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COMPARING THE KIRWAN AND NONCOMMUTATIVE RESOLUTIONS OF QUOTIENT VARIETIES

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ABSTRACT. Let a reductive group G act on a smooth variety X such that a good quotient $X/\!\!/G$ exists. We show that the derived category of a noncommutative crepant resolution (NCCR) of $X/\!\!/G$, obtained from a G-equivariant vector bundle on X, can be embedded in the derived category of the (canonical, stacky) Kirwan resolution of $X/\!\!/G$. In fact the embedding can be completed to a semi-orthogonal decomposition in which the other parts are all derived categories of Azumaya algebras over smooth Deligne-Mumford stacks.

CONTENTS

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1. Introduction

1.1. **Preliminaries.** We fix an algebraically closed ground field k of characteristic 0. Everything is linear over k. Here and below, if $\mathcal{X}$ is an Artin stack and Λ is a quasi-coherent sheaf of rings on $\mathcal{X}$ then $D(\Lambda)$ is the unbounded derived categories of right Λ-modules with quasi-coherent cohomology. We also put $D(\mathcal{X}) := D(\mathcal{O}_{\mathcal{X}})$.

We recall the following definition.

Definition 1.1. [VdB04a] Let R be a normal Gorenstein domain. A *noncommutative crepant resolution* (NCCR) of R is an R-algebra of finite global dimension of the form $\Lambda = \operatorname{End}_R(M)$ which in addition is Cohen-Macaulay as R-module and where M is a non-zero finitely generated reflexive R-module.

In this paper we will say that a sheaf of k-algebras Λ on a scheme X is a NCCR of X if $\Lambda(U)$ is a NCCR of $\Gamma(U)$ for every connected affine open $U \subset X$.

The derived categories of NCCRs are particular instances of "categorical strongly crepant resolutions" and the latter are conjectured to be minimal among all "categorical resolutions" [Kuz08]. In the current paper we provide new evidence for this conjecture. Namely we will show that the NCCRs of quotient singularities for reductive groups, of the type constructed in [ŠVdB17], embed in a particular canonical (stacky) resolution of singularities, constructed by Kirwan in [Kir85].

Remark 1.2. The correct interpretation of the conjecture requires some care since for example if X is a noetherian scheme and $\pi : Y \to X$ is a commutative resolution of singularities (where Y can be a smooth Deligne-Mumford stack) then $D(Y)$

is only a categorical resolution of $D(X)$ if X has rational singularities [Lun10, Example 5.1].[1]

To be able to state our main results we introduce some more notation. Let G be a reductive group and let X be a smooth G-variety such that a good quotient[2] $\pi : X \to X/\!\!/G$ exists. In [Kir85], Kirwan constructed (for projective X) a partial resolution of $X/\!\!/G$ by an inductive procedure involving GIT quotients of repeated G-equivariant blowups of X (see §6). The final quotient variety $\mathbf{X}/\!\!/G$ is then a partial resolution of singularities of $X/\!\!/G$ (finite quotient singularities may remain). We may also view the end result as a smooth Deligne Mumford stack $\mathbf{X}/G$ and therefore we say that $\mathbf{X}/G$ is the *Kirwan (stacky) resolution* of $X/\!\!/G$. In [ER17], Edidin and Rydh generalised the Kirwan (and also Reichstein [Rei89]) procedure to irreducible Artin stacks with stable good moduli spaces. We will heavily use their technical results.

1.2. **Assumptions.** Let $X^u \subset X$ be the locus of points whose stabilizer is not finite or whose orbit is not closed (see §4). Throughout the introduction (and in various parts of the paper) we assume

$$\operatorname{codim}(X^u, X) \geq 2. \tag{H2}$$

Occasionally we will impose the slightly stronger condition that X is *generic* in the sense of [ŠVdB17]; i.e. that G acts in addition freely on an open subset of $X - X^u$ whose complement has codimension ≥ 2 (see §7.2).

1.3. **The embedding of a noncommutative resolution in $D(\mathbf{X}/G)$.** In this paper we consider noncommutative resolutions of $X/\!\!/G$ of the form

$$\Lambda = \mathcal{E}nd_X(\mathcal{U})^G \tag{1.1}$$

where $\mathcal{U}$ is a G-equivariant vector bundle on X. This is a minor generalization with respect to [ŠVdB17] where we exclusively considered the case $\mathcal{U} = U \otimes \mathcal{O}_X$ where U is a finite dimensional representation of G.[3]

A feature of a resolution like (1.1) is that there is an embedding

$$- \overset{L}{\otimes}_\Lambda \mathcal{U} : D(\Lambda) \hookrightarrow D(X/G).$$

Now let $\mathbf{X}/G$ be the Kirwan resolution of $X/\!\!/G$ which we factor as

$$\mathbf{X}/G \xrightarrow{\Xi} X/G \to X/\!\!/G.$$

As $X/\!\!/G$ has rational singularities [Bou87], $D(\mathbf{X}/G)$ is a categorical resolution of $D(X/\!\!/G)$ by Remark 1.2, which implies in particular that pullback provides an embedding

$$D(X/\!\!/G) \hookrightarrow D(\mathbf{X}/G).$$

[1] On the other hand in [KL15] it is shown that an arbitrary commutative resolution can always be suitably modified to yield a categorical resolution.

[2] A good quotient of X is a map $X \to Y$ which is locally (on Y) of the form $\operatorname{Spec} R \to \operatorname{Spec} R^G$. If Y exists then it is unique and we write $X/\!\!/G := Y$.

[3] In the context of the current paper the restriction $\mathcal{U} = U \otimes \mathcal{O}_X$ is unnatural as we will, in any case, forcibly encounter more general equivariant vector bundles when iterating Reichstein transforms.

It is important to observe however that *we do not have an embedding of $D(X/G)$ in $D(\mathbf{X}/G)$*; an indication for this is given in §10 where we provide an example with $\operatorname{rk} K_0(X/G) = \infty$ but $\operatorname{rk} K_0(\mathbf{X}/G) = 8$.

The following is our first main result.

Proposition 1.3 (Proposition 6.5)**.** *Let G be a reductive group acting on a smooth variety X such that a good quotient $\pi : X \to X/\!\!/G$ exists. Assume (H2). Let $\mathbf{X}/G$ be the Kirwan resolution of $X/\!\!/G$ discussed above.*

Let $\mathcal{U}$ be a G-equivariant vector bundle on X and assume that $\Lambda = \mathcal{E}nd_X(\mathcal{U})^G$ is (locally) Cohen-Macaulay as a sheaf of algebras on $X/\!\!/G$ (e.g. if Λ is an NCCR). Then the composition

$$D(\Lambda) \hookrightarrow D(X/G) \xrightarrow{L\Xi^*} D(\mathbf{X}/G)$$

is fully faithful.

1.4. **The Reichstein transform and a naive embedding.** One establishes Proposition 1.3 inductively, following the Edidin and Rydh procedure discussed above. Let us describe the procedure more precisely.

For a point $x \in X$ let G_x be its stabilizer and set $\mu(X) = \max_{x\in X} \dim G_x$. Put

$$Z = \{x \in X \mid \dim G_x = \mu(X)\},$$
$$\bar{Z} = \{x \in X \mid \overline{Gx} \cap Z \neq 0\}.$$

Both Z and $\bar{Z}$ are closed in X (see §5.1). Denote by $(-)'$ the strict transform of a closed subset under a blowup. Put

$$X^R = \operatorname{Bl}_Z X - \bar{Z}'.$$

The resulting map $\xi^R : X^R/G \to X/G$ is called the *Reichstein transform* of X/G. One has $\mu(X^R) < \mu(X)$ and hence by performing a sequence of such transforms the maximal stabilizer dimension becomes zero, yielding $\mathbf{X}$ and the Kirwan resolution $\mathbf{X}/G$.

Let $\mathcal{U}' = \xi^{R*}\mathcal{U}$ be a vector bundle on X^R/G and let $\Lambda' = \mathcal{E}nd(\mathcal{U}')^G$, viewed as a sheaf of algebras on $X^R/\!\!/G$.

We first obtain a naive embedding.

Proposition 1.4 (Corollary 5.3, Lemma 5.4)**.** *Assume* (H2)*. If Λ is Cohen-Macaulay, then so is Λ'. Moreover, pullback for the morphism of ringed spaces $(X^R/\!\!/G, \Lambda') \to (X/\!\!/G, \Lambda)$ induces an embedding of derived categories $D(\Lambda) \hookrightarrow D(\Lambda')$.*

We obtain Proposition 1.3 by successive application of Proposition 1.4.

