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Abelianization and fixed point properties of units in integral group rings

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Summary

Let G be a finite group and $\mathcal{U}(\mathbb{Z}G)$ the unit group of the integral group ring $\mathbb{Z}G$. We prove a unit theorem, namely a characterization of when $\mathcal{U}(\mathbb{Z}G)$ satisfies Kazhdan's property (T), both in terms of the finite group G and in terms of the simple components of the semisimple algebra $\mathbb{Q}G$. Furthermore, it is shown that for $\mathcal{U}(\mathbb{Z}G)$ this property is equivalent to the weaker property FAb (i.e. every subgroup of finite index has finite abelianization), and in particular also to a hereditary version of Serre's property FA, denoted HFA. More precisely, it is described when all subgroups of finite index in $\mathcal{U}(\mathbb{Z}G)$ have both finite abelianization and are not a non-trivial amalgamated product.

A crucial step for this is a reduction to arithmetic groups $\mathrm{SL}_n(\mathcal{O})$, where $\mathcal{O}$ is an order in a finite dimensional semisimple $\mathbb{Q}$-algebra D, and finite groups G which have the so-called cut property. For such groups G we describe the simple epimorphic images of $\mathbb{Q}G$. The proof of the unit theorem fundamentally relies on fixed point properties and the abelianization of the elementary subgroups $\mathrm{E}_n(D)$ of $\mathrm{SL}_n(D)$. These groups are well understood except in the degenerate case of lower rank, i.e. for $\mathrm{SL}_2(\mathcal{O})$ with $\mathcal{O}$ an order in a division algebra D with a finite number of units. In this setting we determine Serre's property FA for $\mathrm{E}_2(\mathcal{O})$ and its subgroups of finite index. We construct a generic and computable exact sequence describing its abelianization, affording a closed formula for its $\mathbb{Z}$-rank.

KEYWORDS
Abelianization, Serre's property FA, Kazhdan's property (T), elementary matrix group, integral group ring, unit

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1 | INTRODUCTION

One of the most natural and important questions in (integral) representation theory is whether a finite group G is determined by its integral group ring $\mathbb{Z}G$ (the so called Isomorphism problem, in short (ISO)). Posed for the first time by Higman [36] in 1940, popularized by Brauer [10] in 1963, it was only in the 1980's that firm indications for a positive solution were obtained. Indeed, these years saw a number of major breakthroughs, starting with Roggenkamp-Scott [58] who obtained an affirmative solution to (ISO) for nilpotent groups. In fact, not only they did prove that $G \cong H$ whenever $\mathbb{Z}G = \mathbb{Z}H$, but also that $G = H^{\alpha}$ for some unit

$\alpha \in \mathcal{U}(\mathbb{Q}G)$; hence explaining why the isomorphism occurs. Here $\mathcal{U}(\mathbb{Q}G)$ denotes the unit group of $\mathbb{Q}G$. In general, this stronger statement is called the second Zassenhaus conjecture (ZC2). The third and strongest Zassenhaus conjecture (ZC3) asserts that any finite subgroup of the group of units of augmentation one of $\mathbb{Z}G$ should be *rationally conjugated* (that is, conjugated in $\mathcal{U}(\mathbb{Q}G)$) to a subgroup of the basis G. Shortly after Roggenkamp-Scott, Weiss obtained in his landmark papers [70, 71] that nilpotent groups even satisfy (ZC3). Around the same time, Roggenkamp-Scott [60] provided a metabelian counterexample to (ZC2). It took ten more years until Hertweck constructed, unexpectedly, in [35] a counterexample to (ISO). His construction is still the only known type of counterexample and the general philosophy remains that the ring $\mathbb{Z}G$ encodes a lot of information on G.

A remarkable property of integral group rings is the following: if G and H are finite groups, then $\mathbb{Z}G \cong \mathbb{Z}H$ if and only $\mathcal{U}(\mathbb{Z}G) \cong \mathcal{U}(\mathbb{Z}H)$. Hence (ISO) is equivalent with G being determined by $\mathcal{U}(\mathbb{Z}G)$. This is one of the reasons why the structure of $\mathcal{U}(\mathbb{Z}G)$ already receives for more than five decades tremendous attention; for an overview on the main advances and open problems we refer to [42, 43, 61, 62]. Several main research directions emerged:

1. the search for generic constructions of subgroups of finite index (preferably torsion-free) in $\mathcal{U}(\mathbb{Z}G)$,
2. the understanding of torsion units in $\mathbb{Z}G$ (for which the Zassenhaus conjectures were a driving force for many years; recently a counter example has been given by Eisele-Margolis [24] for the last of these that remained open, but still many open problems remain on the arithmetic of the torsion structure), and
3. the search for unit theorems, i.e. structure theorems for the unit group $\mathcal{U}(\mathbb{Z}G)$.

A fairly complete account of the first direction can be found in the recent books [42, 43] and for the second we refer to the surveys [48, 53] and the references therein.

This paper contributes mainly to the third direction listed above. A very concrete idea of a unit theorem was given by Kleinert [49] in the context of orders:

> *a unit theorem for a finite dimensional semisimple rational algebra A consists of the definition, in purely group theoretical terms, of a class of groups $\mathbf{C}(A)$ such that almost all generic unit groups of A are members of $\mathbf{C}(A)$.*

Recall that a *generic unit group* of A is a subgroup of finite index in the group of reduced norm 1 elements of an order in A. So far, the finite groups G for which a unit theorem, in the sense of Kleinert, is known for $\mathcal{U}(\mathbb{Z}G)$ are those for which the class of groups considered are either finite groups (Higman), abelian groups (Higman), free groups [40] or direct products of free(-by-free) groups [41, 45]. Remarkably, the latter can also be described in terms of the rational group algebra: every simple quotient of $\mathbb{Q}G$ is either a field, a totally definite quaternion algebra or a 2-by-2 matrix ring $\mathrm{M}_2(K)$, where K is either $\mathbb{Q}$, $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$ or $\mathbb{Q}(\sqrt{-3})$. To our knowledge these results cover all the known unit theorems on $\mathcal{U}(\mathbb{Z}G)$. Surprisingly, one obtains a unit theorem when all the non-commutative simple components of $\mathbb{Q}G$ are two-by-two matrices over a field with finitely many units in any order. This is in contrast with the construction of generators of a subgroup of finite index in $\mathcal{U}(\mathbb{Z}G)$, where one has shown that a collection of explicitly constructed units generate a subgroup B of finite index provided $\mathbb{Q}G$ does not have so called exceptional components, and G does not have non-abelian *fixed point free* (i.e. a group of fixed point free automorphisms of a finite group) images. These *exceptional components* (cf. Definition 6.7) are the non-commutative division algebras other than a totally definite quaternion algebra over a number field (of so called *type (I)*) or $\mathrm{M}_2(D)$ with D a finite dimensional $\mathbb{Q}$-division algebra having an order $\mathcal{O}$ with $\mathcal{U}(\mathcal{O})$ finite (of so called *type (II)*).

Very little is known on the structure of B. However several authors, including Marciniak, Sehgal, Salwa, Gonçalves, Passman, del Río have given explicit constructions of free groups in B (we refer the reader to [31]). More recently Gonçalves and Passman in [32] and Janssens, Jespers and Temmerman in [39] gave explicit constructions (within the group B) that generate a free product $C_p * C_p$ of two cyclic groups of order p in $\mathcal{U}(\mathbb{Z}G)$ in all cases where this is possible, i.e. provided G has a noncentral element of order p.

In this paper we complement this line of research by determining when the unit group $\mathcal{U}(\mathbb{Z}G)$ or the group B can be decomposed into a non-trivial *amalgamated product* or an *HNN extension* (see Definitions 2.3 and 2.4) . In fact we show when these decompositions do not occur. In case of an HNN extension this is equivalent, by Bass-Serre theory, to $\mathcal{U}(\mathbb{Z}G)^{ab}$ being finite. Here $\mathcal{U}(\mathbb{Z}G)^{ab}$ denotes the *abelianization* $\mathcal{U}(\mathbb{Z}G)/\mathcal{U}(\mathbb{Z}G)'$ of $\mathcal{U}(\mathbb{Z}G)$, where $\mathcal{U}(\mathbb{Z}G)'$ denotes the *commutator subgroup*. In particular, by a classical theorem of Serre [63], the absence of an amalgam and HNN can be rephrased in terms of satisfying the so-called *FA property*. A group is said to have

- *property FA* if every action on a simplicial tree has a global fixed point,
- *property F*$\mathbb{R}$ if every isometric action on a real tree has a global fixed point.

See Definition 2.2 for more details. Serre proved that a finitely generated group has property FA exactly when it is neither a HNN extension nor a non-trivial amalgam.

Since unit theorems concern a property on almost all subgroups of finite index, we will consider the hereditary properties, denoted HFA and HF$\mathbb{R}$, and a hereditary finite abelianization property, denoted FAb. One says that a group has

- *HFA* if all its finite index subgroups have property FA,
- *HF*$\mathbb{R}$ if all its finite index subgroups have property F$\mathbb{R}$,
- *property FAb* if every subgroup of finite index has finite abelianization.

It is well-known that properties FA, HFA and FAb follow from *Kazhdan's property (T)* [9, Theorem 2.12.6], .

The main result of this paper is a characterization of when $\mathcal{U}(\mathbb{Z}G)$ satisfies these hereditary properties. Surprisingly, all the mentioned hereditary fixed point properties are equivalent and are controlled both in terms of G and in terms of the Wedderburn decomposition of $\mathbb{Q}G$. Recall that a finite group G is called a *cut group* if and only if $\mathcal{U}(\mathbb{Z}G)$ has only trivial central units, i.e. the center of $\mathcal{U}(\mathbb{Z}G)$ is finite. For example, rational groups are cut. Recently, cut groups gained in interest (see for example [4, 7, 68]), but especially the subclass of rational groups has already a long tradition in classical representation theory (for example, see [37, 50]).

Theorem A. (Theorem 7.1, Corollary 7.5 and Corollary 7.7) Let G be a finite group. The following properties are equivalent:

1. the group $\mathcal{U}(\mathbb{Z}G)$ has property HFA,
2. the group $\mathcal{U}(\mathbb{Z}G)$ has property HF$\mathbb{R}$,
3. the group $\mathcal{U}(\mathbb{Z}G)$ has property (T),
4. the group $\mathcal{U}(\mathbb{Z}G)$ has property FAb,
5. G is cut and $\mathbb{Q}G$ has no exceptional components,
6. G is cut and G does not map onto one of 10 explicitly described groups.

Moreover, if $\mathbb{Q}G$ does not have an exceptional component of type (II), then any of the above properties also is equivalent with any of the following properties:

7. $\mathcal{U}(\mathbb{Z}G)^{ab}$ is finite,
8. G is a cut group.

It is worth noticing that $\mathbb{Q}G$, for G a group of odd order, does not have exceptional components of type (II).

From this theorem it follows that in particular, if these conditions are satisfied and the group G has no non-abelian homomorphic image which is fixed point free, the group B is not a non-trivial amalgamated product nor an HNN extension (see Corollary 7.3).

Crucial to prove Theorem A is to reduce the problem to $\mathbb{Z}$*-orders* (also sometimes simply called *orders*, see Section 2.2) in simple components of the semisimple rational algebra $\mathbb{Q}G$. One writes $\mathbb{Q}G = \mathrm{M}_{n_1}(D_1) \oplus \cdots \oplus \mathrm{M}_{n_k}(D_k)$, with each D_i a division algebra. Let $\mathcal{O}_i$ be an order in D_i. Then both $\mathbb{Z}G$ and $\mathrm{M}_{n_1}(\mathcal{O}_1) \oplus \cdots \oplus \mathrm{M}_{n_k}(\mathcal{O}_k)$ are orders in $\mathbb{Q}G$. Due to classical results in order theory their respective unit groups are commensurable and the hereditary fixed point properties of both orders are related and, as we shall prove, strongly determined by the groups $\mathrm{E}_{n_i}(\mathcal{O}_i)$ generated by *elementary matrices* (these are matrices with 1 on the diagonal and at most one non-zero entry elsewhere).

The first part of this paper is therefore dedicated to the (hereditary) fixed point properties of the groups $\mathrm{E}_{n_i}(\mathcal{O}_i)$ and some related groups.

Recall that $\mathrm{SL}_n(D)$, for a division algebra D, consists of the matrices of degree n of reduced norm one (see Section 2.2 for more details). Due to the celebrated works of many ([6, 8, 67] amongst others) on the subgroup congruence problem and the seminal work of Margulis [54], $\mathrm{SL}_n(D)$ enjoys a rich theory on subgroup and rigidity results whenever certain geometric-arithmetic invariants, such as the (reductive) rank, are large enough. More precisely, in these cases (for example when $n \geq 3$), every *arithmetic subgroup* of $\mathrm{SL}_n(D)$ (i.e. every subgroup which is commensurable to $\mathrm{SL}_n(\mathcal{O})$ for some order $\mathcal{O}$ in D), and consequently also $\mathrm{E}_n(\mathcal{O})$, has property (T), where $\mathcal{O}$ is any order in D.

On the other hand, when $\mathrm{SL}_2(D)$ is of so-called low rank, which amounts to say that D contains an order $\mathcal{O}$ with $\mathcal{U}(\mathcal{O})$ finite (see Section 6.2 for more details), the machinery breaks down and the corresponding landscape reshapes. To illustrate this, if D is commutative it was proven [33] that $\mathrm{E}_2(\mathcal{O})$ and all its finitely generated subgroups have the Haagerup property which is a strong form of non-rigidity, hence opposed to property (T).

The objects $\mathrm{E}_2(\mathcal{O})$ with $\mathcal{U}(\mathcal{O})$ finite are the protagonists of the first part of the paper. For a unital ring R we need for our investigations on abelianization to consider the group $\mathrm{GE}_2(R)$ generated by $\mathrm{E}_2(R)$ and the group of invertible diagonal matrices over R. We extend Cohn's techniques [14, 15] to arbitrary finite dimensional division $\mathbb{Q}$-algebras D containing an order $\mathcal{O}$ with finite unit group. In particular, we deal with orders in totally-definite quaternion algebras with center $\mathbb{Q}$. As a first step we obtain in Section 3.1 finite presentations for the groups $\mathrm{E}_2(\mathcal{O})$ and $\mathrm{GE}_2(\mathcal{O})$, which allow us to connect $\mathrm{E}_2(\mathcal{O})^{ab}$ to the arithmetic structure of $\mathcal{O}$.

Theorem B (Theorem 3.10 and Theorem 3.8)**.** Let $\mathcal{O}$ be an order in a finite dimensional division $\mathbb{Q}$-algebra with finite unit group. There exists an (explicit) additive subgroup M of $(\mathcal{O}, +)$ such that

$$\mathrm{E}_2(\mathcal{O})^{ab} \cong (\mathcal{O}/M, +),$$

and an (explicit) two-sided ideal N of $\mathcal{O}$ such that there is a short exact sequence

$$1 \longrightarrow (\mathcal{O}/N, +) \longrightarrow \mathrm{GE}_2(\mathcal{O})^{ab} \longrightarrow \mathcal{U}(\mathcal{O})^{ab} \longrightarrow 1.$$

More concretely, N is the two-sided ideal generated by the elements $u - 1$ with $u \in \mathcal{U}(\mathcal{O})$. Therefore as a by-product, the exact sequence above yields that $\mathrm{GE}_2(\mathcal{O})^{ab}$ is finite (Corollary 3.7) for any order $\mathcal{O}$ as in Theorem B. This is in sharp contrast with the elementary group case, as is shown by the following theorem where for a finitely generated abelian group G,

$$\mathrm{rank}_{\mathbb{Z}}\, G := \max\{n \mid \mathbb{Z}^n \text{ is, up to isomorphism, a subgroup of } G\},$$

and

$$\mathrm{inv}_{\mathcal{O}} := \max\{|B \cap \mathcal{U}(\mathcal{O})| \mid B \text{ a } \mathbb{Z}\text{-module basis of } \mathcal{O}\}.$$

Theorem C (Theorem 3.14)**.** Let $\mathcal{O}$ be an order in a finite dimensional division $\mathbb{Q}$-algebra with finite unit group. Then,

$$\mathrm{rank}_{\mathbb{Z}}\, \mathrm{E}_2(\mathcal{O})^{ab} = \mathrm{rank}_{\mathbb{Z}}\, \mathcal{O} - \mathrm{inv}_{\mathcal{O}}\,.$$

Moreover, the following properties are equivalent:

1. $\mathrm{E}_2(\mathcal{O})^{ab}$ is finite,
2. $\mathcal{O}$ has a $\mathbb{Z}$-basis consisting of units of $\mathcal{O}$,
3. $\mathcal{O}$ is isomorphic to a maximal order in $\mathbb{Q}$, $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-3})$ or in the quaternion algebras $\left(\frac{-1,-1}{\mathbb{Q}}\right)$, $\left(\frac{-1,-3}{\mathbb{Q}}\right)$ or it is the order of *Lipschitz quaternions* $\left(\frac{-1,-1}{\mathbb{Z}}\right)$.

A detailed explanation on the notation of quaternion algebras is given in Section 2.2. When $\mathcal{U}(\mathcal{O})$ is infinite, the situation is drastically different. Indeed, it is well-known (by Margulis) that then $\mathrm{SL}_2(\mathcal{O})$ and $\mathrm{E}_2(\mathcal{O})$ have property FAb.

As mentioned earlier if $\mathrm{E}_2(\mathcal{O})$ has property FA then $\mathrm{E}_2(\mathcal{O})^{ab}$ is finite. Hence, in this case, the orders $\mathcal{O}$ that can appear are restricted by Theorem C. Investigating further these orders and certain subgroups of finite index in $\mathrm{E}_2(\mathcal{O})$ we determine precisely when $\mathrm{E}_2(\mathcal{O})$ has property FA and HFA.

Theorem D (Theorem 5.1 and Theorem 5.7)**.** Let $\mathcal{O}$ be an order in a finite dimensional division $\mathbb{Q}$-algebra with finite unit group. The following properties are equivalent:

1. $\mathrm{E}_2(\mathcal{O})$ has property FA,
2. $\mathcal{O}$ is isomorphic to a maximal order in $\mathbb{Q}(\sqrt{-3})$, $\left(\frac{-1,-1}{\mathbb{Q}}\right)$ or in $\left(\frac{-1,-3}{\mathbb{Q}}\right)$.

Furthermore, $\mathrm{E}_2(\mathcal{O})$ does not have property HFA.

In Section 5.2, properties FA and HFA are also investigated for the group $\mathrm{GE}_2(\mathcal{O})$. For both groups one first needs to understand the respective Borel subgroups. This is done simultaneously in Section 5.1 by considering a more general type of group. Interestingly, in case of the Borel subgroup $\mathrm{B}_2(\mathcal{O})$ of $\mathrm{GL}_2(\mathcal{O})$ we obtain in Proposition 5.10 that it has property FA if and only if $\mathcal{U}(\mathcal{O})$ is not isomorphic to C_2, the cyclic group of order 2.

As mentioned earlier, we do not only consider actions on simplicial trees, but more generally on real trees for property F$\mathbb{R}$. In all the cases where we obtain property FA we actually have the stronger property F$\mathbb{R}$. In Section 4, we also briefly discuss elementary groups of degree at least 3.

In comparison with HFA in Theorem A, studying property FA for the full unit group $\mathcal{U}(\mathbb{Z}G)$ is even more delicate. In Section 6, we show that if $\mathcal{U}(\mathbb{Z}G)$ has property FA, then G is a cut group. We then further investigate cut groups in that section and prove in Proposition 6.12 that if G is a cut group, then $\mathbb{Q}G$ has no exceptional components of type (I). Finally in Proposition 6.16, we give a complete description of the exceptional components that can appear in the Wedderburn-Artin decomposition of a cut group and we state precisely when such components appear.

A full characterization of when $\mathcal{U}(\mathbb{Z}G)$ has property FA has not been obtained but in Section 8 necessary conditions will be proven and open problems will be formulated. Earlier, in Section 7 we propose a trichotomy result (Question 7.8) about $\mathcal{U}(\mathbb{Z}G)$ having property HFA, having property FA but not HFA or it having a non-trivial amalgamated decomposition and finite abelianization. So finite abelianization and a decomposition as amalgamated product would hence be inextricable for unit groups in $\mathbb{Z}G$. We also prove in Proposition 7.9 that this question is equivalent to two other problems of independent interest.

We point out that in the follow-up paper [5] we focus, among other things, on obtaining explicit non-trivial amalgamated decompositions for subgroups of finite index in $\mathcal{U}(\mathbb{Z}G)$ provided $\mathcal{U}(\mathbb{Z}G)$ does not satisfy HFA. Notably, building on the methods in this paper, a weaker version of the above mentioned trichotomy is proven, namely that $\mathcal{U}(\mathbb{Z}G)$ either has property HFA or is, up to commensurability, a non-trivial amalgamated product.

2 | PRELIMINARIES

In this first section we review facts that are needed in the sequel of the article. As a rule, in this paper, a ring R is always meant to be unital and associative, but not necessarily commutative. Moreover, we use the notation $\mathcal{Z}(R)$ for its *center*. For any group Γ, Γ' denotes its *commutator subgroup* and $\Gamma^{ab} = \Gamma/\Gamma'$ its *abelianization*.

2.1 | Trees

In this subsection we recall some background on the geometric concepts used in the paper, see [13]. We will be considering two kinds of trees: simplicial trees and real trees.

Definition 2.1. A connected, undirected graph is called a *simplicial tree* (or simply a *tree*) if it contains no cycle graph as a subgraph.

A metric space is called a *real tree* (or $\mathbb{R}$-*tree*) if it is a geodesic space with no subspace isomorphic to S^1.

Recall that a metric space (X, d) is *geodesic* if between every two points x and y there exists a curve of length $d(x, y)$. The length $L(\alpha)$ of a curve $\alpha : [0, 1] \to (X, d)$ is defined as $\sup \sum d(\alpha(t_i), \alpha(t_{i+1}))$, where $0 = t_0 < \ldots < t_k = 1$ is a partition of $[0, 1]$ and the supremum is taken over all possible partitions.

This definition of a real tree is equivalent to saying that it is a connected 0-*hyperbolic space*, that is to say all triangles are 0-thin. For more on trees or the definition of a 0-hyperbolic space, we refer the reader to [13].

An isomorphism g of a simplicial tree is called an *inversion* if there exist two adjacent vertices which are mapped to one-another. This is equivalent with the fact that g does not have a fixed vertex, but g^2 does. An isomorphism with the former property does not exist for $\mathbb{R}$-trees. Indeed let $T_{\mathbb{R}}$ be a real tree with an isomorphism g and let $x \in T_{\mathbb{R}}$ be a point fixed under g^2. Considering the geodesic between x and $g(x)$, it is easy to see that the midpoint should be a fixed point for g.

In this paper we will be interested, for various types of linear groups Γ, in the existence of global fixed points when we let Γ act on trees.

Definition 2.2. A group Γ is said to have *property* FA if whenever Γ acts on a tree such that no non-identity element acts as inversion, this action has a globally fixed vertex.

A group Γ is said to have *property* F$\mathbb{R}$ if every isometric action on a real tree has a globally fixed point.

A group Γ is said to have the *hereditary property* FA *(respectively* F$\mathbb{R}$*)* if every finite index subgroup of Γ has property FA (respectively F$\mathbb{R}$). We denote these properties by HFA and HF$\mathbb{R}$.

Property FA was first introduced by Jean-Pierre Serre and the name FA comes from the French "points Fixes sur les Arbres".

A simplicial tree can be considered as a real tree by its *geometric realization* [13, Chapter 2, Section 2]. In this way, an action on a simplicial tree T induces an action on its geometric realization $T_{\mathbb{R}}$. If the action was without inversion, then a point $x_{\mathbb{R}}$ of $T_{\mathbb{R}}$ which is fixed under this induced action, has to correspond to a fixed vertex x of T. Thus property F$\mathbb{R}$ implies property FA. There are however real trees which are not simplicial trees.

In general, FA is a weaker property than F$\mathbb{R}$ and an example of a group satisfying FA but not F$\mathbb{R}$ can be found in [55]. Our interest in property FA originates from the structural properties it implies. In order to be more precise we first recall the definition of an amalgamated product and an HNN extension.

Definition 2.3. Let G_1, G_2 and H be groups and $f_1 : H \to G_1$ and $f_2 : H \to G_2$ be injective homomorphisms. Let N be the normal subgroup of the free product $G_1 * G_2$ generated by the elements $f_1(h)f_2(h)^{-1}$ with $h \in H$. Then the *amalgamated product* $G_1 *_H G_2$ is defined as the quotient

$$(G_1 * G_2)/N.$$

This amalgamated product is said to be trivial if either f_1 or f_2 is an epimorphism.

Definition 2.4. Let Γ be a group with presentation $\langle S \mid R\rangle$, H_1 and H_2 be two isomorphic subgroups of Γ and $\theta : H_1 \to H_2$ an isomorphism. Let $t \notin \Gamma$ be a new element and $\langle t\rangle$ a cyclic group of infinite order. The *HNN extension* of Γ relative to H_1, H_2 and θ is the group

$$\langle S, t \mid R, tgt^{-1} = \theta(g), g \in H_1\rangle.$$

A group theoretical characterisation of property FA was obtained by Serre [63, I.6.1 Theorem 15].

Theorem 2.5 (Serre)**.** A countable group Γ has property FA if and only if it satisfies the following properties

- Γ has finite abelianization,
- Γ has no non-trivial decomposition as amalgamated product,
- Γ is finitely generated.

By classical Bass-Serre theory, a finitely generated group is an HNN extension if and only if it has infinite abelianization (see [63, I.5.1. example 3 and the proof of I.6.1 Theorem 15]). Thus we get the following immediate corollary.

Corollary 2.6. A finitely generated group has property FA if and only if it is neither an HNN extension nor a non-trivial amalgamated product.

Unfortunately for property F$\mathbb{R}$ such a group-theoretical description is still an open problem. The following properties are well-known (in case of FA a proof can be found in [63, I.6.3 Examples 1-4], and the other cases can be handled in a similar fashion) and will (mostly) be used without further notice.

Proposition 2.7. Let Γ be a group, N a subgroup of Γ and $\mathcal{P}$ a property among HF$\mathbb{R}$, HFA, F$\mathbb{R}$ and FA.

- If Γ is finitely generated and torsion, then Γ has property $\mathcal{P}$. In particular, finite groups have property $\mathcal{P}$.
- If N is normal in Γ and both N and Γ/N have property $\mathcal{P}$, then so does Γ.
- If N is a subgroup of finite index in Γ with property $\mathcal{P}$, then Γ has property $\mathcal{P}$.
- If N is normal in Γ and Γ has property $\mathcal{P}$ then so does Γ/N.

In particular, a finite direct product $\prod_{i=1}^{q} G_i$ has property $\mathcal{P}$ if and only if every G_i has property $\mathcal{P}$.

Two subgroups Γ_1, Γ_2 of a group Γ are called *commensurable*, if their intersection is of finite index in both Γ_1 and Γ_2. From Proposition 2.7 it follows that properties HFA and HF$\mathbb{R}$ are actually properties of the commensurability class of a group, meaning that either all or none of the groups in the class have this property.

2.2 | Orders and quaternion algebras

Let A be a finite-dimensional algebra over $\mathbb{Q}$. Recall that a *$\mathbb{Z}$-order* (or for brevity just *order*) is a subring of A that is finitely generated as a $\mathbb{Z}$-module and contains a $\mathbb{Q}$-basis of A. The following property will be primordial and used very regularly in the rest of the paper. For a proof see [42, Lemma 4.6.9]

Proposition 2.8. Let A be a finite dimensional semisimple $\mathbb{Q}$-algebra and let $\mathcal{O}$ and $\mathcal{O}'$ be both orders in A. Then their unit groups $\mathcal{U}(\mathcal{O})$ and $\mathcal{U}(\mathcal{O}')$ are commensurable.

Let K be a field of characteristic 0. Recall that for $u, v \in K \setminus \{0\}$ the *quaternion algebra* $D = \left(\frac{u,v}{K}\right)$ is the central K-algebra D, i.e. $\mathcal{Z}(D) = K \cdot 1$, that is a 4-dimensional K-vector space with basis $\{1, i, j, k\}$ and multiplication determined by

$$i^2 = u, \quad j^2 = v, \quad ij = k = -ji.$$

Due to following classical theorem of Hasse-Brauer-Noether-Albert, a quaternion algebra is uniquely determined by the places at which it ramifies. Recall that a field extension E of a field K is said to be a *splitting field* of a central simple K-algebra A if $E \otimes_K A \cong \mathrm{M}_n(E)$, and $E \otimes_K A$ is said to be the *split extension* of A.

Theorem 2.9. [57, Theorem 32.11] Let K be a number field and D a quaternion algebra over K. Define $\mathrm{Ram}(D)$ as the set of places v of K such that D is *ramified* at v, i.e. such that the completion K_v of K, with respect to v, is not a splitting field of D. Then $\mathrm{Ram}(D)$ is a finite set with an even number of elements. Moreover, for any finite set S of places of K such that $|S|$ is even, there is a unique quaternion algebra D with center K such that $\mathrm{Ram}(D) = S$.

For $K = \mathbb{Q}$, it is well-known that every place corresponds to a prime integer (for a finite place) or ∞ (for the unique infinite place). Thus a quaternion algebra D over $\mathbb{Q}$ is uniquely determined by its *discriminant* $d = \prod_{p \in \mathrm{Ram}(D)\setminus\{\infty\}} p$ which is the product of all finite places at which D is ramified. For simplicity's sake, we will sometimes denote a quaternion algebra $\left(\frac{u,v}{\mathbb{Q}}\right)$ with discriminant d and center $\mathbb{Q}$ by $\mathbb{H}_d$, which is well defined by the above. Later we will frequently encounter the following three quaternion algebras:

$$\mathbb{H}_2 = \left(\frac{-1,-1}{\mathbb{Q}}\right), \quad \mathbb{H}_3 = \left(\frac{-1,-3}{\mathbb{Q}}\right) \quad \text{and} \quad \mathbb{H}_5 = \left(\frac{-2,-5}{\mathbb{Q}}\right).$$

If K is a totally real number field and $\sigma(u), \sigma(v) < 0$ for every embedding $\sigma : K \to \mathbb{R}$, then the quaternion algebra $D = \left(\frac{u,v}{K}\right)$ is called *totally definite*. The *conjugate* $\bar{x}$ of $x = a_1 \cdot 1 + a_2 \cdot i + a_3 \cdot j + a_4 \cdot k \in D$, $a_1, a_2, a_3, a_4 \in K$ is

$$\bar{x} = a_1 \cdot 1 - a_2 \cdot i - a_3 \cdot j - a_4 \cdot k.$$

We now recall the concept of reduced norm. Let A be a finite dimensional central simple algebra over a field K of characteristic 0. Let E be a splitting field of A. The *reduced norm* of $a \in A$ is defined as

$$\mathrm{RNr}_{A/K}(a) = \det(1_E \otimes_K a).$$

Note that $\mathrm{RNr}_{A/K}(\cdot)$ is a multiplicative map, $\mathrm{RNr}_{A/K}(A) \subseteq K$ and $\mathrm{RNr}_{A/K}(a)$ does only depend on K and $a \in A$ (and not on the chosen splitting field E and isomorphism $E \otimes_K A \cong \mathrm{M}_n(E)$), see [42, page 51]. For a subring R of A, put

$$\mathrm{SL}_1(R) = \{\ a \in \mathcal{U}(R) \mid \mathrm{RNr}_{A/K}(a) = 1\ \},$$

which is a (multiplicative) group. If $A = \mathrm{M}_n(A')$ and $R = \mathrm{M}_n(R')$ with A' a finite dimensional central simple algebra over K and R' a subring of A', then we also write $\mathrm{SL}_1(A) = \mathrm{SL}_n(A')$ and $\mathrm{SL}_1(R) = \mathrm{SL}_n(R')$.

