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REPRESENTATIONS OF HOPF-ORE EXTENSIONS OF GROUP ALGEBRAS

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Abstract. In this paper, we study the representations of the Hopf-Ore extensions $kG(\chi^{-1}, a, 0)$ of group algebra kG, where k is an algebraically closed field. We classify all finite dimensional simple $kG(\chi^{-1}, a, 0)$-modules under the assumption $|\chi| = \infty$ and $|\chi| = |\chi(a)| < \infty$ respectively, and all finite dimensional indecomposable $kG(\chi^{-1}, a, 0)$-modules under the assumption that kG is finite dimensional and semisimple, and $|\chi| = |\chi(a)|$. Moreover, we investigate the decomposition rules for the tensor product modules over $kG(\chi^{-1}, a, 0)$ when char(k)=0. Finally, we consider the representations of some Hopf-Ore extension of the dihedral group algebra kD_n, where $n = 2m$, $m > 1$ odd, and char(k)=0. The Grothendieck ring and the Green ring of the Hopf-Ore extension are described respectively in terms of generators and relations.

1. Introduction

During the past years, the classification of Hopf algebras has made great progress. Andruskiewitsch and Schneider [1] classified the finite dimensional pointed Hopf algebras over an algebraically closed field of characteristic zero such that their coradicals are commutative and the prime factors of the dimensions of the coradicals are greater than 7. Beattie et al [3, 4] constructed many pointed Hopf algebras by means of Ore extensions, and answered the tenth Kaplansky's conjecture in the negative. Panov [9] introduced Hopf-Ore extensions, and classified the Hopf-Ore extensions of group algebras and the enveloping algebras of Lie algebras. Krop and Radford [7] defined the rank of a Hopf algebra to measure the complexity of the Hopf algebras H generated by H_1, and showed that a finite dimensional rank one pointed Hopf algebra over an algebraically closed field k with char(k)=0 is isomorphic to a quotient of a Hopf-Ore extension of its coradical. Scherotzke [10] proved such a result for the case of $\mathrm{char}(k) = p > 0$. Wang et al [13] generalized the result to the case that k is an arbitrary field. Brown et al [5] studied the connected Hopf algebras and iterated Ore extensions. You et al [14] studied generalized Hopf-Ore extension, and classified the generalized Hopf-Ore extensions of the enveloping algebras of some Lie algebras. Zhou et al [15] proved that every connected graded Hopf algebra with finite GK-dimension over a field k of characteristic zero is an iterated Ore extensions of k.

In [9], Panov proved that every Hopf-Ore extension $kG[x; \tau, \delta]$ of a group algebra kG is of the form $kG(\chi, a, \delta)$, where a is a central element of the group G and χ is

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a linear character of G over the ground field k. If $\chi(a) \neq 1$ then one can assume $\delta = 0$ by replacing the variable x with $x - \gamma(1-a)$ for some scalar $\gamma \in k$, i.e. $kG(\chi, a, \delta) \cong kG(\chi, a, 0)$, see [13]. Wang et al [13] also studied the representations of $kG(\chi^{-1}, a, 0)$ and its rank one quotient Hopf algebra $kG(\chi^{-1}, a, 0)/I$. They constructed finite dimensional indecomposable weight modules over $kG(\chi^{-1}, a, 0)$ and $kG(\chi^{-1}, a, 0)/I$ and classified them. It was shown that there is a simple weight $kG(\chi^{-1}, a, 0)/I$-module M with $\dim(M) > 1$ only if $|\chi| = |\chi(a)| < \infty$. It is well-known that the finite dimensional representation category mod$H$ of a Hopf algebra H is a tensor category. In [11, 12], we investigated the decomposition rules for the tensor products of finite dimensional indecomposable weight $kG(\chi^{-1}, a, 0)$-modules and described the structure of the Green ring of the category of finite dimensional weight modules over $kG(\chi^{-1}, a, 0)$ for the case that k is an algebraically closed field of characteristic zero. This gives rise to the natural questions: How to classify the finite dimensional indecomposable modules over $kG(\chi^{-1}, a, 0)$? How to describe the Green ring of $kG(\chi^{-1}, a, 0)$?

In this paper, we study the finite dimensional representations of $H = kG(\chi^{-1}, a, 0)$, a Hopf-Ore extension of a group algebra kG, where k is an algebraically closed field. The paper is organized as follows. In Section 2, we recall some notions and notations including Grothendieck ring and Green ring, and the Hopf algebra structure of H. Section 3 deals with the finite dimensional irreducible representations of H. We describe and classify the finite dimensional simple modules over H in two cases: $|\chi| = \infty$ and $|\chi| = |\chi(a)| < \infty$. In Section 4, we construct and classify the finite dimensional indecomposable H-modules under the assumptions that the group algebra kG is semisimple and $|\chi| = |\chi(a)|$. In Section 5, we investigate the decomposition rules for tensor product modules over H under the assumptions: $ch(k) = 0$, $|G| \leq \infty$ and $|\chi(a)| = |\chi|$. In Section 6, we apply the obtained results to some Hopf-Ore extension of the group algebra kD_n, where D_n is the dihedral group of order $2n$, $n = 2m$ with $m > 1$ odd, and char(k)=0. The Grothedieck ring and the Green ring of the Hopf-Ore extension are described by means of generators and relations respectively.

2. **Preliminaries**

Throughout, let k be an algebraically closed field. Unless otherwise stated, all algebras and Hopf algebras are defined over k; all modules are finite dimensional and left modules; dim and $\otimes$ denote $\dim_k$ and $\otimes_k$, respectively. We refer to [2, 6, 8] for the basic concepts and notations of Hopf algebras or those in the representation theory. We use ε, Δ and S to denote the counit, comultiplication and antipode of a Hopf algebra respectively. Let $k^{\times} = k \setminus \{0\}$. For a group G, let $\hat{G}$ denote the group of the linear characters of G over k, and let $Z(G)$ denote the center of G. Let $\mathbb{Z}$ denote all integers. $\mathbb{N}$ stands for all nonnegative integers, and $\mathbb{N}^+$ stands for all positive integers. Denote by $\sharp X$ the number of the elements in a set X.

2.1. **Grothendieck ring and Green ring.**

For an algebra A, we denote by mod$A$ the category of finite dimensional A-modules. For a module $M \in$ mod$A$ and an element $n \in \mathbb{N}$, let nM be the direct sum of n copies of M. Thus $nM = 0$ if $n = 0$.

The *Grothendieck ring* $G_0(H)$ of a Hopf algebra H is defined to be the abelian group generated by the isomorphism classes $[V]$ of V in mod$H$ modulo the relations $[V] = [U] + [W]$ for all short exact sequences $0 \to U \to V \to W \to 0$ in mod$H$. The multiplication of $G_0(H)$ is defined by $[U][V] = [U \otimes V]$, the tensor product of H-modules. The ring $G_0(H)$ is associative and has identity. $G_0(H)$ has a $\mathbb{Z}$-basis $\{[V_i]|i \in I\}$, where $\{V_i|i \in I\}$ are all non-isomorphic simple modules. Moreover, for each $V \in$ mod$A$, we have $[V] = \sum_i [V : V_i][V_i]$ in $G_0(H)$, where $[V : V_i]$ denotes the multiplicity of V_i in a composition series of V.

The *Green ring* $r(H)$ of a Hopf algebra H is defined to be the abelian group generated by the isomorphism classes $[V]$ of V in mod$H$ modulo the relations $[U \oplus V] = [U] + [V]$, $U, V \in$ mod$H$. The multiplication of $r(H)$ is determined by $[U][V] = [U \otimes V]$, the tensor product of H-modules. Then $r(H)$ is an associative ring with identity. Notice that $r(H)$ is a free abelian group with a $\mathbb{Z}$-basis $\{[V]|V \in \text{ind}(H)\}$, where ind$(H)$ denotes the category of indecomposable objects in mod$H$.

Note that there is a canonical ring epimorphism $r(H) \to G_0(H)$, $[V] \mapsto [V]$. If H is a finite dimensional semisimple Hopf algebra, then the epimorphism is a ring isomorphism, i.e., $r(H) = G_0(H)$.

2.2. Hopf-Ore extensions of a group algebra kG.

Let G be a group and $a \in Z(G)$. Let $\chi \in \hat{G}$ with $\chi(a) \neq 1$ and let $q = \chi(a)$. The Hopf-Ore extension $H = kG(\chi^{-1}, a, 0)$ of the group algebra kG can be described as follows. H is generated, as an algebra, by G and x subject to the relations $xg = \chi^{-1}(g)gx$ for all $g \in G$. The coalgebra structure and the antipode are given by

$$\begin{aligned} &\Delta(x) = x \otimes a + 1 \otimes x, && \varepsilon(x) = 0, && S(x) = -xa^{-1}, \\ &\Delta(g) = g \otimes g, && \varepsilon(g) = 1, && S(g) = g^{-1}, \end{aligned}$$

where $g \in G$. H has a k-basis $\{gx^i \mid g \in G, i \in \mathbb{N}\}$.

3. Simple modules

In this and the next two sections, we fix $H = kG(\chi^{-1}, a, 0)$, a Hopf-Ore extension of a group algebra kG as defined in the previous section. Let $q = \chi(a)$.

Let V be a kG-module. Then V becomes an H-module by setting $x \cdot v = 0$, $v \in V$ (see [13, Page 812]). Thus, one obtains an embedding functor $F :$ mod$kG \to$ mod$H$. Obviously, F is a tensor functor. Hence modkG can be regarded as a tensor subcategory of mod$H$.

Let $\{V_i|i \in I\}$ be all non-isomorphic simple kG-module. For any $i \in I$, V_i becomes a simple H-module as above. For any $\lambda \in \hat{G}$, there is a one-dimensional H-module V_λ defined by $g \cdot v = \lambda(g)v$ and $x \cdot v = 0$ for any $g \in G$ and $v \in V_\lambda$ (see [13]). V_λ is also a simple kG-module. Hence we may regard $\hat{G} \subseteq I$. Thus, V_ε is the trivial H-module, where ε is the identity of the group $\hat{G}$. One can easily check that $V_i \otimes V_\lambda \cong V_\lambda \otimes V_i$ is a simple module as well for any $i \in I$ and $\lambda \in \hat{G}$. Hence there exists a permutation σ of I such that $V_\chi \otimes V_i \cong V_{\sigma(i)}$, $i \in I$. Consequently, $V_{\sigma^t(i)} \cong V_{\chi^t} \otimes V_i$, $t \in \mathbb{Z}$. Define a binary relation $\sim$ on I as follows: $i \sim j$ if

$i=\sigma^t(j)$, or equivalently, $V_i=V_{\sigma^t(j)}$, for some $t\in\mathbb{Z}$, where $i,j\in I$. Obviously, $\sim$ is an equivalent relation. Denote by $[i]$ the equivalence class containing i. Let I_0 be the set of all equivalence classes of I with respect to $\sim$.

Clearly, if $|\chi|=s<\infty$ then $\sigma^s(i)=i$ for any $i\in I$. Conversely, we have the following lemma.

Lemma 3.1. *If $\sigma^t(i)=i$ for some $i\in I$ and $t\in\mathbb{Z}$ with $t\neq 0$, then $|\chi|<\infty$.*

Proof. Assume $\sigma^t(i)=i$ for some $i\in I$ and $t\in\mathbb{N}^+$. Then $V_{\chi^t}\otimes V_i\cong V_i$. Let $\phi: V_i\to V_{\chi^t}\otimes V_i$ be a kG-module isomorphism. Let $0\neq v_0\in V_{\chi^t}$. Since $\dim(V_{\chi^t})=1$, there exists a linear automorphism f of V_i such that $\phi(v)=v_0\otimes f(v)$, $v\in V_i$. From $\phi(g\cdot v)=g\cdot\phi(v)$, one gets $f(g\cdot v)=\chi^t(g)g\cdot f(v)$, and so $g^{-1}\cdot f(g\cdot v)=\chi^t(g)f(v)$, where $g\in G$ and $v\in V_i$. This implies $\det(f)=\chi^{t\dim(V_i)}(g)\det(f)$, $g\in G$. It follows that $\chi^{t\dim(V_i)}(g)=1$ for all $g\in G$, and so $|\chi|<\infty$. $\square$

Let $\langle\chi\rangle$ be the subgroup of $\hat{G}$ generated by χ and $\hat{G}/\langle\chi\rangle$ the corresponding quotient group. By [13, Proposition 3.17(a)], one can see that $\hat{G}/\langle\chi\rangle\subseteq I_0$.