1.5. **Semi-orthogonal decomposition of $D(\mathbf{X}/G)$.** In the case that Λ in the statement of Proposition 1.3 is actually a noncommutative crepant resolution of X (and X is generic), another main result of this paper is that the embedding $D(\Lambda) \hookrightarrow D(\mathbf{X}/G)$ can be completed to a semi-orthogonal decomposition.

Here we first encounter an impediment, wishing to proceed in the inductive way using the "naive" embedding given by Proposition 1.4. The hindrance is that the NCCR property is not preserved when passing from Λ to Λ' (see Example 8.5). Therefore we would not be able to proceed inductively, even if we could enhance the embedding in Proposition 1.4 to a semi-orthogonal decomposition.

The most serious issue is that finite global dimension is not preserved. This obstacle we overcome by slightly tweaking $\mathcal{U}'$, and hence Λ'. Let $\mathcal{O}_{\mathrm{Bl}_Z X}(1)$ be the tautological relatively ample line bundle on $\mathrm{Bl}_Z X$ and let $\mathcal{O}_{X^R}(1)$ be its restriction to X^R. For some $N > 0$, $\mathcal{O}_{X^R}(N)$ is the pullback of a line bundle $(\pi^R_* \mathcal{O}_{X^R}(N))^G$ on the quotient $\pi^R : X^R \to X^R /\!\!/ G$ (see Proposition 5.2(6)). From a vector bundle $\mathcal{U}$ on X/G we produce the vector bundle $\mathcal{U}^R$ on X^R/G as

$$\mathcal{U}^R = \bigoplus_{i=0}^{N-1} (\xi^{R*}\mathcal{U})(i).$$

We obtain an *Orlov's type (blow-up) semi-orthogonal decomposition* for $\Lambda^R = \mathcal{E}nd_{X^R}(\mathcal{U}^R)$ with one component corresponding to Λ and the other components corresponding to representatives Z_i for the orbits of the G-action on the connected components of the center Z of the blow-up. Let $G_i \subset G$ be the stabilizer of Z_i, as a connected component.

Proposition 1.5 (Corollary 8.10)**.** *Assume* (H2) *and that Λ is Cohen-Macaulay. Let $\mathcal{U}_{Z_i}$ be the restriction of $\mathcal{U}$ to Z_i and let $\Lambda_{Z_i} = \mathcal{E}nd_{Z_i}(\mathcal{U}_{Z_i})^{G_i}$. There is a semi-orthogonal decomposition*

$$D(\Lambda^R) \cong \langle D(\Lambda), \underbrace{D(\Lambda_{Z_1}), \dots, D(\Lambda_{Z_1})}_{c_1-1}, \dots, \underbrace{D(\Lambda_{Z_t}), \dots, D(\Lambda_{Z_t})}_{c_t-1} \rangle$$

where $c_i = \operatorname{codim}(Z_i, X)$. *Moreover, the components corresponding to different Z_i are orthogonal.*

Unfortunately it turns out that the NCCR property is *still* not preserved by the passage $\Lambda \mapsto \Lambda^R$; the culprit being that the Reichstein transform may produce non-trivial stabilizers in codimension one. We solve this by introducing the following two technical conditions.

(α) Λ is *homologically homogeneous* (see Definition 7.3).

(β) $\mathcal{U}$ is a *generator in codimension one* (see Definition 7.5).

Both of these conditions are satisfied if X is generic and Λ is a NCCR (see Proposition 7.7). Moreover we prove that both properties, along with the (H2) property, are preserved under the passage $X \mapsto X^R$ (see Propositions 5.2,8.13).

The successive applications of semi-orthogonal decompositions as in Proposition 1.5 following successive Reichstein transforms yield a semi-orthogonal decomposition of $D(\mathbf{\Lambda})$ where $\mathbf{\Lambda}$ is obtained on the final step. We will show that if (α, β) hold for $\mathbf{\Lambda}$ then in fact $D(\mathbf{\Lambda}) \cong D(\mathbf{X}/G)$. So by the above discussion we conclude that it is enough to assume (α, β) hold for the initial Λ to obtain a semi-orthogonal decomposition of $D(\mathbf{X}/G)$. This semi-orthogonal decomposition is stated in Theorem 8.15. We will not restate it here as we prefer to give a more geometric version in the next section.

1.6. **Geometric description of the semi-orthogonal decomposition.** We further proceed to give a geometric description of the $D(\Lambda_{Z_i})$ appearing in Proposition 1.5. For simplicity we here state our final result only in the abelian case. For the general case see Theorem 8.15, Corollary 9.9.

Let us assume the Kirwan resolution is obtained by performing n successive Reichstein transforms with Z_j being blown up at the j-th step. Let Z_{ji}, $1 \le i \le t_j$, be representatives for the orbits of the G-action on the connected components of Z_j. Let H_{ji} be the stabilizer of Z_{ji}.

Theorem 1.6 (Theorem 8.15, Remark 8.18, Corollary 9.9)**.** *Assume* (H2). *Assume that Λ is homologically homogeneous and that $\mathcal{U}$ is a generator in codimension* 1. *Let G be abelian (for general G see Theorem 8.15, Corollary 9.9). There is a semi-orthogonal decomposition*

$$D(\mathbf{X}/G) \cong \langle D(\Lambda), D(Z_{ji}/(G/H_{ji}))^{\oplus N_{ji}}_{1\leq j\leq n, 1\leq i\leq t_j, 0\leq k\leq c_{ji}-2}\rangle$$

for some $N_{ji} \in \mathbb{N}_{>0}$, where $c_{ji} := \operatorname{codim}(Z_{ji}, X_j)$, and the terms appear in lexicographic order (according to the label (j, i, k)).

As we have already mentioned in §1.5, by Proposition 7.7 the conditions for this theorem are satisfied if X is generic and Λ is a NCCR of $X/\!\!/G$.

For general G, Z_{ji} will not have a global stabilizer group, however the generic stabilizer is conjugate to a fixed group H_{ji}. Thus, instead of $Z_{ji}/(G/H_{ji})$ we should take $Z^{\langle H_{ji}\rangle}_{ji}/(N_V(H_{ji})/H_{ji})$ where $Z^{\langle H_{ji}\rangle}_{ji}$ is a suitable (smooth) subscheme of $Z^{H_{ji}}_{ji}$ and $N_V(H_{ji})$ is a subgroup of the normalizer group $N(H_{ji})$, and adorn it with a sheaf of (equivariant) Azumaya algebras (see Corollary 9.9).

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3. Notation and conventions

We fix an algebraically closed field k of characteristic 0. Everything is linear over k. In particular $\operatorname{Spec} k$ is the base scheme and unadorned tensor products are over k.

All schemes are separated. The stacks we will use are global quotients stacks X/G for which X is at least separated. We will silently identify G-equivariant sheaves on X and sheaves on X/G. If a good quotient $\pi : X \to X/\!\!/G$ exists we write $\pi_s : X/G \to X/\!\!/G$ for the corresponding stack morphism. On some occasions we sloppily write $(-)^G$ for π_{s*}. We sometimes silently globalize results for X, X/G, $X/\!\!/G, \ldots$ which are available in the literature for X affine and which are seen to be trivially local over $X/\!\!/G$.

All modules are right modules. If Λ is ring then $D(\Lambda)$ is the unbounded derived category of Λ. If $\mathcal{X}$ is an Artin stack and Λ is a quasi-coherent sheaf of rings on $\mathcal{X}$ then $D(\Lambda)$ is the unbounded derived categories of Λ-modules with quasi-coherent cohomology. We also put $D(\mathcal{X}) := D(\mathcal{O}_{\mathcal{X}})$.

For an affine algebraic group H we denote by H_e the identity component of H. and we let $\operatorname{rep}(H)$ be the set of isomorphism classes of irreducible H-representations.

For $U, U' \in \operatorname{Ob}(\mathfrak{a})$ where $\mathfrak{a}$ is a Karoubian category we write $U :=: U'$ to indicate $U \in \operatorname{Ob}(\operatorname{add}(U'))$ and $U' \in \operatorname{Ob}(\operatorname{add}(U))$.

Unless otherwise specified, "graded" means $\mathbb{Z}$-graded and elements of a graded ring are automatically assumed to be homogeneous.

4. Generalities

Unless otherwise specified X is a smooth variety and G is a reductive group acting on X such that a good quotient $\pi : X \to X/\!\!/G$ exists (see e.g. [ŠVdB16, Definition 3.3.1] for the definition of good quotient).

