The Drinfel'd double for group-cograded multiplier Hopf algebras

Peer-reviewed author version

DELVAUX, Lydia & Van Daele, A. (2007) The Drinfel'd double for group-cograded multiplier Hopf algebras. In: ALGEBRAS AND REPRESENTATION THEORY, 10(3). p. 197-221.

DOI: 10.1007/s10468-006-9042-1
Handle: http://hdl.handle.net/1942/3981

The Drinfel'd double for group-cograded multiplier Hopf algebras

Lydia Delvaux

Department of Mathematics, L.U.C., Universiteitslaan, B-3590 Diepenbeek (Belgium)

E-mail: lydia.delvaux@luc.ac.be

Alfons Van Daele

Department of Mathematics, K.U.Leuven, Celestijnenlaan 200B, B-3001 Heverlee (Belgium)

E-mail: alfons.vandaele@wis.kuleuven.ac.be

Abstract

Let G be any group and let $K(G)$ denote the multiplier Hopf algebra of complex functions with finite support in G. The product in $K(G)$ is pointwise. The comultiplication on $K(G)$ is defined with values in the multiplier algebra $M(K(G) \otimes K(G))$ by the formula $(\Delta(f))(p, q) = f(pq)$ for all $f \in K(G)$ and $p, q \in G$. In this paper we consider multiplier Hopf algebras B (over $\mathbb{C}$) such that there is an embedding $I : K(G) \to M(B)$. This embedding is a non-degenerate algebra homomorphism which respects the comultiplication and maps $K(G)$ into the center of $M(B)$. These multiplier Hopf algebras are called *G-cograded multiplier Hopf algebras.* They are a generalization of the Hopf group-coalgebras as studied by Turaev and Virelizier.

In this paper, we also consider an *admissible* action π of the group G on a G-cograded multiplier Hopf algebra B. When B is paired with a multiplier Hopf algebra A, we construct the Drinfel'd double D^π where the coproduct and the product depend on the action π. We also treat the *-algebra case.

If π is the trivial action, we recover the usual Drinfel'd double associated with the pair $\langle A, B\rangle$. On the other hand, also the Drinfel'd double, as constructed by Zunino for a finite-type Hopf group-coalgebra, is an example of the construction above. In this case, the action is non-trivial but related with the adjoint action of the group on itself. Now, the double is again a G-cograded multiplier Hopf algebra.

Mathematics Subject Classification: 16W30, 17B37.

March 2004 (*Version 1.0*)

Introduction

Let A be an algebra over $\mathbb{C}$. If A has no unit, we require that the product in A is non-degenerate as a bilinear map. The multiplier algebra $M(A)$ of A is the largest algebra with unit in which A sits as a dense two-sided ideal. If A has a unit, then $M(A) = A$. Consider a group G and let A be the algebra of complex valued functions with finite support in G, with pointwise product. This algebra has no unit, except when G is finite. The multiplier algebra $M(A)$ is given by the algebra of all complex valued functions on G. We define a comultiplication Δ on A by $(\Delta(f))(p,q) = f(pq)$ where $f \in A$ and $p, q \in G$. If G is finite, then Δ maps A into $A \otimes A$ and makes A into a Hopf algebra. However, when G is not finite, $\Delta(f) \in M(A \otimes A)$ for all $f \in A$. In this case (A, Δ) is a multiplier Hopf algebra, as reviewed in Section 1. In this paper, the multiplier Hopf algebra A associated with a group G as above, is denoted as $(K(G), \Delta)$.

Multiplier Hopf algebras are generalizations of Hopf algebras when the underlying algebra is no longer assumed to have a unit. Integrals on multiplier Hopf algebras are defined as in the Hopf algebra case, see Section 1. If A is a multiplier Hopf algebra with integrals, the dual object can be defined within the same category. This duality generalizes the one for finite-dimensional Hopf algebras, but applies to a much bigger class of (multiplier) Hopf algebras. In fact, the theory of multiplier Hopf algebras is a theory that allows results which are not possible within the framework of usual Hopf algebras. Furthermore, this theory is also a good model for an analytical theory of locally compact quantum groups. The link between these two theories are the multiplier Hopf *-algebras with positive integrals, see [VD2], [VD3] and especially [K-VD].

Let G be any group. In this paper we deal with *G-cograded multiplier Hopf algebras* in the sense of Definition 2.1 (in Section 2 of this paper). Roughly speeking, a multiplier Hopf algebra B is G-cograded if there is a central, non-degenerate embedding $I : K(G) \to M(B)$. Furthermore, this embedding respects the comultiplication, i.e. $\Delta(I(f)) = (I \otimes I)(\Delta(f))$ for all $f \in K(G)$. Remark that we give a meaning to this equation by extending the homomorphism Δ from B to $M(B)$ and $I \otimes I$ from $K(G) \otimes K(G)$ to $M(K(G) \otimes K(G))$ using the fact that the homomorphisms are non-degenerate and hence have unique extensions.

It is shown in [A-D-VD] that a Hopf group-coalgebra, as introduced by Turaev in [T], is a special case of a group-cograded multiplier Hopf algebra. Therefore, we can interpret the results of Turaev, Virelizier and Zunino within the theory of cograded multiplier Hopf algebras. This threws a new light on their results. More precisely, a lot of the results for Hopf group-coalgebras follow from the more general results for multiplier Hopf algebras. Moreover, we can apply the techniques from the theory of multiplier Hopf algebra in the study of Hopf group-coalgebras. We refer to [A-D-VD] for details about this approach to Hopf group-coalgebras.

The main goal of this paper is to apply this point of view when constructing the quantum double for such G-cograded multiplier Hopf algebras. We recover and generalize the work of Zunino on this subject as it is found in [Z].

Let us now summarize the *content* of this paper.

In *Section 1*, we recall the definition of a multiplier Hopf algebra and we review some results which are used in this paper.

In *Section 2*, we first recall the notion of a group-cograded multiplier Hopf algebra as studied in [A-D-VD]. Then we consider an *admissible* action of the group G on a G-cograded multiplier Hopf algebra B. Using this action, we deform the comultiplication of B. This gives rise to a new multiplier Hopf algebra $\widetilde{B} = (B, \widetilde{\Delta})$ where the underlying algebra structure of B is unchanged but with a different comultiplication. If B is regular, so is $\widetilde{B}$. If it is a multiplier Hopf *-algebra and has positive integrals, then the same is true for $\widetilde{B}$, see Theorem 2.11.

In *Section 3*, we start with a multiplier Hopf algebra pairing $\langle A, B\rangle$ where B is a G-cograded multiplier Hopf algebra. Let π be an admissible action of G on B. Then we construct a twisted tensor product multiplier Hopf algebra (as reviewed in Section 1) of the multiplier Hopf algebras (A, Δ^{cop}) and $(B, \widetilde{\Delta})$. This twisted tensor product multiplier Hopf algebra is denoted as $D^\pi = A^{cop} \bowtie \widetilde{B}$. The main results on D^π are given in Theorem 3.8 and Proposition 3.9. If π is the trivial action, we recover the usual Drinfel'd double of the pair $\langle A, B\rangle$, as reviewed in Section 1. The Drinfel'd double, as constructed by Zunino for a finite-type Hopf group-coalgebras in [Z] is considered in Example 3.14.

By the constructions in Section 2 and Section 3, we have further interesting, non-trivial examples of multiplier Hopf (*-)algebras with (positive) integrals.

Let us finish this introduction by mentioning some *basic references*. For multiplier Hopf algebras, these are [VD1] and [VD-Z]. For multiplier Hopf algebras with integrals, we refer to [VD2]. The Drinfel'd double of a pair of multiplier Hopf algebras is studied extensively in [Dr-VD] and in [De-VD]. The Hopf group-coalgebras are introduced in [T]. The new approach to Hopf group-coalgebras is studied and generalized to the case of multiplier Hopf algebras in [A-D-VD].

Acknowledgements

We are indebted to A.T. Abt El-Hafez, A.S. Hegazi and M. Mansour who provided us with their preprint "Multiplier Hopf group-coalgebras" [H-A-M] and in doing so, brought to our attention the works of Turaev, Virelizier and Zunino.

1 Preliminaries on multiplier Hopf (*-)algebras

We begin this section with a short introduction to the theory of multiplier Hopf algebras.

Multiplier Hopf (*-)algebras with (positive) integrals

As mentioned already in the introduction, throughout this paper, all algebras are algebras over the field $\mathbb{C}$ of complex numbers. They may or may not have units, but always should be non-degenerate, i.e. the multiplication maps (viewed as bilinear forms) are non-degenerate. For an algebra A, the multiplier algebra $M(A)$ of A is defined as the largest algebra with unit in which A is a dense ideal, i.e. A has no (left and right) annihilators in $M(A)$.

Now, we recall the definition of a multiplier Hopf algebra (see [VD1] for details). Consider the tensor product $A \otimes A$ which is again an algebra with a non-degenerate product. The embedding of $A \otimes A \hookrightarrow M(A \otimes A)$ factors through $M(A) \otimes M(A)$ in an obvious way as follows: $A \otimes A \hookrightarrow M(A) \otimes M(A) \hookrightarrow M(A \otimes A)$. A comultiplication on A is a homomorphism $\Delta : A \to M(A \otimes A)$ such that $\Delta(a)(1 \otimes b)$ and $(a \otimes 1)\Delta(b)$ are elements of $A \otimes A$ for all $a, b \in A$. We require Δ to be coassociative in the sense that

$$(a \otimes 1 \otimes 1)(\Delta \otimes \iota)(\Delta(b)(1 \otimes c)) = (\iota \otimes \Delta)((a \otimes 1)\Delta(b))(1 \otimes 1 \otimes c)$$

for all $a, b, c \in A$ (where ι denotes the identity map).

1.1 Definition [VD1] A pair (A, Δ) of an algebra A with non-degenerate product and a comultiplication Δ on A is called a multiplier Hopf algebra if the linear maps $T_1, T_2 : A \otimes A \to A \otimes A$, defined by

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b) \qquad T_2(a \otimes b) = (a \otimes 1)\Delta(b)$$

are bijections.

The conditions in Definition 1.1 in fact imply that Δ is a non-degenerate homomorphism. For the convenience of the reader, we recall the notion of a *non-degenerate homomorphism*. It is a homomorphism $\gamma : A \to M(B)$, where A and B are algebras with a non-degenerate product, such that $\gamma(A)B = B\gamma(A) = B$. So, e.g. every element $b \in B$ is a sum of elements of the form $\gamma(a)b'$ with $a \in A$ and $b' \in B$. An important property of such a non-degenerate homomorphism γ is that it has a unique extension, to a unital homomorphism from $M(A)$ to $M(B)$. The extension is still denoted by γ. See the appendix in [VD1]. The homomorphisms $\iota \otimes \Delta$ and $\Delta \otimes \iota$ are also non-degenerate and so have unique extensions to $M(A \otimes A)$ in a natural way. The coassociativity as formulated above means nothing else but $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$ in $M(A \otimes A \otimes A)$.

The bijectivity of the two maps in Definition 1.1 is equivalent with the existence of a counit ε and an antipode S satisfying (and defined) by

$$
\begin{aligned}
(\varepsilon \otimes \iota)(\Delta(a)(1 \otimes b)) &= ab \qquad & m((S \otimes \iota)(\Delta(a)(1 \otimes b))) &= \varepsilon(a)b \\
(\iota \otimes \varepsilon)((a \otimes 1)\Delta(b)) &= ab \qquad & m((\iota \otimes S)((a \otimes 1)\Delta(b))) &= \varepsilon(b)a
\end{aligned}
$$

where $\varepsilon : A \to \mathbb{C}$ is a homomorphism, $S : A \to M(A)$ is an anti-homomorphism and m is the multiplication map, considered as a linear map from $A \otimes A$ to A and extended to $A \otimes M(A)$ and $M(A) \otimes A$.

A multiplier Hopf algebra is called *regular* if (A, Δ^{cop}) is (also) a multiplier Hopf algebra, where Δ^{cop} denotes the co-opposite comultiplication defined as $\Delta^{cop} = \sigma \circ \Delta$ with σ the usual flip map from $A \otimes A$ to itself (and extended to $M(A \otimes A)$). In this case, we also have that $\Delta(a)(b \otimes 1)$ and $(1 \otimes b)\Delta(a)$ are in $A \otimes A$ for all $a, b \in A$. A multiplier Hopf algebra is regular if and only if the antipode is a bijection from A to A (see [VD2, Proposition 2.9]). Any Hopf algebra is a multiplier Hopf algebra. Conversely, a multiplier Hopf algebra with unit is a Hopf algebra.

