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involving free boundaries at the micro scale
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Homogenization of a mineral dissolution and precipitation model involving free boundaries at the micro scale

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Abstract

In this work we present the homogenization of a reaction-diffusion model that includes an evolving microstructure. Such type of problems model, for example, mineral dissolution and precipitation in a porous medium. Hence, we are dealing with a multi-scale problem with free boundaries on the pore scale. In the initial state the microscopic geometry is given by a periodically perforated domain, including spherical solid grains. The radius of each grain is of order ϵ and depends on the unknown (the solute concentration) at its surface. Therefore the radii of the grains change in time, leading to a nonlinear, free boundary problem. In a first step, we transform the evolving micro domain to a fixed, periodically domain. Using the Rothe-method we prove the existence of a weak solution and obtain *a priori* estimates that are uniform with respect to ϵ. Finally, letting $\epsilon \to 0$, we derive a macroscopic model, the solution of which approximates the micro-scale solution. For this, we use the method of two-scale convergence, and obtain strong compactness results enabling to pass to the limit in the nonlinear terms.

1 Introduction

In this paper, we consider a reaction-diffusion model defined in a perforated domain having a heterogeneous microstructure. Due to the reactions taking place at the boundaries of the perforations, these perforations may evolve. This evolution is not known *a priori*, but depends on the solution of the problem, and therefore the model involves free boundaries at the micro scale. The initial state of the domain is isomorphic to a periodically perforated domain, where the periodicity is of order ϵ, with ϵ being a small, positive scale separation parameter. The aim of this work is to establish existence of a weak solution with ϵ-uniform *a priori* estimates, and to derive the corresponding macroscopic model. This is obtained by homogenization methods, after passing to the limit $\epsilon \to 0$. The macroscopic model is derived rigorously, its solution being an approximation of the solution to the microscopic model.

Reactive transport in evolving porous media occur in a variety of real-life applications. We mention here mineral precipitation and dissolution [7, 46, 40], biofilm growth [39], colloid deposition [13], or water diffusion into absorbent particles [15, 41]. A typical example is the precipitation and dissolution taking place in a porous medium, consisting of alternating solid grains and voids (the pore space). We assume that the void space is completely filled by a fluid, say, water, which is stationary. Soluble species can diffuse

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inside the fluid, and can precipitate at the fluid-solid interface to form a solid layer (e.g. salt). The reverse process of dissolution is also possible, by which solute species is released back to the fluid. Assuming that, in general, the precipitate layer has a thickness that is comparable to the typical pore size, when defining the micro-domain one has to exclude the grains and (if present) the precipitate layer. In other words, the boundary of the micro-domain wherein the problem is defined has two parts: the outer part of the boundary, where the entire medium is embedded into, and the inner boundary, defined as the solid-fluid interface. One cannot assume that the micro-domain remains fixed in time, as the thickness of the precipitate layer depends on the species concentration, which, itself, is an unknown in the model. Hence, the precipitation and dissolution processes lead to a variable pore space, and the fluid-solid interface is a free boundary.

In a simplifying setting, we assume here that the porous medium includes spherical solid grains, having a radius of order ϵ, and being periodically-distributed. Furthermore, we assume that the evolution of this grain is radially symmetric, therefore its evolution is well described by the radius of the resulting solid. In this way, the free boundary is reduced to a one-dimensional equation for the radius. Nevertheless, this still leads to a strongly nonlinear problem, defined in time-dependent domains with *a priori* unknown evolution.

In what follows let $T > 0$ stand for a maximal time and consider a heterogeneous medium $\Omega \in \mathbb{R}^d$ ($d \in \{2, 3\}$), consisting of regions occupied by a stationary fluid (the pore space) and of small, solid regions of spherical shape (the grains with the attached mineral layers). These centres of the solid regions are distributed periodically in a d-dimensional (hyper)cube structure. The distance between two successive centres is of ϵ order (the micro-scale length). In this way, the pore space depends on both ϵ and on the time t, and for given a $t \in [0, T]$ we denote the pore space by $\Omega_\epsilon(t)$. Within $\Omega_\epsilon(t)$, we consider a reaction-diffusion equation, with the solute concentration u_ϵ as unknown quantity. For the ease of presentation, and since the aim here is to provide mathematically rigorous derivation of the macroscopic model, only one solute species is considered. In the case of two species, the model can be reduced to the situation here by considering a decoupled component of the model, the total electric charge (see, e.g. [25, 43]).

Recalling the simplifying assumptions made above, the solid phase consists of spherical regions, including the original solid part of the medium, and the radially symmetric mineral layer. These solids are characterised by a radius R_ϵ, which is grain-dependent and also changes in time. More precisely, it is obtained as the solution of an ordinary differential equation, depending in a nonlinear way on the (averaged) solute concentration at the surface of the grain and the radius itself. In this way, the spatial variable x enters in the equation as a parameter, and varies for every microscopic cell. Such a structure can also be found in the models discussed in [13, 15, 40, 41], but for different types of applications and not in the context of homogenization.

As already mentioned, the goal of this contribution is to provide a mathematically rigorous derivation of the macroscopic model approximating the microscopic precipitation-dissolution model defined in evolving microscopic geometries. In doing so, two aspects are essential. First, we transform the problem defined in the evolving micro-domain $\Omega_\epsilon(t)$ into one problem defined in a fixed, periodically perforated domain Ω_ϵ. In doing so, we employ the Hanzawa transformation [23] (we also refer to [37] for an overview of this topic). This leads to a change in the coefficients of the equations, which now depend on the radius of the grains and therefore on the unknown concentration. We prove the existence of a solution pair (u_ϵ, R_ϵ) by using the Rothe-method, and derive *a priori* estimates that are uniform with respect to the parameter ϵ.

Secondly, for the derivation of the macroscopic model, which is obtained in the limit along a sequence $\epsilon \to 0$, we use the method of two-scale convergence. To pass to the limit in the nonlinear terms, we need strong two-scale convergence results for the concentration u_ϵ, as well as for the coefficients depending on the radius R_ϵ and its time derivative $\partial_t R_\epsilon$. For the latter, this is obtained as a consequence of the convergence of the radius function R_ϵ, for which a Kolmogorov-type compactness argument is used. As will follow from below, due to the nonlinear character of the problem and because of the coefficients for the time-derivative appearing in the transformed model, we cannot obtain ϵ-uniform *a priori* estimates for $\partial_t u_\epsilon$, what would guarantee the strong two-scale convergence of u_ϵ, see [19, 30]. We therefore apply an alternative strategy, namely to control the product $J_\epsilon u_\epsilon$, where J_ϵ denotes the Jacobi determinant of the Hanzawa transformation, and to solve an auxiliary approximation problem to establish the strong convergence of u_ϵ. Finally, for the strong convergence of $\partial_t R_\epsilon$ and to identify the limit equation for the radius, we prove a two-scale compactness result for averaged functions on the oscillating surface of the micro cells. With these convergence results, we are able to pass to the limit in the microscopic model.

The outcome is a macroscopic model consisting of a reaction-diffusion equation for the macroscopic concentration, defined in the entire Ω, coupled with an ordinary differential equation for the macroscopic radius depending on the macroscopic concentration. The effective parameters like the diffusion or the porosity are obtained by solving cell problems formulated in evolving reference cells, and accounting for the microscopic evolution of the porous medium. The resulting is therefore a strongly coupled, two-scale mathematical model, defined in the entire domain Ω that does not depend on time.

The analysis and homogenization of reaction-diffusion models defined in evolving microscopic geometries have been addressed in several publications. Various strategies have been adopted, depending also on the particular geometry considered there. The simplest situation appears in one spatial dimension, or if the pore space of a porous medium is a long but thin strip or tube. Then the evolution of the solid-fluid interface can be described by a free boundary function. In this sense we refer to [45] for the one-dimensional case, where the existence and uniqueness of a solution for a precipitation-dissolution model involving multi-valued dissolution rates is proved, using the fixed-domain transformation proposed in [42]. In the multi-dimensional case, we mention [44, 26], where effective models are derived formally by transversal averaging.

For more general situations, as considered here, level set methods have been employed to describe the evolving microscopic geometry. We refer to [46, 7, 8, 22, 40], where dissolution and precipitation processes are modelled, and to [39] for a similar approach in modelling biofilm growth in a porous medium. However, the rigorous homogenization is missing. Without entering into details, we mention that one can consider as an alternative to the free boundaries the phase-field approach. Then, the fluid-solid interface is approximated by a narrow diffuse-interface layer.

Whereas a rich literature exists on the rigorous homogenization of reactive transport processes with nonlinear reaction terms in the bulk domain and the microscopic surface for a complex, but fixed microstructure, see for example [12, 19, 20, 18, 31], the results for an evolving microstructure are scarce. Moreover, in most of the papers dealing with such cases, the microstructural evolution is assumed known *a priori*. In this sense, we mention [35, 36], where homogenized models are derived rigorously for linear reaction-diffusion-advection problems (also see [16]). Close to the present work is [21], where the reaction and adsorption/desorption terms are nonlinear, without requiring a radially symmetric microscopic evolution, and the diffusion is low, leading to a different scaling

for the gradient of the concentration; however, the evolution is assumed known. Some general two-scale results for transformations of locally periodic domains are treated in [48].

In all these works, the models are defined in perforated domains, which resembles well the structure of a porous medium with variable microstructure, but the microscopic evolution is known *a priori*. For the analysis of free boundary models with radially symmetric evolution of the interface we refer to [15]. There, the existence of a solution is proved for a system involving a nonlinear parabolic problem with radially symmetric perforations, in which the rate of change of the radius of these perforations depends linearly on the saturation. Also, the analysis does not include any homogenization results. Also, the two-scale system considered in [13] is the homogenized counterpart of a microscopic problem that is similar to the one considered here, but for linear adsorption/desorption rates. The existence of a solution is obtained for sufficiently small times. We also mention [17], where rigorous upscaling results are proved for the Laplace and the heat equations posed in a domain with a rough/rapidly oscillating boundary. This rough boundary is also a free boundary, as its normal velocity depends (linearly) on the solution, with a time-dependent rate.

Our paper is organized as follows: In Section 2 we formulate the microscopic model on the evolving domain $\Omega_\epsilon(t)$ and transform it to the fixed domain Ω_ϵ. In Section 4 we prove existence for the microscopic model and establish *a priori* estimates depending explicitly on ϵ. In Section 5 we show the two-scale convergence results for the microscopic solution and derive the macroscopic model. A conclusion is given in Section 6. Some elemental calculations concerning the Hanzawa transformation can be found in die Appendix A.

2 The microscopic model

In what follows $T > 0$ is a finite time, $\epsilon > 0$ denotes a small scale separation parameter s.t. $\epsilon^{-1} \in \mathbb{N}$ ($n \in \mathbb{N}_0$ being the spatial dimension), and $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_n) \in \mathbb{N}^n$ is an n-tuple of strictly positive natural numbers. Let $\Omega = (\mathbf{0}, \boldsymbol{\ell}) = (0, \ell_1) \times \cdots \times (0, \ell_n)$ and $Y := (0, 1)^n$. The center of Y is denoted by $\mathfrak{m} := \frac{1}{2}(1, \ldots, 1) \in \mathbb{R}^n$. For the set of n-tuples $K_\epsilon := \{\mathbf{k} \in \mathbb{Z}^n : \epsilon(Y + \mathbf{k}) \subset \Omega\}$ one has

$$\overline{\Omega} = \bigcup_{\mathbf{k} \in K_\epsilon} \epsilon\big(\overline{Y} + \mathbf{k}\big). \tag{1}$$

In other words, we decompose $\overline{\Omega}$ into a set of microscopic (ϵ-sized) hypercubes, two different ones sharing at most one side. Each $x \in \overline{\Omega}$ is contained in a microscopic hypercube having the center

$$\mathfrak{m}_{\epsilon,x} := \epsilon \left(\left[\frac{x}{\epsilon} \right] + \mathfrak{m} \right). \tag{2}$$

To introduce the evolving geometry we first define $\underline{R}, \overline{R}$ representing the minimal, respective maximal admissible radius of an n-dimensional sphere contained in a microscopic hypercube, and satisfying

$$0 < \underline{R} < \overline{R} < \frac{1}{2}. \tag{3}$$

Then, for any $t \in [0, T]$ and $x \in \overline{\Omega}$ we associate a radius

$$r_{\epsilon,t,x} := R_\epsilon\left(t, \mathfrak{m}_{\epsilon,x}\right), \tag{4}$$

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where the function R_ϵ will be a solution component of the problem stated below.

Further, for any $x \in \overline{\Omega}$ a unique $\mathbf{k} \in K_\epsilon$ exists s.t. $x \in \epsilon(Y + \mathbf{k})$, namely $\mathbf{k} = \left[\frac{x}{\epsilon}\right]$. Then, for any $x \in \overline{\Omega}$, with $\mathbf{k}$ as before and $t \in [0, T]$, we consider the n-dimensional ball, respectively sphere

$$B_{\epsilon,\mathbf{k}}(t) := B_{R_\epsilon(t,x)}(\epsilon(\mathfrak{m} + \mathbf{k})), \qquad \Gamma_\epsilon^{\mathbf{k}}(t) := \Gamma_\epsilon\left(t, \epsilon\left[\frac{x}{\epsilon}\right]\right) := \partial B_{\epsilon,\mathbf{k}}(t),$$

both having the radius $r_{\epsilon,t,x} \in (\epsilon\underline{R}, \epsilon\overline{R}) \subset \left(0, \frac{\epsilon}{2}\right)$. With this, at any time t, the evolving microscopic domain and its inner part of the (freely-moving) boundary are defined as

$$\Omega_\epsilon(t) := \Omega \setminus \bigcup_{\mathbf{k} \in K_\epsilon} \overline{B_{\epsilon,\mathbf{k}}(t)}, \text{ respectively } \Gamma_\epsilon(t) := \bigcup_{\mathbf{k} \in K_\epsilon} \Gamma_\epsilon^{\mathbf{k}}(t). \tag{5}$$

Observe that the outer boundary $\partial\Omega$ is fixed. Finally, we define the sets

$$Q_\epsilon^T := \bigcup_{t \in (0,T)} \{t\} \times \Omega_\epsilon(t), \quad \text{and} \quad H_\epsilon^T := \bigcup_{t \in (0,T)} \{t\} \times \Gamma_\epsilon(t). \tag{6}$$

We consider the following microscopic problem that is motivated by microscopic models for precipitation-dissolution processes in a porous medium (see e.g. [45, 44, 46]).

Problem $\mathbf{P}_\epsilon$. Find $u_\epsilon, R_\epsilon : \bar{Q}_\epsilon^T \to \mathbb{R}$ solving

$$\partial_t u_\epsilon - \nabla \cdot \left(D\nabla u_\epsilon\right) = f \qquad\qquad \text{in } Q_\epsilon^T, \tag{7a}$$

$$-D\nabla u_\epsilon \cdot \nu_\epsilon = \partial_t R_\epsilon(u_\epsilon - \rho) \qquad\qquad \text{on } H_\epsilon^T, \tag{7b}$$

$$-D\nabla u_\epsilon \cdot \nu_\epsilon = 0 \qquad\qquad \text{on } (0, T) \times \partial\Omega, \tag{7c}$$

$$\partial_t R_\epsilon = G_\epsilon(t, x, u_\epsilon, R_\epsilon) \qquad\qquad \text{in } Q_\epsilon^T, \tag{7d}$$

$$u_\epsilon(0) = u_\epsilon^{\text{in}} \qquad\qquad \text{in } \Omega_\epsilon(0), \tag{7e}$$

$$R_\epsilon(0) = R_\epsilon^{\text{in}} \qquad\qquad \text{in } \Omega_\epsilon(0). \tag{7f}$$

Here, D is a diffusion tensor (its properties being mentioned in Assumption (A6) below) and $\rho > 0$ is a constant representing the molar density of the precipitate. The unknowns u_ϵ and R_ϵ represent the concentration of the solute species, respectively the thickness of the precipitate layer around a grain centered at $\epsilon\left[\frac{x}{\epsilon}\right]$. The vector ν_ϵ is the unit normal to $\Gamma_\epsilon(t)$ outwards $\Omega_\epsilon(t)$, while u_ϵ^{in} and R_ϵ^{in} are given initial conditions (see Assumptions (A1) and (A2)). The function G_ϵ appearing in (7d) is defined as

$$G_\epsilon(t, x, u_\epsilon, R_\epsilon) = \frac{\epsilon}{\left|\Gamma_\epsilon\left(t, \epsilon\left[\frac{x}{\epsilon}\right]\right)\right|} \int_{\Gamma_\epsilon\left(t, \epsilon\left[\frac{x}{\epsilon}\right]\right)} g\left(u_\epsilon(t, z), \frac{R_\epsilon(t, z)}{\epsilon}\right) dz. \tag{7g}$$

Observe that the domain $\Omega_\epsilon(t)$ depends on the unknown function R_ϵ, and therefore this problem involves a (microscopic) free boundary. Further, for any $t \in (0, T)$, $G_\epsilon(t, \cdot, u_\epsilon, R_\epsilon)$ is constant inside microscopic cells, i.e. inside any $\epsilon(Y + \mathbf{k})$ with $\mathbf{k} \in K_\epsilon$. The perforation of the microscopic cell evolves in a radially symmetric way with radius R_ϵ. From the application point of view, this is a simplified setup, in the sense that we assume that the precipitate layer around a grain has a uniform, but unknown thickness. This is guaranteed by (7d). Based on Assumption (A4), we will prove that R_ϵ remains between two values $\epsilon\underline{R}$ and $\epsilon\overline{R}$, so $\Gamma_\epsilon\left(t, \epsilon\left[\frac{x}{\epsilon}\right]\right)$ can never touch the boundary of the cell, or degenerate into a point.

Remark 1. *Observe that (7d) is stated for all $t \in (0, T)$ and $x \in \Omega_\epsilon(t)$, which implies that R_ϵ is defined in the entire $\Omega_\epsilon(t)$. In fact, R_ϵ is constant inside each microscopic cell $Y_{\epsilon,\mathbf{k}}(t) := \Omega_\epsilon(t) \cap \epsilon(Y + \mathbf{k})$, with $\mathbf{k} \in K_\epsilon$. This follows from (7d) and the definition of G_ϵ in (7g). Hence, in the following we extend the function R_ϵ constantly from the perforated micro cell $Y_{\epsilon,\mathbf{k}}(t)$ to the whole micro cell $\epsilon(Y + k)$. Further, as we will see later, the micro cells $Y_{\epsilon,\mathbf{k}}(t)$ are included in Ω. Especially, we have $\partial \Omega_\epsilon(t) \setminus \Gamma_\epsilon(t) = \partial \Omega$, i.e., the outer boundary is fixed and R_ϵ is defined on the whole fixed domain Ω.*

For the functional spaces used below the notations are standard. We only mention that $L^\infty\left(\Omega, C^{0,1}\left([0,T], \left[\epsilon\underline{R}, \epsilon\overline{R}\right]\right)\right)$ is the set of functions that are $C^{0,1}$ w.r.t. time f.a.e. $x \in \Omega$, s.t. R_ϵ is bounded by $\underline{R}$ and $\overline{R}$.

Definition 1. *The pair of functions (u_ϵ, R_ϵ) with $u_\epsilon \in L^2((0,T), H^1(\Omega_\epsilon(t)))$ and $R_\epsilon \in L^\infty\left(\Omega, C^{0,1}\left([0,T], \left[\epsilon\underline{R}, \epsilon\overline{R}\right]\right)\right)$ is a weak solution of Problem P_ϵ if, for all $\phi_\epsilon \in C^1\left(\overline{Q_\epsilon^T}\right)$ with $\phi_\epsilon(T, \cdot) = 0$, it holds that*

$$-\int_0^T \int_{\Omega_\epsilon(t)} u_\epsilon \partial_t \phi_\epsilon \, dx dt + \int_0^T \int_{\Omega_\epsilon(t)} D\nabla u_\epsilon \cdot \nabla \phi_\epsilon \, dx dt$$

$$= \int_0^T \int_{\Omega_\epsilon(t)} f\phi_\epsilon \, dx dt - \int_0^T \int_{\Gamma_\epsilon(t)} G_\epsilon(u_\epsilon, R_\epsilon)(u_\epsilon - \rho)\phi_\epsilon \, d\sigma dt \qquad (8)$$

$$+ \int_{\Omega_\epsilon(0)} u_\epsilon^{\mathrm{in}} \phi_\epsilon(0) dx + \int_0^T \int_{\Gamma_\epsilon(t)} \partial_t S_\epsilon(t, S_\epsilon^{-1}) \cdot \nu u_\epsilon \phi_\epsilon \, d\sigma dt,$$

where S_ϵ is defined in (9). Furthermore, the function R_ϵ fulfills (7d) and the initial condition (7f) almost everywhere.

Observe that, in the definition above, the function S_ϵ describes the evolution of the moving surface $\Gamma_\epsilon(t)$. The evolution of the evolving domain $\Omega_\epsilon(t)$ is given by (5).

2.1 Transformation to the fixed domain

As mentioned in the introduction, the present analysis relies on the idea of transforming Problem P_ϵ (or its weak form stated in Definition 1, defined on the evolving domain $\Omega_\epsilon(t)$, to a problem defined in a fixed, reference domain Ω_ϵ. To this aim, we employ the Hanzawa transform, [23] (see also [37] for an overview of this topic). With $\overline{R}$ introduced above, see (3), the domain Ω_ϵ is defined by

$$\Omega_\epsilon := \Omega \setminus \bigcup_{\mathbf{k} \in K_\epsilon} \overline{B_{\epsilon\overline{R}}\left(\epsilon(\mathbf{k} + \mathfrak{m})\right)}.$$

Further, we define $\Gamma_\epsilon := \partial\Omega_\epsilon \setminus \partial\Omega$. In other words, the microscopic domain is obtained as the union of scaled and shifted reference elements

$$Y^* := Y \setminus \overline{B_{\overline{R}}(\mathfrak{m})} = (0,1)^n \setminus \overline{B_{\overline{R}}(\mathfrak{m})}.$$

The boundary of $B_{\overline{R}}(\mathfrak{m})$ is denoted by $\Gamma := \partial B_{\overline{R}}(\mathfrak{m})$.

With this, for any $t \in [0,T]$ we consider the Hanzawa transformation $S_\epsilon(t, \cdot) : \Omega_\epsilon \to \Omega_\epsilon(t)$, defined as

$$S_\epsilon(t, x) := x + \left(R_\epsilon(t, x) - \epsilon\overline{R}\right) \chi_0\left(\frac{x}{\epsilon}\right) \nu_0\left(\frac{x}{\epsilon}\right), \qquad (9)$$

where the functions χ_0 and ν_0 are Y-periodic extensions to the entire $\mathbb{R}^n$ of the functions defined in Appendix A. Since R_ϵ is constant for every cell $\epsilon(Y + \mathbf{k})$ with $\mathbf{k} \in K_\epsilon$, and $\chi_0 \in C^\infty_{\text{per}}(Y)$, the function S_ϵ is (strongly) differentiable with respect to x. According to the calculations in Appendix A,

$$\nabla S_\epsilon(t,x) = I_n + \left(\frac{R_\epsilon - \epsilon\overline{R}}{\epsilon}\right) \nabla_y (\chi_0\nu_0) \left(\frac{x}{\epsilon}\right),$$

with $I_n \in \mathbb{R}^{n\times n}$ being the identity matrix, and

$$0 < \left(\frac{\underline{R}}{\overline{R}}\right)^{n-1} \leq \det\left(\nabla S_\epsilon(t,x)\right) \leq 1 + 2\left(\overline{R} - \underline{R}\right). \tag{10}$$

These estimates are crucial for the homogenization, as the functional determinant of the transformation S_ϵ is bounded away from 0, uniformly with respect to ϵ. We emphasize that S_ϵ is defined and smooth on the whole domain Ω, but the Jacobi determinant is not positive anymore.

