On several two-boundary problems for a particular class of Lévy processes

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Running head: Two-boundary problems for certain Lévy processes

Abstract

Several two-boundary problems for the Poisson process with an exponential component are solved in the present article. The integral transforms are obtained of the joint distribution of the epoch of the first exit from interval and the value of the overshoot through boundaries at the epoch of the exit. Also the joint distribution of the epoch of the first entrance into the interval and the value of the process at this epoch are determined in terms of integral transforms. The distributions of the number of upward and downward entrances into the interval are found.

1 Introduction

In the present article we study one particular class of Lévy processes, i.e. a Poisson process with a negative exponential component (for a more rigorous definition see below). Several characteristics of the process are of particular interest, namely the joint distribution of the first exit time from a fixed interval and the value of the overshoot at the epoch of the exit; the joint distribution of the epoch of the first entrance into the interval and the value of the process at this epoch. We also determine the distribution of the number of upwards and downwards entrances of the interval by the process. Our motivation stems from the fact that these boundary characteristics of the process arise in different settings such as queuing theory, financial mathematics, inventory theory etc.

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A lot of work has been done in the area of two-boundary problems for Lévy processes in general and for special cases of them. We give a brief overview about the existing results on these problems. Let $(\Omega,\ \mathfrak{F}, \{\mathfrak{F}_t\}, P)$ be a filtered probability space, where the filtration $\{\mathfrak{F}_t\}$ satisfies the usual conditions of right-continuity and completion. We assume that all random variables and stochastic processes are defined on this probability space. A Lévy process is a $\mathfrak{F}$ -adapted stochastic process $\{\xi(t);\, t \geq 0\}$ which has independent and stationary increments and its paths are right-continuous with left limits [26]. In assumption that $\xi(0) = 0$ the Laplace transform of the process $\{\xi(t);\, t \geq 0\}$ has the form $E[e^{-p\xi(t)}] = e^{t\,k(p)}$, $Rep = 0$, where the function $k(p)$ is called the Laplace exponent and is given by the formula ([28], p.110)

$$k(p) = \frac{1}{t}\ln Ee^{-p\xi(t)} = \frac{1}{2}p^2\sigma^2 - \alpha p + \int_{-\infty}^{\infty}\left(e^{-px} - 1 + \frac{px}{1+x^2}\right)\Pi(dx). \tag{1}$$

Here $\alpha,\, \sigma \in \mathbb{R}$ and $\Pi(\cdot)$ is a measure on the real line. The introduced process is a space homogeneous, strong Markov process. Note, that the distribution of the first exit time from interval plays a crucial role in applications and its knowledge also allows to solve a number of other two-boundary problems. Let us fix $B > 0$ and define the variable

$$\chi(y) = \inf\{\, t :\ y + \xi(t) \notin [0, B]\,\}, \quad y \in [0, B],$$

the first exit time from the interval $[0, B]$ by the process $y + \xi(t)$. The random variable $\chi(y)$ is a Markov time [9] and $P\,[\,\chi(y) < \infty\,] = 1$. Exit from the interval $[0, B]$ can take place either through the upper boundary B, or through the lower boundary 0. Introduce events: $A^B = \{\omega\, :\, \xi(\chi(y)) > B\,\}$, i.e. the exit takes place through the upper boundary; $A_0 = \{\omega : \xi(\chi(y)) < 0\,\}$, i. e. the exit takes place through the lower boundary. Define

$$X(y) = (\xi(\chi(y)) - B)\, I_{A^B} + (-\xi(\chi(y)))\, I_{A_0}, \qquad P\,[\,A^B + A_0\,] = 1,$$

the value of the overshoot through one of the boundaries at the epoch of the exit, where $I_A = I_A(\omega)$ is the indicator of the event A. The first two-boundary problem for Lévy processes with the Laplace exponent of the general form (1) has been solved by Gihman and Skorohod ([9], p.306-311). These authors have determined the joint distribution of $\{\,\xi^-(t), \xi(t), \xi^+(t)\,\}$, where $\xi^+(t) = \sup_{u \leq t} \xi(u)$, $\xi^-(t) = \inf_{u\leq t} \xi(u)$, $t \geq 0$. For a spectrally positive Lévy process (i.e.

a semi-continuous from below process with independent increments) with the Laplace exponent

$$k(p) = \frac{1}{2}p^2\sigma^2 - \alpha p + \int_0^\infty \left(e^{-px} - 1 + \frac{px}{1+x^2}\right)\Pi(dx), \qquad \mathrm{Re}\, p \geq 0$$

the joint distribution of $\{\chi(y), X(y)\}$ has been studied by many authors among which Emery [8], Shurenkov and Suprun [30]. The first exit time for a spectrally one-sided process has been considered by Pistorius [23], [24], Kyprianou [16], Bertoin [2] and others. Kadankov and Kadankova [10] have suggested another approach for determining the joint distribution of $\{\chi(y), X(y)\}$ for the Lévy process with Laplace exponent (1). Their method is based on application of one-boundary functionals $\{\tau^x, T^x\}$, $\{\tau_x, T_x\}$, $x \geq 0$, where

$$\tau^x = \inf\{t : \xi(t) > x\},\ T^x = \xi(\tau^x) - x, \quad \tau_x = \inf\{t : \xi(t) < -x\},\ T_x = -\xi(\tau_x) - x.$$

Integral transforms of these joint distributions have been obtained in 60's in papers of Rogozin [25], Pecherskii [19], Borovkov [4], Zolotarev [31]. Kadankov and Kadankova [10] have used probabilistic methods (the total probability law, space homogeneity and the strong Markov property of the process) to determine the integral transforms $E\,[\, e^{-s\chi(y)};\ X(y) \in du,\, A^B\,]$, $E\,[\, e^{-s\chi(y)};\ X(y) \in du,\, A_0\,]$ $(s > 0,\ u \geq 0)$ of the joint distribution of $\{\chi(y), X(y)\}$. For a spectrally positive Lévy process several two-boundary problems have been solved in [11]-[13].

The paper is organized as follows. First we introduce the process which we are going to study and then state auxiliary results. Further, in Section 3 we derive the integral transforms of the joint distribution of the first exit time from a fixed interval by a Lévy process and the value of the overshoot at the epoch of the exit. We also prove the corresponding results for a Poisson process with a negative exponential component. The Laplace transforms of the joint distribution of the epoch of the first entrance into the interval and the value of the process at this epoch are found in Section 4. Finally, in Section 5 the distributions of the number of upward and downward entrances into the interval are determined.

2 Main definitions and auxiliary results

Let us give a formal definition of the process which we consider. Let $\eta \in (0, \infty)$ be a positive random variable, and γ be an exponential variable with parameter $\lambda > 0$: $P\,[\,\gamma > x\,] = e^{-\lambda x}$,

$x \geq 0$. Introduce the random variable $\xi \in \mathbb{R}$ by its distribution function

$$F(x) = a\, e^{x\lambda}\, I\{x \leq 0\} + (a + (1-a)\, P\,[\,\eta \leq x\,])\, I\{x > 0\}, \qquad a \in (0,1), \quad \lambda > 0.$$

