1.2. [GKZ90] *Assume $\alpha \in Y(H)_{\mathbb{C}}$ is non-resonant. Then $\bar{P}(\alpha)$ is a simple perverse sheaf, in particular it is the* intermediate extension *of its corresponding local system. If $\alpha, \alpha' \in Y(H)_{\mathbb{C}}$ are non-resonant and $\alpha - \alpha' \in Y(H)$ then $\bar{P}(\alpha) \cong \bar{P}(\alpha')$.*

Example 1.3. Let (T, W) be as in Example 1.1. In that case $T^* = \mathbb{C}^* = \mathbb{P}^1 \setminus \{0, \infty\}$. After a suitable identification $Y(H)_{\mathbb{C}} \cong \mathbb{C}^3$ and choosing $\alpha = (a, b, c) \in \mathbb{C}^3$ the corresponding GKZ system on T^* is the *Gaussian hypergeometric equation*

$$z(1-z)\frac{d^2w}{dz^2} + [c - (a+b+1)z]\frac{dw}{dz} - abw = 0.$$

The non-resonant parameters are those a, b, c for which $a, b, a - c, b - c$ are all non-integers.

1.5. **Kite's result.** The singular locus of the GKZ system on T^* is defined by the so-called *principal A-discriminant* E_A [GKZ89]. This paper started with following beautiful observation in [Kit17].

Theorem 1.4 ([Kit17, Proposition 4.1], §4.4). *The image of $\mathcal{H}_{\mathbb{C}}$ (see §1.2) in T^* is, up to a translation $\bar{\tau}$ (see §4.4), equal to $V(E_A)$.*

This immediately suggests the question:

Question 1.5. What is the relationship, if any, between the perverse sheaves $\bar{P}(\alpha)$, $\alpha \in Y(H)_{\mathbb{C}}$, and the perverse sheaf S^c?

Alas we immediately see an issue. S^c does not depend on any continuous parameters whereas $\bar{P}(\alpha)$ does. So to make sense of the question we either have to pick a specific α or we have to change the definition of S^c so that it also depends on continuous parameters. In this paper we take the latter approach.

1.6. **Creating a family.** Since $X(T)$ is a quotient of $X(\mathbb{T})$ we may view $X(\mathbb{T})$ as acting on the vector space $X(T)_{\mathbb{C}}$. We first replace the $X(T)$-equivariant $\mathcal{H}$-schober $\mathcal{S}^c$ by an $X(\mathbb{T})$-equivariant $\mathcal{H}$-schober $\bar{\mathcal{S}}^c$ built from subcategories of $D(W/\mathbb{T})$. More precisely for $C \in \mathcal{C}$, $\bar{\mathcal{S}}_C^c$ is defined as the thick subcategory of $D(W/\mathbb{T})$ spanned by $\chi \otimes \mathcal{O}_W$ for $\chi \in X(\mathbb{T})$ such that the image of χ in $X(T)$ is in $\nu + \Delta$ for $\nu \in -C$. Note that $X(H) = \ker(X(\mathbb{T}) \to X(T))$ acts trivially on $\mathcal{C}$. In other words $X(H)$ acts on $\bar{\mathcal{S}}_C^c$ for every $C \in \mathcal{C}$.

We now change our point of view a bit. We choose a splitting $\mathbb{T} = T \times H$ and we consider $\bar{\mathcal{S}}^c$ as an $X(T)$-equivariant schober on $X(T)_{\mathbb{C}}$, equipped with an additional $X(H)$-action. The reward for doing this is that the decategorification $K_{\mathbb{C}}^0(\bar{\mathcal{S}}^c)$ of $\bar{\mathcal{S}}^c$ is now built from modules over the group ring $\mathbb{C}[X(H)]$ of $X(H)$. Since $\mathbb{C}[X(H)] \cong \mathbb{C}[H]$ (the coordinate ring of H) we may specialize $K_{\mathbb{C}}^0(\bar{\mathcal{S}}^c)$ at an element h of H. Denote the result by $K_h^0(\bar{\mathcal{S}}^c)$. Via [KS16] we obtain a corresponding

perverse sheaf on $X(T)_{\mathbb{C}}$ which we denote by $\tilde{K}_h^0(\bar{\mathcal{S}}^c)$. After descent under $X(T)$ this yields a perverse sheaf of $X(T)_{\mathbb{C}}/X(T) = T^*$ which we denote by $S^c(h)$. We recover S^c defined above as $S^c(\mathbf{1})$ where $\mathbf{1}$ denotes the unit element of H.

1.7. **Main result.** The following result fulfils the promise made at the start of the introduction.

Theorem 1.6 (Theorem 13.1). *Assume that $\alpha \in Y(H)_{\mathbb{C}}$ is non-resonant and put $h = \exp(2\pi i \alpha) \in H$. Then $\bar{P}(\alpha) \cong \bar{\tau}^* S^c(h^{-1})$ where $\bar{\tau}$ is the translation introduced in Theorem 1.4.*

Now note that while Theorem 1.6 is in the spirit of Question 1.5, it does not actually answer the latter. Indeed as we have said $S^c = S^c(\mathbf{1})$ and $\exp(2\pi i \alpha) = \mathbf{1}$ if and only if $\alpha \in Y(H)$ and such α are, in some sense, as far away from being non-resonant as possible. The case of resonant parameters is currently work in progress and will be discussed in a future paper.

Remark 1.7. Theorem 1.6 suggests that it should be possible to parametrize the GKZ system by H, instead of by the covering space $Y(H)_{\mathbb{C}}$. This is indeed possible, analytically, if we restrict ourselves to the non-resonant part of $Y(H)_{\mathbb{C}}$. See §4.7.

1.8. **Discussion.** Theorem 1.6 connects two (families of) perverse sheaves, one defined via algebra and one defined via analysis. A perverse sheaf is generically a (shifted) local system. Restricting ourselves to local systems, results in the spirit of Theorem 1.6, but with different specifics, appear elsewhere. We give some examples.

- In [BH06a] a local system "at infinity" obtained from the derived categories of suitable toric GIT quotients is shown to be the same as a local system at infinity corresponding to a GKZ system. This result is not restricted to the quasi-symmetric case.
- In [ABM15] a result like Theorem 1.6 is proved for the standard resolution of suitable Slodowy slices. The corresponding derived category is equipped with an action of the affine braid group and hence may be regarded as a local system of triangulated categories on the complement of a hyperplane arrangement. After decategorification, this local system is shown to be the same as a local system defined using quantum cohomology (which is equipped with a connection). It would be very interesting to apply these techniques in the setting of Theorem 1.6.
- Bridgeland observes that the moduli space of stability conditions of a triangulated category often arises as a covering space of a nice space such that the central charge map descends to a multivalued map which satisfies an interesting differential equation whose monodromy comes from the action of autoequivalences. For example, [BQS20] (see also [Ike17]) considers the Calabi-Yau completions [Gin, Kel11] of the A_2-quiver. It is shown that the associated spaces of stability conditions, modulo suitable autoequivalences, are given by $\mathfrak{h}^{\mathrm{reg}}/S_3$ where $\mathfrak{h}$ is the two-dimensional irreducible representation of S_3 and $\mathfrak{h}^{\mathrm{reg}} \subset \mathfrak{h}$ is the hyperplane complement on which S_3 acts freely. The central charge, which is a multivalued holomorphic map $\mathfrak{h}^{\mathrm{reg}}/S_3 \to \mathbb{C}^2$, satisfies a hypergeometric equation [BQS20, eq. (14)].

1.9. **Outline of proof.** The proof of Theorem 1.6 consists of three major steps.

Step 1. We compute the monodromy of the perverse sheaves $\bar{P}(\alpha)$. To this end we follow the method introduced by Beukers in [Beu16] but in our setting we are able to get a precise formula for the full monodromy and not just the local monodromy as in loc. cit. We believe that this result is interesting in its own right but it is a bit too technical to state here. See Theorem 6.4.

Step 2. We compute the monodromy of the perverse sheaf $S^c(h)$ and show that it is the same as the monodromy of $\bar{P}(\alpha)$ modulo the reparametrisation indicated in Theorem 1.6. This is done by using suitable complexes introduced in [ŠVdB17a]. See Proposition 12.6.

Step 3. We prove that $S^c(h)$ is the intermediate extension of the local system it defines, which allows us to conclude by combining Proposition 1.2 with Steps 1,2.

To accomplish Step 3 we first use some combinatorics to show $S^c(h)$ has no quotients supported on $\mathcal{H}_{\mathbb{C}}/X(T)$. We then show that $K_h^0(\bar{S}^c)$ is nearly self dual, implying that $S^c(h)$ also has no subobjects supported on $\mathcal{H}_{\mathbb{C}}/X(T)$. Therefore it has the intermediate extension property.

The self duality is shown by defining a subschober $\bar{S}^f \subset \bar{S}^c$ of $\bar{S}^c$ where $\bar{S}^f_C$ consists of those objects in $\bar{S}^c_C$ which are supported on the nullcone of W. Then $K^0(\bar{S}^f_C)$ is both: *isomorphic* to $K^0(\bar{S}^c_C)$, after a suitable localization, and (2) *dual* to $K^0(\bar{S}^c_C)$, via the Euler form. See §13.1.

2. Acknowledgement

3. Notation and conventions

A list of notations is given at the end of the paper.

3.1. **Identifications for tori.** Below we routinely use some identifications associated with algebraic tori. We summarize these below. Let L be a finitely generated free abelian group and let $T := L \otimes_{\mathbb{Z}} \mathbb{C}^*$ be the corresponding algebraic torus. Then we have canonical identifications

$$\mathrm{Lie}(T) = L \otimes_{\mathbb{Z}} \mathbb{C}$$
$$X(T) = L^*$$
$$Y(T) = L.$$

In particular we obtain a canonical isomorphism

$$Y(T)_{\mathbb{C}} \cong \mathrm{Lie}(T).$$

One checks that the resulting composition

$$Y(T) \to Y(T)_{\mathbb{C}} \cong \mathrm{Lie}(T)$$

sends $\lambda : G_m \to T$ to $d\lambda : \mathbb{C} = \mathrm{Lie}(G_m) \to \mathrm{Lie}(T)$, evaluated at 1. The exponential exact sequence

$$0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{e^{2\pi i -}} \mathbb{C}^* \to 0$$

after tensoring with L becomes an exact sequence

$$0 \to Y(T) \to \mathrm{Lie}(T) = Y(T)_{\mathbb{C}} \xrightarrow{e^{2\pi i -}} T \to 0.$$

The dual torus T^* is defined as $L^* \otimes_{\mathbb{Z}} \mathbb{C}^*$. Hence we have identifications

$$\mathrm{Lie}(T^*) = \mathrm{Lie}(T)^*$$
$$X(T^*) = Y(T)^*$$
$$Y(T^*) = X(T)^*.$$

Remark 3.1. Below we will usually silently make the identifications given above.

3.2. **Setting.** Throughout we fix an exact sequence of free abelian groups

$$(3.1) \qquad 0 \to L \xrightarrow{B^*} \mathbb{Z}^d \xrightarrow{A} N \to 0.$$

with $n := \mathrm{rk}\, L$. We do not fix bases for L, N but the middle term $\mathbb{Z}^d$ will be equipped with its tautological basis (denoted by $(e_i)_i$) and this basis plays an essential role in the theory.

The map denoted by B^* is the dual of the corresponding map $B : \mathbb{Z}^d \to L^*$. We write $b_i = B(e_i) \in L^*$, $a_i = A(e_i) \in N$. Following custom we will usually identify A, B with the sequences of elements $(a_i)_i$, $(b_i)_i$.

Tensoring (3.1) with $- \otimes_{\mathbb{Z}} \mathbb{C}^*$ we obtain an exact sequence of algebraic tori

$$(3.2) \qquad 1 \to T \xrightarrow{B^*} \mathbb{T} \xrightarrow{A} H \to 1$$

with $\dim T = n$, $\dim \mathbb{T} = d$. Let $W \cong \mathbb{C}^d$ and let $\mathbb{T} = (\mathbb{C}^*)^d$ act coordinate-wise on W. Then T acts on W via the map B^*.

From (3.2) we obtain:

$$(3.3) \qquad 0 \to X(H) \xrightarrow{A^*} X(\mathbb{T}) \xrightarrow{B} X(T) \to 0.$$

In this interpretation the $b_i \in X(T) \cong L^*$ are the weights for the T-action on W. Dualizing (3.3), we obtain

$$0 \to Y(T) \xrightarrow{B^*} Y(\mathbb{T}) \xrightarrow{A} Y(H) \to 0$$

which is just an avatar of (3.1).

Below we make the identification (see §3.1)

$$(3.4) \qquad X(T)_{\mathbb{C}}/X(T) = Y(T^*)_{\mathbb{C}}/Y(T^*) = \mathrm{Lie}(T^*)/Y(T^*) \overset{e^{2\pi i -}}{\cong} T^*.$$

3.3. **Quasi-symmetry.** We say that W is *quasi-symmetric* if for every line $0 \in \ell \in X(T)_{\mathbb{R}}$, $\sum_{b_i \in \ell} b_i = 0$.

3.4. **Affine hyperplane arrangements.** We will encounter hyperplane arrange-ments in several different contexts so we introduce some general notions related to them.

Below let $\mathcal{H}$ be an affine hyperplane arrangement[2] in an n-dimensional real vector space V. Following custom we will often confuse $H \in \mathcal{H}$ with a specific affine function ("equation") $H : V \to \mathbb{R}$ defining H.

For $H \in \mathcal{H}$ we let H_0 be the parallel translate of H which passes through the origin and we define the *central hyperplane arrangement* $\mathcal{H}_0$ corresponding to $\mathcal{H}$ as

$$\mathcal{H}_0 = \{H_0 \mid H \in \mathcal{H}\}.$$

If $H \in \mathcal{H}$ then the corresponding complex hyperplane $\mathcal{H}_\mathbb{C} \subset V_\mathbb{C}$ is given by the same equation which defines H. To imagine $H_\mathbb{C}$ one may use the following concrete interpretation

$$H_\mathbb{C} = \{x_0 + ix_i \in V + iV \mid H(x_0) = 0, H_0(x_1) = 0\} = H \times iH_0.$$

Note that $H_\mathbb{C}$ has codimension two in $V_\mathbb{C}$. The *complex hyperplane arrangement* $\mathcal{H}_\mathbb{C} \subset V_\mathbb{C}$ is defined by

$$\mathcal{H}_\mathbb{C} = \{H_\mathbb{C} \mid H \in \mathcal{H}\}.$$

A vector $\rho \in V$ is *generic* if it is not parallel to any of the hyperplanes in $\mathcal{H}$.

The closures of the connected components of $V \setminus \mathcal{H}$ are convex polytopes. The collection of the (relatively open) faces of these polytopes is denoted by $\mathcal{C}$. This set is partially ordered by $C_1 \leq C_2$ iff $C_1 \subset \overline{C_2}$. We denote $C_1 \wedge C_2 = \operatorname{relint}(\overline{C_1} \cap \overline{C_2})$. By $\mathcal{C}^0$ we denote the set of chambers, i.e. the polytopes of dimension n, in $\mathcal{C}$. A triple of faces $(C_1, C_2, C_3) \in \mathcal{C}$ is *collinear* if there exists $C' \leq C_1, C_2, C_3$ and there exist points $c_i \in C_i$ such that $c_2 \in [c_1, c_3]$.

3.5. **A hyperplane arrangement in $X(T)_\mathbb{R}$.** Unless otherwise specified we will use the notation $(\mathcal{H}, \mathcal{C})$ in a concrete sense which we now outline. Put

$$(3.5) \qquad \Sigma = \left\{ \sum_i \beta_i b_i \mid \beta_i \in (-1, 0) \right\}, \qquad \Delta = (1/2)\overline{\Sigma}.$$

Denote by $(H_i)_i$ the affine hyperplanes in $X(T)_\mathbb{R}$ spanned by the facets of Δ. Put

$$(3.6) \qquad \mathcal{H} = \bigcup_i (-H_i + X(T)).$$

The following lemma is rather easy to check.

Lemma 3.2. $\rho \in X(T)_\mathbb{R}$ is generic (see §3.4) if and only if ρ does not lie in any (proper) subspace spanned by b_i's. If W is furthermore quasi-symmetric, then this is also equivalent to the condition that ρ does not lie in the boundary of any cone spanned by b_i's.

For $C \in \mathcal{C}$ (defined as in §3.4) we introduce the following notation. Let

$$\mathcal{L}_C = (\nu + \Delta) \cap X(T)$$

for an arbitrary $\nu \in C$ $\left(\mathcal{L}_C \text{ does not depend on } \nu, \text{ see [HLS20][ŠVdB19, §5.1]}\right)$. It follows from loc.cit. (see [ŠVdB19, Lemma 5.3]) that

$$(3.7) \qquad C' < C \implies \mathcal{L}_C \subset \mathcal{L}_{C'}.$$

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[2]We always silently assume that affine hyperplane arrangements are locally finite and that their corresponding central arrangements are finite.

3.6. **Non-resonance condition.** Consider the real central hyperplane arrangement $\mathcal{I}_0$ in $\mathfrak{h} = Y(H)_{\mathbb{R}}$ spanned by the facets of $\mathbb{R}_{\geq 0}A$ and let $\mathcal{I}$ be the real affine hyperplane arrangement $Y(H) + \mathcal{I}_0$. We say that α is *non-resonant* if $\alpha \notin \mathcal{I}_{\mathbb{C}}$. Also let $\mathcal{I}_0'$ be the real central hyperplane arrangement consisting of hyperplanes spanned by subsets of A and let $\mathcal{I}' = Y(H) + \mathcal{I}_0'$. We say that α is *totally non-resonant* if $\alpha \notin \mathcal{I}_{\mathbb{C}}'$. Clearly $\mathcal{I}_0 \subset \mathcal{I}_0'$. So totally non-resonant implies non-resonant.

We will say that $h \in H$ is *(totally) non-resonant* if $(\log h)/(2\pi i) \in \mathfrak{h}$ is (totally) non-resonant. Note that it does not matter which branch of $\log h$ we choose.

We will denote by H^{nres}, $\mathfrak{h}^{\mathrm{nres}}$ the (analytically open) sets of non-resonant elements of H and $\mathfrak{h}$.

4. The GKZ hypergeometric system

We now discuss the celebrated Gelfand-Kapranov-Zelevinsky system of differential equations [GKZ89]. It is well-known that the theory of said system becomes substantially easier under the following assumptions which we now make.

Assumption 4.1. *Except when otherwise specified we assume $\sum_i b_i = 0$. It is easy to see that the latter condition is equivalent to the assumption that there exists $h \in N^*$ such that $\forall i : \langle h, a_i \rangle = 1$.*

Remark 4.2. Assumption 4.1 is implied by quasi-symmetry. It follows from Assumption 4.1 that W is *unimodular*. In our current setup we also have $\mathbb{Z}A = N$ and $\mathbb{Z}B = L^* \cong X(T)$. Furthermore under Assumption 4.1, Δ as introduced in (3.5) is centrally symmetric and in particular the minus sign in the definition of $\mathcal{H}$ in (3.6) is superfluous.

4.1. **The GKZ system of differential equations.** Let x_i, $1 \leq i \leq d$, be the coordinates on $\mathbb{T}^* = (\mathbb{C}^*)^d$, we write $\partial_i = \partial/\partial x_i$. For $l \in L$ write $B^*(l) = (l_i)_{i=1}^d$ and put

$$\square_l = \prod_{l_i > 0} \partial_i^{l_i} - \prod_{l_i < 0} \partial_i^{-l_i}.$$

Let $\mathfrak{h} = \mathrm{Lie}(H)$ so that $\mathfrak{h}^* = \mathrm{Lie}(H^*)$. As $H^* \subset \mathbb{T}^*$, H^* acts by translation on $\mathbb{T}^*$. Therefore $\mathfrak{h}^*$ acts by derivations on $\mathcal{O}_{\mathbb{T}^*}$. If $\phi \in \mathfrak{h}^*$ then we write E_ϕ for the corresponding derivation.

Put $\mathcal{D} = \mathcal{D}_{\mathbb{T}^*}$. The GKZ system of differential equation with parameter $\alpha \in Y(H)_{\mathbb{C}} = \mathfrak{h}$ corresponds to the following $\mathcal{D}$-module

$$(4.1) \qquad \mathcal{P}(\alpha) = \mathcal{D} \bigg/ \left(\sum_{l \in X(T^*)} \mathcal{D}\square_l + \sum_{\phi \in \mathfrak{h}^*} \mathcal{D}(E_\phi - \phi(\alpha)) \right).$$

Theorem 4.3. [Ado94, Theorem 3.9] *The $\mathcal{D}$-module $\mathcal{P}(\alpha)$ is holonomic with regular singularities.*

4.2. **Non-resonance.** The GKZ system is *non-resonant* if the parameter α is non-resonant (see §3.6). Let $\mathfrak{v}$ be the normalized volume[3] of the convex hull of A. In the non-resonant case $\mathfrak{v}$ equals the rank of the GKZ system [GKZ89, GKZ93]. In the quasi-symmetric case (see §3.3) $\mathfrak{v}$ also equals the number of integral points in $\varepsilon + (1/2)\Sigma$ for arbitrary generic (see §3.5) $\varepsilon \in X(T)_{\mathbb{R}}$ (see e.g. [HLS20, Corollary

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[3]The normalized volume is the volume divided by the fundamental volume of the lattice $N \cap \{\langle h, - \rangle = 1\}$ where $h \in N^*$ is as in Assumption 4.1.

4.2, Remark 4.3] together with [BH06b] (explicitly stated in [ŠVdB17b, Theorem A.1])).[4]

4.3. **Descent of the GKZ system.** The GKZ system is a weakly H^*-equivariant $\mathcal{D}$-module on $\mathbb{T}^*$ with character α (see Definition A.1), so it represents an object in $\mathrm{Qch}_\alpha(H^*, \mathcal{D}_{\mathbb{T}^*})$ (see loc. cit.).

By Corollary A.11, choosing a splitting $\iota : \mathbb{T} \to T$ of B^* in (3.2) and denoting by a slight abuse of notation $\iota : T^* \to \mathbb{T}^*$ also adjoint to ι, $\mathcal{D}$-module pullback

$$(4.2) \qquad \iota^* : \mathrm{Qch}_\alpha(H^*, \mathcal{D}_{\mathbb{T}^*}) \to \mathrm{Qch}(\mathcal{D}_{T^*})$$

is an equivalence of (abelian) categories. Below we will be concerned with

$$\overline{\mathcal{P}}(\alpha) := \iota^*\mathcal{P}(\alpha) \in \mathcal{D}_{T^*}.$$

Remark 4.4. $\overline{\mathcal{P}}(\alpha)$ depends on the splitting ι, but in a relatively weak way. The difference of two splittings ι, ι' can be regarded as a map $\delta : N \to L$ which then induces a corresponding map $\delta : T^* \to H^*$. Then it follows from Lemma A.13 that $\iota'^*\mathcal{P}(\alpha)$ differs from $\iota^*\mathcal{P}(\alpha)$ by tensoring with the invertible $\mathcal{D}$-module generated by the multi-valued function $\theta_{\alpha,\delta}$ on T^* which satisfies $\theta_{\alpha,\delta} \circ e^{2\pi i -} = e^{2\pi i \langle \delta(-),\alpha \rangle}$ where we have lifted δ to a linear map $\mathfrak{t}^* \to \mathfrak{h}^*$.