In [Dr-VD], the use of the Sweedler notation for regular multiplier Hopf algebras has been introduced. We will also use this notation in this paper. We will e.g. write $\sum a_{(1)} \otimes a_{(2)} b$ for $\Delta(a)(1 \otimes b)$ and $\sum ab_{(1)} \otimes b_{(2)}$ for $(a \otimes 1)\Delta(b)$.

1.2 Definition [VD1] If A is a *-algebra, we require the comultiplication Δ to be also a *-homomorphism. Then, a multiplier Hopf *-algebra is a *-algebra with a comultiplication, making it into a multiplier Hopf algebra. For a multiplier Hopf *-algebra, regularity is automatic.

Recall that a *-algebra A over $\mathbb{C}$ is an algebra with an involution $a \mapsto a^*$. An involution is a antilinear map satisfying $a^{**} = a$ and $(ab)^* = b^* a^*$ for all a and b in A. The multiplier algebra $M(A)$ is again a *-algebra.

1.3 Example Let G be any group and let A be the *-algebra $K(G)$ of complex, finitely supported functions on G. In this case $M(A)$ consists of all complex functions on G. Moreover $A \otimes A$ can be naturally identified with finitely supported complex functions on $G \times G$ so that $M(A \otimes A)$ is the space of all complex functions on $G \times G$. If we define $\Delta : A \to M(A \otimes A)$ by $(\Delta(f))(p, q) = f(pq)$ for $f \in A$ and $p, q \in G$, we clearly get a *-homomorphism. If $f, g \in A$, then $(p, q) \mapsto f(pq)g(q)$ and $(p, q) \mapsto g(p)f(pq)$ have finite support and so belong to $A \otimes A$. The coassociativity condition on Δ is a consequence of the associativity of the multiplication on G. So Δ is a comultiplication. To obtain that the pair (A, Δ) is a multiplier Hopf algebra in the sense of Definition 1.1, we notice that the bijectivity of the linear maps T_1 and T_2 follows from the fact that the maps $(p, q) \mapsto (pq, q)$ and $(p, q) \mapsto (p, pq)$ are bijective from $G \times G$ to itself (because G is assumed to be a group).

This is a very simple example. Interesting examples of multiplier Hopf algebras are found among the discrete quantum groups (i.e. the duals of compact quantum groups), see e.g. [VD2].

We now discuss the notion of an *integral* on a multiplier Hopf (*-)algebra. It is like the integral on a Hopf algebra. We will restrict to the case of a regular multiplier Hopf algebra (in particular to a multiplier Hopf *-algebra). We first observe the following. If

f is a linear functional on a multiplier Hopf algebra A, we can define for all $a \in A$, the multipliers $(\iota \otimes f)\Delta(a)$ in $M(A)$ in the following way

$$((\iota \otimes f)\Delta(a))b = (\iota \otimes f)(\Delta(a)(b \otimes 1)) \qquad b((\iota \otimes f)\Delta(a)) = (\iota \otimes f)((b \otimes 1)\Delta(a))$$

where $b \in A$. This is well-defined as both $\Delta(a)(b \otimes 1)$ and $(b \otimes 1)\Delta(a)$ are in $A \otimes A$ and we can apply $\iota \otimes f$ mapping $A \otimes A$ to $A \otimes \mathbb{C}$ (which is naturally itendified with A itself). Similarly, we can define $(f \otimes \iota)\Delta(a)$ in $M(A)$. Then, the following definition makes sense.

1.4 Definition [VD2] A linear functional φ on A is called left invariant if $(\iota \otimes \varphi)\Delta(a) = \varphi(a)1$ for all $a \in A$. A left integral is a non-zero left invariant functional on A. Similarly, a non-zero linear functional ψ satisfying $(\psi \otimes \iota)\Delta(a) = \psi(a)1$ for all $a \in A$ is called a right integral.

In Example 1.3, a left integral is given by the formula $\varphi(f) = \sum_{q \in G} f(q)$ (the sum is well-defined as only finitely many entries are non-zero). In this example the left integral is also right invariant. This however is no longer true in general. Multiplier Hopf (*-)algebras with integrals are studied intensively in [VD2]. There are various data (and many relations among them) about left and right integrals. We collect some important results of [VD2].

1.5 Theorem [VD2] Let (A, Δ) be a multiplier Hopf (*-)algebra with left integral φ. Any other left integral is a scalar multiple of φ. There is also a right integral ψ, unique up to a scalar. The left integral is faithful in the sense that when $a \in A$, then $a = 0$ if $\varphi(ab) = 0$ for all b or $\varphi(ba) = 0$ for all b. Similarly, the right integral is faithful. There is an automorphism σ of A such that $\varphi(ab) = \varphi(b\sigma(a))$ for all $a, b \in A$. There is an invertible multiplier δ in $M(A)$ such that $(\varphi \otimes \iota)\Delta(a) = \varphi(a)\delta$ and $(\iota \otimes \psi)\Delta(a) = \psi(a)\delta^{-1}$ for all $a \in A$. $\blacksquare$

One of the main features of a multiplier Hopf algebra A with integrals is the existence of the dual multiplier Hopf algebra $(\widehat{A}, \widehat{\Delta})$. It is constructed in the following way.

1.6 Definition [VD2] Let (A, Δ) be a multiplier Hopf (*-)algebra with left integral φ. Denote by $\widehat{A}$ the space of linear functionals on A of the form $x \mapsto \varphi(xa)$ where $a \in A$. The product (respectively the coproduct) of $\widehat{A}$ is dual to the coproduct (respectively the product) of A.

This dual object $(\widehat{A}, \widehat{\Delta})$ is again a regular multiplier Hopf algebra with integrals. Moreover, the dual of $(\widehat{A}, \widehat{\Delta})$ is canonically isomorphic with the orginial multiplier Hopf algebra (A, Δ). For a finite-dimensional Hopf algebra A, we notice that $\widehat{A}$ equals the usual dual Hopf algebra of A. It is possible to generalize many aspects of harmonic analysis in this general framework. One can define the Fourier transform, one can prove Plancherel's formula, ... For details, see [VD2].

In the case that (A, Δ) is a multiplier Hopf *-algebra, a left integral is called positive if $\varphi(a^*a) \geq 0$ for all $a \in A$. We mention that, when there is a positive left integral, there is also a positive right integral. Multiplier Hopf *-algebras with positive integrals, give rise to dual multiplier Hopf *-algebras with positive integrals. In [K-VD] is explained how any multiplier Hopf *-algebra with positive integrals gives rise to a locally compact quantum group as introduced and studied by Kustermans and Vaes in [K-V].

Pairing and Drinfel'd double of multiplier Hopf algebras

We now recall how the Drinfel'd double is constructed from a pair of multiplier Hopf algebras.

Start with two regular multiplier Hopf algebras (A, Δ) and (B, Δ) together with a non-degenerate bilinear map $\langle \cdot, \cdot \rangle$ from $A \times B$ to $\mathbb{C}$. This is called a pairing if certain conditions are fulfilled. The main property is that the coproduct in A is dual to the product in B and vice versa. There are however certain regularity conditions, needed to give a correct meaning to this statement. The investigation of these conditions is done in [Dr-VD]. We recall some important aspects here.

1.7 Definition For $a \in A$ and $b \in B$, define $a \blacktriangleright b$, $b \blacktriangleleft a$, $b \blacktriangleright a$ and $a \blacktriangleleft b$ as multipliers in the following way. Take $a' \in A$, $b' \in B$ and define

$$\begin{aligned} (b \blacktriangleright a)a' &= \sum \langle a_{(2)}, b\rangle a_{(1)}a' \qquad & (a \blacktriangleright b)b' &= \sum \langle a, b_{(2)}\rangle b_{(1)}b' \\ (a \blacktriangleleft b)a' &= \sum \langle a_{(1)}, b\rangle a_{(2)}a' \qquad & (b \blacktriangleleft a)b' &= \sum \langle a, b_{(1)}\rangle b_{(2)}b'. \end{aligned}$$

The regularity conditions on the pairing say (among other things) that the multipliers $b \blacktriangleright a$ and $a \blacktriangleleft b$ in $M(A)$ (resp. $a \blacktriangleright b$ and $b \blacktriangleleft a$ in $M(B)$) actually belong to A (resp. B).

Then it is possible to state that the product and the coproduct are dual to each other:

$$\begin{aligned} \langle a, bb'\rangle &= \langle b' \blacktriangleright a, b\rangle \qquad & \langle aa', b\rangle &= \langle a, a' \blacktriangleright b\rangle \\ &= \langle a \blacktriangleleft b, b'\rangle & &= \langle a', b \blacktriangleleft a\rangle. \end{aligned}$$

There are four modules involved. All these modules are unital. By definition, e.g. B is a left A-module for the action $A \blacktriangleright B$. That B is unital means that any element $b \in B$ is a linear combination of elements of the form $a \blacktriangleright b'$ with $a \in A$ and $b' \in B$. A stronger result however is possible here, coming from the existence of local units, see [D-VD-Z]. Take e.g. $b \in B$. Then there are elements $\{a_1, a_2, \ldots, a_n\}$ in A and $\{b_1, b_2, \ldots, b_n\}$ in B such that $b = \sum a_i \blacktriangleright b_i$. Because of the existence of local units, there is an $e \in A$ such that $ea_i = a_i$ for all i. It follows easily that $e \blacktriangleright b = b$. So, we have that for all $b \in B$ there exists an element $e \in A$ such that $b = e \blacktriangleright b$.

As an important consequence of this last result, we can use the Sweedler notation in the framework of dual pairs in the following sense. Take $a \in A$ and $b \in B$, and e.g. the element $b \blacktriangleright a = \sum \langle a_{(2)}, b\rangle a_{(1)}$. In the right hand side of this equation, the element $a_{(2)}$ is covered by b through the pairing because $b = e \blacktriangleright b$ for some $e \in A$ and therefore $\sum \langle a_{(2)}, b\rangle a_{(1)} = \langle a_{(2)}, e \blacktriangleright b\rangle a_{(1)} = \sum \langle a_{(2)}e, b\rangle a_{(1)}$.

We also mention that $\langle S(a), b\rangle = \langle a, S(b)\rangle$ and as expected, $\langle a, 1\rangle = \varepsilon(a)$ and $\langle 1, b\rangle = \varepsilon(b)$

where $a \in A$ and $b \in B$. For these formulas, one has to extend the pairing to $A \times M(B)$ and to $M(A) \times B$. This can be done in a natural way using the fact that the four modules $A \blacktriangleright B$, $B \blacktriangleright A$, $A \blacktriangleleft B$ and $B \blacktriangleleft A$ are unital.

If A and B are multiplier Hopf *-algebras. A pairing $\langle A, B\rangle$ is called a multiplier Hopf *-algebra pairing if additionally $\langle a^*, b\rangle = \overline{\langle a, S(b)^*\rangle}$ for all $a \in A$ and $b \in B$.

A pairing of two multiplier Hopf algebras is the natural setting for the construction of the Drinfel'd double and it turns out that the conditions on the pairing $\langle A, B\rangle$ are sufficient to make this construction. This is done in a rigorous way in the papers [Dr-VD], [D] and [De-VD]. We recall some essential ideas.

The main point is that there is (as in the case of finite-dimensional Hopf algebras) an invertible twist map.

1.8 Definition For $a \in A$ and $b \in B$, we set

$$R(b \otimes a) = \sum (b_{(1)} \blacktriangleright a \blacktriangleleft S^{-1}(b_{(3)})) \otimes b_{(2)}.$$

It is proven in [Dr-VD] that this map is well-defined and bijective. Let $D = A \bowtie B$ denote the algebra with the tensor product $A \otimes B$ as the underlying space and with the twisted product given by the twist map R as follows:

$$(a \bowtie b)(a' \bowtie b') = (m_A \otimes m_B)(\iota_A \otimes R \otimes \iota_B)(a \otimes b \otimes a' \otimes b')$$

with $a, a' \in A$ and $b, b' \in B$. The maps $A \to M(D) : a \mapsto a \bowtie 1$ and $B \to M(D) : b \mapsto 1 \bowtie b$ are non-degenerate algebra embeddings. The embedding of A in $M(D)$ gives rise to the embedding of $A \otimes A$ in $M(D \otimes D)$. Similarly, $B \otimes B$ can be embedded in $M(D \otimes D)$. These embeddings can be extended to the multiplier algebras. The comultiplication on D can then be given by the formula $\Delta_D(a \bowtie b) = \Delta^{cop}(a)\Delta(b)$.