Consider a right-continuous compound Poisson process $\xi(t) = \sum_{k=1}^{N(t)} \xi_k, \quad t \geq 0$, where $\{\xi_k;\ k \geq 1\}$ are independent identically distributed with ξ random variables, and $N(t)$ is a homogeneous Poisson process with intensity $c > 0$. Then its Laplace exponent is of the form

$$k(p) = c \int_{-\infty}^{\infty} (e^{-xp} - 1)\, dF(x) = a_1 \frac{p}{\lambda - p} + a_2(E[\,e^{-p\eta}] - 1), \qquad c > 0, \quad \operatorname{Re} p = 0, \tag{2}$$

where $a_1 = ac, \quad a_2 = (1-a)c$. Here and in the sequel we will call such process the Poisson process with a negative exponential component. Note, that jumps of the process $\{\xi(t);\ t \geq 0\}$ occur at the time epochs that are exponentially distributed with parameter c. With probability $1-a$ there occur positive jumps with value distributed as η, and with probability a there occur negative jumps (of which value is γ that is exponentially distributed with parameter λ). The first term of (2) is the simplest case of a rational function, while the second term is nothing but the Laplace exponent of a monotone Poisson process with positive jumps of value η. It is well known fact ([3] or [5]), that in this case the equation $k(p) - s = 0, \quad s > 0$ has a unique root $c(s) \in (0, \lambda)$, in the semi-plane $\operatorname{Re} p > 0$. Denote by ν_s an independent of the process exponentially distributed random variable with parameter $s > 0$, i.e. $P[\,\nu_s > t\,] = \exp\{-st\}$. Then for the integral transforms of the random variables $\xi^+(\nu_s), \quad \xi^-(\nu_s)$ the following formulae hold

$$\begin{aligned} E[\,e^{-p\xi^-(\nu_s)}] &= \frac{c(s)}{\lambda}\,\frac{\lambda - p}{c(s) - p}, \qquad \operatorname{Re} p \leq 0, \\ E[\,e^{-p\xi^+(\nu_s)}] &= \frac{s\lambda}{c(s)}\,(p - c(s))\, R(p,s), \qquad \operatorname{Re} p \geq 0, \end{aligned} \tag{3}$$

where

$$E[e^{-p\,\xi^{\pm}(\nu_s)}] = \exp\left\{\int_0^{\infty} \frac{1}{t}\, e^{-st}\, E\,[\,e^{-p\xi(t)} - 1;\ \pm\xi(t) > 0\,]\, dt\right\}, \qquad \pm\operatorname{Re} p \geq 0;$$

$$R(p,s) = \big(a_1 p + (p - \lambda)[s - a_2(\,E[\,e^{-p\eta}] - 1)]\big)^{-1}, \qquad \operatorname{Re} p \geq 0, \quad p \neq c(s). \tag{4}$$

Observe, that the function $R(p,s)$ is analytic in the semi-plane $\operatorname{Re} p > c(s)$, and $\lim\limits_{p\to\infty} R(p,s) = 0$. Therefore, it allows the representation in the form of an absolutely convergent Laplace integral ([7])

$$R(p,s) = \int_0^\infty e^{-px} R_x(s)\, dx, \qquad \operatorname{Re} p > c(s). \tag{5}$$

We will call the function $R_x(s)$, $x \geq 0$ the resolvent of the Poisson process with a negative exponential component. We assume that $R_x(s) = 0$, for $x < 0$. Note, that $R_0(s) = \lim\limits_{p\to\infty} p\, R(p,s) = (c+s)^{-1}$, and the equalities (3) imply

$$P\,[\,\xi^-(\nu_s) = 0\,] = \frac{c(s)}{\lambda}, \qquad P\,[\,\xi^+(\nu_s) = 0\,] = \frac{\lambda}{c(s)}\,\frac{s}{s+c}.$$

The second formula of (3) yields

$$R(p,s) = \frac{c(s)}{s\lambda}\,\frac{1}{p - c(s)}\, E[\,e^{-p\xi^+(\nu_s)}], \qquad \operatorname{Re} p > c(s). \tag{6}$$

The functions

$$\frac{1}{p - c(s)} = \int_0^\infty e^{-u(p-c(s))} du,\ \operatorname{Re} p > c(s), \quad E[\,e^{-p\xi^+(\nu_s)}] = \int_0^\infty e^{-up} d\,P\,[\,\xi^+(\nu_s) < u\,],\ \operatorname{Re} p \geq 0,$$

which enter the right-hand side of (6), are the Laplace transforms for $\operatorname{Re} p > c(s)$. Therefore, the original functions of the left-hand side and the right-hand side of (6) coincide, and

$$R_x(s) = \frac{c(s)}{s\lambda} \int_{-0}^{x} e^{c(s)(x-u)} d\,P\,[\,\xi^+(\nu_s) < u\,], \qquad x \geq 0. \tag{7}$$

which is the resolvent representation of the Poisson process with a negative exponential component. Note, that the representation for the resolvent of the spectrally one-sided Lévy process similar to (7) was obtained by Shurenkov and Suprun [30]. This representation implies that $R_x(s)$, $x \geq 0$ is positive, monotone, continuous, increasing function of exponential order, i.e. there exists $0 < A(s) < \infty$ such that $R_x(s) < A(s)\exp\{xc(s)\}$, for all $x \geq 0$. Therefore,

$$\int_0^\infty R_x(s) e^{-\alpha x} dx < \infty, \qquad \alpha > c(s).$$

Moreover, in the neighborhood of any $x \geq 0$ the function $R_x(s)$ has bounded variation. Hence, the inversion formula ([18], p. 406) is valid

$$R_x(s) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{xp} R(p, s)\, dp, \qquad \alpha > c(s). \tag{8}$$

The latter equality together with (5) determines the resolvent of the Poisson process with a negative exponential component. To derive the joint distribution of the first exit time and the value of the overshoot at the epoch of the exit for a Poisson process with a negative exponential component we apply a general theorem for Lévy processes which has been proved in [10]. Before stating the theorem we mention the following results

$$\begin{aligned} &E\,[\,e^{-s\tau^x - pT^x}\,] = \left(E\,[e^{-p\xi^+(\nu_s)}]\right)^{-1} E\,[\,e^{-p(\xi^+(\nu_s)-x)};\ \xi^+(\nu_s) > x\,], && \operatorname{Re} p \geq 0, \\ &E\,[\,e^{-s\tau_x - pT_x}\,] = \left(E\,[e^{p\xi^-(\nu_s)}]\right)^{-1} E\,[\,e^{p(\xi^-(\nu_s)+x)};\ -\xi^-(\nu_s) > x\,], && \operatorname{Re} p \geq 0. \end{aligned} \tag{9}$$

The formulae (9) have been obtained by Pecherskii and Rogozin [19]. A simple proof of these equalities is given in [10]. It follows from (3) and (9) after some calculations that the integral transforms of the joint distributions $\{\tau_x,\, T_x\}$, $\{\tau^x,\, T^x\}$ of the Poisson process with a negative exponential component satisfy the equalities

$$\begin{aligned} &E[\,e^{-s\tau_x};\, T_x \in du\,] = (\lambda - c(s))\, e^{-xc(s)}\, e^{-\lambda u}\, du = E[\,e^{-s\tau_x}]\, P\,[\,\gamma \in du\,], \\ &\int_0^\infty e^{-px}\, E[\,e^{-s\tau^x - z\xi(\tau^x)}]\, dx = \frac{1}{p}\left(1 - \frac{p+z-c(s)}{z-c(s)}\ \frac{R(p+z,s)}{R(z,s)}\right), \quad \operatorname{Re} p > 0,\ \operatorname{Re} z \geq 0. \end{aligned} \tag{10}$$

The first equality of (10) yields that τ_x and T_x are independent. Moreover, for all $x \geq 0$ the value of the overshoot through the lower level T_x is exponentially distributed with parameter λ. This fact serves as a characterizing feature of the Poisson process with a negative exponential component. Now we state the main results on two-sided exit problems.

3 The first exit from an interval

We now derive the joint distribution of the first exit time and the value of the overshoot at the epoch of the exit, which will be used for solving another problems. The following theorem is true for general Lévy processes ([10]).