4.4. **The GKZ discriminant locus.** The singular locus of $\overline{\mathcal{P}}(\alpha)$ is given by the principal A-discriminant E_A [GKZ89]. If W is quasi-symmetric then by [Kit17, Proposition 4.1] it is the image of a hyperplane arrangement in $X(T)_{\mathbb{C}}$ under the identification $X(T)_{\mathbb{C}}/X(T) \cong T^*$ (3.4). More precisely,

$$(\mathcal{H}_{\mathbb{C}} + \zeta)/X(T) \overset{e^{2\pi i -}}{\cong} V(E_A)$$

where $\mathcal{H}$ is the hyperplane arrangement defined in §3.5 and $\zeta := -\frac{i}{2\pi}\sum_i(\log|n_j|)b_j$ with $b_j = n_j l_\ell$ for a generator l_ℓ of the rank one lattice $X(T) \cap \ell$ where $\ell \in X(T)_{\mathbb{R}}$ is the line through the origin which contains b_j.

4.5. **Reminder on the Riemann-Hilbert correspondence.** For a smooth quasi-compact separated scheme $X/\mathbb{C}$ of pure dimension d let $\mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X)$ be the full subcategory of $\mathrm{Qch}(\mathcal{D}_X)$ consisting of regular holonomic $\mathcal{D}_X$-modules. Moreover, let $D_{\mathrm{rh}}^b(\mathcal{D}_X)$ be the bounded derived category of $\mathcal{D}_X$-modules with regular holonomic cohomology.

Remark 4.5. The category $D_{\mathrm{rh}}^b(\mathcal{D}_X)$ is compatible with all standard $\mathcal{D}$-module operations (e.g. [HTT08, Theorem 6.1.5]). Moreover, the category $\mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X)$ is closed under subquotients in $\mathrm{Qch}(\mathcal{D}_X)$ (see [HTT08, §6.1]).

Let $D_{\mathrm{cs}}^b(\mathbb{C}_{X^{\mathrm{an}}})$ be the bounded derived category of sheaves of vector spaces on X^{an} with constructible cohomology (with respect to an algebraic stratification). Also let $\mathrm{Perv}(X) \subset D_{\mathrm{cs}}^b(\mathbb{C}_{X^{\mathrm{an}}})$ be the category of perverse sheaves on X^{an}. Recall that $\mathrm{Perv}(X)$ is the heart of a t-structure. We denote the corresponding cohomology by $^pH(-)$. Put

$$\mathrm{Sol}_X : D^b(\mathcal{D}_X) \to D(\mathbb{C}_{X^{\mathrm{an}}}) : \mathcal{M} \mapsto \mathrm{R}\mathcal{H}om_{\mathcal{D}_X^{\mathrm{an}}}(\mathcal{M}^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}).$$

---

[4]Lev Borisov showed us a nice direct combinatorial proof of this result.

By the celebrated Riemann-Hilbert correspondence Sol restricts to an equivalence of triangulated categories[5]

$$\mathrm{Sol}_X : D_{\mathrm{rh}}^b(\mathcal{D}_X) \to D_{\mathrm{cs}}(\mathbb{C}_X)^\circ$$

which sends the natural t-structure on $D_{\mathrm{rh}}^b(\mathcal{D}_X)$ to the d-shifted perverse one on $D_{\mathrm{cs}}(\mathbb{C}_X)^\circ$ and hence restricts further to an equivalence of abelian categories

$$\mathrm{Sol}_X[d] : \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X) \to \mathrm{Perv}(X)^\circ$$

(see e.g. [HTT08, Proposition 4.7.4, Corollary 7.2.4, Theorem 7.2.5]).

Lemma 4.6. *Let $f : Y \to X$ be a morphism between smooth quasi-compact separated schemes over $\mathbb{C}$ of pure dimensions d_Y, d_X and let Lf^* be the unshifted $\mathcal{D}$-module pullback (denoted by Lf° in [BGK+87]). Then*

(4.3) $$\mathrm{Sol}_Y \circ Lf^* = Lf^{\mathrm{an},*} \circ \mathrm{Sol}_X$$

as functors $D_{\mathrm{rh}}^b(\mathcal{D}_X) \to D_{\mathrm{cs}}^b(\mathbb{C}_{X^{\mathrm{an}}})$.

Proof. Put $d_{Y,X} = d_Y - d_X$. Let $\mathrm{DR}_X : D_{\mathrm{rh}}^b(\mathcal{D}_X) \to D_{\mathrm{cs}}^b(\mathbb{C}_{X^{\mathrm{an}}})$ be the De Rham functor [HTT08, §4] and let $f^! : D^b(\mathcal{D}_X) \to D^b(\mathcal{D}_Y)$ be the *shifted* $\mathcal{D}$-module pullback functor; i.e. $f^! = Lf^*[d_{Y,X}]$. By [BGK+87, §14.5(4)],

$$\mathrm{DR}_Y \circ f^! = f^{\mathrm{an},!} \circ \mathrm{DR}_X.$$

Moreover, by [HTT08, Prop 4.7.4], we have $\mathrm{DR}_X \cong \mathrm{Sol}_X(\mathbb{D}_X(-))[d_X]$ where $\mathbb{D}_X = \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(-, \mathcal{D}_X[d_X])$ and by [HTT08, Corollary 4.6.5], $\mathbb{D}_X^{\mathrm{an}}(\mathrm{Sol}_X(-)[d_X]) = \mathrm{Sol}_X(\mathbb{D}_X(-))[d_X]$ where $\mathbb{D}_X^{\mathrm{an}} := \mathrm{R}\mathcal{H}om(-, \mathbb{C}_{X^{\mathrm{an}}}[2d_X])$ is the Verdier duality functor. Finally $f^{\mathrm{an},!}$ and $Lf^{\mathrm{an},*}$ are related in the usual way by $Lf^{\mathrm{an},*} = \mathbb{D}_Y^{\mathrm{an}} \circ f^{\mathrm{an},!} \circ \mathbb{D}_X^{\mathrm{an}}$. By combining these ingredients one obtains the formula (4.3). $\qquad\square$

Remark 4.7. If $f : Y \to X$ is a closed immersion then (4.3) says informally that $\mathcal{D}$-module pullback corresponds to restriction of solutions.

4.6. **The GKZ perverse sheaf.** By Theorem 4.3, $\mathcal{P}(\alpha) \in \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X)$. We put $P(\alpha) = \mathrm{Sol}_{\mathbb{T}^*}(\mathcal{P}(\alpha))[d]$. We call $P(\alpha)$ the *GKZ perverse sheaf*, with parameter α. For further reference we recall the following results.

Proposition 4.8. *Assume that α, α' are non-resonant.*

(1) *$P(\alpha)$ is a simple perverse sheaf. Moreover, $P(\alpha) \cong P(\alpha')$ if and only if $\alpha - \alpha' \in N$.*

(2) *The analogous statements in the category of $\mathcal{D}$-modules hold for $\mathcal{P}(\alpha)$.*

Proof. The statements about perverse sheaves and about $\mathcal{D}$-modules are equivalent by the Riemann-Hilbert correspondence. The first claim (for perverse sheaves) is [GKZ90] (for the explicit statement see §4.7. Proof of Theorem 2.11 in loc.cit.). (See also [SW12, Theorem 4.1] in the context of $\mathcal{D}$-modules.) The second claim (for $\mathcal{D}$-modules) is [Sai01, Corollary 2.6], see also [Dwo90, Thm 6.9.1], [Beu11a, Theorem 2.1]. $\qquad\square$

We may deduce the corresponding claims for $\overline{\mathcal{P}}(\alpha)$ (see §4.3).

Proposition 4.9. *The following holds for $\overline{\mathcal{P}}(\alpha) = \iota^*\mathcal{P}(\alpha)$.*

(1) *$L\iota^*\mathcal{P}(\alpha) = \overline{\mathcal{P}}(\alpha)$.*

(2) *$\overline{\mathcal{P}}(\alpha)$ has regular singularities.*

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[5]$(-)^\circ$ denotes the opposite.

(3) If α is non-resonant then $\overline{\mathcal{P}}(\alpha)$ is a simple $\mathcal{D}_{T^*}$-module.

(4) If α, α' are non-resonant and $\alpha - \alpha' \in N$ then $\overline{\mathcal{P}}(\alpha) \cong \overline{\mathcal{P}}(\alpha')$.

Proof. The properties (except for (1)) are consequences of the preceding results.

(1) This follows from Lemma A.12 below.

(2) See Remark 4.5 and Theorem 4.3.

(3) This follows from Proposition 4.8(2) and (4.2).

(4) Let α, α' be as in the statement of this proposition. By Proposition 4.8(2) we then have $\mathcal{P}(\alpha) \cong \mathcal{P}(\alpha')$. Hence $\iota^* \mathcal{P}(\alpha) \cong \iota^* \mathcal{P}(\alpha')$. $\qquad\square$

Lemma 4.10. *Put $^p\iota^* = {}^pH^0(L\iota^{\mathrm{an},*}) : \mathrm{Perv}(\mathbb{T}^*) \to \mathrm{Perv}(T^*)$. Then one has*

$$(4.4) \qquad\qquad \overline{P}(\alpha) := {}^p\iota^* P(\alpha) = \mathrm{Sol}_{T^*}(\overline{\mathcal{P}}(\alpha))[\dim T].$$

Moreover, $\overline{P}(\alpha)$ satisfies the analogues of Proposition 4.9 (3)(4).

Proof. The formula (4.4) may be deduced from Lemma 4.6 using Proposition 4.9(1). The other claims follow from Proposition 4.9 by the Riemann-Hilbert correspondence. $\qquad\square$

Corollary 4.11. *If $j : T^* \setminus V(E_A) \hookrightarrow T^*$ is the embedding then $j^{\mathrm{an},*}\overline{P}(\alpha) = \mathrm{Sol}_{T^* \setminus V(E_A)}(j^*\overline{\mathcal{P}}(\alpha))[\dim T]$. If α is non-resonant then $\overline{P}(\alpha) \cong j_{!*}(j^*\overline{P}(\alpha))$, where $j_{!*}$ is the intermediate extension.*

Proof. The first claim follows from Lemma A.12. For the second claim first note that $\overline{P}(\alpha)$ is simple by Lemma 4.10. As $\overline{P}(\alpha)$ is not supported on $V(E_A)$, the conclusion then follows from [BBD82, §4.3] (see also [HTT08, Proposition 8.2.5(i), Corollary 8.2.10]). $\qquad\square$

4.7. **Parametric descent of GKZ systems.** The results in this section, which we will not use in the sequel, were announced in Remark 1.7.

It is clear from the definition of $\mathcal{P}(\alpha)$ (see (4.1)) that we may in fact define a relative $\mathcal{D}$-module $\mathcal{P}$ over $\mathbb{T}^* \times \mathfrak{h} \to \mathfrak{h}$ whose fiber in α is equal to $\mathcal{P}(\alpha)$. However Proposition 4.8(2) strongly suggests we should be able to descend the restriction of $\mathcal{P}$ to $\mathfrak{h}^{\mathrm{nres}}$ to a relative $\mathcal{D}$-module for the projection $\mathbb{T}^* \times H^{\mathrm{nres}} \to H^{\mathrm{nres}}$ whose fibers are still $\mathcal{P}(\alpha)$. This is not possible algebraically but it is possible analytically.

Proposition 4.12. *There exists a coherent $\mathcal{D}^{an}_{\mathbb{T}^* \times H^{\mathrm{nres}}/H^{\mathrm{nres}}}$-module $\tilde{\mathcal{P}}$ such that the fiber of $\tilde{\mathcal{P}}$ over $h \in H^{nres}$ is equal to $\mathcal{P}^{an}((\log h)/(2\pi i))$.*

We will start with an algebraic result. Put

$$\mathcal{P} = (\mathcal{D}_{\mathbb{T}^*} \boxtimes_{\mathbb{C}} \mathcal{O}_{\mathfrak{h}}) \Bigg/ \left(\sum_{l \in X(T^*)} \mathcal{D}\square_l + \sum_{\phi \in \mathfrak{h}^*} \mathcal{D}(E_\phi - \phi) \right).$$

Then for $\alpha \in \mathfrak{h}$ we have $\mathcal{P}(\alpha) \cong \mathcal{P}_\alpha$.

For $v \in \mathfrak{h}$ denote by τ_v the translation by v on $\mathfrak{h}$. We use the same symbol for the corresponding translation on $\mathbb{T}^* \times \mathfrak{h}$. We put a $\mathfrak{h}$-grading on the sections of $\mathcal{D}_{\mathbb{T}}$ such that $|\partial_i| = -a_i$. We then have for a section f of $\mathcal{D}_{\mathbb{T}^*}$:

$$[E_\phi, f] = \langle \phi, |f| \rangle f.$$

From this it is easy to see that there are well-defined $\mathcal{D}_{\mathbb{T}^*}$-morphisms

$$(\cdot f) : \mathcal{P}(\alpha) \to \mathcal{P}(\alpha - |f|) : \bar{D} \mapsto \overline{Df}$$

which may be obtained as restrictions of the $\mathcal{D}_{\mathbb{T}^*} \boxtimes_{\mathbb{C}} \mathcal{O}_{\mathfrak{h}}$-morphism given by

$$(\cdot f) : \mathcal{P} \to \tau^*_{-|f|} \mathcal{P} : \bar{D} \mapsto \overline{Df}.$$

Proposition 4.13. *The morphism* $(\cdot \partial_i) : \mathcal{P} \to \tau^*_{a_i} \mathcal{P}$ *becomes an isomorphism when pulled back to*[6] $\mathfrak{h}^{\mathrm{nres}}$.

Proof. This follows from the proof of [Beu11a, Theorem 2.1]. Let $i_\eta : \eta \to \mathbb{T}^*$ be the generic point of $\mathbb{T}^*$. For any specific non-resonant α Beukers constructs an element $P_\alpha \in i^*_\eta \mathcal{D}_{\mathbb{T}^*}$ such that

(4.5) $$P_\alpha \partial_i \equiv 1 \bmod (\text{relations of } \mathcal{P}(\alpha)).$$

Going through the proof one sees:

(1) The constructed P_α is actually a section of $\mathcal{D}_{\mathbb{T}^*}$.
(2) The construction of P_α is polynomial in α except that one has to invert a finite number of times the evaluation of a hyperplane in $\mathfrak{h} \setminus \mathfrak{h}^{\mathrm{nres}}$ on α.

To check these assertions consult the first display on p.34 in loc. cit.

If follows that one may consider the α as variables and so one obtains a section P of $\mathcal{D}_{\mathbb{T}^*} \boxtimes_{\mathbb{C}} \mathcal{O}_{\mathfrak{h}^{\mathrm{nres}}}$ which yields P_α when restricted to $\mathbb{T}^* \times \alpha$ such that one furthermore has

$$P \partial_i \equiv 1 \bmod (\text{relations of } \mathcal{P}).$$

Finally we may assume that P is homogeneous for the $\mathfrak{h}$-grading (by dropping the terms not of degree $-|\partial_i|$). Hence if $\mathcal{P}^{\mathrm{nres}}$ denotes the pullback of $\mathcal{P}$ to $\mathbb{T}^* \times \mathfrak{h}^{\mathrm{nres}}$ then we have morphisms

$$\mathcal{P}^{\mathrm{nres}} \xrightarrow{(\cdot \partial_i)} \tau^*_{a_i} \mathcal{P}^{\mathrm{nres}} \xrightarrow{(\cdot P)} \mathcal{P}^{\mathrm{nres}}$$

whose composition is the identity. It now suffices to invoke Lemma 4.14 below. $\square$

Lemma 4.14. *Let A be a ring and let M be a noetherian left A-module. Let ϕ be an automorphism of A and assume that there are A-module morphisms*

$$M \xrightarrow{u} {}_\phi M \xrightarrow{v} M$$

whose composition is the identity. Then u, v are isomorphisms.

Proof. uv is an idempotent morphism ${}_\phi M \to {}_\phi M$ whose image is isomorphic to M (as $vu = \mathrm{id}$). Hence we obtain an isomorphism ${}_\phi M \cong M \oplus X$ for some left A-module X. Or $M \cong {}_\theta M \oplus Y$ for $\theta := \phi^{-1}$ and $Y := {}_\theta X$. Iterating we get

$$M \cong Y \oplus {}_\theta Y \oplus {}_{\theta^2} Y \oplus \cdots \oplus {}_{\theta^n} M$$

where $Y \subset Y \oplus {}_\theta Y \subset \cdots$ represents an infinite ascending chain of submodules of M, contradicting the noetherianity of M. $\square$

Proof of Proposition 4.12. The isomorphisms $(\cdot \partial_i) : \mathcal{P}^{\mathrm{nres}} \to \tau^*_{a_i} \mathcal{P}^{\mathrm{nres}}$ exhibited in Proposition 4.13 provide descent data for the action of $Y(H) \subset \mathfrak{h}$ on $\mathbb{T}^* \times \mathfrak{h}^{\mathrm{nres}}$ by translation. This descent cannot be done in the Zariski topology (cf. Remark 4.15 below) but it can be done in in the analytic topology since for this topology the action is discrete. This proves the result. $\square$

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[6]Note that while $\mathfrak{h}^{\mathrm{nres}}$ is not an open subscheme of $\mathfrak{h}$ it is still a perfectly good noetherian affine scheme, albeit not of finite type.

Remark 4.15. The relative $\mathcal{D}$-module $\tilde{\mathcal{P}}$ exhibited in Proposition 4.12 is not algebraic. In fact the situation is already interesting in the case that $T = 1$, $\dim H = \dim \mathbb{T} = 1$. In that case $H \times \mathbb{T}^* \cong \mathbb{C}^* \times \mathbb{C}^*$ and the underlying $\mathcal{O}_{\mathbb{T}^* \times H^{\mathrm{nres}}}^{\mathrm{an}}$-module of $\tilde{\mathcal{P}}$ is the restriction of the analytic line bundle $\mathcal{L}$ on $\mathbb{T}^* \times H$ which is locally over an open contractible subset U of H of the form

$$\mathcal{L}_U := \mathcal{L} \mid (\mathbb{T}^* \times U) = x^{\hat{h}} \mathcal{O}_{\mathbb{T}^* \times U}^{\mathrm{an}}$$

where $\hat{h} = (\log h)/(2\pi i) \in \Gamma(U, \mathcal{O}_U^{\mathrm{an}})$, and where $x^{\hat{h}}$ is a symbol satisfying $x^{\hat{h}+1} = xx^{\hat{h}}$ (so that $\mathcal{L}_U$ is canonically independent of the chosen branch of $\log h$). One checks that $\mathcal{L}$ has no global section so it is not algebraic. In fact $c_2(\mathcal{L}) \in H^2(\mathbb{T}^* \times H, \mathbb{Z}) \cong \mathbb{Z}$ is non-trivial so $\mathcal{L}$ is not flat. Hence it cannot be made into a module over $\mathcal{D}_{\mathbb{T}^* \times H}^{\mathrm{an}}$, as one perhaps naively might hope.

5. SOLUTIONS OF THE GKZ SYSTEM

5.1. **Mellin-Barnes solutions.** In [Beu16] Beukers shows that so-called *Mellin-Barnes integrals* satisfy the GKZ system. Let $\gamma \in \mathbb{C}^d = Y(\mathbb{T})_{\mathbb{C}} = X(\mathbb{T}^*)_{\mathbb{C}}$ be such that $\alpha = A(\gamma)$.

Let us first recall the definition in loc. cit. For $\sigma \in Y(T)_{\mathbb{C}}$ such that[7] $\mathrm{Re}(\gamma_j + \langle b_j, \sigma \rangle) \notin \mathbb{N}$ the Mellin-Barnes integral is formally defined as

$$(5.1) \qquad M(v_1, \ldots, v_d) = \int_{\sigma + iY(T)_{\mathbb{R}}} \prod_{j=1}^d \Gamma(-\gamma_j - \langle b_j, s \rangle) v_j^{\gamma_j + \langle b_j, s \rangle} ds.$$

The condition on σ is to guarantee that the integrand does not have poles on the integration domain. For the (in)dependence of σ, γ see Lemma 5.3 below.

Differentiating (5.1) under the integral sign with respect to the v_i yields the relation $(E_\phi - \phi(\alpha))M = 0$ for $\phi \in \mathfrak{h}^*$, which is part of the GKZ system.

5.1.1. *Making the Mellin-Barnes integral single-valued.* Note that M is defined on $\mathbb{T}^*$ (more precisely on its region of convergence, see §5.1.2 below) but it is multi-valued due to the fact that the exponentials $v_j^u := e^{\log(v_j)u}$ are multi-valued. To make M single-valued we write $v = e^{2\pi i \hat{v}}$ for $\hat{v} \in \mathrm{Lie}\,\mathbb{T}^* = Y(\mathbb{T}^*)_{\mathbb{C}} = X(\mathbb{T})_{\mathbb{C}}$ and we express the integrand in terms of $\hat{v}$. Writing the single-valued version of M as $\hat{M}$, i.e. $\hat{M}(\hat{v}) = M(v)$, we have

$$(5.2) \qquad \hat{M}(\hat{v}) = \int_{\sigma + iY(T)_{\mathbb{R}}} e^{2\pi i(\langle \hat{v}, \gamma \rangle + \langle B\hat{v}, s \rangle)} \prod_{j=1}^d \Gamma(-\gamma_j - \langle b_j, s \rangle) ds.$$

5.1.2. *Convergence domain.* By [Beu16, Corollary 4.2] the integral defining $\hat{M}$ converges absolutely if[8] $B(\mathrm{Re}\,\hat{v}) \in (1/2)\Sigma$.

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[7]It is possible to choose suitable σ except when $\mathrm{Re}\,\gamma_i \in \mathbb{N}$ and $b_i = 0$ for some i. Below we will not care about this case. See Convention 5.4.

[8]Formally the description of the convergence domain in [Beu16] is slightly different from ours. However under the standing Assumption 4.1 both descriptions are equivalent.

5.1.3. *Other GKZ relations.*

Theorem 5.1. [Beu16, Theorem 3.1] *If*

$$(5.3) \qquad \operatorname{Re}(\gamma_i + \langle b_i, \sigma \rangle) < 0 \quad \text{for all } 1 \le i \le d,$$

then M also satisfies $\Box_l M = 0$ for all $l \in L$.

Note: in loc. cit. γ is assumed to be real, but the proof works equally well for complex γ.