If we write $\mathcal{O}$ for an order in a finite dimensional division $\mathbb{Q}$-algebra D then, as will be explained in further detail and more generally in Section 6.2, $\mathrm{SL}_n(D)$ is an algebraic $\mathbb{Q}$-group and $\mathrm{SL}_n(\mathcal{O})$ an arithmetic subgroup therein. The properties of $\mathrm{SL}_n(\mathcal{O})$ strongly depend on whether $\mathcal{U}(\mathcal{O})$ is finite or not. If it is infinite, there is a vast literature showing that $\mathrm{SL}_n(\mathcal{O})$ satisfies strong properties as illustrated in the introduction. Therefore in this paper we will consider the case where $\mathcal{U}(\mathcal{O})$ is finite. Interestingly this is not a condition on $\mathcal{O}$ but rather a condition on D and one can classify the division algebras containing such an order. Due to the importance of the following classical result we recall its proof.

Theorem 2.10 (Folklore)**.** Let A be a finite dimensional simple $\mathbb{Q}$-algebra and $\mathcal{O}$ an order in A. Then, $\mathcal{U}(\mathcal{O})$ is finite if and only if one of the following holds:

1. $A = \mathbb{Q}(\sqrt{-d})$ with $d \geq 0$ a non-negative integer,

2. $A = \left(\frac{u,v}{\mathbb{Q}}\right)$ with $u, v < 0$ negative integers.

Proof. We may assume that $A = \mathrm{M}_n(D)$ for D a division algebra containing $\mathbb{Q}$ in its center. Due to Proposition 2.8 we may assume that the order is of the form $\mathcal{O} = \mathrm{M}_n(\mathcal{O}')$, for an order $\mathcal{O}'$ in D. Since $\mathrm{GL}_n(\mathcal{O}')$ is infinite for $n \geq 2$, we have $n = 1$, and $A = D$ a division algebra.

If D is commutative, D is a number field and the statement (in both directions) is a direct consequence of Dirichlet's unit theorem [42, Corollary 5.2.6].

If D is non-commutative and $\mathcal{U}(\mathcal{O})$ is finite, Kleinert's theorem [42, Proposition 5.5.6] implies that D is a totally definite quaternion algebra. However, $\langle \mathrm{SL}_1(\mathcal{O}), \mathcal{U}(\mathcal{Z}(\mathcal{O}))\rangle$ has finite index in $\mathcal{U}(\mathcal{O})$, see [42, Proposition 5.5.1]. In particular, also the unit group of $\mathcal{Z}(\mathcal{O})$, which is an order in $\mathcal{Z}(D)$ by [42, Lemma 4.6.6], must be finite and consequently by the commutative case $\mathcal{Z}(D) = \mathbb{Q}(\sqrt{-d})$, $d \geq 0$. As D is a totally definite quaternion algebra, $\mathcal{Z}(D)$ is a totally real extension of $\mathbb{Q}$. Hence $\mathcal{Z}(D) = \mathbb{Q}$.

Conversely, if $D = \left(\frac{u,v}{\mathbb{Q}}\right)$ with $u, v < 0$ negative integers, then by the previous $\mathcal{U}(\mathcal{Z}(\mathcal{O}))$ is a finite group and by Kleinert's theorem $\mathrm{SL}_1(\mathcal{O})$ is also finite. Hence, the group $\langle \mathrm{SL}_1(\mathcal{O}), \mathcal{U}(\mathcal{Z}(\mathcal{O}))\rangle$, which is of finite index in $\mathcal{U}(\mathcal{O})$, is finite. This in turn implies that $\mathcal{U}(\mathcal{O})$ is finite. □

2.3 | Linear groups

When studying the groups $\mathrm{GL}_n(R)$ and its subgroups, it sometimes helps to consider the groups $\mathrm{GE}_n(R)$ and $\mathrm{E}_n(R)$. Here, $\mathrm{E}_n(R)$ is the subgroup of $\mathrm{GL}_n(R)$ generated by the matrices having 1 on the diagonal and one non-zero entry off the diagonal and $\mathrm{GE}_n(R)$ is the subgroup of $\mathrm{GL}_n(R)$ generated by $\mathrm{E}_n(R)$ and the invertible diagonal matrices. These groups have been thoroughly studied in the literature, see, for example [14, 15]. Note that if R is a subring of a division algebra, then $\mathrm{E}_n(R) \leq \mathrm{SL}_n(R)$.

In the case of $n = 2$, we will be using a special (but equivalent) set of generators for $\mathrm{E}_2(R)$. What follows in this subsection is based on [14]. By I we denote the 2×2 identity matrix.

The group $\mathrm{GE}_2(R)$ is the group generated by all matrices

$$[\mu, \nu] = \begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix}, \ (\mu, \nu \in \mathcal{U}(R)), \qquad E(x) = \begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix}, \ (x \in R).$$

For $\mu \in \mathcal{U}(R)$, put $D(\mu) = [\mu, \mu^{-1}]$. Define the group $\mathrm{D}_2(R) = \langle [\mu, \nu] \mid \mu, \nu \in \mathcal{U}(R)\rangle$. Note that

$$E(0)^{-1}E(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \qquad E(-x)E(0)^{-1} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = E(0).$$

Consequently,

$$\mathrm{E}_2(R) = \langle E(x) \mid x \in R\rangle.$$

A priori, $\mathrm{GE}_2(R) \leq \mathrm{GL}_2(R)$, but it can happen that these groups are equal. In this case, we call the ring R a GE_2*-ring*. The following type of rings will form an important class of examples.

Definition 2.11. Let R be a ring. A *left Euclidean* map on R is a map $\delta : R \setminus \{0\} \to \mathbb{N}$ satisfying

$$\forall\, a, b \in R \text{ with } b \neq 0, \exists\, q, r \in R : a = qb + r \text{ with } \delta(r) < \delta(b) \text{ or } r = 0.$$

A *right Euclidean* map on R is a map $\delta : R \setminus \{0\} \to \mathbb{N}$ satisfying

$$\forall\, a, b \in R \text{ with } b \neq 0, \exists\, q, r \in R : a = bq + r \text{ with } \delta(r) < \delta(b) \text{ or } r = 0.$$

We call the ring R a *left (right) Euclidean* ring if it has a left (right) Euclidean map.

If R is a subring of $\mathbb{C}$ or a quaternion algebra with totally real center, then R is endowed with an algebraic norm $x\bar{x}$, $x \in R$. If this map is a left (right) Euclidean map, then we call R *left (right) norm Euclidean*.

We will omit the proof of the following, since it is the same as in the well known commutative case (see for example [66, Proposition 1.4.1]).

Proposition 2.12. Left or right Euclidean rings are GE_2-rings.

Let G be a group generated by a set of elements X. Then a subset $\mathcal{R}$ of the free group F_X on the elements X is called a *defining set of relations of G with respect to X* if the canonical epimorphism

$$F_X \to G,$$

has as kernel the normal closure of the group generated by $\mathcal{R}$, i.e. $\langle \mathcal{R}^{F_X} \rangle$.

If $S \subseteq X$ and $H = \langle S \rangle \leq G$, then any element of $F_S \leq F_X$ is said to be *expressed in abstract letters of* H.

In the group $\mathrm{GE}_2(R)$ the following relations hold, see [14, (2.2)-(2.4)].

$$E(x)E(0)E(y) = E(0)^2E(x+y), \qquad x, y \in R \tag{R1}$$

$$E(\mu)E(\mu^{-1})E(\mu) = E(0)^2D(\mu), \qquad \mu \in \mathcal{U}(R) \tag{R2}$$

$$E(x)[\mu, \nu] = [\nu, \mu]E(\nu^{-1}x\mu), \qquad x \in R,\ \mu, \nu \in \mathcal{U}(R) \tag{R3}$$

$$E(0)^2 = D(-1). \tag{R4}$$

The ring R is called *universal for* GE_2 if these relations, together with the relations in the group $\mathrm{D}_2(R)$, form a complete set of defining relations of $\mathrm{GE}_2(R)$ with respect to the elements $E(x)$ and $[\mu, \nu]$ for $x \in R$ and $\mu, \nu \in \mathcal{U}(R)$. When the group $\mathrm{GE}_2(R)$ is discussed, we will often omit mentioning these explicit generators. If we talk about a set of defining relations for $\mathrm{E}_2(R)$, the generators are always assumed to be $E(x), x \in R$. The relations (R1)-(R4) together with the relations in the group $\mathrm{D}_2(R)$ are called the *universal relations*. Clearly $E(0)^2 = D(-1) = -I$.

Equation (R3) specializes to

$$E(x)D(\mu) = D(\mu^{-1})E(\mu x\mu), \quad x \in R,\ \mu \in \mathcal{U}(R). \tag{R3'}$$

The inverse of $E(x)$ is given by the formula

$$E(x)^{-1} = E(0)E(-x)E(0), \qquad \forall x \in R, \tag{R5}$$

which follows from (R1) and $D(-1)^2 = I$. From the universal relations one can also derive the following useful formulas, see [14, (2.8), (2.9) and (9.2)]

$$E(x)E(y)^{-1}E(z) = E(x-y+z), \qquad x, y, z \in R, \tag{R6}$$

$$E(x)E(\alpha)E(y) = E(x-\alpha^{-1})D(\alpha)E(y-\alpha^{-1}), \qquad x, y \in R, \alpha \in \mathcal{U}(R), \tag{R7}$$

$$[u^{-1}v^{-1}uv, 1] = D(u^{-1})D(v^{-1})D(uv), \qquad u, v \in \mathcal{U}(R). \tag{R8}$$

Rings that are not universal for GE_2 have to have some additional defining relations. For several results the actual form of these non-universal relations is not of importance, but rather the fact that they can be chosen to have a special form (for example can be expressed in abstract letters of $E_2(R) = \langle E(x) \mid x \in R \rangle$). Hence we introduce the following class of rings.

Definition 2.13. Let R be a ring for which there exists a set Φ of words expressed in abstract letters of $\mathrm{E}_2(R)$ such that Φ together with the universal relations yield a full list of defining relations for $\mathrm{GE}_2(R)$. Then we call R *almost-universal for* GE_2.

In Proposition 3.1 we will prove that orders in totally definite quaternion algebras are almost-universal. The following is a slight generalization of [15, Theorem 1] for $\mathrm{GE}_2(R)$.

Theorem 2.14. Let R be a ring, almost-universal for GE_2 with Φ a set of relations expressed in letters of $\mathrm{E}_2(R)$ such that Φ together with the universal relations is a complete set of defining relations of $\mathrm{GE}_2(R)$. The group $\mathrm{E}_2(R)$ is generated by the symbols $E(x)$, $x \in R$ and if we define $D(u)$ and $[w, 1]$ for $u \in \mathcal{U}(R), w \in \mathcal{U}(R)'$ by the relations (R2) and repeated use of (R8) then a complete set of defining relations for $\mathrm{E}_2(R)$ is given by

$$E(x)E(0)E(y) = E(0)^2E(x+y) \tag{R1}$$

$$E(x)D(u) = D(u^{-1})E(uxu) \tag{R3'}$$

$$E(0)^2 = D(-1) \tag{R4}$$

$$[w_1, 1]\dots[w_n, 1] = I \text{ where } w_j \in \mathcal{U}(R)' \text{ and } w_1 \dots w_n = 1. \tag{R9}$$

$$f = I \text{ for all } f \in \Phi \tag{R10}$$

Proof. From the given relations, it is clear we can still deduce (R7), $D(u^{-1}) = D(u)^{-1}$ and (R5). Thus, using these relations, we can rewrite any relation $w = I$ in $\mathrm{E}_2(R)$ as

$$w' = D(u_1)\dots D(u_k)E(a_1)\dots E(a_r) = I. \tag{2.1}$$

We will show that we may reduce the latter relation to a relation with $r = 0$. Note that $r = 1$ cannot occur as $E(a_1)$ is not a diagonal matrix, $r = 2$ is only possible if $a_1 = a_2 = 0$ and this case can be treated with (R4). So assume $r \geq 3$. From the universal relations and (R7), the relation (2.1) may always be written in such a form that $a_i \notin \mathcal{U}(R) \cup \{0\}$ if $1 < i < r$ and $a_1 \neq 0$.

Remark that the universal relations for GE_2 are equivalent to the relations (R1), (R3'), (R4),

$$E(x)[u, 1] = [1, u]E(xu), \tag{2.2}$$

and those in $\mathrm{DE}_2(R)$. Since R is almost-universal for GE_2, and w' is also a word in $\mathrm{GE}_2(R)$, it is a product of conjugates of relators (R1), (R3'), (R4), (2.2) and (R10). By (R3), conjugates of relators (R1), (R3'), (R4), or (R10) are words in $\mathrm{E}_2(R)$. In particular they are 1 in $\mathrm{E}_2(R)$. Hence we can write w' as a product of conjugates of relators of the form $[1, u]E(xu)[u^{-1}, 1]E(x)^{-1}$ and $D(u)$'s.

Further, the relator $[1, u]E(xu)[u^{-1}, 1]E(x)^{-1}$ can also be expressed in the generators of $\mathrm{E}_2(R)$ as follows:

$$\begin{aligned}[1, u]E(xu)[u^{-1}, 1]E(x)^{-1} &= [1, u]E(xu)[1, u^{-1}][1, u][u^{-1}, 1]E(x)^{-1}\\ &= [1, u][u^{-1}, 1]E(uxu)D(u^{-1})E(x)^{-1}\\ &= D(u^{-1})E(uxu)D(u^{-1})E(x)^{-1}\\ &= D(u^{-1})D(u)E(x)E(x)^{-1}.\end{aligned}$$

This last word is trivial in $\mathrm{E}_2(R)$, so the word w' reduces to the form

$$D(v_1)\dots D(v_l) = I.$$

Note that in the latter form $r = 0$. Moreover, again as R is almost-universal for GE_2, by the relations in the group $\mathrm{D}_2(R)$, we have that $v_1 \dots v_l = 1$. By (R8), $D(u)D(v) = [uvu^{-1}v^{-1}, 1]D(u^{-1}v^{-1})^{-1}$. Now by repeated use of the latter, w can be further rewritten as

$$[v_1v_2v_1^{-1}v_2^{-1}, 1][v_2v_1v_3v_1^{-1}v_2^{-1}v_3^{-1}, 1]\dots[v_{l-1}\dots v_1v_lv_1^{-1}\dots v_l^{-1}, 1]D(v_lv_{l-1}\dots v_1)^{-1} = I.$$

By the above, this is equivalent with $[w_1, 1]\dots[w_t, 1] = I$ with $w_j \in \mathcal{U}(R)'$ and $w_1 \dots w_t = 1$, which is exactly (R9). □

3 | ABELIANIZATION OF $\mathrm{E}_2(\mathcal{O})$ AND $\mathrm{GE}_2(\mathcal{O})$

In this section we will study the abelianization of linear groups of degree 2 over orders $\mathcal{O}$, with a finite unit group, in a rational division algebra. To do so we will first prove in Subsection 3.1 that, similar to the case of rings of integers in number fields, those orders allow an almost universal presentation. This will enable us, in Subsection 3.2, to show that the abelianization of $\mathrm{GE}_2(\mathcal{O})$ fits into a short exact sequence described in Theorem B of the introduction, which will be used to calculate this abelianization. Also in Theorem B, we announced a short exact sequence that describes the abelianization of $\mathrm{E}_2(\mathcal{O})$. This result, and an explicit formula for the $\mathbb{Z}$-rank of $\mathrm{E}_2(\mathcal{O})^{ab}$ (see also Theorem C), forms the main part of Subsection 3.3. It will also allow us to characterize when this abelianization is finite.

3.1 | An (almost) universal presentation

In [15, Lemma page 160] an explicit description of the non-universal relations for $\mathrm{GE}_2(R)$, with R a subring of the complex numbers satisfying certain conditions (including rings of algebraic integers in imaginary quadratic extensions of the rationals), is obtained. For our purposes we need a quaternion variant thereof. To achieve this, we give a carefully adapted, more detailed version of the arguments in [15].

Let $H = \left(\frac{u,v}{\mathbb{Q}}\right)$ be a totally definite quaternion algebra with center $\mathbb{Q}$, i.e. u, v negative integers. Define $|x| := \sqrt{x\overline{x}}$, the positive square root of $x\overline{x}$, for $x \in H$ and recall that $x\bar{x} \in \mathbb{Z}$ for x contained in an order in H. We record the following well known properties of this norm map on H. For all $x, y \in H$, $\lambda \in \mathbb{Q}$:

(N1) $|x| \geq 0$ and $|x| = 0 \Leftrightarrow x = 0$ (N2) $|\lambda x| = |\lambda||x|$

(N3) $|x + y| \leq |x| + |y|$ (N4) $|xy| = |x||y|$

Proposition 3.1. Let $K = \mathbb{Q}(\sqrt{-d})$, with d a non-negative integer, i.e. K is either $\mathbb{Q}$ or a quadratic imaginary extension of $\mathbb{Q}$. Let $H = \left(\frac{u,v}{\mathbb{Q}}\right)$ be a totally definite quaternion algebra with center $\mathbb{Q}$. Let $\mathcal{O}$ be an order in K or H. Then a complete set of defining relations for $\mathrm{GE}_2(\mathcal{O})$ is given by the universal relations together with

$$(E(\overline{a})E(a))^n = E(0)^2, \qquad \text{for each } a \in \mathcal{O} \text{ such that } 1 < |a| = \sqrt{n} < 2. \tag{3.1}$$

We will only give an explicit proof in the case of a quaternion algebra. The case of a quadratic imaginary extension of $\mathbb{Q}$ is an easy adaptation of this proof. We first need an auxiliary lemma. Its proof is straightforward which is why we omit it here.

Lemma 3.2. Let K and H be as in Proposition 3.1 and $z, a \in K$, or $z, a \in H$, $z \neq 0$. Let $1 < |a| = \sqrt{n}$. Then

$$|z - a| < 1 \quad \text{if and only if} \quad \left| z^{-1} - \frac{1}{n-1}\overline{a} \right| < \frac{1}{n-1}. \tag{3.2}$$

It is well known that every relation in $\mathrm{GE}_2(\mathcal{O})$ can be written in the form

$$E(t_1) \dots E(t_l) = D,$$

with $D \in \mathrm{D}_2(\mathcal{O})$ (see for example [14, (2.11)]). We will call l the *length of the relation*. Using the universal relations and (R7) (which follows from them), one may always rewrite this relation to a relation where $t_1 \neq 0$ and $t_i \notin \mathcal{U}(\mathcal{O}) \cup \{0\}$ for $1 < i < l$. We call such a form a *canonical form*.

For the proof of Proposition 3.1, we will introduce the following notation. Starting from a list $t = (t_1, t_2, \dots, t_l)$ of elements of $\mathcal{O}$, one may obtain a list $b(t) = (b_0, b_1, \dots, b_l)$ and two non-negative integers $m(t)$ and $h(t)$ as follows:

$$b_0 = 0, \quad b_1 = 1, \quad b_i = b_{i-1}t_{i-1} - b_{i-2} \text{ when } 2 \leq i \leq l,$$

$$m(t) = \max\{|b_0|, \dots, |b_l|\}, \quad h(t) = \max\{i \mid |b_i| = m(t)\}.$$

We will simply write $m = m(t)$ and $h = h(t)$ when t is clear from the context, and extend this notation to $m' = m(t'), h' = h(t'), b' = b(t')$ when we use a second list $t' = (t'_1, t'_2, \dots, t'_{l'})$. Remark that b_i is the element in the upper-right corner of the product

$$E(t_1) \dots E(t_i),$$

where $b_0 = 0$ is to be interpreted as the upper right corner of the empty product, namely I. On the set $\mathbb{R}^2$, denote the lexicographical order by $\preceq$. In the proof of Proposition 3.1, we will show that one can reduce a relation $E(t_1) \dots E(t_l) = D$ to a different relation $E(t'_1) \dots E(t'_{l'}) = D' \in \mathrm{D}_2(\mathcal{O})$ for which $(m', h') \prec (m, h)$. For this, we first need the following lemma.

From now on we are working with $\mathrm{GE}_2(\mathcal{O})$ as an abstract group in terms of the generators $E(x)$ and $[\alpha, \beta]$. In particular $-I$ is no longer intrinsic, however we will continue to use it as a notation for $D(-1)$.

Lemma 3.3. Let $\mathcal{O}$ be an order in a quaternion algebra. Every relation $E(t_1) \dots E(t_l) = D$ in $\mathrm{GE}_2(\mathcal{O})$ (with $D \in \mathrm{D}_2(\mathcal{O})$ and $t_1 \neq 0$) can be rewritten, using the universal relations of $\mathrm{GE}_2(\mathcal{O})$, into a relation $E(t'_1) \dots E(t'_{l'}) = D'$ in canonical form, such that $(m', h') \preceq (m, h)$. Moreover, $l' \leq l$.

Proof. If the relation is not yet in canonical form, then for some $1 < i < l$, $t_i \in \mathcal{U}(\mathcal{O}) \cup \{0\}$.

If $t_i = 0$, then one can use (R1) to replace $E(t_{i-1})E(t_i)E(t_{i+1})$ by $E(t_{i-1} + t_{i+1})$ and D by $-D$. This reduces the length of the relation by 2. Since b'_{i-1} and b_{i-1} only depend on the t's coming before, we have $b'_j = b_j$ for $j \leq i - 1$. Moreover, since

$$E(t_1) \dots E(t_{i-1} + t_{i+1})E(t_{i+2}) = -E(t_1) \dots E(t_{i-1})E(t_i)E(t_{i+1})E(t_{i+2}),$$

we get that $b'_j = -b_{j+2}$ for $j \geq i$. So, from $\{|b'_0|, \dots, |b'_{t-2}|\} \subseteq \{|b_0|, \dots, |b_t|\}$ then follows that $m' \leq m$ and when $m' = m$, then $h' \leq h$. In other words, $(m', h') \preceq (m, h)$.

If $t_i \in \mathcal{U}(\mathcal{O})$ for some $1 < i < l$, then one can use (R7) (which follows from the universal relations) to replace $E(t_{i-1})E(t_i)E(t_{i+1})$ by $E(t_{i-1} - t_i^{-1})D(t_i)E(t_{i+1} - t_i^{-1})$ and then use (R3') to move $D(t_i^{\pm 1})$ to the right of the equation. This reduces the length of the relation by 1. Similar to the above, $b'_j = b_j$ for $j \leq i - 1$. For the cases $j \geq i$, let us first consider b_{i+1}, the upper right entry of

$$\begin{aligned} E(t_1) \dots E(t_{i-1})E(t_i)E(t_{i+1}) &= E(t_1) \dots E(t_{i-1} - t_i^{-1})D(t_i)E(t_{i+1} - t_i^{-1}), \\ &= E(t_1) \dots E(t_{i-1} - t_i^{-1})E(t'_i)D(t_i^{-1}), \\ &= E(t'_1) \dots E(t'_{i-1})E(t'_i)D(t_i^{-1}) \end{aligned}$$

Since multiplying by $D(t_i^{-1})$ does not change the modulus of the element in the upper right corner, $|b_{i+1}|$ is equal to the modulus of the element in the upper right corner of

$$E(t_1') \dots E(t_{i-1}')E(t_i'),$$

which is $|b_i'|$. A similar proof shows that for each $j \geq i$ holds $|b_j'| = |b_{j+1}|$. This shows that $\{|b_0'|, \dots, |b_{t-1}'|\} \subseteq \{|b_0|, \dots, |b_t|\}$, and so that $m' \leq m$ and when $m' = m$, then $h' \leq h$. In other words, $(m', h') \preceq (m, h)$.

Notice that in both cases, we have reduced the length of the relation. We repeat this process until the relation is in canonical form. □

We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1.. Remark that for $z \in \mathcal{O}$, $|z| \in \{0, 1, \sqrt{2}, \sqrt{3}\}$, if $|z| < 2$. Here we use that $|z|^2 \in \mathbb{Z}$, as z is an algebraic integer.

In order to prove the proposition, we begin with a relation

$$E(t_1) \dots E(t_l) = D \tag{3.3}$$

in $\mathrm{GE}_2(\mathcal{O})$ (with D a diagonal matrix) and will reduce it to a relation implied by the universal relations and (3.1). A relation of length 0 is the trivial relation and a simple calculation shows that a relation of length 2 is impossible, except when $t_1 = t_2 = 0$, but this is the relation (R4), i.e. $E(0)^2 = -I$. A relation of length 1 does not exist since b_1 is always equal to 1.

Assume $l \geq 3$, i.e. assume a relation of length at least 3. Without loss of generality, we may assume that $t_1 \neq 0$. Indeed if $t_1 = 0$ we can conjugate the relation with $E(0)^{-1}$ such that $t_1 \neq 0$. By Lemma 3.3, we furthermore may assume that the relation is written in a canonical form. Let now $m = m(\mathbf{t})$ and $h = h(\mathbf{t})$ where $\mathbf{t} = (t_1, \dots, t_l)$ is the list associated to the relation (3.3).

Strategy of the rest of the proof: the argument we use is adapted from [15] and will use induction on (m, h). We will show that such a canonical relation can be reduced to a relation (not necessarily canonical) for which $(m', h') \prec (m, h)$, by using (3.1). Afterwards, we show that Lemma 3.3 is applicable, which does not increase (m', h'). For this new relation in canonical form, either $l' < 3$, which finishes the proof, or $l' \geq 3$ and we may continue by induction. Since (m', h') takes only discrete values in $\mathbb{R}^2_{\geq 0}$, this shows that at some point the length of the new relation will be less than 3, which finishes the proof.

For notation's sake, write

$$a = b_h, \quad b = b_{h-1} \quad \text{and} \quad t = t_h.$$

Note that $a \neq 0$ since $|b_h| \geq |b_1| = 1$. As $b_l = 0$ and $b_1 = 1$, we get that $h \neq l$. Hence $h < l$, which in turn implies that $|t| > 1$, for else $t \in \mathcal{U}(\mathcal{O}) \cup \{0\}$, which contradicts the assumption that the relation is in canonical form. Also $h \neq 1$. Indeed, for suppose $h = 1$, then $b_2 = b_3 = \dots = b_l = 0$. Hence $0 = b_2 = t_1$, but this is again a contradiction.

Since $b_{h+1} = at - b$ and by definition of h, we have

$$|at - b| < |b_h| = |a|, \tag{3.4}$$

and

$$|b| \leq |b_h| = |a|. \tag{3.5}$$

Furthermore $b \neq 0$, since otherwise $|a||t| = |b_{h+1}| < |a|$, which would imply $|t| < 1$.

Inequality (3.4) is equivalent to

$$|a^{-1}b - t| < 1. \tag{3.6}$$

Note that $|t| \geq 2$ implies $|at - b| \geq 2|a| - |b| \geq |a|$, a contradiction with (3.4). Hence $|t| < 2$ and thus $|t| \in \{\sqrt{2}, \sqrt{3}\}$. We will handle both cases separately.

First suppose that $|t| = \sqrt{2}$. Applying Lemma 3.2 to equation (3.6) one obtains $|b^{-1}a - \bar{t}| < 1$ or equivalently

$$|a - b\bar{t}| < |b|. \tag{3.7}$$

We rewrite (3.1) to

$$\begin{aligned}
E(t) &= E(0)^2 E(\bar{t})^{-1} E(t)^{-1} E(\bar{t})^{-1}, && \text{(basic cancelation rule of groups applied to (3.1))}\\
&= E(0)^2 E(0) E(-\bar{t}) E(0)^2 E(-t) E(0)^2 E(-\bar{t}) E(0), && \text{(using (R5))}\\
&= E(0)^2 E(0) E(-\bar{t}) E(-t) E(-\bar{t}) E(0), && \text{(centrality of } E(0)^2 \text{ and } E(0)^4 = 1)
\end{aligned}$$

which we substitute in the relation (3.3) to obtain (after using (R1) two times to get rid of $E(0)$)

$$E(t_1) \dots E(t_{h-2}) E(t_{h-1} - \bar{t}) E(-t) E(t_{h+1} - \bar{t}) E(t_{h+2}) \dots E(t_l) = D',$$

for some diagonal matrix D'. This is the new relation for which we claim $(m', h') \prec (m, h)$ after setting

$$t'_i = \begin{cases} t_i, & \text{if } i < h-1; \\ t_{h-1} - \bar{t}, & \text{if } i = h-1; \\ -t, & \text{if } i = h; \\ t_{h+1} - \bar{t}, & \text{if } i = h+1; \\ t_i, & \text{if } i > h+1. \end{cases}$$

Then it is easy to see that $b'_i = b_i$ for $i < h$ and $b'_h = a - b\bar{t}$. When $i > h$, one sees that $b'_i = -b_i$. Indeed,

$$b'_{h+1} = b'_h t'_h - b'_{h-1} = (b_h - b_{h-1}\bar{t})(-t) - b_{h-1} = -b_h t + b_{h-1} = -b_{h+1}.$$

Furthermore,

$$\begin{aligned} b'_{h+2} &= b'_{h+1}t'_{h+1} - b'_h = -b_{h+1}(t_{h+1} - \bar{t}) - (b_h - b_{h-1}\bar{t}) \\ &= -b_{h+1}t_{h+1} - b_h + (b_{h+1} + b_{h-1})\bar{t} = -b_{h+2} - 2b_h + (b_{h+1} + b_{h-1})\bar{t} \\ &= -b_{h+2} + (b_{h+1} + b_{h-1} - b_h t)\bar{t} = -b_{h+2}. \end{aligned}$$

For $i > h+2$, $t'_i = t_i$ and by induction it is easily proven that $b'_i = -b_i$. By using (3.7) it follows that, when $|t| = \sqrt{2}$, the relation (3.3) can be reduced to a relation for which $(m', h') \prec (m, h)$

Suppose now that $|t| = \sqrt{3}$. From (3.5) it follows that $|a^{-1}b| \leq 1$. We claim that $|1 - a^{-1}b\bar{t}| < |a^{-1}b|$, or equivalently (see Lemma 3.2) $|a^{-1}b - \frac{t}{2}| < \frac{1}{2}$. Indeed, for suppose $|a^{-1}b - \frac{t}{2}| \geq \frac{1}{2}$ and write $a^{-1}b = x + yi + zj + wk$ and $t = x' + y'i + z'j + w'k$ with $x, y, z, w, x', y', z', w' \in \mathbb{Q}$. To keep notation simple, put $\theta = -1 + xx' + uyy' + vzz' + uvww'$. The inequality $|a^{-1}b - \frac{t}{2}| \geq \frac{1}{2}$ translates to

$$|a^{-1}b|^2 \geq \frac{1}{4} - \frac{|t|^2}{4} + (\theta + 1) = \theta + \frac{1}{2}.$$

On the other hand, from (3.6) it follows that

$$|a^{-1}b|^2 < 1 - |t|^2 + 2(\theta + 1) = 2\theta.$$

These last two inequalities together yield $\theta + \frac{1}{2} < 2\theta$ so $\frac{1}{2} < \theta$. But then the first inequality yields $|a^{-1}b|^2 \geq \theta + \frac{1}{2} > 1$, a contradiction.