The main result is the following.

1.9 Theorem With the notations and definitions above, the pair (D, Δ_D) is a multiplier Hopf algebra, called the quantum double (or Drinfel'd double). If A and B have integrals, then D has integrals too.

More precisely, let φ_A denote a left integral on A and let ψ_B denote a right integral on B, then $\psi_D = \varphi_A \otimes \psi_B$ is a right integral on D.

If $\langle A, B \rangle$ is a multiplier Hopf *-algebra pairing of two multiplier Hopf *-algebras, then $D = A^{cop} \bowtie B$ is again a multiplier Hopf *-algebra. Suppose that φ_A (resp. ψ_B) is a positive left integral on A (resp. right integral on B). In [De-VD] it is shown that there esists a complex number ρ such that $\rho(\varphi_A \otimes \psi_B)$ is a positive right integral on $D = A^{cop} \bowtie B$.

Twisted tensor product construction of multiplier Hopf (*-)algebras

The Drinfel'd double construction, as reviewed above, is a special case of a twisted tensor product of multiplier Hopf (*-)algebras. General twisted tensor products of multiplier Hopf algebras are studied in [D]. Also here, we recall some ideas.

1.10 Assumptions Let A and B be two algebras and suppose that there is given a bijective linear map $R : B \otimes A \to A \otimes B$ such that

$$R(m_B \otimes \iota_A) = (\iota_A \otimes m_B)(R \otimes \iota_B)(\iota_B \otimes R)$$
$$R(\iota_B \otimes m_A) = (m_A \otimes \iota_B)(\iota_A \otimes R)(R \otimes \iota_A).$$

Recall that m_A denotes the product in A, considered as a linear map $m_A : A \otimes A \to A$ and similarly for the product m_B on B.

One can consider the twisted tensor product algebra $A \bowtie B$ in the following way. As a vector space $A \bowtie B$ is $A \otimes B$. The product in $A \bowtie B$ is defined by

$$(a \bowtie b)(a' \bowtie b') = (m_A \otimes m_B)(\iota_A \otimes R \otimes \iota_B)(a \otimes b \otimes a' \otimes b')$$

for $a, a' \in A$ and $b, b' \in B$. As before, we use ι to denote the identity map, in particular we have ι_A and ι_B for the identity maps on A and B respectively.

The above assumptions on R are necessary for the associativity of the product in $A \bowtie B$. Because the the products in A and B are assumed to be non-degenerate, one can prove that the bijectivity of R guaranties that the product in $A \bowtie B$ is again non-degenerate.

1.11 Remarks For details, see [D].

(i) By using the conditions on R and the bijectivity, one can prove that the product in $A \bowtie B$ is also given by the following expressions

$$(a \bowtie b)(a' \bowtie b') = ((\iota_A \otimes m_B) \circ R_{12} \circ (\iota_B \otimes m_A \otimes \iota_B) \circ (R^{-1})_{12})(a \otimes b \otimes a' \otimes b')$$
$$(a \bowtie b)(a' \bowtie b') = ((m_A \otimes \iota_B) \circ R_{34} \circ (\iota_A \otimes m_B \otimes \iota_A) \circ (R^{-1})_{34})(a \otimes b \otimes a' \otimes b')$$

for all $a, a' \in A$ and $b, b' \in B$. Recall that we use the *leg-numbering notation* for the maps R. When we write e.g. R_{12}, we consider R as acting on the first two factors in the tensor product. Similarly R_{34} is the map R as acting on the 3th and the 4th factor. These formulas will be used to justify that the decompositions of the comultiplications in A and in B are well covered when we use the Sweedler notation.

(ii) The maps

$$A \to M(A \bowtie B) : a \mapsto a \bowtie 1 \qquad B \to M(A \bowtie B) : b \mapsto 1 \bowtie b$$

are non-degenerate algebra embeddings and therefore extend, in a natural way, to unital algebra embeddings from $M(A)$ and $M(B)$ respectively to $M(A \bowtie B)$. Therefore, we have the non-degenerate algebra embeddings

$$A \otimes A \to M((A \bowtie B) \otimes (A \bowtie B)) : a \otimes a' \mapsto (a \bowtie 1) \otimes (a' \bowtie 1)$$
$$B \otimes B \to M((A \bowtie B) \otimes (A \bowtie B)) : b \otimes b' \mapsto (1 \bowtie b) \otimes (1' \bowtie b').$$

Also these embeddings extend to the multiplier algebras in a natural way.

(iii) Let A and B be *-algebras with non-degenerate products. Suppose that the twist map $R : B \otimes A \to A \otimes B$ is bijective and satisfies the above conditions. If furthermore $(R \circ *_B \otimes *_A \circ \sigma)(R \circ *_B \otimes *_A \circ \sigma) = \iota_A \otimes \iota_B$, then there is a *-operation on $A \bowtie B$ given as follows: $(a \bowtie b)^* = R(b^* \otimes a^*)$ for all $a \in A, b \in B$. Now the embeddings in Remark (ii) become *-embeddings.

The comultiplications on A and B can be used to define the comultipication on $A \bowtie B$ as usual:

1.12 Definition [**D**] Let A and B be multiplier Hopf algebras. Let $R : B \otimes A \to A \otimes B$ be a bijective map satisfying the Assumptions 1.10. For $a \in A$ and $b \in B$, define

$$\overline{\Delta}(a \bowtie b) = \Delta(a)\Delta(b) \in M((A \bowtie B) \otimes (A \bowtie B)).$$

In the next theorem, we formulate sufficient conditions for $(A \bowtie B, \overline{\Delta})$ to be a regular multiplier Hopf algebra.

1.13 Theorem [**D**] Let A and B be multiplier Hopf algebras with a bijective twist map R, satisfying the following conditions

(1) $R(m_B \otimes \iota_A) = (\iota_A \otimes m_B)(R \otimes \iota_B)(\iota_B \otimes R)$
$R(\iota_B \otimes m_A) = (m_A \otimes \iota_B)(\iota_A \otimes R)(R \otimes \iota_A)$

(2) $\overline{\Delta}(R(b \otimes a)) = \Delta(b)\Delta(a)$ in $M((A \bowtie B) \otimes (A \bowtie B))$ for all $a \in A$ and $b \in B$.

Then $(A \bowtie B, \overline{\Delta}, \overline{\varepsilon}, \overline{S})$ is a regular multiplier Hopf algebra with, $\overline{\varepsilon}$ and $\overline{S}$ given as $\overline{\varepsilon} = \varepsilon_A \otimes \varepsilon_B$ and $\overline{S} = R \circ (S_B \otimes S_A) \circ \sigma$.
Let A and B be multiplier Hopf *-algebras and $(R \circ (*_B \otimes *_A) \circ \sigma)(R \circ (*_B \otimes *_A) \circ \sigma) = \iota_A \otimes \iota_B$. Then $(A \bowtie B, \overline{\Delta})$ is made into a multiplier Hopf *-algebra, if the *-operation is defined by $(a \bowtie b)^* = R(b^* \otimes a^*)$. ■

2 Actions of G on a G-cograded multiplier Hopf algebra

Throughout this section, G is an arbitrary group. Let $K(G)$ be the multiplier Hopf algebra of the complex valued functions with finite support in G, see Example 1.3.

Group-cograded multiplier Hopf algebras

Recall the following definition given in [A-D-VD].

2.1 Definition A *G-cograded multiplier Hopf (*-)algebra* is multiplier Hopf (*-)algebra B so that the following hold:

(1) $B = \bigoplus_{p \in G} B_p$ with $\{B_p\}_{p \in G}$ a family of (*-)subalgebras such that $B_p B_q = 0$ if $p \neq q$,

(2) $\Delta(B_{pq})(1 \otimes B_q) = B_p \otimes B_q$ and $(B_p \otimes 1)\Delta(B_{pq}) = B_p \otimes B_q$ for all $p, q \in G$.

We say that the comultiplication of B is G-graded.

2.2 Proposition The data of a Hopf group-coalgebra, as introduced by Turaev in [T], give an example of a cograded multiplier Hopf algebra.
For the proof of this propisition, we refer to [A-D-VD].

In order to characterize the coalgebra structure of a general G-cograded multiplier Hopf algebra, we first need the following lemma.

2.3 Lemma Let A and B be multiplier Hopf algebras. Let $f : A \to M(B)$ be a non-degenerate algebra homomorphism which respects the comultiplication in the sense that $\Delta_B \circ f = (f \otimes f) \circ \Delta_A$. Then f preserves the unit, the counit and the antipode in the following way. For any $a \in A$ and $b \in B$, we have

$$f(1_A) = 1_B, \quad \varepsilon_B(f(a)) = \varepsilon_A(a) \quad \text{and} \quad S_B(f(a)) = f(S_A(a)).$$

2.4 Proposition A multiplier Hopf (*-)algebra B is G-cograded if and only if there exists an injective, non-degenerate (*-)homomorphism $I : K(G) \to M(B)$ so that

(i) $I(K(G)) \subset Z(M(B))$ when $Z(M(B))$ is the center of $M(B)$,

(ii) $\Delta(I(f)) = (I \otimes I)\Delta(f)$ for all $f \in K(G)$.

We have $I(\delta_p) = 1_p$, where δ_p is the complex valued function on G, given by $\delta_p(q) = 0$ if $p \neq q$ and $\delta_p(p) = 1$ and where 1_p is the unit in $M(B_p)$.
Again, for the proof of these two results, we refer to [A-D-VD].

An immediate consequence of these two results is the following.
2.5 Proposition Let B be a G-cograded multiplier Hopf (*-)algebra in the sense of Definition 2.1. So B has the form $B = \bigoplus_{p \in G} B_p$. Then we have

(i) $\varepsilon(a) = 0$ whenever $a \in B_p$ and $p \neq e$ (where e is the identity in G),

(ii) $S(B_p) \subseteq M(B_{p^{-1}})$ for all p.

Now that we have discussed the notion of a group-cograded multiplier Hopf algebra in general, we are ready to study a special type of actions on these objects.

Admissible actions on group-cograded multiplier Hopf algebras

Here is the main definition.
2.6 Definition Let B be a G-cograded multiplier Hopf (*-)algebra. So B has the form $B = \bigoplus_{p \in G} B_p$. Let $Aut(B)$ denote the group of algebra automorphisms on B. By an *action* of the group G on B, we mean a group homomorphism $\pi : G \to Aut(B)$. If B is a multiplier Hopf *-algebra we assume that π_p is a *-automorphism for all $p \in G$. Further, we require that for all $p \in G$

(1) π_p respects the comultiplication on B in the sense that $\Delta(\pi_p(b)) = (\pi_p \otimes \pi_p)(\Delta(b))$ for all $b \in B$.

We call this action *admissible* if there is an action ρ of G on itself so that

(2) $\pi_p(B_q) = B_{\rho_p(q)}$,

(3) $\pi_{\rho_p(q)} = \pi_{pqp^{-1}}$.

for all p, q in G.

The action ρ of G on itself determines an action $\tilde{\rho}$ of G on $K(G)$ by the formula $(\tilde{\rho}_p(f))(q) = f(\rho_{p^{-1}}(q))$ when $f \in K(G)$ and $p, q \in G$. This is an action of G on the multiplier Hopf algebra $K(G)$ (in the sense of the above definition). Condition (2) says that $I \circ \tilde{\rho}_p = \pi_p \circ I$ for all $p \in G$ where I is the canonical imbedding of $K(G)$ in $M(B)$. So, for an action to be admissible, we first of all need that, on the level of $K(G)$, it comes from an action of G on itself.
If this action is the adjoint action, that is, if $\rho_p(q) = pqp^{-1}$ for all p, q, then condition (3) is automatically fulfilled. If this is not the case, then we want π itself to take care of ρ not being the adjoint action. The other extreme therefore is obtained when π is simply the trivial action. Then we can take for ρ also the trivial action in order to satisfy (1) while (2) is again automatically satisfied, now however for a completely different reason. Condition (2) seems to be quite natural. We will indicate further why we need condition (3). In any case, as we saw above, we have the following example.

2.7 Example Let B be a G-cograded multiplier Hopf algebra with canonical decomposition $B = \sum_{p \in G} \oplus B_p$. Let π be an action of G on B. If $\pi_p(B_q) = B_{pqp^{-1}}$ for all $p, q \in G$, then we have an admissible action.
Also the trivial action is admissible and it is not hard to construct examples combining these two extreme cases.