Now, define the functions

$$\begin{aligned}
\tilde{u}_\epsilon &: (0,T) \times \Omega_\epsilon \to \mathbb{R}, & \tilde{u}_\epsilon(t,x) &:= u_\epsilon(t, S_\epsilon(t,x)), \\
\tilde{R}_\epsilon &: (0,T) \times \Omega_\epsilon \to [\epsilon\underline{R}, \epsilon\overline{R}], & \tilde{R}_\epsilon(t,x) &:= R_\epsilon(t, S_\epsilon(t,x)).
\end{aligned} \tag{11}$$

From Remark 1 we immediately obtain that, in fact, $\tilde{R}_\epsilon$ and R_ϵ coincide. However, to better distinguish the equation on the fixed and the evolving domain, $\tilde{R}_\epsilon$ will appear in the solution pair $(\tilde{u}_\epsilon, \tilde{R}_\epsilon)$ for the problem formulated in the fixed domain.

To simplify the writing, the functions defined below will be used frequently in what follows,

$$F_\epsilon(t,x) := \nabla S_\epsilon(t,x), \ J_\epsilon(t,x) := \det(F_\epsilon(t,x)). \tag{12}$$

The inverse and the transpose of the inverse of F_ϵ are denoted by

$$F_\epsilon^{-1}(t,x) := \left(F_\epsilon(t,x)\right)^{-1}, \quad F_\epsilon^{-T}(t,x) := \left(F_\epsilon^{-1}(t,x)\right)^T. \tag{13}$$

After employing the Hanzawa transformation (9), (8) can be rewritten in terms of the variables $\tilde{u}_\epsilon$ and $\tilde{R}_\epsilon$. More precisely, for all $\phi \in H^1(\Omega_\epsilon)$ and almost every $t \in (0,T)$ it holds that

$$\begin{aligned}
\int_{\Omega_\epsilon} \partial_t\left(J_\epsilon\tilde{u}_\epsilon\right)\phi\,dx &- \int_{\Omega_\epsilon} \tilde{u}_\epsilon\partial_t J_\epsilon\phi\,dx - \int_{\Omega_\epsilon} V_\epsilon \cdot \nabla\tilde{u}_\epsilon\phi\,dx \\
&+ \int_{\Omega_\epsilon} D_\epsilon\nabla\tilde{u}_\epsilon \cdot \nabla\phi\,dx = \int_{\Omega_\epsilon} J_\epsilon f_\epsilon\phi\,dx - \int_{\Gamma_\epsilon} \partial_t\tilde{R}_\epsilon(\tilde{u}_\epsilon - \rho)J_\epsilon\phi\,d\sigma,
\end{aligned} \tag{14}$$

with f_ϵ, V_ϵ and D_ϵ defined as

$$\begin{aligned}
f_\epsilon(t,x) &:= f(t, S_\epsilon(t,x)), \\
V_\epsilon(t,x) &:= J_\epsilon(t,x)F_\epsilon^{-T}(t,x)\partial_t S_\epsilon(t,x), \\
D_\epsilon(t,x) &:= J_\epsilon(t,x)F_\epsilon^{-1}D(S_\epsilon(t,x))F_\epsilon^{-T}(t,x).
\end{aligned} \tag{15}$$

The evolution of the radius $\tilde{R}_\epsilon$ is given by

$$\partial_t\tilde{R}_\epsilon(t,x) = G_\epsilon(t,x,\tilde{u}_\epsilon,\tilde{R}_\epsilon) = \frac{\epsilon}{|\epsilon\Gamma|} \int_{\Gamma_\epsilon\left(\epsilon\left[\frac{x}{\epsilon}\right]\right)} g\left(\tilde{u}_\epsilon(t,z), \frac{\tilde{R}_\epsilon(t,z)}{\epsilon}\right) d\sigma,$$

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where $\Gamma_\epsilon : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$\Gamma_\epsilon(z) := \epsilon\Gamma + z, \text{ for any } z \in \mathbb{R}^n. \tag{16}$$

The initial conditions read

$$\tilde{u}_\epsilon(0) = u_\epsilon^{\text{in}}(S_\epsilon(0, \cdot)) =: \tilde{u}_\epsilon^{\text{in}} \quad \text{in } \Omega_\epsilon,$$
$$\tilde{R}_\epsilon(0) = R_\epsilon^{\text{in}}(S_\epsilon(0, \cdot)) =: \tilde{R}_\epsilon^{\text{in}} \quad \text{in } \Omega_\epsilon.$$

As before, one has $\tilde{R}_\epsilon^{\text{in}} = R_\epsilon^{\text{in}}$ since R_ϵ^{in} is constant on every cell $\epsilon(Y + \mathbf{k})$, $\mathbf{k} \in K_\epsilon$.

In summary, the problem transformed to the fixed domain is

Problem $\tilde{P}_\epsilon$. Find $\tilde{u}_\epsilon, \tilde{R}_\epsilon : [0, T) \times \bar{\Omega}_\epsilon \to \mathbb{R}$ solving

$$\partial_t(J_\epsilon \tilde{u}_\epsilon) - V_\epsilon \cdot \nabla \tilde{u}_\epsilon - \tilde{u}_\epsilon \partial_t J_\epsilon = \nabla \cdot \left(D_\epsilon \nabla \tilde{u}_\epsilon\right) + J_\epsilon f_\epsilon \qquad \text{in } (0, T) \times \Omega_\epsilon, \tag{17a}$$
$$-D_\epsilon \nabla \tilde{u}_\epsilon \cdot \nu_\epsilon = \tilde{R}'_\epsilon(\tilde{u}_\epsilon - \rho)J_\epsilon \qquad \text{on } (0, T) \times \Gamma_\epsilon, \tag{17b}$$
$$\tilde{u}_\epsilon = 0 \qquad \text{on } (0, T) \times \partial\Omega, \tag{17c}$$
$$\partial_t \tilde{R}_\epsilon = G_\epsilon(t, x, \tilde{u}_\epsilon, \tilde{R}_\epsilon) \qquad \text{in } (0, T) \times \Omega_\epsilon, \tag{17d}$$
$$\tilde{u}_\epsilon(0) = \tilde{u}_\epsilon^{\text{in}} \qquad \text{in } \Omega_\epsilon, \tag{17e}$$
$$\tilde{R}_\epsilon(0) = \tilde{R}_\epsilon^{\text{in}} \qquad \text{in } \Omega_\epsilon. \tag{17f}$$

The definition of a weak solution of Problem $\tilde{P}_\epsilon$ is given in Definition 2 below.

We now state the assumptions on the data

Assumptions on the data:

(A1) For the microscopic initial data $u_\epsilon^{\text{in}} \in H^1(\Omega_\epsilon)$ a $C > 0$ exists such that

$$\|u_\epsilon^{\text{in}}\|_{H^1(\Omega_\epsilon)} \leq C$$

uniformly w.r.t ϵ. Further there exists $u^{\text{in}} \in H^1(\Omega)$ such that u_ϵ^{in} converges in the two-scale sense to u^{in}.

(A2) The initial radii $R_\epsilon^{\text{in}} \in L^\infty(\Omega)$ are constant on every cell $\epsilon(Y + \mathbf{k})$, $\mathbf{k} \in K_\epsilon$. Furthermore, there exist $\underline{R}, \overline{R} \in \mathbb{R}$, $0 < \underline{R} < \overline{R} < \frac{1}{2}$, such that

$$\epsilon\underline{R} \leq R_\epsilon^{\text{in}} \leq \epsilon\overline{R} \qquad \text{in a.e. sense.}$$

Also, we assume that a $C > 0$ exists such that for any $0 < h \ll 1$ and $\boldsymbol{\ell} \in \mathbb{Z}^n$ with $|\epsilon\boldsymbol{\ell}| < h$ it holds

$$\epsilon^{-1}\|R_\epsilon^{\text{in}}(\cdot + \boldsymbol{\ell}\epsilon) - R_\epsilon^{\text{in}}\|_{L^2(\Omega^h)} \leq C|\boldsymbol{\ell}\epsilon|,$$

with $\Omega^h := \{x \in \Omega : \text{dist}(\partial\Omega, x) > h\}$. Additionally, there exists $R^{\text{in}} \in L^\infty(\Omega)$, such that

$$\epsilon^{-1}R_\epsilon^{\text{in}} \to R^{\text{in}} \qquad \text{in the two-scale sense}$$

in L^p for every $p \in [1, \infty)$.

(A3) For the (precipitation/dissolution) rate one has $g \in C^1(\mathbb{R}^2) \cap W^{1,\infty}(\mathbb{R}^2)$.

(A4) There exists $0 < \delta_0 < \overline{R} - \underline{R}$, such that for all $u \in \mathbb{R}$ it holds that

(i) $g(u, R) \leq 0$ for all $R \in \left[\overline{R} - \delta_0, \overline{R} \right]$,

(ii) $g(u, R) \geq 0$ for all $R \in \left[\underline{R}, \underline{R} + \delta_0 \right]$.

(A5) The source term is continuous, $f \in C^0\big([0, T] \times \overline{\Omega}\big)$.

(A6) The diffusion tensor $D \in C^0\big(\overline{\Omega}\big)^{n \times n}$ is symmetric and coercive, i.e., there exists $c_0 > 0$ such that

$$D(x) \xi \cdot \xi \geq c_0 \quad \text{for all } x \in \overline{\Omega}, \, \xi \in \mathbb{R}^n.$$

Note that, by the assumption on the shifts of R_ϵ^{in} in Assumption (A2), $\epsilon^{-1} R_\epsilon^{\mathrm{in}}$ is relatively compact in $L^2(\Omega)$, and $R^{\mathrm{in}} \in H^1(\Omega)$, see Remark 5.

2.2 Weak formulation of the micro model

Now, we give the definition of the weak solution of the microscopic model (7). In the following, we suppress the notation $\tilde{\cdot}$, i.e., the solution of Problem $\tilde{\mathrm{P}}_\epsilon$ is denoted (by an abuse of notation) by (u_ϵ, R_ϵ). We consider weak solutions, as defined below.

Definition 2. *The pair (u_ϵ, R_ϵ) with $u_\epsilon \in L^2((0, T), H^1(\Omega_\epsilon))$ such that $\partial_t(J_\epsilon u_\epsilon) \in L^2((0, T), H^1(\Omega_\epsilon)')$ and $R_\epsilon \in W^{1,\infty}((0, T), L^2(\Omega_\epsilon))$ s.t. $R_\epsilon, \partial_t R_\epsilon \in L^\infty((0, T) \times \Omega_\epsilon)$, and $\epsilon \underline{R} \leq R_\epsilon \leq \epsilon \overline{R}$ is a weak solution of Problem $\tilde{\mathrm{P}}_\epsilon$ if, for all $\phi \in H^1(\Omega_\epsilon)$, it holds for almost every $t \in (0, T)$ that*

$$
\begin{aligned}
\langle \partial_t \big(J_\epsilon u_\epsilon \big), \phi \rangle_{H^1(\Omega_\epsilon)', H^1(\Omega_\epsilon)} &- \int_{\Omega_\epsilon} u_\epsilon \partial_t J_\epsilon \phi \, dx - \int_{\Omega_\epsilon} V_\epsilon \cdot \nabla u_\epsilon \phi \, dx \\
&+ \int_{\Omega_\epsilon} D_\epsilon \nabla u_\epsilon \cdot \nabla \phi \, dx = \int_{\Omega_\epsilon} J_\epsilon f_\epsilon \phi \, dx - \int_{\Gamma_\epsilon} \partial_t R_\epsilon (u_\epsilon - \rho) J_\epsilon \phi \, d\sigma,
\end{aligned}
\tag{18}
$$

and

$$
\partial_t R_\epsilon(t, x) = \frac{\epsilon}{|\epsilon \Gamma|} \int_{\Gamma_\epsilon\left(\epsilon\left[\frac{x}{\epsilon}\right]\right)} g\left(u_\epsilon(t, z), \frac{R_\epsilon(t, z)}{\epsilon} \right) d\sigma.
\tag{19}
$$

Additionally, (u_ϵ, R_ϵ) fulfill the initial conditions (17e) and (17f).

Remark 2.

(i) *We emphasize that due to the mean value theorem, see [14, Chapter 5.9, Theorem 2], it holds that $R_\epsilon \in C^{0,1}([0, T], L^\infty(\Omega_\epsilon))$. Therefore, (19) holds for all $t \in [0, T]$ and almost every $x \in \Omega$.*

(ii) *In the following we extend R_ϵ constantly from every perforated micro cell $\epsilon(Y^* + \mathbf{k})$ with $\mathbf{k} \in K_\epsilon$ to the whole cell $\epsilon(Y + \mathbf{k})$, and therefore R_ϵ can be treated as a function defined on the whole domain Ω.*

We will see that $\partial_t u_\epsilon \in L^2((0, T), H^1(\Omega_\epsilon)')$ and the initial condition for u_ϵ is well-defined. However, the norm of $\partial_t u_\epsilon$ is of order ϵ^{-1} and therefore we work with the time-derivative $\partial_t(J_\epsilon u_\epsilon)$ to establish convergence results for the micro solutions to pass to the limit $\epsilon \to 0$.

Remark 3. *In fact,* (18)*can be rewritten as*

$$\langle \partial_t (J_\epsilon u_\epsilon), \phi \rangle_{H^1(\Omega_\epsilon)', H^1(\Omega_\epsilon)} + \int_{\Omega_\epsilon} u_\epsilon V_\epsilon \cdot \nabla \phi_\epsilon dx - \int_{\partial \Omega_\epsilon} u_\epsilon \phi_\epsilon V_\epsilon \cdot \nu d\sigma$$
$$+ \int_{\Omega_\epsilon} D_\epsilon \nabla u_\epsilon \cdot \nabla \phi dx = \int_{\Omega_\epsilon} J_\epsilon f_\epsilon \phi dx - \int_{\Gamma_\epsilon} \partial_t R_\epsilon (u_\epsilon - \rho) J_\epsilon \phi d\sigma, \tag{20}$$

This is a consequence of the equality

$$-\int_{\Omega_\epsilon} \tilde{u}_\epsilon \partial_t J_\epsilon \phi_\epsilon dx - \int_{\Omega_\epsilon} V_\epsilon \cdot \nabla \tilde{u}_\epsilon \phi_\epsilon dx = \int_{\Omega_\epsilon} \tilde{u}_\epsilon V_\epsilon \cdot \nabla \phi_\epsilon dx - \int_{\partial \Omega_\epsilon} \tilde{u}_\epsilon \phi_\epsilon V_\epsilon \cdot \nu d\sigma,$$

all $\tilde{u}_\epsilon, \phi_\epsilon \in H^1(\Omega_\epsilon)$, *where* J_ϵ, V_ϵ *are defined in* (12)–(15). *This holds since*

$$\partial_t J_\epsilon = \nabla \cdot V_\epsilon \tag{21}$$

(see [21, p. 105]), which is a direct consequence of Piola's identity $\nabla \cdot (J_\epsilon F_\epsilon^{-1}) = 0$, *see e.g. [29, p. 117]. More precisely, for the term on the left in* (21) *one has*

$$\partial_t J_\epsilon = J_\epsilon tr (F_\epsilon^{-1} \partial_t F_\epsilon) \tag{22}$$

due to the Jacobi formula. For the term on the right, the product rule gives

$$\nabla \cdot V_\epsilon = tr (J_\epsilon F_\epsilon^{-1} \nabla \partial_t S_\epsilon) + [\nabla \cdot (J_\epsilon F_\epsilon^{-1})] \cdot \partial_t S_\epsilon.$$

Using Piola's identity, (21) *follows immediately.*

3 Main results

The aim of the paper is two-folded. First, we show the existence of a weak solution of the microscopic problem in (17) together with uniform *a priori* estimates with respect to ϵ. The following theorem is proven in Section 4.

Theorem 1. *There exists a weak solution of the (transformed, microscopic) Problem* $\tilde{P}_\epsilon$ *in the sense of Definition 2. This solution fulfills the a priori estimates in Lemma 7.*

In a second step we use the ϵ-uniform estimates to derive two-scale compactness results for the micro solutions, and show that the $\epsilon \to 0$ limit functions $u_0 : (0, T) \times \Omega \to \mathbb{R}$ and $R_0 : (0, T) \times \Omega \to \mathbb{R}$ solve a macroscopic model with homogenized coefficients. In this sense, we start by defining

$$S_0(t, x, y) = y + (R_0(t, x) - \overline{R})(\chi_0 \nu_0)(y),$$

and

$$J_0(t, x, y) = \det(\nabla_y S_0(t, x, y)),$$
$$V_0(t, x, y) = J_0(t, x, y) \nabla_y S_0(t, x, y)^{-1} \partial_t S_0(t, x, y),$$

and the averaged quantities

$$q(t, x) := \int_{Y^*} \nabla_y \cdot V_0(t, x, y) dy,$$
$$\bar{J}_0(t, x) := \int_{Y^*} J_0(t, x, y) dy.$$

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Further we define the moving cell surface as

$$\Gamma(t, x) := \partial B_{R_0(t,x)}(x).$$

Now, the macroscopic model is

Problem P_0. Find $u_0, R_0 : [0, T) \times \bar{\Omega} \to \mathbb{R}$ solving

$$
\begin{aligned}
\partial_t\big(\bar{J}_0 u_0\big) - u_0 q - \nabla \cdot \big(D_0^* \nabla u_0\big) \\
= \int_{Y^*} J_0 f_0 dy - |\Gamma(t, x)| \partial_t R_0 (u_0 - \rho) \quad & \text{in } (0, T) \times \Omega, \\
-D_0^* \nabla u_0 \cdot \nu = 0 \quad & \text{on } (0, T) \times \partial\Omega, \quad (23) \\
\partial_t R_0 = g(u_0, R_0) \quad & \text{in } (0, T) \times \Omega, \\
u_0(0) = u^{\text{in}} \quad & \text{in } \Omega, \\
R_0(0) = R^{\text{in}} \quad & \text{in } \Omega,
\end{aligned}
$$

where D_0^* is the homogenized diffusion coefficient defined in (50) via the cell problems (48).

We proceed with the definition of a weak solution for the macroscopic model.

Definition 3. *A weak solution of the (macroscopic) Problem P_0 is a pair (u_0, R_0) satisfying*

$$
\begin{aligned}
u_0 \in L^2((0, T), H^1(\Omega)) \quad & \text{with } \partial_t\big(\bar{J}_0 u_0\big) \in L^2((0, T), H^1(\Omega)'), \\
R_0 \in W^{1,\infty}((0, T), L^\infty(\Omega)),
\end{aligned}
$$

and for all $\phi \in H^1(\Omega)$ and almost every $t \in (0, T)$ it holds that

$$
\begin{aligned}
\langle \partial_t(\bar{J}_0 u_0), \phi \rangle_{H^1(\Omega)', H^1(\Omega)} - \int_\Omega q u_0 \phi dx + \int_\Omega D_0^* \nabla u_0 \cdot \nabla \phi dx \\
= \int_\Omega \int_{Y^*} J_0 f_0 dy \phi dx - \int_\Omega \partial_t R_0 (u_0 - \rho) \phi |\Gamma(t, x)| dx.
\end{aligned} \quad (24)
$$

Additionally, it holds almost everywhere in $(0, T) \times \Omega$

$$\partial_t R_0 = g(u_0, R_0).$$

Further, there hold the initial conditions $(\bar{J}_0 u_0)(0) = \bar{J}_0(0) u^{\text{in}}$ and $R_0(0) = R^{\text{in}}$.

We emphasize the from the function spaces in the Definition 3 and the fact that $\bar{J}_0 \in L^\infty((0, T), L^\infty(\Omega)) \cap L^\infty((0, T), H^1(\Omega))$, see Remark 7 and Remark 8, it follows immediately that $(\bar{J}_0 u_0) \in C^0([0, T], L^2(\Omega))$, whereas it is not obvious in which sense we can understand $u_0(0)$, see also Remark 8. In the remaining part of this work, we prove that the macroscopic (weak) solution defined above is obtained as the limit $\epsilon \to 0$ of solutions to the microscopic problems in (17).

Theorem 2. *Let (u_ϵ, R_ϵ) be a weak solution of the (microscopic) Problem $\tilde{P}_\epsilon$, in the sense of Definition 2. Then, up to a sequence $\epsilon \to 0$, it holds that*

$$
\begin{aligned}
u_\epsilon \to u_0 \quad & \text{strongly in the two-scale sense in } L^2, \quad (25) \\
\epsilon^{-1} R_\epsilon \to R_0 \quad & \text{in } L^p((0, T) \times \Omega) \quad (26)
\end{aligned}
$$

for all $p \in [1, \infty)$, where the pair (u_0, R_0) is a weak solution of the macroscopic model (23), in the sense of Definition 3. Additionally, we have $u_0 \in C^0([0, T], L^2(\Omega))$ and $u_0(0) = u_0$.

For the definition of the two-scale convergence we refer to the Appendix B. The proof of Theorem 2 is given in Section 5.

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4 Existence of a microscopic solution and *a priori* estimates

In this section we use the Rothe-method to establish the existence of a weak solution of the microscopic model (17), and therefore (1), in the sense of Definition 2. Futher, we show *a priori* estimates for the solution depending explicitly on the parameter ϵ. These *a priori* estimates form the basis for the derivation of the macroscopic model.

4.1 The time-discretized model

We start by formulating the time-discretized model. For $N \in \mathbb{N}$ we define the time step $\Delta t := \frac{T}{N}$ and set $t^j := j\Delta t$ for $j \in \{0, \ldots, N\}$. Then, we are looking for a sequence of solutions $(u_\epsilon^j, R_\epsilon^j)$ of the time-discretized problems. Clearly, for $j = 0$ we take $u_\epsilon^0 = u_\epsilon^{\text{in}}$ and $R_\epsilon^0 = R_\epsilon^{\text{in}}$.

Definition 4. *Let $j \in \{1, \ldots, N\}$ and $(u_\epsilon^{j-1}, R_\epsilon^{j-1}) \in H^1(\Omega_\epsilon) \times L^\infty(\Omega_\epsilon)$ be given. A pair $(u_\epsilon^j, R_\epsilon^j) \in H^1(\Omega_\epsilon) \times L^\infty(\Omega_\epsilon)$ is a solution of the time-discrete problem at time t^j if*

$$\frac{R_\epsilon^j - R_\epsilon^{j-1}}{\Delta t} = G_\epsilon^{j-1} \quad in \ \Omega_\epsilon, \tag{27}$$

and for all $\phi \in H^1(\Omega_\epsilon)$ it holds that

$$\int_{\Omega_\epsilon} \frac{J_\epsilon^j u_\epsilon^j - J_\epsilon^{j-1} u_\epsilon^{j-1}}{\Delta t} \phi dx + \int_{\Omega_\epsilon} D_\epsilon^j \nabla u_\epsilon^j \cdot \nabla \phi dx$$

$$= \int_{\Omega_\epsilon} V_\epsilon^j \cdot \nabla u_\epsilon^j \phi dx + \int_{\Omega_\epsilon} J_\epsilon^j f_\epsilon^j \phi dx + \int_{\Omega_\epsilon} u_\epsilon^j \frac{J_\epsilon^j - J_\epsilon^{j-1}}{\Delta t} \phi dx \tag{28}$$

$$- \int_{\Gamma_\epsilon} G_\epsilon^{j-1}(u_\epsilon^{j-1} - \rho)\phi J_\epsilon^j d\sigma.$$

The time-discrete coefficients above are defined as

$$
\begin{aligned}
G_\epsilon^{j-1}(x) &:= \frac{\epsilon}{|\epsilon\Gamma|} \int_{\Gamma_\epsilon\left(\epsilon\left[\frac{x}{\epsilon}\right]\right)} g\left(u_\epsilon^{j-1}, \frac{R_\epsilon^{j-1}}{\epsilon}\right) d\sigma, \\
S_\epsilon^j(x) &:= x + (R_\epsilon^j - \epsilon\overline{R})\chi_0\left(\frac{x}{\epsilon}\right)\nu_0\left(\frac{x}{\epsilon}\right), \\
F_\epsilon^j(x) &:= I_n + \frac{R_\epsilon^j - \epsilon\overline{R}}{\epsilon}\left[\nabla_y\left(\chi_0\nu_0\right)\right]\left(\frac{x}{\epsilon}\right), \\
J_\epsilon^j(x) &:= \det F_\epsilon^j(x), \\
D_\epsilon^j(x) &:= J_\epsilon^j(x)\left[F_\epsilon^j(x)\right]^{-1} D\left(S_\epsilon^j(x)\right)\left[F_\epsilon^j(x)\right]^{-T}, \\
V_\epsilon^j(x) &:= J_\epsilon^j(x)\left[F_\epsilon^j(x)\right]^{-T}\frac{R_\epsilon^j(x) - R_\epsilon^{j-1}(x)}{\Delta t}\chi_0\left(\frac{x}{\epsilon}\right)\nu_0\left(\frac{x}{\epsilon}\right), \\
f_\epsilon^j(x) &:= f_\epsilon(t^j, x).
\end{aligned} \tag{29}
$$

Remark 4. *Given $u_\epsilon^{j-1}, R_\epsilon^{j-1}$, the existence of R_ϵ^j follows straightforwardly from (27). Moreover, this also gives J_ϵ^j, see (29), which means that, in fact, (28) providing u_ϵ^j is linear. Moreover, as follows from Proposition 1 below, R_ϵ^j is constant on every microcell $\epsilon(Y + \mathbf{k})$, assuming that the same holds for R_ϵ^{j-1}. Therefore we have $F_\epsilon^j = \nabla S_\epsilon^j$ in classical sense.*

The constant $C > 0$ used below is generic and does not depend on Δt or ϵ.

