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FRACTAL ANALYSIS OF HYPERBOLIC SADDLES WITH APPLICATIONS

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Abstract. In this paper we express the Minkowski dimension of spiral trajectories near hyperbolic saddles and semi-hyperbolic singularities in terms of the dimension of intersections of such spirals with transversals near these singularities. We apply these results to hyperbolic saddle-loops and hyperbolic 2-cycles to obtain, using Minkowski dimension of a single spiral trajectory, some known upper bounds on the cyclicity of such limit periodic sets.

1. Introduction

The Minkowski dimension is a fractal dimension that quantifies how the Lebesgue measure of the δ-neighborhood of a bounded set in $\mathbb{R}^N$ behaves as $\delta \to 0$. There are several equivalent ways of calculating this dimension but we mostly use the following one:

Definition 1 ([4])**.** *Let G be a bounded set in N-dimensional Euclidean space $\mathbb{R}^N$. Let*

$$G_\delta :=\{p \in \mathbb{R}^N : dist(p, G) < \delta\}, \delta > 0,$$

be the δ-neighborhood of G and let $|G_\delta|$ be its Lebesgue measure.
The upper and the lower Minkowski dimension of set G are defined as the limits

$$\overline{dim}_B\, G = \limsup_{\delta\to 0}\left[N - \frac{\ln|G_\delta|}{\ln\delta}\right] \quad and \quad \underline{dim}_B\, G = \liminf_{\delta\to 0}\left[N - \frac{\ln|G_\delta|}{\ln\delta}\right]$$

respectively. If these two values are equal, the common value is called the Minkowski dimension of set G and denoted by $\dim_B G$.

The Minkowski dimensions are preserved under bi-Lipschitz transformations, even when the image and the original are not in the same ambient space. More precisely, a transformation $\Psi : A \subset \mathbb{R}^N \to \mathbb{R}^K$ is called bi-Lipschitz if there exist positive constants m and M such that, for any $x, y \in A$,

$$m||\Psi(x) - \Psi(y)|| \le ||x - y|| \le M||\Psi(x) - \Psi(y)||.$$

For a bi-Lipschitz map $\Psi : A \subset \mathbb{R}^N \to \mathbb{R}^K$ we have that

$$\underline{\dim}_B A = \underline{\dim}_B \Psi(A) \quad \text{and} \quad \overline{\dim}_B A = \overline{\dim}_B \Psi(A).$$

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All three Minkowski dimensions are monotone in the sense that, for $G \subseteq H, \dim G \leq \dim H$, when both are defined. In addition, the Minkowski dimension and the upper Minkowski dimension are finitely stable ($\dim(G \cup H) = \max\{\dim G, \dim H\}$). For more on these and other properties of the Minkowski dimension we refer the reader to [4, 13].

It is already known that the Minkowski dimension of spirals around weak foci and limit cycles of planar analytic vector fields yields information on the cyclicity of those limit periodic sets. First results of this type were obtained in [15]. We state two main theorems that describe such connections.

Theorem 1 (Weak focus case, [15, 14]). *Let Γ be a spiral trajectory of the system*

$$\begin{cases} \dot{r} = r(r^{2l} + \sum_{i=0}^{l-1} a_i r^{2i}) \\ \dot{\phi} = 1 \end{cases}$$

near the origin. Then

(*a*) *if $a_0 \neq 0$, then* $\dim_B \Gamma = 1$.
(*b*) *if $a_0 = a_1 = ... = a_{k-1} = 0,\ a_k \neq 0,\ k \geq 1$, then* $\dim_B \Gamma = \frac{4k}{2k+1}$.

Theorem 2 (Limit cycle case, [15, 14]). *Let the system*

$$\begin{cases} \dot{r} = r(r^{2l} + \sum_{i=0}^{l-1} a_i r^{2i}) \\ \dot{\phi} = 1 \end{cases}$$

have a limit cycle $r = a$ of multiplicity $m,\ 1 \leq m \leq l$. Let Γ_1 and Γ_2 be spiral trajectories of this system near the limit cycle from outside or inside respectively. Then $\dim_B \Gamma_1 = \dim_B \Gamma_2 = 2 - \frac{1}{m}$.

Due to the Flow-Box Theorem (see for instance [2, Theorem 1.12]), in order to calculate the dimension of spiral trajectories near limit cycles, it is sufficient to calculate the dimension of a sequence of points obtained by intersecting any such spiral with a transversal to the limit cycle (i.e. of the orbit of the first-return map on the transversal). For the case of foci, there is a variant of the Flow-Box Theorem developped in [14], called *Flow-Sector Theorem*, that allows similar relation. For more details, see [14].

In addition, due to results from [3], there is a direct correspondence between the multiplicity of a fixed point of a line diffeomorphism (the first return map) and the Minkowski dimension of its orbit converging to the fixed point and, as a consequence, with the cyclicity of a focus/limit cycle.

In this paper we deal with spiral trajectories near more complex limit periodic sets: polycycles containing saddles and/or saddle-nodes. For a hyperbolic saddle with eigenvalues $\lambda_- < 0$ and $\lambda_+ > 0$, the hyperbolicity ratio is the quantity $r = -\frac{\lambda_-}{\lambda_+} > 0$. We read an upper bound on cyclicity of those sets in some known cases from the Minkowski dimension of their spiral trajectories. A better understanding of the cyclicity of such limit periodic sets is crucial for tackling the Hilbert's 16th problem. It is not possible to use the Flow-Box Theorem to calculate the Minkowski dimension of spiral trajectories accumulating on such polycycles (from within), because the theorem does not apply near singularities, so we need a new method to calculate the Minkowski dimension of parts of the trajectories near singular points. Our main results are stated in Theorem 3 and Theorem 4 of Section 2 which deal with neighborhoods of a hyperbolic saddle and a semi-hyperbolic singularity respectively.

In Section 3 we apply Theorem 3 to a saddle-loop and find a relation between the codimension of the saddle-loop (an upper bound on the cyclicity of the loop) and the Minkowski dimension of its spiral trajectories. The Minkowski dimension depends only on the codimension of the loop, but the correspondence is 2–1. For a more precise formulation, see Theorem 5.

Finally, in Section 4, we apply Theorem 3 to a hyperbolic 2-cycle, and compare to cyclicity results obtained in [11]. To summarize, for a non-resonant hyperbolic 2-cycle with ratios of hyperbolicity $r_1 < 1 < r_2$ such that $r_1 r_2 = 1$ and $r_1, r_2 \notin \mathbb{Q}$, the cyclicity of the 2-cycle is shown to be at most $\left\lfloor 3 + (1 + r_1)\frac{d-1}{2-d} \right\rfloor$, where d is the Minkowski dimension of any spiral trajectory near the 2-cycle.