Deformation of a group-cograded mutliplier Hopf algebra

Let G be a group. Let B be a G-cograded regular multiplier Hopf algebra and let π be an admissible action of G on B. We will construct a new regular multiplier Hopf algebra on B by deforming the comultiplication, while the algebra structure on B is kept. The deformation of the comultiplication of B, as defined in the following definition, depends on the action π.

2.8 Definition Take B and π as above. For $b \in B$, we define the multiplier $\widetilde{\Delta}(b)$ in $M(B \otimes B)$ by the following formulas. Take $b' \in B_q$, then we define

$$\begin{aligned} \widetilde{\Delta}(b)(1 \otimes b') &= (\pi_{q^{-1}} \otimes \iota)(\Delta(b)(1 \otimes b')) \\ (1 \otimes b')\widetilde{\Delta}(b) &= (\pi_{q^{-1}} \otimes \iota)((1 \otimes b')\Delta(b)) \end{aligned}$$

in $B \otimes B$. By the associativity of the product in $B \otimes B$, we have that $\widetilde{\Delta}(b)$ is a multiplier in $M(B \otimes B)$.

2.9 Proposition Take the notations as in Definition 2.8. For all $b \in B_p$ and $b' \in B_q$, we have

$$\begin{aligned} (b' \otimes 1)\widetilde{\Delta}(b) &= \sum b'\pi_{qp^{-1}}(b_{(1)}) \otimes b_{(2)} \\ \widetilde{\Delta}(b)(b' \otimes 1) &= \sum \pi_{qp^{-1}}(b_{(1)})b' \otimes b_{(2)} \end{aligned}$$

in $B \otimes B$. The map $\widetilde{\Delta} : B \to M(B \otimes B)$ is a non-degenerate homomorphism.

Proof. To prove the first formula, take $r \in G$ and $b'' \in B_r$. Then we have

$$((b' \otimes 1)\widetilde{\Delta}(b))(1 \otimes b'') = (b' \otimes 1)(\widetilde{\Delta}(b)(1 \otimes b'')) = \sum b'\pi_{r^{-1}}(b_{(1)}) \otimes b_{(2)}b''.$$

As $b' \in B_q$, we must have $\pi_{r^{-1}}(b_{(1)}) \in B_q$ and so $b_{(1)} \in \pi_r(B_q) = B_{\rho_r(q)}$ (using condition (2) in Definition 2.6). As $b'' \in B_r$ we must have $b_{(2)} \in B_r$. Finally, because $b \in B_p$ we need $p = \rho_r(q)r$. It follows that $\pi_{pr^{-1}} = \pi_{\rho_r(q)} = \pi_{rqr^{-1}}$ (using condition (3) in Definition 2.6). Therefore, $\pi_p = \pi_{rq}$ and $\pi_{r^{-1}} = \pi_{qp^{-1}}$. This proves the first formula. Similarly, the second one can be proven.

Further, $\widetilde{\Delta} : B \to M(B \otimes B)$ is a homomorphism because $\Delta : B \to M(B \otimes B)$ is a homomorphism and $(\pi_p \otimes \iota_B)$ is a non-degenerate homomorphism on $B \otimes B$ for all $p \in G$. For all $p, q \in G$ we have that $B_p \otimes B_q = \widetilde{\Delta}(B_{\rho_q(p)q})(1 \otimes B_q)$ and $B_p \otimes B_q = (1 \otimes B_q)\widetilde{\Delta}(B_{\rho_q(p)q})$. Therefore, $\widetilde{\Delta} : B \to M(B \otimes B)$ is a non-degenerate homomorphism in the sense that $\widetilde{\Delta}(B)(B \otimes B) = B \otimes B = (B \otimes B)\widetilde{\Delta}(B)$. This completes the proof. ■

2.10 Lemma Take the notations as above. Then $\widetilde{\Delta}$ is coassociative.

Proof. As $\widetilde{\Delta}$ can be extended to $M(B)$ in a natural way, to show that $\widetilde{\Delta}$ is coassociative we need $(\widetilde{\Delta} \otimes \iota)(\widetilde{\Delta}(x)) \overset{(*)}{=} (\iota \otimes \widetilde{\Delta})(\widetilde{\Delta}(x))$ in $M(B \otimes B \otimes B)$ for all $x \in B$.

Let 1_p and 1_q denote the units in $M(B_p)$ and $M(B_q)$ respectively. Then the equation $(*)$ will be satisfied if for all $p, q \in G$ we have

$$((\widetilde{\Delta} \otimes \iota)(\widetilde{\Delta}(x)))(1 \otimes 1_p \otimes 1_q) = ((\iota \otimes \widetilde{\Delta})(\widetilde{\Delta}(x)))(1 \otimes 1_p \otimes 1_q).$$

For the left hand side of the above equation we set

$$\begin{aligned}
((\widetilde{\Delta} \otimes \iota)(\widetilde{\Delta}(x)))(1 \otimes 1_p \otimes 1_q) &= ((\widetilde{\Delta} \otimes \iota)(\widetilde{\Delta}(x)(1 \otimes 1_q)))(1 \otimes 1_p \otimes 1_q) \\
&= ((\widetilde{\Delta} \otimes \iota)(\pi_{q^{-1}} \otimes \iota)(\Delta(x)))(1 \otimes 1_p \otimes 1_q) \\
&= ((\pi_{p^{-1}} \otimes \iota \otimes \iota)((\Delta \otimes \iota)(\pi_{q^{-1}} \otimes \iota)(\Delta(x))))(1 \otimes 1_p \otimes 1_q) \\
&= ((\pi_{p^{-1}}\pi_{q^{-1}} \otimes \pi_{q^{-1}} \otimes \iota)((\Delta \otimes \iota)(\Delta(x))))(1 \otimes 1_p \otimes 1_q).
\end{aligned}$$

Observe that $1_p \otimes 1_q = \widetilde{\Delta}(1_{\rho_q(p)q})(1_p \otimes 1_q)$ because $\pi_{q^{-1}}(B_{\rho_q(p)}) = B_p$. For the right hand side of the above equation we set

$$\begin{aligned}
((\iota \otimes \widetilde{\Delta})(\widetilde{\Delta}(x)))(1 \otimes 1_p \otimes 1_q) &= ((\iota \otimes \widetilde{\Delta})(\widetilde{\Delta}(x)))(1 \otimes \widetilde{\Delta}(1_{\rho_q(p)q}))(1 \otimes 1_p \otimes 1_q) \\
&= ((\iota \otimes \widetilde{\Delta})(\widetilde{\Delta}(x)(1 \otimes 1_{\rho_q(p)q})))(1 \otimes 1_p \otimes 1_q) \\
&= ((\iota \otimes \widetilde{\Delta})((\pi_{q^{-1}\rho_q(p^{-1})} \otimes \iota)(\Delta(x))))(1 \otimes 1_p \otimes 1_q).
\end{aligned}$$

As π is an admissible action of G on B, we have $\pi_{q^{-1}\rho_q(p^{-1})} = \pi_{p^{-1}q^{-1}}$. Therefore, we obtain that the right hand side equals

$$\begin{aligned}
&((\pi_{p^{-1}}\pi_{q^{-1}} \otimes \iota \otimes \iota)((\iota \otimes \widetilde{\Delta})(\Delta(x))))(1 \otimes 1_p \otimes 1_q) \\
&\qquad = ((\pi_{p^{-1}}\pi_{q^{-1}} \otimes \iota \otimes \iota)((\iota \otimes \pi_{q^{-1}} \otimes \iota)((\iota \otimes \Delta)(\Delta(x)))))(1 \otimes 1_p \otimes 1_q) \\
&\qquad = ((\pi_{p^{-1}}\pi_{q^{-1}} \otimes \pi_{q^{-1}} \otimes \iota)((\iota \otimes \Delta)(\Delta(x))))(1 \otimes 1_p \otimes 1_q).
\end{aligned}$$

We see that both expressions are the same. ■

We now prove that the comultiplication $\widetilde{\Delta}$ makes B into a regular multiplier Hopf algebra. We also calculate the counit and the antipode for this new multiplier Hopf algebra.

2.11 Theorem Let B be a regular G-cograded multiplier Hopf algebra. So, as an algebra, B has the form $B = \bigoplus_{p \in G} B_p$. Assume that π is an admissible action of G on B. Let $\widetilde{\Delta}$ denote the comultiplication as defined in Definition 2.8. Then we have the following.

(1) $(B, \widetilde{\Delta})$ is a regular multiplier Hopf algebra. The counit $\widetilde{\varepsilon}$ is the original counit ε. The antipode $\widetilde{S}$ is given by the formula $\widetilde{S}(b) = \pi_{p^{-1}}(S(b))$ for $b \in B_p$.

(2) If B is a G-cograded multiplier Hopf *-algebra, then $(B, \widetilde{\Delta})$ is again a multiplier Hopf *-algebra.

(3) If φ is a left integral on B, then φ is also a left integral on $(B, \widetilde{\Delta})$. However, if ψ is a right integral, it is in general not right invariant on $(B, \widetilde{\Delta})$. It has to be modified. For $b \in B_p$, define $\widetilde{\psi}(b) = \psi_B(\pi_{p^{-1}}(b))$. Then $\widetilde{\psi}$ is a right integral. In the *-case, we have that a positive left integral on B is again a positive left integral on $(B, \widetilde{\Delta})$. A positive right integral ψ on B gives rise to a positive right integral $\widetilde{\psi}$ on $(B, \widetilde{\Delta})$.

Proof.

(1) We will make use of [VD2, Proposition 2.9] to prove that $(B, \widetilde{\Delta})$ is a regular multiplier Hopf algebra. Recall that, as an algebra, $B = \bigoplus_{p \in G} B_p$. The comultiplication

$\widetilde{\Delta} : B \to M(B \otimes B)$ is defined as in Definition 2.8 and we will also use the formulas of Proposition 2.9.

Moreover, we will use that π_p is an isomorphism of B for all $p \in G$ which respects the comultiplication in the sense that $\Delta(\pi_p(b)) = (\pi_p \otimes \pi_p)(\Delta(b))$ for all $b \in B$.

We first consider the counit. We will show that the original counit ε on B is also the counit for $\widetilde{B}$. Take $b \in B_p$ and $b' \in B_q$. Then we have

$$\begin{aligned}
&(\varepsilon \otimes \iota)(\widetilde{\Delta}(b)(1 \otimes b')) = \sum \varepsilon(\pi_{q^{-1}}(b_{(1)}))b_{(2)}b' = \sum \varepsilon(b_{(1)})b_{(2)}b' = bb' \\
&(\iota \otimes \varepsilon)((b' \otimes 1)\widetilde{\Delta}(b)) = (\iota \otimes \varepsilon)((b' \otimes 1)\widetilde{\Delta}(b)(1 \otimes 1_e)) = \\
&\qquad (\iota \otimes \varepsilon)((b' \otimes 1)(\pi_e \otimes \iota)(\Delta(b)(1 \otimes 1_e))) = (\iota \otimes \varepsilon)((b' \otimes 1)\Delta(b)) = b'b.
\end{aligned}$$

Recall that e denotes the identity in G.

Next, we prove the existence of the antipode. Define $\widetilde{S}$ by the formula $\widetilde{S}(b) = \pi_{p^{-1}}(S(b))$ for all $b \in B_p$. Let m denote the multiplication in the algebra B. Take $b \in B_p$ and $b' \in B_q$. Then we have

$$m((\widetilde{S} \otimes \iota)(\widetilde{\Delta}(b)(1 \otimes b'))) = m(\widetilde{S} \otimes \iota)(\sum \pi_{q^{-1}}(b_{(1)}) \otimes b_{(2)}b').$$

As $b_{(2)} \in B_q$, we calculate that $\pi_{q^{-1}}(b_{(1)}) \in B_{\rho_{q^{-1}}(pq^{-1})}$. Notice that $\pi_{\rho_{q^{-1}}(qp^{-1})} = \pi_{q^{-1}}\pi_{q\rho_{q^{-1}}(qp^{-1})} = \pi_{p^{-1}q}$.