Theorem 1. *Let* $\{\xi(t);\, t \geq 0\}$, $\xi(0) = 0$ *be a real-valued Lévy process with Laplace exponent*

(1), $B>0$ *be fixed,* $y\in[0,B]$, $x=B-y$, *and*

$$\chi(y)=\inf\{\,t>0:\ y+\xi(t)\notin[0,B]\,\},\qquad X(y)=(\xi(\chi(y))-B)\,I_{A^B}+(-\xi(\chi(y)))\,I_{A_0}$$

the instant of the first exit by the process $y+\xi(t)$ *from the interval* $[0,B]$ *and the value of the overshoot through a boundary at the epoch of the exit from the interval by the given process. The Laplace transforms of the joint distribution of* $\{\,\chi(y),X(y)\,\}$ *for* $s>0$ *satisfy the following formulae*

$$\begin{aligned} E\,[\,e^{-s\chi(y)};\ X(y)\in du,\ A^B\,]&=f_+^s(x,du)+\int_0^\infty f_+^s(x,dv)\,K_+^s(v,du),\\ E\,[\,e^{-s\chi(y)};\ X(y)\in du,\ A_0\,]&=f_-^s(y,du)+\int_0^\infty f_-^s(y,dv)\,K_-^s(v,du),\end{aligned}\tag{11}$$

where

$$\begin{aligned} f_+^s(x,du)&=E\,[\,e^{-s\tau^x};\ T^x\in du\,]-\int_0^\infty E\,[\,e^{-s\tau_y};\ T_y\in dv\,]\,E\,[\,e^{-s\tau^{v+B}};\ T^{v+B}\in du\,],\\ f_-^s(y,du)&=E\,[\,e^{-s\tau_y};\ T_y\in du\,]-\int_0^\infty E\,[\,e^{-s\tau^x};\ T^x\in dv\,]\,E\,[\,e^{-s\tau_{v+B}};\ T_{v+B}\in du\,];\end{aligned}$$

and $K_\pm^s(v,du)=\sum\limits_{n=1}^{\infty}K_\pm^{(n)}(v,du,s)$, $v\ge0$ *is the series of the successive iterations;*

$$K_\pm^{(1)}(v,du,s)=K_\pm(v,du,s),\quad K_\pm^{(n+1)}(v,du,s)=\int_0^\infty K_\pm^{(n)}(v,dl,s)\,K_\pm(l,du,s)\tag{12}$$

are the successive iterations $(n\in\mathbb{N}=\{1,2,\dots\})$ *of the kernels* $K_\pm(v,du,s)$, *which are given by the defining equalities*

$$\begin{aligned} K_+(v,du,s)&=\int_0^\infty E\,[\,e^{-s\tau_{v+B}};\ T_{v+B}\in dl\,]\,E\,[\,e^{-s\tau^{l+B}};\ T^{l+B}\in du\,],\\ K_-(v,du,s)&=\int_0^\infty E\,[\,e^{-s\tau^{v+B}};\ T^{v+B}\in dl\,]\,E\,[\,e^{-s\tau_{l+B}};\ T_{l+B}\in du\,].\end{aligned}\tag{13}$$

Remark 1. Shurenkov and Suprun [30] have obtained the following representations for the Laplace transforms of the distribution of the first exit time $\chi(y)$ $(s>0)$ for a spectrally one-sided Lévy process:

$$\begin{aligned} E\,[e^{-s\chi(y)};\ A_0]&=\frac{R_s(x)}{R_s(B)},\\ E\,[e^{-s\chi(y)};\ A^B]&=1-\frac{R_s(x)}{R_s(B)}-s\frac{R_s(x)}{R_s(B)}\int_0^B R_s(u)\,du+s\int_0^x R_s(u)\,du.\end{aligned}$$

The function $R_s(x), \quad x \geq 0,\ s > 0,$ which enters these formulae is called a resolvent or a scale function ([1], p. 195). It is determined by its Laplace transform

$$\int_0^\infty e^{-px}\, R_s(x)\, dx = \frac{1}{k(p) - s}, \qquad \operatorname{Re} p > c(s),$$

where $c(s) > 0, \quad s > 0$ is a unique positive root of the equation $k(p) - s = 0$ in the semi-plane $\operatorname{Re} p > 0$. In this paper we will call this function the resolvent. The resolvent function and its properties have been investigated by Borovskih [3], Bertoin [1]-[2], Pistorius [23], Kyprianou [17]. For a Poisson process with positive jumps and a negative drift the resolvent representations for the Laplace transform of the distribution of $\chi(y)$ have been obtained by Korolyuk [15].

We apply now the formulae of Theorem 1 for the case when the underlying process is the Poisson process with an exponentially distributed negative component.

Corollary 1. *Let* $\{\xi(t);\ t \geq 0\},\ \xi(0) = 0$ *be a real-valued Poisson process with a negative exponential component with the Laplace exponent given by (2),* $B > 0, \quad y \in [0, B], \quad x = B - y,$ *and*

$$\chi(y) = \inf\{\, t > 0 :\ y + \xi(t) \notin [0, B]\,\}, \qquad X(y) = (\xi(\chi(y)) - B)\, I_{A^B} + (-\xi(\chi(y)))\, I_{A_0}$$

the instant of the first exit from the interval and the value of the of the overshoot through one of the boundaries. Then for $s > 0$

1) the integral transforms of the joint distribution $\{\, \chi(y),\ X(y)\,\}$ *satisfy the following equalities*

$$E[\, e^{-s\chi(y)};\ X(y) \in du,\ A_0\,] = e^{-\lambda u}(\lambda - c(s))\, e^{-yc(s)} \left(1 - E[\, e^{-s\tau^x - c(s)\xi(\tau^x)}\,]\right) K(s)^{-1} du, \quad (14)$$
$$E[\, e^{-s\chi(y)};\ X(y) \in du,\ A^B\,] = E[\, e^{-s\tau^x};\ T^x \in du\,] - E\,[\, e^{-s\chi(y)};\ A_0\,]\ E\,[\, e^{-s\tau^{\gamma+B}};\ T^{\gamma+B} \in du\,],$$

where

$$K(s) = 1 - E[\, e^{-s\tau_B}]\ E[\, e^{-s\tau^{\gamma+B} - c(s) T^{\gamma+B}}\,],$$
$$E[\, e^{-s\tau^{\gamma+B} - c(s)T^{\gamma+B}}\,] = \lambda \int_0^\infty e^{-\lambda u} E\,[\, e^{-s\tau^{u+B} - c(s) T^{u+B}}\,]\, du,$$

in particular

$$E[\,e^{-s\chi(y)};\; A_0\,] = \left(1 - \frac{c(s)}{\lambda}\right)\, e^{-yc(s)}\left(1 - E[\,e^{-s\tau^x} e^{-c(s)\xi(\tau^x)}\,]\right) K(s)^{-1}, \tag{15}$$
$$E[\,e^{-s\chi(y)};\; A^B\,] = E[\,e^{-s\tau^x}] - E[\,e^{-s\chi(y)};\; A_0\,]\; E[\,e^{-s\tau^{\gamma+B}}];$$

2) for the Laplace transforms of the random variable $\chi(y)$ *the following representations hold*

$$E[\,e^{-s\chi(y)};\; X(y)\in\, du,\, A_0\,] = e^{-\lambda(u+B)}\,\frac{R_x(s)}{\hat R_B(\lambda,s)}\,du, \qquad E[\,e^{-s\chi(y)};\; A_0\,] = \frac{1}{\lambda}\,e^{-\lambda B}\,\frac{R_x(s)}{\hat R_B(\lambda,s)},$$
$$E[\,e^{-s\chi(y)};\; A^B\,] = 1 - \frac{R_x(s)}{\hat R_B(\lambda,s)}\left[\frac{1}{\lambda}\,e^{-\lambda B} + s\lambda\,\hat S_B(\lambda,s)\right] + s\lambda\, S_x(s),$$
$$\int_0^\infty e^{-st} P\,[\,\chi(y) > t\,]\,dt = \lambda\,\frac{R_x(s)}{\hat R_B(\lambda,s)}\,\hat S_B(\lambda,s) - \lambda\, S_x(s), \tag{16}$$

where $R_x(s),\quad x\ge 0$ *the resolvent of the process, defined by (5), (8);*

$$S_x(s) = \int_0^x R_u(s)\,du, \qquad \hat R_B(\lambda,s) = \int_B^\infty e^{-\lambda u} R_u(s)\,du, \qquad \hat S_B(\lambda,s) = \int_B^\infty e^{-\lambda u} S_u(s)\,du.$$