In the sequel we use two notions for the asymptotic behavior of functions as $x \to 0$. For $f(x)$ and $g(x)$ two positive functions with $x \approx 0$ and $x > 0$, we write

$$f(x) \simeq g(x),\ x \to 0,$$

if there exist two positive constants m and M such that $mg(x) \leq f(x) \leq Mg(x)$ for all x sufficiently small. For $f(x)$ and $g(x)$ two positive functions with $x \approx 0$ and $x > 0$, we write

$$f(x) \sim g(x),\ x \to 0,$$

if

$$\lim_{x \to 0+} \frac{f(x)}{g(x)} = 1.$$

2. The main results

Let us explain the basic idea behind our method for calculating the Minkowski dimension of spiral trajectories of a polycycle. Due to the finite stability of the Minkowski dimension, in order to compute the dimension of a spiral trajectory, we consider separately different parts of the spiral: parts near the singular points and parts near the regular sides of the polycycle. The Minkowski dimension of the entire spiral is the maximal dimension of its constituting parts.

The Flow-Box Theorem allows us to calculate the dimension of parts near the regular sides of the polycycle, but we need a new tool to calculate the dimension of the remaining parts. For any transversal to a regular side of a polycycle, the points of intersection of the spiral with the transversal define a sequence $(y_n)_n$. The distance between consecutive points of the sequence eventually starts to decrease. We take two transversals, one on each side of the singular point and sufficiently close to the singular point, in the domain where the saddle can be brought to a simpler normal form (see the proofs of Theorems 3 and 4 in Section 2). Without loss of generality, in the normal form coordinates we assume the transversals $\{x = 1\}$ (vertical) and $\{y = 1\}$ (horizontal), and compute the dimension of the family of curves passing through a given sequence of points on the entry transversal and ending on exit transversal.

Note that planar saddles and saddle-nodes, unlike planar foci or complex saddles, are not monodromic points, so that the first return map around the saddle/saddle-node point is not well-defined before we close the connection (as in the polycyle). Therefore, our first results in Theorems 3 and 4 concern the Minkowski dimension of a union of disjoint local trajectories in the neighborhood of the saddle/saddle-node singularity that correspond to (any) prescribed sequence of points on the entry transversal. The expression for the Minkowski dimension of spiral trajectories accumulating on a polycycle is provided in Corollary 1.

Let s be a hyperbolic saddle or a semi-hyperbolic singularity of an analytic vector field. By t_S and t_U we denote the transversals to the stable and the unstable manifold of s in the saddle case, i.e. the transversals to the stable (up to the time reversal preserving the geometry of the flow) and the center manifold in the semi-hyperbolic case (in the saddle region). We call t_S also the *entry* and t_U the *exit* transversal, due to obvious reasons. Up to the reversal of the time, without loss of generality we assume that the ratio of hyperbolicity of the saddle is greater than 1 in the hyperbolic saddle case. Up to the change of the axes, we additionally assume that the stable, i.e. entry, transversal is the vertical transversal $\{x = 1\}$.

We take $(y_n)_n$ to be any sequence on t_S that converges monotonically to the intersection of t_S with the stable manifold, and such that the distances between consecutive points y_n decrease monotonically (see Figure 1). Let $(x_n)_n$ be the sequence of points where the trajectories $(\Gamma_n)_n$ of the vector field of the saddle/saddle-node s going through $(y_n)_n$ intersect t_U. Under the above assumptions it is not hard to see that $\dim_B (y_n)_n \geq \dim_B (x_n)_n$. We show that the Minkowski dimension of the union of trajectories $(\Gamma_n)_n$ between the points y_n and x_n is

$$\dim_B(\cup_n \Gamma_n) = 1 + \dim_B (y_n)_n.$$

Theorem 3 (Minkowski dimension of the hyperbolic saddle)**.** *Let $s = 0$ be a hyperbolic saddle of an analytic vector field with ratio of hyperbolicity $\frac{1}{\alpha} \geq 1$:*

$$\begin{cases} x' = -x + h.o.t., \\ y' = \alpha y + h.o.t. \end{cases}$$

Let t_S, t_U and $(y_n)_{n\in\mathbb{N}}$ on t_S be defined as above. If the sequence $(y_n)_{n\in\mathbb{N}}$ has Minkowski dimension, $\dim_B (y_n)_n$*, then*

$$\dim_B\ (\cup_{n\in\mathbb{N}}\Gamma_n) = 1 + \dim_B(y_n)_n.$$

Theorem 4 (Minkowski dimension of the semi-hyperbolic singularity)**.** *Let $s = 0$ be a semi-hyperbolic singularity of an analytic vector field:*

$$\begin{cases} x' = -x + h.o.t., \\ y' = \alpha y^m + h.o.t.,\ \alpha > 0,\ m \geq 2. \end{cases}$$

Let t_S, t_U and $(y_n)_{n\in\mathbb{N}}$ on t_S be defined as above. If the sequence $(y_n)_{n\in\mathbb{N}}$ has the Minkowski dimension, $\dim_B (y_n)_n$*, then*

$$\dim_B\ (\cup_{n\in\mathbb{N}}\Gamma_n) = 1 + \dim_B(y_n)_n.$$

In Propositions 1 and 2, to fix the ideas, we first prove the weaker versions of Theorems 3 and 4 for linear saddles:

$$\begin{cases} \dot{x} = -x \\ \dot{y} = \alpha y, \quad 0 < \alpha \leq 1, \end{cases} , \tag{1}$$

and the simplest semi-hyperbolic singularities

$$\begin{cases} \dot{x} = -x \\ \dot{y} = \alpha y^m, \quad m \geq 2,\ \alpha > 0. \end{cases} \tag{2}$$

Proposition 1. *For a linear saddle* (1)*, under notation of Theorem 3, it holds that:*

$$\dim_B(\cup_{n\in\mathbb{N}}\Gamma_n) = 1 + \dim_B(y_n)_n.$$

Proof. Since bi-Lipschitz transformations do not change the Minkowski dimensions, using a rescaling in x and y we may assume that $t_S = \{x = 1\}$ and $t_U = \{y = 1\}$. The rescaled sequence on $t_S = \{x = 1\}$ satisfies the same assumptions as the original one, and we use the same notation $(y_n)_{n\in\mathbb{N}}$. We first present the standard computation of the Minkowski dimension of the sequence $(y_n)_{n\in\mathbb{N}}$ in one-dimensional ambient space $\mathbb{R}$. Let us denote $Y := \{y_n : n \in \mathbb{N}\}$. For $\delta > 0$ small enough, there is a unique critical index n_δ such that $y_{n_\delta} - y_{n_\delta+1} < 2\delta$ and $y_n - y_{n+1} \geq 2\delta$, for all $n < n_\delta$. We now divide the δ-neighborhood Y_δ into two parts. The δ-neighborhoods of points $y_1, y_2, ..., y_{n_\delta-1}$ do not intersect, and we call their union the tail of Y_δ and denote it by T_δ. On the other hand, the δ-neighborhood of the remainder of the sequence is the interval $(-\delta, y_{n_\delta} + \delta)$, and it is refered to as the nucleus of Y_δ and denoted by N_δ (see e.g.[13]). Note that N_δ and T_δ are disjoint. The Lebesgue measure of Y_δ is now equal to:

$$|Y_\delta| = |N_\delta| + |T_\delta| = (y_{n_\delta} + 2\delta) + (n_\delta - 1)\cdot 2\delta = y_{n_\delta} + 2\delta n_\delta.$$