4.1. **(Semi-)stability.** A point in X is *stable* if it has closed orbit and finite stabilizer. We write X^s for the stable locus of X and X^u for its complement. If $\mathcal{L}$ is a line bundle on X which linearises the G-action then by [MFK94, §4] $x \in X$ is ($\mathcal{L}$-)semi-stable if there is $f \in H^0(X, \mathcal{L}^{\otimes n})^G$ for $n > 0$ such that $f(x) \neq 0$ and X_f is affine. We denote the set of $\mathcal{L}$-semi-stable points by $X^{ss,\mathcal{L}}$.

Remark 4.1. By [MFK94, Theorem 1.10], a good quotient $\pi : X^{ss,\mathcal{L}} \to X^{ss,\mathcal{L}}/\!\!/G$ exists. Moreover, there is an $N > 0$ such that the restriction of $\mathcal{L}^{\otimes N}$ to $X^{ss,\mathcal{L}}$ is the pullback of a line bundle on $X^{ss,\mathcal{L}}/\!\!/G$. It follows in particular that any $\mathcal{L}$-semi-stable point x has a G-equivariant saturated[4] neighbourhood on which $\mathcal{L}$ is torsion.

A particular example of a linearisation is given by a line bundle of the form $\mathcal{L} = \chi \otimes \mathcal{O}_X$ for $\chi \in X(G)$. We can sometimes reduce to this case by Lemma 4.6 below.

4.2. **(Semi-)stability and étale maps.**

Lemma 4.2. *Assume that $\phi : Y \to X$ is a G-equivariant étale map. Let $x \in X$ and let $y \in Y$ be a preimage of x. Then the following holds true:*

(1) *$G_y \subset G_x$ and $\dim G_y = \dim G_x$.*
(2) *If Gx is closed then so is Gy.*
(3) *If x is stable then so is y.*

In addition, if ϕ is strongly étale[5] then $G_x = G_y$ and the converse of (2) and (3) holds.

Proof. (1) is clear since ϕ is quasi-finite. For (2) assume that Gx is closed and Gy is not closed. Since the action of G on Gx is transitive and $\overline{Gy} \subset \phi^{-1}(Gx)$, we have $\phi(\overline{Gy} \setminus Gy) = Gx$. Hence $\dim(\overline{Gy} \setminus Gy) = \dim Gx = \dim Gy$ (as ϕ is quasi-finite), which is a contradiction. (3) follows by combining (1) and (2).

Now assume ϕ is strongly étale. By definition, $Y = V \times_{X/\!\!/G} X$ with $V \to X/\!\!/G$ étale. Let $\bar{y}$, $\bar{x}$ be the images of y, x in V, $X/\!\!/G$, respectively, and let $Y_{\bar{y}}$, $X_{\bar{x}}$ be the corresponding fibers. Then ϕ induces an isomorphism $Y_{\bar{y}} \cong X_{\bar{x}}$ and hence $G_x = G_y$. If Gy is closed then it is closed in $Y_{\bar{y}}$, and hence Gx is closed in $X_{\bar{x}}$, and therefore closed in X. This proves the converse of (2). The converse of (3) is again a combination of (1) and the converse of (2). □

Lemma 4.3. *Assume that $\phi : Y \to X$ is a G-equivariant étale map which is moreover affine. Let $x \in X$ and let $y \in Y$ be a preimage of x. Assume $\mathcal{L}$ is a linearisation of the G-action on X and let $\mathcal{M} = \phi^*\mathcal{L}$. If x is $\mathcal{L}$-semi-stable then y is $\mathcal{M}$-semi-stable. If ϕ is strongly étale and Y and X are affine then the converse also holds.*

[4] A G-invariant open subset in X is saturated if it is a pullback of an open subset in $X/\!\!/G$.

[5] A G-equivariant map $\phi : Y \to X$ is strongly étale if it is induced by pullback from an étale map $V \to X/\!\!/G$. In particular the inclusion of a saturated open subset is strongly étale.

Proof. The first part follows by pulling back the section nonvanishing on x to Y. The converse follows by considering the restriction of L and $M = \underline{\operatorname{Spec}} \operatorname{Sym} \mathcal{M}$ to $X_{\bar{x}}$ and $Y_{\bar{y}}$, respectively (with notation as in the proof of Lemma 4.2). $\square$

4.3. **Genericity conditions.** Let $i \in \mathbb{N}$. Below we write (Hi) for the condition $\operatorname{codim}(X^u, X) \geq i$. Note that $(H1)$ is equivalent to $X^s \neq \emptyset$. Furthermore $(Hi) \Rightarrow (Hj)$ for $j \leq i$.

4.4. **Reduction to the linear case.** We will often reduce to the linear case using the Luna slice theorem [Lun73]. Assume that x is a point in X with closed orbit. There there is a smooth affine slice S at x such that there is a strongly étale map $\phi : G \times^{G_x} S \to X$. Furthermore we may assume that there is a strongly étale map $\gamma : S \to T_x S$, sending x to 0. We will usually abuse terminology by simply calling S a slice as x and by calling $(G_x, T_x S)$ the *linearised slice* at x.

Lemma 4.4. *The hypothesis (Hi) holds for (G, X) if and only it holds for $(G_x, T_x S)$ for all points $x \in X$ with closed orbit.*

Proof. Let x be a point in X with closed orbit. We first show that $\operatorname{codim}(X^u, X) \leq \operatorname{codim}((T_x S)^u, T_x S)$, so that if (Hi) holds for (G, X) it holds for $(G_x, T_x S)$.

If $x \notin X^u$ then G_x is finite and hence $(T_x S)^u$ is empty so that there is nothing to prove.

Now assume $x \in X^u$ and let $\phi : G \times^{G_x} S \to X$, $\gamma : S \to T_x S$ be as above. By Lemma 4.2, $\phi^{-1}(X^u) = (G \times^{G_x} S)^u$. In addition one can verify that $(G \times^{G_x} S)^u = G \times^{G_x} S^u$. We deduce $\operatorname{codim}(X^u, X) \leq \operatorname{codim}(G \times^{G_x} S, (G \times^{G_x} S)^u)) = \operatorname{codim}(S^u, S)$ (as ϕ is étale). Let $\operatorname{codim}_x(S^u, S) := \dim S - \dim \operatorname{Spec} \mathcal{O}_{S^u, x}$ be the local codimension at x. Then we compute

$$
\begin{aligned}
\operatorname{codim}(S^u, S) &\leq \operatorname{codim}_x(S^u, S) \\
&= \operatorname{codim}_0((T_x S)^u, T_x S) \\
&= \operatorname{codim}((T_x S)^u, T_x S).
\end{aligned}
$$

For the first equality we use that γ is strongly étale and hence a local homeomorphism, and moreover $S^u = \gamma^{-1}((T_x S)^u)$ by Lemma 4.2. For the second equality we use that $T_x S$ is a G_x-representation and that $(T_x S)^u$ is defined by a homogeneous ideal.

To prove the converse we have to show that $\operatorname{codim}(X^u, X) = \operatorname{codim}((T_x S)^u, T_x S)$ for at least one x. By reversing the above arguments it follows that we may take x to be a point with closed orbit in an irreducible component of X^u of maximal dimension (guaranteeing $\operatorname{codim}(S^u, S) = \operatorname{codim}_x(S^u, S)$). $\square$

4.5. **Equivariant vector bundles.**

Lemma 4.5. *If $\mathcal{V}$ is a G-equivariant vector bundle on X and $x \in X$ is G-invariant point then x has a saturated affine G-invariant neighborhood on which $\mathcal{V}$ is of the form $V \otimes \mathcal{O}_X$ for the G-representation V which is the fiber of $\mathcal{V}$ in x, i.e. $V = \mathcal{V} \otimes_X k(x)$.*

Proof. By taking the pullback of an affine neighborhood of the image of x in $X /\!\!/ G$ we may reduce to the case that X is affine. Choose a G-invariant splitting $\Gamma(X, \mathcal{V}) \to V$ (since X is affine $V = \Gamma(X, \mathcal{V}) \otimes k(x)$). This gives us an G-equivariant map $V \otimes \mathcal{O}_X \to \mathcal{V}$ which is an isomorphism in a neighborhood of x. The maximal neighborhood U on which this is the case must be G-equivariant and open. Then $\pi(X \setminus U) = (X \setminus U)/\!\!/G$ and $\{\pi(x)\} = \{x\}/\!\!/G$ are disjoint closed subsets of $X/\!\!/G$ (see [Bri, Theorem 1.24(iv)]). Finally the saturated neigbourhood of x we want is $X \setminus \pi^{-1}(\pi(X \setminus U)) = \pi^{-1}(X/\!\!/G \setminus \pi(X \setminus U))$; i.e. the maximal saturated subset of U. $\square$