Lemma 3.2. *Let $i\in I$ and $l,r\in\mathbb{Z}$. Then $\sigma^l(i)\neq\sigma^r(i)$ if $|\chi|=\infty$ and $l\neq r$, or $|q|=s<\infty$ and $s\nmid l-r$.*

Proof. If $|\chi|=\infty$ and $l\neq r$, then $\sigma^{l-r}(i)\neq i$ by Lemma 3.1, and hence $\sigma^l(i)\neq\sigma^r(i)$. Now assume $|q|=s<\infty$ and $s\nmid l-r$. If $\sigma^l(i)=\sigma^r(i)$, then $\sigma^{l-r}(i)=i$. By the proof of Lemma 3.1, there is a linear automorphism f of V_i such that $f(av)=\chi^{l-r}(a)af(v)=q^{l-r}af(v)$, $v\in V_i$. Since a is a central element of G and V_i is a simple kG-module, there exists an $\alpha\in k^\times$ such that $av=\alpha v$ for all $v\in V_i$. Hence $\alpha f(v)=q^{l-r}\alpha f(v)$, $v\in V_i$. This implies $q^{l-r}\alpha=\alpha$, and hence $q^{l-r}=1$, a contradiction. This completes the proof. $\square$

For any H-module M, the subspace $M^x=\{m\in M|xm=0\}$ is a submodule of M. If $M^x=M$ then M is called *x-torsion.* If $M^x=0$ then M is called *x-torsionfree.* Obviously, if M is a simple H-module, then M is either *x-torsion* or *x-torsionfree.*

Lemma 3.3. *If there exists a nonzero x-torsionfree H-module, then $|\chi|<\infty$.*

Proof. Suppose that M is a nonzero *x-torsionfree* H-module. Let V be a simple kG-submodule of M. Then $V\cong V_i$ for some $i\in I$. Without loss of generality, we may assume that V_i is a simple kG-submodule of M. Since M is *x-torsionfree*, $x^jV_i\neq 0$ for any $j\geqslant 1$. It is easy to check that x^jV_i is a kG-submodule of M and $x^jV_i\cong V_{\sigma^j(i)}$. Since M is finite dimensional, there is a positive integer n such that $x^nV_i\subseteq\sum_{j=0}^{n-1}x^jV_i$. Since $V_i, xV_i,\cdots,x^nV_i$ are all simple kG-modules, $x^nV_i\cong x^lV_i$ as kG-modules for some $0\leqslant l\leqslant n-1$. This implies $V_{\sigma^n(i)}\cong V_{\sigma^l(i)}$, and hence $\sigma^n(i)=\sigma^l(i)$. By Lemma 3.1, $|\chi|<\infty$. $\square$

Let $i\in I$. Then one can define a module $M(V_i)=H\otimes_{kG}V_i$. Note that H is a free right kG-module with a basis $\{x^l|l\geqslant 0\}$. Hence $M(V_i)=\oplus_{l=0}^{\infty}x^l\otimes_{kG}V_i$ as vector spaces. For any $v\in V_i$, still denote by v the element $1\otimes_{kG}v$ of $M(V_i)$ for simplicity. Then we may view V_i as $1\otimes_{kG}V_i$ in $\subset M(V_i)$.

Assume $|\chi| = s < \infty$. Let $i \in I$. For any $\beta \in k$ and a monic polynomial $f_n(y) = (y-\beta)^n = y^n - \sum_{j=0}^{n-1}\alpha_j y^j$ with $n \geqslant 1$, let $N_\beta^n(i)$ be the submodule of $M(V_i)$ generated by $f_n(x^s)M(V_i)$, and define $V_n(i,\beta) = M(V_i)/N_\beta^n(i)$ to be the corresponding quotient module. Since $f(x^s)$ is a central element of H, $N_\beta^n(i) = f_n(x^s)M(V_i)$ and hence $\dim V_n(i,\beta) = ns\dim V_i$. For any $m \in M(V_i)$, denote still by m the image of m under the canonical epimorphism $M(V_i) \to V_n(i,\beta)$. Then it is easy to see that $V_n(i,\beta)$ is generated, as an H-module, by V_i, and $V_n(i,\beta) = \oplus_{j=0}^{ns-1}x^jV_i$ as vector spaces. Moreover, we have

$$x^{ns}v = \sum_{j=0}^{n-1}\alpha_j x^{js}v, \ v \in V_n(i,\beta).$$

If $n = 1$, then $x^sv = \beta v$ for all $v \in V_1(i,\beta)$. Let $V(i,\beta)$ be the module $V_1(i,\beta)$. Obviously, $V(i,\beta)$ is *x-torsionfree* for any $i \in I$ and $\beta \in k^\times$.

Proposition 3.4. *Assume* $|\chi| = |q| = s < \infty$. *Let* $i,j \in I$ *and* $\alpha,\beta \in k^\times$. *Then* $V(i,\beta)$ *is simple and* $V(i,\beta) \cong V(j,\alpha)$ *if and only if* $[i] = [j]$ *and* $\beta = \alpha$.

Proof. Let N be a nonzero submodule $V(i,\beta)$ and V a simple kG-submodule of N. Since $V(i,\beta) = \oplus_{j=0}^{s-1}x^jV_i$ is *x-torsionfree*, x^jV_i is a kG-submodule of $V(i,\beta)$ and $x_jV_i \cong V_{\sigma^j(i)}$, $0 \leqslant j \leqslant s-1$. Then by Lemma 3.2, $V_i, xV_i, \cdots, x^{s-1}V_i$ are pairwise non-isomorphic simple kG-submodules of $V(i,\beta)$. It follows that $V = x^jV_i$ for some $0 \leqslant j \leqslant s-1$. Thus, $N \supset x^{s-j}V = x^sV_i = \beta V_i = V_i$. Since the H-module $V(i,\beta)$ is generated by V_i, $N = V(i,\beta)$ and so $V(i,\beta)$ is a simple H-module.

Assume $V(i,\beta) \cong V(j,\alpha)$. Then $\dim V_i = \dim V_j$. Let $\phi : V(i,\beta) \to V(j,\alpha)$ be an H-module isomorphism. Pick up a nonzero element $v \in V_j$. Then there exist an $m \in V(i,\beta)$ such that $\phi(m) = v$. In this case, we have $\phi(x^sm) = \phi(\beta m) = \beta v$ and $\phi(x^sm) = x^s\phi(m) = x^sv = \alpha v$. Hence $\beta = \alpha$. Since ϕ is an H-module isomorphism, $\phi(V_i)$ is a kG-submodule of $V(j,\alpha)$ and $\phi(V_i) \cong V_i$ as kG-modules. By the discussion above, $V(j,\alpha) = \oplus_{t=0}^{s-1}x^tV_j$ and $V_j, xV_j, \cdots, x^{s-1}V_j$ are non-isomorphic simple kG-submodules of $V(j,\alpha)$. Hence there exists an integer t with $0 \leqslant t \leqslant s-1$ such that $x^tV_j = \phi(V_i)$ since $\phi(V_i)$ is a simple kG-submodule of $V(j,\alpha)$. Therefore, $V_i \cong V_{\sigma^t(j)}$ as kG-modules, which implies $[i] = [j]$.

Conversely, assume that $[i] = [j]$ and $\beta = \alpha$. Then there exists an integer t with $0 \leqslant t \leqslant s-1$ such that $V_i = V_{\sigma^t(j)}$. Note that $V_i \subset V(i,\beta)$ and $V_j \subset V(j,\beta)$ as stated before. Since x^tV_j is a kG-submodule of $V(j,\beta)$ and $x^tV_j \cong V_{\sigma^t(j)}$, we have $V_i \cong x^tV_j$ as kG-modules. Let $\phi : V_i \to x^tV_j$ be a kG-module isomorphism. Since $V(i,\beta) = \oplus_{l=0}^{s-1}x^lV_i$, we may extend ϕ to a linear map ϕ_0 from $V(i,\beta)$ to $V(j,\beta)$ by letting $\phi_0(x^lv) = x^l\phi(v)$ for all $0 \leqslant l \leqslant s-1$ and $v \in V_i$. It is easy to check that ϕ_0 is an H-module homomorphism. Since both $V(i,\beta)$ and $V(j,\beta)$ are simple, it follows from $\phi_0 \neq 0$ that ϕ_0 is an H-module isomorphism. □

Theorem 3.5. *Let* M *be a simple* H*-module.*

1. *If* M *is* x*-torsion, then* $M \cong V_i$ *for some* $i \in I$.
2. *If* $|\chi| = |q| = s$ *and* M *is* x*-torsionfree, then* $s < \infty$ *and* $M \cong V(i,\beta)$ *for some* $i \in I$ *and* $\beta \in k^\times$.

Proof. (1) If M is *x-torsion*, then M is a simple kG-submodule. Hence there is an $i \in I$ such that $M \cong V_i$ as kG-modules, and so $M \cong V_i$ as H-modules.

(2) Assume that $|\chi| = |q| = s$ and M is *x-torsionfree*. Then $s < \infty$ by Lemma 3.3. Let V be a simple kG-submodule of M. Then $V \cong V_i$ for some $i \in I$. Without loss of generality, we may assume that V_i is a simple kG-submodule of M. Define a liner map $\phi : M(V_i) = H \otimes_{kG} V_i \to M$ by $\phi(h \otimes v) = hv$ for any $h \in H$ and $v \in V_i$. Since M is a simple H-module, it is easy to see that ϕ is an H-module epimorphism. Since x^s is central element in H and M is *x-torsionfree*, there exists a $\beta \in k^{\times}$ such that $x^s m = \beta m$ for any $m \in M$, i.e., $(x^s - \beta)M = 0$. Hence $\phi(N^1_\beta(i)) = \phi((x^s - \beta)M(V_i)) = (x^s - \beta)\phi(M(V_i)) = (x^s - \beta)M = 0$. Thus, ϕ induces an H-module epimorphism $\widetilde{\phi} : V(i, \beta) = M(V_i)/N^1_\beta(i) \to M$, which must be an isomorphism since $V(i, \beta)$ and M are both simple H-modules. □

Corollary 3.6. *The following statements hold.*

1. *If* $|\chi| = \infty$, *then* $\{V_i | i \in I\}$ *is a representative set of isomorphic classes of simple* H*-modules.*
2. *If* $|\chi| = |q| = s < \infty$, *then* $\{V_i, V(j, \beta) | i \in I, [j] \in I_0, \beta \in k^{\times}\}$ *is a representative set of isomorphic classes of simple* H*-modules.*

Proof. It follows from Lemmas 3.2-3.3, Proposition 3.4 and Theorem 3.5. □

4. **Indecomposable modules**

Throughout this section, we assume that the group algebra kG is finite dimensional and semisimple. We will use the notations of last section and let $|\chi| = s$. In this case, $1 < |q| \leqslant s < \infty$. Moreover, I and I_0 are finite sets.

Let A be a k-algebra, and M an A-module. Then the smallest nonnegative integer l with $\mathrm{rad}^l(M) = 0$ is called the *radical length* of M, denoted by $\mathrm{rl}(M)$, and $0 \subset \mathrm{rad}^{l-1}(M) \subset \cdots \subset \mathrm{rad}^2(M) \subset \mathrm{rad}(M) \subset M$ is called the *radical series* of M. By [2, Proposition II.4.7], $\mathrm{rl}(M) = \mathrm{sl}(M)$, the socle length of M, which is sometimes called the *Loewy length* of M. Let $\mathrm{l}(M)$ denote the length of M.

For any $t \in \mathbb{N}^+$ and $i \in I$, let $J_t(i)$ be the submodule of $M(V_i)$ generated by $x^t V_i$, and define $V_t(i) = M(V_i)/J_t(i)$ to be the corresponding quotient module. Note that $J_t(i) = x^t M(V_i) = \oplus_{j=t}^{\infty} x^j V_i$. For simplicity, denote still by z the image of an element $z \in M(V_i)$ under the canonical epimorphism $M(V_i) \to V_t(i)$. Then $V_i \subseteq V_t(i)$, $V_t(i) = \oplus_{j=0}^{t-1} x^j V_i$ as vector spaces, $\mathrm{dim} V_t(i) = t \mathrm{dim} V_i$ and $x^t V_t(i) = 0$. Moreover, $x^j V_i$ is a kG-submodule of $V_t(i)$ and $x^j V_i \cong V_{\sigma^j(i)}$, $0 \leqslant j \leqslant t-1$.

Remark 4.1. $V_1(i) \cong V_i$ *and* $V_t(i, 0) = V_{ts}(i)$ *as* H*-modules, where* $i \in I$ *and* $t \in \mathbb{N}^+$.

The following lemma is obvious.

Lemma 4.2. *Let* $i \in I$ *and* $t \in \mathbb{N}^+$. *Then for any* $0 \leqslant j \leqslant t-1$ *and* $0 \neq v \in V_i$, $x^j v \neq 0$ *in* $V_t(i)$.

Proposition 4.3. *Let $i \in I$ and $t \in \mathbb{N}^+$. Then $V_t(i)$ is an indecomposable uniserial H-module. Moreover,* $\mathrm{l}(V_t(i)) = t$.

Proof. For any $0 \leqslant l \leqslant t-1$, let $M_l = \sum_{j=l}^{t-1} x^j V_i \subseteq V_t(i)$. Then M_j is an H-submodule of $V_t(i)$. Obviously,

$$V_t(i) = M_0 \supset M_1 \supset \cdots \supset M_{t-1} \supset M_t = 0$$

is a composition series of $V_t(i)$ and $M_l/M_{l+1} \cong V_{\sigma^l(i)}$, $0 \leqslant l \leqslant t-1$. Hence $\mathrm{l}(V_t(i)) = t$.