Since, by our assumptions, Y has Minkowski dimension, by definition of Minkowski dimension it holds that

$$\dim_B Y = \lim_{\delta\to 0}\left[1 - \frac{\ln(y_{n_\delta} + 2\delta n_\delta)}{\ln\delta}\right].$$

Let

$$\Gamma := \{\Gamma_n : n \in \mathbb{N}\}.$$

Let us first consider the part of the set Γ of trajectories in the region $\{\frac{1}{2} \leq x \leq 1\}$. Since the Minkowski dimension of the Cartesian product is the sum of Minkowski dimensions, in this region the Flow-Box Theorem allows us to easily calculate the Minkowski dimension to be $1 + \dim_B Y$. Due to monotonicity of $\underline{\dim}_B$, we now have the lower bound

$$1 + \dim_B Y \leq \underline{\dim}_B\Gamma.$$

Therefore, to prove the proposition, it suffices to show that

$$\overline{\dim}_B\Gamma \leq 1 + \dim_B Y. \tag{3}$$

It can be shown that there exists a positive constant $C > 0$ such that, for any $n \in \mathbb{N}$, the Lebesgue measure of $(\Gamma_n)_\delta$ is bounded from above by $C\delta$, as $\delta \to 0$. This bound is uniform both in $\delta \in (0, \delta_0)$, $\delta_0 > 0$, and in $n \in \mathbb{N}$. Therefore, for a given $\delta > 0$, we bound the Lebesgue measure of the δ-neighborhood of the union of trajectories $\cup_{n<n_\delta}\Gamma_n$, arising from the tail of Y_δ, from above by

$$|\cup_{n<n_\delta}(\Gamma_n)_\delta| \leq C\delta(n_\delta - 1). \tag{4}$$

On the other hand, the Lebesgue measure of the δ-neighborhood of the union of trajectories $\cup_{n\geq n_\delta}\Gamma_n$, arising from the nucleus of Y_δ, is bounded from above by

$$|\cup_{n\geq n_\delta}(\Gamma_n)_\delta| \leq y_{n_\delta} + \int_{y_{n_\delta}}^1 x(y)dy + D\delta, \tag{5}$$

where D is a universal positive constant and $y \mapsto x(y)$ is a function whose graph is the curve Γ_{n_δ}. For more details, see Figure 1.

To prove inequality (3), we consider separately two cases.

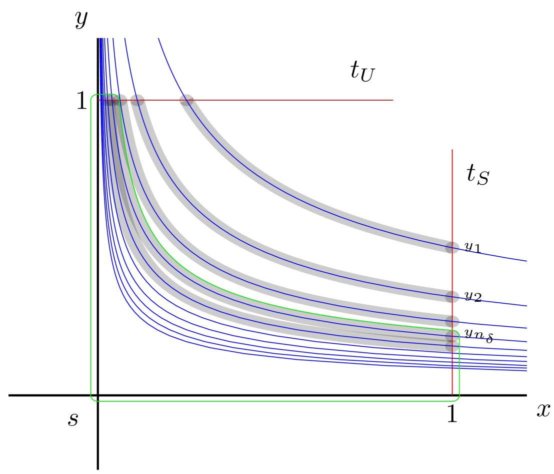

FIGURE 1. δ-neighborhood of Γ

(1) Case: $0 < \alpha < 1$.
By direct integration of the simple vector field (1), we get

$$x(y) = \left(\frac{y_{n_\delta}}{y}\right)^{\frac{1}{\alpha}},$$

so

$$\int_{y_{n_\delta}}^{1} x(y)dy = \frac{\alpha}{1-\alpha}(y_{n_\delta} - y_{n_\delta}^{\frac{1}{\alpha}}). \tag{6}$$

Therefore, using (4), (5) and (6), there exists a universal positive constant $M > 0$ such that

$$|\Gamma_\delta| \leq M(y_{n_\delta} + 2\delta n_\delta),\ \delta > 0.$$

Finally,

$$\begin{aligned}\limsup_{\delta\to 0}\left[2 - \frac{\ln|\Gamma_\delta|}{\ln\delta}\right] &\leq \limsup_{\delta\to 0}\left[2 - \frac{\ln M + \ln(y_{n_\delta} + 2\delta n_\delta)}{\ln\delta}\right] \\ &= \limsup_{\delta\to 0}\left[2 - \frac{\ln(y_{n_\delta} + 2\delta n_\delta)}{\ln\delta}\right] = 1 + \dim_B Y.\end{aligned}$$

Therefore, $\overline{\dim}_B\Gamma \leq 1 + \dim_B Y$.

(2) Case: $\alpha = 1$.
Note that formula (6) cannot be used. However, similarly as in Case 1, by direct integration we get:

$$x(y) = \frac{y_{n_\delta}}{y},\quad \int_{y_{n_\delta}}^{1} x(y)dy = y_{n_\delta}(-\ln y_{n_\delta}).$$

Therefore, there exists $M > 0$ such that: $|\Gamma_\delta| \leq M(y_{n_\delta}(-\ln y_{n_\delta}) + 2\delta n_\delta)$. Now, since both $y_{n_\delta} \to 0$ and $\delta n_\delta \to 0$, as $\delta \to 0$, for every small $\kappa > 0$ there exists δ_κ, such that, for every $0 < \delta < \delta_\kappa$, it holds that

$$y_{n_\delta}(-\ln y_{n_\delta}) < y_{n_\delta}^{1-\kappa},\ 2\delta n_\delta < (2\delta n_\delta)^{1-\kappa},\ \delta < \delta_\kappa.$$

Therefore,

$$|\Gamma_\delta| \le 2M \max\{y_{n_\delta}, 2\delta n_\delta\}^{1-\kappa},\ \ \delta < \delta_\kappa.$$

Now, for every $\kappa > 0$,

$$\limsup_{\delta\to 0}\left[2 - \frac{\ln|\Gamma_\delta|}{\ln\delta}\right] \le \limsup_{\delta\to 0}\left[2 - \frac{\ln(2M) + (1-\kappa)\ln\max\{y_{n_\delta}, 2\delta n_\delta\}}{\ln\delta}\right]$$
$$\le \limsup_{\delta\to 0}\left[2 - (1-\kappa)\frac{\ln(y_{n_\delta} + 2\delta n_\delta)}{\ln\delta}\right] = 1 + \kappa + (1-\kappa)\dim_B Y.$$

Letting $\kappa \to 0$, we get

$$\limsup_{\delta\to 0}\left[2 - \frac{\ln|\Gamma_\delta|}{\ln\delta}\right] \le 1 + \dim_B Y.$$