Lemma 5.2. *For a given α, there exists $\gamma \in \mathbb{C}^d$ with $A\gamma = \alpha$ and $\sigma \in Y(T)_{\mathbb{C}}$ such that* (5.3) *holds if and only if $\operatorname{Re}\alpha$ is in $\mathbb{R}_{<0}A$.*

Proof. We need to determine when there exists $\gamma \in \mathbb{C}^d$ with $A\gamma = \alpha$ and $\sigma \in Y(T)_{\mathbb{C}}$ such that $\forall i : \operatorname{Re}(\gamma_i + \langle b_i, \sigma \rangle) < 0$. Now we have $\sum_i (\gamma_i + \langle b_i, \sigma \rangle) a_i = \sum_i \gamma_i a_i = \alpha$. In other words we may assume $\sigma = 0$ and we have to look for γ satisfying $\forall i : \operatorname{Re}\gamma_i < 0$ and $\sum_i \gamma_i a_i = \alpha$. Such γ exists if and only of α is as in the statement of lemma. $\qquad\square$

Lemma 5.3. *For given α with $\operatorname{Re}\alpha \in \mathbb{R}_{<0}A$, let $\sigma \in Y(T)_{\mathbb{C}}, \gamma \in \mathbb{C}^d$ with $A\gamma = \alpha$. Assume that σ, γ satisfy* (5.3). *Then the corresponding Mellin-Barnes integral only depends on α.*

Proof. We first fix γ and vary $\sigma \to \sigma'$ (preserving (5.3)). Then the integral does not change by the fact that the domain $\{\sigma + t(\sigma' - \sigma) + iY(\mathbb{T})_{\mathbb{R}} \mid t \in [0,1]\}$ does not contain any pole of the integrand. Now we keep σ fixed and we vary $\gamma \to \gamma'$ (keeping $A\gamma' = \alpha$) such that (σ, γ') still satisfies (5.3). Then we can find σ' such that $\forall i : (\gamma_i + \langle b_i, \sigma \rangle)_i = (\gamma_i' + \langle b_i, \sigma' \rangle)_i$ (as $A(\gamma - \gamma') = 0$ and hence $\gamma - \gamma' \in B^*Y(T)$). Hence (γ', σ') satisfies (5.3). Then we use the already established independence of σ. $\qquad\square$

Convention 5.4. *For $\operatorname{Re}\alpha \in \mathbb{R}_{<0}A$ we write M^α (or $\hat{M}^\alpha$) for the MB-integral corresponding to $\gamma \in Y(\mathbb{T})_{\mathbb{C}} \cong \mathbb{C}^d$ such that $\alpha = A\gamma$, and $\sigma \in Y(T)_{\mathbb{C}}$ such that $\forall i : \operatorname{Re}(\gamma_i + \langle b_i, \sigma \rangle) < 0$. This notation is justified by Lemma 5.3, as the MB integral is independent of σ, γ satisfying* (5.3) *for $A\gamma = \alpha$ with α fixed. If we locally fix α then we usually write $M := M^\alpha$ (or $\hat{M} := \hat{M}^\alpha$).*

The next lemma and the ensuing remark will be used below.

Lemma 5.5. *The integral $\hat{M}^\alpha$ depends holomorphically on the parameter $\alpha \in \mathbb{R}_{<0}A$.*

Proof. By Lemma 5.3, it suffices to fix σ satisfying (5.3) for some γ and prove that $\hat{M}^\alpha$ depends holomorphically on the parameter γ on the domain where (5.3) is satisfied for γ. We then write $\hat{M}^\gamma$ instead of $\hat{M}^\alpha$. To accomplish this we use Morera's theorem. Let D be the domain as in the lemma. Then we need to check that the integral of $\hat{M}^\gamma$ as a function of γ vanishes when integrating over a closed piecewise C^1-curve C in D. At this point we apply the estimate [Beu16, §4] of the absolute value of the integrand of $\hat{M}^\gamma$, which allows us to apply the Fubini theorem, and exchange the integrals over C and over $\sigma + iY(T)_{\mathbb{R}}$. As the integrand of $\hat{M}^\gamma$ is an analytic function of γ (on D), its integral over C vanishes. Thus, the condition of Morera's theorem holds. $\qquad\square$

Remark 5.6. Below we will extend Mellin-Barnes integrals analytically outside the convergence domain of their defining integrals. By the generalized Hartogs' lemma

[BM48, Theorem 5, Ch. VII], the conclusion of Lemma 5.5 remains valid for such extended functions.

5.1.4. *Restriction of Mellin-Barnes solutions.* From (5.2) one obtains

$$(5.4) \qquad \hat{M}(\hat{v} + A^*w) = e^{2\pi i \langle w, \alpha \rangle} \hat{M}(\hat{v}).$$

We use the splitting ι from §4.3, which induces a corresponding splitting of B : $X(\mathbb{T})_{\mathbb{C}} \to X(T)_{\mathbb{C}}$. Then $\hat{M} \circ \iota$ becomes a function which is defined on $(1/2)\Sigma \times iX(T)_{\mathbb{R}}$.

Of course $\hat{M} \circ \iota$ depends on the splitting ι. With δ being the difference of splittings ι and ι' as in Remark 4.4 we find

$$(\hat{M} \circ \iota')(\hat{v}) = e^{2\pi i \langle \delta \hat{v}, \alpha \rangle} (\hat{M} \circ \iota)(\hat{v}).$$

So $\hat{M} \circ \iota'$ and $\hat{M} \circ \iota$ differ by the character $e^{2\pi i \langle \delta(-), \alpha \rangle}$ of $X(T)_{\mathbb{C}}$ which agrees nicely with Remark 4.4.

We now fix ι and write $\hat{M}$ for $\hat{M} \circ \iota$. We will also think of $\hat{M}$ as a multi-valued function M on $X(T)_{\mathbb{C}}/X(T) \xrightarrow[\cong]{e^{2\pi i -}} T^*$. It will always be clear from the context on which spaces M and $\hat{M}$ are defined.

5.1.5. *Basis of solutions.* We assume that W is quasi-symmetric. We give a basis of solutions of the restricted GKZ system $\overline{\mathcal{P}}(\alpha)$ on a dense open subset of $T^* \setminus V(E_A)$ (see §4.4). To make the solutions univalued we will work on a dense open subset of the corresponding covering space

$$(X(T)_{\mathbb{R}} \setminus \mathcal{H}) \times iX(T)_{\mathbb{R}} \subset X(T)_{\mathbb{C}} \setminus (\zeta + \mathcal{H}_{\mathbb{C}}) \longrightarrow T^* \setminus V(E_A)$$

which by §3.5 has a cell decomposition

$$(X(T)_{\mathbb{R}} \setminus \mathcal{H}) \times iX(T)_{\mathbb{R}} = \bigcup_{C \in \mathcal{C}^0} C \times iX(T)_{\mathbb{R}}.$$

Proposition 5.7. *Assume that $\alpha \in Y(H)_{\mathbb{C}}$ is non-resonant and $\alpha \in \sum \mathbb{R}_{<0} A$. For $\chi \in X(T)$ put*

$$\hat{M}_\chi(x) := \hat{M}(x - \chi)$$

(see Convention 5.4). Then the collection of functions

$$(5.5) \qquad \mathcal{M}_C := \left\{ \hat{M}_\chi | \chi \in \mathcal{L}_C \right\}$$

where $\mathcal{L}_C$ is as defined in §3.5, gives a basis of solutions of the pullback of the GKZ system $\overline{\mathcal{P}}(\alpha)$ to $C \times iX(T)_{\mathbb{R}}$ for $C \in \mathcal{C}^0$.

Proof. Let $C \in \mathcal{C}^0$. Recall that $\mathcal{L}_C = (\nu + \Delta) \cap X(T)$ for $\nu \in C$. If $\chi \in \mathcal{L}_C$ then $\nu \in \chi + \Delta$ (as $\Delta = -\Delta$ by unimodularity) and hence $C \subset \chi + \Delta$. Moreover, as $C \in \mathcal{C}$ is a chamber it does not lie on the boundary of $\chi + \Delta$. By §5.1.2, $\hat{M}$ is defined on $(1/2)\Sigma \times iX(T)_{\mathbb{R}}$. Hence $\hat{M}_\chi$ is defined on $C \times iX(T)_{\mathbb{R}}$.

The set $\left\{ \hat{M}_\chi | \chi \in \mathcal{L}_C \right\}$ is linearly independent by [NPT19, Lemma 5.3.1] (see also [Beu16, Proposition 4.3)]). As for non-resonant α the rank of the GKZ system equals $|\mathcal{L}_C| = D$ (see §4.2), the conclusion follows by Theorem 5.1 (using the assumption $\alpha \in \mathbb{R}_{<0} A$, and recalling the Convention 5.4). $\qquad \square$

5.2. **Power series solutions.** The GKZ system has a formal solution given by [GKZ89]

$$\Phi_\gamma(v_1, \ldots, v_d) = \sum_{l \in L} \prod_{i=1}^{d} \frac{v_i^{(B^*l)_i + \gamma_i}}{\Gamma((B^*l)_i + \gamma_i + 1)}.$$

This function is multi-valued but it can be made single-valued by considering it as a function $\hat{\Phi}_\gamma$ on $X(\mathbb{T})_\mathbb{C}$, as we did for M. We have the formula

$$\hat{\Phi}_\gamma(\hat{v}) = \sum_{l \in L} \frac{e^{2\pi i \langle \hat{v}, B^*l + \gamma \rangle}}{\prod_{i=1}^{d} \Gamma((B^*l)_i + \gamma_i + 1)}.$$

Let $I = \{i_1, \ldots, i_n\}$ be a subset of $\{1, \ldots, d\}$ such that $b_{i_1}, \ldots, b_{i_n}$ are linearly independent in $X(T)_\mathbb{R}$. We choose γ_I as in §5.1 (i.e. $\alpha = A(\gamma_I)$) such that $\gamma_{I,i} \in \mathbb{Z}$ for $i \in I$. This gives us $|\det((b_i)_{i \in I})|$ choices for γ_I modulo L.

Lemma 5.8. [Beu11b, Proposition 16.2][Sti07, §3.3, §3.4] *Let $\rho \in \sum_{i \in I} \mathbb{R}_{>0} b_i \subset X(T)_\mathbb{R}$. Then Φ_{γ_I} converges on an open neighbourhood of*

$$D_\rho := \{v \in (\mathbb{C}^*)d \mid \exists \tilde{\rho} \in B^{-1}(\rho), 0 < t < t_\rho : \forall i : |v_i| = t^{\tilde{\rho}_i}\}$$

for a suitable $0 < t_\rho \ll 1$.

Corollary 5.9. *Let $\rho \in \sum_{i \in I} \mathbb{R}_{>0} b_i \subset X(T)_\mathbb{R}$. Then $\hat{\Phi}_{\gamma_I}$ converges on an open neighbourhood of*

$$\hat{D}_\rho := \{\hat{v} \in \mathbb{C}^d \mid \exists \tilde{\rho} \in B^{-1}(\rho), u > u_\rho : \operatorname{Im} \hat{v} = u\tilde{\rho}\}$$

for a suitable $u_\rho \gg 0$.

For $\rho \in X(T)_\mathbb{R}$ we define

$$\mathcal{I}_\rho = \left\{ J \subset \{1, \ldots, d\} \mid |J| = n, \mathbb{R}\{b_i \mid i \in J\} = X(T)_\mathbb{R}, \rho = \sum_{j \in J} \beta_j b_j \text{ for } \beta_j > 0 \right\}.$$

If $\mathcal{I}_\rho$ is non-empty then we call ρ a *convergence direction*.

We let $\hat{\mathcal{I}}_\rho$ be the multiset with $\det((b_i)_{i \in I})$ copies of $I \in \mathcal{I}_\rho$. To each copy we associate a distinct γ_I (i.e. γ_I does not depend only on $I \in \mathcal{I}_\rho$, but on $I \in \hat{\mathcal{I}}_\rho$). For simplicity, we denote Φ_I, $\hat{\Phi}_I$ for Φ_{γ_I}, $\hat{\Phi}_{\gamma_I}$, $I \in \hat{\mathcal{I}}_\rho$.

Proposition 5.10. [GKZ89] *(see also [Beu16, §2]) If α is totally non-resonant and ρ is generic (see §3.5, Lemma 3.2), then $\left\{\Phi_I \mid I \in \hat{\mathcal{I}}_\rho\right\}$ is a basis of solutions of the GKZ system on an open neighbourhood of D_ρ.*[9]

5.2.1. *Restriction of power series solutions.* Let us write $\hat{\Phi}_\gamma$ for $\hat{\Phi}_\gamma \circ \iota$ where $\iota : X(T)_\mathbb{C} \to X(\mathbb{T})_\mathbb{C}$ is as in §5.1. Then one checks for $x \in X(T)_\mathbb{C}$

$$\hat{\Phi}_\gamma(x) = \sum_{l \in Y(T)} \frac{e^{2\pi i \langle x, l + \iota\gamma \rangle}}{\prod_{i=1}^{d} \Gamma((B^*l)_i + \gamma_i + 1)}$$

where $\iota\gamma \in Y(T)_\mathbb{C}$, denoting, by a slight abuse of notation, $\iota : Y(\mathbb{T})_\mathbb{C} \to Y(T)_\mathbb{C}$ also the adjoint to ι. Note that for $p \in X(T)$

(5.6) $$\hat{\Phi}_\gamma(x + p) = e^{2\pi i \langle p, \iota\gamma \rangle} \hat{\Phi}_\gamma(x)$$

---

[9]The assumption of total non-resonance enters here, and guarantees that all solutions at infinity are logarithm free, which as it will be seen later, simplifies the computation of the monodromy.

as $\langle p, l \rangle \in \mathbb{Z}$ for $l \in Y(T)$.

Corollary 5.11. *Let the setting be as in Lemma 5.8. Then $\hat{\Phi}_{\gamma_I}$ converges on an open neighbourhood of*

$$\hat{D}_\rho := \{ x \in X(T)_{\mathbb{C}} \mid \operatorname{Im} x = u\rho \text{ for } u > u_\rho \}$$

for a suitable $u_\rho \gg 0$.

Proof. By Corollary 5.9, $\hat{\Phi}_{\gamma_I}$ converges for $x \in X(T)_{\mathbb{C}}$ such that

$$\operatorname{Im}(\iota(x)) \in]u_\rho, \infty[B^{-1}(\rho)$$

for a suitable $u_\rho \gg 0$. This is equivalent to

$$\iota(\operatorname{Im} x) \in B^{-1}(]u_\rho, \infty[\rho)$$

and this is equivalent to $\operatorname{Im} x = B(\iota(\operatorname{Im} x)) \in]u_\rho, \infty[\rho$. $\qquad\square$

6. Monodromy of the GKZ system in the quasi-symmetric case

In this section assume throughout that W is quasi-symmetric. We give a very explicit description of the representation of the fundamental groupoid given by the local system determined by the GKZ system. We do this by connecting the MB integral solutions with the power series solutions. This approach is similar to the one used in [Beu16] from which we took our inspiration.

6.1. **The fundamental groupoid of the complement of a hyperplane arrangement.** Let $(\mathcal{H}, \mathcal{C}, V, \ldots)$ be as in §3.4. There are a number of presentations available [Del72, KS16, HLS20, Sal87] for the fundamental groupoid $\Pi_1(V_{\mathbb{C}} \setminus \mathcal{H}_{\mathbb{C}})$. We will use the presentation from [KS16].

Let $C_1, C_2 \in \mathcal{C}^0$ be chambers with $\dim C_1 \wedge C_2 = n - 1$. Denote $C_0 = C_1 \wedge C_2$ and let H be an equation for the hyperplane in $\mathcal{H}$ containing C_0 which is strictly positive on C_2.

For every $C \in \mathcal{C}^0$, choose $\rho_C \in C$ and put $\rho_1 := \rho_{C_1} \in C_1$, $\rho_2 := \rho_{C_2} \in C_2$ and for $\ell \in V \setminus H_0$ consider the path v_ℓ connecting ρ_1 to ρ_2 via the following line segments

$$\rho_1 \to i\ell + \rho_1 \to i\ell + \rho_2 \to \rho_2.$$

Then v_ℓ, $v_{\ell'}$ are homotopic iff $H_0(\ell), H_0(\ell')$ have the same sign. So up to homotopy this gives us two paths $\rho_1 \to \rho_2$ in $V_{\mathbb{C}} \setminus \mathcal{H}_{\mathbb{C}}$. We pick ℓ such that $H_0(\ell) > 0$ and write $v_{C_1 C_2} = v_\ell$.

Definition 6.1. Consider the abstract groupoid $\Pi(\mathcal{H})$ with the following presentation:

(1) An object for every $C \in \mathcal{C}^0$.
(2) A morphism $v_{C_1 C_2} : C_1 \to C_2$ for every $C_1, C_2 \in \mathcal{C}^0$ with $C_1 \wedge C_2 \neq \emptyset$, such that $v_{CC} = \operatorname{id}$.
(3) Relations of the form $v_{C_1 C_3} = v_{C_2 C_3} v_{C_1 C_2}$ for collinear (§3.4) triples C_1, C_2, $C_3 \in \mathcal{C}^0$.

It is easy to see that $\Pi(\mathcal{H})$ is generated by $v_{C_1 C_2}$ for couples C_1, C_2 which share a facet.

Proposition 6.2. [KS16, Proposition 9.11] *There is an equivalence of groupoids*

$$\Pi(\mathcal{H}) \to \Pi_1(V_{\mathbb{C}} \setminus \mathcal{H}_{\mathbb{C}})$$

sending $C \in \mathcal{C}^0$ to $\rho_C \in C$ and $v_{C_1 C_2} : C_1 \to C_2$ such that $\dim C_1 \wedge C_2 = n-1$ to $v_{C_1 C_2} : \rho_{C_1} \to \rho_{C_2}$.

6.2. **Fundamental groupoids of quotient spaces.** If $\mathcal{M}$ is a groupoid and $\mathcal{G}$ is a group acting on $\mathcal{M}$ then the semi-direct product $\mathcal{M} \rtimes \mathcal{G}$ has the same objects as $\mathcal{M}$ and is obtained by freely adjoining morphisms $g_m : m \to gm$ to $\mathcal{M}$, for $g \in \mathcal{G}$, $m \in \mathrm{Ob}(\mathcal{M})$ subject to the relations

   (1) $e_m = \mathrm{id}_m$, for $e \in \mathcal{G}$ the identity element.
   (2) $h_{gm} \cdot g_m = (hg)_m$ for $g, h \in \mathcal{G}$, $m \in \mathrm{Ob}(\mathcal{M})$.
   (3) $g(f) \cdot g_m = g_n \cdot f$ for $g \in \mathcal{G}$, $f : m \to n$ in $\mathcal{M}$.

It is easy to see that the construction of $\mathcal{M} \rtimes \mathcal{G}$ is compatible with equivalences of groupoids. If $\mathcal{G}$ acts freely and discretely on a topological space Y then there is an equivalence of groupoids

$$(6.1) \qquad \Pi_1(Y) \rtimes \mathcal{G} \cong \Pi_1(Y/\mathcal{G})$$

(see [Bro06, Chapter 11][HLS20, §6]). For use below we make this more concrete.

Recall that a $\mathcal{G}$-equivariant local system L on Y is a local system equipped with *descent data*, i.e. isomorphisms

$$u_g : L \to g^* L$$

for all $g \in \mathcal{G}$, satisfying the standard cocycle condition. Then L descends to a local system $\bar{L}$ on $Y/\mathcal{G}$ such that $\pi^*(\bar{L}) \cong L$ where $\pi : Y \to Y/\mathcal{G}$ is the quotient morphism.

Lemma 6.3. *The $\Pi_1(Y) \rtimes \mathcal{G}$ representation $\mathcal{L}$ corresponding to $\bar{L}$ via the equivalence* (6.1) *is the following:*

   (1) $\mathcal{L}_y = L_y$ *for* $y \in Y = \mathrm{Ob}(\Pi_1(Y) \rtimes \mathcal{G})$.
   (2) *The $\Pi_1(Y)$-action on $\coprod_{y \in Y} \mathcal{L}_y$ is obtained from the fact that L is a local system on Y.*
   (3) *If $g \in \mathcal{G}$ then the corresponding morphism $\mathcal{L}(g_y) : \mathcal{L}_y = L_y \to \mathcal{L}_{gy} = L_{gy}$ is obtained by specializing u_g at y (using the fact that $(g_* L)_y = L_{gy}$).*

Now let $V, \mathcal{H}, \ldots$ be as above and assume that V is equipped with an affine, $\mathcal{H}$-preserving, group action by a group $\mathcal{G}$. In that case $V_{\mathbb{C}} \setminus \mathcal{H}_{\mathbb{C}}$ and $\mathcal{C}$ are of course also preserved. If $\mathcal{G}$ acts freely and discretely then from Proposition 6.2 and (6.1) we obtain equivalences of groupoids

$$(6.2) \qquad \Pi(\mathcal{H}) \rtimes \mathcal{G} \xrightarrow{\cong} \Pi_1(V_{\mathbb{C}} \setminus \mathcal{H}_{\mathbb{C}}) \rtimes \mathcal{G} \xrightarrow{\cong} \Pi_1((V_{\mathbb{C}} \setminus \mathcal{H}_{\mathbb{C}})/\mathcal{G}).$$

6.3. **Statement of the main result.** We now let $\mathcal{H}, \mathcal{C}, \ldots$ have again their standard meaning. After choosing a splitting $\iota : \mathbb{T} \to T$ of $B^* : T \to \mathbb{T}$, the GKZ system defines a local system on $(X(T)_{\mathbb{C}} \setminus (\mathcal{H}_{\mathbb{C}} + \zeta))/X(T)$ (cfr §4.4). By (6.2) we have equivalences

(6.3)

$$\Pi(\mathcal{H}) \rtimes X(T) \underset{\cong}{\longrightarrow} \Pi_1(V_{\mathbb{C}} \setminus \mathcal{H}_{\mathbb{C}}) \rtimes X(T) \xrightarrow[\cong]{\text{translation}} \Pi_1((X(T)_{\mathbb{C}} \setminus (\mathcal{H}_{\mathbb{C}} + \zeta))/X(T)).$$

In other words, the GKZ system yields a representation of $\Pi(\mathcal{H}) \rtimes \mathcal{G}$ which we will now describe explicitly. First we introduce some notation. Let $C' < C$ with

$\dim C = n$, $\dim C' = n - 1$. Let $L \in \mathcal{H}$ be the hyperplane spanned by C' and assume that L is represented by an equation which is strictly positive on C. Then we put

(6.4) $$J_{C'C} = \{i \in \{1, \ldots, d\} \mid L_0(b_i) > 0\}.$$

Below we write the analytic continuation along a path v as $\mathsf{v}(-)$.

Theorem 6.4. *Assume that $\alpha \in Y(H)_{\mathbb{C}}$ is non-resonant and satisfies $\operatorname{Re}\alpha \in \mathbb{R}_{<0}A$ and let M be the representation of $\Pi(\mathcal{H}) \rtimes X(T)$ corresponding to the local system given by the solutions of the GKZ system $\bar{\mathcal{P}}(\alpha)$ via (6.3).*

(1) *For $C \in \mathcal{C}^0$ we have*
$$M(C) = \mathcal{M}_C$$
(using the notation (5.5)).

(2) *For $C_1, C_2 \in \mathcal{C}^0$ such that $\dim C_1 \wedge C_2 = n - 1$ write $C_0 = C_1 \wedge C_2$. The map $\mathsf{v}_{C_1 C_2} : M(C_1) \to M(C_2)$ evaluated on $\hat{M}_\chi \in M(C_1)$ is given by*

(6.5) $\quad M(\mathsf{v}_{C_1 C_2})(\hat{M}_\chi) =$

$$\begin{cases} \hat{M}_\chi & \text{if } \chi \in \mathcal{L}_{C_1} \cap \mathcal{L}_{C_2}, \\ \sum_{\emptyset \neq \mathsf{J} \subset J_{C_0 C_2}} (-1)^{|\mathsf{J}|+1} \left(\prod_{j \in \mathsf{J}} e^{-2\pi i \gamma_j} \right) \hat{M}_{\chi + \sum_{j \in \mathsf{J}} b_j} & \text{if } \chi \in \mathcal{L}_{C_1} \setminus \mathcal{L}_{C_2}, \end{cases}$$

where γ is the unique element of $Y(\mathbb{T})_{\mathbb{C}}$ such that $A\gamma = \alpha$ and $\iota\gamma = 0$.