Proof. For the Poisson process with a negative exponential component, equalities of Theorem 1 take a simplified form. Using the equalities (10) and the defining formulae (13) for the kernels $K_\pm(v,du,s)$, yields

$$K_+(v,du,s) = \left(1 - \frac{c(s)}{\lambda}\right)\, e^{-c(s)(v+B)}\, E\,[\,e^{-s\tau^{\gamma+B}};\; T^{\gamma+B}\in\, du\,],$$
$$K_-(v,du,s) = e^{-\lambda u}(\lambda - c(s))\, e^{-c(s)B}\, E\,[\,e^{-s\tau^{v+B}};\; e^{-c(s)T^{v+B}}\,]\,du,$$

where γ is an exponentially distributed random variable with the parameter λ, independent of the process under the consideration. Using these equalities, the method of the mathematical induction and the formula (12) we obtain the successive iterations $K_\pm^{(n)}(v,du,s),\quad n\in\mathbb{N}$ of the kernels $K_\pm(v,du,s)$:

$$K_-^{(n)}(v,du,s) = E[\,e^{-s\tau^{v+B} - c(s)T^{v+B}}\,]\,\left(E\,e^{-s\tau_B}\right)^n \left(E[\,e^{-s\tau^{\gamma+B} - c(s)T^{\gamma+B}}\,]\right)^{n-1}\,\lambda e^{-\lambda u}\,du,$$
$$K_+^{(n)}(v,du,s) = e^{-vc(s)}\,\left(E\,e^{-s\tau_B}\right)^n \left(E[\,e^{-s\tau^{\gamma+B} - c(s)T^{\gamma+B}}\,]\right)^{n-1}\, E[\,e^{-s\tau^{\gamma+B}};\; T^{\gamma+B}\in\, du\,].$$

The series $K_\pm^s(v,du)$ of the successive iterations $K_\pm^{(n)}(v,du,s)$ are nothing but geometric series and their sums are given by

$$K_-^s(v,du) = \sum_{n=1}^{\infty} K_-^{(n)}(v,du,s) = E[\,e^{-s\tau^{v+B}}\;e^{-c(s)T^{v+B}}\,]\;\;E\,[e^{-s\tau_B}]\;K(s)^{-1}\lambda\,e^{-\lambda u}\,du,$$
$$K_+^s(v,du) = \sum_{n=1}^{\infty} K_+^{(n)}(v,du,s) = e^{-vc(s)}\;E\,[e^{-s\tau_B}]\;K(s)^{-1}\;E[\,e^{-s\tau^{\gamma+B}};\;T^{\gamma+B}\in\;du\,],$$

where

$$K(s) = 1 - E\,e^{-s\tau_B}\;E[\,e^{-s\tau^{\gamma+B}}e^{-c(s)T^{\gamma+B}}\,].$$

Substituting the obtained expressions for $K_\pm^s(v,du)$ into (11), implies the formulae (14) of the corollary. Integrating the formulae (14) with respect to $u\in\mathbb{R}_+$, yields the formula (15) of the corollary. Now, utilizing the definition of the resolvent (5), (8) and the equalities (10), we derive the resolvent representation for the functions $E\,[\,e^{-s\tau^{\,x}-c(s)\xi(\tau^x)}\,]$, $E\,[e^{-s\tau^x}]$:

$$E\,[\,e^{-s\tau^{\,x}-c(s)\xi(\tau^x)}\,] = 1 - e^{-xc(s)}\;R_x(s)\;r(c(s),s),$$
$$E[\,e^{-s\tau^x}] = 1 - \frac{s\lambda}{c(s)}\;R_x(s) + s\lambda\,S_x(s),$$

where

$$S_x(s) = \int_0^x\;R_u(s)\,du, \qquad r(c(s),s) = \frac{d}{dp}R(p,s)^{-1}\bigg|_{p=c(s)}.$$

Substituting the found these resolvent representations into (15), we obtain the representations (16) of the corollary. □

Remark 2. Note, that the resolvent representations similar to (16) were obtained [14] for a integer-valued random walk with the negative geometrical component. It is worth mentioning that the compound Poisson process with linear deterministic decrease between positive and negative jumps has been studied by Perry, Stadje, and Zacks [20]. A martingale approach for solving exit problems has been applied in the article of Perry et al [21].

4 The first entrance time into an interval

The knowledge of the joint distribution $\{\chi(y),\,X(y)\}$) allows us to solve another two-boundary problem, namely to determine the integral transforms of the joint distribution of the epoch of the first entrance into the fixed interval by the Lévy process and the value of the process at this epoch. We prove the theorem with corresponding results and as a corollary from the theorem we obtain these integral transforms for the Poisson process with a negative exponential exponent.

Theorem 2. *Let* $\{\xi(t);\, t\geq 0\}$, $\xi(0)=0$ *be a Lévy process with the Laplace exponent (1),* $B>0$, $\chi(y)\stackrel{\rm def}{=}0$, *for* $y\notin[0,B]$, *and*

$$\overline{\chi}(y)=\inf\{\,t>\chi(y):\; y+\xi(t)\in[0,B]\,\},\qquad \overline{X}(y)=y+\xi(\overline{\chi}(y))\in[0,B],\quad y\in\mathbb{R}$$

be the instant of the first entrance into the interval $[0,B]$ *by the process* $y+\xi(t)$ *and the value of the process at this epoch. Then the integral transforms of the joint distribution* $\{\overline{\chi}(y),\overline{X}(y)\}$, $y\in\mathbb{R}$ *for* $s>0$ *satisfy the following equalities*

$$\begin{aligned}
b^v(du,s)&\stackrel{def}{=}E\,[\,e^{-s\overline{\chi}(v+B)};\;\overline{X}(v+B)\in du\,]=\int_0^\infty Q_+^s(v,dl)\,E[\,e^{-s\tau_l};\;B-T_l\in du\,]\\
&+\int_0^\infty Q_+^s(v,dl)\int_0^\infty E[\,e^{-s\tau_l};\;T_l-B\in d\nu\,]\,E[\,e^{-s\tau^\nu};\;T^\nu\in du\,], && v>0,\\
b_v(du,s)&\stackrel{def}{=}E\,[\,e^{-s\overline{\chi}(-v)};\;\overline{X}(-v)\in du\,]=\int_0^\infty Q_-^s(v,dl)\,E[\,e^{-s\tau^l};\;T^l\in du\,] && (17)\\
&+\int_0^\infty Q_-^s(v,dl)\int_0^\infty E[\,e^{-s\tau^l};\;T^l-B\in d\nu\,]\,E[\,e^{-s\tau_\nu};\;B-T_\nu\in du\,], && v>0,\\
b(y,du,s)&\stackrel{def}{=}E\,[\,e^{-s\overline{\chi}(y)};\;\overline{X}(y)\in du\,]=\int_0^\infty E\,[\,e^{-s\chi(y)};\;X(y)\in dv,\,A^B]\,b^v(du,s)\\
&+\int_0^\infty E\,[\,e^{-s\chi(y)};\;X(y)\in dv,\,A_0]\,b_v(du,s), && y\in[0,B],
\end{aligned}$$

where $\delta(x)$, $x\in\mathbb{R}$ *is the delta function and*

$$Q_\pm^s(v,du)=\delta(v-u)\,du+\sum_{n\in\mathbb{N}}Q_\pm^{(n)}(v,du,s),\qquad v>0; \tag{18}$$

is the series of the successive iterations $Q_\pm^{(n)}(v,du,s)$, $n\in\mathbb{N}$,

$$Q_\pm^{(1)}(v,du,s)=Q_\pm(v,du,s),\qquad Q_\pm^{(n+1)}(v,du,s)=\int_0^\infty Q_\pm^{(n)}(v,dl,s)Q_\pm(l,du,s); \tag{19}$$

the successive iterations of the kernels $Q_\pm(v,du,s)$, *which are given by the defining formulae*

$$\begin{aligned}
Q_+(v,du,s)&=\int_0^\infty E[e^{-s\tau_v};\;T_v-B\in dl]\,E[e^{-s\tau^l};\;T^l-B\in du],\\
Q_-(v,du,s)&=\int_0^\infty E[e^{-s\tau^v};\;T^v-B\in dl]\,E[e^{-s\tau_l};\;T_l-B\in du].
\end{aligned} \tag{20}$$