Let N be a nonzero submodule of $V_t(i)$. Since $x^t N \subseteq x^t V_t(i) = 0$, there is an integer l with $1 \leqslant l \leqslant t$ such that $x^l N = 0$ but $x^{l-1} N \neq 0$. If $l = t$ then $N \subseteq V_t(i) = M_0 = M_{t-l}$. If $l < t$ and $z \in N$, then $z = \sum_{j=0}^{t-1} x^j v_j$ for some $v_j \in V_i$. Since $x^l z = \sum_{j=0}^{t-1} x^{l+j} v_j = \sum_{j=0}^{t-1-l} x^{l+j} v_j = 0$, $x^{l+j} v_j = 0$ for any $0 \leqslant j \leqslant t-1-l$. By Lemma 4.2, $v_j = 0$ for any $0 \leqslant j \leqslant t-1-l$. Hence $z = \sum_{j=t-l}^{t-1} x^j v_j \in M_{t-l}$, and so $N \subseteq M_{t-l}$. Thus, we have proven $N \subseteq M_{t-l}$. Since $x^{l-1} N \neq 0$, we may choose an element $z \in N$ such that $x^{l-1} z \neq 0$. From $N \subseteq M_{t-l}$, we have $z = \sum_{j=t-l}^{t-1} x^j v_j$ for some $v_j \in V_i$. Hence $0 \neq x^{l-1} z = x^{t-1} v_{t-l} \in N \cap (x^{t-1} V_i)$. This implies $v_{t-l} \neq 0$ and $x^{t-1} V_i \subseteq N$ since $x^{t-1} V_i$ is simple as a kG-module. Now suppose that $1 \leqslant r < l$ and $x^j V_i \subseteq N$ for all $t-r \leqslant j \leqslant t-1$. Then $x^{l-r-1} z = \sum_{j=t-l}^{t-1} x^{l-r-1+j} v_j = \sum_{j=t-l}^{t-l+r} x^{l-r-1+j} v_j \in N$. Hence $0 \neq x^{t-r-1} v_{t-l} = x^{l-r-1} z - \sum_{j=t-l+1}^{t-l+r} x^{l-r-1+j} v_j \in N \cap (x^{t-(r+1)} V_i)$, and so $x^{t-(r+1)} V_i \subseteq N$. Thus, we have shown that $x^j V_i \subseteq N$ for all $t-l \leqslant j \leqslant t-1$. Therefore, $M_{t-l} \subseteq N$, and so $N = M_{t-l}$. It follows that $V_t(i)$ is uniserial and indecomposable. □

Corollary 4.4. *Let $i, j \in I$ and $n, t \in \mathbb{N}^+$. Then $V_t(i) \cong V_n(j)$ if and only if $t = n$ and $i = j$.*

Proof. By Proposition 4.3 and its proof, we have $\mathrm{l}(V_t(i)) = t$, $\mathrm{l}(V_n(j)) = n$, $V_t(i)/\mathrm{rad}(V_t(i)) \cong V_i$ and $V_n(j)/\mathrm{rad}(V_n(j)) \cong V_j$. Hence the corollary follows. □

Lemma 4.5. *Let M be an H-module. If each composition factor of M is isomorphic to V_i for some $i \in I$, then $xM = \mathrm{rad}(M)$ and $M^x = \mathrm{soc}(M)$.*

Proof. Assume that each composition factor of M is isomorphic to some V_i. Then $x(M/\mathrm{rad}(M)) = 0$, and hence $xM \subseteq \mathrm{rad}(M)$. On the other hand, it is easy to see that xM is a submodule of M. Let $\overline{M} = M/xM$. Then $x\overline{M} = 0$, and hence each kG-submodule of $\overline{M}$ is an H-submodule of $\overline{M}$. So $\overline{M}$ is a semisimple H-module, which implies that $\mathrm{rad}(M) \subseteq xM$. Therefore $xM = \mathrm{rad}(M)$. Similarly, from $xM^x = 0$, one gets $M^x \subseteq \mathrm{soc}(M)$. By the assumption on M, one knows that each simple submodule of M is contained in M^x. Hence $\mathrm{soc}(M) \subseteq M^x$, and so $M^x = \mathrm{soc}(M)$. □

Theorem 4.6. *Let M be an indecomposable H-module. If each composition factor of M is isomorphic to V_j for some $j \in I$. Then M is isomorphic to some $V_t(i)$, where $i \in I$ and $t \in \mathbb{N}^+$.*

Proof. Assume that each composition factor of M is isomorphic to some V_j. Define a linear endomorphism ϕ of M by $\phi(m) = xm$, $m \in M$. Then using the map ϕ, it follows from Lemma 4.5 and [13, Lemma 4.1] that M is uniserial. Hence the radical series of M is its unique composition series. Let $t = \mathrm{l}(M)$. Then $t \geqslant 1$ and $x^tM = 0$ but $x^{t-1}M \neq 0$ by Lemma 4.5. Since M is semisimple as a kG-module, there is a simple kG-submodule V of M such that $x^{t-1}V \neq 0$. Let $N = \sum_{j=0}^{t-1} x^jV$. From $x^tV \subseteq x^tM = 0$, it is easy to see that N is an H-submodule of M, and consequently N is also uniserial. Hence $\mathrm{l}(N)$ is equal to the radical length of N. Clearly, $x^{t-1}N = x^{t-1}V \neq 0$ and $x^tN = 0$. Thus, by Lemma 4.5, one knows that $\mathrm{l}(N) = t = \mathrm{l}(M)$, which implies $M = N = \sum_{j=0}^{t-1} x^jV$. Since V is a simple kG-submodule of M, there exists an $i \in I$ such that $V \cong V_i$ as kG-modules. Let $f : V_i \to V$ be a kG-module isomorphism. Define a linear map $\psi : M(V_i) \to M$ to be the composition

$$\psi : M(V_i) = H \otimes_{kG} V_i \xrightarrow{\mathrm{id}\otimes f} H \otimes_{kG} V \overset{\cdot}{\to} M.$$

That is, $\psi(h \otimes v) = hf(v)$ for any $h \in H$ and $v \in V_i$. Obviously, ψ is an H-module epimorphism. Now we have $\psi(J_t(i)) = \psi(x^tM(V_i)) = x^t\psi(M(V_i)) = x^tM = 0$. Hence ψ induces an H-module epimorphism $\overline{\psi} : V_t(i) = M(V_i)/J_t(i) \to M$. Since $\mathrm{l}(V_t(i)) = t = \mathrm{l}(M)$, $\overline{\psi}$ is an H-module isomorphism. This completes the proof. □

Let M be an arbitrary H-module. For any monic polynomial $f(y) \in k[y]$, put

$$M^{(f)} = \{m \in M | f(x^s)^r m = 0 \text{ for some integer } r > 0\}.$$

Note that [13, Lemma 4.10, Theorem 4.11, Corollary 4.12, Lemma 4.13] still hold. When $f(y) = y - \beta$ for some $\beta \in k$, we denote $M^{(f)}$ by $M^{(\beta)}$.

In the rest of this section, assume $|q| = |\chi| = s$.

Lemma 4.7. *Let M be an indecomposable H-module. If there exists a scalar $\beta \in k^{\times}$ such that $(x^s - \beta)M = 0$, then M is simple and isomorphic to $V(i, \beta)$ for some $i \in I$.*

Proof. Clearly, $M^x = 0$. Let U_1 be a simple kG-submodule of M and $M_1 = HU_1$ be the H-submodule of M generated by U_1. Then there is an $i_1 \in I$ such that $U_1 \cong V_{i_1}$ as kG-modules. Let $f_1 : V_{i_1} \to U_1$ be a kG-module isomorphism. Then the composition map

$$\phi_1 : M(V_{i_1}) = H \otimes_{kG} V_{i_1} \xrightarrow{\mathrm{id}\otimes f_1} H \otimes_{kG} U_1 \overset{\cdot}{\to} M_1,\ h \otimes v \mapsto hf_1(v)$$

is an H-module epimorphism. Since $(x^s - \beta)M = 0$, $\phi_1(N^1_\beta(i_1)) = \phi_1((x^s - \beta)M(V_{i_1})) = (x^s - \beta)\phi_1(M(V_{i_1})) = (x^s - \beta)M_1 = 0$. Hence ϕ_1 induces an H-module epimorphism $\overline{\phi_1} : V(i_1, \beta) = M(V_{i_1})/N^1_\beta(i_1) \to M_1$, which must be an isomorphism since $V(i_1, \beta)$ is a simple H-module. Now let $l \geqslant 1$ and suppose that we have found simple H-submodules $M_1, \cdots, M_l$ of M such that the sum $\sum_{j=1}^l M_j$ in M is direct and $M_j \cong V(i_j, \beta)$ for some $i_j \in I$, $1 \leqslant j \leqslant l$. If $\sum_{j=1}^l M_j \neq M$, then there is a simple kG-submodule U_{l+1} of M such that $U_{l+1} \nsubseteq \sum_{j=1}^l M_j$. Let $M_{l+1} = HU_{l+1}$ be the H-submodule of M generated by U_{l+1}. Then a similar argument as above shows that $M_{l+1} \cong V(i_{l+1}, \beta)$ for some $i_{l+1} \in I$. Thus, M_{l+1} is simple and $M_{l+1} \nsubseteq \sum_{j=1}^l M_j$. Hence the sum $\sum_{j=1}^{l+1} M_j$ in M is direct. Since M is finite dimensional, there are finitely many simple H-submodules $M_1, \cdots, M_m$ of

M such that $M=\oplus_{j=1}^{m}M_j$ and $M_j\cong V(i_j,\beta)$ for some $i_j\in I$, $1\leqslant j\leqslant m$. Since M is indecomposable, $m=1$. This completes the proof. □

Lemma 4.8. *Assume* $\beta\in k^{\times}$. *Let* M *be an indecomposable* H*-module with* $M=M^{(\beta)}$. *Then each composition factor of* M *is isomorphic to* $V(i,\beta)$ *for some* $i\in I$.

Proof. Let N be the composition factor of M. Then $N=N^{(\beta)}$ by $M=M^{(\beta)}$. By [13, Lemma 4.13], one knows that $(x^s-\beta)N=0$. Then it follows from Lemma 4.7 that $N\cong V(i,\beta)$ for some $i\in I$. □

Lemma 4.9. *Assume* $\beta\in k^{\times}$. *Let* M *be an* H*-module with* $M=M^{(\beta)}$. *Then* $\mathrm{rad}(M)=(x^s-\beta)M$.

Proof. Since $M/\mathrm{rad}(M)$ is semisimple, $(x^s-\beta)(M/\mathrm{rad}(M))=0$ by Lemma 4.8. Hence $(x^s-\beta)M\subseteq\mathrm{rad}(M)$. On the other hand, we have $(x^s-\beta)(M/(x^s-\beta)M)=0$. Hence it follows from the proof of Lemma 4.7 that $M/(x^s-\beta)M$ is semisimple. This implies $\mathrm{rad}(M)\subseteq(x^s-\beta)M$, and so $\mathrm{rad}(M)=(x^s-\beta)M$. □

Proposition 4.10. *Let* $i\in I$, $\beta\in k^{\times}$ *and* $r\in\mathbb{N}^+$. *Then* $V_r(i,\beta)$ *is uniserial and indecomposable. Moreover,* $\mathrm{l}(V_r(i,\beta))=r$ *and the composition factors of* $V_r(i,\beta)$ *are all isomorphic to* $V(i,\beta)$.

Proof. Since $(x^s-\beta)^rV_r(i,\beta)=0$, $V_r(i,\beta)=V_r(i,\beta)^{(\beta)}$. It follows from Lemma 4.9 that $\mathrm{rad}^j(V_r(i,\beta))=(x^s-\beta)^jV_r(i,\beta)$ for any $j\geqslant 0$. Clearly, $(x^s-\beta)^{r-1}V_r(i,\beta)\neq 0$. Hence the series

$$0\subset(x^s-\beta)^{r-1}V_r(i,\beta)\subset(x^s-\beta)^{r-2}V_r(i,\beta)\subset\cdots\subset(x^s-\beta)V_r(i,\beta)\subset V_r(i,\beta)$$

is the radical series of $V_r(i,\beta)$, and so $\mathrm{rl}(V_r(i,\beta))=r$. Let $\pi:M(V_i)\to V_r(i,\beta)$ be the canonical H-module epimorphism. Let $0\leqslant j\leqslant r-1$. Since $x^s-\beta$ is a central element of H, the map $\psi:V_r(i,\beta)\to(x^s-\beta)^jV_r(i,\beta)$, $v\mapsto(x^s-\beta)^jv$ is an H-module epimorphism. Hence the composition map $\phi=\psi\circ\pi$ is an H-module epimorphism from $M(V_i)$ to $(x^s-\beta)^jV_r(i,\beta)$. Since $\phi(N_\beta^1(i))=\phi((x^s-\beta)M(V_i))=(x^s-\beta)\phi(M(V_i))=(x^s-\beta)^{j+1}V_r(i,\beta)$, ϕ induces an H-module epimorphism $\overline{\phi}:V(i,\beta)=M(V_i)/N_\beta^1(i)\to(x^s-\beta)^jV_r(i,\beta)/(x^s-\beta)^{j+1}V_r(i,\beta)$. By Proposition 3.4, $V(i,\beta)$ is simple. Hence $\overline{\phi}$ must be an isomorphism. Thus, the above radical series of $V_r(i,\beta)$ is a composition series. It follows that $V_r(i,\beta)$ is uniserial and indecomposable. Moreover, $\mathrm{l}(V_r(i,\beta))=\mathrm{rl}(V_r(i,\beta))=r$ and each composition factor of $V_r(i,\beta)$ is isomorphic to $V(i,\beta)$. □

Theorem 4.11. *Let* M *be an indecomposable* H*-module. Then* $M\cong V_t(i)$ *for some* $i\in I$ *and* $t\in\mathbb{N}^+$, *or* $M\cong V_r(i,\beta)$ *for some* $i\in I$, $\beta\in k^{\times}$ *and* $r\in\mathbb{N}^+$. *Moreover,* M *is uniserial.*

Proof. By [13, Corollary 4.12], there exists a monic irreducible polynomial $f(y)\in k[y]$ such that $M=M^{(f)}$. Since k is an algebraically closed field, $f(y)=y$ or $f(y)=y-\beta$ for some $\beta\in k^{\times}$.