Proposition 1. *Let $j \in \{1, \dots, N\}$ and $(u_\epsilon^{j-1}, R_\epsilon^{j-1}) \in H^1(\Omega_\epsilon) \times L^\infty(\Omega_\epsilon)$ with $\epsilon \underline{R} \leq R_\epsilon^{j-1} \leq \epsilon \overline{R}$, and assume R_ϵ^{j-1} is constant on every microscopic cell $\epsilon(Y + \mathbf{k})$ with $\mathbf{k} \in K_\epsilon$. Then, for Δt small enough, we have $R_\epsilon^j \in L^\infty(\Omega_\epsilon)$ with $\epsilon \underline{R} \leq R_\epsilon^j \leq \epsilon \overline{R}$. Also, R_ϵ^j is constant on $\epsilon(Y + \mathbf{k})$, and one has*

$$\left\| \frac{R_\epsilon^j - R_\epsilon^{j-1}}{\Delta t} \right\|_{L^\infty(\Omega_\epsilon)} \leq C\epsilon. \tag{30}$$

Proof. From the assumption $g \in L^\infty(\mathbb{R} \times \mathbb{R})$ in (A3) one obtains that $R_\epsilon^j \in L^\infty(\Omega_\epsilon)$, and that (30) holds true. Further, the definition of R_ϵ^j implies that the function is constant on every $\epsilon(Y + \mathbf{k})$ with $\mathbf{k} \in K_\epsilon$. It remains to check the lower and upper bound of R_ϵ^j: We only prove the upper bound, since the lower bound follows by the same arguments. For almost every $x \in \Omega_\epsilon$, if $R_\epsilon^{j-1}(x) \in [\epsilon(\overline{R} - \delta_0), \epsilon \overline{R}]$ we use Assumption (A4) to obtain

$$R_\epsilon^j(x) \leq R_\epsilon^{j-1}(x) \leq \epsilon \overline{R}.$$

If $\epsilon \underline{R} \leq R_\epsilon^{j-1}(x) < \epsilon(\overline{R} - \delta_0)$, taking $\Delta t \leq \delta_0 / \|g\|_{L^\infty(\mathbb{R} \times \mathbb{R})}$ gives

$$R_\epsilon^j = R_\epsilon^{j-1} + \Delta t G_\epsilon^{j-1} \leq R_\epsilon^{j-1} + \epsilon \Delta t \|g\|_{L^\infty(\mathbb{R} \times \mathbb{R})} \leq R_\epsilon^{j-1} + \epsilon \delta_0 < \epsilon \overline{R}.$$

Finally, (30) is a direct consequence of Assumption (A3). $\qquad \square$

The boundedness of R_ϵ^j gives uniform estimates for the time-discrete coefficients, as follows from the lemma below.

Lemma 1. *Let $j \in \{1, \dots, N\}$, and $u_\epsilon^{j-1} \in H^1(\Omega_\epsilon)$, as well as $R_\epsilon^{j-1}, R_\epsilon^j \in L^\infty(\Omega_\epsilon)$ be given. Assume that $\epsilon \underline{R} \leq R_\epsilon^{j-1}, R_\epsilon^j \leq \epsilon \overline{R}$ and $R_\epsilon^{j-1}, R_\epsilon^j$ are constant in every cell $\epsilon(Y + \mathbf{k})$ with $\mathbf{k} \in K_\epsilon$. Then we have $S_\epsilon^j, F_\epsilon^j \in C^\infty(\overline{\Omega_\epsilon})^n$, satisfying*

$$\|S_\epsilon^j\|_{C^1(\overline{\Omega_\epsilon})} = \|S_\epsilon^j\|_{C^0(\overline{\Omega_\epsilon})} + \|F_\epsilon^j\|_{C^0(\overline{\Omega_\epsilon})} \leq C. \tag{31a}$$

Similarly, for the Jacobi-determinant we have $J_\epsilon^j \in C^\infty(\overline{\Omega_\epsilon})$, with

$$\left(\frac{\underline{R}}{\overline{R}} \right)^{n-1} \leq J_\epsilon^j \leq C. \tag{31b}$$

Further, we have $D_\epsilon^j \in C^0(\overline{\Omega_\epsilon})^{n \times n}$ and $V_\epsilon^j \in C^0(\overline{\Omega_\epsilon})^n$, with

$$\|D_\epsilon^j\|_{C^0(\overline{\Omega_\epsilon})} + \|V_\epsilon^j\|_{L^\infty(\Omega_\epsilon)} \leq C. \tag{31c}$$

Additionally, it holds that

$$\left\| \frac{J_\epsilon^j - J_\epsilon^{j-1}}{\Delta t} \right\|_{C^0(\overline{\Omega_\epsilon})} \leq C. \tag{31d}$$

Proof. First, we observe that, since $F_\epsilon^j = \nabla S_\epsilon^j \in C^\infty(\overline{\Omega_\epsilon})^{n \times n}$, the regularity and the estimates for F_ϵ^j are a direct consequence of the results for S_ϵ^j.

Next, since R_ϵ^j is constant on $\epsilon(Y + \mathbf{k})$ for every $\mathbf{k} \in K_\epsilon$, χ_0 and ν_0 are smooth, and $\chi_0\left(\frac{\cdot}{\epsilon}\right)$ has compact support in $\epsilon(Y + \mathbf{k})$. This immediately implies the smoothness of S_ϵ^j, F_ϵ^j, J_ϵ^j, and V_ϵ^j. Further, we have $F_\epsilon^j = E_n$ on the lateral boundary of the micro

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cell $\partial(\epsilon(Y + \mathbf{k}))$, and, Assumption (A6) gives $D_\epsilon^j \in C^0(\overline{\Omega_\epsilon})^{n \times n}$. Inequality (31a) follows immediately from $\epsilon \underline{R} \leq R_\epsilon^j \leq \epsilon \overline{R}$. For the estimate (31b), we first notice that the regularity of the determinant and the essential boundedness of F_ϵ^j implies $\|J_\epsilon^j\|_{L^\infty(\Omega_\epsilon)} \leq C$. The lower bound is obtained from the results in Appendix A.

To prove (31c), one needs to control the inverse of $F_\epsilon^j(x)$. This is achieved by using the following inequality, stating that a $C > 0$ exists such that, for any invertible matrix $A \in \mathbb{R}^{n \times n}$, and for an arbitrary norm in $\mathbb{R}^{n \times n}$,

$$\|A^{-1}\| \leq \frac{C}{|\det(A)|} \|A\|^{n-1}. \tag{32}$$

This inequality is straightforward for the spectral norm in $\mathbb{R}^{n \times n}$, and extends to any norm by the equivalence of norms in finite dimensional vector spaces. Then, by (31b), for almost every $x \in \Omega_\epsilon$ one gets

$$\left\| \left[F_\epsilon^j(x) \right]^{-1} \right\| \leq \frac{C}{J_\epsilon^j(x)} \left\| F_\epsilon^j(x) \right\|^{n-1} \leq C,$$

implying (31c). Finally, (31d) follows from the local Lipschitz continuity of the determinant, the essential boundedness of F_ϵ^j, and the inequality (30). $\quad\square$

We are now able to prove the existence of a weak solution for the time-discrete problems, together with some ϵ-dependent *a priori* estimates.

Proposition 2. *Let $j \in \{1, \ldots, N\}$ and $(u_\epsilon^{j-1}, R_\epsilon^{j-1}) \in H^1(\Omega_\epsilon) \times L^\infty(\Omega)$ be given such that $\epsilon \underline{R} \leq R_\epsilon^{j-1} \leq \epsilon \overline{R}$, and R_ϵ^{j-1} is constant on every cell $\epsilon(Y + \mathbf{k})$, $\mathbf{k} \in K_\epsilon$. For Δt small enough, there exists a weak solution $(u_\epsilon^j, R_\epsilon^j)$ of the time-discrete problem from Definition 4.*

Proof. As mentioned in Remark 4, the existence of R_ϵ^j is straightforward, while u_ϵ^j is the weak solution of a linear problem. Moreover, from the proof of Proposition 1, one gets that choosing $\Delta t \leq \delta_0 / \|g\|_{L^\infty(\mathbb{R} \times \mathbb{R})}$ guarantees that the estimates there, and therefore in Lemma 1 as well, are valid. The variational equation (28) can be rewritten as

$$a_\epsilon^j(u_\epsilon^j, \phi) = l_\epsilon^j(\phi) \qquad \text{for all } \phi \in H^1(\Omega_\epsilon),$$

with $a_\epsilon^j : H^1(\Omega_\epsilon) \times H^1(\Omega_\epsilon) \to \mathbb{R}$ and $l_\epsilon^j : H^1(\Omega_\epsilon) \to \mathbb{R}$ defined by

$$a_\epsilon^j(u, \phi) := \int_{\Omega_\epsilon} \frac{J_\epsilon^{j-1}}{\Delta t} u \phi dx + \int_{\Omega_\epsilon} D_\epsilon^j \nabla u \cdot \nabla \phi dx - \int_{\Omega_\epsilon} V_\epsilon^j \cdot \nabla u \phi dx,$$

$$l_\epsilon^j(\phi) := \int_{\Omega_\epsilon} \frac{J_\epsilon^{j-1}}{\Delta t} u_\epsilon^{j-1} \phi dx + \int_{\Omega_\epsilon} J_\epsilon^j f_\epsilon^j \phi dx - \int_{\Gamma_\epsilon} G_\epsilon^{j-1} \left(u_\epsilon^{j-1} - \rho \right) J_\epsilon^j \phi d\sigma.$$

Obviously, a_ϵ^j is bilinear and l_ϵ^j is linear, and both are continuous. To apply the Lax-Milgram Lemma we have to prove the coercivity of the bilinear-form a_ϵ^j. Since J_ϵ^{j-1} is bounded away from 0 (uniformly w.r.t. ϵ and j) and D is coercive, we get the coercivity of D_ϵ^j, i.e., there exists a constant $d > 0$, such that for every $x \in \Omega_\epsilon$ it holds that

$$D_\epsilon^j(x) \xi \cdot \xi \geq d \quad \text{for all } \xi \in \mathbb{R}^n.$$

Further, the essential boundedness of V_ϵ^j implies the existence of a constant $C_1 > 0$, such that

$$-\int_{\Omega_\epsilon} u \, V_\epsilon^j \cdot \nabla u dx \geq -C_1 \|u\|_{L^2(\Omega_\epsilon)}^2 - \frac{d}{2} \|\nabla u\|_{L^2(\Omega_\epsilon)}^2.$$

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Together, we obtain for all $u \in H^1(\Omega_\epsilon)$

$$a_\epsilon^j(u, u) \geq \int_{\Omega_\epsilon} \left[\frac{J_\epsilon^{j-1}}{\Delta t} - C_1 \right] |u|^2 dx + \frac{d}{2} \|\nabla u\|_{L^2(\Omega_\epsilon)}^2.$$

Recalling the lower bounds for J_ϵ^{j-1}, which are uniform with respect to j, this implies the coercivity of a_ϵ^j for Δt small enough, and the claim is proved. $\quad\square$

Proposition 3. *Assume Δt small enough. Then, a $C > 0$ not depending on of ϵ and Δt exists such that the sequence of time-discrete solutions $(u_\epsilon^j, R_\epsilon^j)$ $(j = 1, \ldots, N)$ satisfies the estimates*

$$\max_{j \in \{1,\ldots,N\}} \|u_\epsilon^j\|_{L^2(\Omega_\epsilon)} + \Delta t \sum_{j=1}^N \|\nabla u_\epsilon^j\|_{L^2(\Omega_\epsilon)}^2 \leq C.$$

Proof. We take $\Delta t u_\epsilon^j$ as test function in (28) to obtain

$$\begin{aligned}
A_\epsilon^1 + A_\epsilon^2 := & \int_{\Omega_\epsilon} \left(J_\epsilon^j u_\epsilon^j - J_\epsilon^{j-1} u_\epsilon^{j-1} \right) u_\epsilon^j dx + \Delta t \int_{\Omega_\epsilon} D_\epsilon^j \nabla u_\epsilon^j \cdot \nabla u_\epsilon^j dx \\
= & \Delta t \int_{\Omega_\epsilon} V_\epsilon^j \cdot \nabla u_\epsilon^j u_\epsilon^j dx + \Delta t \int_{\Omega_\epsilon} J_\epsilon^j f_\epsilon^j u_\epsilon^j dx + \Delta t \int_{\Omega_\epsilon} \left(u_\epsilon^j \right)^2 \frac{J_\epsilon^j - J_\epsilon^{j-1}}{\Delta t} dx \\
& - \Delta t \int_{\Gamma_\epsilon} G_\epsilon^{j-1} \left(u_\epsilon^{j-1} - \rho \right) u_\epsilon^j J_\epsilon^j d\sigma \\
=: & \sum_{i=1}^4 B_\epsilon^i.
\end{aligned}$$

For the first term on the left-hand side we have

$$\begin{aligned}
A_\epsilon^1 = & \frac{1}{2} \left\| \sqrt{J_\epsilon^j} u_\epsilon^j \right\|_{L^2(\Omega_\epsilon)}^2 + \frac{1}{2} \left\| \sqrt{J_\epsilon^j} u_\epsilon^j - \sqrt{J_\epsilon^{j-1}} u_\epsilon^{j-1} \right\|_{L^2(\Omega_\epsilon)}^2 \\
& - \frac{1}{2} \left\| \sqrt{J_\epsilon^{j-1}} u_\epsilon^{j-1} \right\|_{L^2(\Omega_\epsilon)}^2 - \int_{\Omega_\epsilon} \sqrt{J_\epsilon^{j-1}} \left(\sqrt{J_\epsilon^{j-1}} - \sqrt{J_\epsilon^j} \right) u_\epsilon^j u_\epsilon^{j-1} dx.
\end{aligned}$$

We denote the last term above by B_ϵ^5. Now, using the coercivity of D_ϵ^j as in the proof of Proposition 2, we obtain

$$A_\epsilon^2 \geq \Delta t \, d \left\| \nabla u_\epsilon^j \right\|_{L^2(\Omega_\epsilon)}^2.$$

To estimate the terms B_ϵ^i $(i = 1, \ldots, 5)$, we first observe that

$$|B_\epsilon^1| \leq \|V_\epsilon^j\|_{L^\infty(\Omega_\epsilon)} \Delta t \|\nabla u_\epsilon^j\|_{L^2(\Omega_\epsilon)} \|u_\epsilon^j\|_{L^2(\Omega_\epsilon)} \leq C \Delta t \|u_\epsilon^j\|_{L^2(\Omega_\epsilon)}^2 + \frac{d\Delta t}{4} \|\nabla u_\epsilon^j\|_{L^2(\Omega_\epsilon)}^2.$$

For the second and the third term, we use Lemma 1 to obtain

$$\begin{aligned}
|B_\epsilon^2| &\leq C\Delta t \left(\|f_\epsilon^j\|_{L^2(\Omega_\epsilon)}^2 + \|u_\epsilon^j\|_{L^2(\Omega_\epsilon)}^2 \right), \\
|B_\epsilon^3| &\leq C\Delta t \|u_\epsilon^j\|_{L^2(\Omega_\epsilon)}^2.
\end{aligned}$$

For B_ϵ^4 we use that $\|G_\epsilon^{j-1}\|_{L^\infty(\Gamma_\epsilon)} \leq C\epsilon$, and the scaled trace inequality,

$$\epsilon \|u_\epsilon\|_{L^2(\Gamma_\epsilon)}^2 \leq C \|u_\epsilon\|_{L^2(\Omega_\epsilon)}^2 + \frac{d}{4} \epsilon^2 \|\nabla u_\epsilon\|_{L^2(\Omega_\epsilon)}^2 \qquad (33)$$

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for all $u_\epsilon \in H^1(\Omega_\epsilon)$, which can be obtained by a standard decomposition argument for Ω_ϵ. Then, for $\epsilon \leq 1$,

$$
\begin{aligned}
|B_\epsilon^4| &\leq C\Delta t \left(1 + \epsilon \|u_\epsilon^j\|_{L^2(\Gamma_\epsilon)}^2 + \epsilon \|u_\epsilon^{j-1}\|_{L^2(\Gamma_\epsilon)}^2 \right) \\
&\leq C\Delta t \left(1 + \|u_\epsilon^j\|_{L^2(\Omega_\epsilon)}^2 + \|u_\epsilon^{j-1}\|_{L^2(\Omega_\epsilon)}^2 \right) + \frac{d\Delta t}{4} \left(\|\nabla u_\epsilon^j\|_{L^2(\Omega_\epsilon)}^2 + \|\nabla u_\epsilon^{j-1}\|_{L^2(\Omega_\epsilon)}^2 \right).
\end{aligned}
$$

For the last term B_ϵ^5 occuring in the term A_ϵ^1, we use the Lipschitz continuity of $\sqrt{\cdot}$ away from 0, to obtain with (31d)

$$
\begin{aligned}
|B_\epsilon^5| &\leq C\Delta t \int_{\Omega_\epsilon} \frac{\left| \sqrt{J_\epsilon^j} - \sqrt{J_\epsilon^{j-1}} \right|}{\Delta t} |u_\epsilon^j||u_\epsilon^{j-1}| dx \\
&\leq C\Delta t \int_{\Omega_\epsilon} \frac{\left| J_\epsilon^j - J_\epsilon^{j-1} \right|}{\Delta t} \cdot \left(|u_\epsilon^j|^2 + |u_\epsilon^{j-1}|^2 \right) dx \\
&\leq C\Delta t \left(\|u_\epsilon^j\|_{L^2(\Omega_\epsilon)}^2 + \|u_\epsilon^{j-1}\|_{L^2(\Omega_\epsilon)}^2 \right).
\end{aligned}
$$

Altogether, we obtain

$$
\begin{aligned}
&\frac{1}{2} \|\sqrt{J_\epsilon^j} u_\epsilon^j\|_{L^2(\Omega_\epsilon)}^2 + \frac{1}{2} \|\sqrt{J_\epsilon^j} u_\epsilon^j - \sqrt{J_\epsilon^{j-1}} u_\epsilon^{j-1}\|_{L^2(\Omega_\epsilon)}^2 \\
&- \frac{1}{2} \|\sqrt{J_\epsilon^{j-1}} u_\epsilon^{j-1}\|_{L^2(\Omega_\epsilon)}^2 + \frac{d\Delta t}{2} \|\nabla u_\epsilon^j\|_{L^2(\Omega_\epsilon)}^2 \\
&\leq C\Delta t \left(1 + \|f_\epsilon^j\|_{L^2(\Omega_\epsilon)}^2 + \|u_\epsilon^{j-1}\|_{L^2(\Omega_\epsilon)}^2 + \|u_\epsilon^j\|_{L^2(\Omega_\epsilon)}^2 \right) + \frac{d\Delta t}{4} \|\nabla u_\epsilon^{j-1}\|_{L^2(\Omega_\epsilon)}^2.
\end{aligned}
$$

Summing up these inequalities over $j = 1$ to an arbitrary $k \in \{1, \ldots, N\}$, we use again the boundedness of J_ϵ^j proved above and Assumption (A1) to obtain that, for Δt small enough,

$$
\begin{aligned}
&\|u_\epsilon^k\|_{L^2(\Omega_\epsilon)}^2 + \sum_{j=1}^k \|\sqrt{J_\epsilon^j} u_\epsilon^j - \sqrt{J_\epsilon^{j-1}} u_\epsilon^{j-1}\|_{L^2(\Omega_\epsilon)}^2 + \Delta t \sum_{j=1}^k \|\nabla u_\epsilon^j\|_{L^2(\Omega_\epsilon)}^2 \\
&\leq C \left(1 + \|u_\epsilon^{\mathrm{in}}\|_{L^2(\Omega_\epsilon)}^2 + \Delta t \sum_{j=1}^k \|f_\epsilon^j\|_{L^2(\Omega_\epsilon)}^2 + \Delta t \sum_{j=1}^k \|u_\epsilon^j\|_{L^2(\Omega_\epsilon)}^2 + \Delta t \|\nabla u_\epsilon^{\mathrm{in}}\|_{L^2(\Omega_\epsilon)}^2 \right) \\
&\leq C \left(1 + \|f\|_{C^0(\overline{\Omega}, L^2(\Omega))} + \Delta t \sum_{j=1}^{k-1} \|u_\epsilon^j\|_{L^2(\Omega_\epsilon)}^2 \right).
\end{aligned}
$$

Now, the discrete Gronwall-inequality implies the desired result. $\qquad\square$

4.2 The interpolation in time

In this section, we define the piecewise constant and piecewise linear interpolation of the discrete values with respect to time. Let $(u_\epsilon^j, R_\epsilon^j)$ for $j \in \{1, \ldots, N\}$ be a weak solution of the time-discrete problem from Proposition 2 with initial condition $(u_\epsilon^{\mathrm{in}}, R_\epsilon^{\mathrm{in}})$. For

$t \in (t^{j-1}, t^j]$ with $j = 1, \ldots, N$, we define:

$$\overline{U}^{\epsilon}_{\Delta t}(t) := u_{\epsilon}^j, \qquad \widehat{U}^{\epsilon}_{\Delta t}(t) := u_{\epsilon}^{j-1} + \frac{t - t^{j-1}}{\Delta t}\left(u_{\epsilon}^j - u_{\epsilon}^{j-1}\right),$$

$$\overline{R}^{\epsilon}_{\Delta t}(t) := R_{\epsilon}^j, \qquad \widehat{R}^{\epsilon}_{\Delta t}(t) := R_{\epsilon}^{j-1} + \frac{t - t^{j-1}}{\Delta t}\left(R_{\epsilon}^j - R_{\epsilon}^{j-1}\right),$$

$$\overline{J}^{\epsilon}_{\Delta t}(t) := J_{\epsilon}^j, \qquad \widehat{J}^{\epsilon}_{\Delta t}(t) := J_{\epsilon}^{j-1} + \frac{t - t^{j-1}}{\Delta t}\left(J_{\epsilon}^j - J_{\epsilon}^{j-1}\right),$$

$$\overline{W}^{\epsilon}_{\Delta t}(t) := J_{\epsilon}^j u_{\epsilon}^j, \qquad \widehat{W}^{\epsilon}_{\Delta t}(t) := J_{\epsilon}^{j-1} u_{\epsilon}^{j-1} + \frac{t - t^{j-1}}{\Delta t}\left(J_{\epsilon}^j u_{\epsilon}^j - J_{\epsilon}^{j-1} u_{\epsilon}^{j-1}\right).$$

For $t < 0$ we extend the functions constantly by the initial value. We emphasize that for the existence proof of the continuous problem (17) it is not necessary to consider the interpolations $\overline{W}^{\epsilon}_{\Delta t}$ and $\widehat{W}^{\epsilon}_{\Delta t}$. However, we use these functions to show the boundedness of $\partial_t(J_{\epsilon} u_{\epsilon})$ in $L^2((0,T), H^1(\Omega_{\epsilon})')$ uniformly with respect to ϵ, what is obtained directly from the *a priori* estimates of $\partial_t \widehat{W}^{\epsilon}_{\Delta t}$. We are not able to prove such a uniform bound for $\partial_t u_{\epsilon}$. This is caused by the fact that the gradient of J_{ϵ}^j is of order ϵ^{-1}, i.e., we have

$$\|\nabla J_{\epsilon}^j\|_{L^\infty(\Omega_{\epsilon})} \leq C\epsilon^{-1} \quad \text{for all } j \in \{1, \ldots, N\}. \tag{34}$$