So, we have $|1 - a^{-1}b\bar{t}| < |a^{-1}b|$, or equivalently by (3.4)

$$|a - b\bar{t}| < |b| \leq |a|, \tag{3.8}$$

and, applying Lemma 3.2 to $|b^{-1}a - \bar{t}| < 1$, gives

$$|2b - at| < |a|. \tag{3.9}$$

Applying Lemma 3.2 to (3.6) also shows that

$$|2a - b\bar{t}| < |b| \leq |a|. \tag{3.10}$$

We rewrite (3.1) to

$$\begin{aligned} E(t) &= E(0)^2 E(\bar{t})^{-1} E(t)^{-1} E(\bar{t})^{-1} E(t)^{-1} E(\bar{t})^{-1}, && \text{(basic cancelation rule of groups applied to (3.1))} \\ &= E(0)^2 E(0) E(-\bar{t}) E(0)^2 E(-t) E(0)^2 E(-\bar{t}) E(0)^2 E(-t) E(0)^2 E(-\bar{t}) E(0), && \text{(using (R5))} \\ &= E(0)^2 E(0) E(-\bar{t}) E(-t) E(-\bar{t}) E(-t) E(-\bar{t}) E(0), && \text{(centrality of } E(0)^2 \text{ and } E(0)^4 = 1) \end{aligned}$$

which we substitute in the relation (3.3) again to obtain (after using (R1) two times to get rid of $E(0)$)

$$E(t_1) \dots E(t_{h-2}) E(t_{h-1} - \bar{t}) E(-t) E(-\bar{t}) E(-t) E(t_{h+1} - \bar{t}) E(t_{h+2}) \dots E(t_l) = D',$$

for some diagonal matrix D'. This is the new relation for which we claim $(m', h') \prec (m, h)$ after setting

$$t'_i = \begin{cases} t_i, & \text{if } i < h-1; \\ t_{h-1} - \bar{t}, & \text{if } i = h-1; \\ -t, & \text{if } i = h \text{ or } h+2; \\ -\bar{t}, & \text{if } i = h+1; \\ t_{h+1} - \bar{t}, & \text{if } i = h+3; \\ t_{i-2}, & \text{if } i > h+3. \end{cases}$$

Note that the new relation has length $l+2$. We have that $b'_i = b_i$ for $i < h$ and an easy calculation shows that $b'_h = a - b\bar{t}$, $b'_{h+1} = 2b - at$, $b'_{h+2} = 2a - b\bar{t}$. Moreover, when $i > h+2$, then $b'_i = -b_{i-2}$. Indeed,

$$\begin{aligned} b'_{h+3} &= b'_{h+2}t'_{h+2} - b'_{h+1} = (2a - b\bar{t})(-t) - (2b - at) \\ &= b - at = b_{h-1} - b_h t_h = -b_{h+1}, \end{aligned}$$

and

$$\begin{aligned} b'_{h+4} &= b'_{h+3}t'_{h+3} - b'_{h+2} = (b - at)(t_{h+1} - \bar{t}) - (2a - b\bar{t}) \\ &= (b - at)t_{h+1} + a = -b_{h+1}t_{h+1} + b_h = -b_{h+2}. \end{aligned}$$

For $i > h+4$, $t'_i = t_{i-2}$ and by induction it is easily proven that $b'_i = -b_{i-2}$. Because of (3.8)-(3.10) it follows that, also when $|t| = \sqrt{3}$, the relation (3.3) may be reduced to a relation for which $(m', h') \prec (m, h)$.

Now, by Lemma 3.3, the relation obtained in the previous two cases can be reduced to a relation in canonical form for which (m', h') does not further increase. We only need to show that $t'_1 \neq 0$. In the steps above, the only way to get $t'_1 = 0$ is if $h = 2$, $t_2 = \bar{t_1}$, $|t_1| = |t_2| < 2$ and $b_2 = t_1$ has maximal modulus. Clearly $|t_1\bar{t_1} - 1| = ||t_1|^2 - 1| = 1$ or 2 (remember that $|t_1|^2 = |t_2|^2 = 2$ or 3). Since $|t_1\bar{t_1} - 1| = |b_3| < |b_2| = |t_1| \leq \sqrt{3}$ it follows that $||t_1|^2 - 1| = 1$ so $|t_1|^2 = 2$. In this case, $b_3 \neq 0$, so the length of the relation is at least 4. One calculates that $b_4 = t_3 - t_1$. From the assumptions $|b_4| < |b_2| = |t_1| = \sqrt{2}$, so $|b_4| \in \{0, 1\}$.

If $|b_4| = 0$, then $t_3 = t_1$. If the length of the relation is exactly 4, then one can show that $t_4 = \bar{t_1}$, but this is the relation (3.1) and the induction step would stop here. The length cannot be exactly 5. Indeed $|b_4| = 0$ implies that $E(t_1)...E(t_4)$ is a lower-triangular matrix and thus $E(t_1)...E(t_4)E(t_5)$ cannot be a diagonal matrix. So the length of the relation is at least 6. From easy calculations it follows that $b_5 = -1$ and $b_6 = -t_5$. From the maximality of h we can deduce that $|t_5| = |b_6| < |b_2| = \sqrt{2}$, showing that t_5 is either a unit or 0, a contradiction with the fact that the relation was in canonical form.

Thus suppose that $|b_4| = 1$. We will first show that if b_{i-1} and b_i are units, then b_{i+1} is also a unit. Indeed, $|b_{i+1}| = |b_i t_i - b_{i-1}| \geq \big||b_i||t_i| - |b_{i-1}|\big| = \big||t_i| - 1\big| \geq \sqrt{2} - 1 > 0$. The fact that $|t_i| \geq \sqrt{2}$ follows from the fact that the length of the relation is at least $i+1$ and thus t_i is not a unit or 0 from the canonical form. On the other hand, by the minimality of h we need $|b_{i+1}| < |b_2| = \sqrt{2}$, showing that $|b_{i+1}| = 1$ and b_{i+1} is a unit.

Through $|b_3| = |b_4| = 1$ and the repeated use of the result above we obtain that the word should be infinitely long, a contradiction.

In the end, we proved that $t'_1 \neq 0$ and that Lemma 3.3 can be applied. This finishes the proof. □

3.2 | On the abelianization of $\mathrm{GE}_2(\mathcal{O})$ over orders $\mathcal{O}$ with $\mathcal{U}(\mathcal{O})$ finite

Let $\mathcal{O}$ be an order in a finite dimensional division $\mathbb{Q}$-algebra D with $\mathcal{U}(\mathcal{O})$ finite. The main goal of this section is to describe $\mathrm{GE}_2(\mathcal{O})^{ab}$ in a computable and uniform way. More concretely in Theorem 3.8 we obtain a short exact sequence

$$1 \longrightarrow (\mathcal{O}/N, +) \longrightarrow \mathrm{GE}_2(\mathcal{O})^{ab} \longrightarrow \mathcal{U}(\mathcal{O})^{ab} \longrightarrow 1$$

where N is the two-sided ideal generated by the elements $u - 1$ with $u \in \mathcal{U}(\mathcal{O})$. To start we describe $\mathrm{GE}_2(R)/\mathrm{E}_2(R)$ in the more general context of rings which are almost-universal for GE_2. Thereafter we restrict to orders in finite-dimensional division $\mathbb{Q}$-algebras with finite unit group and prove that an exact sequence as stated above exists. This is inspired by the results in [14].

Proposition 3.4. Let R be a ring which is almost-universal for GE_2, then

$$\mathrm{GE}_2(R)/\mathrm{E}_2(R) \cong \mathcal{U}(R)^{ab}.$$

The isomorphism is induced by the map

$$\varphi : \mathrm{GE}_2(R) \to \mathcal{U}(R)^{ab} \text{ by } \varphi(E(x)) = 1 \text{ and } \varphi([\alpha,\beta]) = \widetilde{\alpha\beta}, \tag{3.11}$$

which is a group homomorphism. The map $\widetilde{\ }: \mathcal{U}(R) \to \mathcal{U}(R)^{ab}$ denotes the canonical morphism.

Proof. Since $\mathrm{E}_2(R)$ is in the kernel of φ, it is enough to check that the relations not in $\mathrm{E}_2(R)$ are preserved to prove that φ is a well-defined group homomorphism. The only such relations are those of the form $E(x)[\alpha,\beta] = [\beta,\alpha]E(\beta^{-1}x\alpha)$ and the relations in $\mathrm{D}_2(R)$. As $\mathcal{U}(R)^{ab}$ is abelian, $\widetilde{\alpha\beta} = \widetilde{\beta\alpha}$ and hence the first type of relation is preserved. It is easy to check that φ preserves the relations in $\mathrm{D}_2(R)$.

Now the map φ is onto and $\mathrm{E}_2(R) \subseteq \ker(\varphi)$. For the reverse inclusion, let $A \in \ker(\varphi)$. Using the universal relations we may write $A = [\alpha,\beta]E(x_1)\dots E(x_r)$. Since $A \in \ker(\varphi)$ we have that $\widetilde{\alpha\beta} = \widetilde{\beta\alpha} = \widetilde{1}$, i.e. $\beta\alpha \in \mathcal{U}(R)'$. Hence, by (R8), $[\beta\alpha, 1] \in \mathrm{E}_2(R)$. As $D(\beta)[\alpha,\beta] = [\beta\alpha, 1]$, we have that $[\alpha,\beta] \in \mathrm{E}_2(R)$ and hence $A \in \mathrm{E}_2(R)$. ☐

Corollary 3.5. The following properties hold for a ring R which is almost-universal for GE_2:

1. $\mathrm{GE}_2(R)' \subseteq \mathrm{E}_2(R)$,
2. $\mathrm{D}_2(R) \cap \mathrm{E}_2(R) = \langle D(\mu) \mid \mu \in \mathcal{U}(R)\rangle$.

Proof. The first statement is a direct consequence of Proposition 3.4. For the second statement assume $[\alpha,\beta] \in \mathrm{D}_2(R)\cap\mathrm{E}_2(R)$. Then $\varphi([\alpha,\beta]) = \widetilde{1}$, in particular, $[\alpha,\beta] = D(\beta^{-1})[\beta\alpha, 1]$ with $\beta\alpha \in \mathcal{U}(R)'$. Consequently, writing $\beta\alpha = \prod_{i\in I}\delta_i^{-1}\mu_i^{-1}\delta_i\mu_i$, we see that $[\alpha,\beta] = D(\beta^{-1})\prod_{i\in I} D(\delta_i^{-1})D(\mu_i^{-1})D(\delta_i\mu_i) \in \langle D(\mu) \mid \mu \in \mathcal{U}(R)\rangle$. ☐

Proposition 3.4 also indicates that in order to understand $\mathcal{U}(R)^{ab}$ for some ring R which is almost-universal for GE_2, one may "increase its size" to $\mathrm{GE}_2(R)$ and instead investigate its abelianization (which will be the content of the following subsection). Recall that the *Borel subgroup* of $\mathrm{GE}_2(R)$, denoted $\mathrm{B}_2(R)$, is the subgroup consisting of the upper-triangular matrices with units on the diagonal, i.e. $\mathrm{B}_2(R) = \{\begin{pmatrix}\alpha & x\\ 0 & \beta\end{pmatrix} \mid x \in R, \alpha,\beta \in \mathcal{U}(R)\}$.

Proposition 3.6. Let R be a ring, finitely generated as $\mathbb{Z}$-module, which is almost-universal for GE_2, then the following properties are equivalent:

1. $\mathcal{U}(R)^{ab}$ is finite,
2. $\mathrm{B}_2(R)^{ab}$ is finite,
3. $\mathrm{GE}_2(R)^{ab}$ is finite.

Proof. It is easy to calculate that for any $a \in R$ we get

$$\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}^{-1}\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}^{-1}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}^{2} \in \mathrm{B}_2(R)',$$

and clearly also $D_2(R)' \leq B_2(R)'$. This shows that $B_2(R)^{ab}$ is an epimorphic image of the group $H \times D_2(R)^{ab} \cong H \times \mathcal{U}(R)^{ab} \times \mathcal{U}(R)^{ab}$, where H is some finitely generated abelian group of exponent 2 (and so it is finite). Hence, if $\mathcal{U}(R)^{ab}$ is finite then also $B_2(R)$ has finite abelianization.

For the next implication notice that $\mathrm{GE}_2(R) = \langle E(0), \mathrm{B}_2(R)\rangle$. Since $E(0)$ has finite order we now easily see that $\mathrm{GE}_2(R)^{ab}$ is finite if $\mathrm{B}_2(R)^{ab}$ is finite.

Finally from Proposition 3.4 and Corollary 3.5 it follows that $\mathcal{U}(R)^{ab}$ is an epimorphic image of $\mathrm{GE}_2(R)^{ab}$ and so the remaining implication also follows. ☐

Corollary 3.7. Let $\mathcal{O}$ be an order in a finite dimensional division $\mathbb{Q}$-algebra with $\mathcal{U}(\mathcal{O})$ finite, then $\mathrm{GE}_2(\mathcal{O})^{ab}$ is finite.

Proof. Because $\mathcal{U}(\mathcal{O})$ is finite, we know from Theorem 2.10 and Proposition 3.1 that $\mathcal{O}$ is almost-universal for GE_2. Hence, we may apply Proposition 3.6 and it suffices to show that $\mathcal{U}(\mathcal{O})^{ab}$ is finite. However, this follows readily from the fact that $\mathcal{U}(\mathcal{O})$ is finite. ☐

If R is almost-universal for GE_2, then by Proposition 3.4 we have that $\mathrm{GE}_2(R)' \subseteq \mathrm{E}_2(R)$ and

$$\mathcal{U}(R)^{ab} \cong \mathrm{GE}_2(R)^{ab} / \left(\mathrm{E}_2(R)/\,\mathrm{GE}_2(R)'\right),$$

where the isomorphism is induced by the map

$$\varphi : \mathrm{GE}_2(R) \to \mathcal{U}(R)^{ab} \qquad \text{with} \qquad \varphi(E(x)) = 1 \text{ and } \varphi([\alpha, \beta]) = \widetilde{\alpha\beta}, \tag{3.12}$$

for $\alpha, \beta \in \mathcal{U}(R)$ and $x \in R$. So in order to understand $\mathrm{GE}_2(R)^{ab}$ it remains to describe $\mathrm{E}_2(R)/\,\mathrm{GE}_2(R)'$. For orders $\mathcal{O}$ in a finite dimensional division $\mathbb{Q}$-algebra with a finite number of units this will be achieved through the following map

$$\psi : \mathrm{E}_2(\mathcal{O}) \to (\mathcal{O}/N, +) : E(x) \mapsto x - 1 + N, \tag{3.13}$$

where N is the two-sided ideal of $\mathcal{O}$ generated by the elements $u - 1$ with $u \in \mathcal{U}(\mathcal{O})$. In the following theorem we will prove that the kernel of ψ is exactly $\mathrm{GE}_2(\mathcal{O})'$.

Theorem 3.8. Let $\mathcal{O}$ be an order in a finite dimensional division $\mathbb{Q}$-algebra with $\mathcal{U}(\mathcal{O})$ finite and let N be the two-sided ideal of $\mathcal{O}$ generated by the elements $u - 1$ with $u \in \mathcal{U}(\mathcal{O})$. Then

$$\mathrm{E}_2(\mathcal{O})/\,\mathrm{GE}_2(\mathcal{O})' \cong (\mathcal{O}/N, +).$$

In particular, we have the following short exact sequence of groups:

$$1 \longrightarrow (\mathcal{O}/N, +) \overset{\iota\circ\bar{\psi}^{-1}}{\longrightarrow} \mathrm{GE}_2(\mathcal{O})^{ab} \overset{\bar{\varphi}}{\longrightarrow} \mathcal{U}(\mathcal{O})^{ab} \longrightarrow 1,$$

where $\iota :\ \mathrm{E}_2(\mathcal{O})/\,\mathrm{GE}_2(\mathcal{O})' \hookrightarrow \mathrm{GE}_2(\mathcal{O})/\,\mathrm{GE}_2(\mathcal{O})'$ is induced by the inclusion $\mathrm{E}_2(\mathcal{O}) \hookrightarrow \mathrm{GE}_2(\mathcal{O})$, $\bar{\psi}$ is the isomorphism induced by ψ defined in (3.13) and $\bar{\varphi}$ is induced by φ in (3.12).

Proof. Because $\mathcal{U}(\mathcal{O})$ is finite, we know from Theorem 2.10 and Proposition 3.1 that $\mathcal{O}$ is almost-universal for GE_2. Hence, by the discussion before Theorem 3.8, we know that

$$1 \longrightarrow \mathrm{E}_2(\mathcal{O})/\,\mathrm{GE}_2(\mathcal{O})' \overset{\iota}{\longrightarrow} \mathrm{GE}_2(\mathcal{O})^{ab} \overset{\bar{\varphi}}{\longrightarrow} \mathcal{U}(\mathcal{O})^{ab} \longrightarrow 1,$$

forms an exact sequence of groups. So, in order to prove Theorem 3.8, it suffices to show that $\bar{\psi}$ is well-defined and forms an isomorphism between $\mathrm{E}_2(\mathcal{O})/\,\mathrm{GE}_2(\mathcal{O})'$ and $(\mathcal{O}/N, +)$.

First we show that $\psi :\ \mathrm{E}_2(\mathcal{O}) \to (\mathcal{O}/N, +) : E(x) \mapsto x - 1 + N$ is well defined. For this it is enough to prove that the relations from Theorem 2.14 are preserved. We will use α to denote an element of $\mathcal{U}(\mathcal{O})$.

Remark that, by the definition of N, $D(\alpha) = E(0)^2 E(\alpha)E(\alpha^{-1})E(\alpha)$ is mapped to -2. As $-2 \in N$, $D(\alpha)$ is mapped to zero for every $\alpha \in \mathcal{U}(\mathcal{O})$. In particular, relations (R4) and (R9) are preserved. Further relation (R3') reduces to $\alpha x \alpha \equiv x \bmod N$, for $\alpha \in \mathcal{U}(\mathcal{O})$ and $x \in \mathcal{O}$. Since $\alpha x \alpha - x = (\alpha - 1)x\alpha + x(\alpha - 1) \in N$, it is indeed preserved under ψ. Relation (R1) is trivially preserved under ψ. Finally the only relations left to check are (R10). By Proposition 3.1 and Theorem 2.10, these relations are of the form (3.1). They are easily checked using that $2 \in N$ and $a + \overline{a} = 2\,\mathrm{Tr}(a)$.

We want to show that $\mathrm{GE}_2(\mathcal{O})' \subseteq \ker(\psi)$. To do this, remark that clearly $\mathrm{E}_2(\mathcal{O})' \subseteq \ker(\psi)$ and that $D(\alpha) \in \ker(\psi)$, as proven above. It only remains to prove that, for $x \in \mathcal{O}$ and $\alpha, \beta, \gamma, \delta \in \mathcal{U}(\mathcal{O})$ we have that the commutator between $[\alpha, \beta]$ and $[\gamma, \delta]$ and the commutator between $[\alpha, \beta]$ and $E(x)$ is in the kernel since these elements (together with $\mathrm{E}_2(\mathcal{O})'$) generate $\mathrm{GE}_2(\mathcal{O})'$ as a normal subgroup and the image of ψ is an abelian group.

Clearly the commutator between $[\alpha, \beta]$ and $[\gamma, \delta]$ is a diagonal matrix in $\mathrm{E}_2(\mathcal{O})$ by Proposition 3.4, and thus by the above it is in the kernel of ψ.

For the other commutator we can write

$$\begin{aligned}
[\alpha, \beta]^{-1} E(x)[\alpha, \beta]E(x)^{-1} &= [\alpha^{-1}, \beta^{-1}][\beta, \alpha]E(\beta^{-1}x\alpha)E(x)^{-1}, \\
&= [\alpha^{-1}\beta, \beta^{-1}\alpha]E(\beta^{-1}x\alpha)E(0)E(-x)E(0).
\end{aligned}$$

Since $[\alpha^{-1}\beta, \beta^{-1}\alpha] = D(\alpha^{-1}\beta)$ this commutator is mapped, under ψ, to $\beta^{-1}x\alpha - x - 4$. As $-4 \in N$ and $\beta^{-1}x\alpha - x = \beta^{-1}(x\alpha - \beta x) = \beta^{-1}(x(\alpha - 1) - (\beta - 1)x) \in N$, this commutator is also in $\ker(\psi)$.

Now ψ induces $\bar{\psi} : \mathrm{E}_2(\mathcal{O})/\,\mathrm{GE}_2(\mathcal{O})' \to (\mathcal{O}/N, +)$. Since ψ is surjective, it remains to prove the injectivity of $\bar{\psi}$.

Note that an arbitrary element in $\mathrm{E}_2(R)$ can be written as

$$E(x_1 + 3) \cdots E(x_l + 3), \qquad x_1, \ldots, x_l \in \mathcal{O}. \tag{3.14}$$

Further remark the following crucial identity

$$
\begin{aligned}
E(x)E(y) &\equiv E(x)E(0)E(y)E(0)E(-1)E(0)E(-1)E(0)E(-1)\\
&\equiv E(x+y-3) \mod \mathrm{E}_2(\mathcal{O})' \qquad (3.15)
\end{aligned}
$$

where we used $E(0)^4 = E(-1)^3 = 1$ and (R1). By induction on l then

$$E(x_1+3)\dots E(x_l+3) \equiv E\left(\left(\sum_{i=1}^{l} x_i\right)+3\right) \mod \mathrm{E}_2(\mathcal{O})'. \qquad (3.16)$$

Suppose now that the expression (3.14) is in $\ker(\psi)$. Then

$$0 \equiv (x_1+2)+\dots+(x_l+2) \equiv \sum_{i=1}^{l} x_i \mod N,$$

since $2 \in N$. In particular by (3.16) is it enough to prove that $E(n+3) \equiv 1 \mod \mathrm{GE}_2(\mathcal{O})'$ for all $n \in N$. By the definition of N, it is enough to do so for $E((\alpha-1)x+3)$ and $E(x(\alpha-1)+3)$, where $x \in \mathcal{O}$ and $\alpha \in \mathcal{U}(\mathcal{O})$. Using equation (R6) we obtain

$$
\begin{aligned}
E((\alpha-1)x+3) &= E(\alpha(x-3)-(x-3)+3\alpha)\\
&= E(\alpha(x-3))E(x-3)^{-1}E(3\alpha).
\end{aligned}
$$

Moreover

$$E(3\alpha) \equiv D(\alpha) \bmod \mathrm{E}_2(\mathcal{O})', \qquad (3.17)$$

and

$$D(\alpha) \equiv D(\alpha^{-1}) \bmod \mathrm{E}_2(\mathcal{O})'. \qquad (3.18)$$

Indeed, using (R3'), (R2) and (R5):

$$
\begin{aligned}
E(\alpha x\alpha)E(x)^{-1} &\equiv D(\alpha)^2 = -D(\alpha)D(-\alpha)\\
&= -E(\alpha)E(\alpha^{-1})E(\alpha)E(-\alpha)E(-\alpha^{-1})E(-\alpha)\\
&= -E(\alpha)E(\alpha^{-1})E(\alpha)E(0)E(\alpha)^{-1}E(0)^2E(\alpha^{-1})^{-1}E(0)^2E(\alpha)^{-1}E(0)\\
&\equiv 1 \mod \mathrm{E}_2(\mathcal{O})'
\end{aligned}
$$

and we get that both $D(\alpha) \equiv D(\alpha^{-1}) \mod \mathrm{E}_2(\mathcal{O})'$ and

$$E(\alpha x\alpha) \equiv E(x) \mod \mathrm{E}_2(\mathcal{O})', \qquad (3.19)$$

in particular $E(\alpha) \equiv E(\alpha^{-1}) \bmod \mathrm{E}_2(\mathcal{O})'$. By the latter, (R1) and (R2),

$$E(3\alpha) = E(\alpha)E(0)E(\alpha)E(0)E(\alpha) \equiv E(0)^2E(\alpha)E(\alpha^{-1})E(\alpha) = D(\alpha) \mod \mathrm{E}_2(\mathcal{O})',$$

as claimed. Taking in (R3) the diagonal matrix $[\alpha^{-1},\alpha^{-1}]$ we see that $E(x) \equiv E(\alpha x\alpha^{-1}) \bmod \mathrm{E}_2(\mathcal{O})'$ and so also $E(\alpha x) \equiv E(x\alpha) \bmod \mathrm{E}_2(\mathcal{O})'$, (replace x by $x\alpha$).

Now,

$$
\begin{aligned}
E(\alpha(x-3))E(x-3)^{-1}E(3\alpha) &\equiv E(x-3)^{-1}E(\alpha(x-3))D(\alpha^{-1}),\\
&\equiv E(x-3)^{-1}E(\alpha(x-3))[1,\alpha][\alpha^{-1},1],\\
&\equiv E(x-3)^{-1}[\alpha,1]E(x-3)[\alpha^{-1},1] \equiv 1 \mod \mathrm{E}_2(\mathcal{O})',
\end{aligned}
$$

where in the second to last equality, (R3) is used. So altogether we proved that $E((\alpha-1)x+3) \equiv 1 \bmod \mathrm{E}_2(\mathcal{O})'$. In analogue way one proves that $E(x(\alpha-1)+3) \equiv 1 \bmod \mathrm{E}_2(\mathcal{O})'$, finishing the proof. □

Corollary 3.9. Let $\mathcal{O}$ be an order in a finite dimensional division $\mathbb{Q}$-algebra with $\mathcal{U}(\mathcal{O})$ finite. If $\mathcal{U}(\mathcal{O})$ contains an element of odd order, then $\mathrm{GE}_2(\mathcal{O})^{ab} \cong \mathcal{U}(\mathcal{O})^{ab}$.

Proof. Assume that an element of $\mathcal{U}(\mathcal{O})$ has odd order. Then there exists an element $\alpha \in \mathcal{U}(\mathcal{O})$ of odd prime order, say p. For this element holds $1+\alpha+\dots+\alpha^{p-1} = 0$ and hence $1 = \sum_{i=1}^{p-1}(-1)^i(1-(-\alpha)^i) \in N$. Thus $N = \mathcal{O}$ and $\mathrm{GE}_2(\mathcal{O})^{ab} \cong \mathcal{U}(\mathcal{O})^{ab}$ by Theorem 3.8. □

At the end of the next section we will exploit Corollary 3.9 to give exact descriptions of $\mathrm{GE}_2(\mathcal{O})^{ab}$ for certain orders $\mathcal{O}$.

3.3 | On the abelianization of $\mathrm{E}_2(\mathcal{O})$ over orders $\mathcal{O}$ with $\mathcal{U}(\mathcal{O})$ finite

As in the previous subsection, we will obtain a short exact sequence that will allow us to study the abelianization of $\mathrm{E}_2(\mathcal{O})$ over orders with $\mathcal{U}(\mathcal{O})$ finite.

Theorem 3.10. Let $\mathcal{O}$ be an order in a finite-dimensional division $\mathbb{Q}$-algebra with $\mathcal{U}(\mathcal{O})$ finite. Let M be the additive subgroup of $\mathcal{O}$ generated by the following set of elements:

1. $\alpha x\alpha - x$ with $x \in \mathcal{O}$ and $\alpha \in \mathcal{U}(\mathcal{O})$,
2. $\sum_{i=1}^{m} 3(\alpha_i + 1)(\beta_i + 1)$ with $\alpha_i, \beta_i \in \mathcal{U}(\mathcal{O})$ satisfying $\prod_{i=1}^{m} \alpha_i^{-1}\beta_i^{-1}\alpha_i\beta_i = 1$,
3. the elements $2(x+\bar{x})+6$ for each element $x \in \mathcal{O}$ satisfying $|x|^2 = 2$,
4. the elements $3(x+\bar{x})$ for each element $x \in \mathcal{O}$ satisfying $|x|^2 = 3$.

Then,

$$\tau : \mathrm{E}_2(\mathcal{O}) \to (\mathcal{O}/M, +) : E(x) \mapsto x - 3 + M$$

is an epimorphism with $\ker(\tau) = \mathrm{E}_2(\mathcal{O})'$. In particular

$$\mathrm{E}_2(\mathcal{O})/\mathrm{E}_2(\mathcal{O})' \cong (\mathcal{O}/M, +).$$

Remark 3.11. As $\mathcal{U}(\mathcal{O})$ is finite, $\mathcal{O}$ is an order in $\mathbb{Q}$, in an quadratic imaginary extension of $\mathbb{Q}$ or a totally definite quaternion algebra over $\mathbb{Q}$, by Theorem 2.10 and the norm map appearing in the third and fourth item of the definition of M in Theorem 3.10 is the same as in the beginning of Section 3.1: $|x| = \sqrt{x\bar{x}}$.

Proof. We first prove that the map τ is well-defined and a group homomorphism. For this it is enough to check that τ preserves the defining relations of $\mathrm{E}_2(\mathcal{O})$ stated in Theorem 2.14, with Φ the non-universal set of relations of the form (3.1).

Relation (R1) is trivially preserved. Note that $12 = 3(1+1)(1+1) \in M$. Hence (R4), or equivalently $E(0)^2 = E(0)^2E(-1)^3$, is preserved. Since $\alpha x\alpha \equiv x \mod M$, for any $\alpha \in \mathcal{U}(\mathcal{O})$ and $x \in \mathcal{O}$, we have that

$$\alpha \equiv \alpha^{-1} \mod M.$$

Now, the image of (R3') under τ yields the equation $x - 3 + 2\alpha + \alpha^{-1} - 3 \equiv 2\alpha^{-1} + \alpha - 3 + \alpha x\alpha - 3 \mod M$ or thus

$$\alpha x\alpha - x \equiv \alpha - \alpha^{-1} \equiv 0 \mod M.$$

We now consider the preservation of (R9). Since $\alpha \equiv \alpha^{-1} \mod M$ for any unit α, we immediately obtain also that $\tau(D(\alpha^{-1})) \equiv \tau(D(\alpha)) \equiv 3(\alpha - 1) \mod M$. By definition there is for every $1 \le k \le n$ a decomposition $[w_k, 1] = \prod_{i\in I_k} D(\alpha_{i,k}^{-1})D(\beta_{i,k}^{-1})D(\alpha_{i,k}\beta_{i,k})$. Furthermore by (2),

$$\begin{aligned}\tau\left(\prod_{1\le k\le n}\prod_{i\in I_k} D(\alpha_{i,k}^{-1})D(\beta_{i,k}^{-1})D(\alpha_{i,k}\beta_{i,k})\right) &\equiv \sum_{1\le k\le n}\sum_{i\in I_k} 3(\alpha_{i,k}-1) + 3(\beta_{i,k}-1) + 3(\alpha_{i,k}\beta_{i,k}-1)\\ &\equiv \sum_{1\le k\le n}\sum_{i\in I_k} 3(\alpha_{i,k} + \beta_{i,k} + \alpha_{i,k}\beta_{i,k} + 1)\\ &\equiv \sum_{1\le k\le n}\sum_{i\in I_k} 3(\alpha_{i,k}+1)(\beta_{i,k}+1) \equiv 0 \mod M,\end{aligned}$$

yielding that (R9) is preserved.

Finally consider the relation $(E(\overline{x})E(x))^n = E(0)^2$ from (3.1). If $|x| = \sqrt{2}$ then

$$\tau\left(E(\overline{x})E(x)E(\overline{x})E(x)E(0)^2\right) \equiv 2(\overline{x}+x)+6 \equiv 0 \mod M.$$

Similarly, $\tau\left((E(\overline{x})E(x))^3E(0)^2\right) \equiv 3(\overline{x}+x) \equiv 0 \mod M$, if $|x| = \sqrt{3}$.

Altogether we proved that τ is well-defined and hence defines an epimorphism. Since $(\mathcal{O}/M, +)$ is abelian, $E_2(\mathcal{O})' \subseteq \ker(\tau)$. By (3.16) in the proof of Theorem 3.8 the reverse inclusion follows if $E(m+3) \in \mathrm{E}_2(\mathcal{O})'$ for all additive generators m of M.