Therefore, we get

$$m((\widetilde{S} \otimes \iota)(\widetilde{\Delta}(b)(1 \otimes b'))) = \sum \pi_{p^{-1}}(S(b_{(1)}))b_{(2)}b'.$$

If $\rho_{p^{-1}}(qp^{-1}) \neq q$ (and hence $p \neq e$), then the last expression equals zero. Remark that also $\widetilde{\varepsilon}(b)b' = 0$. If $\rho_{p^{-1}}(qp^{-1}) = q$, then $\pi_q = \pi_{\rho_{p^{-1}}(qp^{-1})} = \pi_{p^{-1}q}$. Therefore we have that in this case $\pi_{p^{-1}} = \pi_e$ and the expression in the right hand side becomes $\sum S(b_{(1)})b_{(2)}b' = \varepsilon(b)b' = \widetilde{\varepsilon}(b)b'$.

We now prove the second equation for $\widetilde{S}$. Take $b \in B_p$ and $b' \in B_q$, then we have

$$m((\iota \otimes \widetilde{S})((b' \otimes 1)\widetilde{\Delta}(b))) = m((\iota \otimes \widetilde{S})(\sum b'\pi_{qp^{-1}}(b_{(1)}) \otimes b_{(2)})).$$

In this formula we calculate that $b_{(2)} \in B_{\rho_{pq^{-1}}(q^{-1})p}$. Notice that $\pi_{p^{-1}\rho_{pq^{-1}}(q)} = \pi_{qp^{-1}}$. Therefore, we obtain

$$m((\iota \otimes \widetilde{S})((b' \otimes 1)\widetilde{\Delta}(b))) = \sum b'\pi_{qp^{-1}}(b_{(1)}S(b_{(2)})) = \widetilde{\varepsilon}(b)b'.$$

By using [VD2, Proposition 2.9], we conclude that $(B, \widetilde{\Delta})$ is a regular multiplier Hopf algebra.

(2) Now assume that B is a G-cograded multiplier Hopf *-algebra and for all $p \in G$, π_p is furthermore a *-isomorphism on B. We prove that $(B, \widetilde{\Delta})$ is also a multiplier Hopf *-algebra. Therefore, we have to show that $\widetilde{\Delta}$ is a *-homomorphism. Take $b \in B_p$ and $b' \in B_q$, then we have

$$\begin{aligned}
\widetilde{\Delta}(b^*)(1 \otimes b') &= (\pi_{q^{-1}} \otimes \iota)(\Delta(b^*)(1 \otimes b')) = (\pi_{q^{-1}} \otimes \iota)(\Delta(b)^*(1 \otimes b')) \\
&= (\pi_{q^{-1}} \otimes \iota)(((1 \otimes b'^*)\Delta(b))^*) = ((\pi_{q^{-1}} \otimes \iota)((1 \otimes b'^*)\Delta(b)))^* \\
&= ((1 \otimes b'^*)\widetilde{\Delta}(b))^* = \widetilde{\Delta}(b)^*(1 \otimes b').
\end{aligned}$$

(3) Let φ be a left integral on B, as reviewed in Section 1. We prove that φ is also a left integral on $(B, \widetilde{\Delta})$. Take $b \in B_p$ and $b' \in B_q$. Then, we have $((\iota \otimes \varphi)\widetilde{\Delta}(b))b' = (\iota \otimes \varphi)(\widetilde{\Delta}(b)(b' \otimes 1)) = \sum \pi_{qp^{-1}}(b_{(1)})b'\varphi(b_{(2)}) = \varphi(b)b'$. Therefore, $(\iota \otimes \varphi)\widetilde{\Delta}(b) = \varphi(b)1$ in $M(B)$ for all $b \in B$.

Let ψ be a right integral on B. Define $\widetilde{\psi}$ on B by the formula $\widetilde{\psi}(b) = \psi_B(\pi_{p^{-1}}(b))$ when $b \in B_p$. We will now show that $\widetilde{\psi}$ is a right integral on $(B, \widetilde{\Delta})$. Take $b \in B_p$ and $b' \in B_q$, then we have

$$((\widetilde{\psi} \otimes \iota)\widetilde{\Delta}(b))b' = (\widetilde{\psi} \otimes \iota)(\widetilde{\Delta}(b)(1 \otimes b')) = (\widetilde{\psi} \otimes \iota)(\sum \pi_{q^{-1}}(b_{(1)}) \otimes b_{(2)}b').$$

As $\pi_{q^{-1}}(b_{(1)}) \in B_{\rho_{q^{-1}}(pq^{-1})}$, the last expression is given as

$$\begin{aligned}
\sum \psi(\pi_{\rho_{q^{-1}}(qp^{-1})}\pi_{q^{-1}}(b_{(1)}))b_{(2)}b' &= \sum \psi(\pi_{p^{-1}}(b_{(1)}))b_{(2)}b' \\
&= \pi_p(\sum \psi(\pi_{p^{-1}}(b_{(1)}))\pi_{p^{-1}}(b_{(2)})\pi_{p^{-1}}(b')) \\
&= \pi_p(\psi(\pi_{p^{-1}}(b))\pi_{p^{-1}}(b')) = \widetilde{\psi}(b)b'.
\end{aligned}$$

If B is a G-cograded multiplier Hopf *-algebra as in (2), then a positive left integral φ on B stays a positive left integral on $(B, \widetilde{\Delta})$ because the *-algebra structure of B and $(B, \widetilde{\Delta})$ is the same. If ψ is a positive right integral on B, then it is easily shown that $\widetilde{\psi}$ is a positive right integral on $(B, \widetilde{\Delta})$. $\blacksquare$

2.12 Notation We will use $\widetilde{B}$ for the deformed multiplier Hopf algebra $(B, \widetilde{\Delta})$. Of course, if π is the trivial action of G on B, we have that $(B, \widetilde{\Delta})$ equals B.

In Proposition 3.12 we will refine the structure of $\widetilde{B}$ in the case where π is a admissible action such that $\pi_p(B_q) = B_{pqp^{-1}}$ for all $p, q \in G$. This case is like the *mirror* construction for a Hopf group-coalgebra as introduced in [T, Section 11].

3 Pairing and Drinfel'd double construction with G-cograded multiplier Hopf (*-)algebras

In this section, we will apply the 'twisted tensor product' construction of multiplier Hopf (*-)algebras as we have explained in Section 1.

Let G be a group and let B be a regular G-cograded multiplier Hopf algebra in the sense of Definition 2.1. We suppose that π is an admissible action of G on B in the sense of Definition 2.6. In Proposition 3.1 below we study a multiplier Hopf algebra pairing $\langle A, B\rangle$ when B is G-cograded. Further in this section, we define Drinfel'd double constructions $D^\pi = A^{cop} \bowtie \widetilde{B}$ where the product as well as the coproduct are depending on the action π.

3.1 Proposition Let $\langle A, B\rangle$ be a pairing of two regular multiplier Hopf algebras. Suppose that B is G-cograded. Then there exist subspaces $\{A_p\}_{p\in G}$ of A such that

(1) $A = \bigoplus_{p\in G} A_p$ and $A_pA_q \subseteq A_{pq}$,

(2) $\langle A_p, B_q\rangle = 0$ whenever $p \neq q$,

(3) $\langle \Delta(A_p), B_q \otimes B_r \rangle = 0$ if $q \neq p$ or $r \neq p$,

where $p, q, r \in G$.

Proof. As B is a G-cograded multiplier Hopf algebra, B has the form $B = \bigoplus_{p \in G} B_p$. Recall from Section 1, that there are four module algebra structures associated to the pairing $\langle A, B \rangle$, denoted as $A \blacktriangleright B$, $B \blacktriangleright A$, $A \blacktriangleleft B$, $B \blacktriangleleft A$. We have seen that these actions are unital and therefore extend to the multiplier algebras. One can show that $A \blacktriangleleft B_p = B_p \blacktriangleright A = A \blacktriangleleft 1_p = 1_p \blacktriangleright A$ for all $p \in G$ where 1_p denotes the unit in $M(B_p)$. Then we define a subspace A_p in A by $A_p = 1_p \blacktriangleright A$ for any $p \in G$. We now prove that these subspaces satisfy the 3 requirements.

(1) Take $a \in A$. Then there exists an element $b \in B$ so that $a = b \blacktriangleright a$. As $B = \bigoplus_{p \in G} B_p$, it easily follows that $A = \bigoplus_{p \in G} A_p$. Furthermore $A_p A_q = (1_p \blacktriangleright A_p)(1_q \blacktriangleright A_q) = 1_{pq} \blacktriangleright (A_p A_q)$. Therefore, $A_p A_q \subseteq A_{pq}$.

(2) This second property follows from the definition of A_p and the multiplication structure of B.

(3) Take $a \in A_p$, $b \in B_q$ and $b' \in B_r$. Then we have that $\langle \Delta(a), b \otimes b' \rangle = \langle a, bb' \rangle$. Now the result follows from the algebra structure on B and the fact that A_p is also given by $A_p = B_p \blacktriangleright A$. ■

3.2 Remark Take a pairing $\langle A, B \rangle$ as in Proposition 3.1.

(1) It easily follows from the definition that $b \blacktriangleright a = 0$ if $a \in A_q$, $b \in B_p$ and $p \neq q$. Indeed, when $a \in A_q$ and $b \in B$, we have $b \blacktriangleright a = (b1_q) \blacktriangleright a$ and $b1_q = 0$ if $b \in B_p$ and $p \neq q$. Furthermore, if $a \in A_q$ then $b \blacktriangleright a \in A_q$ for all $b \in B$. Similar results hold for the module $A \blacktriangleleft B$.

(2) The antipode S of A maps A_p to $A_{p^{-1}}$.

Again take a pairing $\langle A, B\rangle$ as in Proposition 3.1 and now let π be an admissible action of G on B. We will construct a twisted tensor product multiplier Hopf algebra, as reviewed in Section 1. The twist map, defining the non-trivial product structure on $A \otimes B$, will depend on the pairing as well as on the action π. The comultiplication, which is compatible with this product on $A \otimes B$, will also depend on the action π.

We first prove the following lemma.

3.3 Lemma Take the notations and the assumptions as above. Define linear maps R_1 and R_2 on $A \otimes B$ by the formulas

$$\begin{aligned} R_1(a \otimes b) &= \sum(\pi_{qp^{-1}}(b_{(1)}) \blacktriangleright a) \otimes b_{(2)} \\ R_2(a \otimes b) &= \sum(a \blacktriangleleft b_{(2)}) \otimes b_{(1)} \end{aligned}$$

when $a \in A_q$ and $b \in B_p$. Then R_1 and R_2 are bijections on $A \otimes B$ and the inverses are given by

$$\begin{aligned} R_1^{-1}(a \otimes b) &= \sum(\pi_{p^{-1}}(S^{-1}(b_{(1)})) \blacktriangleright a) \otimes b_{(2)} \\ R_2^{-1}(a \otimes b) &= \sum(a \blacktriangleleft S^{-1}(b_{(2)})) \otimes b_{(1)} \end{aligned}$$

when $b \in B_p$.

Proof. We remark that in all the formulas above, the decompositions are well-covered because the modules $B \blacktriangleright A$ and $A \blacktriangleleft B$ are unital.

The proof for the map R_2 is easy and the result is known. Here, we do not really need these restrictions on a and b.

We give the proof for the map R_1. Take $a \in A_q$ and $b \in B_p$. First remark that, when looking closer at the definition of R_1^{-1}, we see that $\pi_{p^{-1}}(S^{-1}(b_{(1)}))$ is forced to lie in B_q as it acts on the element a in A_q; see Remark 3.2 (1). Then $b_{(1)}$ must be in $B_{\rho_p(q^{-1})}$. Because we assume that $b \in B_p$ it follows that that $b_{(2)}$ is forced to lie in $B_{\rho_p(q)p}$. This is used when we apply the map R_1 in the following calculation. We have

$$\begin{aligned}
(R_1 \circ R_1^{-1})(a \otimes b) &= R_1(\sum(\pi_{p^{-1}}(S^{-1}(b_{(1)})) \blacktriangleright a) \otimes b_{(2)}) \\
&= \sum((\pi_{qp^{-1}\rho_p(q^{-1})}(b_{(2)})\pi_{p^{-1}}(S^{-1}(b_{(1)})) \blacktriangleright a) \otimes b_{(3)} \\
&= \sum(\pi_{p^{-1}}(b_{(2)}S^{-1}(b_{(1)})) \blacktriangleright a) \otimes b_{(3)} = a \otimes b.
\end{aligned}$$

Remark that we have used that $\pi_{qp^{-1}\rho_p(q^{-1})} = \pi_{qp^{-1}pq^{-1}p^{-1}} = \pi_{p^{-1}}$ in the above calculation. A similar argument will give

$$\begin{aligned}
(R_1^{-1} \circ R_1)(a \otimes b) &= R_1^{-1}(\sum(\pi_{qp^{-1}}(b_{(1)}) \blacktriangleright a) \otimes b_{(2)}) \\
&= \sum((\pi_{p^{-1}\rho_{pq^{-1}}(q)}(S^{-1}(b_{(2)}))\pi_{qp^{-1}}(b_{(1)})) \blacktriangleright a) \otimes b_{(3)} \\
&= \sum(\pi_{qp^{-1}}(S^{-1}(b_{(2)})b_{(1)}) \blacktriangleright a) \otimes b_{(3)} = a \otimes b. \qquad \blacksquare
\end{aligned}$$

We now define the twist map $R : B \otimes A \to A \otimes B$.