Proof. For the functions $b^v(du,s)$, $b_v(du,s)$, $v>0$ according to the total probability law and the fact that τ_v, τ^v are Markov times, the following system of equations is valid

$$\begin{aligned} b^v(du,s) =& E[\,e^{-s\tau_v};\; B-T_v\in du\,]+\int_0^\infty E[\,e^{-s\tau_v};\; T_v-B\in dl\,]\, b_l(du,s),\\ b_v(du,s) =& E[\,e^{-s\tau^v};\; T^v\in du\,]+\int_0^\infty E[\,e^{-s\tau^v};\; T^v-B\in dl\,]\, b^l(du,s). \end{aligned} \tag{21}$$

This system is similar to a system of linear equations with two variables. Substituting the expression for $b_v(du,s)$ from the right-hand side of the second equation into the first equation yields

$$\begin{aligned} b^v(du,s) &= E\,[\,e^{-s\tau_v};\; B-T_v\in du\,]+\int_0^\infty E\,[\,e^{-s\tau_v};\; T_v-B\in dl\,]\, E[\,e^{-s\tau^l};\; T^l\in du\,]\\ &+\int_{l=0}^\infty E\,[\,e^{-s\tau_v};\; T_v-B\in dl\,]\int_{\nu=0}^\infty E\,[\,e^{-s\tau^l};\; T^l-B\in d\nu\,]\, b^\nu(du,s). \end{aligned}$$

Changing the order of integration in the third term of the second equation implies for the function $b^v(du,s)$, $v>0$

$$\begin{aligned} b^v(du,s) &= \int_0^\infty Q_+(v,d\nu,s)\, b^\nu(du,s) \\ &+E\,[\,e^{-s\tau_v};\; B-T_v\in du\,]+\int_0^\infty E\,[\,e^{-s\tau_v};\; T_v-B\in dl\,]\, E\,[\,e^{-s\tau^l};\; T^l\in du\,], \end{aligned} \tag{22}$$

which is a linear integral equation with the following kernel

$$Q_+(v,du,s)=\int_0^\infty E[e^{-s\tau_v};\; T_v-B\in dl]\, E\,[e^{-s\tau^l};\; T^l-B\in du],\qquad v>0.$$

We now show, that for all $v,u>0$, $s>s_0>0$ this kernel satisfies the estimation

$$Q_+(v,du,s)<\lambda,\qquad \lambda=E[\,e^{-s\tau_B}]\,E[\,e^{-s\tau^B}],\quad s_0>0.$$

Indeed, for all $s>0$ it follows from

$$\begin{aligned} E\,[\,e^{-s\tau^v};\; T^v-B\in du\,] &= E\,[\,e^{-s\tau^{v+B}};\; T^{v+B}\in du\,]\\ &-\int_0^B E\,[\,e^{-s\tau^v};\; T^v\in dl\,]\, E\,[\,e^{-s\tau^{B-l}};\; T^{B-l}\in du\,], \end{aligned}$$

that the following chain of inequalities holds

$$E[\,e^{-s\tau^v};\; T^v - B \in du\,] \le E[\,e^{-s\tau^{v+B}};\; T^{v+B} \in du\,] \le E[\,e^{-s\tau^{v+B}}\,] \le E[\,e^{-s\tau^B}\,].$$

Analogously we establish

$$E[\,e^{-s\tau_v};\; T_v - B \in du\,] \le E[\,e^{-s\tau_{v+B}};\; T_{v+B} \in du\,] \le E[\,e^{-s\tau_{v+B}}\,] \le E[\,e^{-s\tau_B}\,].$$

These chains of the inequalities imply the following estimation for the kernel $Q_+(v, du, s)$, for all $v, u > 0$, $s > s_0 > 0$

$$\begin{aligned} Q_+(v, du, s) &= \int_0^\infty E[e^{-s\tau_v};\; T_v - B \in dl]\, E[e^{-s\tau^l};\; T^l - B \in du] \\ &\le E[\,e^{-s\tau_B}]\, E[\,e^{-s\tau^B}] < \lambda = E[\,e^{-s_0\tau_B}]\, E[\,e^{-s_0\tau^B}], \qquad s_0 > 0. \end{aligned}$$

This estimation and the method of the mathematical induction yield that the successive iterations $Q_+^{(n)}(v, du, s)$ (19) of the kernels $Q_+(v, du, s)$, for all $v, u > 0$, $s > s_0 > 0$ obey the inequality

$$Q_+^{(n+1)}(v, du, s) = \int_0^\infty Q_+^{(n)}(v, dl, s)\, Q_+(l, du, s) < \lambda^{n+1}, \qquad n \in \mathbb{N}.$$

Therefore, the series of successive iterations

$$Q_+^s(v, du) = \delta(v - u)\, du + \sum_{n\in\mathbb{N}} Q_+^{(n)}(v, du, s) < (1 - \lambda)^{-1}$$

converges uniformly for all $v, u > 0$, $s > s_0 > 0$. Utilizing now the method of successive iterations ([22], p. 33) to solve the integral equation (22), yields the first equality of the theorem. The second equality of the theorem can be verified analogously. It is not difficult to establish the third equality of the theorem using the total probability law and the fact that $\chi(y)$ is the Markov time. □

Denote by

$$m_\gamma^s(du) = \int_0^\infty \lambda\, e^{-\lambda x} E[\,e^{-s\tau^x};\; T^x \in du\,]\, dx, \qquad P(\lambda, du) = e^{-\lambda B}(m_\gamma^s(du) + \lambda\, e^{\lambda u} du)$$

The following corollary from Theorem 2 is valid.

Corollary 2. *Let* $\{\xi(t);\ t \geq 0\}$, $\xi(0) = 0$ *be a real-valued Poisson process with a negative exponential component as specified above,* $B > 0$, $\chi(y) \stackrel{\text{def}}{=} 0$, *in case when* $y \notin [0, B]$, *and let*

$$\overline{\chi}(y) = \inf\{\, t > \chi(y) :\ y + \xi(t) \in [0, B]\,\}, \qquad \overline{X}(y) = \xi(\overline{\chi}(y)) \in [0, B], \quad y \in \mathbb{R}$$

be the instant of the first entrance into the interval $[0, B]$ *by the process* $y + \xi(t)$ *and the value of the process at the epoch of the entrance. Then for the integral transforms of the joint distribution of* $\{\overline{\chi}(y), \overline{X}(y)\}$, $y \in \mathbb{R}$ *for* $s > 0$ *the following formulae hold*

$$\begin{aligned}
b^v(du, s) &= e^{-vc(s)} \left(1 - \frac{c(s)}{\lambda}\right) T(s)^{-1} P(\lambda, du), && v > 0,\\
b_v(du, s) &= m_v^s(du) + e^{Bc(s)} \left(1 - \frac{c(s)}{\lambda}\right) \hat{T}_v^s(c(s))\, T(s)^{-1} P(\lambda, du), && v > 0,\\
b(y, du, s) &= \left(1 - \frac{c(s)}{\lambda}\right) T(s)^{-1} \left[e^{c(s)(B-y)} - \frac{R_{B-y}(s)}{\hat{R}_B(\lambda, s)}\, \frac{e^{-B(\lambda - c(s))}}{\lambda - c(s)} \right] P(\lambda, du)\\
&\quad + \frac{1}{\lambda}\, \frac{R_{B-y}(s)}{\hat{R}_B(\lambda, s)} (T(s)^{-1} - 1)\, P(\lambda, du), && y \in [0, B], \quad (23)
\end{aligned}$$

where

$$\begin{aligned}
&m_x^s(du) = E\,[\, e^{-s\tau^x};\ T^x \in\, du\,], \quad \hat{T}_x^s(c(s)) = E\,[\, e^{-s\tau^x - c(s)T^x};\ T^x > B\,], \qquad x \geq 0,\\
&\hat{T}_\gamma^s(c(s)) = \lambda \int_0^\infty e^{-\lambda x} \hat{T}_x^s(c(s))\, dx, \qquad T(s) = 1 - \left(1 - \frac{c(s)}{\lambda}\right) \hat{T}_\gamma^s(c(s)) e^{-B(\lambda - c(s))}.
\end{aligned}$$