(3) *$M(\mu_C)(\hat{M}_\chi) = \hat{M}_{\chi+\mu}$ for $\mu \in X(T)$ and $\chi \in \mathcal{L}_C$, $C \in \mathcal{C}^0$.*

The proof of this theorem will be carried out in the remainder of this section.

6.4. **Reminder on the splitting.** As already stated above, like in §4.3 we choose a splitting $\iota : \mathbb{T} \to T$ of $B^* : T \to \mathbb{T}$ and we consider the pullback of the GKZ system under $\iota : X(T)_{\mathbb{C}} \to X(\mathbb{T})_{\mathbb{C}}$ and we do the same for the Mellin-Barnes solutions (see §5.1.4) and the formal power series solutions (see §5.2.1).

6.5. **Connecting MB solutions to power series solutions.**

Proposition 6.5. *Assume that $\alpha \in Y(H)_{\mathbb{C}}$ is totally non-resonant and $\operatorname{Re}\alpha \in \mathbb{R}_{<0}A$. Let $\rho \in X(T)_{\mathbb{R}}$ be generic (c.f. §3.5) and $C \in \mathcal{C}^0$. We have on $\hat{D}_\rho \cap (C \times iX(T)_{\mathbb{R}})$*

(6.6) $$\hat{M}_\chi = \sum_{I \in \hat{\mathcal{I}}_\rho} e^{-2\pi i \langle \chi, \iota\gamma_I \rangle} \hat{\Phi}_I^s$$

where $\hat{\Phi}_I^s = a\hat{\Phi}_I$ for suitable $a \in \mathbb{C}^*$, depending only on I, α and ρ.

Proof. We have $\hat{M}^\alpha = \sum_{I \in \hat{\mathcal{I}}_\rho} a_I \hat{\Phi}_I$ for $a_I \neq 0$ ($\hat{M}^\alpha$ as in Convention 5.4) by Propositions 5.7, 5.10 and (5.6). Put $\hat{\Phi}_I^s = a_I \hat{\Phi}_I$. Then (6.6) follows from the definition of M_χ and (5.6). $\qquad\square$

Below we write $\mathcal{P}_\rho = \{\hat{\Phi}_I^s \mid I \in \hat{\mathcal{I}}_\rho\}$.

6.6. **Monodromy.** The fundamental groupoid of $X(T)_{\mathbb{C}} \setminus (\zeta + \mathcal{H}_{\mathbb{C}})$ acts on Mellin-Barnes solutions by analytic continuation. Following literally the equivalences in (6.3) we have to carry out the analytic continuation for $\mathsf{v}_{C_1 C_2} + \zeta$ for chambers C_1, C_2 sharing a face. However this turns out to be technically slightly inconvenient. We therefore observe that $\mathsf{v}_{C_1 C_2}$ was defined as v_ℓ for suitable ℓ and by taking $\|\ell\| \gg 0$ we may assume that $\mathsf{v}_{C_1 C_2}$ and $\mathsf{v}_{C_1 C_2} + \zeta$ are homotopy equivalent. So below we assume that ℓ is taken in this way and we will do the analytic continuation for $\mathsf{v}_{C_1 C_2}$, rather than for its translated version. Also for technical reasons we further assume that ℓ is generic (c.f. §3.5).

Lemma 6.6. *The path $\mathsf{v}_{C_1 C_2}$ (for $\|\ell\| \gg 0$ as was assumed above) intersects $\hat{D}_\ell$ and moreover stays within $C_1 \times iX(T)_{\mathbb{R}} \cup \hat{D}_\ell \cup C_2 \times iX(T)_{\mathbb{R}}$. In particular, it intersects the common convergence domain of $\mathcal{P}_\ell$ as well as the domains of definition of $\mathcal{M}_{C_1}$ and $\mathcal{M}_{C_2}$.*

Proof. This follows from Corollary 5.11. $\qquad\qquad\square$

Proof of Theorem 6.4. (1) is simply Proposition 5.7. (3) follows from the fact that $\mu \in X(T)$ acts on $X(T)_{\mathbb{C}}$ by the corresponding translation τ_μ. Unravelling the descent data for the pullback of $\bar{\mathcal{P}}(\alpha)$ to $X(T)_{\mathbb{C}} \setminus \mathcal{H}_{\mathbb{C}}$ we see by Lemma 6.3(3) that $M(\mu_C)(\hat{M}_\chi) = \tau^*_{-\mu}(\hat{M}_\chi) = \hat{M}_\chi \circ \tau_{-\mu} = \hat{M}_{\chi+\mu}$.

Now we concentrate on (2). For $\hat{M}_\chi \in \mathcal{M}_{C_1}$ we set $\hat{M}_\chi^{C_1 C_2} := \mathsf{v}_{C_1, C_2}(\hat{M}_\chi)$ (on $C_2 \times iX(T)_{\mathbb{R}}$).

If $\chi \in \mathcal{L}_{C_1} \cap \mathcal{L}_{C_2}$ then $\hat{M}_\chi$ is defined on $(C_1 \cup C_0 \cup C_2) \times iX(T)_{\mathbb{R}}$ and so no analytic continuation is necessary. This proves the first case of (2).

Now we proceed to the second case: $\chi \in \mathcal{L}_{C_1} \setminus \mathcal{L}_{C_2}$. The fact that $\chi + \sum_{j \in \mathsf{J}} b_j \in \mathcal{L}_{C_2}$ for $\mathsf{J} \neq \emptyset$, so that (6.5) is well-defined, follows from Lemma 6.10 below.

To continue the proof, we first reduce to the case that α is totally non-resonant. Assume that α is non-resonant. Then there exists a sequence of totally non-resonant $(\alpha^i)_i \in \mathbb{R}_{<0} A$ which converges to α (as the set of totally non-resonant parameters is dense in the set of non-resonant parameters). By Remark 5.6, we may compute the expansion $\hat{M}_\chi^{C_1 C_2}$ for $\chi \in \mathcal{L}_{C_1} \setminus \mathcal{L}_{C_2}$ in the basis $\mathcal{M}_{C_2}$, for α, as a limit of the corresponding expansions for α^i (note that we have to adapt[10] γ to α^i). Thus, if (6.5) holds for α^i, it also holds for α.

From now we assume that α is totally non-resonant. Put $J = J_{C_0 C_2}$. Analytically continuing the elements of $\mathcal{M}_{C_1}$, $\mathcal{M}_{C_2}$ along the path $\mathsf{v}_{C_1 C_2}$ we can write them as linear combinations of elements in $\mathcal{P}_\ell$ (recall that we have assumed that ℓ is generic so that Proposition 6.5 applies with $\rho = \ell$). Hence by (6.6) we have to prove

$$\sum_{I \in \hat{\mathcal{I}}_\ell} e^{-2\pi i \langle \chi, \iota \gamma_I \rangle} \hat{\Phi}_I^s = \sum_{\emptyset \neq \mathsf{J} \subset J} (-1)^{|\mathsf{J}|+1} \prod_{j \in \mathsf{J}} e^{-2\pi i \gamma_j} \sum_{I \in \hat{\mathcal{I}}_\ell} e^{-2\pi i \langle \chi + \sum_{j \in \mathsf{J}} b_j, \iota \gamma_I \rangle} \hat{\Phi}_I^s.$$

Equivalently, for all $I \in \hat{\mathcal{I}}_\ell$

$$e^{-2\pi i \langle \chi, \iota \gamma_I \rangle} = \sum_{\emptyset \neq \mathsf{J} \subset J} (-1)^{|\mathsf{J}|+1} \left(\prod_{j \in \mathsf{J}} e^{-2\pi i \gamma_j} \right) e^{-2\pi i \langle \chi + \sum_{j \in \mathsf{J}} b_j, \iota \gamma_I \rangle}.$$

---

[10]We may think of γ as $\kappa \alpha$ where $\kappa : Y(H) \to Y(\mathbb{T})$ is the splitting of A such that $\iota \kappa = 0$.

Or simply

$$1 = \sum_{\emptyset \neq \mathsf{J} \subset J} (-1)^{|\mathsf{J}|+1} \prod_{j \in \mathsf{J}} e^{-2\pi i (\gamma_j + \langle b_j, \iota \gamma_I \rangle)}$$

which may be rewritten as

$$0 = \prod_{j \in J} (1 - e^{-2\pi i (\gamma_j + \langle b_j, \iota \gamma_I \rangle)})$$

and using Lemma 6.7 below this becomes

$$(6.7) \qquad\qquad 0 = \prod_{j \in J} (1 - e^{-2\pi i \gamma_{I,j}}).$$

It follows from Lemma 6.9 below that $I \cap J \neq \emptyset$. If $j \in I \cap J$ then $\gamma_{I,j} \in \mathbb{Z}$. This implies (6.7). □

6.7. Supporting lemmas.
Here we prove some lemmas that were used above. Recall that W was assumed to be quasi-symmetric.

Lemma 6.7. *Let $\gamma, \gamma' \in Y(T)_{\mathbb{C}}$ be such that $A\gamma = \alpha$, $A\gamma' = \alpha$, $\iota\gamma = 0$. Then for all $1 \leq j \leq d$ we have*

$$(6.8) \qquad\qquad \gamma_j' = \gamma_j + \langle b_j, \iota\gamma' \rangle.$$

Proof. Let $\delta = \gamma' - \gamma$. Then (6.8) maybe rewritten as

$$\delta = B^* \iota \delta$$

which follows from the fact ι is a splitting of B^* and $\delta \in \operatorname{im} B^*$. □

To state the next lemma we introduce some notation. For a facet F of Δ let $\lambda_F \in Y(T)_{\mathbb{R}}$, $c_F \in \mathbb{R}$ be such that $\langle \lambda_F, - \rangle - c_F = 0$ is a defining equation for the hyperplane spanned by F, which is positive on Δ.

Lemma 6.8. *Let $C_1, C_2 \in \mathcal{C}^0$, $C_0 = C_1 \wedge C_2$, $\dim C_0 = n - 1$.*

(1) *Put*

$$F = \Delta \setminus \bigcup_{\rho_0 \in C_0, \rho_2 \in C_2} (\rho_2 - \rho_0 + \Delta).$$

   This is a facet of Δ parallel to C_0.
(2) *$\forall \rho_0 \in C_0 : \mathcal{L}_{C_0} \setminus \mathcal{L}_{C_2} \subset \rho_0 + \operatorname{relint} F$.[11]*
(3) *$\forall \rho_0 \in C_0 : \forall \rho_1 \in C_1 : \forall \rho_2 \in C_2 : \langle \lambda_F, \rho_2 - \rho_0 \rangle > 0$, $\langle \lambda_F, \rho_1 - \rho_0 \rangle < 0$.*

Proof.   (1) The affine spaces spanned by $C \in \mathcal{C}$ are intersections of translated hyperplanes spanned by facets of $-\Delta = \Delta$. In particular if $C \in \mathcal{C}$ is a facet then the hyperplane L spanned by it must be parallel to a facet of Δ. Let $\langle \lambda_L, - \rangle - c_L = 0$ be an equation of L which is strictly positive on C_2. It follows that

$$(6.9) \qquad \forall \rho_2 \in C_2 : \rho_1 \in C_1 : \forall \rho_0 \in C_0 : \langle \lambda_L, \rho_2 - \rho_0 \rangle > 0, \quad \langle \lambda_L, \rho_1 - \rho_0 \rangle < 0.$$

Let $\langle \lambda_L, - \rangle - c \geq 0$ be a supporting half space for Δ. We claim

$$(6.10) \qquad\qquad F = \Delta \cap \{\delta \mid \langle \lambda_L, \delta \rangle - c = 0\}.$$

---

[11]We denote by relint the relative interior.

So we have to prove

$$\{\delta \in \Delta \mid \langle \lambda_L, \delta \rangle - c > 0\} = \Delta \cap \left(\bigcup_{\rho_0 \in C_0, \rho_2 \in C_2} (\rho_2 - \rho_0 + \Delta) \right).$$

The $\supset$ inclusion is clear so we prove the $\subset$ inclusion. Assume $\delta \in \Delta$ is such that $\langle \lambda_L, \delta \rangle - c := u > 0$. Then there exists $\delta' \in \Delta$ such that $\langle \lambda_L, \delta' \rangle - c < u$. Then $\langle \lambda_L, \delta - \delta' \rangle > 0$. Hence for $0 < \epsilon \ll 1$ we may write $\epsilon(\delta - \delta') = \rho_2 - \rho_0$ for $\rho_0 \in C_0$, $\rho_2 \in C_2$. So $\delta - (\rho_2 - \rho_0) = (1 - \epsilon)\delta + \epsilon\delta' \in \Delta$.

(2) In the last paragraph we showed that if $\delta \in \Delta \setminus F$ then $\delta \in \rho_2 - \rho_0 + \Delta$, where moreover ρ_0 might be chosen arbitrarily. Hence, $(-\rho_0 + \mathcal{L}_{C_0}) \setminus F \subset -\rho_0 + \mathcal{L}_{C_2}$ which implies $\mathcal{L}_{C_0} \setminus \mathcal{L}_{C_2} \subset \rho_0 + F$. As, by (1),

$$\bigcap_{\rho_0 \in C_0} (\rho_0 + F) = \bigcap_{\rho_0 \in C_0} (\rho_0 + \operatorname{relint} F),$$

this suffices.

(3) This follows from (6.9) as $\lambda_F = \lambda_L$ by (6.10). $\qquad\qquad\square$

Lemma 6.9. *Let $C_1, C_2 \in \mathcal{C}^0$, $C_0 = C_1 \wedge C_2$. Assume that $\dim C_0 = n - 1$. Let $\chi \in \mathcal{L}_{C_0} \setminus \mathcal{L}_{C_2}$. Write*

$$(6.11) \qquad\qquad \chi = \rho_0 - (1/2)\sum_{i \in J} b_i + \sum_{i \in J'} \beta_i b_i,$$

with $\beta_i \in (-1/2, 0)$, $\rho_0 \in C_0$, $J' \subset \{1, \ldots, d\} \setminus J$ and $|J|$ minimal. Then $J = \{i \mid \langle \lambda_F, b_i \rangle > 0\}$ for λ_F as in Lemma 6.8. In particular $J = J_{C_0 C_2}$ (see (6.4)).

Moreover,

$$\forall \rho_1 \in C_1 : \forall \rho_2 \in C_2 : \forall I \in \hat{\mathcal{I}}_{\rho_2 - \rho_1} : I \cap J \neq \emptyset.$$

Proof. Let $\lambda = \lambda_F \in Y(T)_{\mathbb{R}}$ be as in Lemma 6.8. Since F is in particular the unique facet of $\rho_0 + \Delta$ containing χ we have $J = \{i \mid \langle \lambda, b_i \rangle > 0\}$ (see e.g. [ŠVdB17a, Lemma A.7]). Let $\rho_i \in C_i$ for $i = 1, 2$. Lemma 6.8 further implies $\langle \lambda, \rho_2 - \rho_1 \rangle > 0$. Let $I \in \hat{\mathcal{I}}_{\rho_2 - \rho_1}$. By definition of $\hat{\mathcal{I}}_{\rho_2 - \rho_1}$ we can write $\rho_2 - \rho_1 = \sum_{i \in I} \beta_i b_i$ with $\beta_i > 0$. Hence there exists $i \in I$ such that $\langle \lambda, b_i \rangle > 0$, i.e. $i \in J$. $\qquad\square$

Lemma 6.10. *Let the setting be as in Lemma 6.9. Then for any $\emptyset \neq \mathsf{J} \subseteq J$ one has $\chi + \sum_{j \in \mathsf{J}} b_j \in \mathcal{L}_{C_2}$.*

Proof. By Lemma 6.8, there exists a unique facet F of Δ such that $\chi \in \rho_0 + F$ for $\rho_0 \in C_0$. Let $\lambda = \lambda_F$. One has $\chi_{\mathsf{J}} := \chi + \sum_{j \in \mathsf{J}} b_j \notin \rho_0 + F$ for any $\emptyset \neq \mathsf{J} \subseteq J$, as $J = \{i \mid \langle \lambda, b_i \rangle > 0\}$ by Lemma 6.9. On the other hand it follows from (6.11) and quasi-symmetry that $\chi_{\mathsf{J}} \in (\rho_0 + \Delta) \cap X(T) = \mathcal{L}_{C_0}$. So $\chi_{\mathsf{J}} \in \mathcal{L}_{C_0} \setminus (\rho_0 + F) \subset \mathcal{L}_{C_2}$ using Lemma 6.8(1). $\qquad\square$

7. PERVERSE SHEAVES ON AFFINE HYPERPLANE ARRANGEMENTS

7.1. **Kapranov-Schechtman data.** In this section we consider a general affine hyperplane arrangement $(V, \mathcal{H})$ as in §3.4. Kapranov and Schechtman provide in [KS16] a combinatorial description of the abelian category of perverse sheaves on $V_{\mathbb{C}} \setminus \mathcal{H}_{\mathbb{C}}$.

Theorem 7.1. [KS16, Theorem 9.10] *The category of perverse sheaves on $V_{\mathbb{C}}$ with respect to the stratification induced by $\mathcal{H}_{\mathbb{C}}$ is equivalent to the category of diagrams consisting of finite dimensional vector spaces E_C, $C \in \mathcal{C}$, and linear maps $\gamma_{C'C} : E_{C'} \to E_C$, $\delta_{CC'} : E_C \to E_{C'}$ for $C' \leq C$ such that $((E_C)_C, (\gamma_{C'C})_{CC'})$ is a representation of $(\mathcal{C}, \leq)$ and $((E_C)_C, (\delta_{CC'})_{CC'})$ a representation of $(\mathcal{C}, \geq)$, and the following conditions are satisfied:*

- *(m) $\gamma_{C'C}\delta_{CC'} = \mathrm{id}_{E_C}$ for $C' \leq C$. In particular, $\phi_{C_1C_2} := \gamma_{C'C_2}\delta_{C_1C'}$ for $C' \leq C_1, C_2$ is well defined.*
- *(i) $\phi_{C_1C_2}$ is an isomorphism for every $C_1 \neq C_2$ which are of the same dimension k lying, lie in the same k-dimensional affine space and share a facet.*
- *(t) $\phi_{C_1C_3} = \phi_{C_2C_3}\phi_{C_1C_2}$ for collinear (§3.4) triples of faces (C_1, C_2, C_3).*

We denote the category of data introduced in Theorem 7.1, except for the requirement $\dim E_C < \infty$, by $\mathrm{KS}(\mathcal{H})$. It is obviously an abelian category. The full subcategory of $\mathrm{KS}(\mathcal{H})$ such that $\forall C : \dim E_C < \infty$ is denoted by $\mathrm{KS}^c(\mathcal{H})$. If $E \in \mathrm{KS}^c(\mathcal{H})$ then the associated perverse sheaf on $V_{\mathbb{C}}$ is denoted by $\tilde{E}$. Thus we have an equivalence of categories

$$\mathrm{KS}^c(\mathcal{H}) \to \mathrm{Perv}_{\mathcal{H}_{\mathbb{C}}}(V_{\mathbb{C}}) : E \mapsto \tilde{E}$$

where $\mathrm{Perv}_{\mathcal{H}_{\mathbb{C}}}(V_{\mathbb{C}})$ is the abelian category of perverse sheaves on $V_{\mathbb{C}}$ with respect to stratification $\mathcal{H}_{\mathbb{C}}$. For an explicit construction of $\tilde{E}$ starting from E see [KS16, §6.C].

Let $\Pi(\mathcal{H})$ be as §6.1. There is a "restriction" functor

$$(7.1) \qquad\qquad \mathrm{Res} : \mathrm{KS}(\mathcal{H}) \to \mathrm{Rep}(\Pi(\mathcal{H}))$$

which associates to $E \in \mathrm{KS}(\mathcal{H})$ the representation of $\Pi(\mathcal{H})$ given by $C \mapsto E_C$, $\mathrm{v}_{CC'} \mapsto \phi_{CC'}$. We have the following result.

Proposition 7.2. *Let $E \in \mathrm{KS}^c(\mathcal{H})$. Then the representation of $\Pi(\mathcal{H})$ corresponding to $\tilde{E} \mid (V_{\mathbb{C}} \setminus \mathcal{H}_{\mathbb{C}})$ is given by $\mathrm{Res}(E)$.*

Proof. This follows from Proposition 6.2, using the construction of the Kapranov-Schechtman data [KS16, (4.13),(3.5),§4.C]. $\qquad\square$

7.2. **Duality.** If $E = ((E_C)_C, (\delta_{CC'})_{CC'}, (\gamma_{C'C})_{C'C}) \in \mathrm{KS}^c(\mathcal{H})$ then we put

$$(7.2) \qquad \mathbb{D}(E) := ((E_C^*)_C, (\gamma_{C'C}^{\vee})_{C'C}, (\delta_{CC'}^{\vee})_{CC'}) \in \mathrm{KS}^c(\mathcal{H}).$$

Remark 7.3. Let $\mathbb{D}$ denote the Verdier dual. It is asserted in [KS16, Proposition 4.6][12] that $\mathbb{D}(\tilde{E}) \cong \widetilde{\mathbb{D}(E)}$ but it turns out that a slight twist, similar to the twist in [KS19, Proposition 4.6], is in fact needed to make this statement literally correct [KS]. In the sequel we will not use the compatibility with the Verdier dual.

7.3. **Group actions.** Assume that in addition V is equipped with an affine, $\mathcal{H}$-preserving, group action by a group $\mathcal{G}$ as in §6.2. In that case we can routinely define a $\mathcal{G}$-equivariant version $\mathrm{KS}(\mathcal{G}, \mathcal{H})$ of $\mathrm{KS}(\mathcal{H})$. An object in $\mathrm{KS}(\mathcal{G}, \mathcal{H})$ consists of an object $E = ((E_C)_C, (\delta_{CC'})_{CC'}, (\gamma_{C'C})_{C'C})$ in $\mathrm{KS}(\mathcal{H})$ together with isomorphisms

$$\phi_{g,C} : E_C \to E_{gC}$$

---

[12]Note that in [KS16] perverse sheaves are shifted so that local systems live in degree zero. Hence in loc. cit. the Verdier dual is also shifted.

satisfying the standard cocycle condition, the requirement that $\phi_{e,C}$ is the identity for $e \in \mathcal{G}$ the neutral element, and the obvious compatibility with $\delta_{CC'}$ and the $\gamma_{C'C}$. The subcategory $\mathrm{KS}^c(\mathcal{G}, \mathcal{H}) \subset \mathrm{KS}(\mathcal{G}, \mathcal{H})$ is spanned by the objects in $\mathrm{KS}(\mathcal{G}, \mathcal{H})$ which lie in $\mathrm{KS}^c(\mathcal{H})$ if we forget the $\mathcal{G}$-action.