Finally, since

$$1 + \dim_B Y \le \underline{\dim}_B\Gamma \le \overline{\dim}_B\Gamma \le 1 + \dim_B Y,$$

we conclude that

$$\dim_B \Gamma = 1 + \dim_B Y.$$

□

Proposition 2. *For a semi-hyperbolic singularity* (2)*, under notation of Theorem 4, it holds that:*

$$\dim_B(\cup_{n\in\mathbb{N}}\Gamma_n) = 1 + \dim_B (y_n)_n.$$

Proof. We use a similar nucleus-tail approach from the proof of Lemma 1. Again, rescaling x and y, we assume $t_S = \{x = 1\}$ and $t_U = \{y = 1\}$. This changes the constant α in (2), but α remains positive. Again we denote $Y := \{y_n : n \in \mathbb{N}\}$ and $\Gamma = \{\Gamma_n : n \in \mathbb{N}\}$. As in the proof of Lemma 1, $\underline{\dim}_B\Gamma \ge 1 + \dim_B Y$. Again, there exist uniform positive constants C and D such that same bounds (4) and (5) hold. Let us show that the integral in (5) satisfies

$$\int_{y_{n_\delta}}^1 x(y)dy = o(y_{n_\delta}),\ \ \delta \to 0, \tag{7}$$

where $x(y) = \exp\left(\frac{y^{1-m} - y_{n_\delta}^{1-m}}{\alpha(m-1)}\right)$ is the function whose graph is the trajectory Γ_{n_δ} through $(1, y_{n_\delta})$. To prove this it suffices to show that the function

$$y \mapsto \int_y^1 \exp\left(\frac{t^{1-m} - y^{1-m}}{\alpha(m-1)}\right) dt$$

is $o(y)$, as $y \to 0$. Indeed, we have that

$$\lim_{y\to 0}\frac{\int_y^1 \exp\left(\frac{t^{1-m}-y^{1-m}}{\alpha(m-1)}\right)dt}{y} = \lim_{y\to 0}\frac{\int_y^1 \exp\left(\frac{t^{1-m}}{\alpha(m-1)}\right)dt}{y\exp\left(\frac{y^{1-m}}{\alpha(m-1)}\right)} =$$
$$= \lim_{y\to 0}\frac{-\exp\left(\frac{y^{1-m}}{\alpha(m-1)}\right)}{(1 - \frac{y^{1-m}}{\alpha})\exp\left(\frac{y^{1-m}}{\alpha(m-1)}\right)} = 0,$$

where we used the L'Hospital's rule in the second step. Using (7) as in Lemma 1, we get that $|\Gamma_\delta| \le M(y_{n_\delta} + 2\delta n_\delta)$ for some positive constant M. Consequently, $\overline{\dim}_B\Gamma \le 1 + \dim_B Y$. □

Proof of Theorem 3. Consider an analytic vector field with a hyperbolic saddle, and let t_S and t_U be as in the statement of the theorem. Following [2, Theorem 2.15], near the hyperbolic saddle the field can be reduced to the following (smooth) orbital normal form:

$$\begin{cases} \dot{x} = -x \\ \dot{y} = \alpha y + h(x,y), \end{cases} \tag{8}$$

where $\frac{1}{\alpha} \geq 1$ is the hyperbolicity ratio of the saddle and $h(x,y) = O(xy^2)$, $(x,y) \to (0,0)$, is a C^∞ function. Since a smooth local change of coordinates is bi-Lipschitz, it preserves the Minkowski dimension around 0. Therefore, it suffices to compute Minkowski dimension of the saddle in the normal form (8).

For an arbitrary $\beta > 0$, we choose a small enough neighborhood of the saddle such that $|\alpha y + h(x,y)| \leq (1+\beta)y$. Now we choose new transversals t'_S and t'_U that intersect the stable and unstable manifold respectively in this neighborhood. Up to a rescaling, we may assume that $t'_S = \{x = 1\}$ and $t'_U = \{y = 1\}$. Due to the Flow-Box Theorem, the maximal Minkowski dimension of parts of $\Gamma = \cup_{n\in\mathbb{N}} \Gamma_n$ between t_S and t'_S, t_U and t'_U, as well as of those around t'_U and t'_S, is equal to $1 + \dim_B Y$. Therefore,

$$\underline{\dim}_B \Gamma \geq 1 + \dim_B Y. \tag{9}$$

To prove the theorem, it suffices to show that, for every $\beta > 0$,

$$\overline{\dim}_B \Gamma' \leq \frac{2\beta + 1 + \dim_B Y}{1+\beta}, \tag{10}$$

where Γ' is the part of Γ between t'_S and t'_U. Evidently, $\overline{\dim}_B \Gamma' = \overline{\dim}_B \Gamma$. Passing to limit as $\beta \to 0$, we get:

$$\overline{\dim}_B \Gamma \leq 1 + \dim_B Y,$$

which, along with (9), concludes the proof of the theorem.

Let us now prove (10). For $(u,v) \in \Gamma_n$ we have that

$$-\ln u = \int_1^u -\frac{dx}{x} = \int_{y_n}^v \frac{dy}{\alpha y + h(x,y)} \geq \int_{y_n}^v \frac{dy}{(1+\beta)y},$$

that is,

$$u \leq \left(\frac{y_n}{v}\right)^{\frac{1}{1+\beta}}.$$

Similarly as in the proof of Proposition 1, the Lebesgue measure of the δ-neighborhood of trajectories Γ_n arising from the nucleus is bounded above by

$$|\cup_{n\geq n_\delta} (\Gamma_n)_\delta| \leq y_{n_\delta} + \int_{y_{n_\delta}}^1 \left(\frac{y_{n_\delta}}{y}\right)^{\frac{1}{1+\beta}} dy + D\delta.$$

The integral above is of order $O(y_{n_\delta}^{\frac{1}{1+\beta}})$, as $\delta \to 0$. Now we proceed similarly as in the proof of *Case 2.* in Proposition 1. There exists $M > 0$ such that, for sufficiently small $\delta > 0$,

$$|\Gamma_\delta| \leq M(y_{n_\delta}^{\frac{1}{1+\beta}} + 2\delta n_\delta).$$

Due to the fact that, for every $\beta > 0$, $2\delta n_\delta < (2\delta n_\delta)^{\frac{1}{1+\beta}}$ for sufficiently small $\delta < \delta_\beta$, we get

$$|\Gamma_\delta| \leq 2M \max\{y_{n_\delta}, 2\delta n_\delta\}^{\frac{1}{1+\beta}}, \ \delta < \delta_\beta.$$

Using exactly the same procedure as in the proof of Proposition 1, *Case 2.*, we finally get

$$\limsup_{\delta\to 0}\left[2-\frac{\ln|\Gamma_\delta|}{\ln\delta}\right]\le 2-\frac{1}{1+\beta}(1-\dim_B Y)=\frac{2\beta+1+\dim_B Y}{1+\beta}.$$

□

Proof of Theorem 4. Due to [2, Theorem 2.19] we can use the following orbital normal form near the semi-hyperbolic singularity

$$\begin{cases}\dot{x}=-x,\\ \dot{y}=y^m+h(x,y),\ m\ge 2,\end{cases}$$

where $h(x,y)=O(y^{2m-1})$ is a C^∞ function.