3.4 Definition Take a pairing $\langle A, B \rangle$ as in Proposition 3.1. Let π be an admissible action of G on B. We define the twist map $R : B \otimes A \to A \otimes B$ by the composition $R = R_1 \circ R_2^{-1} \circ \sigma$, where σ is the flip map from $B \otimes A$ to $A \otimes B$. So, for all $a \in A_q$ and $b \in B_p$ we have, using arguments as in the proof of the previous lemma,

$$R(b \otimes a) = \sum(\pi_{p^{-1}}(b_{(1)}) \blacktriangleright a \blacktriangleleft S^{-1}(b_{(3)})) \otimes b_{(2)}.$$

As a composition of bijections, R is a bijection.

We see from this formula, that it will be true for all $a \in A$ when $b \in B_p$. When $a \in A_q$ and $b \in B_p$ then $b_{(2)}$ is forced in $B_{\rho_p(q^{-1})pq}$ in the above formula. This will be used, in particular, in the proof of the following lemma where we obtain that R behaves well with respect to the multiplications of A and B.

3.5 Lemma Take the notations and assumptions as before. The twist map R satisfies the following equations

(1) $R(m_B \otimes \iota_A) = (\iota_A \otimes m_B)(R \otimes \iota_B)(\iota_B \otimes R)$ on $B \otimes B \otimes A$,

(2) $R(\iota_B \otimes m_A) = (m_A \otimes \iota_B)(\iota_A \otimes R)(R \otimes \iota_A)$ on $B \otimes A \otimes A$,

where, as before, m_A and m_B are the multiplications in A and B and where ι_A and ι_B are the identity maps on A and B respectively.

Proof. (1) Take $b \in B_p$, $b' \in B_q$ and $a \in A_r$. Let the right hand side of the first equation act on $b \otimes b' \otimes a$. We get

$$\begin{aligned}((\iota_A \otimes m_B)&(R \otimes \iota_B)(\iota_B \otimes R))(b \otimes b' \otimes a)\\ &= \sum(\pi_{p^{-1}}(b_{(1)})\pi_{q^{-1}}(b'_{(1)}) \blacktriangleright a \blacktriangleleft S^{-1}(b'_{(3)})S^{-1}(b_{(3)})) \otimes b_{(2)}b'_{(2)}.\end{aligned}$$

By the remark preceding this lemma, we find that $b_{(2)} \in B_{\rho_p(r^{-1})pr}$ and $b'_{(2)} \in B_{\rho_q(r^{-1})qr}$. Therefore, $\rho_p(r^{-1})pr = \rho_q(r^{-1})qr$ and we obtain that $\pi_p = \pi_q$. The above equation can now be written as

$$\begin{aligned}((\iota_A \otimes m_B)&(R \otimes \iota_B)(\iota_B \otimes R))(b \otimes b' \otimes a)\\ &= \sum(\pi_{p^{-1}}(b_{(1)}b'_{(1)}) \blacktriangleright a \blacktriangleleft S^{-1}(b_{(3)}b'_{(3)})) \otimes b_{(2)}b'_{(2)}.\end{aligned}$$

- If $p \neq q$, this expression equals zero because $bb' = 0$. Clearly the operator $R(m_B \otimes \iota_A)$ applied on $(b \otimes b' \otimes a)$ also equals zero in this case.

- If $p = q$, then $b, b' \in B_p$ and also $bb' \in B_p$. The above equation now becomes

$$((\iota_A \otimes m_B)(R \otimes \iota_B)(\iota_B \otimes R))(b \otimes b' \otimes a) = (R(m_B \otimes \iota_A))(b \otimes b' \otimes a).$$

This completes the proof of the first statement.

(2) To prove the second statement, take $b \in B_p$, $a \in A_q$ and $a' \in A_r$. Let the right hand side of the second equation act on $b \otimes a \otimes a'$. We get

$$\begin{aligned}((m_A \otimes \iota_B)&(\iota_A \otimes R)(R \otimes \iota_A))(b \otimes a \otimes a')\\ &= (m_A \otimes \iota_B)(\iota_A \otimes R)(\sum(\pi_{p^{-1}}(b_{(1)}) \blacktriangleright a \blacktriangleleft S^{-1}(b_{(3)})) \otimes b_{(2)} \otimes a').\end{aligned}$$

Recall that $b_{(2)} \in B_{\rho_p(q^{-1})pq}$. We also have $\pi_{q^{-1}p^{-1}\rho_p(q)} = \pi_{p^{-1}}$. Therefore, the above expression can be written as

$$\begin{aligned}(m_A \otimes \iota_B)&(\sum(\pi_{p^{-1}}(b_{(1)}) \blacktriangleright a \blacktriangleleft S^{-1}(b_{(5)})) \otimes (\pi_{p^{-1}}(b_{(2)}) \blacktriangleright a' \blacktriangleleft S^{-1}(b_{(4)})) \otimes b_{(3)})\\ &= \sum(\pi_{p^{-1}}(b_{(1)}) \blacktriangleright a \blacktriangleleft S^{-1}(b_{(5)})(\pi_{p^{-1}}(b_{(2)}) \blacktriangleright a' \blacktriangleleft S^{-1}(b_{(4)})) \otimes b_{(3)}.\end{aligned}$$

As $B \blacktriangleright A$ and $A \blacktriangleleft B$ are module algebras, this expression equals

$$\sum(\pi_{p^{-1}}(b_{(1)}) \blacktriangleright (aa') \blacktriangleleft S^{-1}(b_{(3)})) \otimes b_{(2)} = R(b \otimes aa') = (R(\iota_B \otimes m_A))(b \otimes a \otimes a'). \qquad \blacksquare$$

As reviewed in Section 1 (Twisted tensor product construction of multiplier Hopf (*-)algebras), the map R defines a non-trivial product on $A \otimes B$ which is non-degenerate. The algebra defined in this way is denoted as $A \bowtie B$. Recall that the product in $A \bowtie B$ is given by the formula

$$(a \bowtie b)(a' \bowtie b') = (m_A \otimes m_B)(\iota_A \otimes R \otimes \iota_B)(a \otimes b \otimes a' \otimes b')$$

for all $a, a' \in A$ and $b, b' \in B$. In Section 1, Remarks 1.11, other expressions are given for the right hand side.

We now consider these algebras with their comultiplications. Let $A^{cop} = (A, \Delta^{cop})$ and $\widetilde{B} = (B, \widetilde{\Delta})$. In Definition 1.12, we saw that $\Delta^{cop}(a)\widetilde{\Delta}(b)$ is a multiplier in $M((A \bowtie B) \otimes (A \bowtie B))$ for all $a \in A$ and $b \in B$ and that we get a comultiplication. So, the following definition is possible here.

3.6 Definition Take the notations and assumptions as above. For $a \in A$ and $b \in B$, we define the multiplier $\overline{\Delta}(a \bowtie b)$ in $M((A \bowtie B) \otimes (A \bowtie B))$ by the formula

$$\overline{\Delta}(a \bowtie b) = \Delta^{cop}(a)\widetilde{\Delta}(b).$$

As we reviewed in Theorem 1.13, for $\overline{\Delta}$ to be a homomorphism on $A \bowtie B$, we need to prove the following compatibility relation between R and $\overline{\Delta}$.

3.7 Proposition Take the notations and assumptions as above. Then, we have

$$\overline{\Delta}(R(b \otimes a)) = \widetilde{\Delta}(b)\Delta^{cop}(a)$$

in $M((A \bowtie B) \otimes (A \bowtie B))$ for all $a \in A$ and $b \in B$.

Proof. Take $a \in A_q$, $a' \in A_{q'}$, and $b \in B_p$, $b' \in B_{p'}$. Then the product in the twisted tensor product algebra $A \bowtie B$ is given by the formula

$$(a \bowtie b)(a' \bowtie b') = \sum \langle a'_{(1)}, S^{-1}(b_{(3)}) \rangle \langle a'_{(3)}, \pi_{p^{-1}}(b_{(1)}) \rangle a a'_{(2)} \bowtie b_{(2)} b'.$$

Observe that in the right hand side, all the decompositions are covered. Recall also that $\overline{\Delta}(a \bowtie b) = \Delta^{cop}(a) \widetilde{\Delta}(b)$.

Take $a \in A_q$, $a' \in A_{q'}$, $a'' \in A_{q''}$, $b \in B_p$, $b' \in B_{p'}$, $b'' \in B_r$ and $y \in B_s$. Then we calculate in $(A \bowtie B) \otimes (A \bowtie B)$ that

$$\begin{aligned} &((a' \bowtie 1) \otimes (a'' \bowtie y))(\widetilde{\Delta}(b)\Delta^{cop}(a))((1 \bowtie b') \otimes (1 \bowtie b'')) \\ &\qquad = \sum ((a' \bowtie \pi_{s^{-1}}(b_{(1)}) \otimes (a'' \bowtie y b_{(2)}))((a_{(2)} \bowtie b') \otimes (a_{(1)} \bowtie b'')) \\ &\qquad = \sum ((a' \bowtie \pi_{s^{-1}}(b_{(1)}))(a_{(2)} \bowtie b')) \otimes ((a'' \bowtie y b_{(2)})(a_{(1)} \bowtie b'')) \end{aligned}$$

Now, observe that $\pi_{s^{-1}}(b_{(1)}) \in B_{\rho_{s^{-1}}(ps^{-1})}$ and $\pi_{\rho_{s^{-1}}(sp^{-1})} = \pi_{p^{-1}s}$. Then we apply the commutation rules to commute the the elements $\pi_{s^{-1}}(b_{(1)})$ and $a_{(2)}$ in the first factor of the tensor product and the elements $b_{(2)}$ and $a_{(1)}$ in the second factor. Using the property of the antipode, we finally obtain that the above expression equals

$$\sum \langle a_{(4)}, \pi_{p^{-1}}(b_{(1)}) \rangle \langle S^{-1}(a_{(1)}), b_{(4)} \rangle (a' a_{(3)} \bowtie \pi_{s^{-1}}(b_{(2)}) b') \otimes ((a'' \bowtie y)(a_{(2)} \bowtie b_{(3)} b'')).$$

In the second factor of this tensor product, we deal with a product in $A \bowtie B$. This product equals zero if $r \neq \rho_s(q^{-1})qs$. If $r = \rho_s(q^{-1})sq$, then $\pi_{r^{-1}} = \pi_{s^{-1}}$.

On the other hand, we also calculate in $(A \bowtie B) \otimes (A \bowtie B)$ that

$$\begin{aligned} &((a' \bowtie 1) \otimes (a'' \bowtie y))\overline{\Delta}(R(b \otimes a))((1 \bowtie b') \otimes (1 \bowtie b'')) \\ &\quad = ((a' \bowtie 1) \otimes (a'' \bowtie y))\overline{\Delta}(\sum (\pi_{p^{-1}}(b_{(1)}) \blacktriangleright a \blacktriangleleft S^{-1}(b_{(3)})) \bowtie b_{(2)})((1 \bowtie b') \otimes (1 \bowtie b'')) \\ &\quad = \sum ((a' \bowtie 1) \otimes (a'' \bowtie y))\Delta_A^{cop}(\pi_{p^{-1}}(b_{(1)}) \blacktriangleright a \blacktriangleleft S^{-1}(b_{(3)}))\widetilde{\Delta}(b_{(2)})((1 \bowtie b') \otimes (1 \bowtie b'')) \\ &\quad = \sum (a'(\pi_{p^{-1}}(b_{(1)}) \blacktriangleright a_{(2)}) \bowtie \pi_{r^{-1}}(b_{(2)}) b') \otimes ((a'' \bowtie y)((a_{(1)} \blacktriangleleft S^{-1}(b_{(4)})) \bowtie b_{(3)} b'') \\ &\quad = \sum \langle a_{(1)}, S^{-1}(b_{(4)}) \rangle \langle a_{(4)}, \pi_{p^{-1}}(b_{(1)}) \rangle (a' a_{(3)} \bowtie \pi_{r^{-1}}(b_{(2)}) b') \otimes ((a'' \bowtie y)(a_{(2)} \bowtie b_{(3)} b'')). \end{aligned}$$

As before, we have a product in $A \bowtie B$ in the second factor of this tensor product and if $r \neq \rho_s(q^{-1})sq$, this last expression equals zero. If $r = \rho_s(q^{-1})sq$, we have $\pi_{r^{-1}} = \pi_{s^{-1}}$. As both calculations give the same result in $(A \bowtie B) \otimes (A \bowtie B)$, we conclude that $\widetilde{\Delta}(b)\Delta^{cop}(a) = \overline{\Delta}(R(b \otimes a))$ in $M((A \bowtie B) \otimes (A \bowtie B))$. ■

We now formulate the main result of this section.