An object $E \in \mathrm{KS}^c(\mathcal{G}, \mathcal{H})$ defines a perverse sheaf on the stack $V_{\mathbb{C}}/\mathcal{G}$ which we denote by $\tilde{E}$. This yields an equivalence of categories

$$\mathrm{KS}^c(\mathcal{G}, \mathcal{H}) \cong \mathrm{Perv}_{\mathcal{H}_{\mathbb{C}}/\mathcal{G}}(V_{\mathbb{C}}/\mathcal{G}).$$

There is a $\mathcal{G}$-equivariant version of the restriction functor

(7.3)
$$\mathrm{Res} : \mathrm{KS}(\mathcal{G}, \mathcal{H}) \to \mathrm{Rep}(\Pi(\mathcal{H}) \rtimes \mathcal{G})$$

which associates to $E \in \mathrm{KS}(\mathcal{G}, \mathcal{H})$ the representation of $\Pi(\mathcal{H}) \rtimes \mathcal{G}$ given by $C \mapsto E_C$, $\mathrm{v}_{CC'} \mapsto \phi_{CC'}$, $g_C \mapsto \phi_{g,C}$.

Corollary 7.4. *Assume that the group $\mathcal{G}$ acts freely and discretely on V and let $E \in \mathrm{KS}(\mathcal{G}, \mathcal{H})$. Then the representation of $\Pi(\mathcal{H}) \rtimes \mathcal{G}$ (see 6.2) associated to the local system $\tilde{E} \mid (V_{\mathbb{C}} \setminus \mathcal{H}_{\mathbb{C}})/\mathcal{G}$ is given by $\mathrm{Res}(E)$.*

For use below we note that the dual (7.2) can be lifted to a functor

$$\mathbb{D} : \mathrm{KS}^c(\mathcal{G}, \mathcal{H}) \to \mathrm{KS}^c(\mathcal{G}, \mathcal{H})$$

where $\phi_{g,C}^{\mathbb{D}(E)}$ is defined as $(\phi_{g,C}^{-1})^{\vee} = \phi_{g^{-1},C}^{\vee}$.

7.4. **R-linear versions.** It is often convenient to consider a version of the category $\mathrm{KS}(\mathcal{H})$ in which the E_C are modules over a commutative ring R. Then we denote the corresponding category by $\mathrm{KS}_R(\mathcal{H})$ and other related notations will be decorated with R as well in a self-explanatory fashion. For a ring extension S/R we will use the obvious change of rings functor

(7.4)
$$- \otimes_R S : \mathrm{KS}_R(\mathcal{H}) \to \mathrm{KS}_S(\mathcal{H}).$$

By $\mathrm{KS}_R^c(\mathcal{H})$ we denote the full subcategory of $\mathrm{KS}_R(\mathcal{H})$ consisting of objects $(E_C)_C$ such that each E_C is a finitely generated projective R-module.

Observation 7.5. *We will encounter the following situation. Assume that in addition V is equipped with an affine, $\mathcal{H}$-preserving, group action by a group $\mathcal{G} = \mathcal{A} \times \mathcal{B}$ as in §7.3 such that $\mathcal{A}$ sends every element of $\mathcal{C}$ to itself. Then there is an isomorphism of categories*

(7.5)
$$\mathrm{KS}_R(\mathcal{G}, \mathcal{H}) \cong \mathrm{KS}_{R[\mathcal{A}]}(\mathcal{B}, \mathcal{H})$$

where $R[\mathcal{A}]$ is the group ring of $\mathcal{A}$.

Remark 7.6. One needs to be a bit careful in using (7.5) since with our current conventions we do not automatically have the corresponding "finite" statement $\mathrm{KS}_R^c(\mathcal{G}, \mathcal{H}) \cong \mathrm{KS}_{R[\mathcal{A}]}^c(\mathcal{B}, \mathcal{H})$.

8. Perverse schobers on affine hyperplane arrangements

8.1. **$\mathcal{H}$-schobers.** Let the setting be as in §7. $\mathcal{H}$-schobers are categorifications of Kapranov-Schechtman data and hence they can be regarded as categorifications of perverse sheaves. The following brief exposition is more or less literally taken from [ŠVdB19, §3] which in turn is based on [BKS18, KS15].

Definition 8.1. An $\mathcal{H}$-schober $\mathcal{E}$ on $V_{\mathbb{C}}$ is given by triangulated categories $\mathcal{E}_C$, $C \in \mathcal{C}$, adjoint pairs of exact functors $(\delta_{CC'} : \mathcal{E}_C \to \mathcal{E}_{C'}, \gamma_{C'C} : \mathcal{E}_{C'} \to \mathcal{E}_C)$ for $C' \leq C$ such that $(\mathcal{E}_C, (\delta_{C'C})_{C'C})$ defines a pseudo-functor[13] from $(\mathcal{C}, \geq)$ to the 2-category of triangulated categories satisfying the following conditions:

(M) The unit of the adjunction $(\delta_{CC'}, \gamma_{C'C})$ defines a natural isomorphism $\mathrm{id}_{\mathcal{E}_C} \xrightarrow{\cong} \gamma_{C'C}\delta_{CC'}$ for $C' \leq C$, and thus $\phi_{C_1 C_2} := \gamma_{C'C_2}\delta_{C_1 C'}$ for $C' \leq C_1, C_2$ is well defined up to canonical natural isomorphism.

(I) $\phi_{C_1 C_2}$ is an equivalence for every $C_1 \neq C_2$ of the same dimension d lying in the same d-dimensional affine space which share a facet.

(T) For collinear triples of faces (C_1, C_2, C_3) with common face C_0 the counit of the adjunction $(\delta_{C_0 C_2}, \gamma_{C_2 C_0})$ defines a natural isomorphism $\phi_{C_2 C_3}\phi_{C_1 C_2} \xrightarrow{\cong} \phi_{C_1 C_3}$.

The 2-category of $\mathcal{H}$-schobers is denoted by $\mathrm{Schob}(\mathcal{H})$. $\mathcal{H}$-schobers are categorifications of Kapranov-Schechtman data in the following sense.

Fact. *Applying $K^0(-)$ to the data defining an $\mathcal{H}$-schober yields a functor*

$$K^0(-) : \mathrm{Schob}(\mathcal{H}) \mapsto \mathrm{KS}_{\mathbb{Z}}(\mathcal{H})$$

which we call the "decategorification" functor.

Below we use the short hand $K_{\mathbb{C}}^0(-)$ for $K^0(-) \otimes_{\mathbb{Z}} \mathbb{C}$. If $K_{\mathbb{C}}^0(\mathcal{E}) \in \mathrm{KS}^c(\mathcal{H})$ then we will also define $\tilde{K}_{\mathbb{C}}^0(\mathcal{E}) = \widetilde{K_{\mathbb{C}}^0(\mathcal{E})}$ and we will refer to $\tilde{K}_{\mathbb{C}}^0(\mathcal{E})$ as a decategorification as well.

If $\mathcal{E}$ is an $\mathcal{H}$-schober then we will define a subschober $\mathcal{E}'$ of $\mathcal{E}$ as a collection of (full) triangulated subcategories $\mathcal{E}'_C \subset \mathcal{E}_C$ which are stable under $(\delta_{CC'}, \gamma_{C'C})$. In this way $\mathcal{E}'$ becomes tautologically an $\mathcal{H}$-schober. We refer to the latter as a subschober as well and sometimes we use the notation $\mathcal{E}' \subset \mathcal{E}$.

8.2. **Group actions.** Assume that in addition V is equipped with an affine, $\mathcal{H}$-preserving, group action by a group $\mathcal{G}$ as in §6.2. In that case we may define a $\mathcal{G}$-equivariant version $\mathrm{Schob}(\mathcal{G}, \mathcal{H})$ of the category $\mathrm{Schob}(\mathcal{H})$. A $\mathcal{G}$-*action* on an $\mathcal{H}$-schober on $V_{\mathbb{C}}$ is a collection of exact functors

$$\phi_{g,C} : \mathcal{E}_C \to \mathcal{E}_{gC}$$

for $g \in \mathcal{G}$, $C \in \mathcal{C}$, enhanced with natural isomorphisms $\phi_{h,gC}\phi_{g,C} \cong \phi_{hg,C}$ satisfying the obvious compatibility for triple products in $\mathcal{G}$, and the requirement that $\phi_{e,C}$ is the identity functor for $e \in \mathcal{G}$. Moreover we should have pseudo-commutative diagrams for every $C' < C$:

(8.1)
$$
\begin{array}{ccc}
\mathcal{E}_C & \xrightarrow{\phi_{g,C}} & \mathcal{E}_{gC} \\
\downarrow{\scriptstyle \delta_{C,C'}} & & \downarrow{\scriptstyle \delta_{gC,gC'}} \\
\mathcal{E}_{C'} & \xrightarrow[\phi_{g,C'}]{} & \mathcal{E}_{gC'}
\end{array}
$$

---

[13]To specify a pseudo-functor one also needs to specify suitable natural isomorphisms. As is customary we have suppressed these from the notations.

so that the implied natural isomorphism $\delta_{gC,gC'}\phi_{g,C} \cong \phi_{g,C'}\delta_{C,C'}$ should again satisfy a number of obvious compatibilities. An $\mathcal{H}$-schober equipped with a $\mathcal{G}$-action will be called a $\mathcal{G}$-*equivariant* $\mathcal{H}$-*schober*. The concept of a $\mathcal{G}$-equivariant subschober is defined in the obvious way.

We think of a $\mathcal{G}$-equivariant $\mathcal{H}$-schober as a perverse schober on the stack $V_{\mathbb{C}}/\mathcal{G}$. Its decategorification is a $\mathcal{G}$-invariant perverse sheaf; i.e. we will use the functors

$$K^0(-) : \mathrm{Schob}(\mathcal{G}, \mathcal{H}) \to \mathrm{KS}_{\mathbb{Z}}(\mathcal{G}, \mathcal{H})$$

and if $K^0_{\mathbb{C}}(\mathcal{E}) \in \mathrm{KS}^c(\mathcal{G}, \mathcal{H})$ then we will also use the notation $\tilde{K}^0_{\mathbb{C}}(\mathcal{E}) := \widetilde{K^0_{\mathbb{C}}(\mathcal{E})}$.

9. AN $\mathcal{H}$-SCHOBER USING GEOMETRIC INVARIANT THEORY

9.1. **Reminder.** From now on $\mathcal{H}$, $\mathcal{C}$ will have again their usual meaning (see §3.5). Assume that W is quasi-symmetric. In [ŠVdB19] we constructed an $\mathcal{H}$-schober on $X(T)_{\mathbb{C}}$ built up from suitable triangulated subcategories of $D(W/T)$.[14] We briefly recall this construction.[15] Let $P_\chi = \chi \otimes \mathcal{O}_W$ and set

$$P_C := \bigoplus_{\chi \in -\mathcal{L}_C} P_\chi, \quad \mathcal{S}_C := \langle P_C \rangle \subset D(W/T),$$

where $\langle S \rangle$ for $S \subset D(W/T)$ denotes the smallest strict, full triangulated subcategory, closed under coproduct, which contains S. Put

$$\Lambda_C = \mathrm{End}_{W/T}(P_C).$$

The functor $\mathrm{RHom}_{W/T}(P_C, -)$ defines an equivalence

$$(9.1) \qquad\qquad \mathcal{S}_C \cong D(\Lambda_C)$$

where $D(\Lambda_C)$ is the derived category of *right* Λ_C-modules. By [ŠVdB19, Theorem 5.6], Λ_C has finite global dimension. For use below we note that the quasi-inverse to (9.1) is given by

$$(9.2) \qquad\qquad D(\Lambda_C) \cong \mathcal{S}_C : M \mapsto M \otimes_{\Lambda_C} P_C.$$

For $C' \leq C$ let $\delta_{CC'} : \mathcal{S}_C \hookrightarrow \mathcal{S}_{C'}$ be the inclusion. Then $\delta_{CC'}$ admits a right adjoint

$$(9.3) \qquad\qquad \gamma_{C'C} = \mathrm{RHom}_{W/T}(P_C, -) \otimes_{\Lambda_C} P_C.$$

Set $\phi_{C_1 C_2} = \gamma_{C'C_2}\delta_{C_1C'}$ for $C' \leq C_1, C_2$. In addition for $\chi \in X(T)$, let $\phi_{\chi,C} : \mathcal{S}_C \to \mathcal{S}_{\chi+C}$ be the functor $M \mapsto (-\chi) \otimes M$.

Proposition 9.1. [ŠVdB19, Proposition 5.1] $\mathcal{S} := ((\mathcal{S}_C)_C, (\gamma_{C'C})_{C'C}, (\delta_{CC'})_{C'C})$ *with the* $X(T)$-*action* $(\phi_{\chi,C})_{\chi,C}$ *defines an* $X(T)$-*equivariant* $\mathcal{H}$-*schober on* $X(T)_{\mathbb{C}}$.

Let $D^c(W/T)$ be the full subcategory of $D(W/T)$ consisting of bounded complexes of T-equivariant coherent $\mathcal{O}_T$-modules. It was shown in [ŠVdB19] that $\delta_{C'C}$, $\gamma_{CC'}$ preserve the categories

$$\mathcal{S}^c_C := \mathcal{S}_C \cap D^c(W/T).$$

We obtain a corresponding $X(T)$-equivariant subschober

$$\mathcal{S}^c \subset \mathcal{S}.$$

---

[14]In loc. cit. we actually used W^*/T. Here, we follow the mirror symmetry convention and use W/T. This entails some sign changes in the definitions and formulas taken from [ŠVdB19].

[15]In loc. cit. we required, in addition to W being quasi-symmetric, that the generic T-stabilizer is finite. This is satisfied here since $\mathbb{Z}B = X(T)$.

We recall the following lemma.

Lemma 9.2. [ŠVdB19, §5.3] *Under the equivalence* (9.1), $\mathcal{S}_C^c$ *corresponds to the subcategory* $D^c(\Lambda_C)$ *of* $D(\Lambda_C)$ *consisting of bounded complexes with finitely generated cohomology.*

In the next section we discuss a subschober of $\mathcal{S}^c$.

9.2. **The finite length subschober.** The nullcone $W^u \subset W$ is defined by

$$W^u = \{x \in W \mid 0 \in \overline{Tx}\}.$$

Let $D^u(W/T)$ be the full triangulated subcategory of complexes whose cohomology is supported on W^u. We put

$$\mathcal{S}_C^u = \mathcal{S}_C \cap D^u(W/T),$$
$$\mathcal{S}_C^f = \mathcal{S}_C^u \cap \mathcal{S}_C^c.$$

Lemma 9.3. $(\mathcal{S}_C^u)_C$, $(\mathcal{S}_C^f)_C$ *define* $X(T)$-*equivariant subschobers* $\mathcal{S}^u$, $\mathcal{S}^f$ *of* $\mathcal{S}$.

Proof. $X(T)$-equivariance is obvious. Furthermore it is sufficient to prove that $(\mathcal{S}_C^u)_C$ is a subschober. Compatibility with $\delta_{CC'}$ is obvious so we have to discuss compatibility with $\gamma_{CC'}$. From the formula (9.3) it follows that $\gamma_{C'C}$ is right exact[16] for the standard t-structure on $D(W/T)$. Then one quickly finds that it is sufficient to prove that for $M \in \mathrm{coh}(W^u/T)$ the cohomology of

$$\mathrm{Hom}_{W/T}(P_C, M) \overset{L}{\otimes}_{\Lambda_C} P_C$$

is supported in W^u. By the hypotheses on M, $F := \mathrm{Hom}_{W/T}(P_C, M)$ is supported in the origin of $W/\!/T$. We now switch to T-equivariant $\mathbb{C}[W]$-modules. Those are supported in W^u if and only if they are supported in the origin of $W/\!/T$ when regarded as $\mathbb{C}[W]^T$-modules. By resolving P_C, it thus follows that $F \overset{L}{\otimes}_{\Lambda_C} P_C$ is supported in W^u. [17]   □

Notation 9.4. In case of possible confusion we decorate the KS-data associated to the $\mathcal{H}$-schobers $\mathcal{S}^c$ and $\mathcal{S}^f$ by $(-)^c$, $(-)^f$.

Let $\mathrm{mod}^f(\Lambda_C)$ be the full subcategory of $\mathrm{Mod}(\Lambda_C)$ consisting of finite dimensional right Λ_C-modules supported in the origin of $W/\!/T$.

Lemma 9.5. *Under the equivalence* (9.1), $\mathcal{S}_C^f$ *corresponds to the full subcategory* $D^f(\Lambda_C)$ *of* $D(\Lambda_C)$ *consisting of bounded complexes with cohomology in* $\mathrm{mod}^f(\Lambda_C)$.

For $\chi \in X(T)$ let S_χ be the corresponding 1-dimensional T-equivariant $\mathcal{O}_W$-module supported at the origin.

Lemma 9.6. *The simple objects in* $\mathrm{mod}^f(\Lambda_C)$ *are given by* $\mathrm{Hom}_{W/T}(P_C, S_\chi)$ *for* $\chi \in \mathcal{L}_{-C}$.

---

[16]It may look strange that a right adjoint is right exact, but this is not a contradiction. The standard t-structure on $D(W/T)$ does not descend to a t-structure on $\mathcal{S}_C$.

[17]We are using that for M, N respectively a right and a left module over a ring A (possibly non-finitely generated) and $R \subset Z(A)$, it is true that $\mathrm{supp}_R(M \otimes_A N) \subset \mathrm{supp}_R(M) \cap \mathrm{supp}_R(N)$. This is an immediate consequence of the definition of support.

Proof. The objects $\text{Hom}_{W/T}(P_C, S_\chi)$ are 1-dimensional and contained in the category $\text{mod}^f(\Lambda_C)$. Hence they are certainly simple. To prove the converse let $R = \mathbb{C}[W]^T$, equipped with its natural $\mathbb{N}$-grading. Then the simple objects of $\text{mod}^f(\Lambda_C)$ are the simple modules of the finite dimensional algebra $\Lambda'_C := \Lambda_C/R_{\geq 1}\Lambda_C$. Since Λ'_C is also $\mathbb{N}$-graded, $(\Lambda'_C)_{\geq 1}$ is nilpotent. Hence the simple Λ'_C-modules correspond to the indecomposable summands of $(\Lambda'_C)_0 = (\Lambda_C)_0$ and they are precisely the $\text{Hom}_{W/T}(P_C, S_\chi)$ for $\chi \in \mathcal{L}_{-C}$. $\qquad\square$

Below we will denote the object in $\mathcal{S}_C^f$ corresponding to $\text{Hom}_{W/T}(P_C, S_\chi)$ by $\mathfrak{s}_{C,\chi}$. Using the formula (9.2) we obtain

$$(9.4) \qquad \mathfrak{s}_{C,\chi} = \text{Hom}_{W/T}(P_C, S_\chi) \overset{L}{\otimes}_{\Lambda_C} P_C.$$

The following is clear.

Lemma 9.7. *For $\chi, \mu \in \mathcal{L}_{-C}$, $\text{Hom}_{W/T}(P_\chi, \mathfrak{s}_{C,\mu}) = \mathbb{C}^{\delta_{\chi,\mu}}$.*

9.3. Autoduality. Below we will need the autoduality functor

$$\mathcal{D} : D^b(\text{coh}(W/T)) \to D^b(\text{coh}(W/T))^\circ,$$

$$\mathcal{D} = \text{R}\mathcal{H}om_{W/T}(-, \mathcal{O}_W).$$

We recall some basic properties from [ŠVdB19, §5.4].

Lemma 9.8. *We have $\mathcal{D}(\mathcal{S}_C^*) = \mathcal{S}_{-C}^*$ for $* \in \{c, f\}$. Moreover, for $C' < C$, $\mathcal{D}\delta_{-C,-C'}\mathcal{D} = \delta_{CC'}$. Thus, $\mathcal{D}\gamma_{-C',-C}\mathcal{D}$ is left adjoint to $\delta_{CC'}$.*

It will be useful to consider the bifunctor

$$(9.5) \quad \text{RHom}_{W/T}(-, -)' := \text{RHom}_{W/T}(\mathcal{D}(-), -) :$$

$$\mathcal{S}_{-C}^c \times \mathcal{S}_C^f \to D^b(\text{mod}(\mathbb{C})) : (M, N) \mapsto \text{RHom}_{W/T}(\mathcal{D}M, N)$$

Lemma 9.9. *Let $C' < C$ in $\mathcal{C}$. Then $(\delta_{-C,-C'}^c, \gamma_{C'C}^f)$ and $(\gamma_{-C',-C}^c, \delta_{CC'}^f)$ are adjoint pairs under the bifunctor $\text{RHom}_{W/T}(-, -)'$. Moreover $(\phi_{\chi,-C}^c, \phi_{\chi,-\chi+C}^f)$ is an adjoint pair as well.*

Proof. This follows from Lemma 9.8. $\qquad\square$

10. Decategorification of the GIT $\mathcal{H}$-schober

10.1. Duality. In this section we will construct a canonical duality isomorphism in $\text{KS}_{\mathbb{Z}}(X(T), \mathcal{H})$,

$$(10.1) \qquad K^0(\mathcal{S}^c)^- \cong \mathbb{D}_{\mathbb{Z}}(K^0(\mathcal{S}^f)),$$

where $(-)^-$ represents the pullback of equivariant KS-data under $(X(T), X(T)_{\mathbb{R}}) \to (X(T), X(T)_{\mathbb{R}}) : (\chi, x) \mapsto (-\chi, -x)$. Concretely,

$$K^0(\mathcal{S}^c)^-(C) = K^0(\mathcal{S}^c)(-C) = K^0(\mathcal{S}_{-C}^c),$$

$$(10.2) \qquad K^0(\delta_{CC'}^c)^- = K^0(\delta_{-C,-C'}^c), \quad K^0(\gamma_{C'C}^c)^- = K^0(\gamma_{-C',-C}^c),$$

$$K^0(\phi_{\chi,C}^c)^- = K^0(\phi_{-\chi,-C}^c).$$

Lemma 10.1. *The following holds for the Grothendieck groups of $\mathcal{S}_C^c$, $\mathcal{S}_C^f$.*

(1) $K^0(\mathcal{S}_C^c)$ is freely generated by the classes $[P_\chi]$ for $\chi \in -\mathcal{L}_C$.
(2) $K^0(\mathcal{S}_C^f)$ is freely generated by the classes $[\mathfrak{s}_{C,\chi}]$ for $\chi \in -\mathcal{L}_C$.

Proof.

(1) We have $K^0(\mathcal{S}_C^c) = K^0(\Lambda_C)$ and as Λ_C is a $\mathbb{N}$-graded ring of finite global dimension we obtain by [Qui73, p. 112, Theorem 7] that $\Lambda_C \otimes_{\Lambda_{C,0}} -$ induces an isomorphism $K^0(\Lambda_{C,0}) \cong K^0(\Lambda_C)$. It now suffices to observe $\Lambda_{C,0}$ is semi-simple and its summands are indexed by $\chi \in -C$. The image of a summand corresponding to χ is $[P_\chi]$.