This can be easily seen from the definition of J_{ϵ}^j and the results from Proposition 1. Let us start with the *a priori* estimates for $\overline{W}^{\epsilon}_{\Delta t}$ and $\widehat{W}^{\epsilon}_{\Delta t}$:

Lemma 2. *The functions $\overline{W}^{\epsilon}_{\Delta t}$ and $\widehat{W}^{\epsilon}_{\Delta t}$ fulfill the following a priori estimates:*

$$\|\overline{W}^{\epsilon}_{\Delta t}\|_{L^\infty((0,T),L^2(\Omega_{\epsilon}))} + \epsilon\|\nabla\overline{W}^{\epsilon}_{\Delta t}\|_{L^2((0,T),L^2(\Omega_{\epsilon}))} \leq C,$$

$$\|\widehat{W}^{\epsilon}_{\Delta t}\|_{L^\infty((0,T),L^2(\Omega_{\epsilon}))} + \epsilon\|\nabla\widehat{W}^{\epsilon}_{\Delta t}\|_{L^2((0,T),L^2(\Omega_{\epsilon}))} + \|\partial_t\widehat{W}^{\epsilon}_{\Delta t}\|_{L^2((0,T),H^1(\Omega_{\epsilon})')} \leq C.$$

Proof. Except for the time derivative, all estimates follow directly from Lemma 1, Proposition 3, and (34). Hence, we only give a detailed proof for the inequality of $\partial_t\widehat{W}^{\epsilon}_{\Delta t}$. Choose $\phi \in H^1(\Omega_{\epsilon})$ with $\|\phi\|_{H^1(\Omega_{\epsilon})} \leq 1$ as a test function in (28), to obtain for $t \in (t^{j-1}, t^j)$

$$\int_{\Omega_{\epsilon}} \partial_t\widehat{W}^{\epsilon}_{\Delta t}\phi dx = -\int_{\Omega_{\epsilon}} D_{\epsilon}^j \nabla u_{\epsilon}^j \cdot \nabla\phi dx + \int_{\Omega_{\epsilon}} V_{\epsilon} \cdot \nabla u_{\epsilon}^j \phi dx + \int_{\Omega_{\epsilon}} J_{\epsilon}^j f_{\epsilon}^j \phi dx$$
$$+ \int_{\Omega_{\epsilon}} u_{\epsilon}^j \frac{J_{\epsilon}^j - J_{\epsilon}^{j-1}}{\Delta t}\phi dx - \int_{\Gamma_{\epsilon}} G_{\epsilon}^{j-1}(u_{\epsilon}^{j-1} - \rho)J_{\epsilon}^j \phi d\sigma. \tag{35}$$

We only consider in more detail the boundary term. From the trace inequality (33) we obtain

$$\|\phi\|_{L^2(\Gamma_{\epsilon})} \leq C\left(\frac{1}{\sqrt{\epsilon}}\|\phi\|_{L^2(\Omega_{\epsilon})} + \sqrt{\epsilon}\|\nabla\phi\|_{L^2(\Omega_{\epsilon})}\right) \leq C\epsilon^{-\frac{1}{2}}.$$

Using again the trace inequality, the estimates $|\Gamma_{\epsilon}| \leq C\epsilon^{-1}$ and $\|G_{\epsilon}^{j-1}\|_{L^\infty(\Gamma_{\epsilon})} \leq C\epsilon$, and Lemma 1, we get

$$-\int_{\Gamma_{\epsilon}} G_{\epsilon}^{j-1}(u_{\epsilon}^{j-1} - \rho)J_{\epsilon}^j \phi d\sigma \leq C\epsilon\left(\|u_{\epsilon}^{j-1}\|_{L^2(\Omega_{\epsilon})}\|\phi\|_{L^2(\Gamma_{\epsilon})} + \epsilon^{-\frac{1}{2}}\|\phi\|_{L^2(\Gamma_{\epsilon})}\right)$$
$$\leq C\left(\|u_{\epsilon}^{j-1}\|_{L^2(\Omega_{\epsilon})} + \|\nabla u_{\epsilon}^{j-1}\|_{L^2(\Omega_{\epsilon})} + 1\right).$$

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Hence, from equation (35) we obtain using again Lemma 1

$$\left\|\partial_t\widehat{W}_{\Delta t}^\epsilon\right\|_{H^1(\Omega_\epsilon)'} \le C\left(1 + \|u_\epsilon^{j-1}\|_{L^2(\Omega_\epsilon)} + \|u_\epsilon^j\|_{L^2(\Omega_\epsilon)} + \|\nabla u_\epsilon^j\|_{L^2(\Omega_\epsilon)} + \|f_\epsilon^j\|_{L^2(\Omega_\epsilon)}\right).$$

The *a priori* estimates from Proposition 3 and the assumption on f and the initial conditions imply

$$\left\|\partial_t\widehat{W}_{\Delta t}^\epsilon\right\|_{L^2((0,T),H^1(\Omega_\epsilon)')}^2 \le C\Delta t\sum_{j=1}^N\left(1 + \|u_\epsilon^j\|_{L^2(\Omega_\epsilon)}^2 + \|\nabla u_\epsilon^{j-1}\|_{L^2(\Omega_\epsilon)}^2 + \|f_\epsilon^j\|_{L^2(\Omega_\epsilon)}^2\right) \le C,$$

what gives us the desired result. $\qquad\square$

Lemma 3. *The functions $\overline{J}_{\Delta t}^\epsilon$ and $\widehat{J}_{\Delta t}^\epsilon$ fulfill the following a priori estimates:*

$$\left\|\overline{J}_{\Delta t}^\epsilon\right\|_{L^\infty((0,T),C^0(\overline{\Omega_\epsilon}))} + \epsilon\left\|\nabla\overline{J}_{\Delta t}^\epsilon\right\|_{L^\infty((0,T),C^0(\overline{\Omega_\epsilon}))} \le C,$$
$$\left\|\widehat{J}_{\Delta t}^\epsilon\right\|_{W^{1,\infty}((0,T),C^0(\overline{\Omega_\epsilon}))} + \epsilon\left\|\nabla\widehat{J}_{\Delta t}^\epsilon\right\|_{C^0([0,T]\times\overline{\Omega_\epsilon})} \le C.$$

Additionally, both functions are bounded from below, i.e., there exits a constant $c_0 > 0$, such that for almost every $t \in (0,T)$ and every $x \in \overline{\Omega_\epsilon}$ it holds that

$$c_0 \le \overline{J}_{\Delta t}^\epsilon(t,x), \quad c_0 \le \widehat{J}_{\Delta t}^\epsilon(t,x).$$

Proof. This is an easy consequence of (34) and Lemma 1. $\qquad\square$

Lemma 4. *We have $\overline{R}_{\Delta t}^\epsilon \in L^\infty((0,T)\times\Omega)$ and $\widehat{R}_{\Delta t}^\epsilon \in W^{1,\infty}((0,T),L^2(\Omega))$ such that $\widehat{R}_{\Delta t}^\epsilon, \partial_t\widehat{R}_{\Delta t}^\epsilon \in L^\infty((0,T)\times\Omega)$. For almost every $t \in (0,T)$ the functions $\overline{R}_{\Delta t}^\epsilon$, $\widehat{R}_{\Delta t}^\epsilon$, and $\partial_t\widehat{R}_{\Delta t}^\epsilon$ are constant on every micro cell $\epsilon(Y+k)$ for $k \in K_\epsilon$. Further, we have the estimate*

$$\left\|\overline{R}_{\Delta t}^\epsilon\right\|_{L^\infty((0,T)\times\Omega)} + \left\|\widehat{R}_{\Delta t}^\epsilon\right\|_{L^\infty((0,T)\times\Omega)} + \left\|\partial_t\widehat{R}_{\Delta t}^\epsilon\right\|_{L^\infty((0,T)\times\Omega)} \le C\epsilon.$$

Proof. This follows directly from Proposition 1. $\qquad\square$

Lemma 5. *For $\overline{U}_{\Delta t}^\epsilon$ and $\widehat{U}_{\Delta t}^\epsilon$ we have the following a priori estimates:*

$$\left\|\overline{U}_{\Delta t}^\epsilon\right\|_{L^\infty((0,T),L^2(\Omega_\epsilon))} + \left\|\nabla\overline{U}_{\Delta t}^\epsilon\right\|_{L^2((0,T),L^2(\Omega_\epsilon))} \le C,$$
$$\left\|\widehat{U}_{\Delta t}^\epsilon\right\|_{L^\infty((0,T),L^2(\Omega_\epsilon))} + \left\|\nabla\widehat{U}_{\Delta t}^\epsilon\right\|_{L^2((0,T),L^2(\Omega_\epsilon))} + \epsilon\left\|\partial_t\widehat{U}_{\Delta t}^\epsilon\right\|_{L^2((0,T),H^1(\Omega_\epsilon)')} \le C.$$

Proof. Again, we only consider the time-derivative, because the other inequalities are direct consequences of Proposition 3. An elemental calculation gives us for $t \in (t^{j-1},t^j)$

$$\partial_t\widehat{U}_{\Delta t}^\epsilon(t) = \frac{1}{J_\epsilon^j}\partial_t\widehat{W}_{\Delta t}^\epsilon(t) - \frac{u_\epsilon^{j-1}}{J_\epsilon^j}\frac{J_\epsilon^j - J_\epsilon^{j-1}}{\Delta t}.$$

Hence, for all $\phi \in H^1(\Omega_\epsilon)$ with $\|\phi\|_{H^1(\Omega_\epsilon)} \le 1$, we obtain

$$\left\langle\partial_t\widehat{U}_{\Delta t}^\epsilon(t),\phi\right\rangle_{H^1(\Omega_\epsilon)',H^1(\Omega_\epsilon)} \le \left\|\partial_t\widehat{W}_{\Delta t}^\epsilon(t)\right\|_{H^1(\Omega_\epsilon)'}\left\|\frac{\phi}{J_\epsilon^j}\right\|_{H^1(\Omega_\epsilon)} + \left\|\frac{u_\epsilon^{j-1}}{J_\epsilon^j}\right\|_{L^2(\Omega_\epsilon)}\left\|\frac{J_\epsilon^j - J_\epsilon^{j-1}}{\Delta t}\right\|_{L^\infty(\Omega_\epsilon)}$$
$$\le C\left(\epsilon^{-1}\left\|\partial_t\widehat{W}_{\Delta t}^\epsilon(t)\right\|_{H^1(\Omega_\epsilon)'} + 1\right),$$

where for the second inequality we used Lemma 1 and Proposition 3, see also (34). From Lemma 2 we obtain the desired result. $\qquad\square$

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Based on these uniform *a priori* estimates with respect to Δt, we pass to the limit $\Delta t \to 0$ to obtain a solution of the continuous microscopic model (17).

Corollary 1. *There exists $u_\epsilon \in L^2((0,T), H^1(\Omega_\epsilon)) \cap H^1((0,T), H^1(\Omega_\epsilon)')$, such that up to a subsequence of $\{dt\}$ it holds for $\frac{1}{2} < \beta < 1$*

$$\widehat{U}^\epsilon_{\Delta t} \rightharpoonup u_\epsilon \qquad\qquad \text{weakly in } L^2((0,T), H^1(\Omega_\epsilon)),$$
$$\widehat{U}^\epsilon_{\Delta t} \to u_\epsilon \qquad\qquad \text{in } L^2((0,T), H^\beta(\Omega_\epsilon)),$$
$$\partial_t \widehat{U}^\epsilon_{\Delta t} \rightharpoonup \partial_t u_\epsilon \qquad\qquad \text{weakly in } L^2((0,T), H^1(\Omega_\epsilon)'),$$
$$\overline{U}^\epsilon_{\Delta t} \to u_\epsilon \qquad\qquad \text{in } L^2((0,T), L^2(\Omega_\epsilon)),$$
$$\nabla \overline{U}^\epsilon_{\Delta t} \rightharpoonup \nabla u_\epsilon \qquad\qquad \text{weakly in } L^2((0,T), L^2(\Omega_\epsilon)),$$
$$\overline{U}^\epsilon_{\Delta t} \to u_\epsilon \qquad\qquad \text{in } L^2((0,T), L^2(\Gamma_\epsilon)),$$

for $\Delta t \to 0$. Further, we have the following a priori estimate:

$$\|u_\epsilon\|_{L^2((0,T), H^1(\Omega_\epsilon))} + \epsilon\|\partial_t u_\epsilon\|_{L^2((0,T), H^1(\Omega_\epsilon)')} \leq C.$$

Additionally, we have $u_\epsilon \in L^\infty((0,T), L^2(\Omega_\epsilon))$ with

$$\|u_\epsilon\|_{L^\infty((0,T), L^2(\Omega_\epsilon))} \leq C.$$

Proof. From the *a priori* estimates in Lemma 5 we obtain the existence of functions $u_\epsilon, u_\epsilon^* \in L^2((0,T), H^1(\Omega_\epsilon))$ and $\partial_t u_\epsilon \in L^2((0,T), H^1(\Omega_\epsilon)')$, such that up to a subsequence

$$\widehat{U}^\epsilon_{\Delta t} \rightharpoonup u_\epsilon \qquad\qquad \text{weakly in } L^2((0,T), H^1(\Omega_\epsilon)),$$
$$\widehat{U}^\epsilon_{\Delta t} \rightharpoonup u_\epsilon \qquad\qquad \text{weakly* in } L^\infty((0,T), L^2(\Omega_\epsilon)),$$
$$\overline{U}^\epsilon_{\Delta t} \rightharpoonup u_\epsilon^* \qquad\qquad \text{weakly in } L^2((0,T), H^1(\Omega_\epsilon)),$$
$$\partial_t \widehat{U}^\epsilon_{\Delta t} \rightharpoonup \partial_t u_\epsilon \qquad\qquad \text{weakly in } L^2((0,T), H^1(\Omega_\epsilon)').$$

The *a priori* estimates follow directly from the lower semi-continuity of the norm with respect to the weak- and weak*-topology. Since for $\frac{1}{2} < \beta < 1$ the embedding $H^1(\Omega_\epsilon) \hookrightarrow H^\beta(\Omega_\epsilon)$ is compact, the Aubin-Lions Lemma implies the strong convergence of $\widehat{U}^\epsilon_{\Delta t}$ in $L^2((0,T), H^\beta(\Omega_\epsilon))$. Let us check $u_\epsilon = u_\epsilon^*$. For $t \in (t^{j-1}, t^j)$ it holds that

$$\big\|\widehat{U}^\epsilon_{\Delta t}(t) - \overline{U}^\epsilon_{\Delta t}(t)\big\|^2_{L^2(\Omega_\epsilon)} \leq 4\|u_\epsilon^j - u_\epsilon^{j-1}\|^2_{L^2(\Omega_\epsilon)}$$
$$= 4\Delta t(\partial_t \widehat{U}^\epsilon_{\Delta t}, u_\epsilon^j - u_\epsilon^{j-1})_{L^2(\Omega_\epsilon)}$$
$$\leq 4\Delta t\big\|\partial_t \widehat{U}^\epsilon_{\Delta t}(t)\big\|_{H^1(\Omega_\epsilon)'}\|u_\epsilon^j - u_\epsilon^{j-1}\|_{H^1(\Omega_\epsilon)}.$$

Integration with respect to time implies

$$\big\|\widehat{U}^\epsilon_{\Delta t} - \overline{U}^\epsilon_{\Delta t}\big\|^2_{L^2((0,T), L^2(\Omega_\epsilon))} \leq 4\Delta t \sum_{j=1}^N \int_{t^{j-1}}^{t^j} \big\|\partial_t \widehat{U}^\epsilon_{\Delta t}(t)\big\|_{H^1(\Omega_\epsilon)'}\|u_\epsilon^j - u_\epsilon^{j-1}\|_{H^1(\Omega_\epsilon)} dt$$

$$\leq 2\Delta t\left(\big\|\partial_t \widehat{U}^\epsilon_{\Delta t}\big\|^2_{L^2((0,T), H^1(\Omega_\epsilon)')} + \Delta t \sum_{j=1}^N \|u_\epsilon^j - u_\epsilon^{j-1}\|^2_{H^1(\Omega_\epsilon)}\right)$$

$$\leq \frac{C\Delta t}{\epsilon} \overset{\Delta t \to 0}{\longrightarrow} 0,$$

where in the last estimate we used the results from Proposition 3 and Lemma 5. The triangle inequality implies $u_\epsilon = u_\epsilon^*$ and the strong convergence of $\overline{U}_{\Delta t}^\epsilon$ in $L^2((0,T), L^2(\Omega_\epsilon))$.

It remains to show the strong convergence of the traces. This follows directly from the following interpolated trace inequality: There exists a constant $C_\epsilon > 0$ (which may depend on ϵ), such that for all $v \in H^1(\Omega_\epsilon)$ it holds that

$$\|v\|_{L^2(\Gamma_\epsilon)}^2 \leq C_\epsilon \left(\|v\|_{L^2(\Omega_\epsilon)}^2 + \|v\|_{L^2(\Omega_\epsilon)} \|v\|_{H^1(\Omega_\epsilon)} \right).$$

Now, the strong convergence of $\overline{U}_{\Delta t}^\epsilon$ in $L^2((0,T), L^2(\Omega_\epsilon))$ implies the strong convergence of the traces in $L^2((0,T), L^2(\Gamma_\epsilon))$. $\square$

Corollary 2. *There exists $R_\epsilon \in W^{1,\infty}((0,T), L^2(\Omega))$ constant on every micro cell $\epsilon(Y+k)$ for $k \in K_\epsilon$ and $R_\epsilon, \partial_t R_\epsilon \in L^\infty(0,T) \times \Omega$, such that up to a subsequence of $\{dt\}$ it holds for every $p \in [1,\infty)$*

$$\begin{aligned} \widehat{R}_{\Delta t}^\epsilon &\overset{*}{\rightharpoonup} R_\epsilon && weakly^* \ in \ W^{1,\infty}((0,T), L^2(\Omega)), \\ \overline{R}_{\Delta t}^\epsilon &\to R_\epsilon && in \ L^\infty((0,T) \times \Omega), \\ \overline{R}_{\Delta t}^\epsilon &\to R_\epsilon && in \ L^\infty((0,T) \times \Gamma_\epsilon), \end{aligned}$$

for $\Delta t \to 0$ and all $\alpha \in (0,1)$. Further, we have

$$\|R_\epsilon\|_{W^{1,\infty}((0,T), L^2(\Omega))} + \|R_\epsilon\|_{L^\infty((0,T) \times \Omega)} + \|R_\epsilon\|_{L^\infty((0,T) \times \Gamma_\epsilon)} \leq C\epsilon.$$

Proof. The first convergence follows directly from the *a priori* estimates in Lemma 4. Further, due to the Arzelà-Ascoli theorem, $\widehat{R}_{\Delta t}^\epsilon$ converges to R_ϵ in $C^{0,\alpha}([0,T], L^\infty(\Omega))$ for arbitrary $\alpha \in (0,1)$. We emphasize that for every $t \in [0,T]$ the set $\{\widehat{R}_{\Delta t}^\epsilon(t)\}_{\Delta t}$ is a finite dimensional subset of $L^\infty(\Omega)$ and is therefore relatively compact. The convergence in $C^{0,\alpha}([0,T], L^\infty(\Omega))$ implies the strong convergence of $\widehat{R}_{\Delta t}^\epsilon$ in $L^\infty((0,T) \times \Omega)$ and, since $\widehat{R}_{\Delta t}^\epsilon$ is constant on every micro cell, in $L^\infty((0,T) \times \Gamma_\epsilon)$. Let us check the strong convergence of $\overline{R}_{\Delta t}^\epsilon$. We have

$$\left\| \overline{R}_{\Delta t}^\epsilon(t,\cdot) - \widehat{R}_{\Delta t}^\epsilon(t,\cdot) \right\|_{L^\infty((0,T) \times \Omega)} \leq 2 \max_{j \in \{1,\dots,N\}} \left\| R_\epsilon^j - R_\epsilon^{j-1} \right\|_{L^\infty(\Omega)} \leq C\Delta t.$$

The triangle inequality implies $\overline{R}_{\Delta t}^\epsilon \to R_\epsilon$ in $L^\infty((0,T) \times \Omega)$. The inequality follows from Lemma 4. $\square$

We will prove later in Lemma 6 that also the time derivative of $\partial_t \widehat{R}_{\Delta t}^\epsilon$ converges strongly in the L^p-sense. Now, let us define the function

$$J_\epsilon(t,x) = \det \left(I_n + \frac{R_\epsilon(t,x) - \epsilon \overline{R}}{\epsilon} \nabla_y (\chi_0 \nu_0) \left(\frac{x}{\epsilon} \right) \right). \tag{36}$$

In the following corollary we prove the convergence of $\overline{J}_{\Delta t}^\epsilon$ and $\widehat{J}_{\Delta t}^\epsilon$ to the function J_ϵ for $\Delta t \to 0$.