Case 1: $f(y)=y$. Since M is finite dimensional, $x^{rs}M=0$ for some integer $r\geqslant 1$. Let V be a composition factor of M. Then $x^{rs}V=0$, and hence $V^x\neq 0$. Since V is simple, $V^x=V$. By Theorem 3.5, $V\cong V_i$ for some $i\in I$. It follows from

Theorem 4.6 that $M \cong V_t(i)$ for some integer $t \geqslant 1$ and $i \in I$. In this case, M is uniserial by Proposition 4.3.

Case 2: $f(y) = y - \beta$. In this case, $M = M^{(\beta)}$. It follows from Lemma 4.8 that each composition factor of M is isomorphic to $V(i, \beta)$ for some $i \in I$. If $rl(M) = 1$, then M is simple, and so $M \cong V(i, \beta)$ for some $i \in I$.

Now assume $\mathrm{rl}(M) = r > 1$. Then $(x^s - \beta)^r M = 0$ and $(x^s - \beta)^{r-1} M \neq 0$ by Lemma 4.9. Define a liner map $\phi : M \to M$ by $\phi(m) = (x^s - \beta)m$, $m \in M$. Then ϕ is a module endomorphism of M since $x^s - \beta$ is a central element of H. For any submodule N of M, $\phi(N) = \mathrm{rad}(N)$ by Lemma 4.9, and $\phi^{-1}(N)$ is obviously a submodule of M. If V is a simple submodule of M, then $(x^s - \beta)V = \mathrm{rad}(V) = 0$ by Lemma 4.9, and hence $V \subseteq \mathrm{Ker}(\phi)$. Thus, $\mathrm{soc}(M) \subseteq \mathrm{Ker}(\phi)$. On the other hand, by Lemma 4.7, $\mathrm{Ker}(\phi)$ is semisimple, and hence $\mathrm{Ker}(\phi) \subseteq \mathrm{soc}(M)$. Therefore, $\mathrm{Ker}(\phi) = \mathrm{soc}(M)$. It follows from [13, Lemma 4.1(c)] that M is uniserial. Hence $\mathrm{l}(M) = \mathrm{rl}(M) = r$. Since M is semisimple as a kG-module, M is equal to a direct sum of some simple kG-submodules of M. Then from $(x^s - \beta)^{r-1} M \neq 0$, one knows that there is a simple kG-submodule V such that $(x^s - \beta)^{r-1} V \neq 0$. From $(x^s - \beta)^r M = 0$, one gets $(x^s - \beta)^r V = 0$. Let $N = HV$ be the H-submodule of M generated by V. Then $(x^s - \beta)^{r-1} N \neq 0$ and $(x^s - \beta)^r N = H(x^s - \beta)^r V = 0$. By Lemma 4.9, $\mathrm{rl}(N) = r$. Since M is uniserial, so is N. Hence $\mathrm{l}(N) = \mathrm{rl}(N) = r = \mathrm{l}(M)$, and so $M = N = HV$. Since V is a simple kG-module, $V \cong V_i$ as kG-modules for some $i \in I$. Let $f : V_i \to V$ be a kG-module isomorphism. Then one gets an H-module epimorphism

$$\phi : M(V_i) = H \otimes_{kG} V_i \xrightarrow{\mathrm{id} \otimes f} H \otimes_{kG} V \overset{\cdot}{\rightarrow} M, \ h \otimes v \mapsto hf(v).$$

Since $\phi((x^s - \beta)^r M(V_i)) = (x^s - \beta)^r M = 0$, ϕ induces an H-module epimorphism $\overline{\phi}$ from $V_r(i, \beta) = M(V_i)/N^r_\beta(i)$ to M. By Proposition 4.10, $\mathrm{l}(V_r(i, \beta)) = r = \mathrm{l}(M)$, and hence $\overline{\phi}$ is an isomorphism. □

Proposition 4.12. *Let* $i, j \in I$, $\alpha, \beta \in k^\times$ *and* $r, t \in \mathbb{N}^+$. *Then* $V_r(i, \alpha) \cong V_t(j, \beta)$ *if and only if* $r = t$, $\alpha = \beta$ *and* $[i] = [j]$.

Proof. If $V_r(i, \alpha) \cong V_t(j, \beta)$, then $r = \mathrm{l}(V_r(i, \alpha)) = \mathrm{l}(V_t(j, \beta)) = t$ and $V(i, \alpha) \cong V(j, \beta)$ by Proposition 4.10, and consequently $\alpha = \beta$ and $[i] = [j]$ by Proposition 3.4. Conversely, assume that $r = t$, $\alpha = \beta$ and $[i] = [j]$. We need to show $V_t(i, \beta) \cong V_t(j, \beta)$. By $[i] = [j]$, $i = \sigma^n(j)$ for some $0 \leqslant n \leqslant s - 1$. Note that $V_t(j, \beta)$ is generated, as an H-module, by V_j, and $V_t(j, \beta) = \oplus_{l=0}^{ts-1} x^l V_j$. From $(x^s - \beta)^t V_t(j, \beta) = 0$, one gets $V_t(j, \beta)^x = 0$. Hence $x^n V_j$ is a nonzero kG-submodule of $V_t(j, \beta)$ and $x^n V_j \cong V_{\sigma^n(j)} = V_i$. Let M be the H-submodule of $V_t(j, \beta)$ generated by $x^n V_j$. Then $M = H(x^n V_j) = x^n H V_j = x^n V_t(j, \beta) = V_t(j, \beta)$ by $V_t(j, \beta)^x = 0$. Let $f : V_i \to x^n V_j$ be a kG-module isomorphism. Then an argument similar to the proof of Theorem 4.11 shows that f can be extended to an H-module isomorphism from $V_t(i, \beta)$ to $M = V_t(j, \beta)$. □

Corollary 4.13. *Assume that* $|q| = |\chi| = s$. *Then*

$$\{V_t(i), V_t(j, \beta) | i \in I, [j] \in I_0, \beta \in k^\times, t \in \mathbb{N}^+\}$$

is a representative set of isomorphic classes of finite dimensional indecomposable H*-modules.*

Proof. It follows from Proposition 4.3, Corollary 4.4, Proposition 4.10, Theorem 4.11 and Proposition 4.12. ☐

5. **Decomposition rules for tensor product modules**

Throughout this section, assume that k is of characteristic zero and G is a finite group. We also assume $|q| = |\chi| = s$. In this case, the group algebra kG is finite dimensional and semisimple, and $1 < s < \infty$. In this section, we investigate the decomposition rules for tensor product modules over H.

By Corollary 4.13, one knows that

$$\{V_t(i), V_t(j, \beta) | i \in I, [j] \in I_0, \beta \in k^\times, t \in \mathbb{N}^+\}$$

is a representative set of isomorphic classes of finite dimensional indecomposable H-modules.

As stated in Section 3, modkG is a tensor subcategory of mod$H$.

Recall from [13] that an H-module M is a *weight module* if $M = \oplus_{\lambda\in\hat{G}} M_{(\lambda)}$, where $M_{(\lambda)} = \{m \in M | gm = \lambda(g)m, \forall g \in G\}$ for any $\lambda \in \hat{G}$. Let wmod$H$ be the full subcategory of mod$H$ consisting of all finite dimensional weight H-modules. Then wmod$H$ is a tensor subcategory of mod$H$ [13]. By [13, Corollary 4.20],

$$\{V_t(\lambda), V_t(\theta, \beta) | \lambda \in \hat{G}, [\theta] \in \hat{G}/\langle\chi\rangle, \beta \in k^\times, t \in \mathbb{N}^+\}$$

is a representative set of isomorphic classes of finite dimensional indecomposable weight modules over H.

Convention 5.1. *For any $i \in I$, there is a scalar $\omega_i \in k^\times$ such that $av = \omega_i v$ for all $v \in V_i$ since $a \in Z(G)$ and V_i is a simple kG-module. When $i = \lambda \in \hat{G}$, $\omega_\lambda = \lambda(a)$.*

Let $N^l_{i,j} = [V_i \otimes V_j : V_l] \in \mathbb{N}$ be the multiplicity of V_l in a composition series of $V_i \otimes V_j$, $i, j, l \in I$. Then $N^l_{j,i} = N^l_{i,j}$ and $V_i \otimes V_j \cong \oplus_{l\in I} N^l_{i,j} V_l$ in modkG *(or equivalently, in* mod$H$*) since kG is semisimple.*

Lemma 5.2. *Let $i \in I$, $t \in \mathbb{N}^+$ and $\beta \in k^\times$. Then*

1. $V_i \otimes V_t(\varepsilon) \cong V_t(\varepsilon) \otimes V_i \cong V_t(i)$;
2. $V_i \otimes V_t(\varepsilon, \beta) \cong V_t(i, \beta)$;
3. $V_t(\varepsilon, \beta) \otimes V_i \cong V_t(i, \omega_i^s \beta)$.

Proof. The proofs of the three isomorphisms are similar. We only prove (3). From Section 3, one knows that $V_t(\varepsilon, \beta) = \oplus_{j=0}^{ts-1} x^j V_\varepsilon$ and $V_t(i, \omega_i^s\beta) = \oplus_{j=0}^{ts-1} x^j V_i$ as kG-modules. Let $0 \neq v_0 \in V_\varepsilon$ and $v_j = x^j v_0$, $1 \leqslant j \leqslant ts-1$. Then $\{v_0, v_1, \cdots, v_{ts-1}\}$ is a k-basis of $V_t(\varepsilon, \beta)$. Hence one can define a k-linear isomorphism $\phi : V_t(\varepsilon, \beta) \otimes V_i \to V_t(i, \omega_i^s\beta)$ by $\phi(v_j \otimes v) = \omega_i^{-j} x^j v$ for all $0 \leqslant j \leqslant ts-1$ and $v \in V_i$. It is easy to check that $\phi(g(v_j \otimes v)) = g\phi(v_j \otimes v)$ for all $g \in G$, $0 \leqslant j \leqslant ts-1$ and $v \in V_i$. Now let $0 \leqslant j < ts-1$ and $v \in V_i$. Then $\phi(x(v_j \otimes v)) = \phi(xv_j \otimes av) = \omega_i \phi(v_{j+1} \otimes v) = \omega_i \omega_i^{-(j+1)} x^{j+1} v = x(\omega_i^{-j} x^j v) = x\phi(v_j \otimes v)$. Let $(y-\beta)^t = y^t - \sum_{l=0}^{t-1} \alpha_l y^l$. Then $(y-\omega_i^s\beta)^t = y^t - \sum_{l=0}^{t-1} \omega_i^{s(t-l)} \alpha_l y^l$. Hence we have $\phi(x(v_{ts-1} \otimes v)) = \phi(xv_{ts-1} \otimes av) =$

$\omega_i\phi(x^{ts}v_0\otimes v)=\omega_i\phi(\sum_{l=0}^{t-1}\alpha_l x^{ls}v_0\otimes v)=\omega_i\sum_{l=0}^{t-1}\alpha_l\phi(v_{ls}\otimes v)=\sum_{l=0}^{t-1}\alpha_l\omega_i^{1-ls}x^{ls}v$ and $x\phi(v_{ts-1}\otimes v)=x(\omega_i^{1-ts}x^{ts-1}v)=\omega_i^{1-ts}x^{ts}v=\omega_i^{1-ts}\sum_{l=0}^{t-1}\omega_i^{s(t-l)}\alpha_l x^{ls}v=\sum_{l=0}^{t-1}\omega_i^{1-sl}\alpha_l x^{ls}v$. This shows that $\phi(x(v_{ts-1}\otimes v))=x\phi(v_{ts-1}\otimes v)$, and so ϕ is an H-module isomorphism. □