3.8 Theorem Let $\langle A, B\rangle$ be a pair of multiplier Hopf algebras and assume that B is a (regular) G-cograded multiplier Hopf algebra. Let π be an admissible action of G on B.

(1) The space $D^\pi = A^{cop} \bowtie \widetilde{B}$ becomes a (regular) multiplier Hopf algebra, called the Drinfel'd double, with the multiplication, the comultiplication, the counit and the antipode, depending on the pairing as well as on the action π, defined in the following way:

- $(a \bowtie b)(a' \bowtie b') = (m_A \otimes m_B)(\iota_A \otimes R \otimes \iota_B)(a \otimes b \otimes a' \otimes b')$ where $R(b \otimes a') = \sum(\pi_{p^{-1}}(b_{(1)}) \blacktriangleright a' \blacktriangleleft S^{-1}(b_{(3)})) \otimes b_{(2)}$ for all $a' \in A$ and $b \in B_p$,
- $\overline{\Delta}(a \bowtie b) = \Delta^{cop}(a)\widetilde{\Delta}(b)$ for all $a \in A$ and $b \in B$ where $\Delta^{cop}(a)$ and $\widetilde{\Delta}(b)$ are considered as multipliers in $M(D^\pi \otimes D^\pi)$,
- $\overline{\varepsilon}(a \bowtie b) = \varepsilon(a)\varepsilon(b)$ for all $a \in A$ and $b \in B$,
- $\overline{S}(a \bowtie b) = R(\pi_{p^{-1}}(S(b)) \otimes S^{-1}(a))$ for all $a \in A$ and $b \in B_p$.

(2) If moreover A is a multiplier Hopf *-algebra, B a G-cograded multiplier Hopf *-algebra and $\langle A, B\rangle$ a *-pairing, then D^π is again a multiplier Hopf *-algebra with the *-operation given by $(a \bowtie b)^* = R(b^* \otimes a^*)$ for all $a \in A$ and $b \in B$.

Proof.

(1) Let R be the twist map, defined in Definition 3.4. Recall that for $a \in A$ and $b \in B_p$, R is given by the formula

$$R(b \otimes a) = \Sigma(\pi_{p^{-1}}(b_{(1)}) \blacktriangleright a \blacktriangleleft S^{-1}(b_{(3)})) \otimes b_{(2)}.$$

This twist map $R : B \otimes A \to A \otimes B$ is bijective and satisfies the compatibility conditions with the multiplications of the algebras A and B, see Lemma 3.5. As reviewed in Section 1, we consider the twisted tensor product algebra $A \bowtie B$ associated with this twist map R. For all $a \in A$ and $b \in B$, we consider the multiplier $\overline{\Delta}(a \bowtie b) = \Delta^{cop}(a)\widetilde{\Delta}(b)$ in $M((A \bowtie B) \otimes (A \bowtie B))$. Then $\overline{\Delta}$ satisfies the compatibility condition with R as proven in Proposition 3.7. Following Theorem 1.13 we can consider the twisted tensor product multiplier Hopf algebra associated to A^{cop}, $\widetilde{B}$ and the twist map R. We denote this multiplier Hopf algebra as $D^\pi = A^{cop} \bowtie \widetilde{B}$. The counit and the antipode on D^π are uniquely determined in this setting and the formulas are given in the formulation of Theorem 1.13. Remember that the antipode in A^{cop} is S^{-1} while the antipode in $\widetilde{B}$ is given by $\widetilde{S}(b) = \pi_{p^{-1}}(S(b))$ when $b \in B_p$, see Theorem 2.11.

(2) From the conditions on the multiplier Hopf *-algebra B, we have that the multiplier Hopf algebra $\widetilde{B} = (B, \widetilde{\Delta})$ is again a multiplier Hopf *-algebra, see Theorem 2.11 (2). Following Theorem 1.13 the twisted tensor product $D^\pi = A^{cop} \bowtie \widetilde{B}$ is a multiplier Hopf *-algebra via the formula $(a \bowtie b)^* = R(b^* \otimes a^*)$ if this operation defines an involution on D^π. To show that this is the case, take $a \in A_q$ and $b \in B_p$. Then we have that

$$\begin{aligned}((a \bowtie b)^*)^* = (R(b^* \otimes a^*))^* &= (\textstyle\sum (\pi_{p^{-1}}(b^*_{(1)}) \blacktriangleright a^* \blacktriangleleft S^{-1}(b^*_{(3)})) \bowtie b^*_{(2)})^* \\ &= \textstyle\sum R(b_{(2)} \otimes (\pi_{p^{-1}}(b^*_{(1)}) \blacktriangleright a^* \blacktriangleleft S^{-1}(b^*_{(3)}))^*).\end{aligned}$$

Because $\langle A, B\rangle$ is a *-pairing of multiplier Hopf *-algebras, we have that $(b \blacktriangleright a \blacktriangleleft b')^* = S^{-1}(b^*) \blacktriangleright a^* \blacktriangleleft S^{-1}(b'^*)$ for all $a \in A$ and $b, b' \in B$. Therefore, we have $((a \bowtie b)^*)^* = \sum R(b_{(2)} \otimes (\pi_{p^{-1}}(S^{-1}(b_{(1)})) \blacktriangleright a \blacktriangleleft b_{(3)}))$. Because $a \in A_q$, we must have that $b_{(2)} \in B_{\rho_p(q)pq^{-1}}$. As π is an admissible action of G on B, we have $\pi_{qp^{-1}\rho_p(q^{-1})} = \pi_{p^{-1}}$. Now we easily obtain that $((a \bowtie b)^*)^* = a \bowtie b$. ∎

Take the notations and assumptions as in Theorem 3.8. Because $D^\pi = A^{cop} \bowtie \widetilde{B}$ is a twisted tensor product multiplier Hopf algebra of A^{cop} and $\widetilde{B}$, it is quite obvious that integrals on A and on $\widetilde{B}$ compose to an integral on $D^\pi = A^{cop} \bowtie \widetilde{B}$ in the following way. Let φ_A be an left integral on A and let ψ_B be a right integral on B. Consider $\widetilde{\psi}$, defined on B by $\widetilde{\psi}(b) = \psi(\pi_{p^{-1}}(b))$ when $b \in B_p$. Then $\varphi_A \otimes \widetilde{\psi}_B$ is a right integral on $D^\pi = A^{cop} \bowtie \widetilde{B}$, see also Theorem 2.11(3).

We now consider the *-situation. From [D, Remark 3.11] we know that in this case, positive integrals on A^{cop} and $\widetilde{B}$ do not compose in a trivial way to a positive integral on $D^\pi = A^{cop} \bowtie \widetilde{B}$. In [De-VD, Theorem 3.4], the problem for the usual Drinfel'd double $D = A^{cop} \bowtie B$, which is associated to the multiplier Hopf *-algebra pairing $\langle A, B\rangle$, is treated as follows. Let δ_A and δ_B denote the modular multipliers in $M(A)$ and $M(B)$ respectively (see Theorem 1.5). In [De-VD] there is given a meaning to the complex number $\langle \delta_A, \delta_B\rangle^{1/2}$. Furthermore, it is proven in [De-VD, Theorem 3.4] that $\langle \delta_A, \delta_B\rangle^{1/2}(\varphi_A \otimes \psi_B)$ is a positive right integral on $D = A^{cop} \bowtie B$ whenever φ_A is a positive left integral on A and ψ_B is a positive right integral on B.
In the following proposition we obtain this result for the Drinfel'd double construction D^π.

3.9 Proposition Let A and B be multiplier Hopf *-algebras as in Theorem 3.8(2). Let φ_A be a positive left integral on A and ψ_B is a positive right integral on B. Define as before $\widetilde{\psi}$ by $\widetilde{\psi}(b) = \psi(\pi_{p^{-1}}(b))$ when $b \in B_p$. Then $\langle \delta_A, \delta_B\rangle^{1/2}(\varphi_A \otimes \widetilde{\psi}_B)$ is a positive right integral on $D^\pi = A^{cop} \bowtie \widetilde{B}$.

Proof. A straightforward calculation shows that

$$(\iota_A \otimes \widetilde{\psi}_B)(R(b \otimes a)) = \widetilde{\psi}_B(b)(\delta_B^{-1} \blacktriangleright a)$$

for all $a \in A$ and $b \in B$. Now, the proof of [De-VD, Theorem 3.4] can be repeated, with the twist map R given by the formula $R(b \otimes a) = \sum(\pi_{p^{-1}}(b_{(1)}) \blacktriangleright a \blacktriangleleft S^{-1}(b_{(3)})) \otimes b_{(2)}$ for all $a \in A$ and $b \in B_p$. $\blacksquare$

Two special cases

The first case is the one with the trivial action. We get the following (expected) result.

3.10 Proposition Take the notations as in Theorem 3.8. If the admissible action π is the trivial action, then D^π, constructed in Theorem 3.8, is nothing else but the usual Drinfel'd double, $D = A^{cop} \bowtie B$, associated with the pair $\langle A, B\rangle$ (as constructed and studied in [Dr-VD] and [De-VD]).

The other case is more interesting. Now, let π an admissible action such that for all $p, q \in G$ we have $\pi_p(B_q) = B_{pqp^{-1}}$. This is the case with the adjoint action as in Example 2.7. In the framework of Hopf group-coalgebras, these actions are called crossings, see [T, Section 11]. We generalize this definition here.

3.11 Definition Let B be a regular G-cograded multiplier Hopf algebra. An admissible action π of G on B is called a *crossing* if for all $p, q \in G$ we have $\pi_p(B_q) = B_{pqp^{-1}}$.

In the following propositions, we describe the multiplier Hopf algebras $\widetilde{B}$ and D^π in more detail for the case where the admissible action is a crossing. We prove that the considered multiplier Hopf algebras are again G-cograded. Furthermore, we show that there is again a crossing of G on $\widetilde{B}$ and on D^π, defined in a natural way.

The following proposition generalizes the *mirror* construction in the framework of crossed Hopf group-coalgebras, see [T, Section 11].

3.12 Proposition Let B be a regular G-cograded multiplier Hopf algebra. Let π be a crossing of G on B. The deformed multiplier Hopf algebra $\widetilde{B} = (B, \widetilde{\Delta})$ is again G-cograded. Furthermore, π is also a crossing of G on $\widetilde{B}$. The deformation $\widetilde{\widetilde{B}} = (B, \widetilde{\widetilde{\Delta}})$

equals the original G-cograded multiplier Hopf algebra B.