Corollary 3. *Up to a subsequence, for every $p \in (1,\infty)$ there holds the following convergence*

$$\begin{aligned} \overline{J}_{\Delta t}^\epsilon &\to J_\epsilon && in \ L^\infty((0,T) \times \Omega_\epsilon), \\ \widehat{J}_{\Delta t}^\epsilon &\rightharpoonup J_\epsilon && weakly \ in \ L^p((0,T), W^{1,p}(\Omega_\epsilon)), \\ \widehat{J}_{\Delta t}^\epsilon &\to J_\epsilon && in \ L^\infty((0,T) \times \Omega_\epsilon), \\ \partial_t \widehat{J}_{\Delta t}^\epsilon &\rightharpoonup \partial_t J_\epsilon && weakly \ in \ L^p((0,T) \times \Omega_\epsilon) \end{aligned}$$

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for $\Delta t \to 0$. We emphasize that, due to the structure of J_ϵ in (36), we have $J_\epsilon \in C^0([0,T] \times \overline{\Omega_\epsilon})$. Further, the following inequality holds

$$\|J_\epsilon\|_{C^0([0,T] \times \overline{\Omega_\epsilon})} + \|\partial_t J_\epsilon\|_{L^\infty((0,T) \times \Omega_\epsilon)} + \epsilon\|\nabla J_\epsilon\|_{C^0([0,T] \times \overline{\Omega_\epsilon})} \leq C.$$

Proof. First of all, let us check the convergence of $\overline{J}_{\Delta t}^\epsilon$. Using the Lipschitz continuity of the determinant and the essential boundedness of $\overline{R}_{\Delta t}^\epsilon$ with respect to Δt, we obtain

$$\left\|J_\epsilon - \overline{J}_{\Delta t}^\epsilon\right\|_{L^\infty((0,T) \times \Omega_\epsilon)} \leq \frac{C}{\epsilon}\left\|R_\epsilon - \overline{R}_{\Delta t}^\epsilon\right\|_{L^\infty((0,T) \times \Omega_\epsilon)} \stackrel{\Delta t \to 0}{\longrightarrow} 0.$$

Further, we have from (31d)

$$\|\widehat{J}_{\Delta t}^\epsilon - \overline{J}_{\Delta t}^\epsilon\|_{L^\infty((0,T) \times \Omega_\epsilon)} \leq 2\|J_\epsilon^j - J_\epsilon^{j-1}\|_{L^\infty((0,T) \times \Omega_\epsilon)} \leq C\Delta t.$$

The weak convergence of $\widehat{J}_{\Delta t}^\epsilon$ in $L^p((0,T), W^{1,p}(\Omega_\epsilon)) \cap W^{1,p}((0,T), L^p(\Omega_\epsilon))$ now follows directly from the *a priori* estimates in Lemma 3. The estimates for J_ϵ are a consequence of Lemma 3, formula (36), and the inequality for R_ϵ in Corollary 2. $\qquad\square$

Corollary 4. *Up to a subsequence, we have the following convergence results:*

$$\widehat{W}_{\Delta t}^\epsilon \rightharpoonup J_\epsilon u_\epsilon \qquad\qquad in\ L^2((0,T), H^1(\Omega_\epsilon)),$$
$$\partial_t \widehat{W}_{\Delta t}^\epsilon \rightharpoonup \partial_t\left(J_\epsilon u_\epsilon\right) \qquad\qquad in\ L^2((0,T), H^1(\Omega_\epsilon)'),$$
$$\overline{W}_{\Delta t}^\epsilon \to J_\epsilon u_\epsilon \qquad\qquad in\ L^2((0,T), L^2(\Omega_\epsilon))$$

for $\Delta t \to 0$. Further, we have the a priori estimate

$$\|J_\epsilon u_\epsilon\|_{L^\infty((0,T), L^2(\Omega_\epsilon))} + \epsilon\|\nabla(J_\epsilon u_\epsilon)\|_{L^2((0,T), L^2(\Omega_\epsilon))} + \|\partial_t(J_\epsilon u_\epsilon)\|_{L^2((0,T), H^1(\Omega_\epsilon)')} \leq C.$$

Proof. The existence of a limit function W_ϵ such that the convergence results above hold (up to a subsequence) for $J_\epsilon u_\epsilon$ replaced by W_ϵ follows as in the proofs above. The identity $W_\epsilon = J_\epsilon u_\epsilon$ follows easily from the pointwise almost everywhere convergence of $\widehat{J}_{\Delta t}^\epsilon \widehat{U}_{\Delta t}^\epsilon$ to $J_\epsilon u_\epsilon$ (for a subsequence) and $\widehat{W}_{\Delta t}^\epsilon = \widehat{J}_{\Delta t}^\epsilon \widehat{U}_{\Delta t}^\epsilon$. $\qquad\square$

4.3 Existence for the continuous model

Now, we want to pass to the limit $\Delta t \to 0$ in the time-discrete problem from Definition 4. Let us define the piecewise constant coefficients in the following way for $t \in (t^{j-1}, t^j)$:

$$G_{\Delta t}^\epsilon(t,x) := G_\epsilon^{j-1}(x), \quad S_{\Delta t}^\epsilon(t,x) := S_\epsilon^j(x), \quad F_{\Delta t}^\epsilon(t,x) := \nabla S_\epsilon^j(x),$$
$$D_{\Delta t}^\epsilon(t,x) := D_\epsilon^j, \quad V_{\Delta t}^\epsilon(t,x) := V_\epsilon^j, \quad f_{\Delta t}^\epsilon := f_\epsilon^j.$$

Multiplying the variational equation (28) with $\phi \in C_0^\infty\left([0,T) \times \overline{\Omega_\epsilon}\right)$ and integrating with respect to time gives us:

$$\int_0^T \int_{\Omega_\epsilon} \partial_t \widehat{W}_{\Delta t}^\epsilon \phi\, dx dt + \int_0^T \int_{\Omega_\epsilon} D_{\Delta t}^\epsilon \nabla\overline{U}_{\Delta t}^\epsilon \cdot \nabla\phi\, dx dt$$
$$= \int_0^T \int_{\Omega_\epsilon} V_{\Delta t}^\epsilon \cdot \nabla\overline{U}_{\Delta t}^\epsilon \phi\, dx dt + \int_0^T \int_{\Omega_\epsilon} \overline{J}_{\Delta t}^\epsilon f_{\Delta t}^\epsilon \phi\, dx dt \qquad (37)$$
$$+ \int_0^T \int_{\Omega_\epsilon} \overline{U}_{\Delta t}^\epsilon \partial_t \widehat{J}_{\Delta t}^\epsilon \phi\, dx dt - \int_{\Gamma_\epsilon} G_{\Delta t}^\epsilon (\overline{U}_{\Delta t}^\epsilon(\cdot - \Delta t) - \rho)\overline{J}_{\Delta t}^\epsilon \phi\, d\sigma dt.$$

To pass to the limit $\Delta t \to 0$ in (37), we need the following convergence results for the interpolated coefficients:

Lemma 6. *Up to a subsequence of $\{dt\}$ we have the following convergence results for arbitrary $p \in [1, \infty)$*

$$\partial_t \widehat{R}^\epsilon_{\Delta t} = G^\epsilon_{\Delta t} \to G_\epsilon(u_\epsilon, R_\epsilon) \qquad\qquad in \ L^p((0,T) \times \Gamma_\epsilon), \qquad (38a)$$

$$D^\epsilon_{\Delta t} \to D_\epsilon \qquad\qquad in \ L^p((0,T) \times \Omega_\epsilon), \qquad (38b)$$

$$V^\epsilon_{\Delta t} \to V_\epsilon \qquad\qquad in \ L^p((0,T) \times \Omega_\epsilon) \qquad (38c)$$

for $\Delta t \to 0$.

Proof. We start with the convergence (38a). We have

$$\left\| G^\epsilon_{\Delta t} - G_\epsilon(u_\epsilon, R_\epsilon) \right\|^2_{L^2((0,T)\times\Gamma_\epsilon)}$$

$$\leq C\epsilon^2 \sum_{j=1}^N \sum_{\mathbf{k} \in K_\epsilon} \int_{t^{j-1}}^{t^j} \int_{\Gamma_\epsilon(\epsilon k)} \left| g\left(u^{j-1}_\epsilon, \frac{R^{j-1}_\epsilon}{\epsilon} \right) - g\left(u_\epsilon, \frac{R_\epsilon}{\epsilon} \right) \right|^2 d\sigma dt.$$

$$= C\epsilon^2 \sum_{j=1}^n \int_{t^{j-1}}^{t^j} \left\| g\left(u^{j-1}_\epsilon, \frac{R^{j-1}_\epsilon}{\epsilon} \right) - g\left(u_\epsilon, \frac{R_\epsilon}{\epsilon} \right) \right\|^2_{L^2(\Gamma_\epsilon)} dt.$$

Using the Lipschitz continuity of g, we obtain

$$\left\| G^\epsilon_{\Delta t} - G_\epsilon(u_\epsilon, R_\epsilon) \right\|^2_{L^2((0,T)\times\Gamma_\epsilon)}$$

$$\leq C\epsilon^2 \left(\left\| \overline{U}^\epsilon_{\Delta t}(\cdot - \Delta t) - \overline{U}^\epsilon_{\Delta t} \right\|^2_{L^2((0,T)\times\Gamma_\epsilon)} + \left\| \overline{R}^\epsilon_{\Delta t}(\cdot - \Delta t) - \overline{R}^\epsilon_{\Delta t} \right\|^2_{L^2((0,T)\times\Gamma_\epsilon)} \right).$$

We emphasize that for $t < 0$ we have $\overline{U}^\epsilon_{\Delta t} = u^{in}_\epsilon$ and $\overline{R}^\epsilon_{\Delta t} = R^{in}_\epsilon$. Because of the convergence results from Corollary 1 and 2, as well as the Kolmogorov compactness theorem, the right-hand side tends to 0 for $\Delta t \to 0$. Hence, we have $\partial_t \widehat{R}^\epsilon_{\Delta t} = G^\epsilon_{\Delta t} \to G_\epsilon(u_\epsilon, R_\epsilon)$ in $L^2((0,T) \times \Gamma_\epsilon)$ for $\Delta t \to 0$. Since $G^\epsilon_{\Delta t} \in L^\infty((0,T) \times \Gamma_\epsilon)$, the dominated convergence theorem of Lebesgue implies that the convergence also holds up to a subsequence in $L^p((0,T) \times \Gamma_\epsilon)$ for all $p \in (1, \infty)$.

For the convergence (38b) we first notice that

$$D^\epsilon_{\Delta t} = \overline{J}^\epsilon_{\Delta t} [F^\epsilon_{\Delta t}]^{-1} D(S^\epsilon_{\Delta t}) [F^\epsilon_{\Delta t}]^{-1},$$

and therefore, due to the convergence results from Corollary 2 and 3, we obtain $D^\epsilon_{\Delta t} \to D_\epsilon$ almost everywhere in $(0,T) \times \Omega_\epsilon$ up to a subsequence. Since $D^\epsilon_{\Delta t}$ is essential bounded, the dominated convergence theorem of Lebesgue implies $D^\epsilon_{\Delta t} \to D_\epsilon$ in $L^p((0,T) \times \Omega_\epsilon)$. The last convergence (38c) follows by similar arguments, where we here additionally use the strong convergence (and therefore the pointwise almost everywhere convergence up to a subsequence) of $\partial_t \widehat{R}^\epsilon_{\Delta t}$ proved above. We emphasize that due to the strong convergence of $\partial_t \widehat{R}^\epsilon_{\Delta t}$ is also valid in $L^p((0,T) \times \Omega)$, since it is constant on every micro cell $\epsilon(Y + \mathbf{k})$. $\qquad \square$

As a consequence of the previous results, we can easily prove the existence result in Theorem 1:

Proof of Theorem 1. This follows directly by passing to the limit in the variational equation (37) and in (19). To establish the initial condition, we just integrate by parts with respect to time in (17) and (37). Since this is standard, we skip the details. $\qquad \square$

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In the following Lemma we summarize the *a priori* estimates for the weak solution (u_ϵ, R_ϵ) which will be necessary for the derivation of the macroscopic model:

Lemma 7. *A constant $C > 0$ not depending on ϵ exists such that the weak solution (u_ϵ, R_ϵ) from Theorem 1 satisfies the a priori estimates*

$$\epsilon\|\partial_t u_\epsilon\|_{L^2((0,T),H^1(\Omega_\epsilon)')} + \|u_\epsilon\|_{L^\infty((0,T),L^2(\Omega_\epsilon))} + \|u_\epsilon\|_{L^2((0,T),H^1(\Omega_\epsilon))} \leq C,$$

$$\frac{1}{\epsilon}\|R_\epsilon\|_{L^\infty((0,T)\times\Omega_\epsilon)} + \frac{1}{\epsilon}\|\partial_t R_\epsilon\|_{L^\infty((0,T)\times\Omega_\epsilon)} \leq C,$$

$$\|\partial_t(J_\epsilon u_\epsilon)\|_{L^2((0,T),H^1(\Omega_\epsilon)')} + \|J_\epsilon u_\epsilon\|_{L^\infty((0,T),L^2(\Omega_\epsilon))}$$
$$+\epsilon\|\nabla(J_\epsilon u_\epsilon)\|_{L^2((0,T),L^2(\Omega_\epsilon))} \leq C.$$

Also, for the Hanzawa transformation S_ϵ and the Jacobi-determinant J_ϵ one has

$$\|S_\epsilon\|_{L^\infty((0,T)\times\Omega_\epsilon)} + \|\nabla S_\epsilon\|_{L^\infty((0,T)\times\Omega_\epsilon)} + \frac{1}{\epsilon}\|\partial_t S_\epsilon\|_{L^\infty((0,T)\times\Omega_\epsilon)} \leq C,$$

$$\|J_\epsilon\|_{L^\infty((0,T)\times\Omega_\epsilon)} + \|\partial_t J_\epsilon\|_{L^\infty((0,T)\times\Omega_\epsilon)} + \epsilon\|\nabla J_\epsilon\|_{L^\infty((0,T)\times\Omega_\epsilon)} \leq C.$$

Finally, there exists a constant $c_0 > 0$ such that (independently of ϵ)

$$c_0 \leq J_\epsilon(t,x) \quad \text{for almost every } (t,x) \in (0,T) \times \Omega_\epsilon.$$

Additionally, the differential equation (7d) for R_ϵ and the boundedness of g immediately implies with the mean value theorem the following uniform estimates:

Corollary 5. *It holds that*

$$\|R_\epsilon\|_{C^{0,1}([0,T],L^\infty(\Omega))} + \|J_\epsilon\|_{C^{0,1}([0,T],L^\infty(\Omega))} \leq C.$$

5 Derivation of the macroscopic model

In this section we show that the sequence of micro solutions (u_ϵ, R_ϵ) converges in a suitable sense to the solution of the macro-model (23). For this we prove two-scale compactness results which allow to pass to the limit in the variational equation for the micro model (18). However, the *a priori* estimates in Lemma 7 only guarantee weak convergence for u_ϵ and R_ϵ, and this is not enough to pass to limit in (18), since also the strong convergence of the coefficients, which depend in a nonlinear way on the micro solution itself, is needed. Also to pass to the limit in the boundary term and in equation (19) strong convergence is necessary. We refer the reader to the Appendix B for a short overview on the two-scale convergence.

5.1 Compactness results

The crucial point is to obtain the strong convergence of u_ϵ, R_ϵ, and $\partial_t R_\epsilon$. Standard arguments for u_ϵ of Aubin-Lions-type, see [30], fail, since we have no uniform bound for the time derivative $\partial_t u_\epsilon$ with respect to ϵ (remember, the norm is of order ϵ^{-1}). Hence, it makes sense to consider also the sequence $J_\epsilon u_\epsilon$ which has bounded time derivative, but which gradient behaves badly with respect to ϵ. We will see that together with the uniform bounds on J_ϵ this is enough to guarantee to strong convergence of u_ϵ via a variational argument. The strong convergence of R_ϵ is obtained by using the Kolmogorov-compactness theorem, see [6], based on uniform bounds for the difference of small shifts.

Combining the strong two-scale convergence of u_ϵ and R_ϵ, we are able to show the strong convergence of ∂R_ϵ and identify its limit. Before proceeding, we observe that, since Ω_ϵ is connected, there exists an extension of u_ϵ to the whole domain Ω (also denoted u_ϵ to avoid an excess of notations) such that ([1, 10], see Section 5.1.3 for more details)

$$\|u_\epsilon\|_{L^2((0,T),H^1(\Omega))} \leq C. \tag{39}$$

5.1.1 Weak two-scale compactness results

We start with some (weak) two-scale convergence results which follow directly from the *a priori* estimates in Lemma 7. For the definition and some basic compactness results for the two-scale convergence see the Appendix B:

Corollary 6. *Up to a subsequence the following convergence results for the microscopic solutions* (u_ϵ, R_ϵ) *of problem* (7) *are valid:*

(i) *There exist* $u_0 \in L^2((0,T), H^1(\Omega))$ *and* $u_1 \in L^2((0,T) \times \Omega, H^1_{\mathrm{per}}(Y^*)/\mathbb{R}))$, *such that*

$$\begin{aligned}
u_\epsilon &\to u_0 && \text{\textit{in the two-scale sense in}} \ L^2, \\
\nabla u_\epsilon &\to \nabla_x u_0 + \nabla_y u_1 && \text{\textit{in the two-scale sense in}} \ L^2, \\
u_\epsilon &\rightharpoonup^* u_0 && \text{\textit{weakly}}^* \ \text{\textit{in}} \ L^\infty((0,T), L^2(\Omega)).
\end{aligned}$$

(ii) *There exists* $R_0 \in W^{1,\infty}((0,T), L^2(\Omega))$ *with* $R_0 \in L^\infty((0,T) \times \Omega)$ *and* $\partial_t R_0 \in L^\infty((0,T) \times \Omega)$, *such that for every* $p \in [1, \infty)$

$$\begin{aligned}
\epsilon^{-1} R_\epsilon &\rightharpoonup^* R_0 && \text{\textit{weakly}}^* \ \text{\textit{in}} \ W^{1,\infty}((0,T), L^2(\Omega)), \\
\epsilon^{-1} R_\epsilon &\to R_0 && \text{\textit{in the two-scale sense in}} \ L^p, \\
\epsilon^{-1} \partial_t R_\epsilon &\to \partial_t R_0 && \text{\textit{in the two-scale sense in}} \ L^p.
\end{aligned}$$

Additionally the two-scale convergences above also hold in the two-scale sense on Γ_ϵ *in* L^p.

(iii) *Defining*

$$S_0(t,x,y) := y + \big(R_0(t,x) - \overline{R}\big)(\chi_0 \nu_0)(y),$$

it holds for every $p \in (1, \infty)$ *that*

$$\begin{aligned}
S_\epsilon &\to x && \text{\textit{in}} \ L^\infty((0,T) \times \Omega), \\
\nabla S_\epsilon &\to \nabla_y S_0 && \text{\textit{in the two-scale sense in}} \ L^p.
\end{aligned}$$

Proof. These results follow directly from the *a priori* estimates in Lemma 7 and standard two-scale compactness results from Lemma 10 in the Appendix B. We point out that R_0 is independent of the microscopic variable $y \in Y$, since R_ϵ is constant on every microscopic cell $\epsilon(Y + \mathbf{k})$ with $\mathbf{k} \in K_\epsilon$. $\qquad\square$

5.1.2 Strong convergence of R_ϵ

Next, we prove the strong convergence of $\epsilon^{-1} R_\epsilon$ in the L^p-sense, where we make use of the Kolmogorov-compactness theorem, see for example [6]. We first introduce the following notation for $0 < h$:

$$\Omega^h := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > h\}.$$

Further, for $\boldsymbol{\ell} \in \mathbb{Z}^n$ we define

$$K_\epsilon^{\boldsymbol{\ell}} := \{\mathbf{k} \in \mathbb{Z}^n \,:\, \epsilon(Y + \mathbf{k} + \boldsymbol{\ell}) \cup \epsilon(Y + \mathbf{k}) \subset \Omega\},$$

and

$$\Omega_\epsilon^{\boldsymbol{\ell}} := \operatorname{int} \bigcup_{\mathbf{k} \in K_\epsilon^{\boldsymbol{\ell}}} \epsilon(\overline{Y^*} + \mathbf{k}), \qquad \Gamma_\epsilon^{\boldsymbol{\ell}} := \bigcup_{\mathbf{k} \in K_\epsilon^{\boldsymbol{\ell}}} \epsilon(\Gamma + \mathbf{k}).$$

We emphasize that $\Omega_\epsilon^{\boldsymbol{\ell}} \subset \Omega^{|\epsilon \boldsymbol{\ell}|}$ and for $|\epsilon \boldsymbol{\ell}| < \frac{h}{2}$ and ϵ small enough it holds that

$$\Omega^h \subset \operatorname{int} \bigcup_{\mathbf{k} \in K_\epsilon^{\boldsymbol{\ell}}} \epsilon(\overline{Y} + \mathbf{k}). \tag{40}$$

Proposition 4. *For all $p \in [1, \infty)$ it holds up to a subsequence that*

$$\epsilon^{-1} R_\epsilon \to R_0 \quad \text{in } L^p((0,T) \times \Omega).$$

Additionally, we have $R_0 \in L^\infty((0,T), H^1(\Omega))$.

Proof. It is enough to show that $R_\epsilon \to R_0$ in $L^2((0,T) \times \Omega)$. Then, the desired result follows from the dominated convergence theorem of Lebesgue. We use the Kolmogorov-compactness theorem. Let $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$, then we have to show:

$$\sup_{\epsilon > 0} \epsilon^{-1} \|R_\epsilon(t + s, x + \xi) - R_\epsilon(t, x)\|_{L^2((0,T) \times \Omega)} \xrightarrow{(s,\xi) \to 0} 0,$$

where we can extend R_ϵ by zero to $\mathbb{R} \times \mathbb{R}^n$. Since $\epsilon^{-1} R_\epsilon$ is uniformly bounded with respect to ϵ in $L^\infty((0,T) \times \Omega)$, it is enough to show for $0 < h \ll 1$ and $|s| + |\xi| < \frac{h}{2}$ that

$$\sup_{\epsilon > 0} \epsilon^{-1} \|R_\epsilon(t + s, x + \xi) - R_\epsilon(t, x)\|_{L^2((h,T-h) \times \Omega^h)} \xrightarrow{(s,\xi) \to 0} 0.$$

First of all, we have

$$\|R_\epsilon(t + s, x + \xi) - R_\epsilon(t, x)\|_{L^2((h,T-h) \times \Omega^h)}$$
$$\leq \|R_\epsilon(t + s, x) - R_\epsilon(t, x)\|_{L^2((h,T-h) \times \Omega)} + \|R_\epsilon(t, x + \xi) - R_\epsilon(t, x)\|_{L^2((0,T) \times \Omega^h)}.$$

For the first term we obtain from the essential boundedness of g:

$$\epsilon^{-2} \|R_\epsilon(t + s, x) - R_\epsilon(t, x)\|_{L^2((h,T-h) \times \Omega)}^2$$
$$= \int_0^{T-s} \frac{1}{|\epsilon \Gamma|^2} \left(\int_t^{t+s} \int_{\Gamma_\epsilon\left(t, \epsilon\left[\frac{x}{\epsilon}\right]\right)} g\left(u_\epsilon(\tau, z), \frac{R_\epsilon(\tau, z)}{\epsilon}\right) dz d\tau \right)^2 dt$$
$$\leq C s^2 \|g\|_{L^\infty(\mathbb{R} \times \mathbb{R}^n)}^2.$$

Let us consider the term including the shifts in the spatial variable. We have (see [19, Proof of Theorem 13]) for $t \in (0, T)$

$$\|R_\epsilon(t, x + \xi) - R_\epsilon(t, x)\|_{L^2(\Omega^h)} \leq \sum_{\mathbf{j} \in \{0,1\}^n} \left\| R_\epsilon\left(t, x + \epsilon \mathbf{j} + \epsilon\left[\frac{\xi}{\epsilon}\right]\right) - R_\epsilon(t, x) \right\|_{L^2(\Omega^h)}. \tag{41}$$

Let $\ell := \ell(\epsilon, \mathbf{j}) := \mathbf{j} + \left[\dfrac{\xi}{\epsilon}\right]$. Then for $t \in (0, T)$ we obtain with (19)

$$\epsilon^{-2}\|R_\epsilon(t, x + \epsilon\ell) - R_\epsilon(t, x)\|_{L^2(\Omega^h)}^2 \leq C\epsilon^{-2}\|R_\epsilon^{\mathrm{in}}(x + \ell\epsilon) - R_\epsilon^{\mathrm{in}}\|_{L^2(\Omega^h)}^2$$

$$+ C \int_{\Omega^h} \left[\frac{1}{|\epsilon\Gamma|} \int_0^t \int_{\Gamma_\epsilon\left(t, \epsilon\left[\frac{x}{\epsilon}\right]\right)} g\left(u_\epsilon(s, z + \epsilon l), \frac{R_\epsilon(s, z + \epsilon l)}{\epsilon}\right)\right.$$

$$\left. - g\left(u_\epsilon(s, z), \frac{R_\epsilon(s, z)}{\epsilon}\right) dz ds\right]^2 dx =: A_\epsilon^1 + A_\epsilon^2.$$

For the first term A_ϵ^1 including the initial value we immediately obtain from the Assumption (A2) that $A_\epsilon^1 \leq C|\ell\epsilon|^2$. For A_ϵ^2, using the trace inequality (33), the Lipschitz continuity of g, (40), and that R_ϵ constant on every micro cell gives

$$A_\epsilon^2 \leq \frac{C}{|\epsilon\Gamma|} \int_{\Omega^h} \int_0^t \int_{\epsilon\Gamma} \left|u_\epsilon\left(s, z + \epsilon\left[\frac{x}{\epsilon}\right] + \epsilon\ell\right) - u_\epsilon\left(s, z + \epsilon\left[\frac{x}{\epsilon}\right]\right)\right|^2$$

$$+ \epsilon^{-2}\left|R_\epsilon\left(s, z + \epsilon\left[\frac{x}{\epsilon}\right] + \epsilon\ell\right) - \left(s, z + \epsilon\left[\frac{x}{\epsilon}\right]\right)\right|^2 d\sigma_z ds dx$$

$$\leq \sum_{\mathbf{k} \in K_\epsilon^\ell} \left\{\frac{C}{|\epsilon\Gamma|}\epsilon^n \int_0^t \int_{\epsilon\Gamma} |u_\epsilon(s, z + \epsilon\mathbf{k} + \epsilon\ell) - u_\epsilon(s, z + \epsilon\mathbf{k})|^2 d\sigma_z ds\right.$$

$$\left. + \frac{C}{|\epsilon\Gamma|\epsilon^2} \int_{\epsilon(Y + \mathbf{k}) \cap \Omega^h} \int_0^t \int_{\epsilon\Gamma} |R_\epsilon(s, z + \epsilon\mathbf{k} + \epsilon\ell) - R_\epsilon(s, z + \epsilon\mathbf{k})|^2 d\sigma_z ds dx\right\}$$

$$= C\epsilon \int_0^t \int_{\Gamma_\epsilon^\ell} |u_\epsilon(s, z + \epsilon\ell) - u_\epsilon(s, z)|^2 d\sigma_z ds$$

$$+ C \int_0^t \int_{\Omega^h} \epsilon^{-2}|R_\epsilon(s, z + \epsilon\ell) - R_\epsilon(s, z)|^2 dz ds.$$