Due to (3.15) and (3.19) in the proof of Theorem 3.8 one also immediately obtains $E(3) \in \mathrm{E}_2(\mathcal{O})'$ and $E(\alpha x\alpha)E(x)^{-1} \in \mathrm{E}_2(\mathcal{O})'$. Consequently, by (R6), $E(\alpha x\alpha - x + 3) = E(\alpha x\alpha)E(x)^{-1}E(3) \in \mathrm{E}_2(\mathcal{O})'$. Next consider an element $\sum_{i\in I} 3(\alpha_i+1)(\beta_i+1)$ such that $\prod_{i=1}^m \alpha_i^{-1}\beta_i^{-1}\alpha_i\beta_i = 1$. By first using (3.16), then consecutively (R1), (3.17), (3.18), (R8) and finally (R9) we obtain

$$\begin{aligned}
E\left(\textstyle\sum_{i=1}^m 3(\alpha_i+1)(\beta_i+1)+3\right) &\equiv \textstyle\prod_{i=1}^m E(3(\alpha_i+1)(\beta_i+1)+3)\\
&\equiv \textstyle\prod_{i=1}^m E(3\alpha_i)E(0)E(3\beta_i)E(0)E(3\alpha_i\beta_i)(E(0)E(3))^2\\
&\equiv \textstyle\prod_{i=1}^m D(\alpha_i)D(\beta_i)D(\alpha_i\beta_i)\\
&\equiv \textstyle\prod_{i=1}^m D(\alpha_i^{-1})D(\beta_i^{-1})D(\alpha_i\beta_i)\\
&\equiv \textstyle\prod_{i=1}^m [\alpha_i^{-1}\beta_i^{-1}\alpha_i\beta_i, 1] \equiv 1 \mod \mathrm{E}_2(\mathcal{O})'.
\end{aligned}$$

Now consider an element $2(x+\overline{x})+6$ with $|x|^2 = 2$. Then, using (R1), the fact that modulo $\mathrm{E}_2(\mathcal{O})'$ all elements commute, that $E(0)^4 = I$, that $E(3) \in \mathrm{E}_2(\mathcal{O})'$ and finally (3.1),

$$\begin{aligned}
&E(2(x+\overline{x})+9)\\
&\quad\equiv E(0)^2E(x+\overline{x})E(0)E(0)^2E(x+\overline{x})E(0)E(0)^2E(3)E(0)E(0)^2E(3)E(0)E(3)\\
&\quad\equiv E(x+\overline{x})^2E(3)^3 \equiv E(x+\overline{x})^2 \equiv E(0)^2E(\overline{x})E(0)E(x)E(0)^2E(\overline{x})E(0)E(x)\\
&\quad\equiv (E(\overline{x})E(x))^2E(0)^2 \equiv E(0)^2E(0)^2 \equiv 1 \mod \mathrm{E}_2(\mathcal{O})'.
\end{aligned}$$

In case of the additive generators $3(x+\bar{x})$ the proof is analogue, hence finishing the proof. □

Remark 3.12. Note the subtle, but crucial, point that M is defined to be the additive subgroup generated by those elements listed in Theorem 3.10, in contrast to N from Theorem 3.8 which is defined as the two-sided ideal generated by the elements in that statement. In fact, since $12 \in M$, one can deduce that M can only be an ideal when $\mathcal{O}/M$ is fnite (which by Theorem 3.14 only is the case when $\mathcal{O}$ contains a $\mathbb{Z}$-basis consisting of units). Also it is interesting to remember that the elements $\alpha x\alpha - x$ exactly encode the image of (R2) under τ, the elements $\prod 3(\alpha_i+1)(\beta_i+1)$ the relations (R9) and the last two elements encode the relations of the form (3.1).

Finally note that if $\mathcal{U}(\mathcal{O})$ is abelian, then the condition $\prod_{i=1}^m \alpha_i^{-1}\beta_i^{-1}\alpha_i\beta_i = 1$ is always satisfied, hence in this case one simply adds all elements $3(\alpha+1)(\beta+1)$.

Remark 3.13. We denote by $\mathcal{I}_d$ the ring of algebraic integers in the imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$ of a positive integer d, e.g. $\mathcal{I}_1 = \mathbb{Z}[\sqrt{-1}]$ and $\mathcal{I}_3 = \mathbb{Z}[\zeta_3]$, where ζ_3 is a primitive complex third root of unity. It is well known that $\mathcal{I}_d$ is Euclidean if and only if $d \in \{1,2,3,7,11\}$. In [28], Fitzgerald showed that the only totally definite quaternion algebras $\mathbb{H}_d$ with center $\mathbb{Q}$ containing a right norm Euclidean order are

$$\mathbb{H}_2 = \left(\frac{-1,-1}{\mathbb{Q}}\right), \quad \mathbb{H}_3 = \left(\frac{-1,-3}{\mathbb{Q}}\right) \quad \text{and} \quad \mathbb{H}_5 = \left(\frac{-2,-5}{\mathbb{Q}}\right).$$

Note that orders that are (right norm) Euclidean are maximal [12, Proposition 2.8]. Furthermore a quaternion algebra having a right norm Euclidean order has class number one [12, Proposition 2.9] and thus also type number one meaning that there is only one conjugacy class of maximal orders. In [28] also a specific representative of that unique conjugacy class, denoted $\mathcal{O}_2 \subseteq \mathbb{H}_2$, $\mathcal{O}_3 \subseteq \mathbb{H}_3$ and $\mathcal{O}_5 \subseteq \mathbb{H}_5$, is constructed. For later use we explicitly state in the table below specific $\mathbb{Z}$-bases $\{b_1,b_2,b_3,b_4\}$ of these orders (which also can be found in [42, Proposition 12.3.2]). The quaternion algebra $\mathbb{H}_2$ also contains the order of *Lipschitz quaternions* $\mathcal{L}$ consisting of all integral linear combinations of the basis elements $1, i, j, k$.

	b_1	b_2	b_3	b_4
$\mathcal{L}$	1	i	j	k
$\mathcal{O}_2$	1	i	j	$\omega_2 = \frac{1+i+j+k}{2}$
$\mathcal{O}_3$	1	i	$\omega_3 = \frac{1+j}{2}$	$\frac{i+k}{2}$
$\mathcal{O}_5$	1	$\frac{1+i+j}{2}$	$\omega_5 = \frac{2+i-k}{4}$	$\frac{2+3i+k}{4}$

(3.20)

When R is a ring which is also freely generated as a $\mathbb{Z}$-module (e.g. R is an order) we define

$$\mathrm{inv}_R = \max\{|B \cap \mathcal{U}(R)| \mid B \text{ a } \mathbb{Z}\text{-module basis of } R\},$$

and for a finitely generated abelian group G one defines

$$\mathrm{rank}_{\mathbb{Z}}\, G = \max\{n \mid \mathbb{Z}^n \text{ is, up to isomorphism, a subgroup of } G\}.$$

Theorem 3.14. Let $\mathcal{O}$ be an order in a finite dimensional division $\mathbb{Q}$-algebra D with $\mathcal{U}(\mathcal{O})$ finite. Then,

$$\operatorname{rank}_{\mathbb{Z}} \mathrm{E}_2(\mathcal{O})^{ab} = \operatorname{rank}_{\mathbb{Z}} \mathcal{O} - \operatorname{inv}_{\mathcal{O}}. \tag{3.21}$$

Moreover, the following properties are equivalent:

(a) $\mathrm{E}_2(\mathcal{O})^{ab}$ is finite,

(b) $\mathcal{O}$ is isomorphic to $\mathbb{Z}, \mathcal{I}_1, \mathcal{I}_3, \mathcal{L}, \mathcal{O}_2$ or $\mathcal{O}_3$,

(c) $\mathcal{O}$ has a $\mathbb{Z}$-basis consisting of units of $\mathcal{O}$,

(d) $\mathcal{O}$ is generated as a ring by $\mathcal{U}(\mathcal{O})$,

(e) $\mathcal{O}$ is generated as a $\mathbb{Z}$-module by $\mathcal{U}(\mathcal{O})$.

Proof. Throughout, we will rely on Theorem 2.10. We start off by proving formula (3.21). We will use the description of $\mathrm{E}_2(\mathcal{O})^{ab}$ given in Theorem 3.10 and the additive subgroup M defined there. Since $3(1+1)(1+1) = 12 \in M$, and for any unit α holds that $3(\alpha+1)(1+1)+3(\alpha+1)(1+1) = 12\alpha+12 \in M$, we readily obtain that $12\alpha \in M$. Consequently, any unit of $\mathcal{O}$ has finite (additive) order in $(\mathcal{O}/M, +)$. As such, $\operatorname{rank}_{\mathbb{Z}} \mathrm{E}_2(\mathcal{O})^{ab} \leq \operatorname{rank}_{\mathbb{Z}} \mathcal{O} - \operatorname{inv}_{\mathcal{O}}$.

If $D = \mathbb{Q}$, then $\mathcal{O} = \mathbb{Z}$ and the statement is correct since $\mathrm{E}_2(\mathbb{Z})^{ab} \cong C_{12}$ is finite. If D is a quadratic imaginary extension of $\mathbb{Q}$, $\mathcal{O}$ is a free $\mathbb{Z}$-module of rank 2. Assume there exists a basis consisting of units for $\mathcal{O}$. Hence $\operatorname{rank}_{\mathbb{Z}} \mathcal{O} - \operatorname{inv}_{\mathcal{O}} = 2-2 = 0 \leq \operatorname{rank}_{\mathbb{Z}} \mathrm{E}_2(\mathcal{O})^{ab}$, showing that the inequality holds trivially. If not, then one may assume the existence of a base of $\mathcal{O}$ of the form $\{1, a\}$ with $a \notin \mathcal{U}(\mathcal{O})$. It is well known that in this case $\mathcal{U}(\mathcal{O}) = \{\pm 1\}$. The generators of type (1) of M (in Theorem 3.10) are then all equal to 0, and the generators of type (2), (3) and (4) are in $\mathbb{Z}$. As such, $12\mathbb{Z} \subseteq M \subseteq \mathbb{Z}$ and thus $\operatorname{rank}_{\mathbb{Z}} \mathrm{E}_2(\mathcal{O})^{ab} = \operatorname{rank}_{\mathbb{Z}}(\mathcal{O}/M, +) = 1 = \operatorname{rank}_{\mathbb{Z}} \mathcal{O} - \operatorname{inv}_{\mathcal{O}}$.

The last situation to consider is when D is a totally definite quaternion algebra over $\mathbb{Q}$. If $\mathcal{O}$ contains a basis of units, similar to before, the inequality is trivially satisfied. In particular, we may assume that $\mathcal{O}$ is not isomorphic to $\mathcal{L}, \mathcal{O}_2$ or $\mathcal{O}_3$. Hence, by [69, Theorem 11.5.12], the unit group $\mathcal{U}(\mathcal{O})$ is cyclic. We will denote the generator by β.

Clearly, the elements of the forms (2), (3) and (4) in Theorem 3.10 are in $\mathbb{Z}[\beta]$. For elements of the form (1) in Theorem 3.10 we do the following. Take any element $\gamma \in \mathcal{O}$ of norm 1. Then γ is a root of a polynomial $(X-\gamma)(X-\bar{\gamma}) = X^2 - tX + 1$ for $t = \gamma + \bar{\gamma} \in \mathcal{Z}(\mathcal{O}) = \mathbb{Z}$, so $\gamma^2 = t\gamma - 1$ and hence for every $x \in \mathcal{O}$

$$\begin{aligned}(\gamma x\gamma - x)\gamma = \gamma x\gamma^2 - x\gamma \;&=\; \gamma x(t\gamma - 1) - x\gamma \;=\; t\gamma x\gamma - \gamma x - x\gamma\\ &= (t\gamma - 1)x\gamma - \gamma x \;=\; \gamma^2 x\gamma - \gamma x \;=\; \gamma(\gamma x\gamma - x).\end{aligned}$$

Thus $\gamma x\gamma - x \in \mathrm{C}_{\mathcal{O}}(\gamma)$, the centralizer of γ in $\mathcal{O}$. A straightforward calculation shows that for $k \geq 2$

$$\gamma^k x\gamma^k - x = t\gamma^{k-2}(\gamma x\gamma - x)\gamma^{k-1} + \gamma^{k-2}x\gamma^{k-2} - x,$$

and hence, by induction on k, $\gamma^k x\gamma^k - x \in \mathrm{C}_{\mathcal{O}}(\gamma)$, for every k. If β has order 2, then $\beta = -1$ and $\mathbb{Z}[\beta] = \mathbb{Z}$. The generators in (1) are 0 and $M \subseteq \mathbb{Z}[\beta] = \mathbb{Z}$. Hence $\mathcal{O}/M$ is of rank at least 3 and a basis of $\mathcal{O}$ can only contain one unit. So the inequality

$$\operatorname{rank}_{\mathbb{Z}} \mathcal{O} - \operatorname{inv}_{\mathcal{O}} = 4 - 1 = 3 \leq \operatorname{rank}_{\mathbb{Z}}(\mathcal{O}/M, +) = \operatorname{rank}_{\mathbb{Z}} \mathrm{E}_2(\mathcal{O})^{ab},$$

holds.

If β has order larger than 2, then β necessarily is not central. As $\mathbb{Z} \subsetneq \mathbb{Z}[\beta] \subseteq \mathrm{C}_{\mathcal{O}}(\beta)$ we obtain in this case that $\operatorname{rank}_{\mathbb{Z}} \mathrm{C}_{\mathcal{O}}(\beta) = 2$ (else by tensoring up with $\mathbb{Q}$ this would mean that β is central in D). Furthermore, by the above, $\alpha x\alpha - x \in \mathrm{C}_{\mathcal{O}}(\beta)$ for every $\alpha \in \mathcal{U}(\mathcal{O})$ and every $x \in \mathcal{O}$. So we get that all generators of M are contained in $\mathrm{C}_{\mathcal{O}}(\beta)$. Hence $\mathcal{O}/M$ maps surjectively onto $\mathcal{O}/\mathrm{C}_{\mathcal{O}}(\beta)$ and therefore is of rank at least 2. Altogether,

$$\operatorname{rank}_{\mathbb{Z}} \mathcal{O} - \operatorname{inv}_{\mathcal{O}} = 4 - 2 = 2 \leq \operatorname{rank}_{\mathbb{Z}}(\mathcal{O}/M, +) = \operatorname{rank}_{\mathbb{Z}} \mathrm{E}_2(\mathcal{O})^{ab},$$

showing the inequality in the last case.

Now we prove that the statements $(a) - (e)$ are equivalent. To start, remark that (a) and (c) are equivalent due to formula (3.21). Hence it remains to prove that (b), (c), (d) and (e) are equivalent. First, for an order $\mathcal{O}$ in an imaginary quadratic number field $\mathcal{U}(\mathcal{O}) = \langle -1 \rangle$, unless $\mathcal{O} \in \{\mathcal{I}_1, \mathcal{I}_3\}$ (e.g. see [34, remark after Th. 240]). In those cases $\mathcal{U}(\mathcal{I}_1) = \langle i \rangle$ and $\mathcal{U}(\mathcal{I}_3) = \langle -\zeta_3 \rangle$, respectively. This implies that the last four conditions are equivalent in the case of orders in number fields with a finite unit group. Second assume that $\mathcal{O}$ is an order in a totally definite quaternion algebra with center $\mathbb{Q}$ and suppose $\mathcal{O}$ is isomorphic to

$\mathcal{L}, \mathcal{O}_2$ or $\mathcal{O}_3$. In all three cases there exists a basis consisting of units of $\mathcal{O}$ given in (3.20). Hence (b) implies (c). Clearly (c) implies (d) which implies (e). For (e) implies (b) note that if $\mathcal{O}$ is an order in a totally definite quaternion algebra with center $\mathbb{Q}$ not isomorphic to $\mathcal{L}, \mathcal{O}_2$ or $\mathcal{O}_3$, then, by [69, Theorem 11.5.12], $\mathcal{U}(\mathcal{O})$ is cyclic, generated by β, say. But then $\mathcal{U}(\mathcal{O})$ is contained in the commutative subring $\mathbb{Z}[\beta] \subseteq \mathcal{O}$, which has $\mathbb{Z}$-rank at most 2, since D is a quaternion algebra. □

Remark 3.15. One can filter from the proof of the previous theorem that when $\mathcal{O}$ is an order in a finite dimensional division $\mathbb{Q}$-algebra D with $\mathcal{U}(\mathcal{O})$ finite, then either $\mathcal{O}$ has a $\mathbb{Z}$-basis consisting of units or $|U(\mathcal{O})| \in \{2, 4, 6\}$.

We can also describe now the abelianization of $\mathrm{GL}_2(\mathcal{O})$ with $\mathcal{O}$ the norm Euclidean maximal orders in quaternion algebras introduced in table (3.20). Note that an element x in such an order $\mathcal{O}$ is a unit if and only if $\mathrm{N}(x) = x\bar{x} \in \mathbb{Z}_{\geq 0}$ equals 1. Then it is not hard to find the units from the description of the orders given in (3.20). If we set $\omega_2 = \frac{1+i+j+k}{2} \in \mathcal{O}_2$, $\omega_3 = \frac{1+j}{2} \in \mathcal{O}_3$ and $\omega_5 = \frac{2+i-k}{4} \in \mathcal{O}_5$, then we have

$$\begin{aligned} \mathcal{U}(\mathcal{O}_2) &= \langle i, \omega_2 \rangle \cong \mathrm{SL}(2,3) \cong Q_8 \rtimes C_3, \\ \mathcal{U}(\mathcal{O}_3) &= \langle i, \omega_3 \rangle \cong C_3 \rtimes C_4, \\ \mathcal{U}(\mathcal{O}_5) &= \langle \omega_5 \rangle \cong C_6. \end{aligned} \tag{3.22}$$

Corollary 3.16. $\mathrm{GL}_2(\mathcal{O}_2)^{ab} \cong C_3$, $\mathrm{GL}_2(\mathcal{O}_3)^{ab} \cong C_4$ and $\mathrm{GL}_2(\mathcal{O}_5)^{ab} \cong C_6$.

Proof. Since $\mathcal{O}_2, \mathcal{O}_3$ and $\mathcal{O}_5$ are Euclidean, they are GE_2-rings by Proposition 2.12. Now $\omega_2 \in \mathcal{U}(\mathcal{O}_2)$, $\omega_3 \in \mathcal{U}(\mathcal{O}_3)$ and $\omega_5 \in \mathcal{U}(\mathcal{O}_5)$ are elements of order 6. Hence, by Corollary 3.9, for $\mathcal{O}$ one of the three orders, $\mathrm{GL}_2(\mathcal{O})^{ab} = \mathrm{GE}_2(\mathcal{O})^{ab} \cong \mathcal{U}(\mathcal{O})^{ab}$. □

4 | PROPERTY HFA_{N-2} AND $\mathrm{HF}\mathbb{R}$ FOR $\mathrm{E}_N(R)$ IF $N \geq 3$

In this section we discuss properties FA and F$\mathbb{R}$ for the groups $\mathrm{E}_n(R)$, where R is a *unital ring* and $n \geq 3$. We prove fixed point properties on higher-dimensional CAT(0) cell complexes for the Steinberg groups $\mathrm{St}_n(R)$, where R is a finitely generated unital ring. This will eventually imply the respective properties for $\mathrm{E}_n(R)$.

For $i \neq j$, let $e_{ij}(r)$ denote the matrix, called *elementary matrix*, in $\mathrm{GL}_n(R)$ having 1 on the diagonal and r in the (i,j)-entry. Recall that $\mathrm{E}_n(R) = \langle e_{ij}(r) \mid 1 \leq i \neq j \leq n, r \in R \rangle$ denotes the elementary subgroup of $\mathrm{GL}_n(R)$. In case $n \geq 3$ it will turn out that the elementary matrices $\mathrm{E}_n(R)$ over a finitely generated ring do not only have global fixed points on simplicial trees but also on 'higher dimensional trees'. More precisely they will have property FA_{n-2} (in the sense of [26]).

Definition 4.1. A group Γ is said to have *property* FA_n if any isometric action, without inversion, on an n-dimensional CAT(0) cell complex has a fixed point.

For definitions and a more in-depth discussion of CAT(0) spaces and cell complexes, we refer the reader to [11, Chapter II]. This definition is indeed a generalization of FA since a simplicial tree is exactly a 1-dimensional CAT(0) cell complex. As such, FA and FA_1 are the same property. Similar to the classical notation, we will say a group has *property* HFA_n if every finite index subgroup has FA_n. Note that if a group has property FA_n for an $n \in \mathbb{N}$, then it has FA_m for every $n > m \in \mathbb{Z}_{\geq 1}$.

In [72, Theorem 1.2] Ye proved that, for a finitely generated ring R and $n \geq 3$, $\mathrm{E}_n(R)$ has property FA_{n-2} and in [25, Theorem 1.1] Ershov and Jaikin-Zapirain proved that it also has property (T), which we know to imply F$\mathbb{R}$ (see [20, Chapter 6., Proposition 11]). The purpose of this section is to prove the following result.

Theorem 4.2. Let $n \geq 3$. Let R be a unital ring which is finitely generated as $\mathbb{Z}$-module, then the group $\mathrm{E}_n(R)^{(m)}$ satisfies property F$\mathbb{R}$ and FA_{n-2} for each $m \geq 1$.

The groups $\mathrm{E}_n(R)^{(m)}$ are subgroups of $\mathrm{E}_n(R)$ that will suit our purposes to study hereditary fixed point properties. They are defined below, just before Theorem 4.6. Actually we will consider the so-called Steinberg groups $\mathrm{St}_n(R)$ and prove in Theorem 4.8 (and the remark thereafter) the above statement for these groups. The construction of $\mathrm{St}_n(R)$ is such that it maps onto $\mathrm{E}_n(R)$ and hence, since property F$\mathbb{R}$ and FA_{n-2} are preserved under quotients, $\mathrm{E}_n(R)$ will inherit these properties from $\mathrm{St}_n(R)$. From now on, throughout this section we assume $n \geq 3$.

Straightforward calculations show that over any ring the elementary matrices satisfy the following relations (where $(a,b) = a^{-1}b^{-1}ab$ is the multiplicative commutator).

Lemma 4.3. Let R be a ring. Then in $\mathrm{E}_n(R)$ we have that

$$\big(e_{kl}(s), e_{ij}(r)\big) = \begin{cases} 1 & \text{if } j \neq k \text{ and } i \neq l, \\ e_{il}(-rs) & \text{if } j = k \text{ and } i \neq l, \\ e_{kj}(sr) & \text{if } j \neq k \text{ and } i = l, \end{cases}$$

for $s, r \in R$ and $1 \leq i, j, k, l \leq n$ with $i \neq j$, $k \neq l$ and $|\{i, j, k, l\}| > 2$.

In general $\mathrm{E}_n(R)$ may satisfy more relations as those above and this deficiency can be quantified by introducing a kind of 'free model of $\mathrm{E}_n(R)$'.

Definition 4.4. Let $n \geq 3$ and J an ideal in R. The *Steinberg group* $\mathrm{St}_n(J)$ is the abstract group generated by the symbols $\{x_{ij}(r) \mid 1 \leq i \neq j \leq n, r \in J\}$ subject to the following relations:

$$x_{ij}(r)x_{ij}(s) = x_{ij}(r+s),$$
$$\big(x_{ij}(r), x_{kl}(s)\big) = 1 \text{ if } j \neq k \text{ and } i \neq l,$$
$$\big(x_{ij}(r), x_{jk}(s)\big) = x_{ik}(rs) \text{ for } i, j, k \text{ pairwise different},$$
$$\big(x_{ij}(r), x_{ki}(s)\big) = x_{kj}(-sr) \text{ for } i, j, k \text{ pairwise different}.$$

The indices will always be taken modulo n.

Clearly there is a natural epimorphism $\pi_n : \mathrm{St}_n(J) \to \mathrm{E}_n(J)$ defined by $\pi_n(x_{ij}(r)) = e_{ij}(r)$ and $\ker(\pi_n)$ measures 'how many' relations essentially different from those in Theorem 4.3 $\mathrm{E}_n(J)$ satisfies.

The proof of the version of Theorem 4.2 for the Steinberg groups consists in obtaining a 'nice' generating set in the sense of [26, Theorem 5.1]. Therefore we start now with providing a first smaller generating set. We will use the left-normed convention for the iterated commutator, i.e. inductively we define $(a_1, a_2, \dots, a_n) := ((a_1, a_2, \dots, a_{n-1}), a_n)$.

Lemma 4.5. Let $n \geq 3$. Let J be an ideal in R and let T_J and T be a set of ring generators of J and R, respectively. Then we have the following.

1. $\mathrm{St}_n(J) = \langle x_{ij}(t) \mid 1 \leq i \neq j \leq n, t \in T_J \rangle$.
2. If T contains 1 or generates R as a $\mathbb{Z}$-module, then
$$\mathrm{St}_n(R) = \langle x_{i,i+1}(r) \mid r \in T, 1 \leq i \leq n \rangle.$$
3. $\mathrm{St}_n(R)$ is a perfect group.

Proof. We first prove statement (1). Assume to begin with that J is generated as $\mathbb{Z}$-module by T_J. Let $r \in J$ be an arbitrary element and write $r = \sum_{s=1}^{k} a_s t_s$ for $a_s \in \mathbb{Z} \setminus \{0\}$ and $t_s \in T_J$. Then clearly $x_{ij}(r) = x_{ij}(t_1)^{a_1} \dots x_{ij}(t_k)^{a_k}$, proving the first part.

Since T_J generates J as ring, the set $\mathcal{T}_J$ consisting of finite products of elements of T_J generates J as $\mathbb{Z}$-module. By using the defining relations of $\mathrm{St}_n(J)$ and the case considered in the previous paragraph, we get

$$\mathrm{St}_n(J) = \langle x_{ij}(t) \mid 1 \leq i \neq j \leq n, t \in \mathcal{T}_J \rangle \leq \langle x_{ij}(t) \mid 1 \leq i \neq j \leq n, t \in T_J \rangle \leq \mathrm{St}_n(J).$$

To prove (2), we first assume that $1 \in T$. Let $S = \{x_{i,i+1}(r) \mid r \in T, 1 \leq i \leq n\}$. Recall that the indices are taken modulo n. According to (1), it suffices to show that S generates the $x_{ij}(t)$ for every $t \in T$. This is similar to what we did earlier:

$$x_{i,j}(t) = (x_{i,i+1}(t), x_{i+1,i+2}(1), \dots, x_{j-1,j}(1)),$$

an iterated commutator of elements of S. Now assume that T is an arbitrary generating set for R as $\mathbb{Z}$-module. Similar arguments as above can be used to express $x_{i,i+1}(1)$ as a product of the elements in S. Thus the previous argument can be applied.

Finally (3) follows immediately from the third defining relation of $\mathrm{St}_n(R)$. □

Let T be a generating set for a ring R as a $\mathbb{Z}$-module. We work with the following subgroups of $\mathrm{St}_n(R)$, for $m \in \mathbb{Z}_{\geq 1}$,

$$\mathrm{St}_n(R)^{(m)} := \langle x_{i,i+1}(r)^m \mid r \in T, 1 \leq i \leq n \rangle = \langle x_{i,i+1}(r) \mid r \in mT, 1 \leq i \leq n \rangle.$$

We will show (in Theorem 4.6) that this subgroup is well-defined, i.e. independent of the generating set T. Unfortunately if T is a set of ring generators of R the definition would in general depend on T. Note that $\mathrm{St}_n(R) = \mathrm{St}_n(R)^{(1)}$. The groups $\mathrm{E}_n(R)^{(m)}$ are analogously defined.

Lemma 4.6. Let $n \geq 3$ and m a non-zero positive integer. Then

1. the group $\mathrm{St}_n(R)^{(m)}$ is well-defined, i.e. independent of the generating set of R as additive group,
2. $\mathrm{St}_n(m^{n-1}R) \leq \mathrm{St}_n(R)^{(m)\prime} \leq \mathrm{St}_n(R)^{(m)} \leq \mathrm{St}_n(mR)$.

Proof. Let T and $\tilde{T}$ be generating sets for R as $\mathbb{Z}$-module. Similar to the proof of Theorem 4.5, it is clear that every element $x_{i,i+1}(r)^m = x_{i,i+1}(mr)$ for r in $\tilde{T}$ can be made from the elements $x_{i,i+1}(t)^m = x_{i,i+1}(mt)$ where $t \in T$, and vice versa. This proves (1).

For (2), note that the second inclusion is trivial. The rightmost inclusion follows from Theorem 4.5 applied to the ideal mR and generating set mT. From the same lemma it also follows that $\mathrm{St}_n(m^{n-1}R) = \langle x_{ij}(m^{n-1}r) \mid 1 \leq i \neq j \leq n, r \in T\rangle$. Using the defining relations as before, we obtain

$$x_{i,i+k}(m^{n-1}r) \;=\; (x_{i,i+1}(mr), x_{i+1,i+2}(m), \dots, x_{i+k-2,i+k-1}(m), x_{i+k-1,i+k}(m^{n-k})).$$

So, the elements that generate $\mathrm{St}_n(m^{n-1}R)$ can be constructed from the generators $x_{i,i+1}(r)^m = x_{i,i+1}(mr)$ of $\mathrm{St}_n(R)^{(m)}$ by taking commutators, which proves the remaining inclusion. □

Remark 4.7. The proof of Theorem 4.6 only uses the relations from Theorem 4.3 and hence the corresponding statements also hold for $\mathrm{E}_n(R)$.

We now have the necessary ingredients to prove the following fixed point properties for $\mathrm{St}_n(R)^{(m)}$. We were only recently informed that property FA_{n-2} had already been proven for the group $\mathrm{E}_n(R)$ in [72, Theorem 2.1] in case R is a finitely generated ring. In hindsight both proofs follow the same line and use [26, Theorem 5.1]. Note that the group $\mathrm{St}_n(R)^{(m)}$ is not the group generated by all the mth powers of the generators of the Steinberg group, but rather is a suitably chosen subgroup of the latter in order to be able to use [26, Theorem 5.1] and still derive the desired hereditary property.

Theorem 4.8. Let $n \geq 3$. Suppose R is finitely generated as $\mathbb{Z}$-module. Then the group $\mathrm{St}_n(R)^{(m)}$ satisfies properties $\mathrm{F}\mathbb{R}$ and FA_{n-2} for each $m \geq 1$.

Proof. Let T be a finite generating set of R as a $\mathbb{Z}$-module which we assume to contain 1.

By definition, $\mathrm{St}_n(R)^{(m)}$ is finitely generated by $S = \{x_{i,i+1}(r) \mid r \in mT, 1 \leq i \leq n\}$. By a theorem of Farb [26, Theorem 5.1], to prove FA_{n-2} it suffices to find a set of finitely generated nilpotent subgroups $C := \{\Gamma_1, \dots, \Gamma_n\}$ of $\mathrm{St}_n(R)^{(m)}$ such that

1. the group generated by the subgroups in C is of finite index in $\mathrm{St}_n(R)^{(m)}$,
2. any proper subset of C generates a nilpotent group,
3. there exists a positive integer z such that for all $1 \leq i \leq n$ and for all $r \in \Gamma_i$, there exists a nilpotent subgroup $N \leq \mathrm{St}_n(R)^{(m)}$ with $r^z \in N'$.

We define these groups to be the finitely generated abelian groups

$$\Gamma_i = \langle x_{i,i+1}(mr) \mid r \in T\rangle, \quad \text{for } 1 \leq i \leq n.$$

Clearly $\langle \Gamma_1, \dots, \Gamma_n\rangle = \mathrm{St}_n(R)^{(m)}$, so the first requirement is satisfied.

Let now $\widehat{\Gamma}_i$ be the group generated by the subgroups of $C \setminus \{\Gamma_i\}$. To prove (2), it is sufficient to prove that each $\widehat{\Gamma}_i$ is nilpotent. Clearly $\pi : \mathrm{St}_n(R) \to \mathrm{St}_n(R), x_{ij}(r) \mapsto x_{i+1,j+1}(r)$ is an isomorphism such that $\pi\left(\Gamma_i\right) = \Gamma_{i+1}$ and $\pi\left(\widehat{\Gamma}_i\right) = \widehat{\Gamma_{i+1}}$. Hence all $\widehat{\Gamma}_i$ are isomorphic to $\widehat{\Gamma}_n$. It is well known (see for example [59, Lemma 4.2.3]) that $\widehat{\Gamma}_n$ is nilpotent. Hence the second requirement is satisfied.

We will show the last requirement for r a generator of Γ_i and $z = m$. This is sufficient, since the Γ_i are finitely generated abelian groups. Consider $x_{i,i+1}(mt)^m = x_{i,i+1}(m^2t)$ with $t \in T$. Applying π^{2-i} to this element, it suffices to show the last requirement for $x_{2,3}(mt)^m = x_{2,3}(m^2t)$.

Using the defining relations we write

$$x_{2,3}(mt)^m = x_{2,3}(m^2t) = (x_{2,1}(m), x_{1,3}(mt)).$$

Now applying the isomorphism of $\mathrm{St}_n(mR)$ which interchanges in $\mathrm{St}_n(mR)$ the indices 1 and 2, $x_{2,1}(m)$ and $x_{1,3}(mt)$ are mapped to elements of $\widehat{\Gamma_n}$, a nilpotent group, proving the statement. Here we used that we may assume $1 \in T$ and thus m and $mr \in mT$. Hence conditions (1) to (3) are satisfied and we conclude that $\mathrm{St}_n(R)^{(m)}$ has property FA_{n-2}.