(2) This follows from Lemmas 9.5, 9.6. $\qquad\square$

For $M \in D^c(W/T)$ and $N \in D^f(W/T)$ the Euler pairing between the classes of M, N is defined as

$$(10.3) \qquad \langle [M], [N] \rangle = \sum_i (-1)^i \dim H^i(\mathrm{RHom}_{W/T}(M, N)).$$

Lemma 10.2. *The Euler pairing yields a perfect duality between the Grothendieck groups $K^0(\mathcal{S}_C^c)$ and $K^0(\mathcal{S}_C^f)$.*

Proof. This follows from Lemma 9.7. $\qquad\square$

We now define a twisted version of the Euler pairing:

$$(10.4) \qquad \langle -, - \rangle' := \langle K^0(\mathcal{D})(-), - \rangle : K^0(\mathcal{S}_{-C}^c) \times K^0(\mathcal{S}_C^f) \to \mathbb{Z}.$$

Lemma 10.3. *Let $C' < C$ in $\mathcal{C}$. Then $\langle -, - \rangle'$ is a perfect duality between $K^0(\mathcal{S}_{-C}^c)$ and $K^0(\mathcal{S}_C^f)$. Moreover, $(K^0(\delta_{-C,-C'}^c), K^0(\gamma_{C'C}^f))$ and $(K^0(\gamma_{-C',-C}^c), K^0(\delta_{CC'}^f))$ are adjoint pairs for $\langle -, - \rangle'$. Finally $(K^0(\phi_{\chi,-C}^c), K^0(\phi_{\chi,-\chi+C}^f))$ is an adjoint pair as well.*

Proof. This follows by combining Lemma 10.2 with Lemma 9.9. $\qquad\square$

Proof of (10.1). The map (10.1) is obtained from the pairing $\langle -, - \rangle'$. The fact that it a well defined map of equivariant KS-data follows from Lemma 10.3. Note: the fact that a minus sign appears in the $X(T)$-action follows from the inverse that appears in the description of $\phi_{g,C}$ for the equivariant dual. See §7.3. $\qquad\square$

Remark 10.4. The need for $(-)^-$ in (10.1) and the associated proofs leads to some foundational musing about the definition of $\mathcal{H}$-schobers. As mentioned in [ŠVdB19, Observation 3.6] the $\mathcal{H}$-schober $\mathcal{S}$ has some favorable properties not shared by all $\mathcal{H}$-schobers. In particular, as we observed in Lemma 9.8, $\delta_{CC'}$ also admits a left adjoint. Carrying this further, one may think of the $\mathcal{H}$-schobers as introduced in [KS15] as *right $\mathcal{H}$-schobers* and then introduce the dual concept of *left $\mathcal{H}$-schobers* by requiring that $\gamma_{C'C}$ be a left adjoint to $\delta_{CC'}$. Both left and right $\mathcal{H}$-schobers are uniquely determined by the δ's and hence it makes sense to use the notations $\mathcal{S}_l$, $\mathcal{S}_r$ to refer to a left and/or a right $\mathcal{H}$-schober with given δ's (if both $\mathcal{S}_l$, $\mathcal{S}_r$ exist, as in our case, then it even makes sense to refer to $\mathcal{S}$ itself as an *$\mathcal{H}$-bischober*). With these notations the formula (10.1) could have been written more elegantly as

$$K^0(\mathcal{S}_l^c) \cong \mathbb{D}_{\mathbb{Z}}(K^0(\mathcal{S}_r^f)).$$

where the duality is now realized via the Euler form $\langle -, - \rangle$ instead of the twisted version $\langle -, - \rangle'$.

10.2. **Explicit description of the monodromy isomorphisms.** Let us recall the complex $C_{\lambda,\chi}$ introduced in [ŠVdB17a, (11.3)], and used in [ŠVdB19], in order to construct $\gamma_{C'C}$ for $C' < C$. For $\lambda \in Y(T)$ define

$$W^{\lambda,+} = \{x \in W \mid \lim_{t \to 0} \lambda(t)x \text{ exists}\}.$$

We also put $K_\lambda = W/W^{\lambda,+}$, $d_\lambda = \dim K_\lambda$. Then by definition $C_{\lambda,\chi}$ is the Koszul resolution of $\mathcal{O}_{W^{\lambda,+}}$ tensored with $\chi \in X(T)$; i.e. $C_{\lambda,\chi}$ equals the complex (with the right-most term in degree 0)

$$(10.5) \quad 0 \to \chi \otimes \wedge^{d_\lambda} K_\lambda^* \otimes \mathcal{O}_W \to \chi \otimes \wedge^{d_\lambda - 1} K_\lambda^* \otimes \mathcal{O}_W \to \cdots \to \chi \otimes \mathcal{O}_W.$$

Lemma 10.5. *Assume that $C_1, C_2 \in \mathcal{C}^0$ share a facet C_0 and let $\chi \in \mathcal{L}_{-C_1} \setminus \mathcal{L}_{-C_2}$. Let $\langle \lambda, - \rangle - c$ for $\lambda \in Y(T)$ be a defining equation of the hyperplane spanned by $-C_0$ which is strictly positive on $-C_2$. Then $\phi_{C_1 C_2}(P_\chi) = \mathrm{cone}(P_\chi \to C_{\lambda,\chi})$.*

Proof. This follows from [ŠVdB17a, §5]. Note that $\mathcal{S}_C$ was called $\mathcal{S}_{-C}$ in loc. cit. and here our space is called W instead of W^*. Neither of these changes has any serious implications as we are proving an intrinsic statement.

We have

$$\phi_{C_1 C_2}(P_\chi) = \gamma_{C_0 C_2} \delta_{C_1 C_0}(P_\chi) = \gamma_{C_0 C_2}(P_\chi).$$

By Proposition 5.12 in loc. cit. we have a semi-orthogonal decomposition

$$(10.6) \qquad\qquad\qquad \mathcal{S}_{C_0} = \langle \mathcal{S}_{C_0 C_2}, \mathcal{S}_{C_2} \rangle.$$

By Lemma 6.8(2) $\mathcal{L}_{-C_0} \setminus \mathcal{L}_{-C_2}$ is contained in the relative interior of the translate of a single facet of Δ. It then follows from the discussion in loc. cit. that

$$\mathcal{S}_{C_0 C_2} = \langle (\chi \otimes \mathcal{O}_{W^{\lambda,+}})_{\chi \in \mathcal{L}_{-C_0} \setminus \mathcal{L}_{-C_2}} \rangle.$$

Using the fact that $C_{\lambda,\chi} \cong \mathcal{O}_{W^{\lambda,+}}$ it follows from Lemma 10.6 below and (10.6) that we have

$$\mathrm{cone}(P_\chi \to C_{\lambda,\chi}) \in \mathcal{S}_{C_0} \cap {}^\perp \mathcal{S}_{C_0 C_2} \subset \mathcal{S}_{C_2}.$$

This implies $\gamma_{C_0 C_2}(P_\chi) = \mathrm{cone}(P_\chi \to C_{\lambda,\chi})$. $\qquad\qquad\qquad\qquad\qquad \square$

We have used the following lemma.[18]

Lemma 10.6. *Let $\lambda \in Y(T)$, $\chi \in X(T)$.*

(1) The weights μ of $\chi \otimes \mathcal{O}_{W^{\lambda,+}}$ satisfy $\langle \lambda, \mu \rangle \le \langle \lambda, \chi \rangle$.
(2) All terms P_μ in $C_{\lambda,\chi}$, except P_χ, satisfy $\langle \lambda, \mu \rangle > \langle \lambda, \chi \rangle$.

Proof. First we note that $W^{\lambda,+}$ is spanned by the weight vectors e_j such that $\langle \lambda, b_j \rangle \ge 0$. Hence $\mathbb{C}[W^{\lambda,+}] = \mathrm{Sym}((W^{\lambda,+})^*)$ is generated by elements of weight $-b_j$ for $\langle \lambda, b_j \rangle \ge 0$. This proves the first claim.

For the second claim we note that K_λ is generated by weight vectors $\bar{e}_j$ such that $\langle \lambda, b_j \rangle < 0$ and now we look at the weights of $\wedge^i K_\lambda^*$ for $i > 0$. $\qquad \square$

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[18]Versions of this lemma have already been used numerous times in our previous work. We give the proof since we need some details about the complexes in the proof of Proposition 12.4.

11. The GIT $\mathcal{H}$-schober in the $X(\mathbb{T})$-equivariant setting

11.1. More splittings. In the rest of this section we will choose a splitting $\kappa :$ $H \to \mathbb{T}$ of $A : \mathbb{T} \to H$ (see §3.2) such that $\iota\kappa = 1$ where $\iota : \mathbb{T} \to T$ is the splitting of $B^* : T \to \mathbb{T}$ introduced in §4.3. Summarizing we now have the following maps

$$T \xrightarrow[\iota]{B^*} \mathbb{T} \xrightarrow[\kappa]{A} H$$

where the composable maps form short exact sequences. In other words with our choices we have defined an isomorphism

$$(11.1) \qquad\qquad\qquad \mathbb{T} \cong T \times H$$

where A, B^*, ι, κ are given by the appropriate inclusion and projection maps.

11.2. Lift of the $\mathcal{H}$-schober. The $X(T)$-equivariant $\mathcal{H}$-schober we have constructed lives on the hyperplane arrangement $\mathcal{H}$ in the real vector spaces $X(T)_{\mathbb{R}}$. Now observe that $X(T)_{\mathbb{R}}$ may be trivially equipped with a $X(\mathbb{T})$-action via the map $B :$ $X(\mathbb{T}) \to X(T)$. It turns out that all the constructions given in §9 have natural $\mathbb{T}$-equivariant versions. Throughout we follow the convention that such $\mathbb{T}$-equivariant versions are indicated by overlining and sometimes we even omit explicit definitions when they are obvious.

First of all a variation of the construction of $\mathcal{S}$ which yields an object in the category $\mathrm{Schob}(X(\mathbb{T}), \mathcal{H})$:

$$(11.2) \qquad \bar{\mathcal{S}} = ((\bar{\mathcal{S}}_C)_C, (\bar{\delta}_{CC'})_{CC'}, (\bar{\gamma}_{C'C})_{C'C}, (\bar{\phi}_{\chi,C})_{\chi,C})$$

on $X(T)_{\mathbb{R}}$, built up from triangulated subcategories $(\bar{\mathcal{S}}_C)_{C \in \mathcal{C}} \subset D(W/\mathbb{T})$ which are just the natural lifts of $(\mathcal{S}_C)_C \subset D(W/T)$ under the pushforward functor for the stack morphism $W/\mathbb{T} \to W/T$. More precisely for $\chi \in X(\mathbb{T})$ put $\bar{P}_\chi := \chi \otimes \mathcal{O}_W$ and for $C \in \mathcal{C}$ put

$$(11.3) \qquad\qquad \bar{\mathcal{L}}_C := B^{-1}(\mathcal{L}_C) \subset X(\mathbb{T}),$$

and

$$\bar{P}_C := \bigoplus_{\chi \in -\bar{\mathcal{L}}_C} \bar{P}_\chi, \qquad \bar{\mathcal{S}}_C := \langle \bar{P}_C \rangle \subset D(W/\mathbb{T}).$$

The functors $(\bar{\delta}_{CC'})_{CC'}, (\bar{\gamma}_{C'C})_{C'C}$ are defined in exactly the same way as for $\mathcal{S}$, and $\bar{\phi}_{\chi,C} : \bar{\mathcal{S}}_C \to \bar{\mathcal{S}}_{B\chi+C}$ with $\mathcal{M} \mapsto (-\chi) \otimes \mathcal{M}$. It is clear that $\bar{\mathcal{S}}$ has subschobers $\bar{\mathcal{S}}^c, \bar{\mathcal{S}}^f$ which are again defined like $\mathcal{S}^c$ and $\mathcal{S}^f$.

12. Decategorification of the $X(\mathbb{T})$-equivariant GIT $\mathcal{H}$-schober

We will now summarize the $\mathbb{T}$-equivariant versions of the results in §9.

12.1. Duality. Below for $M \in D^c(W/\mathbb{T})$, $N \in D^f(W/\mathbb{T})$, we denote

$$\langle [M], [N] \rangle = \sum_{i \in \mathbb{Z}, \chi \in X(H)} (-1)^i \dim(H^i(\mathrm{RHom}_{W/\mathbb{T}}(M, \chi^{-1} \otimes N)))\chi \in \mathbb{Z}[X(H)],$$

noting that the dimensions are finite, and the sum has a finite number of terms. Moreover, let $\mathcal{D} = \mathrm{R}\mathcal{H}om_{W/\mathbb{T}}(-, \mathcal{O}_W)$ and $\langle -, - \rangle' = \langle K^0(\mathcal{D})(-), - \rangle$. We note that since $X(H) \subset X(\mathbb{T})$ stabilizes $\mathcal{C}$, $K^0(\bar{\mathcal{S}}_C^c)$ and $K^0(\bar{\mathcal{S}}_C^f)$ are $\mathbb{Z}[X(H)]$-modules.

Lemma 12.1.

(1) The form $\langle -,- \rangle'$ is $\mathbb{Z}[X(H)]$-linear; i.e. for $a \in K^0(\bar{\mathcal{S}}^c_{-C})$, $b \in K^0(\bar{\mathcal{S}}^f_C)$, $f \in \mathbb{Z}[X(H)]$ we have

$$f\langle a, b \rangle' = \langle a, fb \rangle' = \langle fa, b \rangle'.$$

(2) $K^0(\bar{\mathcal{S}}^c_C)$ is a free $\mathbb{Z}[X(H)]$-module with basis $[\bar{P}_{\iota\chi}]$ for $\chi \in \mathcal{L}_{-C}$.

(3) $K^0(\bar{\mathcal{S}}^f_C)$ is a free $\mathbb{Z}[X(H)]$-module with basis $[\bar{\mathfrak{s}}_{C,\iota\chi}]$ for $\chi \in \mathcal{L}_{-C}$.

(4) The classes of $[\bar{P}_{\iota\chi}]_{\chi \in \mathcal{L}_C} \in K^0(\bar{\mathcal{S}}^c_{-C})$ and $[\bar{\mathfrak{s}}_{C,\iota\chi}]_{\chi \in \mathcal{L}_{-C}} \in K^0(\bar{\mathcal{S}}^f_C)$ are dual $\mathbb{Z}[X(H)]$-bases for $\langle -,- \rangle'$. Whence $\langle -,- \rangle'$ is a perfect duality between $K^0(\overline{\mathcal{S}}^c_{-C})$ and $K^0(\overline{\mathcal{S}}^f_C)$.

(5) For $C' < C$ in $\mathcal{C}$ the pairs $(K^0(\bar{\delta}^c_{-C,-C'}), K^0(\bar{\gamma}^f_{C',C}))$ and $(K^0(\bar{\gamma}^c_{-C',-C}), K^0(\bar{\delta}^f_{C,C'}))$ are adjoint pairs for $\langle -,- \rangle'$. The same holds for the pair $(K^0(\bar{\phi}^c_{\chi,-C}), K^0(\bar{\phi}^f_{\chi,-B\chi+C}))$ for $\chi \in X(\mathbb{T})$, $C \in \mathcal{C}$.

(6) We have

$$K^0(\mathcal{S}^c_C) \cong K^0(\bar{\mathcal{S}}^c_C) \otimes_{\mathbb{Z}[X(H)],\mathbf{1}} \mathbb{Z}, \qquad K^0(\mathcal{S}^c_C)_{\mathbb{C}} \cong K^0(\bar{\mathcal{S}}^c_C) \otimes_{\mathbb{Z}[X(H)],\mathbf{1}} \mathbb{Z}$$

where $\mathbf{1}$ is the ring homomorphism $\mathbb{Z}[X(H)] \to \mathbb{Z} : \chi \mapsto 1$.

Proof. (2,3,4,5) are proved like Lemmas 10.1, 10.2, 10.3. (1,6) are clear. $\square$

Convention 12.2. *From now on we will consider $K^0(\bar{\mathcal{S}}^*)$ for $* \in \{c, f\}$ as objects in the category $\mathrm{KS}_{\mathbb{Z}[H]}(X(T), \mathcal{H})$ via Observation 7.5 and the decomposition (11.1).*

Corollary 12.3. *From Lemma 12.1(5) we obtain a canonical duality isomorphism in $\mathrm{KS}_{\mathbb{Z}[H]}(X(T), \mathcal{H})$*

$$(12.1) \qquad K^0(\bar{\mathcal{S}}^c)^- \cong {}_\tau\mathbb{D}_{\mathbb{Z}[X(H)]}(K^0(\bar{\mathcal{S}}^f)),$$

where $(-)^-$ is like in (10.2), and τ denotes the twist of the $\mathbb{Z}[X(H)]$-action by the automorphism $\mu \mapsto \mu^{-1}$ for $\mu \in X(H)$.

12.2. **Monodromy.**

Proposition 12.4. *Let $N = \mathrm{Res}_{\mathbb{Z}[X(H)]}(K^0(\bar{\mathcal{S}}^c)) \in \mathrm{Rep}_{\mathbb{Z}[X(H)]}(\Pi(\mathcal{H}) \rtimes X(T))$ (see (7.3)) where we consider $K^0(\bar{\mathcal{S}}^c)$ as an object in $\mathrm{KS}_{\mathbb{Z}[X(H)]}(X(T), \mathcal{H})$ as above.*

(1) *For $C \in \mathcal{C}^0$ we have*
$$N(C) = K^0(\bar{\mathcal{S}}^c_C).$$

(2) *For $C_1, C_2 \in \mathcal{C}^0$ such that $\dim C_1 \wedge C_2 = n - 1$, denote $C_0 = C_1 \wedge C_2$. Set $J = J_{-C_0,-C_2}$ (see (6.4)). The map $v_{C_1 C_2}$ evaluated on $[\bar{P}_\chi]$ is given by*

$$N(v_{C_1 C_2})([\bar{P}_\chi]) = K^0(\bar{\phi}_{C_1 C_2})[\bar{P}_\chi] = [\bar{\phi}_{C_1 C_2}(\bar{P}_\chi)] =$$

$$\begin{cases} [\bar{P}_\chi] & \text{if } \chi \in \bar{\mathcal{L}}_{-C_1} \cap \bar{\mathcal{L}}_{-C_2}, \\ \sum_{\emptyset \neq J \subset J}(-1)^{|J|+1}[\bar{P}_{\chi - \sum_{j \in J} e_j}] & \text{if } \chi \in \bar{\mathcal{L}}_{-C_1} \setminus \bar{\mathcal{L}}_{-C_2}. \end{cases}$$

(3) $N(\mu_C)([\bar{P}_\chi]) = K^0(\bar{\phi}_{\iota\mu,C})([\bar{P}_\chi]) = [\bar{P}_{\chi - \iota\mu}]$ *for $\mu \in X(T)$ and $\chi \in \mathcal{L}_{-C}$, $C \in \mathcal{C}^0$.*

Proof. Everything follows from the definition of the functor Res.

(1) This is a direct application of the definition.

(2) This follow from Lemma 10.5 together with the fact, asserted in the proof of Lemma 10.6, that $W^{\lambda,+}$ (with λ as in Lemma 10.5) is spanned by the weight vectors e_j such that $\langle \lambda, b_j \rangle \geq 0$ which in the current setting is precisely the set J.

(3) This follows by using the $\mathbb{T}$-equivariant version of $\phi_{\mu,C}$ (see Proposition 9.1) combined with the splitting $\mathbb{T} \cong T \times H$ (see (11.1)) together with Observation 7.5. $\qquad\square$

12.3. **Specialisation.** For $h \in H$ and $* \in \{c, f\}$ we put

$$K_h^0(\bar{\mathcal{S}}^*) = K^0(\bar{\mathcal{S}}^*) \otimes_{\mathbb{Z}[X(H)],h} \mathbb{C} \in \mathrm{KS}(X(T), \mathcal{H})$$

where

(1) as above we view $K^0(\bar{\mathcal{S}}^*)$ as objects in $\mathrm{KS}_{\mathbb{Z}[X(H)]}(X(T), \mathcal{H})$ using Observation 7.5 and (11.1);

(2) we use the base extension functor (7.4) for the ring morphism

$$h : \mathbb{Z}[X(H)] \to \mathbb{C} : \chi \mapsto \chi(h).$$

We record the following trivial lemma.

Lemma 12.5. *Let* $M \in \mathrm{KS}_{\mathbb{Z}[X(H)]}(X(T), \mathcal{H})$. *Then* $(M^-)_h = (M_h)^-$. *If* $M(C)$ *is a finitely generated projective* $\mathbb{Z}[X(H)]$ *module for all* $C \in \mathcal{C}$ *then* $(\mathbb{D}_{\mathbb{Z}[X(H)]}M)_h = \mathbb{D}(M_h)$.

12.4. **Specialisation and monodromy.**

Proposition 12.6. *For* $h \in H$ *let* $N_h = \mathrm{Res}(K_h^0(\bar{\mathcal{S}}^c)) \in \mathrm{Rep}(\Pi(\mathcal{H}) \rtimes X(T))$ *(see* (7.3)*). We write* $[\overline{P}_\chi]_h$ *for the image of* $[\overline{P}_\chi]$ *in* $K_h^0(\bar{\mathcal{S}}^c)$. *Choose* $\alpha \in Y(H)_{\mathbb{C}}$ *in such a way that* $e^{-2\pi i \alpha} = h$.

(1) *For* $C \in \mathcal{C}^0$ *we have*

$$N_h(C) = K_h^0(\bar{\mathcal{S}}_C^c).$$

(2) *For* $C_1, C_2 \in \mathcal{C}^0$ *such that* $\dim C_1 \wedge C_2 = n - 1$, *denote* $C_0 = C_1 \wedge C_2$. *Set* $J = J_{-C_0,-C_2}$ *(see* (6.4)*). The map* $v_{C_1 C_2}$ *evaluated on* $[\bar{P}_{\iota\chi}]_h$ *is given by*

$$N_h(v_{C_1 C_2})([\bar{P}_{\iota\chi}]_h) = K_f^0(\bar{\phi}_{C_1 C_2})[\bar{P}_{\iota\chi}]_h = [\bar{\phi}_{C_1 C_2}(\bar{P}_{\iota\chi})]_h =$$

$$\begin{cases} [\bar{P}_{\iota\chi}]_h & \text{if } \chi \in \mathcal{L}_{-C_1} \cap \mathcal{L}_{-C_2}, \\ \sum_{\emptyset \neq \mathsf{J} \subset J}(-1)^{|\mathsf{J}|+1}\left(\prod_{j \in \mathsf{J}} e^{-2\pi i \gamma_j}\right)[\bar{P}_{\iota\chi - \sum_{j \in \mathsf{J}} \iota b_j}]_h & \text{if } \chi \in \mathcal{L}_{-C_1} \setminus \mathcal{L}_{-C_2}. \end{cases}$$

where γ *is the unique element of* $Y(\mathbb{T})_{\mathbb{C}}$ *such that* $A\gamma = \alpha$ *and* $\iota\gamma = 0$.

(3) $N_h(\mu_C)([\bar{P}_{\iota\chi}]_h) = K^0(\bar{\phi}_{\iota\mu,C})([\bar{P}_{\iota\chi}]_h) = [\bar{P}_{\iota\chi - \iota\mu}]_h$ *for* $\mu \in X(T)$ *and* $\chi \in \mathcal{L}_{-C}$, $C \in \mathcal{C}^0$.