Now, similarly as in the proof of Theorem 3, we find a sufficiently small neighborhood of the singularity where $|h(x,y)|\le y^m$. Now, with this bound, we proceed similarly as in the proof of Proposition 2 to show that $\overline{\dim}_B\Gamma\le 1+\dim_B Y$. The other inequality, $\underline{\dim}_B\Gamma\ge 1+\dim_B Y$, is deduced analogously as in Theorem 3. □

Corollary 1 (Minkowski dimension of a monodromic polycycle)**.** *Let P be a monodromic N-polycycle of an analytic vector field, with hyperbolic saddles and semi-hyperbolic singularities as vertices, $N\in\mathbb{N}$. Let S be a spiral trajectory accumulating to P. Let $t_1,t_2,....,t_N$ be transversals to (all) regular sides of the polycycle. By Y_k, $k\in\{1,2,...,N\}$, we denote the intersections of S with transversals t_k respectively. Then,*

$$\dim_B\ S=1+\max\left\{\dim_B\ Y_k\colon k\in\{1,2,...,N\}\right\}.$$

Proof. This Corollary is a direct consequence of Theorems 3 and 4 and the finite stability of the Minkowski dimension. □

3. Applications to hyperbolic saddle-loops

The simplest polycycle configuration is a hyperbolic saddle-loop of an analytic vector field. A saddle-loop is an invariant set in which the unstable manifold of the saddle extends to its stable manifold (i.e. there exists a homoclinic connection of the saddle).

Let $r>0$ be the hyperbolicity ratio of the saddle. It is well-known (see e.g. [12, pg. 109]) that the first return map P on any transversal to the regular part of the loop (parametrized regularly by $x\in[0,T[$, where $x=0$ corresponds to the point on the loop) satisfies:

(1) (codimension 1 case, $r\in\mathbb{R}^+\setminus\{1\}$)

$$P(x)\sim Ax^r,\ A>0,$$

(2) (higher finite codimension cases, $r=1$)

$$P(x)=x+\delta(x),$$

where

$$\delta(x)=\beta_1x+\alpha_2x^2(-\ln x)+\beta_2x^2+...+\beta_{k-1}x^{k-1}+\alpha_kx^k(-\ln x)+O(x^k).$$

The saddle loop is said [12] to be of *codimension* $2k$ if $\delta(x) \sim \beta_k x^k$, as $x \to 0$, $\beta_k \neq 0$. It is said to be of *codimension* $2k+1$ if $\delta(x) \sim \alpha_{k+1} x^{k+1}(-\ln x)$, as $x \to 0$, with $\alpha_{k+1} \neq 0$, $k \geq 1$. We exclude here *infinite codimension* cases when $P \equiv \mathrm{id}$, that is, when the loop is of *center type*. In other words, in *finite codimension* cases there is an accumulating spiral trajectory to the loop. In [12] it is shown that the codimension of the saddle loop corresponds to its cyclicity in generic unfoldings.

In the following Theorem 5, we apply our results from Section 2 to give a correspondence between the codimension of the loop and the Minkowski content of its (any) spiral trajectory. The correspondence between the codimension of the saddle-loop and the Minkowski dimension of spiral trajectories is 2–1.

Theorem 5 (Minkowski dimension of a saddle-loop)**.** *The Minkowski dimension of a spiral trajectory S in an analytic vector field that has a finite-codimension saddle-loop as its α/ω-limit set depends only on the codimension of the saddle-loop. More precisely, if $k \geq 1$ is the codimension of the saddle loop, then:*

$$\dim_B S = \begin{cases} 2 - \frac{2}{k}, & k \text{ even}, \\ 2 - \frac{2}{k+1}, & k \text{ odd}. \end{cases}$$

Proof. In codimension 1 and 2 cases, the first return map P is hyperbolic, so the Minkowski dimension of the set of points of intersection of any spiral with a regular transversal to the saddle-loop (i.e. of an orbit of P) is 0 (see [3, Lemma 1]). Now, due to the Flow-Box Theorem, Theorem 3 and the finite stability of the Minkowski dimension, the dimension of the spiral is *trivial*, $\dim_B S = 1$.

For even codimensions $k > 2$, the Minkowski dimension of the set of points of intersection of any spiral with a transversal to the saddle-loop (i.e. of an orbit of P) is $1 - \frac{2}{k}$ due to [3, Theorem 1]. Again, using the Flow-Box Theorem and Theorem 3 we conclude that $\dim_B S = 2 - \frac{2}{k}$.

For odd codimensions $k > 2$, the Minkowski dimension of the set of points of intersection of any spiral with a transversal to the saddle-loop is $1 - \frac{2}{k+1}$ (see [8, Theorem 2]). Therefore, $\dim_B S = 2 - \frac{2}{k+1}$. □

4. Applications to hyperbolic 2-cycles

In this section we focus on analytic vector fields with a hyperbolic 2-saddle polycycle Γ_2 with hyperbolicity ratios r_1 and r_2 (see Figure 2). We apply our fractal methods to the classical results from [11] (and references therein) about the cyclicity of the 2-cycle in the *non-degenerate* and the *degenerate* case.

4.1. ***Non-degenerate* 2-cycles.**

Theorem 6 (Cyclicity of non-degenerate 2-cycles, [11])**.** *If the conditions*

$$r_1 \neq 1, \quad r_2 \neq 1, \quad r_1 r_2 \neq 1 \tag{11}$$

hold, then the polycycle Γ_2 is of cyclicity less than or equal to 2 in any C^∞-unfolding. If, moreover,

$$(r_1 - 1)(r_2 - 1) < 0,$$

there exists a two-parameter C^∞-versal unfolding (X_λ) in which Γ_2 is of cyclicity 2*. Otherwise, there exists a two-parameter C^∞-versal unfolding in which Γ_2 is of cyclicity* 1*.*

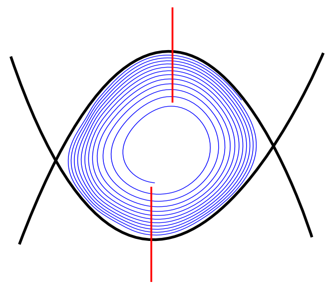

FIGURE 2. A non-trivial hyperbolic 2-cycle

We now state our fractal result for non-degenerate 2-cycles from Theorem 6. Note that, since $r_1 r_2 \neq 1$, the 2-cycle of Theorem 7 is not of *center-type* ($P \neq \mathrm{id}$), but has accumulating spiral trajectories (*focus-type*).