Proposition 5.3. *Let $p,t\in\mathbb{N}^+$, $i,j\in I$ and $\beta\in k^\times$. Let $p=us+r$ with $u\geqslant 0$ and $0\leqslant r<s$. Then*

$$\begin{aligned}
V_p(i)\otimes V_t(j,\beta) &\cong (\oplus_{l\in I}\oplus_{1\leqslant m\leqslant \min(\mathrm{t,u})}N_{i,j}^l(s-r)V_{2m-1+|t-u|}(l,\beta))\\
&\quad\oplus(\oplus_{l\in I}\oplus_{m=1}^{\min(\mathrm{t,u+1})}N_{i,j}^l rV_{2m-1+|t-u-1|}(l,\beta)),\\
V_t(j,\beta)\otimes V_p(i) &\cong (\oplus_{l\in I}\oplus_{1\leqslant m\leqslant \min(\mathrm{u,t})}N_{j,i}^l(s-r)V_{2m-1+|t-u|}(l,\omega_i^s\beta))\\
&\quad\oplus(\oplus_{l}\oplus_{m=1}^{\min(\mathrm{u+1,t})}N_{j,i}^l rV_{2m-1+|t-u-1|}(l,\omega_i^s\beta)).
\end{aligned}$$

Proof. By Convention 5.1, Lemma 5.2 and [11, Theorem 3.6], we have

$$\begin{aligned}
V_p(i)\otimes V_t(j,\beta) &\cong V_i\otimes V_p(\varepsilon)\otimes V_j\otimes V_t(\varepsilon,\beta)\\
&\cong V_i\otimes V_j\otimes V_p(\varepsilon)\otimes V_t(\varepsilon,\beta)\\
&\cong (\oplus_{l\in I}N_{i,j}^l V_l)\otimes((\oplus_{1\leqslant m\leqslant\min(\mathrm{t,u})}(s-r)V_{2m-1+|t-u|}(\varepsilon,\beta))\\
&\quad\oplus(\oplus_{m=1}^{\min(\mathrm{t,u+1})}rV_{2m-1+|t-u-1|}(\varepsilon,\beta)))\\
&\cong (\oplus_{l\in I}\oplus_{1\leqslant m\leqslant\min(\mathrm{t,u})}N_{i,j}^l(s-r)V_{2m-1+|t-u|}(l,\beta))\\
&\quad\oplus(\oplus_{l\in I}\oplus_{m=1}^{\min(\mathrm{t,u+1})}N_{i,j}^l rV_{2m-1+|t-u-1|}(l,\beta)).
\end{aligned}$$

Similarly, one can show the second isomorphism. □

Proposition 5.4. *Let $p,t\in\mathbb{N}^+$, $i,j\in I$ and $\beta\in k^\times$. Then*

$$V_p(i,\alpha)\otimes V_t(j,\beta)\cong\oplus_{l\in I}\oplus_{m=0}^{s-1}\oplus_{u=1}^{\min\{p,t\}}N_{i,j}^l V_{2u-1+|p-t|}(\sigma^m(l),\omega_j^s\alpha+\beta).$$

Moreover, $V_{2u-1+|p-t|}(\sigma^m(l),\omega_j^s\alpha+\beta)\cong V_{s(2u-1+|p-t|)}(\sigma^m(l))$ when $\omega_j^s\alpha+\beta=0$ and $V_{2u-1+|p-t|}(\sigma^m(l),\omega_j^s\alpha+\beta)\cong V_{2u-1+|p-t|}(l,\omega_j^s\alpha+\beta)$ when $\omega_j^s\alpha+\beta\neq 0$.

Proof. The first assertion follows from Convention 5.1, Lemma 5.2, [11, Theorem 3.7] and an argument similar to the proof of Proposition 5.3. The second assertion follows from Remark 4.1 and Proposition 4.12. □

Proposition 5.5. *Let $i,j\in I$, $n,t\in\mathbb{Z}$ with $n\geqslant t\geqslant 1$. Assume that $n=r's+p'$ and $t=rs+p$ with $0\leqslant p',p\leqslant s-1$.*
(1) *Suppose that $p+p'\leqslant s$. If $p\leqslant p'$ then*

$$\begin{aligned}
&V_n(i)\otimes V_t(j)\cong V_t(j)\otimes V_n(i)\\
\cong\ &(\oplus_{l\in I}\oplus_{m=0}^{r}\oplus_{0\leqslant u\leqslant p-1}N_{i,j}^l V_{n+t-1-2ms-2u}(\sigma^u(l)))\\
&\oplus(\oplus_{l\in I}\oplus_{0\leqslant m\leqslant r-1}\oplus_{p\leqslant u\leqslant p'-1}N_{i,j}^l V_{(r+r'-2m)s}(\sigma^u(l)))\\
&\oplus(\oplus_{l\in I}\oplus_{0\leqslant m\leqslant r-1}\oplus_{p'\leqslant u\leqslant p+p'-1}N_{i,j}^l V_{n+t-1-2ms-2u}(\sigma^u(l)))\\
&\oplus(\oplus_{l\in I}\oplus_{0\leqslant m\leqslant r-1}\oplus_{p+p'\leqslant u\leqslant s-1}N_{i,j}^l V_{(r+r'-1-2m)s}(\sigma^u(l))),
\end{aligned}$$

and if $p \geqslant p'$ *then*

$$\begin{array}{rl} & V_n(i) \otimes V_t(j) \cong V_t(j) \otimes V_n(i) \\ \cong & (\oplus_{l\in I} \oplus_{m=0}^{r} \oplus_{0\leqslant u\leqslant p'-1} N_{i,j}^{l} V_{n+t-1-2ms-2u}(\sigma^u(l))) \\ & \oplus(\oplus_{l\in I} \oplus_{m=0}^{r} \oplus_{p'\leqslant u\leqslant p-1} N_{i,j}^{l} V_{(r+r'-2m)s}(\sigma^u(l))) \\ & \oplus(\oplus_{l\in I} \oplus_{0\leqslant m\leqslant r-1} \oplus_{p\leqslant u\leqslant p+p'-1} N_{i,j}^{l} V_{n+t-1-2ms-2u}(\sigma^u(l))) \\ & \oplus(\oplus_{l\in I} \oplus_{0\leqslant m\leqslant r-1} \oplus_{p+p'\leqslant u\leqslant s-1} N_{i,j}^{l} V_{(r+r'-1-2m)s}(\sigma^u(l))). \end{array}$$

(2) *Suppose that* $p+p' \geqslant s+1$ *and let* $\overline{m} = p+p'-s-1$. *If* $p \leqslant p'$ *then*

$$\begin{array}{rl} & V_n(i) \otimes V_t(j) \cong V_t(j) \otimes V_n(i) \\ \cong & (\oplus_{l\in I} \oplus_{m=0}^{r} \oplus_{u=0}^{\overline{m}} N_{i,j}^{l} V_{(r+r'+1-2m)s}(\sigma^u(l))) \\ & \oplus(\oplus_{l\in I} \oplus_{m=0}^{r} \oplus_{u=\overline{m}+1}^{p-1} N_{i,j}^{l} V_{n+t-1-2ms-2u}(\sigma^u(l))) \\ & \oplus(\oplus_{l\in I} \oplus_{0\leqslant m\leqslant r-1} \oplus_{p\leqslant u\leqslant p'-1} N_{i,j}^{l} V_{(r+r'-2m)s}(\sigma^u(l))) \\ & \oplus(\oplus_{l\in I} \oplus_{0\leqslant m\leqslant r-1} \oplus_{u=p'}^{s-1} N_{i,j}^{l} V_{n+t-1-2ms-2u}(\sigma^u(l))), \end{array}$$

and if $p \geqslant p'$ *then*

$$\begin{array}{rl} & V_n(i) \otimes V_t(j) \cong V_t(j) \otimes V_n(i) \\ \cong & (\oplus_{l\in I} \oplus_{m=0}^{r} \oplus_{u=0}^{\overline{m}} N_{i,j}^{l} V_{(r+r'+1-2m)s}(\sigma^u(l))) \\ & \oplus(\oplus_{l\in I} \oplus_{m=0}^{r} \oplus_{u=\overline{m}+1}^{p'-1} N_{i,j}^{l} V_{n+t-1-2ms-2u}(\sigma^u(l))) \\ & \oplus(\oplus_{l\in I} \oplus_{m=0}^{r} \oplus_{p'\leqslant u\leqslant p-1} N_{i,j}^{l} V_{(r+r'-2m)s}(\sigma^u(l))) \\ & \oplus(\oplus_{l\in I} \oplus_{0\leqslant m\leqslant r-1} \oplus_{u=l}^{s-1} N_{i,j}^{l} V_{n+t-1-2ms-2u}(\sigma^u(l))). \end{array}$$

Proof. It follows from Convention 5.1, Lemma 5.2, [11, Theorem 3.15] and an argument similar to the proof of Proposition 5.3. □

Remark 5.6. *Let* $r_w(H)$ *denote the Green ring of* wmod$H$. *Since* modkG *and* wmod$H$ *are both tensor subcategories of* mod$H$, $r(kG)$ *and* $r_w(H)$ *are subrings of* $r(H)$. *The structure of* $r_w(H)$ *has been described in* [12]. *By Lemma 5.2,* $r(H) = r(kG)r_w(H) = r_w(H)r(kG)$. *The injective map* $\hat{G} \to r(H)$, $\lambda \mapsto [V_1(\lambda)] = [V_\lambda]$ *induces a ring embedding* $\mathbb{Z}\hat{G} \hookrightarrow r(H)$ [12]. *In this case,* $r(kG) \cap r_w(H) = \mathbb{Z}\hat{G}$. *Similarly, we have* $\mathbb{Z}\hat{G} \subseteq G_0(kG) \subseteq G_0(H)$. *Moreover,* $G_0(kG) = r(kG)$ *since* kG *is semisimple.*

6. **An example**

In this section, we apply the results of the previous sections to investigate the representations of the Hopf-Ore extensions of the group algebras of dihedral groups.

For any positive integer $n \geqslant 2$, the dihedral group D_n of order $2n$ is defined by

$$D_n = \langle a, b | a^n = b^2 = (ba)^2 = 1\rangle.$$

Throughout this section, assume that $n = 2m$ is even and m is odd with $m > 1$. We also assume char$(k) = 0$. Let $\omega \in k$ be a root of unity with the order $|\omega| = n$.

In this case, kD_n is semisimple and $a^m \in Z(D_n)$.

Let $\lambda, \chi \in \hat{D_n}$ be given by $\lambda(a) = 1$, $\lambda(b) = -1$, $\chi(a) = -1$ and $\chi(b) = 1$. Then $\hat{D_n} = \{\varepsilon, \lambda, \chi, \lambda\chi\}$ and $\hat{D_n}$ is isomorphic to the Klein group K_4. Therefore, kD_n has 4 non-isomorphic one-dimensional simple modules $\{V_\varepsilon, V_\lambda, V_\chi, V_{\lambda\chi}\}$, where V_ε is the trivial kD_n-module. There are $m-1$ non-isomorphic two-dimensional simple kD_n-modules V_l, $1 \leqslant l \leqslant m-1$, their corresponding matrix representations $\rho_l : kD_n \to M_2(k)$ are given by

$$\rho_l(a) = \begin{pmatrix} w^l & 0 \\ 0 & w^{-l} \end{pmatrix} \text{ and } \rho_l(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $I = \{\varepsilon, \lambda, \chi, \lambda\chi, 1, 2, \cdots, m-1\}$. Then $\{V_i | i \in I\}$ is a representative set of isomorphic classes of simple kD_n-modules. The following lemma is well-known.

Lemma 6.1. *Let $1 \leqslant l, t \leqslant m-1$ with $l \neq t$. Then the following hold:*

(1) $V_\lambda \otimes V_\lambda \cong V_\chi \otimes V_\chi \cong V_\varepsilon$ *and* $V_\lambda \otimes V_\chi \cong V_{\lambda\chi}$;
(2) $V_\lambda \otimes V_l \cong V_l$ *and* $V_\chi \otimes V_l \cong V_{m-l}$;
(3) *if* $l + t < m$ *then* $V_l \otimes V_t \cong V_{|l-t|} \oplus V_{l+t}$;
(4) *if* $l + t = m$ *then* $V_l \otimes V_t \cong V_{|l-t|} \oplus V_\chi \oplus V_{\lambda\chi}$;
(5) *if* $l + t > m$ *then* $V_l \otimes V_t \cong V_{|l-t|} \oplus V_{n-l-t}$;
(6) *if* $2l < m$ *then* $V_l \otimes V_l \cong V_\varepsilon \oplus V_\lambda \oplus V_{2l}$;
(7) *if* $2l > m$ *then* $V_l \otimes V_l \cong V_\varepsilon \oplus V_\lambda \oplus V_{n-2l}$.

Since m is odd, $|\chi(a^m)| = 2 = |\chi|$. One can form a Hopf-Ore extension $kD_n(\chi, a^m, 0)$. Note that $\chi^{-1} = \chi$.

Throughout the rest of this section, let $H = kD_n(\chi, a^m, 0)$.

Let $I_0 = \{\varepsilon, \lambda, 1, 2, \cdots, \frac{m-1}{2}\}$. Then it follows from Lemma 6.1(1, 2) and Corollary 3.6(2) that the following set is a representative set of isomorphic classes of finite dimensional simple H-modules:

$$\{V_i, V(j, \beta) | i \in I, j \in I_0, \beta \in k^\times\}.$$

By Lemma 6.1(1, 2) and Corollary 4.13, the following set is a representative set of isomorphic classes of finite dimsensional indecomposable H-modules:

$$\{V_t(i), V_t(j, \beta) | t \in \mathbb{N}^+, i \in I, j \in I_0, \beta \in k^\times\}.$$

Moreover, $V_t(\chi, \beta) \cong V_t(\varepsilon, \beta)$, $V_t(\lambda\chi, \beta) \cong V_t(\lambda, \beta)$ and $V_t(j, \beta) \cong V_t(m-j, \beta)$ for any $t \geqslant 1$, $1 \leqslant j \leqslant m-1$ and $\beta \in k^\times$.