To prove that $\mathrm{St}_n(R)^{(m)}$ has property F$\mathbb{R}$, we will check that for every pair of generators $x_{i,i+1}(s)$ and $x_{j,j+1}(r)$ in S, their commutator $\left(x_{i,i+1}(s), x_{j,j+1}(r)\right)$ commutes with $x_{j,j+1}(r)$. This will indeed suffice, by a result of Culler and Vogtmann [18, Corollary 2.5] since $\mathrm{St}_n(R)^{(m)}$ already has finite abelianization (recall that it has property FA_{n-2} by the first part of the proof).

First, if $j \neq i+1$, $\left(x_{i,i+1}(s), x_{j,j+1}(r)\right) = 1$ which of course commutes with $x_{j,j+1}(r)$. So suppose now $j = i+1$, then $\left(x_{i,i+1}(s), x_{i+1,i+2}(r)\right) = x_{i,i+2}(sr)$ (here we used that $i \neq i+2$, or the fact that $n \neq 2$) which commutes with $x_{i+1,i+2}(r)$, proving the theorem. □

Remark 4.9. As a matter of fact, the reasoning in the previous proof also provides an alternative and elementary proof for the fact that $\mathrm{St}_n(R)$ has F$\mathbb{R}$ when R is finitely generated as a unital ring. Indeed, by taking $m = 1$ in the proof of F$\mathbb{R}$, we may provide the same argument when T generates R as a ring. This proof circumvents the use of the much more general result [25, Theorem 6.2] which states that $\mathrm{St}_n(R)$ satisfies property (T).

As explained earlier, since property FA_n and F$\mathbb{R}$ are preserved under quotients, Theorem 4.2 now follows from the previous theorem and remark.

5 | PROPERTY F$\mathbb{R}$ AND HF$\mathbb{R}$ FOR $\mathrm{E}_2(\mathcal{O})$

In this section we discuss properties FA, F$\mathbb{R}$ and HF$\mathbb{R}$ for the groups $\mathrm{E}_2(R)$, where R is a suitable ring (which will always at least again be associative and unital). Since not every $\mathrm{E}_2(R)$ has property FA, the situation is significantly different from the case $n \geq 3$ which was the setting of the previous section. For R an order in a simple $\mathbb{Q}$-algebra having a finite unit group, we classify exactly when $\mathrm{E}_2(R)$ has property FA and F$\mathbb{R}$. With a view on the latter we consider first Borel type subgroups.

It is well known when $\mathrm{SL}_2(\mathcal{I})$, for $\mathcal{I}$ a $\mathbb{Z}$-order in a field with finite unit group, has property FA. Indeed, by [63, Exercise I.6.5, pg 66] and [29, Theorems 2.1 and 2.4] the only such $\mathcal{I}$ for which $\mathrm{SL}_2(\mathcal{I})$ has F$\mathbb{R}$ (or equivalently FA) is $\mathcal{I}_3$.

The main goal of this section is to generalize this result to all orders $\mathcal{O}$ (not necessarily commutative) in division $\mathbb{Q}$-algebras with $\mathcal{U}(\mathcal{O})$ finite. We will prove the following theorem, which is also Theorem D from the introduction.

Theorem 5.1. Let $\mathcal{O}$ be an order in a finite dimensional division $\mathbb{Q}$-algebra with $\mathcal{U}(\mathcal{O})$ finite. Then the following properties are equivalent:

1. $\mathrm{E}_2(\mathcal{O})$ has property F$\mathbb{R}$,
2. $\mathrm{E}_2(\mathcal{O})$ has property FA,
3. $\mathcal{O}$ is isomorphic to $\mathcal{I}_3, \mathcal{O}_2$ or $\mathcal{O}_3$.

Furthermore, $\mathrm{GL}_2(\mathcal{O})$ has property F$\mathbb{R}$ if $\mathcal{O}$ has a basis of units and $\mathcal{O} \ncong \mathbb{Z}$.

The proof of Theorem 5.1 will be given later in Subsection 5.1 (on page 27) and will strongly require the results obtained in Subsection 3.3. Moreover we first need to understand the connections between $\mathrm{E}_2(\mathcal{O})$, the diagonal matrices therein and the Borel subgroup. The latter will be the content of Proposition 5.4. Next, in Subsection 5.2, we conjecture when $\mathrm{GE}_2(\mathcal{O})$ has property FA and F$\mathbb{R}$ and lay the first stone towards a proof by understanding completely the situation for the Borel subgroup $\mathrm{B}_2(\mathcal{O})$ of $\mathrm{GL}_2(\mathcal{O})$.

Remark 5.2. While in Section 4, property F$\mathbb{R}$ for $\mathrm{E}_n(\mathcal{O})$, $n \geq 3$, is a consequence of the same property for the Steinberg groups $\mathrm{St}_n(\mathcal{O})$, this is no longer true for the cases in Theorem 5.1. Indeed, if one defines $\mathrm{St}_2(\mathcal{O})$ in a similar way, then the only non-trivial defining relation is $x_{ij}(r)x_{ij}(s) = x_{ij}(r+s)$, hence $\mathrm{St}_2(\mathcal{O})$ is the free product of two copies of the additive group of $\mathcal{O}$ and hence cannot have property F$\mathbb{R}$.

5.1 Property F$\mathbb{R}$ for the groups $G_{R,K}$ with applications to F$\mathbb{R}$ for $\mathrm{E}_2(\mathcal{O})$

We will now investigate $\mathrm{E}_2(R)$ and $\mathrm{GE}_2(R)$ simultaneously by defining a more general type of groups, denoted $G_{R,K}$. Consider a subgroup K of $\mathrm{D}_2(R)$ (the group of invertible diagonal 2×2-matrices over the ring R; recall that we always assume our rings to be unital).

Definition 5.3. The group generated by K and $N = \left(\begin{smallmatrix} 1 & R \\ 0 & 1 \end{smallmatrix}\right)$ consisting of the unimodular upper triangular matrices will be denoted by $G_{R,K}$.

Note that for the choice $K = \mathrm{D}_2(R)$ we have that $G_{R,K}$ is the *Borel subgroup* $\mathrm{B}_2(R)$ of $\mathrm{GL}_2(R)$, i.e. the group consisting of invertible upper triangular 2×2-matrices over R.

Notation 1. If K consists of the matrices of the form $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$ with $\alpha\beta \in \mathcal{U}(R)'$ we will instead use the notation $\mathrm{DE}_2(R)$ for K and the notation $\mathrm{BE}_2(R)$ for $G_{R,K}$.

If R is almost-universal, using the determinant like map φ defined in (3.11), one can check that

$$\mathrm{BE}_2(R) = \mathrm{B}_2(R) \cap \mathrm{E}_2(R).$$

Indeed, if one restricts φ to the subgroup $\mathrm{B}_2(R)$, then its kernel coincides with $\mathrm{BE}_2(R) = \mathrm{B}_2(R) \cap \mathrm{E}_2(R)$. Also $\mathrm{DE}_2(R)$ equals $\langle\, D(\mu) \mid \mu \in \mathcal{U}(R) \,\rangle$ by (R8) on page 9, recall that $D(\mu)$ denoted the diagonal matrix $[\mu, \mu^{-1}]$. Note that the group $\mathrm{DE}_2(R)$ already appeared in Corollary 3.5.

Now note that $N = \left(\begin{smallmatrix} 1 & R \\ 0 & 1 \end{smallmatrix}\right)$ is a normal subgroup of $G_{R,K}$. Thus we have the following split short exact sequence

$$1 \longrightarrow N \longrightarrow G_{R,K} \longrightarrow G_{R,K}/N \cong K \longrightarrow 1. \tag{5.1}$$

Hence $G_{R,K}$ is isomorphic to the semi-direct product $N \rtimes_\alpha K$, where $\alpha : K \to \mathrm{Aut}(N)$ and

$$\alpha([u_1, u_2]) : N \to N : n \mapsto [u_1^{-1}, u_2^{-1}]\, n\, [u_1, u_2]$$

is conjugation by $[u_1, u_2]$. Furthermore, N is isomorphic to the additive group of R and hence abelian.

Remark that $\alpha(\lambda)$ for $\lambda \in K$, while in general an automorphism of the abstract group N, can here be considered as a matrix over $\mathbb{Z}$ since $(R, +) \cong N$ is a finitely generated free $\mathbb{Z}$-module by choosing an arbitrary basis of R. Via this identification one may speak of the *eigenvalues of* $\alpha(\lambda)$. Note that its eigenvalues are independent of the chosen basis.

Property FA of extensions (5.1) has been considered by Serre [63, I.6.5., Exercise 4] and Cornulier-Kar [16, Proposition 3.2] by means of sufficient group theoretical restrictions on $G_{R,K}$. We will now provide a linear algebra criterion which will turn out to be easy to check in our setting.

Proposition 5.4. Let R be a ring, which is finitely generated and free as $\mathbb{Z}$-module. Then the following properties hold:

1. If K is countable and has property F$\mathbb{R}$ (resp. FA) and there exists $\lambda \in K$ with finite order such that $\alpha(\lambda)$ (where α was defined above) has only non-rational eigenvalues, then $G_{R,K}$ has property F$\mathbb{R}$ (resp. FA).
2. If $G_{R,K}$ has property F$\mathbb{R}$ (resp. FA), then K has property F$\mathbb{R}$ (resp. FA).
3. Suppose R has a $\mathbb{Z}$-module basis consisting of units and $\mathrm{DE}_2(R) \leq K$. If $G_{R,K}$ has property F$\mathbb{R}$ (resp. FA), then also $\langle \mathrm{E}_2(R), K\rangle$ has property F$\mathbb{R}$ (resp. FA).

We will first need the following lemma which is inspired by [63, I.6.5., Exercise 4]. The exercise is about simplicial trees. We state the lemma for real trees and for the sake of completeness we provide a proof.

Lemma 5.5. Let B be a finitely generated group and $N \trianglelefteq B$ nilpotent and finitely generated. Suppose there is no subgroup M of N that is normal in B and such that $N/M \cong \mathbb{Z}$. Then B has property F$\mathbb{R}$ if B/N has property F$\mathbb{R}$.

Proof. We will in fact show that if B acts on a real tree X, then N has a fixed point on this tree. This implies of course that B has property F$\mathbb{R}$.

Clearly, if B acts on a real tree X, then N does so as well. Now from [17, Proposition 3.8] it follows that exactly one of the following happens:

- the action of N on X has a fixed point,
- there exists a *unique* line T in X, stable under the action of N, on which N acts by translation.

Suppose that the latter happens. Then we have a non-trivial morphism $\varphi : N \to \mathrm{Aut}(T) \cong Iso(\mathbb{R})$. Then T is also stable under the action of B. Indeed let $t \in T$, the invariant tree for N, and $g \in B$. Now $gt \in gT$ and for every $n \in N$ it holds that $n(gt) = (ng)t = (gn')t = g(n't) \in gT$ (for some $n' \in N$), hence gT is invariant under N and thus by the uniqueness $gT = T$, as needed.

We may thus extend the morphism above to a morphism $\varphi : B \to \mathrm{Aut}(T) \cong \mathrm{Iso}(\mathbb{R})$. Since B is finitely generated and $\mathrm{Iso}(\mathbb{R})$ consists of reflections and translations, it is easy to see that $\varphi(B)$ is isomorphic to $(\mathbb{Z}^n) \rtimes C_2$ or $\mathbb{Z}^n$, for some $n \in \mathbb{Z}_{\geq 1}$. Indeed, if the finite number of generators for $\varphi(B)$ are all translations, clearly $\varphi(B) \cong \mathbb{Z}^n$. If some of them are reflections, since a product of two reflections is a translation, one may change the generating set to only contain translations and 1 reflection. This reflection acts by inversion on the translations, so $\varphi(B) \cong \mathbb{Z}^n \rtimes C_2$ in this case. This also implies that every subgroup of $\mathbb{Z}^n$, the subgroup generated by translations, is normal in $\varphi(B)$. Moreover, since N is nilpotent and acts via translation on T, $\varphi(N) \cong \mathbb{Z}^k$ for some $n \geq k \in \mathbb{Z}_{\geq 1}$. All this implies that we may compose $\varphi|^{\varphi(B)}$, the corestriction of φ to $\varphi(B)$, with another morphism to obtain $\psi : B \to \mathbb{Z} \rtimes C_2$ such that $\psi(N) \cong \mathbb{Z}$ (for example, by modding out all the components of $\mathbb{Z}^n$ except for exactly one which has non-zero intersection with $\varphi(N)$). As such, there exists a normal subgroup H of B for which $N/(H \cap N) \cong \mathbb{Z}$, a contradiction with the assumptions. □

Proof of Proposition 5.4. The second statement immediately follows from the short exact sequence (5.1) and Proposition 2.7.

Assume now that K has property F$\mathbb{R}$ and that there exists $\lambda \in K$ such as in the first statement. To prove that $G_{R,K}$ has property F$\mathbb{R}$, we verify the conditions of Lemma 5.5. Assume that $N = \begin{pmatrix} 1 & R \\ 0 & 1 \end{pmatrix} \leq G_{R,K}$ has a subgroup M, normal in $G_{R,K}$ such that $N/M \cong \mathbb{Z}$. Take $H = \langle \lambda \rangle \leq K$. Then we may restrict α to H and consider $\mathbb{Q}[N] := \mathbb{Q} \otimes_{\mathbb{Z}} N$ as a $\mathbb{Q}H$-module. The subgroup M, being normal in $G_{R,K}$, is invariant under the action of the restriction of α. Thus under this identification $\mathbb{Q}[M] := \mathbb{Q} \otimes_{\mathbb{Z}} M$ is a $\mathbb{Q}H$-submodule of $\mathbb{Q}[N]$. Since H is finite, by Maschke's Theorem, $\mathbb{Q}[M]$ has to have a complement, i.e. there is a $\mathbb{Q}H$-submodule V of $\mathbb{Q}[N]$ such that $\mathbb{Q}[N] = \mathbb{Q}[M] \oplus V$ and then necessarily $\dim_{\mathbb{Q}} V = 1$ (since $N/M \cong \mathbb{Z}$). This means in particular that each of the matrices corresponding to an $\alpha(\mu)$, $\mu \in H$, has to have a rational eigenvalue. However, this is in contradiction with the assumptions.

Now, since $G_{R,K}/N \cong K$ has property F$\mathbb{R}$, the first statement follows from Lemma 5.5 if we show that $G_{R,K}$ is finitely generated. For this we need to show that K and N are finitely generated. For the latter let $\{r_1, \ldots, r_l\}$ be a finite $\mathbb{Z}$-basis of R, which exists by the assumptions on R, then $\{\begin{pmatrix} 1 & r_i \\ 0 & 1 \end{pmatrix} \mid 1 \leq i \leq l\}$ is a finite generating set of N. Also K is finitely generated due to Theorem 2.5.

Finally, assume that R has a $\mathbb{Z}$-module basis $\mathcal{B}$ consisting of units and that $G_{R,K}$ satisfies property F$\mathbb{R}$. Note that $\langle \mathrm{E}_2(R), K \rangle = \langle w, G_{R,K} \rangle$ where $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let T be a tree and assume $\langle \mathrm{E}_2(R), K \rangle$ acts on it. Now due to [63, I.6.5., Proposition 26][1], since w is of finite order (hence $\langle w \rangle$ has property F$\mathbb{R}$) and $G_{R,K}$ has property F$\mathbb{R}$, $\langle w, G_{R,K} \rangle$ has property F$\mathbb{R}$ if there exists a generating set $\mathcal{G}$ of $G_{R,K}$ such that wx has a fixed point for all $x \in \mathcal{G}$. For this purpose define $x_\mu = \begin{pmatrix} \mu^{-1} & 1 \\ 0 & \mu \end{pmatrix}$ for $\mu \in \mathcal{U}(R)$. Then,

$$\langle w, G_{R,K} \rangle = \langle w, K, N \rangle = \langle K, \mathrm{E}_2(R) \rangle = \langle w, x_\mu, K \mid \mu \in \mathcal{B} \rangle.$$

Indeed, $[\mu, \mu^{-1}]x_\mu = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$ and $\left\{\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} : \mu \in \mathcal{B}\right\}$ generates the subgroup N. It can easily be seen that the generating set $\mathcal{G} = \{x_\mu, K \mid \mu \in \mathcal{B}\}$ satisfies the condition of [63, I.6.5., Proposition 26]. Indeed,

$$wx_\mu = \begin{pmatrix} 0 & -\mu \\ \mu^{-1} & 1 \end{pmatrix}, \qquad (wx_\mu)^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This implies that wx_μ is of order 6 and hence has a fixed point. Next take $d = [\alpha, \beta] \in K$. Note that also $[\beta, \alpha] = [\alpha, \beta].[\alpha^{-1}\beta, \beta^{-1}\alpha] \in K$, where we used that $\mathrm{DE}_2(R) \leq K$. Consequently $(wd)^4 = \begin{pmatrix} (\beta\alpha)^2 & 0 \\ 0 & (\alpha\beta)^2 \end{pmatrix}$ is an element of K . Since $G_{R,K}$ has F$\mathbb{R}$ by assumption, K does so as well by the second statement. As now $(wd)^4 \in K$ has a fixed point on T, wd needs to have a fixed point as well since the group K does not act via inversions. More precisely, if $(wd)^4$ has a fixed point, but $(wd)^2$ does not, then this implies there is an action by inversion (see the paragraph before Definition 2.2), so $(wd)^2$ needs to have a fixed point and similarly wd needs to have a fixed point. For property F$\mathbb{R}$ the claim follows in a similar way[2].

Thus altogether we have proven that $\langle w, G_{R,K} \rangle = \langle \mathrm{E}_2(R), K \rangle$ has property F$\mathbb{R}$. The result for FA can be obtained similarly by taking T a simplicial tree and using [63, I.6.5., Exercise 4] instead of Lemma 5.5 to prove (1). □

[1]Note that Serre states [63, I.6.5., Proposition 26] for simplicial trees, but the proof stays exactly the same for real trees

[2]If $(wd)^4$ has a fixed point x, then the midpoint y between x and $(wd)^2(x)$ is a fixed point for $(wd)^2$, and similarly the midpoint between y and $(wd)(y)$ is a fixed point for wd.

Remark 5.6. Unfortunately, the converse of the third statement in Proposition 5.4 is not true. It fails already in the (trivial) case where $K = 1$. For this, consider any ring R for which $E_2(R)$ has F$\mathbb{R}$ (such as $\mathcal{I}_3$) and notice that $G_{R,1} \cong N$ is a finitely generated torsion-free abelian group. Also the converse of the first statement in Proposition 5.4 is not true as we will explain after Proposition 5.10.

We are finally ready to prove the main theorem of this section.

Proof of Theorem 5.1. Recall that property F$\mathbb{R}$ implies property FA which on its turn implies finite abelianization by Theorem 2.5. Thus thanks to Theorem 3.14 we need to understand for which isomorphism type $\{\mathbb{Z}, \mathcal{I}_1, \mathcal{I}_3, \mathcal{L}, \mathcal{O}_2, \mathcal{O}_3\}$ of $\mathcal{O}$ the group $E_2(\mathcal{O})$ has property F$\mathbb{R}$ or FA.

In the literature it was obtained that $E_2(\mathbb{Z})$ is isomorphic to the free product $SL_2(\mathbb{Z}) \cong C_4 *_{C_2} C_6$ (see [63, I, 4.2. (c)]) and also that $E_2(\mathcal{I}_1)$ and $E_2(\mathcal{L})$ have non-trivial amalgamated products (see respectively [27, Theorem 4.4.1] and [5, Theorem 7.8]). Therefore using again Theorem 2.5 we already know the implications (1) ⇒ (2) ⇒ (3).

Therefore it remains to prove that all the groups mentioned in the statement have property F$\mathbb{R}$. This will be achieved by verifying the conditions from Proposition 5.4. We will use the notations and results from Remark 3.13 throughout this proof without always explicitly mentioning it. We first claim the following.

Claim: The groups $BE_2(\mathcal{O}_2)$, $BE_2(\mathcal{O}_3)$, $BE_2(\mathcal{I}_3)$, $B_2(\mathcal{L})$ and $B_2(\mathcal{I}_1)$ have property F$\mathbb{R}$ as the eigenvalue condition from Proposition 5.4 (1) is fulfilled.

Once the claim is established Proposition 5.4 (3) implies that also $E_2(\mathcal{O}_2)$, $E_2(\mathcal{O}_3)$, $E_2(\mathcal{I}_3)$, $GE_2(\mathcal{L})$ and $GE_2(\mathcal{I}_1)$ have property F$\mathbb{R}$. In particular we would have proven (3) implies (1), as needed.

Proof of the claim: to check the condition of the existence of an element $\lambda \in K$ of finite order such that $\alpha(\lambda)$ has no rational eigenvalue (where $K = D_2(\mathcal{O})$ or $K = DE_2(\mathcal{O})$) it suffices to calculate the impact of it to a basis of $N = \left(\begin{smallmatrix} 1 & \mathcal{O} \\ 0 & 1 \end{smallmatrix}\right)$. By fixing a basis of the $\mathbb{Z}$-module N, we identify $\mathrm{Aut}(N) \cong GL(2, \mathbb{Z})$ or $\mathrm{Aut}(N) \cong GL(4, \mathbb{Z})$ respectively.

We will first carry out the proof for $\mathcal{L}$, the Lipschitz quaternions and $\mathcal{O}_2$, the Hurwitz quaternions. The Lipschitz quaternions have a basis $\{1, i, j, k\}$ consisting of units and $\mathcal{U}(\mathcal{L}) \cong Q_8$. Take $\lambda_1 = [i, 1] \in D_2(\mathcal{L})$. Then $\alpha(\lambda_1)$ is just left multiplication by $-i$ on $\mathcal{L}$ and this has (complex) eigenvalues $i, -i$ both with multiplicity 2, in particular it does not have any rational eigenvalue. $\mathcal{O}_2$ has a basis $\{1, i, j, \omega\}$ consisting of units and $\mathcal{U}(\mathcal{O}_2) = \langle i, \omega \rangle \cong Q_8 \rtimes C_3$, so $\mathcal{U}(\mathcal{O})' = \langle i, j \rangle$ (see table (3.20) and (3.22)). If we set $\lambda_2 = [i, 1]$, then in this case even $\lambda_2 \in DE_2(\mathcal{O}_2)$ and $\alpha(\lambda_2)$ has the same eigenvalues as $\alpha(\lambda_1)$, hence none of them is rational.

Also the rings of integers $\mathcal{I}_1$ in $\mathbb{Q}(\sqrt{-1})$ and $\mathcal{I}_3$ in $\mathbb{Q}(\sqrt{-3})$, considered as $\mathbb{Z}$-module, have a basis consisting of units. Indeed one can take $\{1, i\}$ and $\{1, \frac{1+\sqrt{-3}}{2}\}$ respectively. Also here the non-rational eigenvalue condition is satisfied, but the matrices we use are $[i, 1]$ and $[\frac{1+\sqrt{-3}}{2}, \left(\frac{1+\sqrt{-3}}{2}\right)^{-1}]$. They are both in D_2 and in the last case even in DE_2 of their respective orders.

Now consider the maximal order $\mathcal{O}_3$ in $\left(\frac{-1,-3}{\mathbb{Q}}\right)$. Take $\omega_3 = \frac{1+j}{2} \in \mathcal{O}_3$ and note that $\omega_3^6 = 1$. Then $\mathcal{O}_3$ has a basis $\{1, i, \omega_3, i\omega_3\}$ consisting of units. Set $\tau = \omega_3^2$, a unit of order 3, then $\mathcal{U}(\mathcal{O}_3) = \langle \tau, i \rangle \cong C_3 \rtimes C_4$ and $\mathcal{U}(\mathcal{O}_3)' = \langle \tau \rangle$. Then $\lambda_3 = [\tau, 1] \in DE_2(\mathcal{O}_3)$ and $\alpha(\lambda_3)$ has eigenvalues ζ_3 and ζ_3^2, both with multiplicity 2, where ζ_3 denotes a complex primitive third root of unity.

So from Proposition 5.4 (1) the claim follows.

Finally, since $E_2(\mathcal{O}_2)$, $E_2(\mathcal{O}_3)$ and $E_2(\mathcal{I}_3)$ are of finite index in the GE_2 of the respective rings (see Proposition 3.4), by Proposition 2.7, also the GE_2's of these orders have property F$\mathbb{R}$. Since $\mathcal{O}_2$, $\mathcal{O}_3$, $\mathcal{I}_1$ and $\mathcal{I}_3$ are left Euclidean rings, $GL_2 = GE_2$ by Proposition 2.12. On the other hand, $\mathcal{L}$ is neither right nor left Euclidean, but one can still directly prove it to be a GE_2-ring (see [5, Proposition 7.10]). Thus the last line of the statement follows by Theorem 3.14. □

Next, we join all the pieces in order to proof that $E_2(\mathcal{O})$ always contains a subgroup of finite index not enjoying property FA.

Theorem 5.7. Let $\mathcal{O}$ be an order in a finite dimensional division $\mathbb{Q}$-algebra with $\mathcal{U}(\mathcal{O})$ finite. Suppose $\mathcal{O} \ncong \mathcal{O}_3$. Then $E_2(\mathcal{O})$ does not satisfy property HFA. In particular also $GE_2(\mathcal{O})$ does not satisfy HFA.

Proof. If $E_2(\mathcal{O})$ has property FA, then by Theorem 5.1, $\mathcal{O}$ is isomorphic to $\mathcal{I}_3$, $\mathcal{O}_2$ or $\mathcal{O}_3$. It remains to prove that $E_2(\mathcal{I}_3)$ and $E_2(\mathcal{O}_2)$ do not satisfy property HFA. We will do this by exhibiting concrete subgroups of finite index not having property FA.

To start we claim that $E_2(\mathbb{Z}[\sqrt{-3}])$ is a subgroup of finite index in $E_2(\mathcal{I}_3)$ with infinite abelianization. Indeed $\mathbb{Z}[\sqrt{-3}]$ is a GE_2-ring [21] and hence $E_2(\mathbb{Z}[\sqrt{-3}]) = SL_2(\mathbb{Z}[\sqrt{-3}])$ which is of finite index in $SL_2(\mathcal{I}_3) = E_2(\mathcal{I}_3)$ because $GL_2(\mathbb{Z}[\sqrt{-3}])$ is of finite index in $GL_2(\mathcal{I}_3)$ using that $\mathcal{I}_3$ is an Euclidean ring. By Theorem 3.14, $E_2(\mathbb{Z}[\sqrt{-3}])$ has infinite abelianization.

Finally in [5, Theorem 7.8] it is proven that $E_2(\mathcal{L})$ is a subgroup of finite index in $E_2(\mathcal{O}_2)$ with a non-trivial decomposition as amalgamated product and thus does not have property FA. □

Remark 5.8. It is possible to prove the same statement as in Theorem 5.7 for the group $\mathrm{SL}_2(\mathcal{O})$ for $\mathcal{O}$ an order in a finite dimensional $\mathbb{Q}$-algebra with $\mathcal{U}(\mathcal{O})$ finite, via geometric methods. Indeed $\mathrm{SL}_2(\mathcal{O})$ has a discontinuous action on the hyperbolic space $\mathbb{H}^3$ or $\mathbb{H}^5$ of dimension 3 or 5. One can construct a reflection acting on this hyperbolic space, and a congruence subgroup Γ of $\mathrm{SL}_2(\mathcal{O})$ which is normalized by the latter reflection. Then by [51, Corollary 3.6], Γ has a virtually free quotient. The latter implies that Γ has a finite index subgroup with infinite abelianization. As Γ has finite index in $\mathrm{SL}_2(\mathcal{O})$, this proves the result. Note that for this method, the order $\mathcal{O}_3$ does not have to be excluded. Moreover, as $\mathrm{E}_2(\mathcal{O}_3)$ has finite index in $\mathrm{SL}_2(\mathcal{O}_3)$, this also shows that the condition $\mathcal{O} \ncong \mathcal{O}_3$ is not necessary in Theorem 5.7. However, note that $\mathrm{E}_2(\mathcal{O})$ is not of finite index in $\mathrm{SL}_2(\mathcal{O})$ for any order $\mathcal{O}$ such that $\mathcal{U}(\mathcal{O})$ is finite, [56]. Hence this remark does not yield an alternative proof of Theorem 5.7.

5.2 | Property F$\mathbb{R}$ for the Borel subgroup with a view on $\mathrm{GE}_2(\mathcal{O})$

Now it is logical to ask, in the same setting as Theorem 5.1, when $\mathrm{GE}_2(\mathcal{O})$ has property FA. We expect a similar theorem to be true.

Question 5.9. Let $\mathcal{O}$ be an order in a finite dimensional division $\mathbb{Q}$-algebra D with $\mathcal{U}(\mathcal{O})$ finite. Are the following properties equivalent?

1. $\mathrm{GE}_2(\mathcal{O})$ has property F$\mathbb{R}$,
2. $\mathrm{GE}_2(\mathcal{O})$ has no non-trivial decomposition as an amalgamated product,
3. $\mathcal{O}$ isomorphic to $\mathcal{I}_1, \mathcal{I}_3, \mathcal{L}, \mathcal{O}_2$ or $\mathcal{O}_3$.

In case that D is a field and $\mathcal{O}$ is not isomorphic to $\mathcal{I}_1$ and $\mathcal{I}_3$ it is proven in [5] that $\mathrm{GE}_2(\mathcal{O})$ indeed has a non-trivial decomposition as an amalgamated product. Hence combined with Theorem 5.1, using that $\mathcal{I}_1$ and $\mathcal{I}_3$ are GE_2-rings, we see that the above question is indeed true for D a field. For the general case, the missing fact is that $\mathrm{GE}_2(\mathcal{O})$ for $\mathcal{O}$ an order in a totally definite quaternion algebra of the form $\left(\frac{a,b}{\mathbb{Q}}\right)$ only has property FA for the orders $\mathcal{L}, \mathcal{O}_2$ and $\mathcal{O}_3$. To achieve this, in view of the proof of Theorem 5.1, it is natural to first fully understand the situation for $\mathrm{B}_2(\mathcal{O})$.

Proposition 5.10. Let $\mathcal{O}$ be an order in a finite dimensional division $\mathbb{Q}$-algebra D with $\mathcal{U}(\mathcal{O})$ finite. Then the following properties are equivalent:

1. $\mathrm{B}_2(\mathcal{O})$ has property F$\mathbb{R}$,
2. $\mathrm{B}_2(\mathcal{O})$ has property FA,
3. $\mathrm{B}_2(\mathcal{O})$ has no non-trivial decomposition as an amalgamated product,
4. $\mathcal{U}(\mathcal{O}) \ncong C_2$.

Proof. If $\mathrm{B}_2(\mathcal{O})$ has property F$\mathbb{R}$, then it has also property FA and consequently, by Serre's algebraic characterisation, it cannot be an amalgamated product.

Next, by contraposition, suppose $\mathcal{U}(\mathcal{O}) \cong C_2$ and write $\mathcal{U}(\mathcal{O}) = \langle u : u^2 = 1\rangle$ and $\mathrm{B}_2(\mathcal{O}) = \begin{pmatrix} \langle u\rangle & \mathcal{O} \\ 0 & \langle u\rangle \end{pmatrix}$ is isomorphic to $(\mathcal{O},+) \rtimes (C_2 \times C_2)$, where the action of $(u,1)$ and $(1,u)$ on $(\mathcal{O},+)$ is via taking the opposite (i.e. sends $x \in \mathcal{O}$ on $-x$). Furthermore $(\mathcal{O},+)$ is a free $\mathbb{Z}$-module of rank $1, 2$ or 4. Thus the group $\mathrm{B}_2(\mathcal{O})$ clearly has an epimorphism to $\mathbb{Z} \rtimes C_2 \cong D_\infty \cong C_2 * C_2$. Since this last group is a free product, also $\mathrm{B}_2(\mathcal{O})$ has a non-trivial amalgamated decomposition.