Proof. From Theorem 2.11 we have that $\widetilde{B} = (B, \widetilde{\Delta})$ is a regular multiplier Hopf algebra. Put $\widetilde{B}_p = B_{p^{-1}}$. Then $\widetilde{B} = \bigoplus_{p\in G} \widetilde{B}_p$ and we have

$$\begin{aligned}\widetilde{\Delta}(\widetilde{B}_p)(1 \otimes \widetilde{B}_q) = \widetilde{\Delta}(B_{p^{-1}})(1 \otimes B_{q^{-1}}) &= (\pi_q \otimes \iota)(\Delta(B_{p^{-1}})(1 \otimes B_{q^{-1}})) \\ &= (\pi_q \otimes \iota)(B_{p^{-1}q} \otimes B_{q^{-1}}) = B_{qp^{-1}} \otimes B_{q^{-1}} = \widetilde{B}_{pq^{-1}} \otimes \widetilde{B}_q.\end{aligned}$$

Similarly,

$$\begin{aligned}(\widetilde{B}_q \otimes 1)\widetilde{\Delta}(\widetilde{B}_p) &= (B_{q^{-1}} \otimes 1)\widetilde{\Delta}(B_{p^{-1}}) = (B_{q^{-1}} \otimes 1)((\pi_{q^{-1}p} \otimes \iota)\Delta(B_{p^{-1}})) \\ &= (\pi_{q^{-1}p} \otimes \iota)((B_{p^{-1}q^{-1}p} \otimes 1)\Delta(B_{p^{-1}})) = (\pi_{q^{-1}p} \otimes \iota)(B_{p^{-1}q^{-1}p} \otimes B_{p^{-1}q}) \\ &= B_{q^{-1}} \otimes B_{p^{-1}q} = \widetilde{B}_q \otimes \widetilde{B}_{q^{-1}p}.\end{aligned}$$

As the algebra structure of $\widetilde{B}$ is the same as the algebra structure of B, π is a crossing on $\widetilde{B}$ if for all $p \in G$ we have that π_p respects the comultiplication $\widetilde{\Delta}$ in the sense that $\widetilde{\Delta}(\pi_p(b)) = (\pi_p \otimes \pi_p)(\widetilde{\Delta}(b))$ for all $b \in B$. To show this, take $b \in B$ and $b' \in B_q$, then we have

$$\begin{aligned}\widetilde{\Delta}(\pi_p(b))(1 \otimes b') &= (\pi_{q^{-1}} \otimes \iota)(\Delta(\pi_p(b))(1 \otimes b')) = (\pi_{q^{-1}p} \otimes \pi_p)(\Delta(b)(1 \otimes \pi_{p^{-1}}(b'))) \\ &= (\pi_p \otimes \pi_p)((\pi_{p^{-1}q^{-1}p} \otimes \iota)(\Delta(b)(1 \otimes \pi_{p^{-1}}(b')))) = (\pi_p \otimes \pi_p)(\widetilde{\Delta}(b)(1 \otimes \pi_{p^{-1}}(b'))) \\ &= ((\pi_p \otimes \pi_p)\widetilde{\Delta}(b))(1 \otimes b').\end{aligned}$$

So $\widetilde{B} = (B, \widetilde{\Delta})$ is a G-cograded multiplier Hopf algebra and π is a crossing of G on $\widetilde{B}$. Therefore we can consider the deformation $\widetilde{\widetilde{B}} = (B, \widetilde{\widetilde{\Delta}})$. We will now show that this is the original G-cograded multiplier Hopf algebra B. Take $b \in B$ and $b' \in \widetilde{B}_q = B_{q^{-1}}$. Then we have

$$\widetilde{\widetilde{\Delta}}(b)(1 \otimes b') = (\pi_{q^{-1}} \otimes \iota)(\widetilde{\Delta}(b)(1 \otimes b')) = (\pi_{q^{-1}} \otimes \iota)((\pi_q \otimes \iota)(\Delta(b)(1 \otimes b')) = \Delta(b)(1 \otimes b').$$

Therefore we have $(B, \widetilde{\widetilde{\Delta}}) = B$. ■

Next we consider the Drinfel'd double construction $D^\pi = A^{cop} \bowtie \widetilde{B}$ when π is a crossing.

3.13 Proposition Take the notations and assumptions as in Theorem 3.8. Assume furthermore that the admissible action π of G on B is a crossing. Then $D^\pi = A^{cop} \bowtie \widetilde{B}$ is again a G-cograded multiplier Hopf algebra. Furthermore, there is a crossing of G on D^π, defined in a natural way.

Proof. Let $D^\pi_p = A \bowtie B_{p^{-1}}$ for any $p \in G$. Because π is assumed to be a crossing, one can show, with the techniques used before, that $R(B_p \otimes A) = A \otimes B_p$ for all $p \in G$. It follows easily that D^π_p is a subalgebra of D^π and that $D^\pi_p D^\pi_q = 0$ if $p \neq q$. Also $D^\pi = \bigoplus_{p\in G} D^\pi_p$. This gives the desired decomposition of the algebra D^π.

Next we prove that $\overline{\Delta}(D^\pi_p)((1 \bowtie 1) \otimes D^\pi_q) = D^\pi_{pq^{-1}} \otimes D^\pi_q$. Take $a, a' \in A$ and $b \in B_{p^{-1}}$, $b' \in B_{q^{-1}}$. Then, we have

$$\overline{\Delta}(a \bowtie b)((1 \bowtie 1) \otimes (a' \bowtie b')) = \sum (a_{(2)} \bowtie \pi_q(b_{(1)})) \otimes ((a_{(1)} \bowtie b_{(2)})(a' \bowtie b')).$$

As π is a crossing of G on B, we have that $b_{(2)} \in B_{q^{-1}}$ and $\pi_q(b_{(1)}) \in B_{qp^{-1}}$. Similarly, to prove that also $(D^\pi_p \otimes (1 \bowtie 1))\overline{\Delta}(D^\pi_q) = D^\pi_p \otimes D^\pi_{p^{-1}q}$, take a, a', b and b' as above and write

$$((a \bowtie b) \otimes (1 \bowtie 1))\overline{\Delta}(a' \bowtie b') = \sum ((a \bowtie b)(a'_{(2)} \bowtie \pi_{p^{-1}q}(b'_{(1)}))) \otimes (a'_{(1)} \bowtie b'_{(2)})).$$

Now we find that $b'_{(2)} \in B_{q^{-1}p}$ and $\pi_{p^{-1}q}(b'_{(1)}) \in B_{p^{-1}}$. So, we have shown that D^π is G-cograded.

Next, we will define a crossing of G on D^π. First consider the action π' of G on A defined in the following way. Take $p \in G$ and define the linear map π'_p on A by the formula $\langle \pi'_p(a), b\rangle = \langle a, \pi_{p^{-1}}(b)\rangle$ for all $a \in A$ and $b \in B$. Clearly π'_p is a linear isomorphism such that $(\pi'_p)^{-1} = \pi'_{p^{-1}}$. From the definition of π'_p, it easily follows that π'_p is an algebra isomorphism on A and $\Delta(\pi'_p(a)) = (\pi'_p \otimes \pi'_p)(\Delta(a))$. Furthermore we have $\pi'_{pq} = \pi'_p\pi'_q$ for all $p, q \in G$.

Next, for $p \in G$, consider the linear isomorphism $\pi'_p \otimes \pi_p$ on $D^\pi = A^{cop} \bowtie \widetilde{B}$. It is not difficult to see that $\pi'_p \otimes \pi_p$ is an algebra isomorphism on D^π. Furthermore $\pi'_{pq} \otimes \pi_{pq} = (\pi'_p \otimes \pi_p)(\pi'_q \otimes \pi_q)$ for all $p, q \in G$. Moreover $(\pi'_p \otimes \pi_p)(D^\pi_q) = D^\pi_{pqp^{-1}}$ for all $p, q \in G$.

To complete the proof, we show that for all $p \in G$, the isomorphism $\pi'_p \otimes \pi_p$ on D^π respects the comultiplication of D^π. For any $a \in A$ and $b \in B$ we have

$$\overline{\Delta}((\pi'_p \otimes \pi_p)(a \bowtie b)) = \Delta^{cop}(\pi'_p(a))\widetilde{\Delta}(\pi_p(b))$$
$$= (\pi'_p \otimes \pi'_p)(\Delta^{cop}(a))(\pi_p \otimes \pi_p)(\widetilde{\Delta}(b)) = ((\pi'_p \otimes \pi_p) \otimes (\pi'_p \otimes \pi_p))(\overline{\Delta}(a \bowtie b)).$$

We conclude that the isomorphisms $\pi'_p \otimes \pi_p$, with $p \in G$, define a crossing of G on D^π. ■

3.14 Example Let G be any group. Consider a Hopf G-coalgebra as given in [T-Section 11]. In Proposition 2.2 we saw that we can associate a regular G-cograded multiplier Hopf algebra. We use the notations and assumptions of this proposition and we put $B = \bigoplus_{p \in G} B_p$. We now suppose that each algebra B_p is finite-dimensional. Next let $B^* = \bigoplus_{p \in G} (B_p)'$ where $(B_p)'$ the dual space of B_p. This is called the reduced dual space of B and in general, B^* is smaller than the (full) dual of B. It can be shown that B^* is a Hopf algebra. The product is defined dual to the coproduct as follows. Take $f \in (B_p)'$ and $g \in (B_q)'$, then fg in $(B_{pq})'$ is defined by the formula $(fg)(x) = (f \otimes g)\Delta_{p,q}(x)$ for all $x \in B_{pq}$. The unit in B^* is ε (on B_1). The coproduct is defined dual to the product. For $f \in (B_p)'$ we have $\Delta(f) \in (B_p)' \otimes (B_p)'$ when Δ is the dual to the multiplication in B_p. For $f \in (B_p)'$ we have $\varepsilon(f) = f(1_p)$ and $S(f) = f \circ S$.

The evaluation map defines a pairing $\langle B^*, B\rangle$ between B^* and B of the type considerd in Proposition 3.1. A crossing in the sense of [T - Section 11] gives a crossing of G on B in the sense of Definition 3.11. The Drinfel'd double $D^\pi = (B^*)^{cop} \bowtie \widetilde{B}$ is a G-cograded multiplier Hopf algebra, $D^\pi = \bigoplus_{p \in G} D^\pi_p$ with $D^\pi_p = B^* \bowtie B_{p^{-1}}$. For each $p \in G$, we define the isomorphism $\pi'_p \in Aut(B^*)$ by the formula $\langle \pi'_p(f), b\rangle = \langle f, \pi_{p^{-1}}(b)\rangle$. From Proposition 3.12 we have that for all $p \in G$, the isomorphisms $\pi'_p \otimes \pi_p$ on $D^\pi = (B^*)^{cop} \bowtie \widetilde{B}$ provide a crossing of the group G on D^π.

This Drinfel'd double D^π is the same as the one constructed in [Z, Section 5] in the framework of Hopf group-coalgebras.

References

[A-D-VD] A.T. Abd El-Hafez, L. Delvaux and A. Van Daele, *Group-cograded multiplier Hopf (*-)algebras*, Preprint (2004). Sever version math.QA/04????.

[D] L. Delvaux, *Twisted tensor product of multiplier Hopf (*-)algebras*, J. Algebra **269**(1) (2003), 285–316.

[De-VD] L. Delvaux and A. Van Daele, *The Drinfel'd double of multiplier Hopf algebras*, J. Algebra **272** (1) (2004), 273–291.

[Dr-VD] B. Drabant, A. Van Daele, *Pairing and quantum double of multiplier Hopf algebras*, Algebras and Representative Theory **4**(2) (2001), 109–132.

[D-VD-Z] B. Drabant, A. Van Daele and Y. Zhang, *Actions of multiplier Hopf algebras*, Commun. Algebra **27**(9) (1999), 4117–4127.

[H-A-M] A.S. Hegazi, A.T. Abt El-Hafez and M. Mansour, *Multiplier Hopf group-coalgebras*, Preprint Mansoura University (2002).

[K-V] J. Kustermans and S. Vaes, *Locally compact quantum groups*, Ann. Sci. Ec. Norm. Sup. **33** (2000), 837–934.

[K-VD] J. Kustermans and A. Van Daele, *C^*-algebraic quantum groups arising from algebraic quantum groups*, Int. J. Math. **8** (1997), 1067–1139.

[T] V.G. Turaev, *Homotopy field theory in dimension 3 and crossed group-categories*, Preprint GT/0005291.

[VD1] A. Van Daele, *Multiplier Hopf algebras*, Trans. Am. Math. Soc. **342**(2) (1994), 917–932.

[VD2] A. Van Daele, *An algebraic framework for group duality*, Advances in Mathematics **140** (1998), 323–366.

[VD3] A. Van Daele, *Multiplier Hopf *-algebras with positive integrals: A laboratory for locally compact quantum group*. In "Locally Compact Quantum Groups and Groupoids", Proceedings of the Meeting of Theoretical Physicists and Mathematicians, Strasbourg, February 21-23, 2003 (editor L. Vainerman).

[VD-Z] A. Van Daele and Y. Zhang, *A survey on multiplier Hopf algebras*, in: S. Caenepeel, F. Van Oystayen (Eds.), Hopf Algebras and Quantum Groups, Dekker, New York, 1998 pp.259-309.

[Z] M. Zunino, *Double construction for crossed Hopf coalgebras*, Preprint QA/0212192.