Theorem 7 (Minkowski dimension of a non-degenerate 2-polycycle)**.** *Let Γ_2 be a monodromic 2-polycycle of an analytic vector field, non-degenerate in the sense of (11). The Minkowski dimension of any spiral trajectory accumulating on Γ_2 is* trivial*:*

$$\dim_B S = 1.$$

Proof. Let t_1 and t_2 be any regularly parametrized transversals to heteroclinic connections of Γ_2 not intersecting the saddles. The first return maps $P_i : t_i \to t_i,\ i \in \{1, 2\}$, as compositions of regular diffeomorphisms and corner maps of the saddles, satisfy

$$P_i(x) \simeq x^{r_1 r_2},\ x \to 0.$$

Due to [3, Lemma 1], the Minkowski dimension of its orbit (i.e. of the intersection of the spiral with t_1 and t_2) is 0. By Corollary 1, the Minkowski dimension of the entire spiral is 1. ☐

4.2. ***Degenerate* 2-cycles.** In [11], Mourtada distinguishes between three families of *degenerate* 2-cycles (when some of non-degeneracy conditions $r_1 \neq 1$, $r_2 \neq 1$ or $r_1 r_2 \neq 1$ do not hold):

- $\mathcal{C}_1 = \{\Gamma_2 \colon r_1 r_2 \neq 1\}$,
- $\mathcal{C}_2 = \{\Gamma_2 \colon r_1 r_2 = 1,\ r_1 \notin \mathbb{Q}\}$,
- $\mathcal{C}_3 = \{\Gamma_2 \colon r_1 r_2 = 1,\ r_1 \in \mathbb{Q}\}$.

In the remainder of this paper we focus only on fractal analysis on families $\mathcal{C}_1$ and $\mathcal{C}_2$ and comparison with their known cyclicities. The cyclicity of polycycles in $\mathcal{C}_3$ is more complicated due to the presence of independent *Ecalle-Roussarie compensators* from two resonant saddles (see [12]).

4.2.1. *Family $\mathcal{C}_1$.*

Theorem 8 (Cyclicity of $\mathcal{C}_1$, Theorem 1 in [11])**.** *Let Γ_2 be a 2-cycle belonging to $\mathcal{C}_1$ and tangent to a planar vector field X_0. Then Γ_2 is of cyclicity less than or equal to 2 in every C^∞ family (X_λ) unfolding X_0. Furthermore, there exists a three-parameter C^∞-versal unfolding (X_λ) in which Γ_2 is of cyclicity 2.*

Again, same as in the case of non-degenerate 2-cycles in Theorem 7, it is clear here (by form of $P \neq \mathrm{id}$) that $\Gamma_2 \in \mathcal{C}_1$ is not of center-type, but has spiral trajectories.

Theorem 9 (Minkowski dimension of $\mathcal{C}_1$)**.** *Let a degenerate* 2*-cycle* Γ_2 *of an analytic vector field belong to the family* $\mathcal{C}_1$*. The Minkowski dimension of any spiral trajectory* S *accumulating on* Γ_2 *is* trivial*,* $\dim_B S = 1$*.*

Proof. The proof is analogous to the proof of Theorem 7. □

4.2.2. *Family* $\mathcal{C}_2$*.* First note that here, since $r_1 r_2 = 1$, an extra assumption of *non-trivial* polycycles is requested in Theorems 10 and 11 to exclude identity first return maps, that is, 2-cycles that are of center type. Therefore, we consider only monodromic cases of polycycles with spiraling trajectories.

Let $\Gamma_2 \in \mathcal{C}_2$ be a 2-cycle of an analytic vector field. The following analysis of the first return maps on transversals to both heteroclinic connections is due to Mourtada [11]. Since the ratios of hyperbolicity r_1 and r_2 of the saddles are irrational, the saddles are analytically linearizable, and there are C^∞ transversals σ_i, τ_i near the saddles such that the associated corner Dulac maps $D_i : \sigma_i \to \tau_i$ are given by

$$y_i = D_i(x_i) = x_i^{r_i},\ i = 1, 2.$$

For more details see [11, pg. 78]. On the other hand, regular transition maps $R_1 : \tau_2 \to \sigma_1$ and $R_2 : \tau_1 \to \sigma_2$ are given by

$$x_1 = R_1(y_2) = \beta_{2,1} y_2 \left[1 + \alpha_1 y_2^{k_1-1} + o(y_2^{k_1-1})\right]$$

and

$$x_2 = R_2(y_1) = \beta_{1,2} y_1 \left[1 + \alpha_2 y_1^{k_2-1} + o(y_1^{k_2-1})\right]$$

where $\beta_{1,2}, \beta_{2,1} > 0$ and $2 \leq k_i \in \mathbb{N} \cup \{\infty\}$, and where $k_i = \infty$ implies $\alpha_i = 0$. The first return maps $P_1 = R_1 \circ D_2 \circ R_2 \circ D_1 : \sigma_1 \to \sigma_1$ and $P_2 = R_2 \circ D_1 \circ R_1 \circ D_2 : \sigma_2 \to \sigma_2$ are then given by:

$$\begin{aligned} P_1(x_1) = \beta_{2,1}\beta_{1,2}^{r_2} x_1 \Big[1 + &\Big(\alpha_1 \beta_{1,2}^{r_2(k_1-1)} x_1^{k_1-1} + o(x_1^{k_1-1})\Big) + \\ &+ \Big(r_2\alpha_2 x_1^{r_1(k_2-1)} + o(x_1^{r_1(k_2-1)})\Big)\Big], \end{aligned} \tag{12}$$

and

$$\begin{aligned} P_2(x_2) = \beta_{1,2}\beta_{2,1}^{r_1} x_2 \Big[1 + &\Big(\alpha_2 \beta_{2,1}^{r_1(k_2-1)} x_2^{k_2-1} + o(x_2^{k_2-1})\Big) + \\ &+ \Big(r_1\alpha_1 x_2^{r_2(k_1-1)} + o(x_2^{r_2(k_1-1)})\Big)\Big]. \end{aligned} \tag{13}$$

Notice that P_1 is hyperbolic/tangent to identity if and only if P_2 is hyperbolic/tangent to identity. Indeed,

$$\beta_{1,2}\beta_{2,1}^{r_1} = \beta_{1,2}^{r_1 r_2}\beta_{2,1}^{r_1} = \left(\beta_{2,1}\beta_{1,2}^{r_2}\right)^{r_1}.$$

Moreover, exactly one of the inequalities $k_1 - 1 < r_1(k_2 - 1)$ or $k_2 - 1 < r_2(k_1 - 1)$ holds. On the contrary, if we assume that both hold, we get

$$k_1 - 1 < r_1(k_2 - 1) < r_1 r_2(k_1 - 1) = k_1 - 1,$$

which is obviously a contradiction. On the other hand, if we assume that neither one holds, we have

$$k_1 - 1 \geq r_1(k_2-1) \geq r_1 r_2(k_1-1) = k_1 - 1.$$

which would imply $k_1 - 1 = r_1(k_2-1)$ and $k_2 - 1 = r_2(k_1-1)$. As a consequence, $r_1,\, r_2 \in \mathbb{Q}$, which is a contradiction.