Proof. We apply now the equalities (17) of Theorem 2 to obtain the formulae (23). For this we have to calculate for the Poisson process with a negative component the kernels $Q_\pm(v, dl, s)$, and the successive iterations $Q_\pm^{(n)}(v, dl, s)$, $n \in \mathbb{N}$, and the series $Q_\pm^s(v, dl)$. Utilizing the defining formula of the kernels (20) and the formulae (10), yields

$$\begin{aligned}
Q_+(v, dl, s) &= e^{-vc(s)} \left(1 - \frac{c(s)}{\lambda}\right) e^{-\lambda B} E\,[\, e^{-s\tau^\gamma};\ T^\gamma - B \in dl\,], && v > 0,\\
Q_-(v, dl, s) &= \hat{T}_v^s(c(s))\, e^{-B(\lambda - c(s))} \left(1 - \frac{c(s)}{\lambda}\right) \lambda\, e^{-\lambda l}\, dl, && v > 0, \qquad (24)
\end{aligned}$$

where $\hat{T}_x^s(c(s)) = E\,[\, e^{-s\tau^x - c(s)T^x};\ T^x > B\,]$, $x \geq 0$. Using the defining formula (19) for the successive iterations and the method of the mathematical induction it follows from (24) that

$$Q_+^{(n)}(v, dl, s) = e^{-vc(s)} \left(1 - \frac{c(s)}{\lambda}\right) e^{-\lambda B} \left(\tilde{T}_\gamma^s(c(s))\right)^{n-1} E\,[\,e^{-s\tau^\gamma};\; T^\gamma - B \in dl\,], \qquad n \in \mathbb{N},$$

$$Q_-^{(n)}(v, dl, s) = \hat{T}_v^s(c(s))\, e^{-B(\lambda - c(s))} \left(1 - \frac{c(s)}{\lambda}\right) \left(\tilde{T}_\gamma^s(c(s))\right)^{n-1} \lambda\, e^{-\lambda l}\, dl, \qquad n \in \mathbb{N},$$

where

$$\tilde{T}_\gamma^s(c(s)) = e^{-B(\lambda - c(s))}\, (\lambda - c(s)) \int_0^\infty e^{-\lambda x} \hat{T}_x^s(c(s))\, dx.$$

The series $Q_\pm^s(v, dl)$ of the successive iterations $Q_\pm^{(n)}(v, dl, s)$ (see (18)) are just the geometrical series and their sums are given by

$$Q_+^s(v, dl) = \delta(v - l)\, dl + e^{-vc(s)} \left(1 - \frac{c(s)}{\lambda}\right) e^{-\lambda B} T(s)^{-1} E\,[\,e^{-s\tau^\gamma};\; T^\gamma - B \in dl\,], \qquad v > 0,$$

$$Q_-^s(v, dl) = \delta(v - l)\, dl + \hat{T}_v^s(c(s))\, e^{-B(\lambda - c(s))} \left(1 - \frac{c(s)}{\lambda}\right) T(s)^{-1} \lambda\, e^{-\lambda l}\, dl, \qquad v > 0,$$

where $T(s) = 1 - \tilde{T}_\gamma^s(c(s))$. Substituting in the equalities (17) of Theorem 2 the expressions for the functions $Q_\pm^s(v, dl)$, and the expressions for the functions $E\,[\,e^{-s\chi(y)};\; X(y) \in dv,\, A^B]$, $E\,[\,e^{-s\chi(y)};\; X(y) \in dv,\, A_0]$, which are given by the formulae of Corollary 1, we obtain the formulae (23) of the corollary. □

5 Number of the entrances into an interval

Now we determine the distribution of the number of the entrances into the interval $[0, B]$ through the upper boundary by the Poisson process with a negative exponential component. Let $B > 0$ be fixed, $B_+ = (B,\, \infty)$, and for all $y \in \mathbb{R}$, $n \in \mathbb{N} \cup 0$ we define the sequence

$$\overline{\chi}_0^+(y) = 0, \qquad \overline{\chi}_{n+1}^+(y) = \inf\{\, t > \overline{\chi}_n^+(y) :\; y + \xi(t - 0) \in B_+,\, y + \xi(t) \in [0, B]\,\}$$

of the epochs of the entrances into the interval $[0, B]$ through the upper boundary B (from the set B_+) by the process $y + \xi(\cdot)$. We set per definition $\overline{\chi}_n^+(y) = \infty$ for all $n \geq n_0$ on the sample paths for which there exists $n_0 \in \mathbb{N} \cup 0$ such that the set in the braces is empty. Introduce the random variable

$$\beta_t^+(y) = \max\{\, n \in \mathbb{N} \cup 0 :\; \overline{\chi}_n^+(y) \leq t\,\}, \qquad y \in R, \quad t > 0$$

the number of the entrances into the interval $[0, B]$ through the upper boundary B (from the set B_+) by the process $y+\xi(\cdot)$ up to the moment t. Define

$$E_x^s(c(s)) = \begin{cases} (1-c(s)/\lambda)\, E\,[e^{-s\tau^x - c(s)T^x}], & x \ge 0, \\ \\ E[\,e^{-s\tau_{-x}}\,] = (1-c(s)/\lambda)\, e^{x\,c(s)}, & x < 0 \end{cases} \tag{25}$$

and

$$\check{E}^{\,s}(\lambda, c(s)) = \int_0^B \lambda\, e^{-\lambda x} E_x^s(c(s))\, dx, \qquad \hat{E}^{\,s}(\lambda, c(s)) = \int_B^\infty \lambda\, e^{-\lambda x} E_x^s(c(s))\, dx.$$

The following statement is true.

Theorem 3. *Let $\{\xi(t);\ t \ge 0\}$, $\xi(0) = 0$ be a Poisson process with a negative exponential component with Laplace exponent (2). Then for the generating functions of the distribution of the number of the downward entrances $\beta_{\nu_s}^+(y)$, $y \in \mathbb{R}$ into the interval $[0, B]$ by the process $y+\xi(\cdot)$ (from the set B_+)on the exponential time interval $[0, \nu_s]$, for $s > 0$ the following equalities hold*

$$E[\,\theta^{\,\beta_{\nu_s}^+(y)}] = 1 - \frac{(1-\theta)(1-e^{-\lambda B})}{1-\hat{E}^{\,s}(\lambda, c(s)) - \theta\, \check{E}^{\,s}(\lambda, c(s))}\, E_{B-y}^s(c(s)), \qquad y \in \mathbb{R}, \quad \theta \in [0,1], \tag{26}$$

where the function $E_x^s(c(s))$ is defined by (25).