Therefore,

$$A_\epsilon^2 \leq C \int_0^t \int_{\Omega_\epsilon^\ell} |u_\epsilon(s, z + \epsilon\ell) - u_\epsilon(s, z)|^2 + \epsilon^2|\nabla u_\epsilon(s, z + \epsilon\ell) - \nabla u_\epsilon(s, z)|^2 dz ds$$

$$+ \int_0^t \int_{\Omega^h} \epsilon^{-2}|R_\epsilon(s, z + \epsilon\ell) - R_\epsilon(s, z)|^2 dz ds$$

$$\leq C \int_0^t \int_{\Omega^{|\epsilon\ell|}} \int_0^1 |\nabla u_\epsilon(s, z + \lambda\epsilon\ell)|^2 \cdot |\epsilon\ell|^2 d\lambda dz ds$$

$$+ C\epsilon^2 + C\epsilon^{-2}\|R_\epsilon(\cdot, \cdot + \epsilon\ell) - R_\epsilon\|_{L^2((0,t)\times\Omega^h)}^2.$$

We can now use the *a priori* estimate for ∇u_ϵ, as proved in Lemma 7. Although these are obtained for the perforated domain Ω_ϵ, they remain valid for the extension of u_ϵ to Ω, see (39). From the above, we obtain

$$A_\epsilon^2 \leq C\left(\|\nabla u_\epsilon\|_{L^2((0,t)\times\Omega)}^2 |\epsilon\ell|^2 + C\epsilon^2 + \epsilon^{-2}\|R_\epsilon(\cdot, \cdot + \epsilon\ell) - R_\epsilon\|_{L^2((0,t)\times\Omega^h)}^2\right)$$

$$\leq C\left(|\epsilon\ell|^2 + C\epsilon^2 + \epsilon^{-2}\|R_\epsilon(\cdot, \cdot + \epsilon\ell) - R_\epsilon\|_{L^2((0,t)\times\Omega^h)}^2\right).$$

The Gronwall-inequality gives

$$\epsilon^{-2}\|R_\epsilon(t, x + \epsilon\ell) - R_\epsilon(t, x)\|_{L^\infty((0,T), L^2(\Omega^h))}^2 \leq C\left(|\epsilon\ell|^2 + \epsilon^2\right).$$

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With $\ell = \mathbf{j} + \left[\dfrac{\xi}{\epsilon}\right]$ and (41) we obtain

$$\epsilon^{-1}\|R_\epsilon(t, x+\xi) - R_\epsilon(t, x)\|_{L^\infty((0,T),L^2(\Omega^h))} \le C\left(|\xi| + \epsilon\right). \tag{42}$$

Hence, for arbitrary $\rho > 0$ there exists $\epsilon_0, \delta_0 > 0$, such that, for all $\xi \in \mathbb{R}^n$ with $|\xi| < \delta_0$ it holds that

$$\sup_{\epsilon \le \epsilon_0} \epsilon^{-1}\|R_\epsilon(t, x+\xi) - R_\epsilon(t, x)\|_{L^\infty((0,T),L^2(\Omega^h))} \le \rho. \tag{43}$$

Since the sequence ϵ is countable there are only finitely many ϵ (denoted by $\epsilon_1, \ldots, \epsilon_N$) such that $\epsilon_i > \epsilon_0$ for $i = 1, \ldots, N$. Due to the Kolmogorov-compactness theorem there exists $\delta_i > 0$ such that for all $|\xi| < \delta_0$

$$\epsilon^{-1}\|R_{\epsilon_i}(t, x+\xi) - R_{\epsilon_i}(t, x)\|_{L^\infty((0,T),L^2(\Omega^h))} \le \rho.$$

Hence, (43) is also valid for all $|\xi| < \delta := \min\{\delta_0, \ldots, \delta_N\}$ and if we take the supremum over all ϵ. This gives the strong convergence of $\epsilon^{-1}R_\epsilon$ to R_0 in $L^2((0,T) \times \Omega)$.

It remains to establish the higher regularity of R_0 with respect to x. This is an easy consequence of (42). In fact, using the strong convergence (up to a subsequence) of R_ϵ in $L^p((0,T) \times \Omega)$ for $p \in [1, \infty)$ showed above, we obtain with $|\xi| < \frac{h}{2}$ for $\epsilon \to 0$ in (42)

$$\left\|\frac{R_0(t, x+\xi) - R_0}{|\xi|}\right\|_{L^p((0,T),L^2(\Omega^h))} \le C,$$

with a constant $C > 0$ which can be chosen independently of p. This implies $R_0 \in L^p((0,T), H^1(\Omega))$ with L^p-norm bounded uniformly with respect to p and the proposition is proved. $\qquad\square$

Remark 5.

(i) *The above proof shows that we also have the pointwise convergence*

$$\epsilon^{-1}R_\epsilon(t, \cdot) \to R_0(t, \cdot) \quad \text{in } L^p(\Omega)$$

for almost every $t \in (0,T)$ and $1 \le p < \infty$.

(ii) *In the same way as we proved the regularity of R_0, we obtain the regularity of the initial condition $R^0 \in H^1(\Omega)$ from the Assumption (A2).*

Corollary 7. *For all $p \in [1, \infty)$ it holds up to a subsequence that*

$$\epsilon^{-1}R_\epsilon \to R_0 \quad \text{strongly in the two-scale sense on } \Gamma_\epsilon \text{ in } L^p.$$

Proof. We use the equivalent characterization of the strong two-scale convergence via the strong convergence of the unfolded sequence $\mathcal{T}_\epsilon(\epsilon^{-1}R_\epsilon)$ in $L^p((0,T) \times \Omega \times \Gamma)$, see Lemma 13 in the Appendix B. Since R_ϵ is constant on every micro cell, $\mathcal{T}_\epsilon(R_\epsilon)$ is constant with respect to $y \in Y$ and therefore $\nabla_y \mathcal{T}_\epsilon(R_\epsilon) = 0$. Hence, we have with the trace inequality

$$\|\mathcal{T}_\epsilon(\epsilon^{-1}R_\epsilon) - R_0\|_{L^p((0,T) \times \Omega \times \Gamma)}$$

$$\le C\left(\|\mathcal{T}_\epsilon(\epsilon^{-1}R_\epsilon) - R_0\|_{L^p((0,T) \times \Omega \times Y^*)} + \|\underbrace{\nabla_y(\mathcal{T}_\epsilon(\epsilon^{-1}R_\epsilon) - R_0)}_{=0}\|_{L^p((0,T) \times \Omega \times Y^*)}\right)$$

$$\le C\|\mathcal{T}_\epsilon(\epsilon^{-1}R_\epsilon) - R_0\|_{L^p((0,T) \times \Omega \times Y)} \xrightarrow{\epsilon \to 0} 0,$$

where the convergence follows from Proposition 4 and Lemma 13. $\qquad\square$

Let us define

$$J_0(t, x, y) := \det\big(\nabla_y S_0(t, x, y)\big),$$
$$D_0(t, x, y) := J_0(t, x, y)\nabla_y S_0(t, x, y)^{-1} D(x)\nabla_y S_0(t, x, y)^{-T}.$$

For the definition of S_0 see Corollary 6. We emphasize that $0 < c_0 \leq J_0(t, x, y)$ for a constant $c_0 > 0$ and almost every $(t, x, y) \in (0, T) \times \Omega \times Y^*$, and therefore $\nabla_y S(t, x, y)^{-1}$ exists almost everywhere. From the strong convergence of R_ϵ, proved in Proposition 4, we immediately obtain:

Corollary 8. *For* $1 \in [1, \infty)$ *it holds up to a subsequence that*

$$\nabla S_\epsilon \to \nabla_y S_0 \qquad\qquad \text{strongly in the two-scale sense in } L^p,$$
$$J_\epsilon \to J_0 \qquad\qquad \text{strongly in the two-scale sense in } L^p,$$
$$D_\epsilon \to D_0 \qquad\qquad \text{strongly in the two-scale sense in } L^p.$$

Additionally, the strong two-scale convergence of ∇S_ϵ *and* J_ϵ *is also valid on* Γ_ϵ *in* L^p.

Proof. Again, we use the unfolding operator. We have

$$\|\mathcal{T}_\epsilon(\nabla S_\epsilon) - \nabla_y S_0\|_{L^p((0,T)\times\Omega\times Y^*)} = \|(\mathcal{T}_\epsilon(\epsilon^{-1} R_\epsilon) - R_0)\chi_0\nu_0\|_{L^p((0,T)\times\Omega\times Y^*)}$$
$$\leq C\|(\mathcal{T}_\epsilon(\epsilon^{-1} R_\epsilon) - R_0)\|_{L^p((0,T)\times\Omega\times Y^*)}$$

The right-hand side converges to zero for $\epsilon \to 0$ because of Proposition 4. For J_ϵ we use the essential boundedness of $\mathcal{T}_\epsilon(J_\epsilon)$ and J_0 on $(0, T) \times \Omega \times Y^*$ uniformly with respect to ϵ, to obtain with the local Lipschitz continuity of the determinant (using $\mathcal{T}_\epsilon(J_\epsilon) = \det(\mathcal{T}_\epsilon(\nabla S_\epsilon))$)

$$\|\mathcal{T}_\epsilon(J_\epsilon) - J_0\|_{L^p((0,T)\times\Omega\times Y^*)} \leq C\|\mathcal{T}_\epsilon(\nabla S_\epsilon) - \nabla_y S_0\|_{L^p((0,T)\times\Omega\times Y^*)} \xrightarrow{\epsilon\to 0} 0.$$

To show the convergence of D_ϵ, we first notice that due to (32) the inverse gradients $\mathcal{T}_\epsilon(\nabla S_\epsilon^{-1})$ and $\nabla_y S_0^{-1}$ are essential bounded on $(0, T) \times \Omega \times Y^*)$ uniformly with respect to ϵ. This gives

$$\|\mathcal{T}_\epsilon(\nabla S_\epsilon^{-1}) - \nabla_y S_0^{-1}\|_{L^p((0,T)\times\Omega\times Y^*)} \leq C\|\mathcal{T}_\epsilon(\nabla S_\epsilon) - \nabla_y S_0\|_{L^p((0,T)\times\Omega\times Y^*)}$$
$$\xrightarrow{\epsilon\to 0} 0,$$

where we have used the straightforward equality

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1},$$

valid for any invertible matrices $A, B \in \mathbb{R}^{n\times n}$. Using now (15), the convergence above yields the strong two-scale convergence of D_ϵ.

The convergence results for ∇S_ϵ and J_ϵ on Γ_ϵ follow by similar arguments and using Corollary 7. $\qquad\square$

5.1.3 Strong convergence of u_ϵ

Next, we prove the strong convergence of u_ϵ in the two-scale sense to the limit function u_0. Since we have no uniform bound of $\partial_t u_\epsilon$ with respect to ϵ, standard methods (see

[30]) fail. Therefore, we first introduce a regularized auxiliary problem: We consider for $t \in (0, T)$ and $\phi_0 \in C^\infty([0, T] \times \overline{\Omega})$ the problem

$$
\begin{aligned}
-\Delta w_\epsilon + w_\epsilon &= J_\epsilon(u_\epsilon - \phi_0) && \text{in } \Omega_\epsilon, \\
-\nabla w_\epsilon \cdot \nu &= 0 && \text{on } \partial\Omega_\epsilon
\end{aligned}
\tag{44}
$$

Obviously, there exists a unique weak solution $w_\epsilon \in L^2((0, T), H^1(\Omega_\epsilon))$. We have the following additional regularity with respect to time and uniform *a priori* estimates with respect to ϵ.

Lemma 8. *It holds that $w_\epsilon \in H^1((0, T), H^1(\Omega_\epsilon))$ with*

$$
\|\partial_t w_\epsilon\|_{L^2((0,T),H^1(\Omega_\epsilon))} + \|w_\epsilon\|_{L^\infty((0,T),H^1(\Omega_\epsilon))} \le C,
\tag{45}
$$

with a constant $C > 0$ independent of ϵ.

Proof. Testing the weak formulation of (44) with w_ϵ we obtain for a constant $c_0 > 0$ almost everywhere in $(0, T)$ with Lemma 7

$$
\begin{aligned}
\|w_\epsilon\|^2_{L^2(\Omega_\epsilon)} + \|\nabla w_\epsilon\|^2_{L^2(\Omega_\epsilon)} &= \int_{\Omega_\epsilon} J_\epsilon(u_\epsilon - \phi_0) w_\epsilon \, dx \\
&\le \|J_\epsilon\|_{L^\infty(\Omega_\epsilon)} \|u_\epsilon - \phi_0\|_{L^2(\Omega_\epsilon)} \|w_\epsilon\|_{L^2(\Omega_\epsilon)}. \\
&\le C\|w_\epsilon\|_{L^2(\Omega_\epsilon)} \\
&\le C + \frac{1}{2}\|w_\epsilon\|^2_{L^2(\Omega_\epsilon)}.
\end{aligned}
$$

This implies

$$
\|w_\epsilon\|_{L^\infty((0,T),H^1(\Omega_\epsilon))} \le C,
$$

where the constant C is independent of ϵ, but depends on the $L^\infty((0, T), L^2(\Omega))$-norm of ϕ_0.

Let us check the estimate for the time-derivative. We define for $0 < h \ll T$ the difference quotient with respect to time for a function $\phi_\epsilon : (0, T) \times \Omega_\epsilon \to \mathbb{R}$ by

$$
\partial_t^h \phi_\epsilon(t, x) := \frac{\phi_\epsilon(t + h, x) - \phi_\epsilon(t, x)}{h} \quad \text{for } (t, x) \in (0, T - h) \times \Omega_\epsilon.
$$

Applying ∂_t^h to the equation (44), we obtain

$$
\begin{aligned}
-\Delta \partial_t^h w_\epsilon + \partial_t^h w_\epsilon &= \partial_t^h[J_\epsilon(u_\epsilon - \phi_0)] && \text{in } \Omega, \\
-\nabla \partial_t^h w_\epsilon \cdot \nu &= 0 && \text{on } \partial\Omega_\epsilon.
\end{aligned}
$$

Testing the weak formulation of the equation above with $\partial_t^h w_\epsilon$ and integrating with respect to time, we obtain

$$
\begin{aligned}
\int_0^{T-h} \int_{\Omega_\epsilon} \nabla \partial_t^h w_\epsilon \cdot \nabla \partial_t^h w_\epsilon + |\partial_t^h w_\epsilon|^2 \, dx \, dt &= \int_0^{T-h} \int_{\Omega_\epsilon} \partial_t^h[J_\epsilon(u_\epsilon - \phi_0)] \partial_t^h w_\epsilon \, dx \, dt \\
&\le \|\partial_t^h(J_\epsilon(u_\epsilon - \phi_0))\|_{L^2((0,T-h),H^1(\Omega_\epsilon)')} \|\partial_t^h w_\epsilon\|_{L^2((0,T-h),H^1(\Omega_\epsilon))} \\
&\le C(\theta) + \theta\|\partial_t^h w_\epsilon\|^2_{L^2((0,T-h),H^1(\Omega_\epsilon))}
\end{aligned}
$$

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for all $\theta > 0$ and a constant $C(\theta) > 0$ depending on θ, where in the last inequality we used the *a priori* bounds for $\partial_t(J_\epsilon u_\epsilon)$, J_ϵ, and $\partial_t J_\epsilon$ from Lemma 7. The constant $C(\theta)$ depends on the $L^2((0,T) \times \Omega)$-norm of $\partial_t \phi_0$ and ϕ_0. For θ small enough the last term can be absorbed from the left-hand side and we obtain

$$\|\partial_t^h w_\epsilon\|_{L^2((0,T-h),H^1(\Omega_\epsilon))} \leq C.$$

This gives the desired result. $\qquad\square$

Remark 6.

(i) By a density argument we want to choose $\phi_0 = u_0$. However, in the proof of the Lemma above we have to work with smooth ϕ_0, since we need the time derivative $\partial_t(J_\epsilon \phi_0)$ and it is not clear whether the time-derivative $\partial_t(J_\epsilon u_0)$ exists and is bounded (not even clear $\partial_t u_0 \in L^2((0,T),H^1(\Omega)')$!).

(ii) The inequality (45) depends on the norm

$$\|\phi_0\|_{L^2((0,T),H^1(\Omega))} + \|\partial_t\phi_0\|_{L^2((0,T)\times\Omega)}.$$

In the proof of the following Proposition 5 we will choose a sequence $\phi_0 = \phi_k$ which converges only in $L^2((0,T),H^1(\Omega))$, i.e., the norm of the time-derivative $\partial_t u_0$ is in general not bounded. We will see in the proof that this has no influence on the result.

(iii) Due to the Sobolev-embedding and since Ω_ϵ is connected, we also have

$$\|w_\epsilon\|_{L^2((0,T),L^q(\Omega_\epsilon))} \leq C$$

for $q = \frac{2n}{n-2}$ if $n \neq 2$ and $q \in [1,\infty)$ if $n = 2$.

Corollary 9. *There exists $w_0 \in L^2((0,T),H^1(\Omega))$ and an extension $\tilde{w}_\epsilon \in L^2((0,T),H^1(\Omega))$ of w_ϵ, such that up to a subsequence it holds that*

$$\begin{aligned}
\tilde{w}_\epsilon &\rightharpoonup w_0 & \text{weakly in } L^2((0,T),H^1(\Omega)), \\
\tilde{w}_\epsilon &\to w_0 & \text{in } L^2((0,T),L^2(\Omega)).
\end{aligned}$$

Especially, we obtain the strong two-scale convergence of w_ϵ to w_0.

Proof. We refer to [1] for the extension and for the strong convergence see [30, Theorem 2.1] or [19, Lemma 10]. $\qquad\square$

Obviously, $w_0 \in L^2((0,T),H^1(\Omega))$ solves the following macroscopic problem:

$$\begin{aligned}
-\nabla \cdot (A\nabla w_0) + w_0 &= \bar{J}_0(u_0 - \phi_0) & \text{in } (0,T) \times \Omega, \\
-A\nabla w_0 \cdot \nu &= 0 & \text{on } (0,T) \times \partial\Omega,
\end{aligned} \qquad (46)$$

with

$$\bar{J}_0(t,x) := \int_{Y^*} J_0(t,x,y)dy \quad \text{for almost every } (t,x) \in (0,T) \times \Omega,$$

and $A \in \mathbb{R}^{n\times n}$ is defined by

$$A_{ij} = \int_{Y^*} \big(\nabla\chi_i + e_i\big) \cdot \big(\nabla\chi_j + e_j\big)dy,$$

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and $\chi_i \in H^1_{\text{per}}(Y^*)/\mathbb{R}$ are the solutions of the following cell problems:

$$-\nabla \cdot \left(\nabla \chi_i + e_i\right) = 0 \qquad\qquad \text{in } Y^*,$$
$$-\left(\nabla \chi_i + e_i\right) \cdot \nu = 0 \qquad\qquad \text{on } \Gamma,$$
$$\chi_i \text{ is } Y\text{-periodic,} \ \int_{Y^*} \chi_i dy = 0.$$

Proposition 5. *Up to a subsequence it holds that*

$$\chi_{\Omega_\epsilon} u_\epsilon \to \chi_{Y^*} u_0 \qquad\qquad \text{strongly in the two-scale sense in } L^2,$$
$$u_\epsilon|_{\Gamma_\epsilon} \to u_0 \qquad\qquad \text{strongly in the two-scale sense on } \Gamma_\epsilon \text{ in } L^2.$$

Proof. We choose a sequence $\phi_k \in C_0^\infty((0,T) \times \overline{\Omega})$, such that $\phi_k \to u_0$ in $L^2((0,T), H^1(\Omega))$ for $k \to \infty$. We denote by w_ϵ^k the solutions of (44) for $\phi_0 = \phi_k$ and w_0^k is the solution of (46) for $\phi_0 = \phi_k$. Testing (44) with $u_\epsilon - \phi_k$, we obtain with $0 < c_0 \leq J_\epsilon$

$$
\begin{aligned}
c_0 \|u_\epsilon - \phi_k\|^2_{L^2(\Omega_\epsilon)} &\leq \int_{\Omega_\epsilon} J_\epsilon (u_\epsilon - \phi_k)^2 dx \\
&= \int_{\Omega_\epsilon} \nabla(u_\epsilon - \phi_k) \cdot \nabla w_\epsilon^k dx + \int_{\Omega_\epsilon} (u_\epsilon - \phi_k) w_\epsilon^k dx \\
&\leq \|u_\epsilon - \phi_k\|_{H^1(\Omega_\epsilon)} \|w_\epsilon^k\|_{H^1(\Omega_\epsilon)}.
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
\frac{c_0}{2} \|u_\epsilon - u_0\|^2_{L^2((0,T) \times \Omega_\epsilon)} &\leq c_0 \|u_\epsilon - \phi_k\|^2_{L^2((0,T) \times \Omega_\epsilon)} + c_0 \|u_0 - \phi_k\|^2_{L^2((0,T) \times \Omega_\epsilon)} \\
&\leq \|u_\epsilon - \phi_k\|_{L^2((0,T),H^1(\Omega_\epsilon))} \|w_\epsilon^k\|_{L^2((0,T),H^1(\Omega_\epsilon))} + c_0 \|u_0 - \phi_k\|^2_{L^2((0,T) \times \Omega_\epsilon)}.
\end{aligned}
\tag{47}
$$

The second term on the right-hand side tends to zero for $k \to \infty$, due to the choice of ϕ_k. For the first term we notice that due to the strong convergence of ϕ_k in $L^2((0,T), H^1(\Omega))$ and Lemma 7, we have

$$\|u_\epsilon - \phi_k\|_{L^2((0,T),H^1(\Omega))} \leq C$$

for a constant $C > 0$ independent of k and ϵ. We have to estimate $\|w_\epsilon^k\|_{L^2((0,T),H^1(\Omega_\epsilon))}$. From (44) we obtain by testing with w_ϵ^k after integration with respect to time:

$$
\begin{aligned}
\|w_\epsilon^k\|^2_{L^2((0,T),H^1(\Omega_\epsilon))} &= \int_0^T \int_{\Omega_\epsilon} J_\epsilon (u_\epsilon - \phi_k) w_\epsilon^k dx dt \\
&= \int_0^T \int_{\Omega_\epsilon} J_\epsilon (u_\epsilon - u_0) w_\epsilon^k dx dt + \int_0^T \int_{\Omega_\epsilon} J_\epsilon (u_0 - \phi_k) w_\epsilon^k dx dt \\
&\leq \int_0^T \int_{\Omega_\epsilon} J_\epsilon (u_\epsilon - u_0) w_\epsilon^k dx dt \\
&\qquad + \|J_\epsilon\|_{L^\infty((0,T) \times \Omega_\epsilon)} \|u_0 - \phi_k\|_{L^2((0,T) \times \Omega_\epsilon)} \|w_\epsilon^k\|_{L^2((0,T) \times \Omega_\epsilon)} \\
&\leq \int_0^T \int_{\Omega_\epsilon} J_\epsilon (u_\epsilon - u_0) w_\epsilon^k dx dt + C \|u_0 - \phi_k\|^2_{L^2((0,T) \times \Omega_\epsilon)} + \frac{1}{2} \|w_\epsilon^k\|^2_{L^2((0,T) \times \Omega_\epsilon)}.
\end{aligned}
$$

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Hence, we obtain

$$\|w_\epsilon^k\|_{L^2((0,T),H^1(\Omega_\epsilon))}^2 \le C \left(\int_0^T \int_{\Omega_\epsilon} J_\epsilon (u_\epsilon - u_0) w_\epsilon^k \, dx \, dt + \|u_0 - \phi_k\|_{L^2((0,T)\times\Omega_\epsilon)}^2 \right).$$