Lemma 4.6. *Let $\mathcal{V}$ be a G-equivariant vector bundle on X. We may choose the slice S as in §4.4 in such a way that the pullback of $\mathcal{V}$ to $G \times^{G_x} S$ coincides with $G \times^{G_x} (V \otimes \mathcal{O}_S)$ for $V = \mathcal{V} \otimes k(x)$.*

Proof. First take an arbitrary slice S at x. We pull back $\mathcal{V}$ to $G \times^{G_x} S$ and replace X by $G \times^{G_x} S$. Now the G-equivariant vector bundle $\mathcal{V}$ on $G \times^{G_x} S$ restricts to a G_x-equivariant vector bundle $\mathcal{V}_S$ on S, such that $\mathcal{V} = G \times^{G_x} \mathcal{V}_S$. We then apply Lemma 4.5 to obtain a saturated affine G_x-invariant open subset $x \in S' \subset S$ such that $\mathcal{V}_S \mid S' \cong V \otimes \mathcal{O}_{S'}$. Since a saturated open immersion is a special case of a strongly étale map, $G \times^{G_x} S' \hookrightarrow G \times^{G_x} S \to X$ is strongly étale and so we may replace S by S'. $\square$

4.6. **The canonical sheaf on X^s/G.** The following lemma gives the precise relation between the canonical sheaf of the stack X^s/G and ω_{X^s} considered as a G-equivariant sheaf.

Lemma 4.7. *There is an isomorphism $\omega_{X^s/G} \cong \wedge^{\dim G}\mathfrak{g} \otimes \omega_{X^s}$.*

Proof. The lemma follows from the fact that the cotangent complex on X/G is given by the complex $\Omega_X \to \mathfrak{g}^* \otimes \mathcal{O}_X$ with Ω_X in degree zero. Since the map is surjective on X^s by the definition of X^s, we get the exact sequence $0 \to \Omega_{X^s/G} \to \Omega_{X^s} \to \mathfrak{g}^* \otimes \mathcal{O}_{X^s} \to 0$. Taking determinants we get the desired equality. $\square$

Remark 4.8. Note that $\alpha = \wedge^{\dim G}\mathfrak{g}$ can only be nontrivial in the case when G is nonconnected. Furthermore α^2 is always trivial (see e.g. [Kno89, p.41]).

5. The Kirwan resolution

Let X be as in §4. We assume in addition that X satisfies (H1), i.e. $X^s \neq \emptyset$.

5.1. **The Reichstein transform.** The steps in the partial resolution of $X/\!\!/G$ described in [Kir85] were reinterpreted by Reichstein [Rei89], and generalized by Edidin and More [EM12] and Edidin and Rydh [ER17]. They are now known as "Reichstein transforms" [EM12]. We will use $(-)^R$ for notations related to the Reichstein transform.

Let μ be the maximal dimension of the stabilizers of the G-action on X and for simplicity we put $Z = X_\mu := \{x \in X \mid \dim G_x = \mu\}$, which is closed and smooth (see e.g. [ER17, Proposition B.2]). Assume $\mu > 0$. Put

$$\bar{Z} = \{x \in X \mid \overline{Gx} \cap Z \neq 0\}.$$

Then $\bar{Z} = \pi^{-1}(Z/\!\!/G)$, so it is closed as well. Let $\xi : \tilde{X} \to X$ be the blowup of X in Z and let $\tilde{Z} \subset \tilde{X}$ be the strict transform of $\bar{Z}$. Let ξ^R be the restriction of ξ to $X^R := \tilde{X} - \tilde{Z}$. The resulting map $\xi^R_s : X^R/G \to X/G$ is called the *Reichstein transform* of X/G.

Remark 5.1. In [ER17], X^R/G is denoted by $R_G(X, Z)$.

Let $\mathcal{O}_{\tilde{X}}(1)$ be the tautological relatively ample line bundle on $\tilde{X}$ and let $\mathcal{O}_{X^R}(1)$ be its restriction to X^R. Let E^R denote the exceptional divisor in X^R. The following can be extracted from [ER17].

Proposition 5.2. *The following properties hold for X^R:*

(1) *X^R has a good quotient $\pi^R : X^R \to X^R /\!\!/ G$.*
(2) *The induced map $X^R /\!\!/ G \to X /\!\!/ G$ is proper.*
(3) *X^R satisfies (H1).*
(4) *If X satisfies (H2) then X^R also satisfies (H2).*
(5) *We have $\mu(X^R) < \mu(X)$.*
(6) *For some $N > 0$, $\mathcal{O}_{X^R}(N)$ is the pullback of the line bundle $(\pi^R_* \mathcal{O}_{X^R}(N))^G$ on $X^R /\!\!/ G$.*

Proof. (1), (2), (5) follow from [ER17, Theorem 2.11 (2a),(2c),(3)]. (6) follows by Remark 4.1. The fact that (H1) is true (asserted in (3)) follows from the assumption that X satisfies (H1) and the fact that $\xi^R : X^R - E^R = (\xi^R)^{-1}(X - Z) \to X - Z$ is an isomorphism. For (4) we observe that X and X^R differ in codimension 1 by the exceptional divisor E^R. We have to prove that a generic point of E^R is stable. To this end we use reduction to the linear case made possible by Lemma 4.4 and Lemma 5.3 below, see Lemma 5.4. As we will now switch to the notations introduced in those lemmas, the reader is advised to consult §5.3 first.

Note that G-stability and G_e-stability are equivalent. The exceptional divisor E^R is given by $W_0 \times \mathbb{P}(W_1)^{ss}$ (see Lemma 5.4). Let $(w_0, w_1) \in W_0 \times W_1$ be a generic point. It is G_e-stable by (H1). Since G_e acts trivially on W_0 this implies in particular that w_1 is G_e-stable. By [Bri, Proposition 1.31], $\mathbb{P}(W_1)^s = \mathbb{P}(W_1^s)$ and hence $[w_1]$ is G_e-stable. Thus, $(w_0, [w_1])$ is G_e-stable as well. $\square$

In the following commutative diagram we summarize the notations that have been introduced up to now and we also introduce some additional ones which should be self explanatory.

$$
\begin{tikzcd}
X^R \arrow[r] \arrow[d, "\xi^R"'] \arrow[rr, bend left, "\pi^R"] & X^R/G \arrow[r, "\pi^R_s"] \arrow[d, "\xi^R_s"'] & X^R /\!\!/ G \arrow[d, "\overline{\xi^R}"] \\
X \arrow[r] \arrow[rr, bend right, "\pi"'] & X/G \arrow[r, "\pi_s"'] & X /\!\!/ G
\end{tikzcd}
\tag{5.1}
$$

5.2. **The Kirwan resolution.** By repeatedly applying the Reichstein transform, the maximal stabilizer dimension ultimately becomes 0 by Proposition 5.2(5). Hence we arrive at a commutative diagram

$$
\begin{tikzcd}
\mathbf{X}/G \arrow[r, "\pi'_s"] \arrow[d, "\Xi"'] & \mathbf{X} /\!\!/ G \arrow[d, "\bar{\Xi}"] \\
X/G \arrow[r, "\pi_s"'] & X /\!\!/ G
\end{tikzcd}
$$

where $\mathbf{X}/G$ is a DM stack and hence $\mathbf{X} /\!\!/ G$ has finite quotient singularities. We call $\mathbf{X} /\!\!/ G$ (or perhaps $\mathbf{X}/G$) the Kirwan (partial) resolution of $X /\!\!/ G$.

5.3. **Reduction of the Reichstein transform to the linear case.** Let H be a reductive group. For a H-representation W we choose a decomposition $W = W_0 \oplus W_1$ of H-representations, where $W_0 = W^{H_e}$.

Lemma 5.3. *With notation as in §4.4, 5.1, let x be a point with maximal stabilizer dimension. Let S be a smooth affine slice at x. We have $(G \times^{G_x} S)^R = G \times^{G_x} S^{R,G_x}$, where S^{R,G_x} is the Reichstein transform of S with respect to G_x, and there is a G-equivariant cartesian diagram*

$$\begin{tikzcd} G \times^{G_x} S^{R,G_x} \arrow[r] \arrow[d] & X^R \arrow[d] \\ G \times^{G_x} S \arrow[r] & X \end{tikzcd}$$

in which the horizontal arrows are strongly étale.

We have $X_\mu \cap S = S^{G_{x,e}}$. Denote $W = T_x S$. Then $W_\mu = W_0$ and there is a G_x-equivariant cartesian diagram

$$\begin{tikzcd} S^{R,G_x} \arrow[r] \arrow[d] & W^{R,G_x} \arrow[d] \\ S \arrow[r] & W \end{tikzcd}$$

in which the horizontal maps are strongly étale.