In particular, for all $y \in \mathbb{R}$

$$\begin{aligned} P\,[\,\beta_{\nu_s}^+(y) = n\,] &= I_{\{n=0\}} \left(1 - \frac{1-e^{-\lambda B}}{1-\hat{E}^{\,s}(\lambda, c(s))} E_{B-y}^s(c(s))\right) \\ &\quad + I_{\{n\in\mathbb{N}\}}\, \frac{1-e^{-\lambda B}}{1-\hat{E}^{\,s}(\lambda, c(s))} \frac{1-E^{\,s}(\lambda, c(s))}{1-\hat{E}^{\,s}(\lambda, c(s))} \left(\frac{\check{E}^{\,s}(\lambda, c(s))}{1-\hat{E}^{\,s}(\lambda, c(s))}\right)^{n-1} E_{B-y}^s(c(s)), \end{aligned}$$

where $E^{\,s}(\lambda, c(s)) = \check{E}^{\,s}(\lambda, c(s)) + \hat{E}^{\,s}(\lambda, c(s))$.

Proof. Introduce

$$A^v(s,\theta) = E[\,\theta^{\,\beta_{\nu_s}^+(B+v)}], \quad v > 0, \qquad A_v(s,\theta) = E[\,\theta^{\,\beta_{\nu_s}^+(B-v)}], \quad v \ge 0.$$

With the total probability law and due to the Markov property of τ_x, τ^x, $x \ge 0$, the introduced generating functions satisfy the following system of the equations

$$
\begin{aligned}
A_v(s,\theta) &= 1 - E[\,e^{-s\tau^v}\,] + \int_0^\infty E\,[\,e^{-s\tau^v};\; T^v \in dl\,]\, A^l(s,\theta), \qquad v \ge 0,\\
A^v(s,\theta) &= 1 - E\,e^{-s\tau_v} + \theta \int_0^B E\,[\,e^{-s\tau_v};\; T_v \in dl\,]\, A_l(s,\theta) \\
&\qquad + \int_B^\infty E\,[\,e^{-s\tau_v};\; T_v \in dl\,]\, A_l(s,\theta), \qquad v > 0.
\end{aligned} \tag{27}
$$

Substituting the expression for the function $A^v(s,\theta)$, $v>0$, from the second equation of the system into the first one we obtain the equation with respect to $A_v(s,\theta)$, $v \ge 0$

$$
\begin{aligned}
A_v(s,\theta) = 1 &- \int_0^\infty E\,[\,e^{-s\tau^v};\; T^v \in dl\,]\, E\,e^{-s\tau_l} \\
&+ \theta \int_0^\infty E\,[\,e^{-s\tau^v};\; T^v \in dl\,] \int_0^B E\,[\,e^{-s\tau_l};\; T_l \in d\nu\,]\, A_\nu(s,\theta) \\
&+ \int_0^\infty E\,[\,e^{-s\tau^v};\; T^v \in dl\,] \int_B^\infty E\,[\,e^{-s\tau_l};\; T_l \in d\nu\,]\, A_\nu(s,\theta), \qquad v \ge 0,
\end{aligned}
$$

which is the linear integral equation with two kernels. In general it is not obvious how to solve it, but the nice features of the Poisson process with negative exponential component suggest a straightforward solution. Substituting the integral transform of $\{\tau_x,\, T_x\}$ from the first formula of (10) into this equation and utilizing the function $E_x^s(c(s))$ defined by (25) yields

$$
A_v(s,\theta) = 1 - E_v^s(c(s))\left(1 - \theta\,\check{A}^s(\lambda,\theta) - \hat{A}^s(\lambda,\theta)\right), \qquad v \ge 0, \tag{28}
$$

where

$$
\check{A}^s(\lambda,\theta) = \int_0^B \lambda\, e^{-\lambda v}\, A_v(s,\theta)\, dv, \qquad \hat{A}^s(\lambda,\theta) = \int_B^\infty \lambda\, e^{-\lambda v}\, A_v(s,\theta)\, dv.
$$

Multiplying (28) by $\lambda\, e^{-\lambda v}$, and integrating this equality with respect to $v \ge 0$, implies

$$
\begin{aligned}
\check{A}^s(\lambda,\theta) &= 1 - e^{-\lambda B} - \check{E}^s(\lambda, c(s))\left(1 - \theta\,\check{A}^s(\lambda,\theta) - \hat{A}^s(\lambda,\theta)\right),\\
\hat{A}^s(\lambda,\theta) &= e^{-\lambda B} - \hat{E}^s(\lambda, c(s))\left(1 - \theta\,\check{A}^s(\lambda,\theta) - \hat{A}^s(\lambda,\theta)\right),
\end{aligned}
$$

which is the system of two linear equations with respect to unknown functions $\hat{A}^s(\lambda,\theta)$, $\check{A}^s(\lambda,\theta)$, where

$$
\check{E}^s(\lambda, c(s)) = \lambda \int_0^B e^{-\lambda x} E_x^s(c(s))\, dx, \qquad \hat{E}^s(\lambda, c(s)) = \lambda \int_B^\infty e^{-\lambda x} E_x^s(c(s))\, dx,
$$

Solving this system we find

$$\theta\, \check{A}^{s}(\lambda, \theta) + \hat{A}^{s}(\lambda, \theta) = 1 - \frac{(1-\theta)(1-e^{-\lambda B})}{1-\theta\, \check{E}^{s}(\lambda, c(s)) - \hat{E}^{s}(\lambda, c(s))}.$$

Substituting the right-hand side of the latter expression into (28), yields

$$A_v(s, \theta) = 1 - \frac{(1-\theta)(1-e^{-\lambda B})}{1-\theta\, \check{E}^{s}(\lambda, c(s)) - \hat{E}^{s}(\lambda, c(s))}\; E_v^{s}(c(s)), \qquad v \geq 0.$$

Now, substituting the found expressions for the generating function $A_v(s,\theta)$ into the second equation of the system (27) implies

$$A^{v}(s, \theta) = 1 - \frac{(1-\theta)(1-e^{-\lambda B})}{1-\theta\, \check{E}^{s}(\lambda, c(s)) - \hat{E}^{s}(\lambda, c(s))} \left(1 - \frac{c(s)}{\lambda}\right) e^{-vc(s)}, \qquad v > 0.$$

Taking into account (25), from these two formulae we obtain the first equality of the corollary. Comparing the coefficients of θ^{n}, $n \in \mathbb{N} \cup 0$ in this equality yields the distribution of the random variable $\beta_{\nu_s}^{+}(y)$, $y \in \mathbb{R}$, and the second equality of the corollary. $\square$

Now we determine the distribution of the number of the entrances into the interval through the lower boundary. Let $B > 0$ be fixed, $B_- = (-\infty, 0)$, and for all $y \in \mathbb{R}$, $n \in \mathbb{N} \cup 0$ we define the sequence

$$\overline{\chi}_0^{-}(y) = 0, \qquad \overline{\chi}_{n+1}^{-}(y) = \inf\{\, t > \overline{\chi}_n^{-}(y) :\ y + \xi(t-0) \in B_-,\, y + \xi(t) \in [0, B]\,\}$$

of the epochs of the entrances into the interval $[0, B]$ through the lower boundary 0 (from the set B_-) by the process $y + \xi(\cdot)$. On sample paths of the process for which there exist $n_0 \in \mathbb{N} \cup 0$ such that the set in the braces is empty, we set per definition $\overline{\chi}_n^{-}(y) = \infty$ for all $n \geq n_0$. Introduce the random variable

$$\beta_t^{-}(y) = \max\{\, n \in \mathbb{N} \cup 0 :\ \overline{\chi}_n^{-}(y) \leq t\,\}, \qquad y \in R, \quad t > 0$$

the number of the entrances of the interval $y + \xi(\cdot)$ through the lower boundary 0 (from the set B_-) by the process $y+\xi(\cdot)$ on the time interval $[0,t]$. Denote

$$\check{m}_\gamma^s = \int_0^\infty \lambda\, e^{-\lambda x}\, E\,[\, e^{-s\tau^x};\; T^x \in [0,B]\,]\, dx, \qquad \hat{m}_\gamma^s = \int_0^\infty \lambda\, e^{-\lambda x}\, E\,[\, e^{-s\tau^x};\; T^x > B\,]\, dx,$$
$$\check{M}_x^s(c(s)) = \left(1 - \frac{c(s)}{\lambda}\right) E\,[\, e^{-s\tau^x}\, e^{-c(s)T^x};\; T^x \in [0,B]\,], \qquad \check{M}^{\,s}(\lambda) = \int_0^\infty \lambda\, e^{-\lambda x}\, \check{M}_x^s(c(s))\, dx,$$
$$\hat{M}_x^s(c(s)) = \left(1 - \frac{c(s)}{\lambda}\right) E\,[\, e^{-s\tau^x}\, e^{-c(s)T^x};\; T^x > B\,], \qquad \hat{M}^{\,s}(\lambda) = \int_0^\infty \lambda\, e^{-\lambda x}\, \hat{M}_x^s(c(s))\, dx. \tag{29}$$