Moreover, in the case $\beta_{1,2}\beta_{2,1}^{r_1} = 1$, $|\alpha_1| + |\alpha_2| \neq 0$. Otherwise $P_1 = P_2 = \mathrm{id}$, which is the trivial case that is not considered here. This is a consequence of the *quasi-analyticity* of the first return maps around hyperbolic saddle polycycles in analytic planar vector fields [6], that states that the Taylor map that associates to a Dulac germ its Dulac asymptotic expansion is injective (i.e., trivial expansion implies the trivial germ).

Theorem 10 (Cyclicity in $\mathcal{C}_2$, [11], p. 83)**.** *Let X_0 be an analytic vector field with a non-trivial (in the sense that the first return map is not equal to the identity) 2-cycle $\Gamma_2 \in \mathcal{C}_2$ tangent to X_0. In the notation as above,*

(1) *If $\beta_{1,2}\beta_{2,1}^{r_1} \neq 1$, then the cyclicity of Γ_2 (in any C^∞ unfolding (X_λ)) is not greater than 3.*
(2) *If $\beta_{1,2}\beta_{2,1}^{r_1} = 1$ and $|\alpha_1| + |\alpha_2| \neq 0$, then the cyclicity of Γ_2 (in any C^∞ unfolding (X_λ)) is not greater than ϵ, where:*

$$\epsilon := \begin{cases} 2 + k_1 + \lfloor \frac{k_1-1}{r_1} \rfloor & \text{if } k_1 - 1 < r_1(k_2-1), \\ 2 + k_2 + \lfloor \frac{k_2-1}{r_2} \rfloor & \text{if } k_2 - 1 < r_2(k_1-1). \end{cases}$$

Note the surprising fact that the cyclicity, unlike in all the previous cases, cannot be read only from a single first return map.

We now state our 'fractal' version of Theorem 10. The goal is to read the Mourtada's upper bound on the cyclicity of the 2-cycle $\Gamma_2 \in \mathcal{C}_2$ from the Minkowski dimension of only one trajectory accumulating to Γ_2. However, as can be expected in the light of the comment before Theorem 10, the dimension of the spiral trajectory will not suffice. In order to read Mourtada's upper bound, we need additional fractal data, see Corollary 2 below.

Theorem 11 (Fractal version of Theorem 10)**.** *Let X_0 be an analytic vector field with a non-trivial 2-cycle $\Gamma_2 \in \mathcal{C}_2$. Let $r := \min\{r_1,\, r_2\}$ be the minimal hyperbolicity ratio. Any spiral trajectory accumulating to the polycycle has the same Minkowski dimension $d \in [1, 2)$. Moreover, the cyclicity of Γ_2 in C^∞ unfoldings of X_0 is at most*

$$\left\lfloor 3 + (1+r)\frac{d-1}{2-d} \right\rfloor . \tag{14}$$

Proof. By Theorem 10, if $\beta_{1,2}\beta_{2,1}^{r_1} \neq 1$ then the cyclicity of the polycycle is at most 3. On the other hand, by (12) and (13), the first return maps P_1 and P_2 are hyperbolic and, therefore, the intersections of any spiral with a transversal to the polycycle has Minkowski dimension 0 (see [3, Lemma 1]). By Corollary 1 we conclude that $d = 1$.

Consider now the case when $\beta_{1,2}\beta_{2,1}^{r_1} = 1$. By (12) and (13), it follows that $|\alpha_1| + |\alpha_2| \neq 0$. Indeed, if $|\alpha_1| + |\alpha_2| = 0$, the first return maps are equal to the identity and the polycycle is trivial (of center type), which is a contradiction with the assumption. Therefore, in (12) and (13), at least one of k_1 and k_2 is finite. Without loss of generality (see the discussion at the beginning of the section) we assume that

$$k_1 - 1 < r_1(k_2 - 1). \tag{15}$$

The Minkowski dimension of an orbit of P_1 is $1-\frac{1}{k_1}$ and the Minkowski dimension of an orbit of P_2 is $1-\frac{1}{r_2(k_1-1)+1}$ (see [3, Theorem 1]). Now we distinguish two cases:

(1) $r_1 > 1$
 Since $r_1 r_2 = 1$, it follows that $r_2 < 1$, so $r_2(k_1-1) < k_1 - 1$. By Corollary 1 we conclude that $d = 2 - \frac{1}{k_1} \in \mathbb{Q}$.
 By Theorem 10, cyclicity in case (15) is at most

$$2 + k_1 + \left\lfloor \frac{k_1 - 1}{r_1} \right\rfloor = \left\lfloor 2 + k_1 + \frac{k_1 - 1}{r_1} \right\rfloor = \left\lfloor 3 + \frac{d-1}{2-d} + r_2 \frac{d-1}{2-d} \right\rfloor.$$

(2) $r_1 < 1$
 It follows that $r_2 > 1$, so $r_2(k_1-1) > k_1 - 1$. Similarly as in the previous case we get that $d = 2 - \frac{1}{1+r_2(k_1-1)} \notin \mathbb{Q}$. The cyclicity of the polycycle is at most

$$2 + k_1 + \left\lfloor \frac{k_1 - 1}{r_1} \right\rfloor = \left\lfloor 2 + k_1 + \frac{k_1 - 1}{r_1} \right\rfloor = \left\lfloor 3 + r_1 \frac{d-1}{2-d} + \frac{d-1}{2-d} \right\rfloor.$$

In the symmetrical case $k_2 - 1 < r_2(k_1 - 1)$ in (15) similar conclusions hold, and the statement of the theorem follows. □

The following corollary covers previous results in non-degenerate and degenerate $\mathcal{C}_1$ and $\mathcal{C}_2$ cases.

Corollary 2. *Let Γ_2 be a non-trivial 2-saddle polycycle of an analytic vector field such that $\Gamma_2 \notin \mathcal{C}_3$. Let S be its one accumulating spiral trajectory.*

(1) *If $\dim_B S = 1$, then the cyclicity is at most 3.*
(2) *If $d := \dim_B S \in (1,2)$, then the upper bound on cyclicity is given by formula (14) of Theorem 11, and*

$$r = \min\left\{\frac{d_1 - 1}{d_2 - 1}, \frac{d_2 - 1}{d_1 - 1}\right\},$$

 $d_1 \in (0,1)$ and $d_2 \in (0,1)$ Minkowski dimensions of sequences obtained as intersections of spiral S with transversals to the two heteroclinic connections. Note also that $d = 1 + \max\{d_1, d_2\}$.