Proof. The equality $(G \times^{G_x} S)^R = G \times^{G_x} S^{R,G_x}$ is an easy verification using the equivalence between the categories of G-equivariant schemes over G/G_x and G_x-equivariant schemes.

Both diagrams follow from [ER17, Proposition 6.6, Diagram (6.6.1)] and the Luna slice theorem §4.4 (as strongly étale morphism is strong [ER17, Definition 6.4]). The observation that both strong morphisms and étale morphisms are preserved under pullback yields that the upper arrows in the diagrams are strongly étale. □

In the case of a representation the Reichstein transform has a more concrete description recorded in the following lemma.

Let $\mathbb{P}(W_1)^{ss}$, $\mathbb{P}(W_1)^{ns}$ be respectively the semi-stable part of $\mathbb{P}(W_1)$ and its complement, corresponding to the G-linearisation $\mathcal{O}(1)$. As $\mathbb{P}(W_1) = (W_1 - \{0\})/G_m$ we alternatively have $\mathbb{P}(W_1)^{ss} = W_1^{ss,1}/G_m$, where $1 \in X(G_m) = \mathbb{Z}$ is the identity character. If W_1^{null} is the G-nullcone in W_1 then $W_1^{ss,1} = W_1 \setminus W_1^{\text{null}}$.

Lemma 5.4. *If $X = W$ is a representation, $X^R = \underline{\mathrm{Spec}}(\mathrm{Sym}_{W_0 \times \mathbb{P}(W_1)^{ss}}(\mathcal{O}(1)))$.*

Proof. We have $\tilde{X} = \underline{\mathrm{Spec}}(\mathrm{Sym}_{W_0 \times \mathbb{P}(W_1)}(\mathcal{O}(1)))$, $\bar{Z} = W_0 \times W_1^{\text{null}}$ and $\tilde{Z} = \underline{\mathrm{Spec}}(\mathrm{Sym}_{W_0 \times \mathbb{P}(W_1)^{ns}}(\mathcal{O}(1)))$, which implies the claim. □

We obtain the following diagram

$$\begin{tikzcd} E^R/G \arrow[r, bend right, "s"'] \arrow[d, "\pi^R_{E,s}"] & X^R/G \arrow[l, dotted, "\theta"'] \arrow[r, "\xi^R"] \arrow[d, "\pi^R_s"] & X/G \arrow[d, "\pi_s"] \\ E^R/\!\!/G \arrow[r, bend right, "\bar{s}"'] & X^R/\!\!/G \arrow[l, dotted, "\bar{\theta}"'] \arrow[r, "\overline{\xi^R}"] & X/\!\!/G \end{tikzcd} \tag{5.2}$$

where s, $\bar{s}$ are obtained from the inclusion of E^R in X^R and where θ and $\bar{\theta}$ only exist in the linear case and are obtained from the projection of the line bundle $W^R = \underline{\operatorname{Spec}}(\operatorname{Sym}_{W_0\times\mathbb{P}(W_1)^{ss}}(\mathcal{O}(1)))$ to the base $W_0 \times \mathbb{P}(W_1)^{ss}$.

From Lemma 5.4 we obtain a very concrete description of the Reichstein transform in the linear case.

Lemma 5.5. *Put $S = \operatorname{Sym} W^\vee$ considered as a graded ring by giving $W_i^\vee$ degree i, $i \in \{0,1\}$. Then $W^R/\!\!/G$ is covered by affine charts of the form $\operatorname{Spec}((S_f^G)_{\geq 0})$ for homogeneous $f \in S_{>0}^G$.*

Proof. It follows from Lemma 5.4 that W^R is covered by affine charts $\operatorname{Spec}((S_f)_{\geq 0})$ for homogeneous $f \in S_{>0}^G$ (by the definition of semi-stable points), and hence $W^R/\!\!/G$ is covered by affine charts as stated. □

Remark 5.6. Elaborating on Lemma 5.5 we obtain yet another concrete description of the Reichstein transform in the linear case as a weighted blowup. Let R be a $\mathbb{Z}$-graded ring and put $R^\dagger = \oplus_{n\geq 0} R_{\geq n}$, where the right-hand side is $\mathbb{N}$-graded by the index n. Then the *weighted blowup* of $\operatorname{Spec} R$ is defined as $\operatorname{Proj} R^\dagger$. One easily checks that $W^R/\!\!/G$ is given by the weighted blowup of $\operatorname{Spec} S^G = W/\!\!/G$.

6. The embedding of an NCCR in the Kirwan resolution

Let X be as in §4 and assume that X moreover satisfies (H2). Assume we are in the setting of §5.1. Let $\mathcal{U}$ be a G-equivariant vector bundle on X and define

$$\Lambda := \pi_{s*}\,\mathcal{E}nd_X(\mathcal{U}), \quad \Lambda' := \pi_{s*}^R\,\mathcal{E}nd_{X^R}(\xi^{R*}\mathcal{U}).$$

Lemma 6.1. *If Λ is Cohen-Macaulay (as a sheaf of $\mathcal{O}_{X/\!\!/G}$-algebras) then the canonical morphism*

$$\Lambda \to R\overline{\xi_*^R}\Lambda'$$

is an isomorphism.

Proof. This statement is local for the étale topology and hence Lemma 5.3 allows us to reduce to the case that $X = W$ is a representation of G. Moreover we may assume by Lemma 4.6 that $\mathcal{U} = U \otimes \mathcal{O}_W$. By Lemma 5.4 we then have $W^R = \underline{\operatorname{Spec}}(\operatorname{Sym}_{W_0\times\mathbb{P}(W_1)^{ss}}(\mathcal{O}(1)))$. Using the diagram (5.1) we see that

$$R\overline{\xi_*^R}\Lambda' = \pi_{s*}R\xi_{s*}^R(\operatorname{End}(U)\otimes\mathcal{O}_{W^R})\,.$$

Write $\xi_s : \tilde{W}/G \to W/G$ for the map induced from $\xi : \tilde{W} \to W$ and similarly j_s for the inclusion $W^R/G \to \tilde{W}/G$. Then we have

$$\begin{aligned}\pi_{s*}R\xi_{s*}^R(\operatorname{End}(U)\otimes\mathcal{O}_{W^R}) &= \pi_{s*}R(\xi_s j_s)_*(\operatorname{End}(U)\otimes\mathcal{O}_{W^R})\\ &= \pi_{s*}(\operatorname{End}(U)\otimes R\xi_{s*}Rj_{s*}\mathcal{O}_{W^R}).\end{aligned}$$

We will use the standard distinguished triangle for cohomology with support

$$\mathcal{R}\Gamma_{\tilde{W}-W^R}(\tilde{W},\mathcal{O}_{\tilde{W}}) \to \mathcal{O}_{\tilde{W}} \to Rj_{s*}\mathcal{O}_{W^R} \to$$

which after applying $\pi_{s*}(\operatorname{End}(U)\otimes R\xi_{s*}(-))$ yields a distinguished triangle

$$\pi_{s*}(\operatorname{End}(U)\otimes R\xi_{s*}\mathcal{R}\Gamma_{\tilde{W}-W^R}(\tilde{W},\mathcal{O}_{\tilde{W}})) \to \Lambda \to R\overline{\xi_*^R}\Lambda' \to .$$

It follows that we need to show

$$\pi_{s*}(\operatorname{End}(U)\otimes R\xi_{s*}\mathcal{R}\Gamma_{\tilde{W}-W^R}(\tilde{W},\mathcal{O}_{\tilde{W}})) = 0. \tag{6.1}$$

We may as well compute $\Gamma(W/\!\!/G, \pi_{s*}(\mathrm{End}(U) \otimes R\xi_{s*}\mathcal{R}\Gamma_{\tilde{W}-W^R}(\tilde{W}, \mathcal{O}_{\tilde{W}})))$ since $W/\!\!/G$ is affine. We have

$$\Gamma(W/\!\!/G, \pi_{s*}(\mathrm{End}(U) \otimes R\xi_{s*}\mathcal{R}\Gamma_{\tilde{W}-W^R}(\tilde{W}, \mathcal{O}_{\tilde{W}}))) = (\mathrm{End}(U) \otimes R\Gamma(\tilde{W}, \mathcal{R}\Gamma_{\tilde{W}-W^R}(\tilde{W}, \mathcal{O}_{\tilde{W}})))^G. \tag{6.2}$$