We show that the first term on the right-hand side converges to zero for fixed k and $\epsilon \to 0$. We have

$$\int_0^T \int_{\Omega_\epsilon} J_\epsilon (u_\epsilon - u_0) w_\epsilon^k \, dx \, dt = \int_0^T \int_{\Omega_\epsilon} \left(J_\epsilon - J_0 \left(x, \frac{x}{\epsilon} \right) \right) (u_\epsilon - u_0) w_0^k \, dx \, dt$$

$$+ \int_0^T \int_{\Omega_\epsilon} J_0 \left(x, \frac{x}{\epsilon} \right) (u_\epsilon - u_0) w_0^k \, dx \, dt$$

$$+ \int_0^T \int_{\Omega_\epsilon} J_\epsilon (u_\epsilon - u_0)(w_\epsilon^k - w_0^k) \, dx \, dt$$

$$=: A_\epsilon^1 + A_\epsilon^2 + A_\epsilon^3.$$

For the third term A_ϵ^3 we obtain with the *a priori* estimates from Lemma 7 and the strong convergence of $\widetilde{w}_\epsilon^k$ from Corollary 9

$$|A_\epsilon^3| \le \|J_\epsilon\|_{L^\infty((0,T)\times\Omega_\epsilon)} \|u_\epsilon - u_0\|_{L^2((0,T)\times\Omega_\epsilon)} \|w_\epsilon^k - w_0^k\|_{L^2((0,T)\times\Omega_\epsilon)} \xrightarrow{\epsilon \to 0} 0.$$

Since $J_0 \left(x, \frac{x}{\epsilon} \right) w_0^k$ is an admissible test-function in the two-scale sense, we obtain from the (weak) two-scale convergence of u_ϵ from Corollary 6 that $A_\epsilon^2 \to 0$ for $\epsilon \to 0$. It remains to estimate A_ϵ^1. From Remark 6 we have $w_0^k \in L^2((0,T), L^q(\Omega))$ for $q > 2$. Hence, there exists $p \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. From the Hölder-inequality we get with Lemma 7

$$|A_\epsilon^1| \le C \left\| J_\epsilon - J_0 \left(x, \frac{x}{\epsilon} \right) \right\|_{L^p((0,T)\times\Omega_\epsilon)} \|u_\epsilon - u_0\|_{L^\infty((0,T),L^2(\Omega_\epsilon))} \|w_0^k\|_{L^2((0,T),L^q(\Omega_\epsilon))}$$

$$\le C \left\| J_\epsilon - J_0 \left(x, \frac{x}{\epsilon} \right) \right\|_{L^p((0,T)\times\Omega_\epsilon)} \xrightarrow{\epsilon \to 0} 0,$$

where at the end we used the strong two-scale convergence of J_ϵ, see also Remark 9 in the Appendix B. Altogether, we get:

$$\limsup_{\epsilon \to 0} \|w_\epsilon^k\|_{L^2((0,T),H^1(\Omega_\epsilon))} \le C \|u_0 - \phi_k\|_{L^2((0,T)\times\Omega)}.$$

Altogether, we obtain from (47)

$$\limsup_{\epsilon \to 0} \|u_\epsilon - u_0\|_{L^2((0,T)\times\Omega_\epsilon)} \le C \left(\|u_0 - \phi_k\|_{L^2((0,T)\times\Omega)}^2 + \|u_0 - \phi_k\|_{L^2((0,T)\times\Omega)} \right).$$

For $k \to \infty$ we get the first convergence in the statement. To prove the strong two-scale convergence on the surface Γ_ϵ we use the trace inequality (33) to obtain with Lemma 7

$$\sqrt{\epsilon} \|u_\epsilon - u_0\|_{L^2((0,T)\times\Gamma_\epsilon)} \le C \left(\|u_\epsilon - u_0\|_{L^2((0,T)\times\Omega_\epsilon)} + \epsilon \|\nabla u_\epsilon - \nabla u_0\|_{L^2((0,T)\times\Omega_\epsilon)} \right)$$

$$\le C \|u_\epsilon - u_0\|_{L^2((0,T)\times\Omega_\epsilon)} + C\epsilon,$$

which tends to 0 for $\epsilon \to 0$. $\qquad\square$

5.1.4 Strong convergence of $\partial_t R_\epsilon$

In this section we identify the limit of $\partial_t R_\epsilon$, i.e., we show $\partial_t R_0 = g(u_0, R_0)$.

Lemma 9. *Let $w_\epsilon \in L^1((0,T) \times \Gamma_\epsilon)$ and $w_0 \in L^1((0,T), W^{1,1}(\Omega))$ such that $w_\epsilon \to w_0$ strongly in the two-scale sense on Γ_ϵ in L^1. Then we have*

$$\frac{1}{|\epsilon\Gamma|} \int_{\Gamma_\epsilon\left(\epsilon\left[\frac{\cdot}{\epsilon}\right]\right)} w_\epsilon(t,z)d\sigma \to w_0 \quad in \ L^1((0,T) \times \Omega).$$

Proof. First of all we extend the function w_0 to a function in $L^1((0,T), W^{1,1}(\mathbb{R}^n))$ and use the same notation w_0 for the extended function. Now, we define for all $(t,x) \in (0,T) \times \Omega$

$$h_\epsilon(t,x) := \frac{1}{|\epsilon\Gamma|} \int_{\Gamma_\epsilon\left(\epsilon\left[\frac{x}{\epsilon}\right]\right)} w_\epsilon(t,z)d\sigma.$$

Hence, we have to show that

$$h_\epsilon \to w_0 \quad in \ L^1((0,T) \times \Omega).$$

We also define for $(t,x) \in (0,T) \times \Omega$ (see (16) for the definition of $\Gamma_\epsilon(x)$)

$$g_\epsilon^0(t,x) := \frac{1}{|\epsilon\Gamma|} \int_{\Gamma_\epsilon(x)} w_0(t,z)d\sigma,$$

$$h_\epsilon^0(t,x) := g_\epsilon^0\left(t, \epsilon\left[\frac{x}{\epsilon}\right]\right).$$

Then we have

$$\|h_\epsilon - w_0\|_{L^1((0,T)\times\Omega)}$$
$$\leq \|h_\epsilon - h_\epsilon^0\|_{L^1((0,T)\times\Omega)} + \|h_\epsilon^0 - g_\epsilon^0\|_{L^1((0,T)\times\Omega)} + \|g_\epsilon^0 - w_0\|_{L^1((0,T)\times\Omega)}$$
$$=: A_\epsilon^1 + A_\epsilon^2 + A_\epsilon^3.$$

For the first term A_ϵ^1 we have

$$A_\epsilon^1 = \int_0^T \int_\Omega \left| \frac{1}{|\epsilon\Gamma|} \int_{\Gamma_\epsilon\left(\epsilon\left[\frac{x}{\epsilon}\right]\right)} w_\epsilon(t,z) - w_0(t,z)d\sigma_z \right| dxdt$$

$$\leq \frac{1}{|\epsilon\Gamma|} \sum_{k \in K_\epsilon} \int_0^T \int_{\epsilon(Y+k)} \int_{\Gamma_\epsilon(\epsilon k)} |w_\epsilon(t,z) - w_0(t,z)| dzdxdt$$

$$\leq \frac{\epsilon^n}{|\epsilon\Gamma|} \int_0^T \int_{\Gamma_\epsilon} |w_\epsilon(t,z) - w_0(t,z)| d\sigma dt$$

$$\leq C\epsilon \|w_\epsilon - w_0\|_{L^1((0,T)\times\Gamma_\epsilon)} \xrightarrow{\epsilon \to 0} 0,$$

due to the strong two-scale convergence of w_ϵ, see also Lemma 13 in the Appendix B. For the term A_ϵ^2 we use a density argument together with the mean value theorem. More precisely, let $w_\ell \in C_0^\infty((0,T) \times \mathbb{R}^n)$ be a sequence with $w_\ell \to w_0$ for $\ell \to \infty$ in $L^1((0,T) \times \mathbb{R}^n)$. Remember that w_0 is extended by zero to the whole $\mathbb{R}^n$. We mention

that we can also choose a sequence converging in $L^1((0,T), W^{1,1}(\Omega))$, but this is not necessary. We define

$$g_\epsilon^\ell(t,x) := \frac{1}{|\epsilon\Gamma|} \int_{\Gamma_\epsilon(x)} w_\ell(t,z) d\sigma,$$

$$h_\epsilon^\ell(t,x) := g_\epsilon^\ell \left(t, \epsilon \left[\frac{x}{\epsilon} \right] \right).$$

Then we have

$$A_\epsilon^2 \leq \|h_\epsilon^0 - h_\epsilon^\ell\|_{L^1((0,T)\times\Omega)} + \|h_\epsilon^\ell - g_\epsilon^\ell\|_{L^1((0,T)\times\Omega)} + \|g_\epsilon^\ell - g_\epsilon^0\|_{L^1((0,T)\times\Omega)}$$
$$=: B_{\epsilon,\ell}^1 + B_{\epsilon,\ell}^2 + B_{\epsilon,\ell}^3.$$

For the first term we get with similar decomposition arguments as for the term A_ϵ^1 above and the trace inequality (33)

$$B_{\epsilon,\ell}^1 \leq C\epsilon \|w_\ell - w_0\|_{L^1((0,T)\times\Gamma_\epsilon)}$$
$$\leq C \left(\|w_0 - w_\ell\|_{L^1((0,T)\times\Omega)} + \epsilon\|\nabla w_0 - \nabla w_\ell\|_{L^1((0,T)\times\Omega)} \right).$$

The first term on the right-hand side converges to 0 for $\ell \to \infty$ uniformly with respect to ϵ. The second term on the right-hand side goes to 0 for $\epsilon \to 0$ (for fixed $\ell \in \mathbb{N}$). For the term $B_{\epsilon,\ell}^2$ we get

$$B_{\epsilon,\ell}^2 \leq \frac{1}{|\epsilon\Gamma|} \int_0^T \int_\Omega \int_{\Gamma_\epsilon(x)} \left| w_\ell \left(t, z - \epsilon \left\{ \frac{x}{\epsilon} \right\} \right) - w_\ell(t,z) \right| d\sigma_z dx dt$$
$$\leq \frac{\epsilon}{|\epsilon\Gamma|} \int_0^T \int_\Omega \int_{\Gamma_\epsilon(x)} \int_0^1 \left| \nabla w_\ell \left(t, z - s \, \epsilon \left\{ \frac{x}{\epsilon} \right\} \right) \right| ds d\sigma_z dx dt$$
$$\leq C\epsilon \|\nabla w_\ell\|_{C^0([0,T]\times\overline{\Omega})}.$$

Hence, $B_{\epsilon,\ell}^2 \to 0$ for $\epsilon \to 0$ for every fixed $\ell \in \mathbb{N}$. For $B_{\epsilon,\ell}^3$ we have

$$B_{\epsilon,\ell}^3 \leq \frac{1}{|\epsilon\Gamma|} \int_0^T \int_\Omega \int_{\Gamma_\epsilon(0)} |w_\ell(t,x+z) - w_0(t,x+z)| d\sigma_z dx dt$$
$$= \frac{1}{|\epsilon\Gamma|} \int_0^T \int_{\Gamma_\epsilon(0)} \int_{\Omega+z} |w_\ell(t,x) - w_0(t,x)| dx d\sigma_z dt$$
$$\leq C\|w_\ell - w_0\|_{L^1((0,T)\times\mathbb{R}^n)},$$

and therefore $B_{\epsilon,\ell}^3 \to 0$ for $\ell \to \infty$, uniformly with respect to ϵ.

It remains to estimate the term A_ϵ^3. With the notation from above we obtain

$$A_\epsilon^3 \leq \|g_\epsilon^0 - g_\epsilon^\ell\|_{L^1((0,T)\times\Omega)} + \|g_\epsilon^\ell - w_\ell\|_{L^1((0,T)\times\Omega)} + \|w_\ell - w_0\|_{L^1((0,T)\times\Omega)}.$$

The first term was already considered above, and the last term obviously tends to 0 for $\ell \to \infty$. For the second term, we argue as for $B_{\epsilon,\ell}^2$ to obtain

$$\|g_\epsilon^\ell - w_\ell\|_{L^1((0,T)\times\Omega)} \leq \frac{1}{|\epsilon\Gamma|} \int_0^T \int_\Omega \int_{\Gamma_\epsilon(x)} |w_\ell(t,z) - w_\ell(t,x)| d\sigma_z dx dt$$
$$\leq C\epsilon \|\nabla w_\ell\|_{C^0([0,T]\times\overline{\Omega})}.$$

Hence, A_ϵ^3 goes to zero for $\epsilon \to 0$. Altogether, we obtain the desired result. $\qquad\square$

As an immediate consequence we obtain the strong convergence of $\epsilon^{-1}\partial_t R_\epsilon$:

Proposition 6. *Up to a subsequence $\epsilon \to 0$, it holds that*

$$\epsilon^{-1}\partial_t R_\epsilon \to \partial_t R_0 \quad in \ L^p((0,T) \times \Omega)$$

for $p \in [1,\infty)$, with R_0 satisfying the equation

$$\partial_t R_0 = g(u_0, R_0) \quad a.e. \ in \ (0,T) \times \Omega.$$

Especially, we have up to a subsequence

$$\epsilon^{-1}\partial_t R_\epsilon \to \partial_t R_0 \quad strongly \ in \ the \ two\text{-}scale \ sense \ on \ \Gamma_\epsilon \ in \ L^p.$$

Proof. The convergence of $\epsilon^{-1}\partial_t R_\epsilon$ in $L^1((0,T) \times \Omega)$ is a direct consequence of Lemma 9 by choosing

$$w_\epsilon(t,x) := g\left(u_\epsilon(t,x), \epsilon^{-1}R_\epsilon(t,x)\right),$$
$$w_0(t,x) := g(u_0(t,x), R_0(t,x)).$$

The strong convergence of u_ϵ and $\epsilon^{-1}R_\epsilon$, see Proposition 4 and 5, and the Lipschitz continuity of g imply the strong two-scale convergence of w_ϵ on Γ_ϵ in L^1. Further it holds that $w_0 \in L^2((0,T), H^1(\Omega))$, due to the Lipschitz-continuity of g and the regularity of u_0 and R_0, see Proposition 4. Then, Lemma 9 implies the convergence of $\epsilon^{-1}\partial_t R_\epsilon$ in $L^1((0,T) \times \Omega)$, and the result for arbitrary $p \in [1,\infty)$ follows again from the dominated convergence theorem of Lebesgue. Now, from $\partial_t R_0 = g(u_0, R_0)$, we obtain

$$\nabla \partial_t R_0 = \partial_1 g(u_0, R_0)\nabla u_0 + \partial_2 g(u_0, R_0)\nabla R_0 \in L^2((0,T) \times \Omega)$$

Using that R_ϵ is constant on each micro cell and the trace inequality (33), we obtain with the same arguments as in the proof of Corollary 7

$$\sqrt{\epsilon}\|\epsilon^{-1}\partial_t R_\epsilon - \partial_t R_0\|_{L^2((0,T)\times\Gamma_\epsilon)}$$
$$\leq C\left(\|\epsilon^{-1}\partial_t R_\epsilon - \partial_t R_0\|_{L^2((0,T)\times\Omega_\epsilon)} + \sqrt{\epsilon}\|\nabla\partial_t R_0\|_{L^2((0,T)\times\Omega_\epsilon)}\right)$$
$$\overset{\epsilon\to 0}{\longrightarrow} 0.$$

Arguing again as in Corollary 7, the convergence is also valid for all $p \in [1,\infty)$, and the proposition is proved. $\square$

Corollary 10. *Up to a subsequence $\epsilon \to 0$, one has for every $p \in [1,\infty)$ that*

$$\partial_t J_\epsilon \to \partial_t J_0 \quad strongly \ in \ the \ two\text{-}scale \ sense \ in \ L^p.$$

Proof. This is a direct consequence of the Jacobi formula (22) and the strong two-scale convergence results in Corollary 8 and Proposition 6. $\square$

5.2 Derivation of the macroscopic equation

In this section we derive the macroscopic model in the limit $\epsilon \to 0$. In a first step, we derive the cell problems on the reference element Y^*. In the second step, we derive the macroscopic equation on Ω. We emphasize that we already derived the limit equation for the radius in Proposition 6.

First of all, choose $\phi_\epsilon(t, x) = \epsilon\phi\left(t, x, \frac{x}{\epsilon}\right)$ for $\phi \in C_0^\infty((0, T) \times \Omega, C_{per}^\infty(Y^*))$ as a test-function in (18). Due to our *a priori* estimates in Lemma 7, all the terms except the diffusion term are of order ϵ. Hence, for $\epsilon \to 0$ we obtain from the strong two-scale convergence of D_ϵ from Corollary 8 and the two-scale convergence of ∇u_ϵ from Corollary 6

$$\int_0^T \int_\Omega \int_{Y^*} D_0(t, x, y)\left[\nabla_x u_0(t, x) + \nabla_y u_1(x, y)\right] \cdot \nabla_y \phi(t, x, y) dy dx dt = 0.$$

From the linearity of the problem we obtain for u_1 the representation

$$u_1(t, x, y) = \sum_{i=1}^n \partial_{x_i} u_0(t, x) w_i(t, x, y),$$

where $w_i \in L^2((0, T) \times \Omega, H^1(Y^*)/\mathbb{R})$ are the solutions of the cell problems

$$
\begin{aligned}
-\nabla_y \cdot \left(D_0\left(\nabla_y w_i + e_i\right)\right) &= 0 && \text{in } (0, T) \times \Omega \times Y^*, \\
-D_0\left(\nabla_y w_i + e_i\right) \cdot \nu &= 0 && \text{on } (0, T) \times \Omega \times \Gamma, \\
w_i \text{ is } Y\text{-periodic}, \quad \int_{Y^*} w_i(t, x, y) dy &= 0 && \text{f.a.e. } (t, x) \in (0, T) \times \Omega.
\end{aligned}
\tag{48}
$$

We emphasize that the L^∞-regularity of D_0 with respect to (t, x) implies the L^∞-regularity of $\nabla_y w_i$ with respect to (t, x).

Next, we derive the macroscopic equation. We choose $\phi_\epsilon = \phi$ with $\phi \in C_0^\infty\left((0, T) \times \overline{\Omega}\right)$ as a test-function in (18) and integrate with respect to time:

$$
\begin{aligned}
\int_0^T &\langle \partial_t(J_\epsilon u_\epsilon), \phi \rangle_{H^1(\Omega_\epsilon)', H^1(\Omega_\epsilon)} dt - \int_0^T \int_{\Omega_\epsilon} u_\epsilon \partial_t J_\epsilon \phi dx dt \\
&- \int_0^T \int_{\Omega_\epsilon} V_\epsilon \cdot \nabla u_\epsilon \phi dx dt 0 + \int_0^T \int_{\Omega_\epsilon} D_\epsilon \nabla u_\epsilon \cdot \nabla \phi dx dt \\
&= \int_0^T \int_{\Omega_\epsilon} J_\epsilon f_\epsilon \phi dx dt - \int_0^T \int_{\Gamma_\epsilon} \partial_t R_\epsilon(u_\epsilon - \rho) J_\epsilon \phi d\sigma dt.
\end{aligned}
\tag{49}
$$

Let us pass to the limit in every single term. For the term including the time-derivative we use the following fact: First of all, using the strong two-scale convergence of J_ϵ from Corollary 8 and u_ϵ from Proposition 5, it is easy to check that $J_\epsilon u_\epsilon$ converges (even strongly by the dominated convergence theorem of Lebesgue [4, Theorem 3.25]) in the two-scale sense to $J_0 u_0$ in L^2. Then, due to the *a priori* estimate of $J_\epsilon u_\epsilon$ in Lemma 7, we obtain that for the zero extension $\widetilde{J_\epsilon u_\epsilon}$ of $J_\epsilon u_\epsilon$ to the whole domain Ω it holds that (see also Lemma 11 in the appendix)

$$\partial_t\left(\widetilde{J_\epsilon u_\epsilon}\right) \rightharpoonup \partial_t\left(\bar{J}_0 u_0\right) \quad \text{weakly in } L^2((0, T), H^1(\Omega)'),$$

with

$$\bar{J}_0(t, x) := \int_{Y^*} J_0(t, x, y) dy.$$

Remark 7. *Since J_0 is polynomial in R_0 with coefficients depending on y in a smooth way, we easily obtain that $\bar{J}_0 \in L^\infty((0, T), H^1(\Omega))$.*

Now, we obtain

$$\int_0^T \langle \partial_t(J_\epsilon u_\epsilon), \phi \rangle_{H^1(\Omega_\epsilon)', H^1(\Omega_\epsilon)} dt = \int_0^T \left\langle \partial_t \left(\widetilde{J_\epsilon u_\epsilon} \right), \phi \right\rangle_{H^1(\Omega)', H^1(\Omega)} dt$$

$$\xrightarrow{\epsilon \to 0} \int_0^T \langle \partial_t(\bar{J}_0 u_0), \phi \rangle_{H^1(\Omega)', H^1(\Omega)} dt.$$

We emphasize that $\bar{J}_0$ is the volume of the moving cell $Y^*(t, x)$ defined by

$$Y^*(t, x) := Y \setminus \overline{B_{R_0(t,x)}(x)},$$

i.e., we have $J_0(t, x) = |Y^*(t, x)|$. For the second term on the left-hand side in (49) we use the strong two-scale convergence of u_ϵ from Proposition 5, as well as the two-scale convergence of $\partial_t J_\epsilon$ to $\partial_t J_0$, see Proposition 10, to obtain

$$\int_0^T \int_{\Omega_\epsilon} u_\epsilon \partial_t J_\epsilon \phi dx dt \xrightarrow{\epsilon \to 0} \int_0^T \int_\Omega \int_{Y^*} u_0(t, x) \partial_t J_0(t, x, y) \phi(t, x) dy dx dt.$$

Since the L^2-norm of V_ϵ is of order ϵ, the third term on the left-hand side in (49) vanishes for $\epsilon \to 0$. Using the strong two-scale convergence of D_ϵ from Corollary 8 and the two-scale convergence of ∇u_ϵ from Corollary 6, we obtain

$$\lim_{\epsilon \to 0} \int_0^T \int_{\Omega_\epsilon} D_\epsilon \nabla u_\epsilon \cdot \nabla \phi dx dt = \int_0^T \int_\Omega \int_{Y^*} D_0 \left[\nabla u_0 + \nabla_y u_1 \right] \cdot \nabla \phi dy dx dt$$

$$= \int_0^T \int_\Omega D_0^*(t, x) \nabla u_0 \cdot \nabla \phi dx dt,$$

with the homogenized diffusion coefficient $D_0^* \in L^\infty((0, T) \times \Omega)^{n \times n}$ defined by

$$\left(D_0(t, x)^* \right)_{ij} := \int_{Y^*} D_0(t, x, y) \left[\nabla_y w_i(t, x, y) + e_i \right] \cdot \left[\nabla_y w_j(t, x, y) + e_j \right] dy, \qquad (50)$$

where w_i are the solutions of the cell problems (48). For the first term on the right-hand side of (49) we have

$$\lim_{\epsilon \to 0} \int_0^T \int_{\Omega_\epsilon} J_\epsilon f_\epsilon \phi dx dt = \int_0^T \int_\Omega \int_{Y^*} J_0 f_0 \phi dy dx dt,$$

where we used the two-scale convergence of J_ϵ and S_ϵ, and the continuity of f, see Assumption (A5). It remains to pass to the limit in the boundary term in (49). We only consider in more detail the term including u_ϵ. The strong two-scale convergence of $\epsilon^{-1} \partial_t R_\epsilon$ and J_ϵ on Γ_ϵ in L^p from Proposition 6 and Corollary 8, as well as the strong two-scale convergence of u_ϵ on Γ_ϵ in L^2 from Proposition 5 imply the strong two-scale convergence of the product $\epsilon^{-1} \partial_t R_\epsilon u_\epsilon J_\epsilon$ on Γ_ϵ in L^q for $q = \frac{2p}{4+p} \in (1, 2)$ (with $p > 4$). In fact, this is an immediate consequence of the characterization of the two-scale convergence via the unfolding operator, see Lemma 13 in the Appendix B. Hence, we obtain

$$\lim_{\epsilon \to 0} \int_0^T \int_{\Gamma_\epsilon} \partial_t R_\epsilon (u_\epsilon - \rho) J_\epsilon \phi d\sigma dt = \int_0^T \int_\Omega \partial_t R_0 (u_0 - \rho) \phi \int_\Gamma J_0 d\sigma_y dx dt.$$