In what follows, we will frequently use the above two classifications, but not mention them for simplicity.

For any $i \in I$, it follows from Convention 5.1 that there is a scale $\omega_i \in k^\times$ such that $a^m v = \omega_i v$, $v \in V_i$. It is easy to see that either $\omega_i = 1$ or $\omega_i = -1$. Since $s = |\chi| = 2$, $\omega_i^s = \omega_i^2 = 1$. Thus, by Propositions 5.3-5.5, we have the following corollary.

Corollary 6.2. *For any $M, N \in \text{mod}H$, $M \otimes N \cong N \otimes M$. Consequently, $G_0(H)$ and $r(H)$ are both commutative rings.*

Convention 6.3. *For any $V \in \text{mod}H$ and $l \in \mathbb{N}$, define $V^{\otimes l}$ by $V^{\otimes 0} = V_1(\varepsilon) \cong V_\varepsilon$ for $l = 0$, $V^{\otimes 1} = V$ for $l = 1$, and $V^{\otimes l} = V \otimes V \otimes \cdots \otimes V$, the tensor product of l-folds of V, for $l > 1$.*

6.1. **The Grothendieck ring of H.** In this subsection, we will investigates the Grotendieck ring $G_0(H)$. From Remark 5.6, $\mathbb{Z}\hat{D_n} \subseteq G_0(kD_n) \subset G_0(H)$. Moreover, $\varepsilon = 1$, the identity of $G_0(H)$, and $\mathbb{Z}\hat{D_n} \cong \mathbb{Z}K_4$.

Lemma 6.4. *Let $1 \leqslant l \leqslant m-1$. Then the decomposition of $V_1^{\otimes l}$ is given as follows:*

1. *if $l = 2r-1$ is odd then $V_1^{\otimes(2r-1)} \cong \oplus_{j=1}^{r}\binom{2r-1}{r-j}V_{2j-1}$;*
2. *if $l = 2r$ is even then $V_1^{\otimes 2r} \cong \binom{2r-1}{r-1}(V_\varepsilon \oplus V_\lambda) \oplus (\oplus_{j=1}^{r}\binom{2r}{r-j}V_{2j})$.*

Proof. We prove the lemma by induction on l. For $l = 1$, it is trivial. For $l = 2$, it follows from Lemma 6.1(6). Now let $2 < l \leqslant m-1$. If $l = 2r-1$ is odd, then by the induction hypothesis and Lemma 6.1, we have

$$\begin{array}{rcl} V_1^{\otimes(2r-1)} & \cong & V_1 \otimes V_1^{\otimes(2r-2)} \\ & \cong & V_1 \otimes (\binom{2r-3}{r-2}(V_\varepsilon \oplus V_\lambda) \oplus (\oplus_{j=1}^{r-1}\binom{2r-2}{r-1-j}V_{2j}) \\ & \cong & \binom{2r-3}{r-2}(V_1 \otimes V_\varepsilon \oplus V_1 \otimes V_\lambda) \oplus (\oplus_{j=1}^{r-1}\binom{2r-2}{r-1-j}V_1 \otimes V_{2j}) \\ & \cong & 2\binom{2r-3}{r-2}V_1 \oplus (\oplus_{j=1}^{r-1}\binom{2r-2}{r-1-j}(V_{2j-1} \oplus V_{2j+1})) \\ & \cong & \oplus_{j=1}^{r}\binom{2r-1}{r-j}V_{2j-1}. \end{array}$$

Similarly, if $l = 2r$ is even then $V_1^{\otimes 2r} \cong \binom{2r-1}{r-1}(V_\varepsilon \oplus V_\lambda) \oplus (\oplus_{j=1}^{r}\binom{2r}{r-j}V_{2j})$. This completes the proof. □

Let $x = [V_1]$ in $G_0(kD_n)$. Then we have the following lemma.

Lemma 6.5. *Let $1 \leqslant l \leqslant m-1$. Then the following hold in $G_0(kD_n)$:*

1. *$\lambda x = x$;*
2. *if $l = 2r-1$ is odd, then*
$$[V_{2r-1}] = \sum_{i=0}^{r-1}(-1)^i\frac{2r-1}{2r-1-2i}\binom{2r-2-i}{i}x^{2r-1-2i};$$
3. *if $l = 2r$ is even, then*
$$[V_{2r}] = \sum_{i=0}^{r-1}(-1)^i\frac{2r}{2r-i}\binom{2r-i}{i}x^{2r-2i} + (-1)^r(\lambda+1).$$

Proof. Note that $\frac{2r-1}{2r-1-i}\binom{2r-2-i}{i}$ and $\frac{2r}{2r-i}\binom{2r-i}{i}$ are integers for all $0 \leqslant i \leqslant r-1$. Part (1) follows from Lemma 6.1(2). For Parts (2) and (3), we prove them by induction on l. If $l = 1$ then it is trivial. If $l = 2$ then it follows from Lemma 6.1(6). Now let $2 < l \leqslant m-1$. If $l = 2r-1$ is odd, then by Lemma 6.1(3), the induction hypothesis and Part (1), we have

$$\begin{array}{rl} [V_{2r-1}] = & x[V_{2r-2}] - [V_{2r-3}] \\ = & x(\sum_{i=0}^{r-2}(-1)^i\frac{2r-2}{2r-2-i}\binom{2r-2-i}{i}x^{2r-2-2i} + (-1)^{r-1}(\lambda+1)) \\ & -\sum_{i=0}^{r-2}(-1)^i\frac{2r-3}{2r-3-2i}\binom{2r-4-i}{i}x^{2r-3-2i} \\ = & \sum_{i=0}^{r-2}(-1)^i\frac{2r-2}{2r-2-i}\binom{2r-2-i}{i}x^{2r-1-2i} + (-1)^{r-1}2x \\ & +\sum_{i=1}^{r-1}(-1)^i\frac{2r-3}{2r-1-2i}\binom{2r-3-i}{i-1}x^{2r-1-2i} \\ = & x^{2r-1} + \sum_{i=1}^{r-1}(-1)^i(\frac{2r-2}{2r-2-i}\binom{2r-2-i}{i} + \frac{2r-3}{2r-1-2i}\binom{2r-3-i}{i-1})x^{2r-1-2i} \\ = & \sum_{i=0}^{r-1}(-1)^i\frac{2r-1}{2r-1-2i}\binom{2r-2-i}{i}x^{2r-1-2i}. \end{array}$$

If $l = 2r$ is even, then a similar argument shows that

$$[V_{2r}] = \sum_{i=0}^{r-1}(-1)^i \frac{2r}{2r-i}\binom{2r-i}{i}x^{2r-2i} + (-1)^r(\lambda+1).$$

This completes the proof. ☐

Corollary 6.6. *The following hold:*

1. *$G_0(kD_n)$ has a $\mathbb{Z}$-basis $X_1 := \{1, \lambda, \chi, \lambda\chi, x, x^2, \cdots, x^{m-1}\}$;*
2. *$G_0(kD_n)$ is generated, as a ring, by its subring $\mathbb{Z}\hat{D_n}$ and the element x.*

Proof. (1) Since $\{[V_i] | i \in I\}$ is a $\mathbb{Z}$-basis of $G_0(kD_n)$, it follows from Lemma 6.5(2, 3) that $G_0(kD_n)$ is generated, as a $\mathbb{Z}$-module, by X_1. Since $\sharp\{[V_i] | i \in I\} = \sharp X_1$, X_1 is also a $\mathbb{Z}$-basis of $G_0(kD_n)$.

(2) It follows from (1). ☐

Corollary 6.7. *The following hold in $G_0(kD_n)$:*

1. $x^m = \sum_{i=1}^{\frac{m-1}{2}}(-1)^{i-1}\frac{m}{m-2i}\binom{m-1-i}{i}x^{m-2i} + (1+\lambda)\chi$;
2. $\chi x = \sum_{i=0}^{\frac{m-3}{2}}(-1)^{i}\frac{m-1}{m-1-i}\binom{m-1-i}{i}x^{m-1-2i} + (-1)^{\frac{m-1}{2}}(1+\lambda)$.

Proof. (1) By Lemma 6.1(4), $x[V_{m-1}] = [V_{m-2}] + \chi + \lambda\chi$. Since m is odd, one gets from Lemma 6.5 that

$$\begin{aligned} x[V_{m-1}] = &\ x(\sum_{i=0}^{\frac{m-3}{2}}(-1)^i\frac{m-1}{m-1-i}\binom{m-1-i}{i}x^{m-1-2i} + (-1)^{\frac{m-1}{2}}(\lambda+1)) \\ = &\ \sum_{i=0}^{\frac{m-3}{2}}(-1)^i\frac{m-1}{m-1-i}\binom{m-1-i}{i}x^{m-2i} + (-1)^{\frac{m-1}{2}}2x \\ = &\ \sum_{i=0}^{\frac{m-1}{2}}(-1)^i\frac{m-1}{m-1-i}\binom{m-1-i}{i}x^{m-2i} \end{aligned}$$

and

$$\begin{aligned} [V_{m-2}] = &\ \sum_{i=0}^{\frac{m-3}{2}}(-1)^i\frac{m-2}{m-2-2i}\binom{m-3-i}{i}x^{m-2-2i} \\ = &\ \sum_{i=1}^{\frac{m-1}{2}}(-1)^{i-1}\frac{m-2}{m-2i}\binom{m-2-i}{i-1}x^{m-2i}. \end{aligned}$$

It follows that

$$\begin{aligned} &\sum_{i=0}^{\frac{m-1}{2}}(-1)^i\frac{m-1}{m-1-i}\binom{m-1-i}{i}x^{m-2i} \\ = &\ \sum_{i=1}^{\frac{m-1}{2}}(-1)^{i-1}\frac{m-2}{m-2i}\binom{m-2-i}{i-1}x^{m-2i} + \chi + \lambda\chi. \end{aligned}$$

Hence we have

$$\begin{aligned} x^m = &\ \sum_{i=1}^{\frac{m-1}{2}}(-1)^{i-1}(\frac{m-1}{m-1-i}\binom{m-1-i}{i} + \frac{m-2}{m-2i}\binom{m-2-i}{i-1})x^{m-2i} + \chi + \lambda\chi \\ = &\ \sum_{i=1}^{\frac{m-1}{2}}(-1)^{i-1}\frac{m}{m-2i}\binom{m-1-i}{i}x^{m-2i} + \chi + \lambda\chi. \end{aligned}$$

(2) It follows from Lemmas 6.1(2) and 6.5(3). ☐

Let $\mathbb{Z}\hat{D_n}[x]$ be the polynomial ring in one variable x over $\mathbb{Z}\hat{D_n}$. Define $f(x), g(x) \in \mathbb{Z}\hat{D_n}[x]$ by

$$\begin{aligned} f(x) &= \sum_{i=0}^{\frac{m-3}{2}}(-1)^i\frac{m-1}{m-1-i}\binom{m-1-i}{i}x^{m-1-2i} + (-1)^{\frac{m-1}{2}}(1+\lambda), \\ g(x) &= \sum_{i=1}^{\frac{m-1}{2}}(-1)^{i-1}\frac{m}{m-2i}\binom{m-1-i}{i}x^{m-2i} + (1+\lambda)\chi. \end{aligned}$$

Let J be the ideal of $\mathbb{Z}\hat{D_n}[x]$ generated by $\lambda x - x$, $\chi x - f(x)$ and $x^m - g(x)$. Then we have the following proposition.

Proposition 6.8. *$G_0(kD_n)\cong\mathbb{Z}\hat{D_n}[x]/J$, the factor ring of $\mathbb{Z}\hat{D_n}[x]$ modulo J.*

Proof. By Corollaries 6.2 and 6.6(2), the ring embedding $\mathbb{Z}\hat{D_n}\hookrightarrow G_0(kD_n)$ can be extended to a ring epimorphism $\phi:\mathbb{Z}\hat{D_n}[x]\to G_0(kD_n)$ by $\phi(x)=[V_1]$. By Lemma 6.5(1) and Corollary 6.7, $\phi(J)=0$. Hence ϕ induces a ring epimorphism $\overline{\phi}:\mathbb{Z}\hat{D_n}[x]/J\to G_0(kD_n)$ given by $\overline{\phi}(\overline{z})=\phi(z)$, where $\overline{z}$ denotes the image of $z\in\mathbb{Z}\hat{D_n}[x]$ under the canonical epimorphism $\mathbb{Z}\hat{D_n}[x]\to\mathbb{Z}\hat{D_n}[x]/J$. By the definition of J, $\mathbb{Z}\hat{D_n}[x]/J$ is generated, as a $\mathbb{Z}$-module, by $U:=\{\overline{1},\overline{\lambda},\overline{\chi},\overline{\lambda\chi},\overline{x},\overline{x}^2,\cdots,\overline{x}^{m-1}\}$. By Corollary 6.6(1), $\overline{\phi}(U)$ is a $\mathbb{Z}$-basis of $G_0(kD_n)$. It follows that U is a $\mathbb{Z}$-basis of $\mathbb{Z}\hat{D_n}[x]/J$ and $\overline{\phi}$ is a ring isomorphism. □

Remark 6.9. *From Proposition 6.8, $G_0(kD_n)$ is a commutative ring generated by its subring $\mathbb{Z}\hat{D_n}$ and an element $x(:=[V_1])$ subject to the three relations given in Lemma 6.5(1) and Corollary 6.7.*

Let $y_\beta=[V(\varepsilon,\beta)]$ in $G_0(H)$ for any $\beta\in k^\times$. Then by Lemma 5.2, one gets the following lemma.