Let $E^{ns} = W_0 \times \mathbb{P}(W_1)^{ns}$, $\tilde{E} = W_0 \times \mathbb{P}(W_1)$. Since $\tilde{W} - W^R = \theta^{-1}(E^{ns})$ we have

$$R\Gamma(\tilde{W}, \mathcal{R}\Gamma_{\tilde{W}-W^R}(\tilde{W}, \mathcal{O}_{\tilde{W}})) = \bigoplus_{n\geq 0} R\Gamma(\tilde{E}, \mathcal{R}\Gamma_{E^{ns}}(\tilde{E}, \mathcal{O}_{\tilde{E}}(n))). \tag{6.3}$$

Put $S = \mathrm{Sym}(W^\vee) = \Gamma(W, \mathcal{O}_W)$. We put a grading on S by giving $W_i^\vee$ degree i for $i \in \{0, 1\}$. Let ω denote the composition

$$\mathrm{Gr}(S) \xrightarrow{\tilde{?}} \mathrm{Qch}(\tilde{E}) \xrightarrow{\Gamma_*} \mathrm{Gr}(S)$$

where $\Gamma_*(\tilde{E}, \mathcal{M}) = \bigoplus_{n\in\mathbb{Z}} \Gamma(\tilde{E}, \mathcal{M}(n))$ and the first (exact) functor is the usual correspondence between graded S-modules and quasi-coherent sheaves on $\tilde{E}$. It is easy to see that $\tilde{?}$ preserves injectives and hence $R\omega = R\Gamma_* \circ \tilde{?}$. We have

$$\mathcal{R}\Gamma_{E^{ns}}(\tilde{E}, \mathcal{O}_{\tilde{E}}(n)) = R\Gamma_{W_0\times W_1^{\mathrm{null}}}(W, \mathcal{O}_W)(n)\tilde{}.$$

Hence the part of degree n of the right-hand side of (6.3) equals

$$\begin{aligned}(R\Gamma_*((R\Gamma_{W_0\times W_1^{\mathrm{null}}}(W, \mathcal{O}_W)(n))\tilde{}))_0 &= (R\omega(R\Gamma_{W_0\times W_1^{\mathrm{null}}}(W, \mathcal{O}_W)(n)))_0 \\ &= (R\omega R\Gamma_{W_0\times W_1^{\mathrm{null}}}(W, \mathcal{O}_W))_n.\end{aligned} \tag{6.4}$$

There is a distinguished triangle in $D(\mathrm{Gr}(S))$

$$R\Gamma_{S_{>0}}(M) \to M \to R\omega(M) \to,$$

for every $M \in D(\mathrm{Gr}(S))$, which applied to $M = R\Gamma_{W_0\times W_1^{\mathrm{null}}}(W, \mathcal{O}_W)$ gives the distinguished triangle

$$\begin{aligned}R\Gamma_{W_0} R\Gamma_{W_0\times W_1^{\mathrm{null}}}(W, \mathcal{O}_W) \to R\Gamma_{W_0\times W_1^{\mathrm{null}}}&(W, \mathcal{O}_W) \\ &\to R\omega R\Gamma_{W_0\times W_1^{\mathrm{null}}}(W, \mathcal{O}_W) \to .\end{aligned}$$

Since $W_0 \subset W_0 \times W_1^{\mathrm{null}}$ the first term equals

$$R\Gamma_{W_0} R\Gamma_{W_0\times W_1^{\mathrm{null}}}(W, \mathcal{O}_W) = R\Gamma_{W_0}(W, \mathcal{O}_W) \tag{6.5}$$

which is 0 in degrees ≥ 0. Thus, the right-hand side of (6.3) equals (using (6.4)) $(R\Gamma_{W_0\times W_1^{\mathrm{null}}}(W, \mathcal{O}_W))_{\geq 0}$. So (6.1) translates (via (6.2)) into

$$(\mathrm{End}(U) \otimes R\Gamma_{W_0\times W_1^{\mathrm{null}}}(W, \mathcal{O}_W))^G_{\geq 0} = 0$$

which by e.g. [VdB89, Lemma 4.1] is equivalent to

$$(R\Gamma_{W_0/\!\!/G}\Lambda)_{\geq 0} = 0.$$

By local duality (see Corollary A.3) and Cohen-Macaulayness of Λ this reduces to showing $H_{\leq 0} = 0$ for $H := \mathrm{Hom}_{W/\!\!/G}(\Lambda, \omega_{W/\!\!/G})$. Note that H is reflexive and localization commutes with Hom, so we may reduce to codimension 1 and replace W by W^s due to (H2). As $W^s/G \to W^s/\!\!/G$ is finite, it is also proper. Therefore we can apply Grothendieck duality for DM stacks [Nir08, Corollary 2.10]. Setting $d_1 = \dim W_1$, $d = \dim W$ we obtain

$$\begin{aligned} H &= \operatorname{Hom}_{W^s/\!\!/G}(\pi_{s*}(\operatorname{End}(U) \otimes \mathcal{O}_W), \omega_{W^s/\!\!/G}) \\ &= \operatorname{Hom}_{W^s/G}(\operatorname{End}(U) \otimes \mathcal{O}_{W^s}, \pi_s^! \omega_{W^s/\!\!/G}) \\ &= \operatorname{Hom}_{W^s/G}(\operatorname{End}(U) \otimes \mathcal{O}_{W^s}, \omega_{W^s/G}) \\ &= \operatorname{Hom}_{W^s/G}(\operatorname{End}(U) \otimes \mathcal{O}_{W^s}, \wedge^{\dim G}\mathfrak{g} \otimes \wedge^d W \otimes \mathcal{O}_{W^s}(-d_1)) \\ &= (\operatorname{End}(U) \otimes \wedge^{\dim G}\mathfrak{g} \otimes \wedge^d W \otimes S)^G(-d_1) \end{aligned} \tag{6.6}$$

where the third equality is [Nir08, Theorem 2.22][6], the fourth equality follows from Lemma 4.7, and the fifth equality from the hypothesis (H2). Hence $H_{\le 0} = (\operatorname{End}(U) \otimes \wedge^{\dim G}\mathfrak{g} \otimes \wedge^d W \otimes S)^G_{\le -d_1}$. Since $(\operatorname{End}(U) \otimes \wedge^{\dim G}\mathfrak{g} \otimes \wedge^d W \otimes S)^G$ lives in nonnegative degrees, the conclusion follows. □

Below we let $\hat{\xi}$ be the morphism of ringed spaces

$$\hat{\xi} : (X^R/\!\!/G, \Lambda') \to (X/\!\!/G, \Lambda)$$

obtained from $\overline{\xi^R}$.

Corollary 6.2. *Assume Λ is Cohen-Macaulay. Then $L\hat{\xi}^* : D(\Lambda) \to D(\Lambda')$ is a full faithful embedding.*

Proof. Note that on the level of $\mathcal{O}_{X^R}$-modules, $R\hat{\xi}_*$ is just $R\overline{\xi^R_*}$. It is sufficient to prove that for any $\mathcal{F} \in D(\Lambda)$ the canonical morphism $\mathcal{F} \to R\hat{\xi}_* L\hat{\xi}^* \mathcal{F}$ is an isomorphism.

This is a local statement and hence we may assume that X is affine. We may then replace $\mathcal{F}$ by a K-projective resolution $\mathcal{P}^\bullet$ with projective terms. Then $L\hat{\xi}^*\mathcal{F} = \hat{\xi}^*\mathcal{P}^\bullet$ and moreover by Lemma 6.1 $\hat{\xi}^*\mathcal{P}^\bullet$ consists of objects acyclic for $\overline{\xi^R_*}$. Since $\overline{\xi^R_*}$ has finite homological dimension we obtain $R\overline{\xi^R_*} L\hat{\xi}^*\mathcal{F} = \overline{\xi^R_*}\hat{\xi}^*\mathcal{P}^\bullet = \mathcal{P}^\bullet$ where for the last equality we use again Lemma 6.1. □

As an immediate corollary of Corollary 6.2 we get the following embedding of $D(\Lambda)$ to $D(X^R/G)$.

Corollary 6.3. *Assume that Λ is Cohen-Macaulay. There is a commutative diagram of derived categories*

$$\begin{tikzcd} D(\Lambda) \arrow[r, hook, "-\overset{L}{\otimes}_\Lambda \mathcal{U}"] \arrow[d, hook, "L\hat{\xi}^*"'] & D(X/G) \arrow[d, "L\xi^{R*}"] \\ D(\Lambda') \arrow[r, hook, "-\overset{L}{\otimes}_{\Lambda'} \xi^{R*}\mathcal{U}"'] & D(X^R/G) \end{tikzcd} \tag{6.7}$$

Lemma 6.4. *If Λ is Cohen-Macaulay then Λ' is Cohen-Macaulay.*

[6][Nir08, Theorem 2.22] is strictly speaking for proper DM stacks. However, this assumption can be circumvented by first applying compactification (with smooth DM stacks). See the first paragraph of the proof of [Ber18, Theorem 1.1].