There only remains one implication to be checked. So suppose that $\mathcal{U}(\mathcal{O}) \ncong C_2$. We will prove that $\mathrm{B}_2(\mathcal{O})$ has property F$\mathbb{R}$. For this we will use Lemma 5.5, applied to the group $\mathrm{B}_2(\mathcal{O})$ with $N \cong (\mathcal{O},+)$ the free abelian subgroup of unimodular upper triangular matrices. Since $\mathrm{B}_2(\mathcal{O})/N \cong \mathcal{U}(\mathcal{O}) \times \mathcal{U}(\mathcal{O})$ has property F$\mathbb{R}$ (indeed, it is finite), it will suffice to prove there is no subgroup M of N which is normal in $\mathrm{B}_2(\mathcal{O})$ and such that $N/M \cong \mathbb{Z}$.

Suppose such an M does exist. Let M' be the subgroup of the additive group of $\mathcal{O}$ such that $M = \{\left(\begin{smallmatrix} 1 & y \\ 0 & 1\end{smallmatrix}\right) \mid y \in M'\}$. Now by assumption and Theorem 2.10, D is equal to $\mathbb{Q}$, $\mathbb{Q}(\sqrt{-d})$ with $d > 0$ or a totally definite quaternion algebra $\left(\frac{a,b}{\mathbb{Q}}\right)$. Therefore, by Remark 3.15, either $\mathcal{O}$ has a $\mathbb{Z}$-module basis consisting of units or $\mathcal{U}(\mathcal{O})$ is isomorphic to C_2, C_4 or C_6. In the former case $\mathrm{B}_2(\mathcal{O})$ has property F$\mathbb{R}$ as proven in Theorem 5.1 (see the claim in its proof). So we may now suppose that $\mathcal{U}(\mathcal{O})$ is isomorphic to C_4 or C_6. First, for an order $\mathcal{O}$ in an imaginary quadratic number field $\mathcal{U}(\mathcal{O}) = \langle -1\rangle$, unless $\mathcal{O} \in \{\mathcal{I}_1, \mathcal{I}_3\}$ and in these cases $\mathcal{O}$ has a basis of units. Therefore it remains to consider the case where $\mathcal{O}$ is an order in $\left(\frac{a,b}{\mathbb{Q}}\right)$ with $a, b < 0$. Assume α is a generator of $\mathcal{U}(\mathcal{O})$. Notice that $\mathbb{Z}[\alpha] \cong \mathbb{Z}[i]$ or $\mathbb{Z}[\zeta_3]$ where ζ_3 is a primitive third root of unity. In both cases, $\mathbb{Z}[\alpha]$ is a principal ideal domain and $\mathcal{O}$ is a finitely generated torsion-free $\mathbb{Z}[\alpha]$-module. Using the fundamental theorem of finitely generated modules over PID's, we obtain that $\mathcal{O} = \mathbb{Z}[\alpha] \oplus b\mathbb{Z}[\alpha]$, for some $b \in \mathcal{O}$. Hence we obtain a $\mathbb{Z}$-basis $\{1, \alpha, b, b\alpha\}$ for $\mathcal{O}$.

We will now go through the proof in the case α is of order 6, but the order 4 case is similar.

Since M is normal in $\mathrm{B}_2(\mathcal{O})$, taking the conjugate with $\begin{pmatrix} \beta^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}$ for $\beta \in \mathcal{U}(\mathcal{O})$, yields $\beta M' \subseteq M'$ and $M'\beta \subseteq M'$ respectively. Additionally, since α is of order 6, $\alpha^2 = \alpha - 1$. We will use these facts throughout.

Our first claim is that $\mathbb{Z} + \mathbb{Z}\alpha \subseteq M'$, or equivalently $M' \cap \langle \alpha \rangle \neq \{0\}$. Indeed, if we suppose the opposite, namely for all non-zero $\beta \in \langle \alpha \rangle$ that $\beta \notin M'$, then also for all $r \in \mathbb{Z} \setminus \{0\}$ and $\beta \in \langle \alpha \rangle$ the element $r\beta \notin M'$ (else N/M is no longer torsion free). However, since $N/M \cong \mathbb{Z}$, then we may find some integers $n, m \in \mathbb{Z} \setminus \{0\}$ such that $m\alpha \equiv n1 \mod M'$. This would imply that $m\alpha - n \in M'$, but then also $(m\alpha - n)\alpha = m(\alpha - 1) - n\alpha = (m-n)\alpha - m \in M'$. This shows that

$$m(m\alpha - n) - n((m-n)\alpha - m) = (m^2 - nm + n^2)\alpha \in M'.$$

Since $m^2 - nm + n^2 \neq 0$, this yields a contradiction.

Suppose now that $b \in M'$. Then also $b\alpha \in M'$ and thus a whole basis of $\mathcal{O}$ is in M'. This contradicts $N/M \cong \mathbb{Z}$. Similarly, suppose $b\alpha \in M'$. Then $b\alpha^2 = b\alpha - b \in M'$, which implies $b \in M'$. This gives again a contradiction. Hence we have that $b \notin M'$ and $b\alpha \notin M'$ and thus also $rb \notin M'$ and $rb\alpha \notin M'$ for every $r \in \mathbb{Z} \setminus \{0\}$, for else N/M would not be torsion free. In the same way as above, we again find two integers $n, m \in \mathbb{Z} \setminus \{0\}$ such that $mb \equiv nb\alpha \mod M'$. By a similar calculation, this gives again a contradiction. This shows that the set M' does not exist and hence also M does not exist. So Lemma 5.5 finishes the proof. □

From Proposition 5.10 we see that also for $\mathrm{B}_2(\mathcal{O})$ property F$\mathbb{R}$ and FA are equivalent. Furthermore we see that $\mathrm{B}_2(\mathcal{O}_5)$ has property F$\mathbb{R}$. However it can be directly checked that there exists no $\lambda \in \mathrm{D}_2(\mathcal{O}_5)$ of finite order such that $\alpha(\lambda)$ has only non-rational eigenvalues. So $\mathcal{O}_5$ yields a counterexample to the converse of the first statement in Proposition 5.4.

6 | FIXED POINT PROPERTIES, EXCEPTIONAL COMPONENTS AND CUT GROUPS

For the remainder of the paper G *will denote a finite group.* In the sequel we aim at describing property FA and HFA for $\mathcal{U}(\mathbb{Z}G)$ both in terms of G and the Wedderburn-Artin components of $\mathbb{Q}G$. In Proposition 6.1 we will see that if $\mathcal{U}(\mathbb{Z}G)$ has FA, then G must be a so-called cut group. Therefore in Section 6.3 we investigate the possible simple algebras $\mathrm{M}_n(D)$ that arise as a component of $\mathbb{Q}G$ for G a cut group.

Earlier we recalled the concepts of reduced norm and SL_1 for a subring of a central simple algebra in Section 2.2. In this part we will frequently need the notion $\mathrm{SL}_1(R)$ for R a subring in a semisimple $\mathbb{Q}$-algebra A. Let $A = \prod \mathrm{M}_{n_i}(D_i)$ be the Wedderburn-Artin decomposition of A and h_i the projections onto the i-th component. Then

$$\mathrm{SL}_1(R) := \{\ a \in R \mid \forall\, i :\ \mathrm{RNr}_{\mathrm{M}_{n_i}(D_i)/\mathcal{Z}(D_i)}(h_i(a)) = 1\ \}.$$

6.1 | FA and cut groups

We start by proving that the size of $\mathcal{U}(\mathbb{Z}G)^{ab}$ restricts the size of the center. More generally the following is true. As before the rank of a finitely generated abelian group A means the rank of its free part and will be denoted by $\mathrm{rank}_{\mathbb{Z}}$.

Proposition 6.1. Let $\mathcal{O}$ be an order in a finite dimensional semisimple $\mathbb{Q}$-algebra A. Then

$$\mathrm{rank}_{\mathbb{Z}}\left(\mathcal{U}(\mathcal{O})^{ab}\right) \geq \mathrm{rank}_{\mathbb{Z}}\left(\mathcal{U}(\mathcal{Z}(\mathcal{O}))\right).$$

Proof. For any n one can embed $\mathrm{GL}_n(\mathcal{O})$ into $\mathrm{GL}_{n+1}(\mathcal{O})$ by sending $B \in \mathrm{GL}_n(\mathcal{O})$ to $\left(\begin{smallmatrix} B & 0 \\ 0 & 1 \end{smallmatrix}\right)$. So (the direct limit) $\mathrm{GL}(\mathcal{O}) = \bigcup \mathrm{GL}_n(\mathcal{O})$ is equipped with the obvious group structure whose identity element we denote I_∞. Recall that $K_1(\mathcal{O}) := \mathrm{GL}(\mathcal{O})^{ab}$ and let

$$i : \mathcal{U}(\mathcal{O}) \to K_1(\mathcal{O}) : u \mapsto \left(e_{11}(u-1) + I_\infty\right)\mathrm{GL}(\mathcal{O})'$$

be the canonical map, where $e_{11}(u-1)$ is the matrix with the value $u-1$ in the entry $(1,1)$ and zero elsewhere. So $i(u)$ is the image inside $K_1(\mathcal{O})$ of the $\mathbb{N} \times \mathbb{N}$-identity matrix but with value u instead of 1 at place $(1,1)$.

By [42, Corollary 9.5.10], $i(\mathcal{U}(\mathcal{Z}(\mathcal{O})))$ is of finite index in $K_1(\mathcal{O})$. In particular also $i(\mathcal{U}(\mathcal{O}))$ has finite index. Since $K_1(\mathcal{O})$ is abelian, $\mathcal{U}(\mathcal{O})' \subseteq \ker(i)$ and we have an induced map $\bar{i} : \mathcal{U}(\mathcal{O})^{ab} \to K_1(\mathcal{O})$ whose image is still of finite index in $K_1(\mathcal{O})$. Therefore $\operatorname{rank}_{\mathbb{Z}}\left(\mathcal{U}(\mathcal{O})^{ab}\right) \geq \operatorname{rank}_{\mathbb{Z}}\left(K_1(\mathcal{O})\right)$. The statement now follows by [42, Corollary 9.5.10] which says that $\operatorname{rank}_{\mathbb{Z}}\left(K_1(\mathcal{O})\right) = \operatorname{rank}_{\mathbb{Z}}\left(\mathcal{U}(\mathcal{Z}(\mathcal{O}))\right)$. □

Due to Proposition 6.1 and Theorem 2.5, if $\mathcal{U}(\mathbb{Z}G)$ has FA, then $\mathcal{U}(\mathcal{Z}(\mathbb{Z}G))$ is finite. The latter is the content of so-called cut groups, a class of groups that was studied in its own right (the term "cut" was introduced in [7]).

Definition 6.2. A finite group G is called a *cut group*, if $\mathcal{U}(\mathcal{Z}(\mathbb{Z}G))$ is finite.

The word cut derives from "central units of the integral group ring trivial". In fact by a classical theorem of Berman and Higman [42, Proposition 7.1.4] each central unit in $\mathbb{Z}G$ not in $\pm\mathcal{Z}(G)$ has infinite order. Hence if $\mathcal{U}(\mathcal{Z}(\mathbb{Z}G))$ is finite, all central units must be trivial (i.e. in $\pm\mathcal{Z}(G)$).

Corollary 6.3. Let G be a finite group such that $\mathcal{U}(\mathbb{Z}G)$ has finite abelianization. Then G is a cut group.

This also implies that when $\mathcal{U}(\mathbb{Z}G)$ has FA, then G is a cut group. The converse is however not true as the following example shows.

Example 6.4. Denote the subgroup of units with augmentation one of $\mathbb{Z}G$ by $\mathrm{V}(\mathbb{Z}G)$. Then $\mathcal{U}(\mathbb{Z}S_3) = \pm\mathrm{V}(\mathbb{Z}S_3)$ and $\mathrm{V}(\mathbb{Z}S_3) = \langle s,t,b \mid s^2, t^3, t^s = t^{-1}, b^s = b^{-1}\rangle$ (see for example [52]). Clearly the latter group is an amalgamated product of the groups $\langle b,s\rangle \cong C_\infty \rtimes C_2 = C_2 * C_2$ and $\langle s,t\rangle = S_3$ over the subgroup $\langle s\rangle \cong C_2$. So, also $\mathcal{U}(\mathbb{Z}S_3)$ has a non-trivial decomposition as amalgamated product and thus does not have property FA. On the other hand, the amalgamated subgroup is finite, this shows that S_3 is a cut group.

For a finite group G the rational group algebra $\mathbb{Q}G$ is semisimple and thus has a Wedderburn-Artin decomposition $\mathbb{Q}G \cong \prod_{i=1}^m \mathrm{M}_{n_i}(D_i)$, where all the D_i are rational division algebras. If $\mathcal{O}_i$ is an order in D_i, then $\mathbb{Z}G$ and $\prod_{i=1}^m \mathrm{M}_{n_i}(\mathcal{O}_i)$ are both orders in $\mathbb{Q}G$ and $\mathcal{Z}(\mathbb{Z}G)$ and $\prod_{i=1}^m \mathcal{Z}(\mathrm{M}_{n_i}(\mathcal{O}_i)) \cong \prod_{i=1}^m \mathcal{Z}(\mathcal{O}_i)$ are orders in $\mathcal{Z}(\mathbb{Q}G)$. Hence Proposition 2.8 implies the following fact that we will use in the sequel without further reference: G is a cut group if and only if all the centres $\mathcal{Z}(\mathcal{O}_i)$ have a finite unit group, that is, $\mathcal{Z}(D_i)$ is the field of rational numbers or an imaginary quadratic extension of $\mathbb{Q}$ (cf. Theorem 2.10).

It would be particularly interesting whether equality holds in Proposition 6.1. By [42, Lemma 9.5.6] for two orders $\mathcal{O}_1, \mathcal{O}_2$ in a finite dimensional semisimple $\mathbb{Q}$-algebra A, $K_1(\mathcal{O}_1)$ and $K_1(\mathcal{O}_2)$ are commensurable. Furthermore, as stated above, $\operatorname{rank}_{\mathbb{Z}}(\mathcal{U}(\mathcal{Z}(\mathcal{O}_1))) = \operatorname{rank}_{\mathbb{Z}}(K_1(\mathcal{O}_1))$, and similarly for $\mathcal{O}_2$, and hence for orders "having a center with finitely many units" is a property defined on commensurability classes. In particular if equality in Proposition 6.1 holds, one would also have a positive answer to the following question.

Question 6.5. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two orders in a finite dimensional semisimple $\mathbb{Q}$-algebra. Is $\mathcal{U}(\mathcal{O}_1)^{ab}$ finite if and only if $\mathcal{U}(\mathcal{O}_2)^{ab}$ is finite?

As in general, finite abelianization and property FA do not descend to subgroups of finite index, a positive answer to the above question cannot be given right away. In contrast, property HFA does descend to subgroups of finite index and therefore the following holds.

Proposition 6.6. Let G be a finite group, $\mathbb{Q}G \cong \prod_{i=1}^{m} \mathrm{M}_{n_i}(D_i)$ the Wedderburn-Artin decomposition of its rational group algebra $\mathbb{Q}G$ and $\mathcal{O}_i$ an order in D_i. Then the following properties are equivalent:

1. $\mathcal{U}(\mathbb{Z}G)$ has property HF$\mathbb{R}$ (resp. HFA),
2. $\mathrm{GL}_{n_i}(\mathcal{O}_i)$ has property HF$\mathbb{R}$ (resp. HFA) for all $1 \leq i \leq m$,
3. $\mathrm{SL}_{n_i}(\mathcal{O}_i)$ has property HF$\mathbb{R}$ (resp. HFA) for all $1 \leq i \leq m$ and G is a cut group.

Proof. We prove the equivalences for property HF$\mathbb{R}$. The proofs for property HFA are the same.

First note that $\mathbb{Z}G$ and $\prod_{i=1}^m \mathrm{M}_{n_i}(\mathcal{O}_i)$ are both orders in $\mathbb{Q}G$. Hence by Proposition 2.8, $\mathcal{U}(\mathbb{Z}G)$ and $\prod_{i=1}^m \mathrm{GL}_{n_i}(\mathcal{O}_i)$ are commensurable. This shows the equivalence between (1) and (2), see Proposition 2.7 and the remark thereafter.

For any order $\mathcal{O}$ in a finite dimensional semisimple $\mathbb{Q}$-algebra, $\langle \mathrm{SL}_1(\mathcal{O}), \mathcal{U}(\mathcal{Z}(\mathcal{O}))\rangle$ has finite index in $\mathcal{U}(\mathcal{O})$ and $\mathrm{SL}_1(\mathcal{O}) \cap \mathcal{U}(\mathcal{Z}(\mathcal{O}))$ is finite by [42, Proposition 5.5.1]. Hence $\mathcal{U}(\mathcal{O})$ has property HF$\mathbb{R}$ if and only if $\mathrm{SL}_1(\mathcal{O})$ and $\mathcal{U}(\mathcal{Z}(\mathcal{O}))$ both have property HF$\mathbb{R}$.

Suppose that (1) and hence also (2) hold. By Corollary 6.3, G is cut. Now consider $\mathcal{O} = \prod_{i=1}^{m} \mathrm{M}_{n_i}(\mathcal{O}_i)$. By the previous paragraph, and the definition of SL_1 for semisimple algebras, all $\mathrm{SL}_{n_i}(\mathcal{O}_i)$ have property HFℝ. This gives (3).

Now assume (3). By the discussion following Corollary 6.3, all the $\mathcal{U}(\mathcal{Z}(\mathcal{O}_i))$ are finite and thus have property HFℝ. By the paragraph above, (2) follows. □

6.2 | Higher rank and exceptional components

Due to Proposition 6.6, property HFA for an order in a finite dimensional semisimple $\mathbb{Q}$-algebra depends on its Wedderburn-Artin components. It will turn out that the main obstruction for HFA lies in the following type of components.

Definition 6.7. Let D be a finite dimensional division algebra over $\mathbb{Q}$. The algebra $\mathrm{M}_n(D)$ is called *exceptional* if it is of one of the following types:

(I) a non-commutative division algebra other than a totally definite quaternion algebra over a number field,

(II) a 2×2-matrix ring $\mathrm{M}_2(D)$ such that D has an order $\mathcal{O}$ with $\mathcal{U}(\mathcal{O})$ finite.

Recall that by a theorem of Kleinert [42, Proposition 5.5.6] the non-commutative division algebras excluded in type (I) are exactly those having an order $\mathcal{O}$ with $\mathrm{SL}_1(\mathcal{O})$ finite. Also recall that, by Theorem 2.10, the condition in type (II) is a condition which can be formulated in terms of D. The name "exceptional component" was coined in [46] because under the presence of such a component the known generic constructions of units do not necessarily generate a subgroup of finite index in $\mathcal{U}(\mathbb{Z}G)$ [44, 42]. The crux of that failure is that these components are exactly those where respectively $\mathrm{SL}_1(D)$ and $\mathrm{SL}_2(D)$ have 'bad' (arithmetic) properties as algebraic group (for the meaning of 'bad' see Remark 6.8 below). Therefore we will now review the structure of $\mathrm{SL}_2(D)$ as an algebraic group and subsequently interpret certain algebraic groups results into the language of exceptional components. Alternatively, the reader might opt to accept the content of Theorem 6.10 and go further to the next section.

Let D be a finite dimensional division algebra over $\mathbb{Q}$ of degree d and denote $\mathcal{Z}(D)$ by K. Further, let E be a splitting field of D, i.e. $D \otimes_K E \cong \mathrm{M}_d(E)$. Call the latter isomorphism φ. Then φ restricts to an embedding of D, viewed as $D \otimes 1$, into $\mathrm{M}_d(E)$ and

$$\mathrm{SL}_2(D) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(D) \;\middle|\; \det \begin{pmatrix} \varphi(a) & \varphi(b) \\ \varphi(c) & \varphi(d) \end{pmatrix} = 1 \right\}.$$

So, in the above, using φ, we identify $\mathrm{M}_2(D)$ with a K-subspace of $\mathrm{M}_{2d}(E)$. Then we see that $\mathrm{SL}_2(D)$ actually is the Zariski closed subspace of the affine space $K^{4d^2} \cong \mathrm{M}_2(D)$ defined by the polynomial equation $\mathrm{RNr}_{\mathrm{M}_2(D)/K} = 1$. Due to this, $\mathrm{SL}_2(D)$ can be viewed as the K-rational points of an algebraic group Γ defined over K.

More generally let Γ be a linear algebraic K-group. Then, by $\mathrm{rank}_K\, \Gamma(K)$ we denote the dimension of a maximal K-split torus of $\Gamma(K)$, called *reductive K-rank* of Γ. Recall that a *K-split torus* is a commutative algebraic subgroup T of $\Gamma(K)$ which is diagonalizable over K, i.e. T is defined over K and K-isomorphic to $\prod_{1 \le i \le q} K^*$, where $q = \dim T = \mathrm{rank}_K\, \Gamma(K)$. All maximal split K-tori of Γ are conjugate over K [65, 15.2.6.] and hence $\mathrm{rank}_K\, \Gamma(K)$ is independent of the choice of T.

In our case, $\Gamma(K) = \mathrm{SL}_2(D)$ with K a global field of characteristic 0 (i.e. a finite extension of $\mathbb{Q}$). Consequently, for every valuation v of K the completion of K with respect to v, denoted K_v, is a local field. Note that $\Gamma(K_v) = \mathrm{SL}_2(D \otimes_K K_v) = \mathrm{SL}_1(\mathrm{M}_2(D \otimes_K K_v))$. It is not hard to see that $\mathrm{rank}_K\, \mathrm{SL}_n(D) = n - 1$, where the diagonal matrices with entries in K and determinant 1 form a maximal K-split torus. With this terminology at hand we can be more precise about the 'bad' behaviour of $\mathrm{SL}_1(D)$ and $\mathrm{SL}_2(D)$.

Remark 6.8. For $n \geq 2$, $\mathrm{M}_n(D)$ is exceptional exactly when $\mathrm{SL}_n(D)$ has a negative answer to the Congruence Subgroup Problem. The reason for the failure is that the type (II) components are exactly those where $\mathrm{SL}_n(D)$ is an algebraic group with $S - \mathrm{rank}(\mathrm{SL}_n(D)) := \sum_{v \in S} \mathrm{rank}(\mathrm{SL}_n(D \otimes_K K_v))$, called S-rank, equal to 1, where S is the set of Archimedean places of $\mathcal{Z}(D)$. In the case of a type (I) exceptional component, still very little is known about the answer to the Congruence Subgroup Problem for $\mathrm{SL}_1(D)$. One reason for this is the lack of unipotent elements in $\mathrm{SL}_1(D)$, which is also an obstruction for the construction of generic units contributing to a subgroup of finite index in these components. Since those excluded in type (I) are such that $\mathrm{SL}_1(\mathcal{O})$ is finite, for any order $\mathcal{O}$, such orders do not pose a problem.

A precise rank computation yields the following well known result for which we unfortunately could not find a concrete reference. Therefore, for the convenience of the reader, we sketch a proof.

Lemma 6.9 (Folklore)**.** Let D be a finite dimensional division $\mathbb{Q}$-algebra and suppose $n \geq 2$. Then there exists an Archimedean place v of $K := \mathcal{Z}(D)$ such that $\operatorname{rank}_{K_v} \mathrm{SL}_n(D \otimes_K K_v) = 1$ if and only if

- $n = 2$ and
- D is a (number) field or $D = \left(\frac{a,b}{K}\right)$ with $a, b < 0$ and K is not totally imaginary.

Proof. Note that for $n \geq 2$, $\mathrm{SL}_n(D \otimes_K K_v)$ contains a K_v-split torus. Hence $\operatorname{rank}_{K_v}(\mathrm{SL}_n(D \otimes_K K_v)) \geq 1$ for any valuation v of K. Suppose first that $D = K$ is a number field. Then $D \otimes_K K_v \cong K_v$ for any place v and it is clear that $\operatorname{rank}_{K_v}(\mathrm{SL}_n(K_v)) = n-1$. In particular the rank equals 1 if and only if $n = 2$.

Now suppose that $D \neq K$. As D is a central simple algebra over K, and K_v is a simple K-algebra, one has, by [42, Proposition 2.1.8], that $D \otimes_K K_v$ is a central simple algebra over K_v, say $\mathrm{M}_d(D')$. Now, $\mathrm{SL}_n(D \otimes_K K_v) = \mathrm{SL}_1(\mathrm{M}_n(D \otimes_K K_v)) \cong \mathrm{SL}_1(\mathrm{M}_{nd}(D')) = \mathrm{SL}_{nd}(D')$. As before, a K_v-split torus of $\mathrm{SL}_{nd}(D')$ consists of the diagonal matrices with values in K_v and determinant 1. Therefore $\operatorname{rank}_{K_v} \mathrm{SL}_{nd}(D') \geq nd - 1$ (actually the former torus is maximal and hence equality even holds). In particular, $\operatorname{rank}_{K_v} \mathrm{SL}_{nd}(D') = 1$ if and only if $n = 2$ and $d = 1$. The latter implies that K_v does not split D and thus $K_v \neq \mathbb{C}$. Consequently we may assume that K is not totally imaginary. Let v be a real place and $D \otimes_K \mathbb{R} = D'$ a non-commutative real division algebra. Then by Frobenius theorem $D' \cong \left(\frac{-1,-1}{\mathbb{R}}\right) \cong \left(\frac{a,b}{\mathbb{R}}\right)$ for any $a, b < 0$. This finishes the proof because D' was obtained by tensoring D with K_v over its center. ☐

Let S be a finite set of places of $K = \mathcal{Z}(D)$ containing the Archimedean ones. In case $\mathrm{SL}_2(D)$ is of so-called higher rank, i.e. $\operatorname{rank}_{K_v} \mathrm{SL}_2(D \otimes_K K_v) \geq 2$ for all $v \in S$, strong fixed point properties hold such as property (T). By Delorme-Guichardet's Theorem [9, Theorem 2.12.4] a countable discrete group Γ has *property* (T) if and only if every affine isometric action of Γ on a real Hilbert space has a fixed point. For background on property (T) we refer the reader to the nicely written book [9]. In particular [9, Theorem 2.12.6] shows that property (T) implies property FA and, since property (T) descends to finite index subgroups, also HFA.

In [54, Theorem (5.8), page 131], Margulis showed that S-arithmetic subgroups of connected semisimple K-groups of higher rank have property (T). In [26, Theorem 1.1.], Farb showed that S-arithmetic subgroups of almost simple simply-connected connected K-groups of K-rank $n \geq 2$ have property HFA_{n-1}. In the following theorem we restrict these results to our context and explain in the proof how to deduce it from the original theorems.

Theorem 6.10 (Margulis-Farb)**.** Let D be a finite dimensional division $\mathbb{Q}$-algebra with $\mathcal{Z}(D) = \mathbb{Q}(\sqrt{-d})$ where $d \geq 0$ and let $\mathcal{O}$ be an order in D. Suppose that $\mathrm{M}_n(D)$ is non-exceptional. Then $\mathrm{SL}_n(\mathcal{O})$ has property (T). If moreover $n \geq 3$ then it also has property HFA_{n-2}.

Proof. Set $K := \mathcal{Z}(D) = \mathbb{Q}(\sqrt{-d})$. Recall that $\mathrm{SL}_n(D)$ is a connected almost K-simple algebraic group (i.e. all proper connected algebraic K-subgroups are finite) due to the assumption on K. Furthermore $\mathrm{SL}_n(D)$ is also simply connected (i.e. any central isogeny $\varphi : H \to \mathrm{SL}_n(D)$, with H a connected algebraic group, is an algebraic group isomorphism). Due to the form of K, it has a unique (up to equivalence) Archimedean valuation, say v. Note that $K_v = \mathbb{R}$ if $K = \mathbb{Q}$ and $K_v = \mathbb{C}$ if $K = \mathbb{Q}(\sqrt{-d})$ with $d > 0$. Taking $S = \{v\}$, the set of all Archimedean places, we see that an S-arithmetic subgroup of $\mathrm{SL}_n(D)$ is simply an arithmetic subgroup of $\mathrm{SL}_n(D)$ of which $\mathrm{SL}_n(\mathcal{O})$ is an example. To obtain property (T) we invoke the celebrated theorem of Margulis [54, Theorem (5.8), page 131]. In order to apply the latter we need that $\operatorname{rank}_{K_v} \mathrm{SL}_n(D \otimes_K K_v) \geq 2$ for the unique Archimedean place v, which by Lemma 6.9 and the form of the center amounts to say that $\mathrm{SL}_n(D) \notin \left\{\mathrm{SL}_2(\mathbb{Q}(\sqrt{-d})), \mathrm{SL}_2\left(\left(\frac{a,b}{\mathbb{Q}}\right)\right)\right\}$, where $d \geq 0$ and $a, b < 0$. By Theorem 2.10 these are exactly the division algebras having an order with finite unit group. In other words [54, Theorem (5.8), page 131] can be applied if $\mathrm{M}_n(D)$ is non-exceptional.

Now if $n \geq 3$, then $\operatorname{rank}_K \mathrm{SL}_n(D) = n - 1 \geq 2$. Hence all the conditions of [26, Th. 1.1.] with $S = \{v\}$ are also satisfied, implying property HFA_{n-2}. ☐

If $\mathrm{M}_n(D)$ is non-exceptional, the groups $\mathrm{E}_n(J)^{(m)}$, for any $m \geq 1$ and J a non-zero ideal in an order $\mathcal{O}$ of D, have finite index in $\mathrm{SL}_n(\mathcal{O})$. By [42, Theorem 11.2.3 and Theorem 12.4.3] and Theorem 4.8 one can obtain, in case $n \geq 3$, an alternative proof that $\mathrm{SL}_n(\mathcal{O})$ and $\mathrm{E}_n(J)^{(m)}$ satisfy property $\mathrm{HF}\mathbb{R}$ and HFA_{n-2} without use of Theorem 6.10 (or more precisely independent of the deep theorems [54, Th. (5.8), page 131] and [26, Th. 1.1.]).

Corollary 6.11. Let $\mathcal{O}$ be an order in a finite dimensional division $\mathbb{Q}$-algebra and J a non-zero ideal in $\mathcal{O}$. Then $\mathrm{E}_n(J)^{(m)}$, for any $m \geq 1$, has property $\mathrm{HF}\mathbb{R}$ and HFA_{n-2} if $n \geq 3$. In particular, $\mathrm{SL}_n(\mathcal{O})$ has property $\mathrm{HF}\mathbb{R}$ and HFA_{n-2} if $n \geq 3$.

Proof. By [42, Theorem 11.2.3 and Theorem 12.4.3] $\mathrm{E}_n(J)^{(m)}$ has finite index in $\mathrm{SL}_n(\mathcal{O})$. In particular the former has property HF$\mathbb{R}$ and HFA_{n-2} if and only if the latter does. Let H be a subgroup of $\mathrm{SL}_n(\mathcal{O})$ of index $[\mathrm{SL}_n(\mathcal{O}) : H] = m < \infty$. Then clearly $\mathrm{E}_n(\mathcal{O})^{(m)} \leq H$ and $\mathrm{E}_n(\mathcal{O})^{(m)}$ is of finite index in H by the above. Now since $\mathcal{O}$ is a finitely generated $\mathbb{Z}$-module, using Theorem 4.2, $\mathrm{E}_n(\mathcal{O})^{(m)}$ has property F$\mathbb{R}$ and FA_{n-2}. Consequently by Proposition 2.7 also H has property F$\mathbb{R}$ and FA_{n-2}. □

6.3 | Describing exceptional components of cut groups

In order to describe when $\mathcal{U}(\mathbb{Z}G)$ has property HFA, by Proposition 6.6, one has to investigate the components of $\mathbb{Q}G$. By Theorem 6.10 we are left with the exceptional components. Moreover, by Corollary 6.3, whenever $\mathcal{U}(\mathbb{Z}G)$ has property (HFA), G must be a cut group. Therefore, we now investigate the possible exceptional Wedderburn-Artin components of $\mathbb{Q}G$ in case G is a cut group.