Proof. Most of the claims follow immediately from Proposition 12.4. The only claim that requires some thoughts is the case $\chi \in \mathcal{L}_{-C_1} \setminus \mathcal{L}_{-C_2}$ in (2). We may write the corresponding equation in Proposition 12.4(2) as, denoting for simplicity $\{\mu\} := K^0(\bar{\phi}_{\mu,C})$ for $\mu \in X(\mathbb{T})$,

$$N(v_{C_1 C_2})([\bar{P}_{\iota\chi}]) = \sum_{\emptyset \neq \mathsf{J} \subset J}(-1)^{|\mathsf{J}|+1}[\bar{P}_{\iota\chi - \sum_{j \in \mathsf{J}} e_j}]$$

(12.2)
$$= \sum_{\emptyset \neq \mathsf{J} \subset J}(-1)^{|\mathsf{J}|+1}\left(\prod_{j \in \mathsf{J}}\{\iota b_j - e_j\}\right)[\bar{P}_{\iota\chi - \sum_{j \in \mathsf{J}} \iota b_j}]$$

$$= \sum_{\emptyset \neq \mathsf{J} \subset J}(-1)^{|\mathsf{J}|+1}\left(\prod_{j \in \mathsf{J}}\{-A^*\kappa e_j\}\right)[\bar{P}_{\iota\chi - \sum_{j \in \mathsf{J}} \iota b_j}]$$

Here the factor $\{-A^* \kappa e_j\}$ represents the action of $\kappa e_j \in X(H)$ (the origin of the sign change is the fact that $\bar{\phi}_{\mu,C} = (-\mu) \otimes -$). Now we compute

$$(12.3) \qquad (\kappa e_j)(h) = e^{-2\pi i \langle \kappa e_j, \alpha \rangle} = e^{-2\pi i \langle e_j, \kappa \alpha \rangle} = e^{-2\pi i \langle e_j, \gamma \rangle}$$

where in the last equality we have used the easily proved fact that $\gamma = \kappa \alpha$. Specialising (12.2) at h and substituting (12.3) yields what we want. $\qquad\square$

12.5. **Specialisation and autoduality.**

Proposition 12.7. *Let $h \in H$. Then $K_h^0(\bar{\mathcal{S}}^c)^- \cong \mathbb{D}(K_{h^{-1}}^0(\bar{\mathcal{S}}^f))$ in* $\mathrm{KS}(X(T), \mathcal{H})$.

Proof. This follows from Corollary 12.3, using Lemma 12.5. $\qquad\square$

Now we state our crucial technical result.

Proposition 12.8. *Let $h \in H$. The map*

$$K_h^0(\bar{\mathcal{S}}^f) \to K_h^0(\bar{\mathcal{S}}^c)$$

obtained by applying the functor $K_h^0(-)$ to the inclusion $\bar{\mathcal{S}}^f \subset \bar{\mathcal{S}}^c$ is an isomorphism if $h \in H^{\mathrm{nres}}$.

Proof. We will prove that

$$(12.4) \qquad K^0(\bar{\mathcal{S}}^f) \to K^0(\bar{\mathcal{S}}^c)$$

becomes an isomorphism after inverting an element $F \in \mathbb{C}[X(H)] \cong \mathbb{C}[H]$ which defines the non-resonant locus. Note that using our general conventions $\mu \in X(H)$ acts by $\mu^{-1} \otimes -$ on both sides of (12.4) but in this context using μ^{-1} is irrelevant since it follows from the definition of the non-resonant locus that it is invariant under $\mu \mapsto \mu^{-1}$. So in the rest of this proof we reduce the amount of sign confusion by having $X(H)$ act via $\mu \mapsto \mu \otimes -$.

If $M, N \in D(W/\mathbb{T})$ then $\mathrm{Hom}_{W/\mathbb{T}}(M, N)$ is a $\mathbb{T}/T = H$-representation, or equivalently a $X(H)$-graded vector space which is moreover finite dimensional in every degree whenever M and N are coherent. If we put

$$\bar{\Lambda}_C := \mathrm{Hom}_{W/T}\left(\bigoplus_{\chi \in -\mathcal{L}_C} \bar{P}_{\iota\chi} \right)$$

then we see that $\bar{\Lambda}_C$ is an $X(H)$-graded ring which is isomorphic to Λ_C if we forget the $X(H)$-grading. Since Λ_C has finite global dimension by [ŠVdB19, Theorem 5.6], we deduce from [NVO79, Corollary I.7.8] that $\bar{\Lambda}_C$ has finite $X(H)$-graded global dimension.

We discuss the $X(H)$-grading in more detail. Note that $\bar{\Lambda}_C$ is itself a finitely generated $X(H)$-graded $\mathbb{C}[W]^T = \mathrm{Sym}(W^*)^T$-module. Let σ be the cone in $Y(H)_{\mathbb{R}}$ spanned by $(a_i)_{i=1,\dots,d}$ and let $\sigma^\vee \subset X(H)_{\mathbb{R}}$ be the dual cone. Then by standard toric geometry (as σ is full dimensional and hence $\sigma^\vee$ is strongly convex), see e.g. [CLS11, (5.1.4)],

$$(12.5) \qquad \{\mu \in X(H) \mid \mathrm{Sym}(W^*)_\mu^T \neq 0\} = -\sigma^\vee \cap X(H)$$

(the $-$ sign is because the weights of W^* are $(-b_i)_{i=1,\dots,d}$) and moreover

$$(12.6) \qquad \dim \mathrm{Sym}(W^*)_\mu^T \in \{0, 1\} \text{ for } \mu \in X(H).$$

For use below we let $\mathrm{m}_1, \ldots \mathrm{m}_\ell$ be monomials in $\mathbb{C}[W]^T$ (with respect to the canonical basis of $W^* \cong \mathbb{C}^d$) whose degrees $m_j := |\mathrm{m}_j| \in X(H)$ are generators for the one dimensional cones in the boundary of $-\sigma^\vee$. By (12.5), (12.6), $\mathbb{C}[W]^T$ is finitely generated over $\mathbb{C}[\mathrm{m}_1, \ldots, \mathrm{m}_d]$.

Below we need to be able to associate a Poincare series to a finitely generated $X(H)$-graded $\bar{\Lambda}_C$-module. Choose $\theta \in \mathrm{relint}\,\sigma$ (e.g. $\theta = \sum_i a_i$) and let $\mathbb{Z}[X(H)]\hat{}$ be the completion of $\mathbb{Z}[X(H)]$ for the filtration on $X(H)$ induced by $-\langle \theta, - \rangle$. For M a finitely generated $X(H)$-graded $\bar{\Lambda}_C$ module we define its "Poincare series" as

$$H(M) = \sum_{\mu \in X(H)} \dim M_\mu\, \mu \in \mathbb{Z}[X(H)]\hat{}.$$

One easily sees that (9.1) can be enhanced to

$$\bar{\mathcal{S}}_C \cong D(\mathrm{Gr}_{X(H)}(\bar{\Lambda}_C))$$

where $\mathrm{Gr}_{X(H)}(\bar{\Lambda}_C)$ denotes the category of $X(H)$-graded $\bar{\Lambda}_C$-modules. Moreover we have analogues of Lemmas 9.2 and 9.5

$$\bar{\mathcal{S}}_C^c \cong D^b(\mathrm{gr}_{X(H)}(\bar{\Lambda}_C)), \qquad \bar{\mathcal{S}}_C^f \cong D^b(\mathrm{gr}_{X(H)}^f(\bar{\Lambda}_C))$$

where $\mathrm{gr}_{X(H)}(\bar{\Lambda}_C)$ and $\mathrm{gr}_{X(H)}^f(\bar{\Lambda}_C)$ denote respectively the categories of $X(H)$-graded $\bar{\Lambda}_C$-modules with finitely generated and finite dimensional cohomology.[19] Hence we have to prove that

(12.7) $$K^0(\mathrm{gr}_{X(H)}^f(\bar{\Lambda}_C)) \to K^0(\mathrm{gr}_{X(H)}(\bar{\Lambda}_C))$$

becomes an isomorphism when restricted to $h \in H^{\mathrm{nres}}$. We may use the projective resolutions of the simple objects in $\mathrm{gr}_{X(H)}^f(\bar{\Lambda}_C)$ to express a basis of simples of $K^0(\mathrm{gr}_{X(H)}^f(\bar{\Lambda}_C))$ in terms of a basis projectives of $K^0(\mathrm{gr}_{X(H)}(\bar{\Lambda}_C))$. The resulting base change matrix Φ has entries in $\mathbb{Z}[X(H)]$ and is equal to the inverse of the matrix of $X(H)$-graded Poincare series:

(12.8) $$\Psi := H(\mathrm{Hom}_{W/T}(\bar{P}_{\iota\chi_1}, \bar{P}_{\iota\chi_2}))_{\chi_1, \chi_2 \in -\mathcal{L}_C} \in M_{|\mathcal{L}_C| \times |\mathcal{L}_C|}(\mathbb{Z}[X(H)]\hat{}).$$

Put $F = \prod_i (1 - m_i) \in \mathbb{Z}[X(H)]$. Since every $\mathrm{Hom}_{W/T}(P_{\chi_1}, P_{\chi_2})$ is a finitely generated $\mathbb{C}[W]^T$-module, and hence a finitely generated $\mathbb{C}[\mathrm{m}_1, \ldots, \mathrm{m}_\ell]$-module it follows that

$$\Phi^{-1} = \Psi \in M_{|\mathcal{L}_C| \times |\mathcal{L}_C|}(\mathbb{Z}[X(H)]_F).$$

Hence the specialisation of Φ at h will be invertible whenever $F(h) \neq 0$. Let α be a lift of h under the map $Y(H)_\mathbb{C} \to H : \alpha \mapsto \exp(2\pi i\alpha)$. Then we have

$$|\mathrm{m}_i|(h) = e^{2\pi i \langle |\mathrm{m}_i|, \alpha \rangle}.$$

It now suffices to invoke Lemma 12.10 below. $\qquad\qquad\square$

Corollary 12.9. *If* $h \in H^{\mathrm{nres}}$ *then* $\mathbb{D}(K_h^0(\bar{\mathcal{S}}^c)) \cong (K_{h^{-1}}^0(\bar{\mathcal{S}}^c))^-$ *in* $\mathrm{KS}(X(T), \mathcal{H})$.

Proof. This follows from by combining Propositions 12.7 and 12.8. $\qquad\square$

The following lemma was used in the proof of Proposition 12.8.

---

[19]Note that finite dimensional $X(H)$-graded $\mathbb{C}[W]^T$-modules are automatically supported in the origin of $W/\!/T$. So the definition of $\mathrm{gr}_{X(H)}^f(\bar{\Lambda}_C)$ is a bit simpler than the definition of $\mathrm{mod}^f(\Lambda_C)$.

Lemma 12.10. $\alpha \in Y(H)_{\mathbb{C}}$. *Then α is non-resonant if and only if $\langle m_j, \alpha \rangle \notin \mathbb{Z}$ for $j = 1, \ldots, \ell$.*

Proof. Put $M = X(H)$, $N = Y(H)$. Using (12.5) we identify the monomials in $\mathbb{C}[W]^T$ with $m \in M$ such that $\langle m, a_i \rangle \leq 0$, $1 \leq i \leq d$.

Assume first that α is resonant. Then there exist $m \in M$, $n \in N$, $S \subset \{1, \ldots, d\}$, $\gamma_i' \in \mathbb{C}$ s.t. $\alpha = n + \sum_{i \in S^c} \gamma_i' a_i$, $\langle m, a_i \rangle = 0$ for $i \in S^c$, $\langle m, a_i \rangle < 0$ for $i \in S$. Then $\langle m, a_i \rangle \leq 0$, $1 \leq i \leq d$. Therefore we can identify m with an element of $\mathbb{C}[W]^T$. Choose m_j in such a way that m_j divides pm in $\mathbb{C}[W]^T$ for $p \in \mathbb{N}_{>0}$. Since $0 \geq \langle m_j, a_i \rangle \geq p\langle m, a_i \rangle$ we have $\langle m_j, a_i \rangle = 0$ for $i \in S^c$. Hence, $\langle m_j, \alpha \rangle = \langle m_j, n \rangle \in \mathbb{Z}$.

On the other hand assume there exists m_j such $\langle m_j, \alpha \rangle := z \in \mathbb{Z}$. Since m_j is a generator of a 1-dimensional cone, m_j is is not a multiple of a $m' \in M$, hence there exists $n \in N$ such that $\langle m, n \rangle = z$. Then

$$\langle m_j, \alpha - n \rangle = 0.$$

Since by duality m_j defines a supporting hyperplane of σ this implies that α lies in $N + \mathcal{I}_0$; i.e. α is resonant. $\qquad\square$

13. The decategorified GIT $\mathcal{H}$-schober and the GKZ system

Now for $h \in H$ we put

$$\tilde{K}_h^0(\bar{\mathcal{S}}^c) := K_h^0(\bar{\mathcal{S}}^c)^{\tilde{}} \in \mathrm{Perv}(X(T)_{\mathbb{C}})$$

using the notation $\tilde{?}$ introduced in §7.3 and we let $S^c(h)$ be the corresponding perverse sheaf on $X(T)_{\mathbb{C}}/X(T) \cong T^*$. The following theorem, which is the main result of this paper, shows that $S^c(h)$, obtained by decategorifying and specializing the GIT $\mathcal{H}$-schober, is, up to a suitable translation, equal to the solution sheaf of a GKZ system, whenever h is non-resonant.

Denote $\bar{\zeta} = e^{2\pi i \zeta}$, where ζ is as introduced in §4.4, let $\bar{\tau}_{\bar{\zeta}^{-1}}$ denote the translation $T^* \to T^*$, $x \mapsto \bar{\zeta}^{-1} x$.

Theorem 13.1. *Assume that $\alpha \in \mathfrak{h} = Y(H)_{\mathbb{C}}$ is non-resonant. Then*

$$\overline{P}(\alpha) \cong \bar{\tau}_{\bar{\zeta}^{-1}}^* S^c(e^{-2\pi i \alpha}).$$

The proof of this theorem will be given in the rest of this section. As before $V(E_A) \subset T^*$ is the GKZ discriminant locus defined in §4.4 and $j : T^* \setminus V(E_A) \hookrightarrow T^*$ is the inclusion.

13.1. Perverse sheaves as intermediate extension. We now show that $S^c(h)$ for $h \in H^{\mathrm{nres}}$ is obtained as the intermediate extension of its restriction to the regular locus, $T^* \setminus V(E_A)$.

Corollary 13.2. *One has $\bar{\tau}_{\bar{\zeta}^{-1}}^* S^c(h) \cong j_{!*}(j^* \bar{\tau}_{\bar{\zeta}^{-1}}^* S^c(h))$ on T^*.*

Proof. Write $\bar{E}_C^c = K^0(\bar{\mathcal{S}}_C^c)$. From Lemma 13.3 below, for $C \in \mathcal{C} \setminus \mathcal{C}^0$ we have $\bar{E}_C^c = \sum_{C < C^0 \in \mathcal{C}^0} \bar{E}_{C^0}^c$, which remains true after the specialisation. Hence, $S^c(h)$ has no *perverse quotient sheaf* supported on $\bar{\zeta} V(E_A)$. Since the cokernel of the map $^p j_!(j^* \bar{\tau}_{\bar{\zeta}^{-1}}^* S^c(h)) \to \bar{\tau}_{\bar{\zeta}^{-1}}^* S^c(h)$ is supported on $V(E_A)$ it follows that this must be an epimorphism of perverse sheaves. Applying Proposition 12.8 with h replaced by h^{-1} (using the assumption that $h \in H^{\mathrm{nres}}$ and hence h^{-1} in H^{nres}), it follows

that $S^c(h)$ has no *perverse subsheaf* supported on $\bar{\zeta}V(E_A)$, which implies that $\bar{\tau}^*_{\bar{\zeta}^{-1}}S^c(h) \to {}^p j_*(j^*\bar{\tau}^*_{\bar{\zeta}^{-1}}S^c(h))$ is also a monomorphism. Hence

$$j_{!*}(j^*\bar{\tau}^*_{\bar{\zeta}^{-1}}S^c(h)) := {}^p\operatorname{im}({}^p j_!(j^*\bar{\tau}^*_{\bar{\zeta}^{-1}}S^c(h)) \to {}^p j_*(j^*\bar{\tau}^*_{\bar{\zeta}^{-1}}S^c(h))) \cong \bar{\tau}^*_{\bar{\zeta}^{-1}}S^c(h). \quad \square$$

We have used the following combinatorial lemma.

Lemma 13.3. *Let $C \in \mathcal{C} \setminus \mathcal{C}^0$. Then*

$$\mathcal{L}_C = \bigcup_{C_0 \in \mathcal{C}^0, C_0 > C} \mathcal{L}_{C_0},$$

$$\bar{\mathcal{L}}_C = \bigcup_{C_0 \in \mathcal{C}^0, C_0 > C} \bar{\mathcal{L}}_{C_0}.$$

Proof. The second claim follows from the first one by taking inverse images under B. So we now check the first claim. The inclusion $\bigcup_{C_0 \in \mathcal{C}^0, C_0 > C} \mathcal{L}_{C_0} \subset \mathcal{L}_C$ follows from (3.7).

To show the converse let $\rho \in C$. Assume that $\chi \in \mathcal{L}_C = (\rho + \Delta) \cap X(T)$. We have $\rho \in \chi + \Delta$ (using $\Delta = -\Delta$). Since $\chi + \Delta$ is convex of dimension n it must intersect some $C_0 > C$ for $C_0 \in \mathcal{C}^0$. Let $\rho_0 \in (\chi + \Delta) \cap C_0$. Then $\chi \in \rho_0 + \Delta$ (using again $\Delta = -\Delta$) and hence $\chi \in \mathcal{L}_{C_0}$. $\square$

13.2. **Comparison of the monodromy.** We first compare the corresponding local systems on $T^* \setminus V(E_A) = (X(T)_{\mathbb{C}} \setminus (\mathcal{H}_{\mathbb{C}} + \zeta))/X(T)$.

Proposition 13.4. *Assume that $\alpha \in \sum_i \mathbb{R}_{<0}a_i$ is non-resonant. Then $j^*\overline{P}(\alpha) \cong j^*\bar{\tau}^*_{\bar{\zeta}^{-1}}S^c(e^{-2\pi i\alpha})$. The isomorphism corresponds to the isomorphism of the corresponding $\Pi(\mathcal{H}) \rtimes X(T)$-representations M (Theorem 6.4), N_h (Proposition 12.6) given by $\hat{M}^\alpha_\chi \mapsto [\bar{P}_{-\iota\chi}]_h$ for $h = e^{-2\pi i\alpha}$.*

Proof. This follows from Theorem 6.4 and Proposition 12.6. $\square$

13.3. **Proof of Theorem 13.1.** Let $\alpha' \in (\alpha + N) \cap \sum_i \mathbb{R}_{<0}a_i$. From Proposition 4.9(4) and Proposition 13.4 we obtain isomorphisms

$$j^*\overline{P}(\alpha) \cong j^*\overline{P}(\alpha') \cong j^*\bar{\tau}^*_{\bar{\zeta}^{-1}}S^c(e^{-2\pi i\alpha'}) = j^*\bar{\tau}^*_{\bar{\zeta}^{-1}}S^c(e^{-2\pi i\alpha}).$$

Since $\overline{P}(\alpha)$ and $S^c(e^{-2\pi i\alpha})$ are intermediate extensions of the restrictions to $T^* \setminus V(E_A)$ by Corollaries 4.11, 13.2, the result follows.

APPENDIX A. DESCENT FOR WEAKLY EQUIVARIANT $\mathcal{D}$-MODULES

In order to have a convenient reference in the body of the paper, we summarize a few facts on weakly equivariant $\mathcal{D}$-modules. No originality is intended. All schemes are separated of finite type over an algebraically closed field of characteristic zero. The main result is Corollary A.11.

A.1. **Weakly equivariant $\mathcal{D}$-modules.**

A.1.1. *Generalities.* Let Y/k be smooth scheme and let G be a reductive group acting on Y. Put $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\alpha \in \mathfrak{g}^*$ be a character of $\mathfrak{g}$, i.e. $\alpha([\mathfrak{g}, \mathfrak{g}]) = 0$.

If $\mathcal{M}$ is a G-equivariant $\mathcal{O}_Y$-module then differentiating the G-action yields a Lie algebra action[20]

$$\gamma : \mathfrak{g} \otimes \mathcal{M} \to \mathcal{M}.$$

If $\mathcal{M} = \mathcal{O}_Y$ then this action is by derivations and hence we obtain an "anchor" morphism $\rho : \mathfrak{g} \to \Gamma(Y, \mathcal{T}_Y)$. Finally the $\mathfrak{g}$-action and the $\mathcal{O}_Y$-action on $\mathcal{M}$ are related by the Leibniz identity.

Definition A.1. A *weakly G-equivariant $\mathcal{D}_Y$-module* is a quasi-coherent $(G, \mathcal{D}_Y)$-module $\mathcal{M}$. For such $\mathcal{M}$ we say that it has character α if for every $v \in \mathfrak{g}$ and for every local section m of $\mathcal{M}$ one has $\gamma(v, m) = (\rho(v) - \alpha(v))m$ where on the right-hand side we used the action obtained via the inclusion $\mathcal{O}_Y \oplus \mathcal{T}_Y \subset \mathcal{D}_Y$.

We will denote the category of weakly G-equivariant $\mathcal{D}_Y$-modules by $\mathrm{Qch}(G, \mathcal{D}_Y)$. The full subcategory of weakly G-equivariant $\mathcal{D}_Y$-modules with character α is denoted by $\mathrm{Qch}_\alpha(G, \mathcal{D}_Y)$.

A.2. **The canonical weakly equivariant $\mathcal{D}$-module.** We keep notation as in §A.1.1. Here and below we write $\mathfrak{g} - \alpha(\mathfrak{g})$ for the vector space $\{v - \alpha(v) \mid v \in \mathfrak{g}\} \subset k \oplus \mathfrak{g}$. We identify $\mathfrak{g} - \alpha(\mathfrak{g})$ with the corresponding sections of $\mathcal{O}_Y \oplus \mathcal{T}_Y$, via the map ρ introduced above (which does not have to be injective).

Lemma A.2. *The $\mathcal{D}_Y$-module $\mathcal{D}_{Y,\alpha} = \mathcal{D}_Y/\mathcal{D}_Y(\mathfrak{g} - \alpha(\mathfrak{g}))$ is weakly G-equivariant with character α.*

Proof. Put $\mathcal{M} = \mathcal{D}_Y/\mathcal{D}_Y(\mathfrak{g} - \alpha(\mathfrak{g}))$. It is clear that $\mathcal{M}$ is weakly G-equivariant. Let D be a local section of $\mathcal{D}_Y$ and let $\bar{D}$ be the corresponding local section of $\mathcal{M}$.