Proof. Since we may check Cohen-Macaulayness étale locally, we can reduce by Lemma 5.3 to the linear case. We use the same notations as in the proof of Lemma 6.1. Using Lemmas 5.5, Lemma 4.6, locally Λ' is of the form $(\Lambda_f)_{\geq 0}$ for homogeneous $f \in (S^G)_{>0}$.

We then need to prove that $(\Lambda_f)_{\geq 0}$ is Cohen-Macaulay. Let $m = \deg f$. We put $A = \Lambda_f$. A is Cohen-Macaulay since it is localization of Λ which is Cohen-Macaulay. Note that A contains a Laurent polynomial ring $A' = A_0[f, f^{-1}]$ as a direct summand. As an ascending chain of one sided ideals in A' may be extended to an ascending chain of one sided ideals in A we see that A' is noetherian. A similar argument shows that the $A_i[f, f^{-1}]$, for $0 \leq i < m$, are noetherian A'-modules and so they are finitely generated. In particular, $A = \oplus_{0\leq i<m} A_i[f, f^{-1}]$ is finitely generated as an A'-module.

Since A is Cohen-Macaulay over A, it is Cohen-Macaulay over A'. As $A = \oplus_{0\leq i<m} A_i[f, f^{-1}]$ the A'-summands $A_i[f, f^{-1}]$ of A are also Cohen-Macaulay A'-modules. Quotienting by the nonzero divisor $f - 1$ we see that A_i is a Cohen-Macaulay A_0-module for $0 \leq i < m$. Thus, $A_i[f]$ is a Cohen-Macaulay $A_0[f]$-module for $0 \leq i < m$. Therefore $A_{\geq 0} = \oplus_{0\leq i<m} A_i[f]$ is Cohen-Macaulay $A_0[f]$-module and thus $A_{\geq 0}$ is Cohen-Macaulay (as it is a finitely generated $A_0[f]$-module). $\square$

In the following proposition we show that Λ as in Lemma 6.4 can be embedded in the smooth Deligne-Mumford stack obtained by the Kirwan resolution.

Proposition 6.5. *Let X be a smooth G-scheme with a good quotient $\pi : X \to X/\!\!/G$ which satisfies in addition (H2).*[7] *Let $\Xi : \mathbf{X}/G \to X/G$ be the Kirwan resolution (see §5.2).*

Let $\mathcal{U}$ be a G-equivariant vector bundle on X and assume that $\Lambda = \pi_{s*} \mathcal{E}nd_X(\mathcal{U})$ is Cohen-Macaulay on $X/\!\!/G$. Put $\mathcal{U}' = \Xi^*\mathcal{U}$, $\Sigma = \pi'_{s*} \mathcal{E}nd_{\mathbf{X}}(\mathcal{U}')$. There is a commutative diagram of derived categories

$$\begin{tikzcd} D(\Lambda) \arrow[r, hook, "-\overset{L}{\otimes}_\Lambda \mathcal{U}"] \arrow[d, hook, "L\hat{\Xi}^*"'] & D(X/G) \arrow[d, "L\Xi^*"] \\ D(\Sigma) \arrow[r, hook, "-\overset{L}{\otimes}_\Sigma \mathcal{U}'"'] & D(\mathbf{X}/G) \end{tikzcd} \tag{6.8}$$

where $\hat{\Xi}$ is the induced morphism of ringed spaces $(\mathbf{X}/\!\!/G, \Sigma) \to (X/\!\!/G, \Lambda)$.

Proof. The commutativity of the diagram and full faithfulness of the horizontal arrows are straightforward. It remains to show full faithfulness of the left most vertical arrow. By construction, $L\hat{\Xi}^*$ is the composition of $L\hat{\xi}^*$'s which correspond to a single Reichstein transform. By Lemma 6.4 we are reduced to showing full faithfulness for a single Reichstein transform. In that case the conclusion follows by Corollary 6.2. $\square$

Remark 6.6. Note that the rightmost vertical map in the diagram (6.8) is in general not fully faithful.

[7] (H2) was imposed on in the beginning of §6 and it was used explicitly in the proof of Lemma 6.1 and implicitly (via Lemma 6.1) in Corollaries 6.2, 6.3.

6.1. **Embedding of $D(\Lambda_Z)$ in $D(X^R/G)$.** In this subsection for use below (c.f. §8) we extract some consequences of the above results and their proofs.

From the proof of Lemma 6.1 we can extract the following.

Lemma 6.7. *Assume that Λ is Cohen-Macaulay. Let $\mathcal{U}_Z, \mathcal{U}_{E^R}$ denote the restrictions of $\mathcal{U}$, $\xi^{R*}\mathcal{U}$ to Z, E^R, respectively. Let $\pi_{Z,s} : Z/G \to Z/\!\!/G$ be the quotient map and denote the restriction/ corestriction of $\overline{\xi^R}$ to a map $E^R/\!\!/G \to Z/\!\!/G$ by $\overline{\xi^R_E}$ so that we have a commutative diagram*

$$\begin{tikzcd} E^R/\!\!/G \arrow[r, hook, "\bar{s}"] \arrow[d, "\overline{\xi^R_E}"'] & X^R/\!\!/G \arrow[d, "\overline{\xi^R}"] \\ Z/\!\!/G \arrow[r] & X/\!\!/G \end{tikzcd}$$

Put

$$\Lambda_Z := \pi_{Z,s,*}\,\mathcal{E}nd_Z(\mathcal{U}_Z), \quad \Lambda'_Z := \pi^R_{E,s,*}\,\mathcal{E}nd_{E^R}(\mathcal{U}_{E^R}).$$

Then

$$R\overline{\xi^R_{E*}}\Lambda'_Z = \Lambda_Z,$$

and moreover on every connected component Z_i of Z

$$R\overline{\xi^R_{E*}}\pi^R_{E,s,*}(\mathcal{E}nd_{E^R}(\mathcal{U}_{E^R})(l)) \mid_{Z_i/\!\!/G_i} = 0 \tag{6.9}$$

for $-c_i < l < 0$ with $c_i = \operatorname{codim}(Z_i, X)$ and where G_i is the stabilizer of Z_i for the action of G on the connected components of Z.

Proof. This is proved in a similar (but easier) way as Lemma 6.1. We pass to the linear case for a point $x \in Z_i$. In this case $c_i = d_1 = \dim W_1$ (with the notation as in §5.3). Following the steps of the proof one is reduced to showing that $(\operatorname{End}(U) \otimes R\omega R\Gamma_{W_0 \times W_1^{\text{null}}}(W, \mathcal{O}_W))^G_l = 0$ for $-d_1 < l \leq 0$. Then we use the extra fact (after (6.5)) that $R\Gamma_{W_0}(W, \mathcal{O}_W)$ is zero in degrees $> -d_1$ (in the proof of Lemma 6.1 it was only needed that it is 0 in degrees ≥ 0). The proof then further proceeds as the proof of the lemma, where the bound on l again comes in at the end of the proof. □

Lemma 6.7 (together with the proof of Corollary 6.2) makes it possible to construct an embedding of $D(\Lambda_Z)$ in $D(X^R/G)$. Let

$$\Lambda_{E^R} = \pi^R_{s*}\,\mathrm{R}\mathcal{E}nd_{X^R}(s_*\mathcal{U}_{E^R}).$$

Let $\xi^R_E : E^R \to Z$ denote the restriction/corestriction of $\xi^R : X^R \to X$.

Corollary 6.8. *Assume Λ is Cohen-Macaulay. Then $R\overline{\xi^R_*}\Lambda_{E^R} \cong \Lambda_Z$. Denote by $\hat{\xi}_E$ the following morphism of DG-ringed spaces*[8] *(obtained for $\overline{\xi^R}$)*[9]

$$\hat{\xi}_E : (X^R/\!\!/G, \Lambda_{E^R}) \to (X/\!\!/G, \Lambda_Z).$$

[8] Λ_{E^R} is a sheaf of DG-algebras.

[9] We silently identify Λ_Z with $i_*\Lambda_Z$ for $i : Z/\!\!/G \hookrightarrow X/\!\!/G$.