We first consider components of type (I), i.e. the exceptional 1×1 components, of $\mathbb{Q}G$ for G a cut group. Surprisingly, it turns out that there none. This result will be crucial in the representation theoretical applications later on.

Proposition 6.12. Let G be a finite cut group. Then $\mathbb{Q}G$ has no exceptional components of type (I).

Suppose that D is a 1×1 component of $\mathbb{Q}G$. Then the proof of Proposition 6.12 consists of the following steps.

1. There exists a primitive central idempotent e such that $D = \mathbb{Q}Ge$. The group $H = Ge$ is a finite subgroup of $\mathcal{U}(D)$, hence a Frobenius complement [64, 2.1.2, page 45]. If G is cut, also its epimorphic image H is cut. Frobenius complements that are cut were classified by Bächle [2, Proposition 4.2].

2. Some of the groups H obtained in (1) are indeed subgroups of a division algebra. This can be decided using Amitsur's classification [1], but we will give a direct argument.

3. For all remaining H, the smallest division algebra generated by H and hence also D is determined.

These steps will be realized in Proposition 6.13 and, as just explained, Proposition 6.12 follows immediately from this.

Proposition 6.13. A finite group G is both cut and isomorphic to a subgroup of units of a division $\mathbb{Q}$-algebra D if and only if G is one of the following groups

1. $1, C_2, C_3, C_4, C_6$,
2. $C_3 \rtimes C_4$, where the action is by inversion,
3. Q_8,
4. $\mathrm{SL}(2,3)$.

Moreover, the $\mathbb{Q}$-span of these groups in any division algebra is, respectively,

(I) $\mathbb{Q}, \mathbb{Q}, \mathbb{Q}(\zeta_3), \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\zeta_3)$,

(II) $\left(\frac{-1,-3}{\mathbb{Q}}\right)$,

(III) $\left(\frac{-1,-1}{\mathbb{Q}}\right)$,

(IV) $\left(\frac{-1,-1}{\mathbb{Q}}\right)$.

Proof. Note that all the groups listed in (1) - (4) are cut groups[3]. Further, they are also subgroups of division algebras. Indeed the cyclic groups are subgroups of $\mathcal{U}(\mathbb{Q}(\zeta_{12}))$, $C_3 \rtimes C_4 \cong \mathcal{U}(\mathcal{O}_3)$ is a subgroup of $\mathcal{U}(\left(\frac{-1,-3}{\mathbb{Q}}\right))$ and $Q_8 \cong \langle i, j\rangle$ and $\mathrm{SL}(2,3) \cong \langle i, \frac{1+i+j+k}{2}\rangle$ are subgroups of $\mathcal{U}(\left(\frac{-1,-1}{\mathbb{Q}}\right))$.

To prove the last statement (which we will use in the converse implication), we will consider $\mathbb{Q}[G]$ (the subring of D generated by the subgroup G) in any division $\mathbb{Q}$-algebra D containing G, for each group listed in (1) to (4). For the cyclic groups, it is clear

[3]This can be checked easily using the characterisations [2, Proposition 2.2 (iii) & (v)]. Alternatively one may use GAP. An example of a code for this is added in Appendix I.

these are the fields listed in (I). In case of the other groups, we will follow the following strategy: $\mathbb{Q}[G]$ is a simple, epimorphic image of the rational group algebra $\mathbb{Q}G$, so it has to be a division algebra appearing in the Wedderburn-Artin decomposition of $\mathbb{Q}G$. Moreover, since these groups are not abelian, the division algebra also has to be non-commutative. Using GAP [30], it is easy to see that the only non-abelian division algebras appearing in the decomposition of the rational group algebra for the groups (2), (3) and (4) are respectively (II), (III) and (IV).

Lastly, from [2, Proposition 4.2] it follows that the Frobenius complements that are cut groups are exactly the groups in (1) - (4) together with the group $C_3 \times Q_8$. Since finite subgroups of division algebras are Frobenius complements it suffices to prove that $C_3 \times Q_8$ is not embeddable in a skew field. This can be done via Amitsur's famous classification theorem, but we will provide a more direct proof. Let D be a division $\mathbb{Q}$-algebra such that $G := C_3 \times Q_8 \leq \mathcal{U}(D)$. Recall now that if B is a finite-dimensional central simple F-algebra, contained in an F-algebra A, then $A = B \otimes \mathrm{C}_A(B)$ (for example see [38, Theorem 4.7]), hence $\mathbb{Q}[G] = \mathbb{Q}[Q_8] \otimes_{\mathbb{Q}} \mathrm{C}_{\mathbb{Q}[G]}(\mathbb{Q}[Q_8])$. Since the centeralizer of Q_8 in G is $C_3 \times C_2$, it follows that $\mathbb{Q}[G]$ contains

$$\mathbb{Q}[Q_8] \otimes_{\mathbb{Q}} \mathbb{Q}[\mathrm{C}_G(Q_8)] = \mathbb{Q}[Q_8] \otimes_{\mathbb{Q}} \mathbb{Q}[C_3 \times C_2] \cong \left(\frac{-1,-1}{\mathbb{Q}}\right) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_3) \cong \left(\frac{-1,-1}{\mathbb{Q}(\zeta_3)}\right).$$

It is well known that this last algebra is split (for example using [69, Theorem 5.4.4]), which is in contradiction with the fact that D is a division algebra. □

Let us now consider components of type (II). Surprisingly if one assumes $\mathrm{M}_2(D)$ to be an exceptional component of $\mathbb{Q}G$, then the parameters d and (a, b) of $D = \mathbb{Q}(\sqrt{-d})$ (resp. $\left(\frac{a,b}{\mathbb{Q}}\right)$) are very limited. It was proven by Eisele, Kiefer and Van Gelder [23, Corollary 3.6] that only a finite number of division algebras can occur and moreover the possible parameters were described.

Theorem 6.14 (Eisele, Kiefer, Van Gelder)**.** Let G be a finite group and let e be a primitive central idempotent of $\mathbb{Q}G$ such that $\mathbb{Q}Ge$ is an exceptional component of type (II). Then $\mathbb{Q}Ge$ is isomorphic to one of the following algebras

(i) $\mathrm{M}_2(\mathbb{Q})$,

(ii) $\mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ with $d \in \{1, 2, 3\}$,

(iii) $\mathrm{M}_2(\mathbb{H}_d)$ with $d \in \{2, 3, 5\}$.

Remark 6.15. All the fields and division algebras appearing in Theorem 6.14 have the peculiar property to contain a norm Euclidean order $\mathcal{O}$ which therefore is maximal and unique up to conjugation [12, Section 2.3]. In view of [57, (21.6), page 189], this yields that also all the 2×2-matrix algebras in Theorem 6.14 have, up to conjugation, a unique maximal order, namely $\mathrm{M}_2(\mathcal{O})$. Recall that in case of $\mathbb{Q}(\sqrt{-d})$, with $d \in \{0, 1, 2, 3\}$, the unique maximal order is their respective ring of integers $\mathcal{I}_d$ and in case of $\mathbb{H}_2, \mathbb{H}_3, \mathbb{H}_5$ the respective maximal orders where described in table (3.20). Note that being norm Euclidean implies that these orders are also GE_2-rings (see Proposition 2.12).

Furthermore in [23] the authors classified the possibilities for a finite group to admit a faithful embedding in an exceptional component of type (II). More precisely, they found 55 possible groups[4], see [23, Table 2]. In Appendix I we add the aforementioned table, along with the information on all the exceptional type (II) components of $\mathbb{Q}H$, for each H in the list, and certain group theoretical properties of H. By $G_{m,\ell}$ we denote the group with SmallGroupID (m, ℓ) in the Small Groups Library of GAP [30]. For a presentation of the groups appearing see Appendix J. Using the table from Appendix I it is easy to check the following.

Proposition 6.16. Let G be a finite cut group. Then the following properties hold.

1. If there exists a primitive central idempotent e_1 of $\mathbb{Q}G$ such that $\mathbb{Q}Ge_1 \cong \mathrm{M}_2\left(\left(\frac{-1,-3}{\mathbb{Q}}\right)\right)$, then there exists another primitive central idempotent e_2 such that $\mathbb{Q}Ge_2 \cong \mathrm{M}_2(F)$ with $F = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-1})$.
2. If there exists a primitive central idempotent e_1 of $\mathbb{Q}G$ such that $\mathbb{Q}Ge_1 \cong \mathrm{M}_2(\mathbb{Q}(\sqrt{-2}))$, then there exists another primitive central idempotent e_2 such that $\mathbb{Q}Ge_2 \cong \mathrm{M}_2(\mathbb{Q})$.
3. There exists a primitive central idempotent e of $\mathbb{Q}G$ such that $\mathbb{Q}Ge \cong \mathrm{M}_2(\mathbb{Q})$ if and only if G maps onto D_8 or S_3.

[4] Note that the group with SmallGroupID [24, 1] and structure description $C_3 \rtimes C_8$ also has a faithful embedding in $\mathrm{M}_2(\mathbb{Q}(\sqrt{-1}))$, but is missing in [23, Table 2]. However it is included in the table in Appendix I.

4. There exists a primitive central idempotent e of $\mathbb{Q}G$ such that $\mathbb{Q}Ge \cong \mathrm{M}_2\left(\left(\frac{-2,-5}{\mathbb{Q}}\right)\right)$ if and only if G maps onto $G_{240,90} \cong SL(2,5) \rtimes 2 \cong 2 \cdot S_5^+$, the Schur cover of S_5 of plus type.
5. If G is solvable, there exists no primitive central idempotent e of $\mathbb{Q}G$ such that $\mathbb{Q}Ge \cong \mathrm{M}_2\left(\left(\frac{-2,-5}{\mathbb{Q}}\right)\right)$.
6. If G is nilpotent, there also exists no primitive central idempotent e of $\mathbb{Q}G$ such that $\mathbb{Q}Ge \cong \mathrm{M}_2\left(\left(\frac{-1,-3}{\mathbb{Q}}\right)\right)$.

7 | UNIT THEOREMS FOR UNITS OF INTEGRAL GROUP RINGS

In this section we will prove our main result, Theorem A from the introduction: a characterization of property HFA for $\mathcal{U}(\mathbb{Z}G)$ both in terms of G and the Wedderburn-Artin components of $\mathbb{Q}G$. First we observe that the problem can be reduced to the groups $\mathrm{GL}_n(\mathcal{O})$, for $\mathcal{O}$ some order in a finite dimensional rational division algebra.

Due to the results obtained so far, we are now able to give a short proof of the following characterization of when $\mathcal{U}(\mathbb{Z}G)$ has property HFA, both in ring theoretical terms and in function of the quotients of G.

Theorem 7.1. Let G be a finite group. Then the following properties are equivalent:

1. The group $\mathcal{U}(\mathbb{Z}G)$ has property HFA,
2. G is cut and $\mathbb{Q}G$ has no exceptional components,
3. G is cut and G does not map onto one of the following 10 groups

$$\begin{array}{ccccc} D_8, & G_{16,6}, & G_{16,13}, & G_{32,50}, & \text{or} \quad Q_8 \times C_3, \\ S_3, & \mathrm{SL}(2,3), & G_{96,202}, & G_{240,90}, & \text{or} \quad G_{384,618}. \end{array}$$

Proof. Let $\mathbb{Q}G = \prod_{i=1}^n \mathrm{M}_{n_i}(D_i)$ be the Wedderburn-Artin decomposition of $\mathbb{Q}G$. For each i, let $\mathcal{O}_i$ be a maximal order in D_i. By Proposition 6.6, $\mathcal{U}(\mathbb{Z}G)$ has property HFA if and only if G is a cut group and all $\mathrm{SL}_{n_i}(\mathcal{O}_i)$ have property HFA. So in (1) we may also assume that G is cut.

If $n_i \geq 3$, then $\mathrm{SL}_{n_i}(\mathcal{O}_i)$ has property HFA by Corollary 6.11. Next, if $n_i = 1$, then D_i is a number field or a totally definite quaternion $\mathbb{Q}$-algebra by Proposition 6.12. Furthermore, $\mathcal{Z}(D_i) = \mathbb{Q}(\sqrt{-d})$ with $d \geq 0$, since we may assume G is cut by the above. Consequently, by Theorem 2.10, $\mathrm{SL}_1(\mathcal{O}_i)$ is finite and hence has property HFA. At this stage we have that $\mathcal{U}(\mathbb{Z}G)$ has property HFA if and only if G is a cut group and for each 2×2-component $\mathrm{M}_2(D_i)$ of $\mathbb{Q}G$, the corresponding $\mathrm{SL}_2(\mathcal{O}_i)$ has property HFA.

If $\mathrm{M}_2(D_i)$ is non-exceptional, then $\mathrm{SL}_2(\mathcal{O}_i)$ has property HFA by Theorem 6.10. Suppose now there exists a primitive central idempotent e_{i_0} in $\mathbb{Q}G$ such that $\mathbb{Q}Ge_{i_0} \cong \mathrm{M}_2(D_{i_0})$ is an exceptional component of type (II). By Proposition 6.16 there is a primitive central idempotent e_i such that $\mathbb{Q}Ge_i = \mathrm{M}_2(D_i)$ is also an exceptional component of type (II), but $\mathbb{Q}Ge_i \ncong \mathrm{M}_2\left(\left(\frac{-1,-3}{\mathbb{Q}}\right)\right)$. Then, by Theorem 5.7, $\mathrm{E}_2(\mathcal{O}_i)$ does not have property HFA. As $\mathcal{O}_i$ is a GE_2-ring (cf. Remark 6.15), it follows that $\mathrm{E}_2(\mathcal{O}_i)$ has finite index in $\mathrm{SL}_2(\mathcal{O}_i)$ and hence $\mathrm{SL}_2(\mathcal{O}_i)$ also does not have property HFA.

By the above, it remains to describe the condition "no exceptional 2×2-components" in terms of forbidden quotients of G. If $\mathbb{Q}Ge$ is an exceptional component then $H = Ge$ must appear in the table in APPENDIX I. In a first instance one has to filter out the non-cut groups. In this list, certain groups have another smaller (in size) group in the remaining list as epimorphic image. These groups may also be filtered out. Eventually, one is left with the groups listed in the statement. □

Note that, by Proposition 6.12 we may substitute statement (2) in Theorem 7.1 by the statement

(2′) G is cut and $\mathbb{Q}G$ has no exceptional components of type (II).

It would be interesting to have a characterisation in terms of the internal structure of G, instead of in terms of quotients.

Remark 7.2. One can obtain a similar result as above for $\mathbb{Q}(\zeta_n)G$, where ζ_n is a primitive complex nth root of unity. Due to Proposition 6.1 and Dirichlet's unit theorem, one obtains readily that if $\mathcal{U}(\mathbb{Z}(\zeta_n)G)$ has FA, then n divides 4 or 6. For all of these values n there will be again no exceptional 1×1 component. Furthermore if $n = 4$ only $\mathrm{M}_2(\mathbb{Q}(\sqrt{-1}))$ can occur as 2×2 exceptional component and if $n = 3$, then only $\mathrm{M}_2(\mathbb{Q}(\sqrt{-3}))$; recall that $\mathbb{Z}[\zeta_6] = \mathbb{Z}[\zeta_3]$. Using the table in [3] one can again describe, in terms of quotients of G, when such components occur.

As a consequence of Theorem 7.1, we get a result on the subgroup of $\mathcal{U}(\mathbb{Z}G)$ generated by bicyclic units. Recall that a bicyclic unit of an integral group ring $\mathbb{Z}G$, for a finite group G, is a unit of the type

$$b(g, \tilde{h}) = 1 + (1 - h)g\tilde{h} \text{ or } b(\tilde{h}, g) = 1 + \tilde{h}g(1 - h),$$

where $g, h \in G$ and $\tilde{h} = \sum_{g \in \langle h \rangle} g \in \mathbb{Z}G$. Also recall that a finite group G is called *fixed point free* if[5] it has a complex representation ρ such that 1 is not an eigenvalue of $\rho(g)$ for all $1 \neq g \in G$.

Corollary 7.3. If the properties of Theorem 7.1 are satisfied and G has no non-abelian homomorphic image which is fixed point free, then the subgroup of $\mathcal{U}(\mathbb{Z}G)$ generated by all the bicyclic units is neither a non-trivial amalgamated product nor an HNN extension.

Proof. Let $\{e_i \mid 1 \leq i \leq q\}$ be the primitive central idempotents of $\mathbb{Q}G$. Denote by B the group generated by the bicyclic units. By the properties of Theorem 7.1, $\mathcal{U}(\mathbb{Z}G)$ has no exceptional components and G is cut.

In particular the center of $\mathcal{U}(\mathbb{Z}G)$ is finite and, by [42, Corollary 5.5.3], $\prod U_i$ is a subgroup of finite index in $\mathcal{U}(\mathbb{Z}G)$, where U_i is a subgroup in $\mathcal{U}(\mathbb{Z}G)$ such that $1 - e_i + U_i e_i$ is of finite index in $\mathrm{SL}_1(\mathbb{Z}Ge_i)$ for every $1 \leq i \leq q$. If e_i is such that $\mathbb{Q}Ge_i \cong \mathrm{M}_{n_i}(D_i)$ with $n_i \geq 2$, then by [42, Theorem 11.2.5] and the proof of [42, Theorem 11.1.3] such a group U_i exists within the group B. If $\mathbb{Q}Ge_i \cong D_i$, then (since no exceptional components exist) D_i is a field or a totally definite quaternion algebra. If D_i is a field, then, since G is cut, it is $\mathbb{Q}$ or an imaginary quadratic extension of $\mathbb{Q}$. Hence $\mathrm{SL}_1(D_i)$ is always finite and thus for these e_i the trivial subgroup U_i can be taken. Hence we have found a subgroup of B that is of finite index in $\mathcal{U}(\mathbb{Z}G)$.

Again by the conditions of Theorem 7.1 B has property FA. Hence, by Corollary 2.6, it is neither a non-trivial amalgamated product nor an HNN extension. □

Another property of interest, which is weaker than HFA, is the so-called FAb property.

Definition 7.4. A group Γ is said to have *property* FAb if every subgroup of finite index has finite abelianization.

Clearly property FAb is also defined on commensurability classes. We can now deduce the following.

Corollary 7.5. Let G be a finite group. Then the following properties are equivalent:

1. $\mathcal{U}(\mathbb{Z}G)$ has property (T),
2. $\mathcal{U}(\mathbb{Z}G)$ has property $\mathrm{HF}\mathbb{R}$,
3. $\mathcal{U}(\mathbb{Z}G)$ has property HFA,
4. $\mathcal{U}(\mathbb{Z}G)$ has property FAb.

Moreover, in these cases, if $\mathbb{Q}G$ has no 2×2-components, $\mathcal{U}(\mathbb{Z}G)$ has property HFA_{m-2} with

$$m = \min\{\ n \neq 1 \mid \mathrm{M}_n(D) \text{ is an epimorphic image of } \mathbb{Q}G, \text{ with } D \text{ a finite-dimensional } \mathbb{Q}\text{-division algebra }\}.$$

Proof. Suppose $\mathcal{U}(\mathbb{Z}G)$ has property FAb, then, by Corollary 6.3, G is a cut group. We will show $\mathcal{U}(\mathbb{Z}G)$ has property (T). Since property (T) is defined on commensurability classes it is enough to prove that $\Gamma = \prod_{i \in I} \mathrm{GL}_{n_i}(\mathcal{O}_i)$ has property (T), where $\mathbb{Q}G = \prod_{i \in I} \mathrm{M}_{n_i}(D_i)$ and $\mathcal{O}_i$ is an order in D_i a finite dimensional division $\mathbb{Q}$-algebra. The group Γ has property (T) if and only if all factors do. Since G is a cut group, whenever $n_i = 1$, we have, by Proposition 6.12 and Theorem 2.10 that $\mathcal{U}(\mathcal{O}_i)$ is finite, in particular it has property (T). Furthermore, $\mathcal{Z}(D) = \mathbb{Q}(\sqrt{-d})$ with $d \geq 0$, for every Wedderburn-Artin component $\mathrm{M}_n(D)$ of $\mathbb{Q}G$. Therefore Theorem 6.10, all the non-exceptional components have property (T).

Next we show that no non-exceptional components of type (II) appear as component of $\mathbb{Q}G$. Recall that those exceptional components are described by Theorem 6.14. Since FAb is a property of commensurability classes, we know that all $\mathrm{SL}_{n_i}(\mathcal{O}_i)$ have FAb. However, by Remark 5.8, no exceptional component of type (II) has property FAb. Hence no exceptional component of type (II) exists as a component of $\mathbb{Q}G$, which finishes the proof of (4) ⇒ (1).

Since property (T) implies property $\mathrm{HF}\mathbb{R}$, cf. [20, Chapter 6., Proposition 11], property $\mathrm{HF}\mathbb{R}$ implies property HFA and HFA implies property FAb, this finishes the proof of the four equivalences.

The last part of the result follows from Theorem 6.10, the assumption on $\mathbb{Q}G$ and the well behaviour of the property under direct products. □

[5]This class of groups coincide with the Frobenius complements [42, Proposition 11.4.6] and hence this could serve as a group theoretical definition.

Remark 7.6. Property (T), HF$\mathbb{R}$, HFA and FAb are all properties defined on commensurability classes. In particular, Corollary 7.5 and Theorem 7.1 are valid for arbitrary orders in $\mathbb{Q}G$.

Corollary 7.7. Let G be a group without exceptional components of type (II) (e.g. $|G|$ odd). Then the following properties are equivalent:

1. $\mathcal{U}(\mathbb{Z}G)$ has property HFA,
2. $\mathcal{U}(\mathbb{Z}G)$ has property FAb,
3. $\mathcal{U}(\mathbb{Z}G)^{ab}$ is finite,
4. G is a cut group.

Proof. Due to Proposition 6.1 it only remains to prove that if G is cut, then $\mathcal{U}(\mathbb{Z}G)$ has property HFA. By assumption, G has no 2×2 exceptional components and due to the cut property, cf. Proposition 6.12, also no exceptional 1×1 components. Hence Theorem 7.1 applies. □

Let $\mathrm{M}_2(D)$ be an exceptional component of type (II) actually appearing in $\mathbb{Q}G$ for a finite group G (see Theorem 6.14) and let $\mathcal{O}$ be an order in D. Then $\mathrm{GL}_2(\mathcal{O})$ has finite abelianization by Corollary 3.7. So also in the presence of exceptional components of type (II) one might anticipate that (3) and (4) in Corollary 7.7 are still equivalent. Interestingly, as proven in Proposition 7.9, this is equivalent to the following trichotomy.

Question 7.8. Let G be a finite cut group. Does exactly one of the following properties hold?

1. $\mathcal{U}(\mathbb{Z}G)$ has property HFA.
2. $\mathcal{U}(\mathbb{Z}G)$ has property FA but not HFA.
3. $\mathcal{U}(\mathbb{Z}G)$ has a non-trivial amalgamated decomposition and finite abelianization.

In [5, Theorem 8.5 and Remark 8.6] we prove that a dichotomy holds for $\mathcal{U}(\mathbb{Z}G)$: for G a finite cut group that is solvable or has an order not divisible by 5, $\mathcal{U}(\mathbb{Z}G)$ has property HFA or it is commensurable with a non-trivial amalgamated product.

Proposition 7.9. Let G be a finite group and $\mathcal{O}$ a maximal order of $\mathbb{Q}G$ containing $\mathbb{Z}G$. Then the following properties are equivalent:

1. If $\mathcal{U}(\mathcal{O})^{ab}$ is finite, then $\mathcal{U}(\mathbb{Z}G)^{ab}$ is finite,
2. If G is cut, then $\mathcal{U}(\mathbb{Z}G)^{ab}$ is finite,
3. Question 7.8 has a positive answer.

For the proof we will need the following proposition.

Proposition 7.10. Let D be a finite dimensional division algebra over $\mathbb{Q}$ and $\mathcal{O}$ an order in D. If $\mathrm{M}_n(D)$ is non-exceptional, then the following are equivalent:

1. $\mathcal{Z}(D) = \mathbb{Q}(\sqrt{-d})$ with $d \geq 0$,
2. $\mathrm{GL}_n(\mathcal{O})^{ab}$ is finite,
3. $\mathcal{U}(\mathcal{O}')^{ab}$ is finite for *some* order $\mathcal{O}'$ in $\mathrm{M}_n(D)$
4. $\mathcal{U}(\mathcal{O}')^{ab}$ is finite for *every* order $\mathcal{O}'$ in $\mathrm{M}_n(D)$.

Proof. To start note that $\mathrm{SL}_n(\mathcal{O})^{ab}$ is finite when $\mathrm{M}_n(D)$ is non-exceptional. Indeed by Theorem 6.10 (or alternatively Corollary 6.11 if $n \geq 3$) and the fact that property (T) implies finite abelianization [9, Corollary 1.3.6].

Now suppose that $\mathcal{Z}(D) = \mathbb{Q}(\sqrt{-d})$, with $d \geq 0$, then, by Theorem 2.10, $\mathcal{U}(\mathcal{Z}(\mathcal{O}))$ is finite. Hence in this case, $\mathrm{SL}_n(\mathcal{O})$ has finite index in $\mathrm{GL}_n(\mathcal{O})$ (using [42, Proposition 5.5.1]) and consequently $\mathrm{GL}_n(\mathcal{O})^{ab}$ is finite due to the finiteness of $\mathrm{SL}_n(\mathcal{O})^{ab}$. In short, (1) indeed implies (2).

As $\mathrm{M}_n(\mathcal{O})$ is an example of an order in $\mathrm{M}_n(D)$, (2) implies (3). We will now prove that (3) implies (1). So let $\mathcal{O}$' be an order in $\mathrm{M}_n(D)$ such that $\mathcal{U}(\mathcal{O}')^{ab}$ is finite. Then from Proposition 6.1 it follows that $\mathcal{U}(\mathcal{Z}(\mathcal{O}'))$ is also finite. Consider now the order $\mathrm{M}_n(\mathcal{O})$. Then $\mathcal{Z}(\mathcal{O}')$ and $\mathcal{Z}(\mathrm{M}_n(\mathcal{O}))$ are both orders in the finite dimensional semisimple $\mathbb{Q}$-algebra $\mathcal{Z}(\mathrm{M}_n(D))$ and hence by Proposition 2.8 the unit group of the two orders are commensurable. In particular also $\mathcal{U}(\mathcal{Z}(\mathrm{M}_n(\mathcal{O}))) = \mathcal{U}(\mathcal{Z}(\mathcal{O}))$ is finite and thus by Theorem 2.10, $\mathcal{Z}(D) = \mathbb{Q}(\sqrt{-d})$ with $d \geq 0$, as desired. Hence the first three items are equivalent.

We now prove (2) implies (4). Suppose $\mathrm{GL}_n(\mathcal{O})^{ab}$ is finite and let $\mathcal{O}'$ be an arbitrary order in $\mathrm{M}_n(D)$. As we have already shown that the first three conditions are equivalent, we already know that $\mathcal{Z}(D) = \mathbb{Q}(\sqrt{-d})$ for some $d \geq 0$. Recall that $\langle \mathrm{SL}_1(\mathcal{O}'), \mathcal{U}(\mathcal{Z}(\mathcal{O}'))\rangle$ is of finite index in $\mathcal{U}(\mathcal{O}')$ (see [42, Proposition 5.5.1]). Also $\mathcal{U}(\mathcal{O}' \cap \mathrm{M}_n(\mathcal{O}))$ equals $\mathcal{U}(\mathcal{O}') \cap \mathrm{GL}_n(\mathcal{O})$ and it is of finite index in both $\mathcal{U}(\mathcal{O}')$ and $\mathrm{GL}_n(\mathcal{O})$, since the unit groups of two orders are commensurable by Proposition 2.8. So altogether we obtain that $\mathcal{U}(\mathcal{O}') \cap \mathrm{SL}_n(\mathcal{O})$ is of finite index in $\mathcal{U}(\mathcal{O}')$ and $\mathrm{SL}_n(\mathcal{O})$, which has property (T) by Theorem 6.10. In particular, $\mathcal{U}(\mathcal{O}') \cap \mathrm{SL}_n(\mathcal{O})$ also has property (T) and thus finite abelianization. This implies that also $\mathcal{U}(\mathcal{O}')^{ab}$ is finite, as desired. The remaining implication (4) to (3) is trivial. □

Note that Proposition 7.10 yields a positive answer to Question 6.5 for non-exceptional finite dimensional simple $\mathbb{Q}$-algebras.

Proof of Proposition 7.9. First we prove that (1) implies (2). Let $\mathbb{Q}G = \prod_{i\in\mathcal{I}} \mathrm{M}_{n_i}(D_i)$ be the Wedderburn-Artin decomposition of $\mathbb{Q}G$, e_i the primitive central idempotent corresponding to $\mathrm{M}_{n_i}(D_i)$ and $\mathcal{O}_i$ an order in $\mathrm{M}_{n_i}(D_i)$. Write $\mathcal{I}$ as the (disjoint) union of three sets I_1, I_2 and I_3 where I_1 are those indices corresponding to 1×1 components, I_2 those with exceptional 2×2 components and I_3 consisting of the remaining components. Suppose G is cut, then there are no exceptional 1×1 component by Proposition 6.12. Consequently, by Theorem 2.10, $\mathcal{U}(\mathcal{O}_i)^{ab}$ is finite for all $i \in I_1$. Also $\mathcal{U}(\mathcal{O}_i)^{ab}$ is finite for any order $\mathcal{O}_i$ in $\mathrm{M}_n(D_i)$ with $i \in I_3$ by Proposition 7.10 (recall that G cut implies that $\mathcal{Z}(D_i) = \mathbb{Q}(\sqrt{-d})$ with $d \geq 0$).

Let now $i \in I_2$. Then, by Theorem 6.14 and Remark 6.15, D_i has up to conjugation a unique maximal order, say $\mathcal{O}_{max,i}$, which is right norm Euclidean and hence $\mathrm{GE}_2(\mathcal{O}_{max,i}) = \mathrm{GL}_2(\mathcal{O}_{max,i})$. By Corollary 3.7, $\mathrm{GE}_2(\mathcal{O}_{max,i})^{ab}$ and hence also $\mathrm{GL}_2(\mathcal{O}_{max,i})^{ab}$ are finite. Altogether, if G is cut, then for any choice of orders $\mathcal{O}_i$ in $\mathrm{M}_{n_i}(D_i)$, when $i \in I_1 \cup I_3$ we have that

$$\left| \prod_{i\in I_1\cup I_3} \mathcal{U}(\mathcal{O}_i)^{ab} \times \prod_{j\in I_2} \mathrm{GL}_2(\mathcal{O}_{max,j})^{ab} \right| < \infty. \tag{7.1}$$

As, by assumption, $\mathcal{O}$ is a maximal order of $\mathbb{Q}G$ containing $\mathbb{Z}G$, $\mathcal{O} \cong \prod_{i\in\mathcal{I}} \mathcal{O}_i$ with $\mathcal{O}_i = \mathcal{O}e_i$ a maximal order in $\mathrm{M}_{n_i}(D_i)$. As mentioned above, for $i \in I_2$, the maximal order $\mathcal{O}_i$ is conjugate to $\mathrm{M}_2(\mathcal{O}_{max,i})$. Since the size of the abelianization is preserved under conjugation we may assume that $\mathcal{O} = \prod_{i\in I_1\cup I_3} \mathcal{O}_i \times \prod_{j\in I_2} \mathrm{M}_2(\mathcal{O}_{max,j})$ which has a unit group with finite abelianization by (7.1). Consequently, by (1), also $\mathcal{U}(\mathbb{Z}G)^{ab}$ is finite. This finishes the proof of (1) implies (2).

We now prove that (2) implies (1). Hence assume statement (2) is true. Let $\mathcal{O}$ be a maximal order of $\mathbb{Q}G$ and suppose that $\mathcal{U}(\mathcal{O})^{ab}$ is finite. Then by Proposition 6.1, $\mathcal{U}(\mathcal{Z}(\mathcal{O}))$ is finite. Consequently, since $\mathcal{Z}(\mathbb{Z}G)$ and $\mathcal{Z}(\mathcal{O})$ are both orders in $\mathcal{Z}(\mathbb{Q}G)$, also $\mathcal{U}(\mathcal{Z}(\mathbb{Z}G))$ is finite. Hence, G is cut and thus by (2), $\mathcal{U}(\mathbb{Z}G)$ is finite, as desired. So, we have proved that (1) and (2) are equivalent.