Lemma 6.10. *$G_0(H)$ is generated, as a ring, by $G_0(kD_n)\cup\{y_\beta|\beta\in k^\times\}$.*

Lemma 6.11. *Let $\alpha,\beta\in k^\times$ with $\alpha\neq-\beta$. Then the following hold in $G_0(H)$:*

$$\chi y_\beta=y_\beta;\ y_\alpha y_\beta=2y_{\alpha+\beta};\ y_\beta y_{-\beta}=2(1+\chi).$$

Proof. By Lemma 5.2 and Proposition 3.4, one knows that $V_\chi\otimes V(\varepsilon,\beta)\cong V(\chi,\beta)\cong V(\varepsilon,\beta)$. Hence $\chi y_\beta=y_\beta$. By Propositions 5.4 and 3.4, we have $V(\varepsilon,\alpha)\otimes V(\varepsilon,\beta)\cong V(\varepsilon,\alpha+\beta)\oplus V(\chi,\alpha+\beta)\cong 2V(\varepsilon,\alpha+\beta)$. Hence $y_\alpha y_\beta=2y_{\alpha+\beta}$. By Proposition 5.4, Remark 4.1 and $|\chi|=2$, one gets $V(\varepsilon,\beta)\otimes V(\varepsilon,-\beta)\cong V(\varepsilon,0)\oplus V(\chi,0)\cong V_2(\varepsilon)\oplus V_2(\chi)$. Then it follows from the proof of Proposition 4.3 that $y_\beta y_{-\beta}=[V_2(\varepsilon)]+[V_2(\chi)]=2(1+\chi)$. □

Lemma 6.12. *The set $X_2:=\{1,\lambda,\chi,\lambda\chi,x^l,\chi x^l|1\leqslant l\leqslant\frac{m-1}{2}\}$ is also a $\mathbb{Z}$-basis of $G_0(kD_n)$.*

Proof. Let N be the $\mathbb{Z}$-submodule of $G_0(kD_n)$ generated by X_2. Then by Lemma 6.5(1), $\lambda N=N$. Clearly, $\chi N=N$. By Lemma 6.1(2), $\chi[V_{\frac{m-1}{2}}]=[V_{\frac{m+1}{2}}]$. Then it follows from Lemma 6.5(2, 3) that $x^{\frac{m+1}{2}}\in N$. This implies $xN\subseteq N$ by $\lambda N=N$ and $\chi N=N$. Therefore, it follows from Corollary 6.6(2) that N is an ideal of $G_0(kD_n)$, and so $N=G_0(kD_n)$ by $1\in N$. Thus, the lemma follows from $\sharp X_2=\sharp\{[V_i]|i\in I\}$. □

Lemma 6.13. *$G_0(H)$ has a $\mathbb{Z}$-basis $X_1\cup X_3$, where X_1 is the $\mathbb{Z}$-basis of $G_0(kD_n)$ given in Corollary 6.6(1) and $X_3:=\{\lambda y_\beta,x^ly_\beta|0\leqslant l\leqslant\frac{m-1}{2},\beta\in k^\times\}$.*

Proof. Let L be the $\mathbb{Z}$-submodules of $G_0(H)$ generated by $\{[V(i,\beta)]|i\in I,\beta\in k^\times\}$. Then $G_0(H)=G_0(kD_n)\oplus L$ as $\mathbb{Z}$-modules and L has a $\mathbb{Z}$-basis $\{[V(i,\beta)]|i\in I_0,\beta\in k^\times\}$. By Lemma 5.2, L is a $G_0(kD_n)$-submodule of $G_0(H)$, and is generated, as a $G_0(kD_n)$-module, by $\{y_\beta|\beta\in k^\times\}$. Hence it follows from Lemmas 6.12 and 6.11, L is generated, as a $\mathbb{Z}$-module, by X_3. It is left to show that X_3 is linearly independent over $\mathbb{Z}$. Note that $y_\beta=[V(\varepsilon,\beta)]$. By Lemmas 5.2(2) and 6.4, $\lambda y_\beta=$

$[V(\lambda,\beta)]$, $xy_\beta = [V(1,\beta)]$ and $x^l y_\beta \equiv [V(l,\beta)]$ modulo $\mathbb{Z}[V(\varepsilon,\beta)] + \mathbb{Z}[V(\lambda,\beta)] + \sum_{i=1}^{l-1}\mathbb{Z}[V(i,\beta)]$ for all $2 \leqslant l \leqslant \frac{m-1}{2}$. Since $\{[V(\varepsilon,\beta)],[V(\lambda,\beta)],[V(l,\beta)] | 1 \leqslant l \leqslant \frac{m-1}{2}, \beta \in k^\times\}$ is linearly independent over $\mathbb{Z}$, so is $\{\lambda y_\beta, x^l\omega_\beta | 0 \leqslant l \leqslant \frac{m-1}{2}, \beta \in k^\times\}$. This is completes the proof. □

Let $Y = \{y_\beta | \beta \in k^\times\}$ and $G_0(kD_n)[Y]$ the polynomial ring in variables Y over $G_0(kD_n)$. Put

$$U := \{\chi y_\beta - y_\beta, y_\alpha y_\beta - 2y_{\alpha+\beta}, y_\beta y_{-\beta} - 2(1+\chi) | \alpha,\beta \in k^\times \text{ with } \alpha \neq -\beta\},$$

and let (U) be the ideal of $G_0(kD_n)[Y]$ generated by U.

Theorem 6.14. *$G_0(H)$ is isomorphic to the factor ring $G_0(kD_n)[Y]/(U)$.*

Proof. By Corollary 6.2 and Lemma 6.10, the ring embedding $G_0(kD_n) \hookrightarrow G_0(H)$ can be extended to a ring epimorphism $\phi : G_0(kD_n)[Y] \to G_0(H)$ by $\phi(y_\beta) = [V(\varepsilon,\beta)]$ for all $\beta \in k^\times$. By Lemma 6.11, $\phi(U) = 0$. Hence ϕ induces a ring epimorphism $\overline{\phi} : G_0(kD_n)[Y]/(U) \to G_0(H)$ given by $\overline{\phi}(\pi(z)) = \phi(z)$ for any $z \in G_0(kD_n)[Y]$, where $\pi : G_0(kD_n)[Y] \to G_0(kD_n)[Y]/(U)$ is the canonical ring epimorphism. Clearly, $\pi(\chi y_\beta) = \pi(y_\beta)$ and $G_0(kD_n)[Y]/(U) = \pi(G_0(kD_n)) + \sum_{\beta\in k^\times}\pi(G_0(kD_n)y_\beta)$. Then by Lemma 6.12, $\sum_{\beta\in k^\times}\pi(G_0(kD_n)y_\beta)$ is generated, as a $\mathbb{Z}$-module, by $Y_1 := \{\pi(\lambda y_\beta), \pi(x^l y_\beta) | 0 \leqslant l \leqslant \frac{m-1}{2}, \beta \in k^\times\}$. Hence $G_0(kD_n)[Y]/(U)$ is generated, as a $\mathbb{Z}$-module, by $\pi(X_1) \cup Y_1$, where X_1 is the $\mathbb{Z}$-basis of $G_0(kD_n)$ given in Corollary 6.6(1). It is easy to check that $\overline{\phi}(z_1) \neq \overline{\phi}(z_2)$ for any $z_1 \neq z_2$ in $\pi(X_1) \cup Y_1$ and that $\overline{\phi}(\pi(X_1) \cup Y_1)$ is a $\mathbb{Z}$-basis of $G_0(H)$ by Lemma 6.13. Hence $\pi(X_1) \cup Y_1$ is $\mathbb{Z}$-basis of $G_0(kD_n)[Y]/(U)$ and $\overline{\phi}$ is a ring isomorphism. □

6.2. The Green ring of H.

In this subsection, we will investigate the Green ring $r(H)$. By Remark 5.6, $\mathbb{Z}\hat{D_n} \subset G_0(kD_n) = r(kD_n) \subset r(H)$. Moreover, $\varepsilon = 1$, the identity of $r(H)$, and $\mathbb{Z}\hat{D_n} \cong \mathbb{Z}K_4$.

Let R be the $\mathbb{Z}$-submodule of $r(H)$ generated by $\{[V_t(i)] | i \in I, t \geqslant 1\}$. By Proposition 5.5(1), R is a subring of $r(H)$. Clearly, $G_0(kD_n) \subset R$. By Proposition 5.2(1), we have the following lemma.

Lemma 6.15. *R is a free $G_0(kD_n)$-module with a basis $\{[V_t(\varepsilon)] | t \geqslant 1\}$.*

By Proposition 5.5(1) or [11, Theorem 3.15], we have the following lemma.

Lemma 6.16. *Let $t \geqslant 2$. Then the following hold:*

1. *$V_2(\varepsilon) \otimes V_1(\varepsilon) \cong V_2(\varepsilon)$ and $V_3(\varepsilon) \otimes V_1(\varepsilon) \cong V_3(\varepsilon)$;*
2. *if t is even, then $V_2(\varepsilon) \otimes V_t(\varepsilon) \cong V_t(\varepsilon) \oplus V_t(\chi)$;*
3. *if t is odd, then $V_2(\varepsilon) \otimes V_t(\varepsilon) \cong V_{t+1}(\varepsilon) \oplus V_{t-1}(\chi)$;*
4. *if $t \geqslant 3$, then $V_3(\varepsilon) \otimes V_t(\varepsilon) \cong V_{t+2}(\varepsilon) \oplus V_{t-2}(\varepsilon) \oplus V_t(\chi)$.*

Let $y = [V_2(\varepsilon)]$ and $z = [V_3(\varepsilon)]$ in $r(H)$. Then $y, z \in R$. For any $t \geqslant 1$, let M_t be the $G_0(kD_n)$-submodule of R generated by $\{[V_l(\varepsilon)] | 1 \leqslant l \leqslant t\}$. Let $M_{-1} = M_0 = 0 \subset R$. Then $M_{t-1} \subset M_t$ for all $t \geqslant 0$.

Corollary 6.17. *Let $t \geqslant 1$. Then the following hold:*

1. *M_t has a $\mathbb{Z}$-basis $\{[V_l(i)]|i \in I, 1 \leqslant l \leqslant t\}$;*
2. *$yM_t \subseteq M_{t+1}$ if t is odd and $yM_t \subseteq M_t$ if t is even;*
3. *$zM_t \subseteq M_{t+2}$.*

Proof. (1) follows from Lemma 6.15, Proposition 5.2(1) and the fact that $\{[V_i]|i \in I\}$ is a $\mathbb{Z}$-basis of $G_0(kD_n)$. (2) and (3) follows from Lemma 6.16 and (1). □

Lemma 6.18. *The following hold:*

1. *$y^2 = (1+\chi)y$ in R (or $r(H)$);*
2. *R is generated, as a ring, by $G_0(kD_n) \cup \{y, z\}$.*

Proof. (1) It follows from Lemma 6.16(2) and Proposition 5.2(1).

(2) Let R' be the subring of R generated by $G_0(kD_n) \cup \{y, z\}$. By Lemma 6.15, we only need to show $[V_t(\varepsilon)] \in R'$ for all $t \geqslant 1$. Clearly, $[V_t(\varepsilon)] \in R'$ for $1 \leqslant t \leqslant 3$. By Lemma 6.16(3) and Proposition 5.2(1), $[V_4(\varepsilon)] = yz - \chi y \in R'$. Now let $t > 4$ and assume $[V_l(\varepsilon)] \in R'$ for all $1 \leqslant l \leqslant t-1$. By Lemma 6.16(4), $V_3(\varepsilon) \otimes V_{t-2}(\varepsilon) \cong V_t(\varepsilon) \oplus V_{t-4}(\varepsilon) \oplus V_{t-2}(\chi)$. Hence $[V_t(\varepsilon)] = (z-\chi)[V_{t-2}(\varepsilon)] - [V_{t-4}(\varepsilon)] \in R'$ by Proposition 5.2(1) and the induction hypothesis. This completes the proof. □

Lemma 6.19. *Let $t \geqslant 0$. Then the following hold:*

1. *$z^t \equiv [V_{2t+1}(\varepsilon)]$ modulo M_{2t-1};*
2. *$yz^t \equiv [V_{2t+2}(\varepsilon)]$ modulo M_{2t};*
3. *$\{z^t, yz^t | t \geqslant 0\}$ is a $G_0(kD_n)$-basis of R.*

Proof. (1) It is trivial for $t = 0, 1$. Now let $t > 1$ and assume $z^{t-1} \equiv [V_{2t-1}(\varepsilon)]$ modulo M_{2t-3}. Then $z^{t-1} = [V_{2t-1}(\varepsilon)] + u$ for some $u \in M_{2t-3}$. Thus, by Lemma 6.16(4) and Corollary 6.17(3),

$$\begin{aligned} z^t &= z[V_{2t-1}(\varepsilon)] + zu \\ &= [V_{2t+1}(\varepsilon)] + [V_{2t-3}(\varepsilon)] + [V_{2t-1}(\chi)] + zu \\ &\equiv [V_{2t+1}(\varepsilon)] \text{ modulo } M_{2t-1}. \end{aligned}$$

(2) By Corollary 6.17(2), $yM_{2t-1} \subseteq M_{2t}$. Then by (1) and Lemma 6.16(1, 3), $yz^t \equiv y[V_{2t+1}(\varepsilon)] \equiv [V_{2t+2}(\varepsilon)]$ module M_{2t}.