Let us define the moving cell surface $\Gamma(t, x)$ by

$$\Gamma(t, x) := \partial B_{R_0(t,x)}(\mathfrak{m}).$$

Using the identity (52) we obtain $|\nabla_y S_0^{-T}\nu| = |\nu| = 1$ on Γ, and we get that

$$\int_\Gamma J_0 d\sigma_y = \int_{\Gamma(t,x)} d\sigma = |\Gamma(t,x)|$$

is the surface area of $\Gamma(t,x)$. Altogether we obtain in the limit $\epsilon \to 0$:

$$\int_0^T \langle \partial_t(\bar{J}_0 u_0), \phi \rangle_{H^1(\Omega)',H^1(\Omega)} dt - \int_0^T \int_\Omega \partial_t \bar{J}_0 u_0 \phi \, dxdt + \int_0^T \int_\Omega D_0^* \nabla u_0 \cdot \nabla\phi \, dxdt$$
$$= \int_0^T \int_\Omega \int_{Y^*} J_0 f_0 dy \phi \, dxdt - \int_0^T \int_\Omega \partial_t R_0 (u_0 - \rho) \phi |\Gamma(t,x)| dxdt$$

for all $\phi \in C_0^\infty((0,T) \times \overline{\Omega})$ and by density for all $\phi \in L^2((0,T), H^1(\Omega))$. Using the relation $\partial_t J_0 = \nabla_y \cdot V_0$ and the notation $q := \int_{Y^*} \nabla_y \cdot V_0 dy$, we obtain the the variatioanl equation (24). To finish the proof of Theorem 2 we have to establish the initial condition $u(0) = u^{\mathrm{in}}$, where we have to show $u^0 \in C^0([0,T], L^2(\Omega))$. Of course, due to Lemma 11, it holds that $(\bar{J}_0 u_0)(0) = \bar{J}_0 u^{\mathrm{in}}$. The spatial regularity of R_0 from Proposition 4, the equality $\partial_t R_0 = g(u_0, R_0)$ almost everywhere in $(0,T) \times \Omega$ and the boundedness of g imply

$$R_0 \in C^{0,1}([0,T], L^\infty(\Omega)) \cap L^\infty((0,T), H^1(\Omega)).$$

Together with the positivity of J_0, we obtain

$$\bar{J}_0^{-1} \in C^{0,1}([0,T], L^\infty(\Omega)) \cap L^\infty((0,T), H^1(\Omega)).$$

Hence, for $q > n$ we obtain

$$\bar{J}_0^{-1} \partial_t(\bar{J}_0 u_0) - \bar{J}_0^{-1} u_0 \partial_t \bar{J}_0 \in L^2((0,T), W^{1,q}(\Omega)').$$

Now, the product rule implies

$$\partial_t u_0 = \bar{J}_0^{-1} \partial_t(\bar{J}_0 u_0) - \bar{J}_0^{-1} u_0 \partial_t \bar{J}_0 \in L^2((0,T), W^{1,q}(\Omega)').$$

Especially, due to [38, Lemma 7.3], we have $u_0 \in C^0([0,T], L^2(\Omega))$ and the initial condition $u_0(0) \in L^2(\Omega)$ makes sense. Using $(\bar{J}_0 u_0)(0) = \bar{J}_0 u^{\mathrm{in}}$ and $\bar{J}_0 \in C^0([0,T], L^\infty(\Omega))$, we get the initial condition $u_0(0) = u^{\mathrm{in}}$.

Remark 8.

(i) *The proof above gives the following regularity results*

$$\bar{J}_0 \in C^{0,1}([0,T], L^\infty(\Omega)) \cap L^\infty((0,T), H^1(\Omega)),$$
$$\partial_t u_0 \in L^2((0,T), W^{1,q}(\Omega)')$$

for $q > n$.

(ii) *The macroscopic equation is formulated on the fixed domain Ω. The moving boundary is in the cell problems. In fact, if we define*

$$\tilde{w}_i(t,x,y) := w_i(t,x,S_0^{-1}(t,x,y)) \quad \textit{for } (t,x,y) \in (0,T) \times \Omega \times Y^*(t,x),$$

then $\tilde{w}_i$ solves the following problem on the moving cell $Y^*(t,x)$

$$-\nabla_y \cdot \left(D(\nabla_y \tilde{w}_i + \nabla_y (S_0^{-1})^T e_i)\right) = 0 \qquad in\ Y^*(t,x),$$
$$-D(\nabla_y \tilde{w}_i + \nabla_y (S_0^{-1})^T e_i) \cdot \nu = 0 \qquad on\ \Gamma(t,x),$$
$$\int_{Y^*(t,x)} J_0^{-1}(t,x,S_0^{-1}(t,x,y))\tilde{w}_i(t,x,y)dy = 0, \qquad \tilde{w}_i\ is\ Y\text{-}periodic.$$

The evolution of $Y^*(t,x)$ is given by S_0 and therefore coupled via the ODE for R_0 to the macroscopic solution u_0.

6 Conclusion

We analyzed a reaction-diffusion problem in an evolving micro-domain depending on the solution of the equation, and derived a macroscopic model. The homogenized model is only depending on the macroscopic variable $x \in \Omega$. The information about the evolving microstructure are contained in the effective coefficients. The homogenized diffusion coefficient is given via solutions of cell problems on evolving reference element, depending on the limit functions u_0 and R_0. Hence, in every macroscopic point $x \in \Omega$, we have to solve a cell problem depending on the solution itself, what leads to a strongly coupled problem which numerical treatment is highly challenging.

We emphasize that the methods in this paper are not restricted to the scalar case and linear reaction kinetics. In fact, the results simply extend to systems with Lipschitz-continuous reaction rates for example of Michaelis-Menton-type.

Our results depend highly on the assumption that the evolution of the surface is given by an ordinary equation. In general, one has to consider for example an hyperbolic level set problem coupled to the transport equation, and even global-in-time solutions for the microscopic solutions are not guaranteed and even cannot be expected. However, for the treatment of more realistic applications on has to take into account such kind of models.

In our results we made no statements about the uniqueness of the micro- and the macro-model. For the homogenization process, uniqueness for the upscaled model is important to obtain the convergence of the whole sequence, which is not guaranteed in our case. However, the low regularity of the product $\bar{J}_0 u_0$ and $\partial_t u_0$ causes trouble for the application of standard energy arguments. In the microscopic model, especially the nonlinear boundary term including the time-derivative $\partial_t R_\epsilon$ makes things complicated. Here, one should take into account entropy methods, see for example [28, 34].

From a physical point of point of view we would expect nonegativity and essential boundedness for a solution. An upper bound can be obtained for the micro-model under additional assumptions on the data, e.g., boundedness of u_ϵ^{in} and $f = 0$ (or f depending on u_ϵ with suitable growth conditions). Nonnegativity for the micro-model can be obtained for growing grains, i.e., $\partial_t R_\epsilon \geq 0$. In the present model, using the average over the boundary of the perforation in the differential equation of R_ϵ complicates the proof for nonnegativity, as there is no pointwise relation between u_ϵ and $\partial_t R_\epsilon$. On the other hand, this structure guarantees that the perforations remain radially symmetric. The treatment of these problems are part of our ongoing work.

Acknowledgments

A Some elemental calculations

Recalling (3), we choose a fixed $\delta_0 \in \left(0, \min\{\frac{1}{2} - \overline{R}, \overline{R}\}\right)$ and define the symmetric cut-off function $\chi \in C_0^\infty(\mathbb{R})$ such that

$$0 \leq \chi(z) \leq 1 \quad \text{for all } z \in \mathbb{R},$$
$$\chi(z) = 1 \quad \text{if } |z| \leq \frac{\delta_0}{2},$$
$$\chi(z) = 0 \quad \text{if } |z| \geq \delta_0,$$
$$z\chi'(z) < 0 \quad \text{if } |z| \in \left(\frac{\delta_0}{2}, \delta_0\right).$$

Further, with $\Gamma = \partial B_{\overline{R}}(\mathfrak{m})$ being the sphere of radius $\overline{R}$ centered in $\mathfrak{m}$, we let d_Γ denote the signed-distance function to Γ

$$d_\Gamma(y) = \begin{cases} \text{dist}(y, \Gamma) & \text{if } |y| \geq \overline{R}, \\ -\text{dist}(y, \Gamma) & \text{if } |y| < \overline{R}. \end{cases}$$

Clearly, for any $y \in \mathbb{R}^n$ one has $d_\Gamma(y) = \|y - \mathfrak{m}\| - \overline{R}$ which is a smooth function for all arguments $y \neq \mathfrak{m}$.

With this, we consider the function

$$\chi_0 : Y \to \mathbb{R}, \quad \chi_0(y) = \chi(d_\Gamma(y)), \tag{51}$$

which is Y-periodic and smooth (also in $\mathfrak{m}$, since it vanishes in a neighborhood of $\mathfrak{m}$), having a compact support in Y.

Further, for any $y \in \Gamma$ we denote by $\nu_0(y)$ the unit normal in y to Γ pointing outwards $B_{\overline{R}}(\mathfrak{m})$. In this simplified setting we have $\nu_0(y) = \frac{y - \mathfrak{m}}{|y - \mathfrak{m}|} = \nabla d_\Gamma(y)$, and we use the same expression to extend ν_0 to the set $Y \setminus \{\mathfrak{m}\}$.

With r s.t. $0 < r < \frac{1}{2}$ we let Y_r^* be the perforated cell

$$Y_r^* := Y \setminus \overline{B_r(\mathfrak{m})}.$$

Recalling that the radius in (7d) is time-dependent, given the function $R_0 : [0, T] \to [\underline{R}, \overline{R}]$ we define the Hanzawa transform $S_0 : [0, T] \times Y_{\overline{R}}^* \to Y$ by

$$S_0(t, y) := y + (R_0(t) - \overline{R})\chi_0(y)\nu_0(y).$$

Clearly, an equivalent form of S_0 reads

$$S_0(t, y) := \mathfrak{m} + \frac{1}{\|y - \mathfrak{m}\|}\left[\|y - \mathfrak{m}\| + (R_0(t) - \overline{R})\chi(\|y - \mathfrak{m}\| - \overline{R})\right](y - \mathfrak{m}).$$

As will be seen below, for any $t \in [0, T]$, $S_0(t, \cdot) : Y_{\overline{R}}^* \to Y_{R_0(t)}^*$ is a bijective mapping. Note that the function S_0 is defined on the entire cell Y, but is only bijective in a neighborhood of Γ.

With I_n denoting the unit matrix in $\mathbb{R}^{n \times n}$, the derivatives of S_0 are

$$
\nabla S_0(t, y) = I_n + \left(R_0(t) - \overline{R} \right) \left[\nu_0(y) \otimes \nabla_y \chi_0(y) + \chi_0(y) \nabla_y \nu_0(y) \right],
$$
$$
\partial_t S_0(t, y) = R_0'(t) \chi_0(y) \nu_0(y).
$$

Also, for the functions χ_0 and ν_0 we have

$$
\nabla \chi_0(y) = \frac{y - \mathfrak{m}}{|y - \mathfrak{m}|} \chi' \left(|y - \mathfrak{m}| - \overline{R} \right) = \nu_0(y) \chi' \left(|y - \mathfrak{m}| - \overline{R} \right),
$$
$$
\nabla \nu_0(y) = \frac{1}{|y - \mathfrak{m}|} I_n - \frac{1}{|y - \mathfrak{m}|^3} (y - \mathfrak{m}) \otimes (y - \mathfrak{m}) = \frac{1}{|y - \mathfrak{m}|} I_n - \frac{1}{|y - \mathfrak{m}|} \nu_0(y) \otimes \nu_0(y).
$$

This gives

$$
\nabla S_0 = \left(1 + \frac{\chi_0}{|y - \mathfrak{m}|} (R_0 - \overline{R}) \right) I_n + \left[(R_0 - \overline{R}) \left(\chi'(|y - \mathfrak{m}| - \overline{R}) - \frac{\chi_0}{|y - \mathfrak{m}|} \right) \right] \nu_0 \otimes \nu_0.
$$
$$
\tag{52}
$$

To compute the Jacobian determinant of S_0 we use the matrix determinant lemma (see [24], Theorem 13.3.8., or [11]) stating that

$$
\det(A + u \otimes v) = (1 + v^T A^{-1} u) \det(A),
$$

for any invertible matrix $A \in \mathbb{R}^{n \times n}$ and column vectors $u, v \in \mathbb{R}^n$. After an elemental calculation, one has

$$
\det(\nabla S_0(t, y)) = \left(1 + \chi'(|y - \mathfrak{m}| - \overline{R})(R_0(t) - \overline{R}) \right) \cdot \left(1 + \frac{\chi_0(y)}{|y - \mathfrak{m}|} (R_0(t) - \overline{R}) \right)^{n-1}.
$$

Using the definition of χ and with $R_0 \in [\underline{R}, \overline{R}]$, for all $y \in Y_{\overline{R}}^*$ one has

$$
1 \le 1 + \chi'(|y - \mathfrak{m}| - \overline{R})(R_0 - \overline{R}) \le 1 + \|\chi'\|_{L^\infty(\mathbb{R})}(\overline{R} - \underline{R}),
$$
$$
0 < \frac{\underline{R}}{\overline{R}} \le \frac{R_0}{\overline{R}} \le 1 + \frac{\chi_0(y)}{|y - \mathfrak{m}|}(R_0 - \overline{R}) \le 1,
$$

as $|y - \mathfrak{m}| \ge \overline{R}$. Therefore

$$
0 < \left(\frac{\underline{R}}{\overline{R}} \right)^{n-1} \le \det(\nabla S_0(t, y)) \le 1 + \|\chi'\|_{L^\infty(\mathbb{R})}(\overline{R} - \underline{R}).
$$

In particular, this shows that $S_0(t, \cdot)$ is a bijection from $Y_{\overline{R}}^*$ to $Y_{R_0(t)}^*$.

B   Two-scale convergence and unfolding

In this section we briefly summarize the concept of the two-scale convergence and the unfolding operator. These methods provide the basic techniques to pass to the limit $\epsilon \to 0$ in the microscopic problem.

B.1 Two-scale convergence

We start with the definition of the two-scale convergence, which was first introduced and analyzed in [33] and [2], see also [27].

Definition 5. *A sequence $u_\epsilon \in L^p((0,T) \times \Omega)$ for $p \in [1, \infty)$ is said to converge in the two-scale sense in L^p to the limit function $u_0 \in L^p((0,T) \times \Omega \times Y)$, if for every $\phi \in L^{p'}((0,T) \times \Omega, C_{\mathrm{per}}(Y))$ the following relation holds*

$$\lim_{\epsilon \to 0} \int_0^T \int_\Omega u_\epsilon(t,x)\phi\left(t,x,\frac{x}{\epsilon}\right) dxdt = \int_0^T \int_\Omega \int_Y u_0(t,x,y)\phi(t,x,y)dydxdt.$$

A two-scale convergent sequence u_ϵ convergences strongly in the two-scale sense to u_0, if

$$\lim_{\epsilon \to 0} \|u_\epsilon\|_{L^p((0,T)\times\Omega)} = \|u_0\|_{L^p((0,T)\times\Omega\times Y)}.$$

Remark 9. *Let $u_0 \in L^p((0,T)\times\Omega, C^0_{\mathrm{per}}(Y))$. Then u_ϵ converges strongly in the two-scale sense to u_0 in L^p if and only if*

$$\lim_{\epsilon \to 0} \left\|u_\epsilon - u_0\left(x, \frac{x}{\epsilon}\right)\right\|_{L^p((0,T)\times\Omega)} = 0.$$

In [3, 32] the method of two-scale convergence was extended to oscillating surfaces:

Definition 6. *A sequence of functions $u_\epsilon \in L^p((0,T) \times \Gamma_\epsilon)$ for $p \in [1, \infty)$ is said to converge in the two-scale sense on the surface Γ_ϵ in L^p to a limit $u_0 \in L^p((0,T) \times \Omega \times \Gamma)$, if for every $\phi \in C\left([0,T] \times \overline{\Omega}, C_{per}(\Gamma)\right)$ it holds that*

$$\lim_{\epsilon \to 0} \epsilon \int_0^T \int_{\Gamma_\epsilon} u_\epsilon(t,x)\phi\left(t,x,\frac{x}{\epsilon}\right) d\sigma dt = \int_0^T \int_\Omega \int_\Gamma u_0(t,x,y)\phi(t,x,y)d\sigma_y dxdt.$$

We say a two-scale convergent sequence u_ϵ converges strongly in the two-scale sense, if additionally it holds that

$$\lim_{\epsilon \to 0} \epsilon^{\frac{1}{p}} \|u_\epsilon\|_{L^p((0,T)\times\Gamma_\epsilon)} = \|u_0\|_{L^p((0,T)\times\Omega\times\Gamma)}.$$

We have the following compactness results (see e.g., [2, 27, 32]:

Lemma 10. *For every $p \in (1, \infty)$ we have:*

(i) *For every bounded sequence $u_\epsilon \in L^p((0,T) \times \Omega)$ there exists $u_0 \in L^p((0,T) \times \Omega \times Y)$ such that up to a subsequence*

$$u_\epsilon \to u_0 \qquad \text{in the two-scale sense in } L^p.$$

(ii) *For every bounded sequence $u_\epsilon \in L^p((0,T), W^{1,p}(\Omega))$ there exist $u_0 \in L^p((0,T) \times \Omega)$ and $u_1 \in L^p((0,T) \times \Omega, W^{1,p}_{\mathrm{per}}(Y)/\mathbb{R})$, such that up to a subsequence*

$$u_\epsilon \to u_0 \qquad\qquad \text{in the two-scale sense in } L^p,$$
$$\nabla u_\epsilon \to \nabla_x u_0 + \nabla_y u_1 \qquad \text{in the two-scale sense in } L^p.$$

(iii) *For every sequence $u_\epsilon \in L^p((0,T), W^{1,p}(\Omega))$ with u_ϵ and $\epsilon\nabla u_\epsilon$ bounded in $L^p((0,T)\times \Omega)$, there exists $u_0 \in L^p((0,T)\times\Omega, W^{1,p}_{\mathrm{per}}(Y))$ such that up to a subsequence it holds that*

$$u_\epsilon \to u_0 \qquad\qquad \text{in the two-scale sense in } L^p,$$
$$\epsilon\nabla u_\epsilon \to \nabla_y u_0 \qquad\qquad \text{in the two-scale sense in } L^p.$$

(iv) *For every sequence $u_\epsilon \in L^p((0,T)\times\Gamma_\epsilon)$ with*

$$\epsilon^{\frac{1}{p}}\|u_\epsilon\|_{L^p((0,T)\times\Gamma_\epsilon)} \leq C,$$

there exists $u_0 \in L^p((0,T)\times\Omega\times\Gamma)$ such that up to a subsequence

$$u_\epsilon \to u_0 \quad \text{in the two-scale sense on } \Gamma_\epsilon \text{ in } L^p.$$

Concerning the time derivative we have the following result:

Lemma 11. *Let $w_\epsilon \in L^2((0,T), H^1(\Omega_\epsilon)) \cap H^1((0,T), H^1(\Omega_\epsilon)')$ with*

$$\|\partial_t w_\epsilon\|_{L^2((0,T),H^1(\Omega_\epsilon)')} + \|w_\epsilon\|_{L^2((0,T)\times\Omega_\epsilon)} + \epsilon\|\nabla w_\epsilon\|_{L^2((0,T)\times\Omega_\epsilon)} \leq C.$$

Denote by $\widetilde{w}_\epsilon$ the zero extension to the whole domain Ω. Let $w_0 \in L^2((0,T)\times\Omega, H^1_{\mathrm{per}}(Y))$ denote the two-scale limit of $\widetilde{w}_\epsilon$ (up to a subsequence) from Lemma 10 (vanishing on $Y \setminus Y^*$) and define $\bar{w}_0 := \int_Y w dy$. Then, again up to a subsequence, it holds that

$$\partial_t \widetilde{w}_\epsilon \to \partial_t \bar{w}_0 \quad \text{weakly in } L^2((0,T), H^1(\Omega)').$$

If additionally $\widetilde{w}_\epsilon(0)$ converges in the two-scale sense to $w^0 \in L^2(\Omega\times Y)$, then we have

$$\bar{w}_0(0) = \int_Y w^0 dy. \tag{53}$$

Proof. This follows by standard two-scale arguments and integration by parts in time, so we skip the details. □

B.2 The unfolding operator

When dealing with nonlinear problems it is helpful to work the unfolding method which gives a characterization for the weak and strong convergence, see Lemma 13 below. For a detailed investigation of the unfolding operator and its properties we refer to [9]. For a perforated domain (here we also allow the case $Y^* = Y$, i.e., $\Omega_\epsilon = \Omega$) we define the unfolding operator for $p \in [1,\infty]$ by

$$\mathcal{T}_\epsilon : L^p((0,T)\times\Omega_\epsilon) \to L^p((0,T)\times\Omega\times Y^*), \quad \mathcal{T}_\epsilon(\phi_\epsilon)(t,x,y) = \phi_\epsilon\left(t, \epsilon\left[\frac{x}{\epsilon}\right] + \epsilon y\right).$$

In the same way, we define the boundary unfolding operator for the oscillating surface Γ_ϵ via

$$\mathcal{T}_\epsilon : L^p((0,T)\times\Gamma_\epsilon) \to L^p((0,T)\times\Omega\times\Gamma), \quad \mathcal{T}_\epsilon(\phi_\epsilon)(t,x,y) = \phi_\epsilon\left(t, \epsilon\left[\frac{x}{\epsilon}\right] + \epsilon y\right).$$

We emphasize that we use the same notation for the unfolding operator on Ω_ϵ and the boundary unfolding operator Γ_ϵ. We summarize the basic properties of the unfolding operator, see [9]:

Lemma 12. *Let $p \in [1, \infty]$.*

(i) For $\phi_\epsilon \in L^p((0,T) \times \Omega_\epsilon)$ it holds that

$$\|\mathcal{T}_\epsilon(\phi_\epsilon)\|_{L^p((0,T) \times \Omega \times \Omega_\epsilon)} = \|\phi_\epsilon\|_{L^p((0,T) \times Y^*)}.$$

(ii) For $\phi_\epsilon \in L^p((0,T), W^{1,p}(\Omega_\epsilon))$ it holds that

$$\nabla_y \mathcal{T}_\epsilon(\phi_\epsilon) = \epsilon \mathcal{T}_\epsilon(\nabla_x \phi_\epsilon).$$

(iii) For $\phi_\epsilon \in L^p((0,T) \times \Gamma_\epsilon)$ it holds that

$$\|\mathcal{T}_\epsilon \phi_\epsilon\|_{L^p((0,T) \times \Omega \times \Gamma)} = \epsilon^{\frac{1}{p}} \|\phi_\epsilon\|_{L^p((0,T) \times \Gamma_\epsilon)}.$$

The following lemma gives a relation between the unfolding operator and the two-scale converges. Its proof is quite standard and we refer the reader to [5] and [9] for more details (see also [47, Proposition 2.5]).

Lemma 13. *Let $p \in [1, \infty)$.*

(a) For a sequence $u_\epsilon \in L^p((0,T) \times \Omega)$, the following statements are equivalent:

  (i) $u_\epsilon \to u_0$ weakly/strongly in the two-scale sense in L^p,

  (ii) $\mathcal{T}_\epsilon u_\epsilon \to u_0$ weakly/strongly in $L^p((0,T) \times \Omega \times Y)$.

(b) For a sequence $u_\epsilon \in L^p((0,T) \times \Gamma_\epsilon)$ the following statements are equivalent:

  (i) $u_\epsilon \to u_0$ weakly/strongly in the two-scale sense on Γ_ϵ in L^p,

  (ii) $\mathcal{T}_\epsilon u_\epsilon \to u_0$ weakly/strongly in $L^p((0,T) \times \Omega \times \Gamma)$.

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