One checks that the differential of the G-action on $\mathcal{D}_Y$ is given by $\gamma(v, D) = [\rho(v), D]$. Whence we also have $\gamma(v, \bar{D}) = \overline{[\rho(v), D]}$. So we have to prove for all $v \in \mathfrak{g}$, $\overline{[\rho(v), D]} = \overline{(\rho(v) - \alpha(v))D}$. We compute

$$\begin{aligned}
[\rho(v), D] &= \rho(v)D - D\rho(v) \\
&\cong \rho(v)D - D\alpha(v) \qquad \text{modulo } \mathcal{D}_Y(\mathfrak{g} - \alpha(\mathfrak{g})) \\
&= (\rho(v) - \alpha(v))D. \qquad\qquad\qquad\qquad \square
\end{aligned}$$

A.3. **Descent for weakly equivariant $\mathcal{D}$-modules.**

A.3.1. *Principal homogeneous spaces.* In the rest of this appendix we keep the notations as in §A.1.1 but now we consider a principal homogeneous G-space $\pi : Y \to X$. We put $\mathcal{D}_{X,\alpha} := (\pi_* \mathcal{D}_{Y,\alpha})^G$. Note that we have a short exact sequence

$$(A.1) \qquad 0 \to (\pi_* \mathcal{D}_Y(\mathfrak{g} - \alpha(\mathfrak{g})))^G \to (\pi_* \mathcal{D}_Y)^G \to \mathcal{D}_{X,\alpha} \to 0.$$

Lemma A.3. *$(\pi_* \mathcal{D}_Y(\mathfrak{g} - \alpha(\mathfrak{g})))^G$ is a two sided ideal in $(\pi_* \mathcal{D}_Y)^G$. Hence multiplication of local sections in $\mathcal{D}_Y$ induces a multiplication on $\mathcal{D}_{X,\alpha}$ via (A.1).*

Proof. Let $v \in \mathfrak{g}$ and let D be a local section of $(\pi_* \mathcal{D}_Y)^G$. We have to show that $(\rho(v) - \alpha(v))D$ is a local section of $(\pi_* \mathcal{D}_Y(\mathfrak{g} - \alpha(\mathfrak{g})))^G$. This follows from the fact that the differentiated G-action on $\mathcal{D}_Y$ is given by $v \mapsto [\rho(v), -]$ (as was already used in the proof of Lemma A.2), and hence, since D is G-invariant, we have $[\rho(v), D] = 0$. $\square$

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[20]One way to obtain this action is to extend the base ring to $k[\epsilon]/(\epsilon^2)$ and to use that $\mathfrak{g} = \ker(G(k[\epsilon]) \to G(k))$. The $\mathfrak{g}$ action is then obtained from the $G(k[\epsilon])$-action on $\mathcal{M}[\epsilon]$.

We observe that $\mathcal{D}_{Y,\alpha}$ is a G-equivariant $(\mathcal{D}_Y, \pi^{-1}\mathcal{D}_{X,\alpha})$-bimodule.

Proposition A.4. *There are inverse equivalences*

$$(A.2) \qquad \mathrm{Qch}_\alpha(G, \mathcal{D}_Y) \to \mathrm{Qch}(\mathcal{D}_{X,\alpha}) : \mathcal{M} \mapsto (\pi_*\mathcal{M})^G,$$

$$(A.3) \qquad \mathrm{Qch}(\mathcal{D}_{X,\alpha}) \to \mathrm{Qch}_\alpha(G, \mathcal{D}_Y) : \mathcal{N} \to \mathcal{D}_{Y,\alpha} \otimes_{\pi^{-1}\mathcal{D}_{X,\alpha}} \pi^{-1}\mathcal{N}.$$

Proof. We have for $\mathcal{N} \in \mathrm{Qch}(\mathcal{D}_{X,\alpha})$:

$$(\pi_*(\mathcal{D}_{Y,\alpha} \otimes_{\pi^{-1}\mathcal{D}_{X,\alpha}} \pi^{-1}\mathcal{N}))^G = (\pi_*\mathcal{D}_{Y,\alpha})^G \otimes_{\mathcal{D}_{X,\alpha}} \mathcal{N} = \mathcal{N}$$

(to check this it is convenient to reduce to the case that X, and hence Y are affine).

Now assume $\mathcal{M} \in \mathrm{Qch}_\alpha(G, \mathcal{D}_Y)$. We have to prove that the canonical map

$$\mathcal{D}_{Y,\alpha} \otimes_{\pi^{-1}\mathcal{D}_{X,\alpha}} \pi^{-1}(\pi_*\mathcal{M})^G \to \mathcal{M}$$

is an isomorphism. Let $\mathcal{K}, \mathcal{C}$ be its kernel and cokernel. Applying the exact functor $\pi_*(-)^G$ we obtain by the first part of the proof $(\pi_*\mathcal{K})^G = (\pi_*\mathcal{C})^G = 0$. It then follows by descent that $\mathcal{K} = \mathcal{C} = 0$. $\qquad\square$

A.3.2. The special case that X is a point.

Lemma A.5. *Assume $Y = G$ and $X = \mathrm{Spec}\, k$. Then the inclusion $\mathcal{O}_G \hookrightarrow \mathcal{D}_G$ induces an isomorphism $\mathcal{O}_G \cong \mathcal{D}_{G,\alpha}$. In particular $\mathcal{D}_{X,\alpha} = (\pi_*\mathcal{O}_G)^G = \mathcal{O}_X$.*

Proof. The anchor map $\mathcal{O}_G \otimes_k \mathfrak{g} \to \mathcal{T}_G$ is an isomorphism in this case. If we filter $\mathcal{D}_G$ by order of differential operators we obtain from this $\mathrm{gr}\,\mathcal{D}_{G,\alpha} = \mathcal{O}_G$. This yields the claim in the lemma. $\qquad\square$

Corollary A.6. *The equivalences in Proposition A.4 specialize to an equivalence* $\mathrm{Qch}_\alpha(G, \mathcal{D}_G) \cong \mathrm{Mod}(k)$. *Moreover the functor $\mathrm{Qch}_\alpha(G, \mathcal{D}_G) \to \mathrm{Mod}(k)$ defined by* (A.2) *is naturally isomorphic to i_e^* where $e \in G$ is the unit element.*

Proof. The first statement is just the specialisation of Proposition A.4 to the case $Y = G$, $X = \mathrm{Spec}\, k$. For the second claim note that both functors, the one defined by (A.2) and i_e^*, factor through $\mathrm{Qch}_\alpha(G, \mathcal{D}_G) \to \mathrm{Qch}(G, \mathcal{O}_G)$. Hence it is sufficient to prove that $(-)^G$ and i_e^* define naturally isomorphic functors on $\mathrm{Qch}(G, \mathcal{O}_G)$. This is standard descent. $\qquad\square$

Corollary A.7. $i_e^* : \mathrm{Qch}_\alpha(G, \mathcal{D}_G) \to \mathrm{Qch}(\mathcal{O}_e) \cong \mathrm{Mod}(k)$ *is also an equivalence of categories.*

Remark A.8. The fact that $\mathcal{D}_{G,\alpha} \cong \mathcal{O}_G$ in $\mathrm{Qch}(G, \mathcal{O}_G)$ shows in particular that $\mathcal{D}_{G,\alpha}$ is invertible for the tensor product of $\mathcal{D}$-modules. Write θ_α for the image of $1 \in \Gamma(G, \mathcal{D}_G)$. Then $\mathcal{D}_{G,\alpha} = \mathcal{O}_G\theta_\alpha$ where θ_α is G-invariant and satisfies $\rho(v)\theta_\alpha = \alpha(v)\theta_\alpha$ for $v \in \mathfrak{g}$.

Let us specialize to the case that T is the torus $(\mathbb{C}^*)^n$. In that case $\mathfrak{t} := \mathrm{Lie}(T) = \mathbb{C}^n$ and $\rho(e_i) = \partial/\partial x_i$ for e_i the canonical i'th basis vector of $\mathbb{C}^n$. Let $\alpha \in \mathfrak{t}^* = \mathbb{C}^n$ be given by $(\alpha_1, \ldots, \alpha_n)$. Then θ_α is the multi-valued function given by $\theta_\alpha(t_1, \ldots, t_n) = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ for $t_i \in \mathbb{C}^*$. More intrinsically we have $\theta_\alpha \circ e^{2\pi i -} = e^{2\pi i \langle \alpha, - \rangle}$.

A.3.3. *The case of split principal homogeneous spaces.*

Lemma A.9. *Assume $Y = G \times X$. Then the inclusion $\mathcal{O}_G \boxtimes \mathcal{D}_X \hookrightarrow \mathcal{D}_{G \times X}$ induces an isomorphism $\mathcal{O}_G \boxtimes \mathcal{D}_X \cong \mathcal{D}_{G \times X, \alpha}$. In particular, $\mathcal{D}_{X,\alpha} \cong (\pi_*(\mathcal{O}_G \boxtimes \mathcal{D}_X))^G \cong \mathcal{D}_X$.*

Proof. This follows from Lemma A.5, taking into account $\mathcal{D}_{G \times X} = \mathcal{D}_G \boxtimes \mathcal{D}_X$, $\mathcal{D}_{G \times X, \alpha} = \mathcal{D}_{G,\alpha} \boxtimes \mathcal{D}_X$. $\square$

Corollary A.10. *Assume $Y = G \times X$. The equivalences in Proposition A.4 specialize to an equivalence $\mathrm{Qch}_\alpha(G, \mathcal{D}_{G \times X}) \cong \mathrm{Qch}(\mathcal{D}_X)$. Moreover, the associated functor $\mathrm{Qch}_\alpha(G, \mathcal{D}_G) \to \mathrm{Qch}(\mathcal{D}_X)$ is naturally isomorphic to i_e^* where $e : X \to G \times X$ is the unit section.*

Proof. This is a generalization of Corollary A.6 with an extra $\mathcal{D}_X$-action. $\square$

Corollary A.11. *Assume that $\pi : Y \to X$ is a split principal homogeneous G-space. Let $i : X \to Y$ be a splitting for Y. Then the $\mathcal{D}$-module inverse image functor $i^* : \mathrm{Qch}_\alpha(G, \mathcal{D}_Y) \to \mathrm{Qch}(\mathcal{D}_X)$ is an equivalence of (abelian) categories.*

Proof. This follows from Corollary A.10, after making the identification $Y \cong G \times X$ using the splitting i. Then $i = i_e$. We use [BGK$^+$87, §4.5] for the fact that i_e^* does indeed compute the usual $\mathcal{D}$-module inverse image for $X \to G \times X$ (note that in loc. cit. the $\mathcal{D}$-module inverse image is denoted by $(-)^\circ$). $\square$

Corollary A.11 implies in particular that i^* is exact. This also follows from the following lemma.

Lemma A.12. *In the setting of Corollary A.11 one has $L^j i^* \mathcal{M} = 0$ for $j > 0$ and $\mathcal{M} \in \mathrm{Qch}(G, \mathcal{O}_Y)$.*

Proof. We may assume $Y = G \times X$ with $i = i_e$. By descent $\mathcal{M} = \pi^* \mathcal{N}$ for $\mathcal{N} \in \mathrm{Qch}(X)$. Hence $Li^*(\mathcal{M}) = (Li^* \circ \pi^*)(\mathcal{N}) = (Li^* \circ L\pi^*)(\mathcal{N}) = \mathcal{N}$, where we have used that Y/X is flat and $\pi \circ i$ is the identity. $\square$

A.4. **Changing the section.**

Lemma A.13. *Let $\pi : Y \to X$ be a split G-principal homogeneous space and let $i_1, i_2 : X \to Y$ be two sections of π. Define $\delta : X \to G$ via $\delta(x)i_1(x) = i_2(x)$ for $x \in X$. Then for $\mathcal{M} \in \mathrm{Qch}_\alpha(G, \mathcal{D}_Y)$ we have*

$$i_2^* \mathcal{M} \cong \delta^* \mathcal{D}_{G,\alpha} \otimes_{\mathcal{O}_X} i_1^* \mathcal{M}.$$

Proof. We may assume $Y = G \times X$ and $i_1 = i_e$, $i_2(x) = (\delta(x), x) := i_\delta(x)$. By Corollary A.10 we have $\mathcal{M} = \mathcal{D}_{G,\alpha} \boxtimes \mathcal{N}$ for $\mathcal{N} \in \mathrm{Qch}(\mathcal{D}_X)$ so that

$$i_\delta^* \mathcal{M} = i_\delta^*(\mathcal{D}_{G,\alpha} \boxtimes \mathcal{N})$$
$$= i_\delta^*(\mathcal{D}_{G,\alpha} \boxtimes \mathcal{O}_X) \otimes_{\mathcal{O}_X} i_\delta^* \mathrm{pr}_2^* \mathcal{N}.$$

The composition $\mathrm{pr}_2 \circ i_\delta$ is the identity so that $i_\delta^* \mathrm{pr}_2^* \mathcal{N} = \mathcal{N} = i_e^* \mathcal{M}$ in $\mathcal{D}_X$. On the other hand $i_\delta^*(\mathcal{D}_{G,\alpha} \boxtimes \mathcal{O}_X) = i_\delta^* \mathrm{pr}_1^* \mathcal{D}_{G,\alpha} = \delta^* \mathcal{D}_{G,\alpha}$. $\square$

Remark A.14. Following up on Remark A.8 we can think of $\delta^* \mathcal{D}_{G,\alpha}$ as $\mathcal{O}_X \theta_{\alpha,\delta}$ with $\theta_{\alpha,\delta} = \theta_\alpha \circ \delta$ where the derivatives of $\theta_\alpha \circ \delta$ are determined by the chain rule. In the torus example where we represented θ_α with an explicit multi-valued function, the notation $\theta_\alpha \circ \delta$ can be taken literally.

List of Notations

$X(-)$	characters	§3.1
$Y(-)$	1-parameter subgroups (cocharacters)	§3.1
L, N	free abelian groups such that $N = \mathbb{Z}^d/L$	§3.2
$T, \mathbb{T}, H$	associated tori such that $\mathbb{T} \cong (\mathbb{C}^*)^d$, $H = \mathbb{T}/T$	§3.2
d, n	$\dim \mathbb{T}$, $\dim T$	§3.2
W	$\cong \mathbb{C}^d$, tautological representation of $\mathbb{T}$	§3.2
A, B	toric data, many incarnations	§3.2
$(b_i)_i$	T-weights of W, describing B	§3.2
V	a real vector space	§3.4
$\mathcal{H}$	a hyperplane arrangement	§3.4
$\mathcal{H}$	a hyperplane arrangement constructed from B	§3.5
$\mathcal{H}_0$	central hyperplane arrangement corresponding to $\mathcal{H}$	§3.4
$\mathcal{C}$	the faces of the connected components of $V \setminus \mathcal{H}$	§3.4
$\mathcal{C}^0$	connected components of $V \setminus \mathcal{H}$ (i.e. chambers in $\mathcal{C}$)	§3.4
Σ	a zonotope constructed from B	§3.5
Δ	$(1/2)\bar{\Sigma}$	§3.5
$(H_i)_i$	affine hyperplanes defining the facets of Δ	§3.5
C	an arbitrary element in $\mathcal{C}$	§3.5
$\mathcal{L}_C$	$(\nu + \Delta) \cap X(T)$ for $\nu \in C$	§3.5
$\mathfrak{h}$	$\mathrm{Lie}(H)$ ($\cong Y(H)_{\mathbb{C}}$)	§3.6
$\mathcal{I}_0, \mathcal{I}_0'$	central arrangements in $Y(H)_{\mathbb{R}}$ built from A	§3.6
$\mathcal{I}, \mathcal{I}'$	associated affine hyperplane arrangements	§3.6
$\mathfrak{h}^{\mathrm{nres}}$	the complement of $\mathcal{I}_{\mathbb{C}}$ (the "non-resonant" part of $\mathfrak{h}$)	§3.6
H^{nres}	the image of $\mathfrak{h}^{\mathrm{nres}}$ under $\exp(2\pi i-)$	§3.6
$\mathcal{D}_X$	the sheaf of differential operators on X	§4.1
E_ϕ	a derivation of $\mathcal{O}_{\mathbb{T}^*}$ corresponding to $\phi \in \mathfrak{h}^*$	§4.1
α	an element of $Y(H)_{\mathbb{C}}$ (often implicit)	§4.1
$\mathcal{P}(\alpha)$	the GKZ $\mathcal{D}_{\mathbb{T}^*}$-module with parameter α	§4.1
$\mathfrak{v}$	the normalised volume of the convex hull of A	§4.2
ι	a splitting of $B^* : T \to \mathbb{T}$ (also used for derived maps)	§4.3
$\bar{\mathcal{P}}(\alpha)$	$\iota^*\mathcal{P}(\alpha)$	§4.3
E_A	the principal A-discriminant	§4.4
ζ	$-\frac{i}{2\pi}\sum_i(\log|n_j|)b_j \in X(T)_{\mathbb{C}}$ for suitable $n_j \in \mathbb{Z}$	§4.4
$\mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X)$	regular holonomic $\mathcal{D}_X$-modules	§4.5
$D^b_{\mathrm{rh}}(\mathcal{D}_X)$	corresponding bounded derived category	§4.5
$\mathrm{Perv}(X)$	the category of perverse sheaves on X^{an}	§4.5
${}^pH(-)$	perverse cohomology	§4.5
Sol_X	the solution functor	§4.5
$P(\alpha)$	$\mathrm{Sol}_{\mathbb{T}^*}(\mathcal{P}(\alpha))[d]$	§4.6
${}^p\iota^*$	${}^pH^0(L\iota^{\mathrm{an},*}(?))$	§4.6
$\bar{P}(\alpha)$	${}^p\iota^*P(\alpha)$	§4.6
j	the inclusion $T^* \setminus V(E_A) \hookrightarrow T^*$	§4.6
$j_{!*}$	intermediate extension functor corresponding to j	§4.6
γ	an element in $Y(\mathbb{T})_{\mathbb{C}} = \mathbb{C}^d$ with $\alpha = A\gamma$	§5.1
M	the Mellin-Barnes integral (a solution to GKZ)	§5.1
$\hat{M}$	the single-valued version of M	§5.1.1

M^α, $\hat{M}^\alpha$	MB integrals corresponding to α	§5.1.3
$\hat{M}_\chi(x)$	$\hat{M}(x - \chi)$	§5.1.5
$\mathcal{M}_C$	$\{\hat{M}_\chi \vert \chi \in \mathcal{L}_C\}$	§5.1.5
Φ_γ	a power series solution for the GKZ system	§5.2
$\hat{\Phi}_\gamma$	the single-valued power series solution	§5.2
γ_I	an element in $\mathbb{C}^d$ with $\alpha = A\gamma_I$, $\gamma_{I,i} \in \mathbb{Z}$, $i \in I$	§5.2
D_ρ	a domain in $(\mathbb{C}^*)^d$	§5.2
$\hat{D}_\rho$	a domain in $\mathbb{C}^d$	§5.2
$\mathcal{I}_\rho, \hat{\mathcal{I}}_\rho$	(multi)sets of subsets of $\{1, \ldots, d\}$	§5.2
Φ_I, $\hat{\Phi}_I$	Φ_{γ_I}, $\hat{\Phi}_{\gamma_I}$	§5.2
$\mathrm{v}_{C_1 C_2}$	a path from ρ_1 to ρ_2, $\rho_i \in C_i$	§6.1
$\Pi(\mathcal{H})$	an incarnation of the fundamental groupoid of $\mathcal{H}_\mathbb{C}^c$	§6.1
$\mathcal{M} \rtimes \mathcal{G}$	semi-direct product of a groupoid $\mathcal{M}$ and a group $\mathcal{G}$	§6.2
$J_{C'C}$	a specific subset of $\{1, \ldots, d\}$	§6.3
M	the representation of $\Pi(\mathcal{H}) \rtimes X(T)$ given by $\bar{\mathcal{P}}(\alpha)$	§6.3
$\hat{\Phi}_I^s$	a normalisation of $\hat{\Phi}_I$	§6.5
$\mathcal{P}_\rho$	$\{\hat{\Phi}_I^s \mid I \in \hat{\mathcal{I}}_\rho\}$	§6.5
$\mathrm{Perv}_{\mathcal{H}_\mathbb{C}}(V_\mathbb{C})$	perverse sheaves on $V_\mathbb{C}$, stratified using $\mathcal{H}_\mathbb{C}$	§7.1
$E_C, \delta_{CC'}, \gamma_{C'C}$	KS-data for objects in $\mathrm{Perv}_{\mathcal{H}_\mathbb{C}}(V_\mathbb{C})$	§7.1
$\phi_{C_1 C_2}$	$\gamma_{C'C_2}\delta_{C_1 C'}$ for $C' \le C_1, C_2$	§7.1
$\mathrm{KS}(\mathcal{H})$	the category of KS-data (allowing $\dim E_C = \infty$)	§7.1
$\mathrm{KS}^c(\mathcal{H})$	$\{E \in \mathrm{KS}(\mathcal{H}) \vert \dim(E_C) < \infty\}$	§7.1
$\tilde{E}$	the perverse sheaf associated to $E \in \mathrm{KS}^c(\mathcal{H})$	§7.1
Res	functor $\mathrm{KS}(\mathcal{H}) \to \mathrm{Rep}(\Pi(\mathcal{H}))$ computing monodromy	§7.1
$\mathbb{D}(E)$	dual KS-data for $E \in \mathrm{KS}^c(\mathcal{H})$	§7.1
$\phi_{g,C}$	isomorphism $E_C \to E_{gC}$, part of equivariant KS-data	§7.3
$\mathrm{KS}(\mathcal{G}, \mathcal{H})$, $\mathrm{KS}^c(\mathcal{G}, \mathcal{H})$	$\mathcal{G}$-equivariant KS-data	§7.3
Res	functor $\mathrm{KS}(\mathcal{G}, \mathcal{H}) \to \mathrm{Rep}(\Pi(\mathcal{H}) \rtimes \mathcal{G})$ (equivariant Res)	§7.3
$\mathbb{D}$	functor $\mathrm{KS}^c(\mathcal{G}, \mathcal{H}) \to \mathrm{KS}^c(\mathcal{G}, \mathcal{H})^\circ$ (equivariant $\mathbb{D}$)	§7.3
$\mathrm{KS}_R(\mathcal{H})$, $\mathrm{KS}_R^c(\mathcal{H})$	R-linear KS-data	§7.4
$\mathcal{E}$	a general $\mathcal{H}$-schober	§8.1
$\mathcal{E}_C, \delta_{CC'}, \gamma_{C'C}$	$\mathcal{H}$-schober data (categorified KS-data)	§8.1
$\phi_{C_1 C_2}$	$\gamma_{C'C_2}\delta_{C_1 C'}$ for $C' \le C_1, C_2$	§8.1
$\mathrm{Schob}(\mathcal{H})$	the 2-category of $\mathcal{H}$-schobers	§8.1
$K_\mathbb{C}^0(-)$	$K^0(-) \otimes_\mathbb{Z} \mathbb{C}$ (decategorification)	§8.1
$\tilde{K}_\mathbb{C}^0(\mathcal{E})$	$\widetilde{K_\mathbb{C}^0(\mathcal{E})}$ (decategorification)	§8.1
$\phi_{g,C}$	equivalence $\mathcal{E}_C \to \mathcal{E}_{gC}$ (for equivariant schobers)	§8.2
$\mathrm{Schob}(\mathcal{G}, \mathcal{H})$	$\mathcal{G}$-equivariant $\mathcal{H}$-schobers	§8.2
$\mathcal{S}$	a particular $\mathcal{H}$-schober (the "GIT $\mathcal{H}$-schober")	§9.1
P_χ	$\chi \otimes \mathcal{O}_W$ (for $\chi \in X(T)$)	§9.1
P_C	$\oplus_{\chi \in \mathcal{L}_{-C}} P_\chi$	§9.1
$\mathcal{S}_C$	$\langle P_C \rangle$	§9.1
Λ_C	$\mathrm{End}_{W/T}(P_C)$	§9.1
$D(\Lambda_C)$	the derived category of right Λ_C-modules	§9.1
$\mathcal{S}_C, \gamma_{C'C}, \delta_{CC'}$	$\mathcal{H}$-schober data for $\mathcal{S}$	§9.1
$\phi_{\chi,C}$	the functor $\mathcal{S}_C \to \mathcal{S}_{\chi+C}$, $M \mapsto (-\chi) \otimes M$	§9.1
$D^c(W/T)$	bounded coherent complexes in $D(W/T)$	§9.1

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