Proof. By Theorems 7, 9 and 11, $\dim_B S = 1$ for non-degenerate cycles, for family $\mathcal{C}_1$ and in the case $\beta_{1,2}\beta_{2,1}^{r_1} \neq 1$ in $\mathcal{C}_2$. In all those cases the first return maps are strongly hyperbolic or hyperbolic. In all these families, by Theorems 6, 8 and 10 of Mourtada, the upper bound on cyclicity is 3. On the other hand, if $r_1 \cdot r_2 = 1$ and r_1, $r_2 \notin \mathbb{Q}$, the first return maps on both transversals are tangent to the identity, with multiplicities γ_1 and γ_2 striclty greater than 1. It is easy to check by e.g. [3] that $r_1 = \frac{\gamma_1}{\gamma_2}$ and $r_2 = \frac{\gamma_2}{\gamma_1}$. By [3], $d_1 = 1 - \frac{1}{\gamma_1}$ and $d_2 = 1 - \frac{1}{\gamma_2}$, and the above formula for $r := \min\{r_1, r_2\}$ follows. □

Remark 1. *Note that a trivial saddle polycycle is of center type (no spiral trajectories). The first return map on transversals to heteroclinic connections is equal to the identity. The Minkowski dimension of just one periodic trajectory close to the polycycle is* 1 *(moreover, of finite length). On the other hand, the continuum of periodic trajectories accumulating on the polycycle is an open set of non-zero area and its Minkowski dimension is equal to* 2*. None of the two makes much sense to consider. Therefore, we exclude this case from our fractal considerations.*

In all non-trivial cases, by quasi-analyticity of first return maps around hyperbolic polycycles of planar analytic vector fields, the first return map on transversals is never the identity, but either tangent to the identity or (strongly) hyperbolic. Therefore its orbits on transversals to heteroclinic connections have Minkowski dimension belonging to $[0, 1)$*, by e.g.* [3]*. By Corollary 1, the Minkowski dimension of a spiral trajectory* S *around the non-trivial hyperbolic saddle polycycle then satisfies* $\dim_B S \in [1, 2)$*.*

5. Conclusion

In Sections 3 and 4 we have discussed the relation between cyclicity and box dimension only for hyperbolic 1- and 2-cycles. Our results rely on known cyclicity results of such polycycles by Mourtada [11]. However, Corollary 1 gives the box dimension of spiral trajectories of *any* monodromic saddle (saddle-node) polycycle.

For more vertices polycycles, there are some results on cyclicity, always under some genericity assumptions. By [9], for hyperbolic n-polycycles, $n \in \mathbb{N}$, the non-degeneracy (and genericity) conditions that correspond to non-degeneracy conditions of Theorem 6 for $n = 2$, comprise of finitely many algebraic equations in ratios of hyperbolicity of the vertices, $r_1, \ldots, r_n$, among which are all conditions of the form:

$$\prod_{j\in I} r_j \neq 1, \tag{16}$$

for all $I \subseteq \{1, \ldots, n\}$. For such polycycles, there is an explicit upper bound $c(n)$ on the cyclicity of the polycycle (given explicitely and inductively in [9]), which is finite, but *increases* with the number of vertices n, and $c(1) = 1$, $c(2) = 2$ (see Section 3 and Theorem 6).

Theorem 12 (Non-degenerate n-polycycles, [9])**.** *There exists a finite set* $E(n)$ *of algebraic conditions on ratios of hyperbolicity of a* n*-polycycle* Γ*, containing all conditions of the form* (16)*, under which there exists an explicit upper bound* $c(n) < \infty$ *on the cyclicity of any* C^∞*-unfolding of any* n*-polycle* Γ *with ratios of hyperbolicity* $r_1, \ldots, r_n$*.*

Our Corollary 1 is applicable for computing the box dimension of all monodromic polycyles, by computing their first return maps on transversals. Note that, in all non-degenerate cases, since $P(x) \simeq x^{r_1\cdots r_n}$, and $r_1 \cdots r_n \neq 1$, the box dimension of the spiral Γ is trivial, i.e. $\dim_B \Gamma = 1$. By the same argument, it is also trivial in more degenerate cases, when $r_1 \cdots r_n \neq 1$ (the other products of ratios of hyperbolicity are irrelevant). In general, for $n \geq 3$, there are no general cyclicity results as in Theorem 12 only under this weaker condition on the product of all ratios, and even less in cases when $r_1 \cdots r_n = 1$. In Mourtada's proof [9, 10], the first return map, up to some non-essential term, is equal to

$$P_\lambda(x) \sim \left(\cdots \left((x^{r_1(\lambda)} + \beta_1(\lambda))^{r_2(\lambda)} + \beta_2(\lambda) \right)^{r_3(\lambda)} \cdots \right)^{r_n(\lambda)} + \beta_n(\lambda),$$

where $\lambda \approx 0$ is bifurcation parameter, $\beta_i(\lambda)$, $\beta_i(0) = 0$, are the connection breaking parameters, and $r_i(0) = r_i$. Algebraic equations in ratios of hyperbolicity at $\lambda = 0$ introduce compensator variables in the Chebyshev scale for the unfolding, so the problems arise when there is more than one independent compensator. There is a generalization for saddle and also saddle-node polycycles in [5], where the authors prove the existence of an upper bound on cyclicity depending on the number of parameters in generic unfoldings (under some other generic conditions) of elementary polycycles, but the bound is not explicit. Also, later, in [1], the authors provide a (non-optimal) upper bound on the cyclicity of only Hamiltonian systems and thus solve the infinitesimal Hilbert's problem. Nonexistence of general cyclicity results for unfoldings of elementary polycycles comprise the main obstacle for solving the Hilbert's 16th problem.

Also, a partial result is a finite cyclicity result of [7] for analytic unfoldings of non-resonant polycycles where all ratios $r_i(\lambda) \notin \mathbb{Q}$, $i = 1, \ldots, n$, are irrational and constant in the unfolding, and the polycycle does not break in the unfolding ($b_i(\lambda) = 0$), proven by definability of such unfoldings in an o-minimal structure. In such cases it can happen that $r_1 \cdots r_n = 1$, and $P(x) \simeq x + \text{h.o.t.}$, so the box dimension is in general strictly bigger than 1.

Based on these two observations, even for saddle n-polycycles, the relation between the box dimension and the cyclicity in the unfoldings of an n-polycycle is far from $1-1$. This is not surprising, since the asymptotic scale for the first return maps of general polycycles contains generalized powers (from non-resonant saddles), logarithms (from resonant saddles) or even exponentials and iterated logarithms in case of saddle-node vertices. At the same time, by its definition, the box dimension compares the asymptotic behavior of the epsilon-neighborhood of orbits *only to powers* of ε. The Minkowski dimension is well adapted for unfoldings in power scales, while in the case of polycycles one deals with more general, Chebyshev scales for the unfolding [12]. For more details, see [8]. The problem of this non-bijectiveness is visible already for resonant saddle loops in Theorem 5, but becomes even more complicated for more vertices. In addition, the exact relation between cyclicity and box dimension cannot be established since cyclicity results for general polycycles are not known except in some particular cases, and even then are not necessarily optimal.

Declarations

Ethical Approval Not applicable.
Competing interests The authors declare that they have no conflict of interest.
Availability of data and materials Not applicable.

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