Corollary 3. *Let* $\{\xi(t);\, t \geq 0\}$, $\xi(0)=0$ *be a Poisson process with a negative exponential component as before. Then the generating function of the number of the upward entrances* $\beta_{\nu_s}^-(y)$, $y \in \mathbb{R}$ *into the interval* $[0,B]$ *by the process* $y+\xi(\cdot)$ *up to the moment* ν_s *for* $s>0$ *satisfies the equalities*

$$E[\,\theta^{\beta_{\nu_s}^-(y)}] = 1 - \frac{(1-\theta)\,\check{m}_\gamma^s}{1-\hat{M}^{\,s}(\lambda) - \theta\,\check{M}^{\,s}(\lambda)} \left(1 - \frac{c(s)}{\lambda}\right) e^{-yc(s)}, \qquad y \geq 0,$$
$$E[\,\theta^{\beta_{\nu_s}^-(-y)}] = 1 - (1-\theta)\,\check{m}_y^s - \frac{(1-\theta)\,\check{m}_\gamma^s}{1-\hat{M}^{\,s}(\lambda) - \theta\,\check{M}^{\,s}(\lambda)} \left(\hat{M}_y^s(c(s)) + \theta\,\check{M}_y^s(c(s))\right), \qquad y > 0.$$

In particular, for $y \geq 0$

$$P[\,\beta_{\nu_s}^-(y) = n\,] = I_{\{n=0\}} \left(1 - \frac{\check{m}_\gamma^s}{1-\hat{M}^{\,s}(\lambda)} \left(1 - \frac{c(s)}{\lambda}\right) e^{-yc(s)}\right)$$
$$+ I_{\{n\in\mathbb{N}\}} \frac{\check{m}_\gamma^s}{1-\hat{M}^{\,s}(\lambda)} \frac{1-E^{\,s}(\lambda)}{1-\hat{M}^{\,s}(\lambda)} \left(\frac{\check{M}^{\,s}(\lambda)}{1-\hat{M}^{\,s}(\lambda)}\right)^{n-1} \left(1 - \frac{c(s)}{\lambda}\right) e^{-yc(s)},$$

where $M^{\,s}(\lambda) = \hat{M}^{\,s}(\lambda) + \check{M}^{\,s}(\lambda)$.

Proof. Introduce

$$B^{\,v}(s,\theta) = E[\,\theta^{\beta_{\nu_s}^-(v)}], \quad v \geq 0, \qquad B_v(s,\theta) = E[\,\theta^{\beta_{\nu_s}^-(-v)}], \quad v > 0.$$

For the introduced generating functions, according to the total probability law and the Markov property of τ_x, τ^x, $x \geq 0$, we write

$$B^{\,v}(s,\theta) = 1 - E\,[e^{-s\tau_v}] + \int_0^\infty E\,[\, e^{-s\tau_v};\; T_v \in dl\,]\, B_l(s,\theta), \qquad v \geq 0,$$
$$B_v(s,\theta) = 1 - E\,[e^{-s\tau^{\,v}}] + \theta \int_0^B E\,[\, e^{-s\tau^{\,v}};\; T^{\,v} \in dl\,]\, B^{\,l}(s,\theta)$$
$$+ \int_B^\infty E\,[\, e^{-s\tau^{\,v}};\; T^{\,v} \in dl\,]\, B^{\,l}(s,\theta), \qquad v > 0.$$

Substituting the expressions for the joint distribution $\{\tau_v, T_v\}$ from the first formula of (10) in these equations implies

$$\begin{aligned} B^{\,v}(s,\theta) &= 1 - \left(1 - \frac{c(s)}{\lambda}\right) e^{-vc(s)} + B(\lambda, s) \left(1 - \frac{c(s)}{\lambda}\right) e^{-vc(s)} && v \geq 0, \\ B_v(s,\theta) &= 1 - m_v^s + \theta \int_0^B m_v^s(dl)\, B^{\,l}(s,\theta) + \int_B^\infty m_v^s(dl)\, B^{\,l}(s,\theta), && v > 0. \end{aligned} \tag{30}$$

where $B(\lambda, s) = \int_0^\infty \lambda e^{-\lambda x} B_x(s,\theta)\, dx$. Now if we determine the function $B(\lambda, s)$ then the functions $B^{\,v}(s,\theta)$, $B_v(s,\theta)$ will be determined by the equalities (30). Let us find the function $B(\lambda, s)$. Substituting into the second equality of (30) the expression for the function $B^{\,v}(s,\theta)$ from the first equality implies

$$B_v(s,\theta) = 1 - (1-\theta)\check{m}_v^s - \hat{M}_v^s(c(s)) - \theta\, \check{M}_v^s(c(s) + B(\lambda, s) \left(\hat{M}_v^s(c(s)) - \theta\, \check{M}_v^s(c(s)\right), \tag{31}$$

where the functions $\check{M}_v^s(c(s))$, $\hat{M}_v^s(c(s))$ are given by the formulae (29). Multiplying the both sides of the latter equality by $\lambda e^{-\lambda v}$, and integrating it with respect to $v \geq 0$, yields the following equation

$$B(\lambda, s) = 1 - (1-\theta)\check{m}_\gamma^s - \hat{M}^{\,s}(\lambda) - \theta\, \check{M}^{\,s}(\lambda) + B(\lambda, s) \left(\hat{M}^{\,s}(\lambda) - \theta\, \check{M}^{\,s}(\lambda)\right),$$

which is a linear equation with respect to $B(\lambda, s)$. Solving it yields

$$B(\lambda, s) = 1 - \frac{(1-\theta)\, \check{m}_\gamma^s}{1 - \hat{M}^{\,s}(\lambda) - \theta\, \check{M}^{\,s}(\lambda)}, \qquad \theta \in [0,1]. \tag{32}$$

Substituting the right-hand side of this formula into the first equality of (30), we obtain

$$B^{\,v}(s,\theta) = 1 - \frac{(1-\theta)\, \check{m}_\gamma^s}{1 - \hat{M}^{\,s}(\lambda) - \theta\, \check{M}^{\,s}(\lambda)} \left(1 - \frac{c(s)}{\lambda}\right) e^{-vc(s)}, \qquad v \geq 0, \tag{33}$$

the function $B^{\,v}(s,\theta)$, $v \geq 0$, and the first equality of the corollary. Substituting the right-hand side of the formula (32) into (31) implies

$$B_v(s,\theta) = 1 - (1-\theta)\, \check{m}_v^s - \frac{(1-\theta)\, \check{m}_\gamma^s}{1 - \hat{M}^{\,s}(\lambda) - \theta\, \check{M}^{\,s}(\lambda)} \left(\hat{M}_v^s(c(s)) + \theta\, \check{M}_v^s(c(s))\right), \qquad v > 0,$$

the expression for the function $B_v(s,\theta)$, $v > 0$, and the second equality of the corollary. Comparing the coefficients of $\theta^{\,n}$, $n \in \mathbb{N} \cup 0$ in the both sides of (33) we obtain the distribution of the random variable $\beta^-_{\nu_s}(v), \quad v \geq 0,$ and the third equality of the corollary. $\square$

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