(3) It follows from (1), (2) and Lemma 6.15. □

Proposition 6.20. *Let $G_0(kD_n)[y, z]$ be the polynomial ring in two variables y, z over $G_0(kD_n)$. Then $R \cong G_0(kD_n)[y, z]/(y^2 - (1+\chi)y)$, where $(y^2 - (1+\chi)y)$ is the ideal of $G_0(kD_n)[y, z]$ generated by $y^2 - (1+\chi)y$.*

Proof. By Corollary 6.2 and Lemma 6.18(2), the ring embedding $G_0(kD_n) \hookrightarrow R$ can be extended to a ring epimorphism $\phi : G_0(kD_n)[y, z] \to R$ such that $\phi(y) = [V_2(\varepsilon)]$ and $\phi(z) = [V_3(\varepsilon)]$. By Lemma 6.18(1), ϕ induces a ring epimorphism $\overline{\phi} : G_0(kD_n)[y, z]/(y^2 - (1+\chi)y) \to R$ such that $\overline{\phi}(\pi(u)) = \phi(u)$ for any $u \in G_0(kD_n)[y, z]$, where $\pi : G_0(kD_n)[y, z] \to G_0(kD_n)[y, z]/(y^2 - (1+\chi)y)$ is the canonical epimorphism. In an obvious way, $G_0(kD_n)[y, z]/(y^2 - (1+\chi)y)$ becomes a $G_0(kD_n)$-module. In this case, $\overline{\phi}$ is a $G_0(kD_n)$-module map.

Clearly, $G_0(kD_n)[y,z]/(y^2-(1+\chi)y)$ is generated, as a $G_0(kD_n)$-module, by $X_4 := \{\pi(z^t), \pi(yz^t)|t \geqslant 0\}$. By Lemma 6.19(3), $\overline{\phi}(X_4)$ is a $G_0(kD_n)$-basis of R. This implies that $\overline{\phi}$ is injective, and so it is a ring isomorphism. $\square$

For any $\beta \in k^\times$, let $w_\beta = [V(\varepsilon,\beta)]$ in $r(H)$. Then we have the following lemma.

Lemma 6.21. *$r(H)$ is generated, as a ring, by $R \cup \{w_\beta|\beta \in k^\times\}$.*

Proof. Let R' be the subring of $r(H)$ generated by $R \cup \{w_\beta|\beta \in k^\times\}$. Then $G_0(kD_n) \subset R \subset R'$. By Lemma 5.2(2) and the classification of finite dimensional indecomposable H-modules, it is enough to show that $[V_t(\varepsilon,\beta)] \in R'$ for all $t \geqslant 1$ and $\beta \in k^\times$. We prove it by induction on t. For $t=1$, it is trivial. Now assume $t \geqslant 1$ and assume $[V_l(\varepsilon,\beta)] \in R'$ for all $1 \leqslant l \leqslant t$ and $\beta \in k^\times$. By Proposition 5.3 (or [11, Theorem 3.6]), $V_3(\varepsilon) \otimes V_t(\varepsilon,\beta) \cong V_t(\varepsilon,\beta) \oplus V_{t-1}(\varepsilon,\beta) \oplus V_{t+1}(\varepsilon,\beta)$, where $V_0(\varepsilon,\beta)=0$. Hence $[V_{t+1}(\varepsilon,\beta)] = (z-1)[V_t(\varepsilon,\beta)] - [V_{t-1}(\varepsilon,\beta)] \in R'$. $\square$

Lemma 6.22. *Let $\alpha, \beta \in k^\times$ with $\alpha \neq -\beta$. Then the following hold in $r(H)$:*

$$\chi w_\beta = w_\beta;\ w_\alpha w_\beta = 2w_{\alpha+\beta};\ w_\beta w_{-\beta} = (1+\chi)y;\ yw_\beta = 2w_\beta.$$

Proof. The first three equations follow from an argument similar to the proofs of Lemma 6.11 and Lemma 5.2(1). By Proposition 5.3 (or [11, Theorem 3.6]), $V_2(\varepsilon) \otimes V(\varepsilon,\beta) \cong 2V(\varepsilon,\beta)$, and hence $yw_\beta = 2w_\beta$. $\square$

Let P be the $\mathbb{Z}$-submodule of $r(H)$ generated by $\{[V_t(i,\beta)]|t \geqslant 1, i \in I, \beta \in k^\times\}$. Then $r(H) = R \oplus P$ and P has a $\mathbb{Z}$-basis $\{[V_t(i,\beta)]|t \geqslant 1, i \in I_0, \beta \in k^\times\}$.

For $t \geqslant 1$ and $\beta \in k^\times$, let P^β and P_t^β be the $\mathbb{Z}$-submodules of $r(H)$ generated by $\{[V_l(i,\beta)]|l \geqslant 1, i \in I\}$ and $\{[V_l(i,\beta)]|1 \leqslant l \leqslant t, i \in I\}$, respectively. Then P^β and P_t^β have the $\mathbb{Z}$-bases $\{[V_l(i,\beta)]|l \geqslant 1, i \in I_0\}$ and $\{[V_l(i,\beta)]|1 \leqslant l \leqslant t, i \in I_0\}$, respectively. Clearly, $P_t^\beta \subset P_{t+1}^\beta$, $P^\beta = \sum_{t\geqslant 1} P_t^\beta = \cup_{t\geqslant 1} P_t^\beta$ and $P = \oplus_{\beta \in k^\times} P^\beta$. By Proposition 5.3, P and P^β are R-submodules of $r(H)$.

Let $t \geqslant 1$ and $\beta \in k^\times$. By Lemma 5.2(2), P_t^β is a $G_0(kD_n)$-module generated by $\{[V_l(\varepsilon,\beta)]|1 \leqslant l \leqslant t\}$. By Proposition 5.3, $V_3(\varepsilon) \otimes V_t(i,\beta) \cong V_t(i,\beta) \oplus V_{t-1}(i,\beta) \oplus V_{t+1}(i,\beta)$ for any $i \in I$, where $V_0(i,\beta)=0$. Hence $zP_t^\beta \subseteq P_{t+1}^\beta$. We claim that

$$z^l w_\beta \equiv [V_{l+1}(\varepsilon,\beta)] \text{ modulo } P_l^\beta, \forall l \geqslant 1.$$

For $l=1$, $zw_\beta = [V_2(\varepsilon,\beta)] + [V_1(\varepsilon,\beta)] \equiv [V_2(\varepsilon,\beta)]$ modulo P_1^β. Let $l>1$ and assume $z^{l-1}w_\beta \equiv [V_l(\varepsilon,\beta)]$ modulo P_{l-1}^β. Then $z^{l-1}w_\beta = [V_l(\varepsilon,\beta)] + u$ for some $u \in P_{l-1}^\beta$. Hence $z^l w_\beta = z[V_l(\varepsilon,\beta)] + zu = [V_{l+1}(\varepsilon,\beta)] + [V_l(\varepsilon,\beta)] + [V_{l-1}(\varepsilon,\beta)] + zu \equiv [V_{l+1}(\varepsilon,\beta)]$ modulo P_l^β. Thus, we have shown the claim. Therefore, P_t^β is generated, as a $G_0(kD_n)$-module, by $\{z^l w_\beta|0 \leqslant l \leqslant t-1\}$. Then by Lemmas 6.12 and 6.22, P_t^β is generated, as a $\mathbb{Z}$-module, by $X_t^\beta := \{\lambda z^l w_\beta, x^i z^l w_\beta|0 \leqslant i \leqslant \frac{m-1}{2}, 0 \leqslant l \leqslant t-1\}$. Since $\sharp X_t^\beta = \sharp\{[V_l(i,\beta)]|1 \leqslant l \leqslant t, i \in I_0\}$, X_t^β is also a $\mathbb{Z}$-basis of P_t^β. It follows that $X^\beta := \{\lambda z^l w_\beta, x^i z^l w_\beta|0 \leqslant i \leqslant \frac{m-1}{2}, l \geqslant 0\}$ is a $\mathbb{Z}$-basis of P^β. Summarizing the above discussion, we have the following lemma.

Lemma 6.23. *P has a $\mathbb{Z}$-basis $B_P := \{\lambda z^t w_\beta, x^l z^t w_\beta|0 \leqslant l \leqslant \frac{m-1}{2}, t \geqslant 0, \beta \in k^\times\}$.*

Theorem 6.24. *Let $R[Z]$ be the polynomial ring in the variables $Z = \{w_\beta | \beta \in k^\times\}$ over R. Let (W) be the ideal of $R[Z]$ generated by*

$$W := \left\{ \begin{array}{c|c} \chi w_\beta - w_\beta,\ w_\alpha w_\beta - 2w_{\alpha+\beta}, & \alpha, \beta \in k^\times \\ w_\beta w_{-\beta} - (1+\chi)y,\ yw_\beta - 2w_\beta & \textit{with } \alpha \neq \beta \end{array} \right\}.$$

Then $r(H)$ is isomorphic to the factor ring $R[Z]/(W)$.

Proof. By Lemma 6.21, the ring embedding $R \hookrightarrow r(H)$ can be extended to a ring epimorphism $\phi : R[Z] \to r(H)$ such that $\phi(w_\beta) = [V(\varepsilon, \beta)]$ for all $\beta \in k^\times$. By Lemma 6.22, ϕ induces a ring epimorphism $\overline{\phi} : R[Z]/(W) \to r(H)$ such that $\overline{\phi}(\pi(u)) = \phi(u)$ for any $u \in R[Z]$, where $\pi : R[Z] \to R[Z]/(W)$ is the canonical epimorphism. By the definition of W, $R[Z]/(W) = \pi(R) + \sum_{\beta \in k^\times} \pi(Rw_\beta)$. Let X_2 be the $\mathbb{Z}$-basis of $G_0(kD_n)$ given in Lemma 6.12. Then by Lemma 6.19(3), R has a $\mathbb{Z}$-basis $B_R := \{rz^t, ryz^t | r \in X_2, t \geqslant 0\}$. Again by the definition of W, $\sum_{\beta \in k^\times} \pi(Rw_\beta)$ is generated, as a $\mathbb{Z}$-module, by

$$S_R := \{\pi(\lambda z^t w_\beta), \pi(x^l z^t w_\beta) | 0 \leqslant l \leqslant \frac{m-1}{2}, t \geqslant 0, \beta \in k^\times\}.$$

Hence $R[Z]/(W)$ is generated, as a $\mathbb{Z}$-module, by $B := \pi(B_R) \cup S_R$. From $\overline{\phi}(\pi(w_\beta)) = [V(\varepsilon, \beta)]$ and $\overline{\phi}(\pi(r)) = r$ for any $r \in R$, one can check that $\overline{\phi}(a) \neq \overline{\phi}(b)$ for any $a, b \in B$ with $a \neq b$, and that $\overline{\phi}(B) = B_R \cup B_P$, which is a $\mathbb{Z}$-basis of $r(H)$ by Lemma 6.23. It follows that $\overline{\phi}$ is a ring isomorphism. □

Let $X := \{x, y, z, w_\beta | \beta \in k^\times\}$ and $\mathbb{Z}\hat{D_n}[X]$ the polynomial ring in variables X over $\mathbb{Z}\hat{D_n}$. Let (Q) be the ideal of $\mathbb{Z}\hat{D_n}[X]$ generated by the following set

$$Q := \left\{ \begin{array}{c|c} \chi x - f(x),\ x^m - g(x),\ y^2 - (1+\chi)y, & \\ \lambda x - x,\ \chi w_\beta - w_\beta,\ yw_\beta - 2w_\beta, & \begin{array}{c} \alpha, \beta \in k^\times \\ \text{with } \alpha \neq \beta \end{array} \\ w_\alpha w_\beta - 2w_{\alpha+\beta},\ w_\beta w_{-\beta} - (1+\chi)y & \end{array} \right\},$$

where $f(x), g(x) \in \mathbb{Z}\hat{D_n}[x] \subset \mathbb{Z}\hat{D_n}[X]$ are given before Proposition 6.8. Then by Propositions 6.8, 6.20 and Theorem 6.24, one gets the following corollary.

Corollary 6.25. *$r(H)$ is isomorphic to the factor ring $\mathbb{Z}\hat{D_n}[X]/(Q)$.*

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