We will now prove that (2) implies a positive answer to Question 7.8. Suppose that G is cut and hence that $\mathcal{U}(\mathbb{Z}G)^{ab}$ is finite by (2). Then if $\mathcal{U}(\mathbb{Z}G)$ does not have property (FA), it must have a non-trivial amalgamated decomposition by Theorem 2.5 as desired. Conversely, in all the cases when Question 7.8 has a positive answer, the abelianization of $\mathcal{U}(\mathbb{Z}G)$ is finite, so (3) clearly implies (2). □

8 | UNIT GROUPS OF GROUP RINGS AND PROPERTY FA

In this section we consider when $\mathcal{U}(\mathbb{Z}G)$ has FA and when it has FA but not HFA.

Theorem 8.1. Let G be a finite solvable group and assume that $\mathcal{U}(\mathbb{Z}G)$ has FA. Then the following properties hold:

1. G does not map epimorphically on D_8 and S_3.

2. $\mathcal{U}(\mathbb{Z}G)$ does not satisfy HFA if and only if G maps onto one of the following 7 groups

$$G_{16,6}, \quad G_{16,13}, \quad Q_8 \times C_3, \quad \mathrm{SL}(2,3), \quad G_{32,50}, \quad G_{96,202}, \quad \text{and} \quad G_{384,618}.$$

Proof. Let $\{e_i \mid i \in I\}$ be a full set of primitive central idempotents of $\mathbb{Q}G$ with $\mathbb{Q}Ge_i \cong \mathrm{M}_{n_i}(D_i)$ for $i \in I$ and decompose $I = I_1 \cup I_2$ in such a way that $\mathbb{Q}Ge_i$ is an exceptional component for all $i \in I_1$ and non-exceptional for all $i \in I_2$. Since, by assumption, $\mathcal{U}(\mathbb{Z}G)$ has property FA, G must be cut by Corollary 6.3 and consequently, by Proposition 6.12, all exceptional components are 2×2 matrix rings. The latter have, by Theorem 6.14 and Remark 6.15, up to conjugation in $\mathbb{Q}G$, a unique maximal order. So, without loss of generality, we may assume that $\mathbb{Z}G$ is a subring of the order $\prod_{i\in I_1} \mathrm{M}_2(\mathcal{O}_i) \times \prod_{j\in I_2} \mathbb{Z}Ge_j$, where $\mathcal{O}_i$ is a maximal order of D_i. Since orders have commensurable unit groups, $\mathcal{U}(\mathbb{Z}G)$ has finite index in $\prod_{i\in I_1} \mathrm{GL}_2(\mathcal{O}_i) \times \prod_{j\in I_2} \mathcal{U}(\mathbb{Z}Ge_j)$. Therefore, the latter also enjoys property FA, and thus $\mathrm{GL}_2(\mathcal{O}_i)$ has FA for $i \in I_1$ (also $\mathcal{U}(\mathbb{Z}Ge_j)$ has FA for all $j \in I_2$, however as G is cut, it follows from Theorem 6.10 that they all even have HFA and hence they do not add any restriction).

Now recall that $\mathrm{GL}_2(\mathbb{Z})$ is a non-trivial amalgamated product, see [22, Proposition 25]. Thus for all $i \in I_1$, $\mathcal{O}_i \ncong \mathbb{Z}$ and so by Proposition 6.16, G cannot map onto S_3 or D_8, proving (1). Now (2) follows from Theorem 7.1 and the fact that $G_{240,90}$ is not solvable. □

In case of nilpotent groups a more precise statement can be given.

Corollary 8.2. Let G be a finite nilpotent group and assume that $\mathcal{U}(\mathbb{Z}G)$ has FA. Then the following properties hold:

1. G does not map epimorphically on D_8.
2. $\mathcal{U}(\mathbb{Z}G)$ does not satisfy HFA if and only if G has $G_{16,6}$, $G_{16,13}$, $G_{32,50}$ or $Q_8 \times C_3$ as epimorphic image.

Proof. For the first statement simply apply Theorem 8.1 and note that S_3 is not nilpotent and hence cannot be a quotient of the nilpotent group G. A similar reasoning can be given for the second statement. Indeed $\mathrm{SL}(2,3)$, $G_{96,202}$ and $G_{384,618}$ are not nilpotent. □

It is natural to ask whether in Theorem 8.1 and its corollary the converse of the first statement holds. This problem is connected to the problem whether FA for $\mathcal{U}(\mathbb{Z}G)$ is fully determined by the Wedderburn-Artin components of $\mathbb{Q}G$ (as in the hereditary case). In order to formulate some concrete questions we fix the following notations: $\mathbb{Q}G \cong \prod \mathrm{M}_{n_i}(D_i)$, $\mathrm{M}_{n_i}(D_i) = \mathbb{Q}Ge_i$ with e_i a central primitive idempotent of $\mathbb{Q}G$ and $\mathcal{O}_i$ a maximal order in $\mathrm{M}_{n_i}(D_i)$.

By the proof of Theorem 8.1 we know that if $\mathcal{U}(\mathbb{Z}G)$ has property FA, then $\mathcal{U}(\mathcal{O}_i)$ has property FA for all i.

Question 8.3. With notations as above:

1. Does $\mathcal{U}(\mathbb{Z}G)$ have property FA if and only if $\mathcal{U}(\mathcal{O}_i)$ has property FA for all i?
2. Does $\mathcal{U}(\mathbb{Z}G)$ have property FA if and only if $\mathcal{U}(\mathbb{Z}Ge_i)$ has property FA for all i?

Note that the previous questions are connected to all the properties in Proposition 7.9. Unfortunately, in general property FA is dependant on the chosen order (in contrast to having finite center or finite K_1). Indeed $\mathrm{M}_2(\mathbb{Z}[\sqrt{-3}])$ and $\mathrm{M}_2(\mathcal{I}_3)$ are both orders in $\mathrm{M}_2(\mathbb{Q}(\sqrt{-3}))$ but $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-3}])$ is an amalgamated product by [5, Theorem 4.2] whereas $\mathrm{SL}_2(\mathcal{I}_3)$ has property FA by the remark just before Theorem 5.1.

Remark 8.4. We expect Question 8.3 (1) to not be true in general. For example suppose that the only exceptional components are of type $\mathrm{M}_2(D)$ with $D \in \{\mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3}), \mathbb{H}_2, \mathbb{H}_3\}$ and let $\mathcal{O}$ be the unique maximal order in D. The projection $\mathcal{U}(\mathbb{Z}Ge_i)$ of $\mathcal{U}(\mathbb{Z}G)$ into that exceptional component $\mathbb{Q}Ge_i = \mathrm{M}_2(D)$ will be a subgroup of finite index in $\mathrm{GL}_2(\mathcal{O})$, however usually not of index 1. Now the group $\mathrm{GL}_2(\mathcal{O})$ has a subgroup of very small index which has not property FA. So it seems very plausible that even with exceptional components as above, $\mathcal{U}(\mathbb{Z}Ge_i)$ sometimes will not have property FA and hence also not $\mathcal{U}(\mathbb{Z}G)$.

This last remark also ties into the following very natural question.

Question 8.5. Is there a finite cut group G, such that $\mathcal{U}(\mathbb{Z}G)$ has property FA, but does not have property HFA?

An explicit positive answer to this last question could also be used to study Question 8.3. A negative answer on the other hand has consequences on Question 7.8, making the trichotomy into a dichotomy.

In order to show some properties of unit groups $\mathcal{U}(R)$, it is common in the literature (and in our Section 5) to blow up the group to a significantly larger group. For example, $\mathrm{GE}_2(R)$ and $\mathrm{B}_2(R)$ often help in studying properties for $\mathcal{U}(R)$. In the case of $\mathcal{U}(\mathbb{Z}G)$ however, this will not work.

Proposition 8.6. Let G be a finite group. Then $\mathrm{B}_2(\mathbb{Z}G)$, $\mathrm{E}_2(\mathbb{Z}G)$ and $\mathrm{GE}_2(\mathbb{Z}G)$ do not have property FA.

Proof. The augmentation map

$$\omega : \mathbb{Z}G \to \mathbb{Z} : \sum_{g\in G} a_g g \mapsto \sum_{g\in G} a_g,$$

is an epimorphism of rings. We may extend this morphism to an epimorphism of groups $\Omega : \mathrm{GE}_2(\mathbb{Z}G) \to \mathrm{GE}_2(\mathbb{Z})$. Since $\mathrm{GE}_2(\mathbb{Z}) = \mathrm{GL}_2(\mathbb{Z})$ does not have property FA, $\mathrm{GE}_2(\mathbb{Z}G)$ also does not have property FA. The same reasoning works for $\mathrm{E}_2(\mathbb{Z}G)$ and $\mathrm{E}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})$. The augmentation map ω also induces an epimorphism from $\mathrm{B}_2(\mathbb{Z}G)$ to $\mathrm{B}_2(\mathbb{Z})$ by letting ω act entry wise. However $\mathrm{B}_2(\mathbb{Z})$ is non-trivially an amalgamated product by Proposition 5.10. Hence also $\mathrm{B}_2(\mathbb{Z}G)$ does not have property FA □

Remark that for $\mathrm{E}_2(\mathbb{Z}G)$ and $\mathrm{GE}_2(\mathbb{Z}G)$ we only used that $\mathbb{Z}G$ has a ring epimorphism to $\mathbb{Z}$, so in those cases the proof works for any ring R with a ring epimorphism to $\mathbb{Z}$. More generally the following holds.

Proposition 8.7. Let R be a unital ring that has a finite basis as $\mathbb{Z}$-module consisting of units. Then $\mathrm{B}_2(R)$ has property FA if and only if $\mathcal{U}(R)$ has property FA and R has no ring epimorphism onto $\mathbb{Z}$.

Proof. For the same reasons as in the proof of Proposition 8.6, the conditions that R has no ring epimorphism onto $\mathbb{Z}$ is necessary. Moreover, $\mathcal{U}(R)$ is an epimorphic image of $\mathrm{B}_2(R)$, so also this condition is necessary. We will now prove that they are also sufficient.

Note that in the implication (1) ⇒ (2) of the proof of Proposition 3.6, the fact that the ring is almost-universal is not used. So, since $\mathcal{U}(R)$ has FA, it has finite abelianization which in turn implies that $\mathrm{B}_2(R)$ has finite abelianization.

Suppose $(R,+) \cong (\mathbb{Z}^n,+)$, i.e. R has $\mathbb{Z}$-module basis of cardinality n. Recall that $\mathrm{B}_2(R) \cong N \rtimes \mathrm{D}_2(R)$ is a semi-direct product where N is isomorphic to the additive group of R. As in the proof of Proposition 3.6 one proves that $N^2 \leq \mathrm{B}_2(R)'$. Since also $D_2(R)' \leq \mathrm{B}_2(R)'$, this shows that $\mathrm{B}_2(R)^{ab}$ is an epimorphic image of the group $C_2^n \times D_2(R)^{ab} \cong C_2^n \times \mathcal{U}(R)^{ab} \times \mathcal{U}(R)^{ab}$. Hence, the assumption provides that $\mathrm{B}_2(R)$ has finite abelianization. In order to prove it has property FA, it suffices thus to show that it is not an amalgam. Suppose, by contradiction, that $\mathrm{B}_2(R) \cong A *_U B$ for some subgroups $A, B, U \leq \mathrm{B}_2(R)$. We will show this is impossible by considering the abelian normal subgroup N.

This subgroup N, being abelian, is contained in the maximal normal subgroup of $\mathrm{B}_2(R)$ not containing any free subgroup, denoted by $NF(\mathrm{B}_2(R))$ (see [19]) and is well-defined. This implies that $N \leq U$, by [19, Proposition 7] or that the amalgam decomposition is so-called degenerate meaning that U has index 2 in both A and B and thus is normal in the whole group.

Assume the first. Now we can on the one hand consider $\mathrm{B}_2(R)/N \cong A/N *_{U/N} B/N$ as a non-trivial amalgamated product, but on the other hand $\mathrm{B}_2(R)/N \cong \mathrm{D}_2(R) \cong \mathcal{U}(R) \times \mathcal{U}(R)$, which is a group having FA by assumption. This is a contradiction.

If $N \nleq U$, then U is of index 2 in A and B and thus also normal in the whole group. Since N is not a non-trivial amalgamated product (indeed, it is abelian), we know by the work of Karass and Solitar [47, Corollary of Theorem 6] that N is one of the following three types of groups.

1. N is contained in a conjugate of A or B,
2. $N = \bigcup_{i=1}^{\infty} (U^{\alpha_i} \cap N)$ is an infinite ascending union for some $\alpha_i \in \mathrm{B}_2(R)$,
3. $N = \langle z \rangle \times M$, with z an element of infinite order and $M = N \cap U \cong C_\infty^{n-1}$.

If N is contained in a conjugate of A or B, it should be in A or B since it is normal. Using the fact that it is normal would even imply that $N \leq U$, which is a contradiction.

Suppose the second case is true, then $N = \bigcup_{i=1}^{\infty} (U^{\alpha_i} \cap N) = \bigcup_{i=1}^{\infty} (U \cap N) = U \cap N$ since U is normal, but this again contradicts the fact that $N \nleq U$.

In the last case, the subgroup M is moreover normal in $\mathrm{B}_2(R)$. Denote $M = \begin{pmatrix} 1 & \widetilde{M} \\ 0 & 1 \end{pmatrix}$. Then $\widetilde{M} \cong \mathbb{Z}^{n-1}$ and $(R,+) \cong \mathbb{Z} \oplus \widetilde{M}$. Now since R has a $\mathbb{Z}$-module basis consisting of units and M is normal in $\mathrm{B}_2(R)$, we get that $\widetilde{M}$ is a two-sided ideal of R. Indeed, it suffices to remark that, for any units g and h of R:

$$\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}^{-1} M \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} = \begin{pmatrix} 1 & g^{-1}\widetilde{M}h \\ 0 & 1 \end{pmatrix}.$$

Therefore we may form $R/\widetilde{M}$ which is easily seen to be also isomorphic to $\mathbb{Z}$ as rings. However this contradicts the fact that there is no ring epimorphism from R to $\mathbb{Z}$. So $\mathrm{B}_2(R)$ is also not an amalgamated product and thus altogether has property FA, as needed. □

Remark 8.8. Let R be a ring as in Proposition 8.7. Then R is an epimorphic image of the group ring $\mathbb{Z}\mathcal{U}(R)$. We can extend this morphism to a group morphism from $\mathrm{GE}_2(\mathbb{Z}\mathcal{U}(R))$ (or $\mathrm{E}_2(\mathbb{Z}\mathcal{U}(R))$) to $\mathrm{GE}_2(R)$ (or $\mathrm{E}_2(R)$). It might thus be tempting to deduce property FA for $\mathrm{GE}_2(R)$ and $\mathrm{E}_2(R)$ from the same properties of the same groups over the universal object $\mathbb{Z}\mathcal{U}(R)$. However, Proposition 8.6 shows that this argument is too simple and should $\mathrm{GE}_2(R)$ and $\mathrm{E}_2(R)$ have property FA, then it is for more subtle reasons. This also shows why, in Section 5, we did not use this universal object.

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APPENDIX I | GROUPS WITH FAITHFUL EXCEPTIONAL 2×2 COMPONENTS

In this appendix we reproduce [23, Table 2] listing those finite groups G that have a faithful exceptional component of type (II) in the Wedderburn-Artin decomposition of the rational group algebra $\mathbb{Q}G$ (see Definition 6.7)[6]. We also add certain attributes relevant for us. For each group G such that $\mathbb{Q}G$ has at least one exceptional 2×2 component in which G embedds ("faithful component") the columns of the table contain the following information:

SMALLGROUP ID:	the identifier of the group G in the small group library
Structure:	the structure description of the group. Colons indicate split extensions, a period an extension that is (possibly) non-split
cut:	indicates whether the group is a cut group (see Definition 6.2)
dℓ:	derived length of the group; ∞ for non-solvable groups
cℓ:	the nilpotency class of the group; ∞ indicates that the group is not nilpotent (omitted for non-solvable groups)
exceptional components of type (II):	exceptional components of type (II) (not necessarily faithful) of the group algebra $\mathbb{Q}G$ (with multiplicity)
quotients:	small group IDs of non-trivial quotients of G that also appear in this table.

Recall that we use the following shorthands for some algebras appearing in the table:

$$\mathbb{Q}(i) = \mathbb{Q}(\sqrt{-1}), \quad \mathbb{H}_2 = \left(\frac{-1,-1}{\mathbb{Q}}\right), \quad \mathbb{H}_3 = \left(\frac{-1,-3}{\mathbb{Q}}\right) \quad \text{and} \quad \mathbb{H}_5 = \left(\frac{-2,-5}{\mathbb{Q}}\right).$$

One way to check whether a group has the cut-property is via the following GAP code:

```
IsCutGroup := function(G)
return
  ForAll( List( ConjugacyClasses(G) , Representative ),
    x ->
      ForAll( Filtered( [2..Order(x)-1], j -> Gcd(j, Order(x)) = 1 ),
        j -> IsConjugate (G, x^j, x) or IsConjugate(G, x^j, x^-1)
      )
  );
end;
```

[6] including the group with SMALLGROUPID `[24, 1]` that was accidentally omitted in the original table

cut	SMALLGROUP ID	Structure	cl	dℓ	exceptional components	quotients
✓	[6, 1]	S_3	∞	2	$1\times M_2(\mathbb{Q})$,	
✓	[8, 3]	D_8	2	2	$1\times M_2(\mathbb{Q})$,	
✓	[12, 4]	D_{12}	∞	2	$2\times M_2(\mathbb{Q})$,	[6, 1],
✓	[16, 6]	$C_8 : C_2$	2	2	$1\times M_2(\mathbb{Q}(i))$,	
✓	[16, 8]	$QD16$	3	2	$1\times M_2(\mathbb{Q})$, $1\times M_2(\mathbb{Q}(\sqrt{-2}))$,	[8, 3],
✓	[16, 13]	$(C_4 \times C_2) : C_2$	2	2	$1\times M_2(\mathbb{Q}(i))$,	
✓	[18, 3]	$C_3 \times S_3$	∞	2	$1\times M_2(\mathbb{Q})$, $1\times M_2(\mathbb{Q}(\sqrt{-3}))$,	[6, 1],
×	[24, 1]	$C_3 : C_8$	∞	2	$1\times M_2(\mathbb{Q})$, $1\times M_2(\mathbb{Q}(i))$,	[6, 1],
✓	[24, 3]	$\mathrm{SL}(2,3)$	∞	3	$1\times M_2(\mathbb{Q}(\sqrt{-3}))$,	
✓	[24, 5]	$C_4 \times S_3$	∞	2	$2\times M_2(\mathbb{Q})$, $1\times M_2(\mathbb{Q}(i))$,	[6, 1], [12, 4],
✓	[24, 8]	$(C_6 \times C_2) : C_2$	∞	2	$3\times M_2(\mathbb{Q})$, $1\times M_2(\mathbb{Q}(\sqrt{-3}))$,	[6, 1], [8, 3], [12, 4],
✓	[24, 10]	$C_3 \times D_8$	2	2	$1\times M_2(\mathbb{Q})$, $1\times M_2(\mathbb{Q}(\sqrt{-3}))$,	[8, 3],
✓	[24, 11]	$C_3 \times Q_8$	2	2	$1\times M_2(\mathbb{Q}(\sqrt{-3}))$,	
✓	[32, 8]	$(C_2 \times C_2).(C_4 \times C_2)$	3	2	$2\times M_2(\mathbb{Q})$, $1\times M_2(\mathbb{H}_2)$,	[8, 3],
✓	[32, 11]	$(C_4 \times C_4) : C_2$	3	2	$2\times M_2(\mathbb{Q})$, $2\times M_2(\mathbb{Q}(i))$,	[8, 3],
✓	[32, 44]	$(C_2 \times Q_8) : C_2$	3	2	$2\times M_2(\mathbb{Q})$, $1\times M_2(\mathbb{H}_2)$,	[8, 3],
✓	[32, 50]	$(C_2 \times Q_8) : C_2$	2	2	$1\times M_2(\mathbb{H}_2)$,	
×	[36, 6]	$C_3 \times (C_3 : C_4)$	∞	2	$1\times M_2(\mathbb{Q})$, $2\times M_2(\mathbb{Q}(\sqrt{-3}))$,	[6, 1], [18, 3],
✓	[36, 12]	$C_6 \times S_3$	∞	2	$2\times M_2(\mathbb{Q})$, $2\times M_2(\mathbb{Q}(\sqrt{-3}))$,	[6, 1], [12, 4], [18, 3],
×	[40, 3]	$C_5 : C_8$	∞	2	$1\times M_2(\mathbb{H}_5)$,	
✓	[48, 16]	$(C_3 : Q_8) : C_2$	∞	2	$3\times M_2(\mathbb{Q})$, $1\times M_2(\mathbb{Q}(\sqrt{-2}))$, $1\times M_2(\mathbb{Q}(\sqrt{-3}))$, $1\times M_2(\mathbb{H}_2)$,	[6, 1], [8, 3], [12, 4], [16, 8], [24, 8],
×	[48, 18]	$C_3 : Q_{16}$	∞	2	$3\times M_2(\mathbb{Q})$, $1\times M_2(\mathbb{Q}(\sqrt{-3}))$, $1\times M_2(\mathbb{H}_3)$,	[6, 1], [8, 3], [12, 4], [24, 8],
×	[48, 28]	$\mathrm{SL}(2,3).C_2$	∞	4	$1\times M_2(\mathbb{Q})$, $1\times M_2(\mathbb{H}_3)$,	[6, 1],
✓	[48, 29]	$\mathrm{GL}(2,3)$	∞	4	$1\times M_2(\mathbb{Q})$, $1\times M_2(\mathbb{Q}(\sqrt{-2}))$,	[6, 1],
×	[48, 33]	$((C_4 \times C_2) : C_2) : C_3$	∞	3	$1\times M_2(\mathbb{Q}(i))$,	
✓	[48, 39]	$(C_4 \times S_3) : C_2$	∞	2	$4\times M_2(\mathbb{Q})$, $1\times M_2(\mathbb{Q}(i))$, $1\times M_2(\mathbb{H}_3)$,	[6, 1], [12, 4], [16, 13],
✓	[48, 40]	$Q_8 \times S_3$	∞	2	$4\times M_2(\mathbb{Q})$, $1\times M_2(\mathbb{H}_2)$,	[6, 1], [12, 4],
✓	[64, 37]	$(C_4 \times C_2).(C_4 \times C_2)$	4	2	$2\times M_2(\mathbb{Q})$, $2\times M_2(\mathbb{H}_2)$,	[8, 3],
✓	[64, 137]	$(C_4 : Q_8) : C_2$	3	2	$6\times M_2(\mathbb{Q})$, $2\times M_2(\mathbb{H}_2)$,	[8, 3],
×	[72, 19]	$(C_3 \times C_3) : C_8$	∞	2	$2\times M_2(\mathbb{H}_3)$,	
✓	[72, 20]	$(C_3 : C_4) \times S_3$	∞	2	$4\times M_2(\mathbb{Q})$, $1\times M_2(\mathbb{Q}(i))$, $1\times M_2(\mathbb{H}_3)$,	[6, 1], [12, 4], [24, 5],
✓	[72, 22]	$(C_6 \times S_3) : C_2$	∞	2	$5\times M_2(\mathbb{Q})$, $2\times M_2(\mathbb{Q}(\sqrt{-3}))$, $1\times M_2(\mathbb{H}_3)$,	[6, 1], [8, 3], [12, 4], [24, 8],
×	[72, 24]	$(C_3 \times C_3) : Q_8$	∞	2	$4\times M_2(\mathbb{Q})$, $1\times M_2(\mathbb{H}_3)$,	[6, 1], [12, 4],
✓	[72, 25]	$C_3 \times \mathrm{SL}(2,3)$	∞	3	$4\times M_2(\mathbb{Q}(\sqrt{-3}))$,	[24, 3],
✓	[72, 30]	$C_3 \times ((C_6 \times C_2) : C_2)$	∞	2	$3\times M_2(\mathbb{Q})$, $6\times M_2(\mathbb{Q}(\sqrt{-3}))$,	[6, 1], [8, 3], [12, 4], [18, 3], [24, 8], [24, 10], [36, 12],
✓	[96, 67]	$\mathrm{SL}(2,3) : C_4$	∞	4	$1\times M_2(\mathbb{Q})$, $2\times M_2(\mathbb{Q}(i))$,	[6, 1],
✓	[96, 190]	$(C_2 \times \mathrm{SL}(2,3)) : C_2$	∞	4	$2\times M_2(\mathbb{Q})$, $1\times M_2(\mathbb{H}_2)$,	[6, 1], [12, 4],
×	[96, 191]	$\mathrm{SL}(2,3).C_2) : C_2$	∞	4	$2\times M_2(\mathbb{Q})$, $1\times M_2(\mathbb{H}_2)$,	[6, 1], [12, 4],
✓	[96, 202]	$((C_2 \times Q_8) : C_2) : C_3$	∞	3	$1\times M_2(\mathbb{H}_2)$,	
×	[120, 5]	$\mathrm{SL}(2,5)$		∞	$1\times M_2(\mathbb{H}_3)$,	
✓	[128, 937]	$(Q_8 \times Q_8) : C_2$	4	3	$6\times M_2(\mathbb{Q})$, $4\times M_2(\mathbb{H}_2)$,	[8, 3],
×	[144, 124]	$\mathrm{SL}(2,3).C_2)$	∞	4	$4\times M_2(\mathbb{Q})$, $4\times M_2(\mathbb{H}_3)$,	[6, 1], [48, 28],
✓	[144, 128]	$S_3 \times \mathrm{SL}(2,3)$	∞	3	$1\times M_2(\mathbb{Q})$, $3\times M_2(\mathbb{Q}(\sqrt{-3}))$, $1\times M_2(\mathbb{H}_2)$,	[6, 1], [18, 3], [24, 3],
✓	[144, 135]	$(C_3 \times C_3) : (C_8 : C_2)$	∞	2	$1\times M_2(\mathbb{Q}(i))$, $4\times M_2(\mathbb{H}_3)$,	[16, 6],
✓	[144, 148]	$(C_3 \times C_3) : ((C_4 \times C_2) : C_2)$	∞	2	$8\times M_2(\mathbb{Q})$, $1\times M_2(\mathbb{Q}(i))$, $4\times M_2(\mathbb{H}_3)$,	[6, 1], [12, 4], [16, 13], [48, 39],
×	[160, 199]	$((C_2 \times Q_8) : C_2) : C_5$	∞	3	$1\times M_2(\mathbb{H}_2)$,	
✓	[192, 989]	$(\mathrm{SL}(2,3) : C_4) : C_2$	∞	4	$3\times M_2(\mathbb{Q})$, $1\times M_2(\mathbb{Q}(\sqrt{-3}))$, $2\times M_2(\mathbb{H}_2)$,	[6, 1], [8, 3], [12, 4], [24, 8],
×	[240, 89]	$\mathrm{SL}(2,5).C_2$		∞	$1\times M_2(\mathbb{H}_5)$,	
✓	[240, 90]	$\mathrm{SL}(2,5) : C_2$		∞	$1\times M_2(\mathbb{H}_5)$,	
✓	[288, 389]	$(C_3 \times C_3) : ((C_4 \times C_4) : C_2)$	∞	3	$2\times M_2(\mathbb{Q})$, $2\times M_2(\mathbb{Q}(i))$, $2\times M_2(\mathbb{H}_3)$,	[8, 3], [32, 11],
×	[320, 1581]	$(((C_2 \times Q_8) : C_2) : C_5).C_2$	∞	4	$2\times M_2(\mathbb{H}_2)$,	
✓	[384, 618]	$((Q_8 \times Q_8) : C_2) : C_3$	∞	3	$1\times M_2(\mathbb{H}_2)$,	
✓	[384, 18130]	$((Q_8 \times Q_8) : C_3) : C_2$	∞	4	$1\times M_2(\mathbb{Q})$, $1\times M_2(\mathbb{H}_2)$,	[6, 1],
×	[720, 409]	$\mathrm{SL}(2,9)$		∞	$2\times M_2(\mathbb{H}_3)$,	
✓	[1152, 155468]	$(((Q_8 \times Q_8) : C_3) : C_2) : C_3$	∞	4	$1\times M_2(\mathbb{Q})$, $1\times M_2(\mathbb{Q}(\sqrt{-3}))$, $1\times M_2(\mathbb{H}_2)$,	[6, 1], [18, 3],
×	[1920, 241003]	$C_2.((C_2 \times C_2 \times C_2 \times C_2) : A_5)$		∞	$1\times M_2(\mathbb{H}_2)$,	

APPENDIX J | SOME GROUP PRESENTATIONS

We give the presentations of certain groups appearing in Theorem 7.1 (the indices indicate their SMALLGROUP IDs). We start with the following nilpotent groups:

$$\begin{aligned} G_{16,6} &= \langle\, a, b \mid a^8 = b^2 = 1,\ a^b = a^5 \,\rangle \cong C_8 \rtimes C_2, \\ G_{16,13} &= \langle\, a, b, c \mid a^4 = b^2 = c^2 = 1 = (a,b) = (a,c),\ b^c = a^2 b \,\rangle \cong (C_4 \times C_2) \rtimes C_2, \\ G_{32,50} &= \langle\, i, j, a, b \mid i^4 = 1,\ i^2 = j^2,\ i^j = i^{-1},\ a^2 = 1, (i,a) = (j,a) = 1, \\ &\qquad b^2 = 1,\ i^b = i^{-1},\ j^b = j^{-1},\ a^b = i^2 a \,\rangle \cong (Q_8 \times C_2) \rtimes C_2. \end{aligned}$$

The group $G_{16,13} \cong D_8 \curlyvee C_4$ is the central product of D_8 and C_4 (central subgroups of order 2 identified) and $G_{32,50} \cong Q_8 \curlyvee D_8$ is the central product of Q_8 with D_8. The group $G_{16,6}$ is sometimes called the modular group of order 16.

We also need the following non-nilpotent groups:

$$\begin{aligned} G_{96,202} &= \langle\, i, j, b, t, a \mid i^4 = 1,\ i^2 = j^2, i^j = i^{-1},\ b^3 = 1,\ i^b = j,\ j^b = ij, \\ &\qquad t^2 = 1,\ (i,t) = (j,t) = (b,t) = 1, \\ &\qquad a^2 = 1,\ (i,a) = (j,a) = (b,a) = 1,\ t^a = i^2 t \,\rangle, \\ G_{240,90} &= \langle\, x, y, z, a \mid x^3 = y^5 = z^2 = 1,\ (x,z) = (y,z) = 1,\ (xy)^2 = z, \\ &\qquad a^2 = 1,\ (z,a) = 1,\ x^a = x^2,\ y^a = (xy^3)^2 \,\rangle, \\ G_{384,618} &= \langle\, i_1, j_1, i_2, j_2, a \mid i_1^4 = 1,\ i_1^2 = j_1^2,\ i_1^{j_1} = i_1^{-1},\ i_2^4 = 1,\ i_2^2 = j_2^2,\ i_2^{j_2} = i_2^{-1}, \\ &\qquad (i_1, i_2) = (i_1, j_2) = (j_1, i_2) = (j_1, j_2) = 1,\ a^6 = 1, \\ &\qquad i_1^a = j_2^{-1},\ j_1^a = (i_2 j_2)^{-1},\ i_2^a = j_1^{-1},\ j_2^a = (i_1 j_1)^{-1} \,\rangle. \end{aligned}$$

They have the following structures: $G_{96,202} \cong (\mathrm{SL}(2,3) \times C_2) \rtimes C_2$, $G_{240,90} \cong SL(2,5) \rtimes 2 \cong 2 \cdot S_5^+$, the Schur cover of S_5 of plus type, and $G_{384,618} \cong (Q_8 \times Q_8) \rtimes C_6$.

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