Then there is a commutative diagram of derived categories

$$
\begin{tikzcd}
D(\Lambda_Z) \arrow[r, hook, "-\overset{L}{\otimes}_{\Lambda_Z}\mathcal{U}_Z"] \arrow[d, hook, "L\hat{\xi}_E^*"'] & D(Z/G) \arrow[d, "Rs_* L\xi_E^{R*}"] \\
D(\Lambda_{E^R}) \arrow[r, hook, "-\overset{L}{\otimes}_{\Lambda_{E^R}} s_*\mathcal{U}_{E^R}"'] & D(X^R/G)
\end{tikzcd}
\tag{6.10}
$$

Proof. As in Corollary 6.2, it follows from $R\overline{\xi^R_*}\Lambda_{E^R} \cong \Lambda_Z$ that $L\hat{\xi}_E^*$ is fully faithful. Moreover, the horizontal arrows are fully faithful and the diagram commutes. Thus, it suffices to prove that $R\overline{\xi^R_*}\Lambda_{E^R} \cong \Lambda_Z$.

There is a standard exact sequence on X^R/G

$$0 \to \xi^{R*}\mathcal{U}(1) \xrightarrow{t} \xi^{R*}\mathcal{U} \to s_*\mathcal{U}_{E^R} \to 0 \tag{6.11}$$

obtained by restriction to X^R of a similar sequence valid for any blowup.

Applying $\mathrm{R}\mathcal{H}om_{X^R}(-, s_*\mathcal{U}_{E^R})$ to (6.11) we get the distinguished triangle

$$\mathrm{R}\mathcal{E}nd_{X^R}(s_*\mathcal{U}_{E^R}) \to \mathrm{R}\mathcal{H}om_{X^R}(\xi^{R*}\mathcal{U}, s_*\mathcal{U}_{E^R}) \to \mathrm{R}\mathcal{H}om_{X^R}(\xi^{R*}\mathcal{U}, s_*\mathcal{U}_{E^R})(-1) \to . \tag{6.12}$$

By adjointness,

$$\mathrm{R}\mathcal{H}om_{X^R}(\xi^{R*}\mathcal{U}, s_*\mathcal{U}_{E^R}) = s_*\,\mathrm{R}\mathcal{H}om_{E^R}(\mathcal{U}_{E^R}, \mathcal{U}_{E^R}). \tag{6.13}$$

Applying $R\overline{\xi^R_*}\pi^R_{s*}$ to (6.12) we obtain by Lemma 6.7 that $R\overline{\xi^R_*}\Lambda_{E^R} \cong \Lambda_Z$ as desired. □

7. Homologically homogeneous endomorphism sheaves

Endomorphism sheaves of vector bundles appeared in §6 above. In this section we discuss the local properties of vector bundles whose endomorphism sheaves have good homological properties. This will be used in subsequent sections. More precisely the "fullness" property will be used in the proof of semi-orthogonal decomposition for the Kirwan resolution given in Theorem 8.15 and the "saturation" property will be important for the associated geometric interpretation obtained in Corollary 9.9.

7.1. Equivariant vector bundles in the case of constant stabilizer dimension.

Let Z be a G-equivariant smooth k-scheme with a good quotient $\pi : Z \to Z/\!\!/G$. We assume that the stabilizers $(G_x)_{x\in Z}$ have dimension independent of x. In particular all orbits in Z are closed (as otherwise the closure of a nonclosed orbit would contain a (closed) point with higher dimensional stabilizer) and hence all G_x are reductive.

For $x \in Z$ we let $H_x \subset G_x$ be the pointwise stabilizer of $T_xZ/T_x(Gx)$. This is a normal subgroup of G_x. Using the Luna slice theorem one checks that H_x has finite index in G_x and in particular is reductive.

Definition 7.1. Let $\mathcal{U}$ be a G-equivariant vector bundle on Z.

(1) $\mathcal{U}$ is *saturated* if for every $x \in Z$ we have that the G_x-representation $\mathcal{U}_x$ is up to nonzero multiplicities induced from H_x.

(2) Assume that G acts with finite stabilizers. Then we say that $\mathcal{U}$ is *full* if for all $x \in Z$, $\mathcal{U}_x$ contains all irreducible G_x-representations.

Lemma 7.2. *Assume that G acts with finite stabilizers and that $\mathcal{U}$ is full. Then $\mathcal{U}$ is a projective generator of* $\operatorname{Qch}(Z/G)$*, locally over* $Z/\!\!/G$*,*[10] *and hence in particular*

$$\operatorname{Qch}(Z/G) \cong \operatorname{Qch}(\Lambda)$$

for $\Lambda = \pi_{s*}\,\mathcal{E}nd_Z(\mathcal{U})$*.*

Proof. We may check this on strong étale neighbourhoods of $x \in Z$. Therefore we may assume by Luna slice theorem that $Z/G = S/G_x$ where S is a smooth connected affine slice at x, G_x is finite, and that $\mathcal{U} = U \otimes S$, where $U = \mathcal{U}_x$ by Lemma 4.5. Since $\mathcal{U}$ is full, U contains all irreducible representations of G_x, hence $\mathcal{U}$ is projective generator. □

7.2. Homologically homogeneous sheaves of algebras.

Definition 7.3 ([BH84, SVdB08])**.** A prime affine k-algebra Λ is *homologically homogeneous* if it is finitely generated as a module over its centre R and if all simple Λ-modules have the same (finite) projective dimension.

A coherent sheaf $\mathcal{A}$ of algebras on a k-scheme X is homologically homogeneous if $\mathcal{A}(U)$ is homologically homogeneous for every connected affine $U \subset X$.

We refer to the foundational paper [BH84] as a general reference for homologically homogeneous rings. We also recall from [VdB04a, Lemma 4.2]

Lemma 7.4. *Assume that X is normal with Gorenstein singularities. A NCCR on X is homologically homogeneous.*

We now assume that X is a smooth k-scheme, G is a reductive group acting with a good quotient $\pi : X \to X/\!\!/G$. We do not assume that X satisfies $(H1)$.

Definition 7.5. Let $\mathcal{U}$ be a G-equivariant vector bundle on X. $\mathcal{U}$ is *generator in codimension one* of $\operatorname{Qch}(X/G)$ if $\pi_{s*}\,\mathcal{H}om_X(\mathcal{U},\mathcal{M}) = 0$ for $\mathcal{M} \in \operatorname{Qch}(X/G)$ implies $\operatorname{codim}\operatorname{Supp}_X \mathcal{M} \geq 2$.

Theorem 7.6. *Let $Z \subset X$ is the locus of maximal stabilizer dimension. Let $\mathcal{U}$ be a G-equivariant vector bundle on X such that $\pi_{s*}\,\mathcal{E}nd_X(\mathcal{U})$ is homologically homogeneous on $X/\!\!/G$. Assume that $\mathcal{U}$ is a generator in codimension one. Then*

(1) *$\mathcal{U} \mid Z$ is saturated.*
(2) *If G acts with finite stabilizers (and hence $Z = X$) then $\mathcal{U}$ is full.*

Proof. Using Lemmas 4.4, 4.5 we may reduce to the linear case. The result then follows from Lemma 7.9 below. □

The next proposition says that generation in codimension one is automatic if X is particularly nice. We say that X is *generic* [ŠVdB17], if the set $X^{\mathbf{s}}$ of stable points with trivial stabilizers satisfies $\operatorname{codim}(X \setminus X^{\mathbf{s}}, X) > 1$.

Proposition 7.7. *If X is generic then the following holds true.*

(1) *Every nonzero G-equivariant vector bundle $\mathcal{U}$ on X is a generator in codimension* 1.

[10] $\pi_{s*}\operatorname{Hom}_Z(\mathcal{U},\mathcal{M}) = 0$ for $\mathcal{M} \in \operatorname{Qch}(Z/G)$ implies $\mathcal{M} = 0$ (see e.g. [VdB04b]).

(2) $\Lambda = \pi_{s*}\,\mathcal{E}nd_X(\mathcal{U})$ *is an NCCR if and only if* Λ *is homologically homogeneous and* $X/\!\!/G$ *is Gorenstein.*

Proof. (1) By definition G acts freely on $X^{\mathbf{s}}$. If $\pi_{s*}\,\mathcal{H}om_X(\mathcal{U},\mathcal{M}) = 0$ then we may restrict to obtain $\pi_{s*}(\mathcal{H}om_X(\mathcal{U},\mathcal{M}) \mid X^{\mathbf{s}}) = 0$ and by descent we get $\mathcal{M} \mid X^{\mathbf{s}} = 0$. We now use $\operatorname{codim}(X \setminus X^{\mathbf{s}}, X) \geq 2$.

(2) For ($\Rightarrow$) we use Lemma 7.4 and the fact that an NCCR (in [VdB04a]) is defined for Gorenstein schemes. For ($\Leftarrow$) we moreover use that X is generic and therefore $\Lambda \cong \mathcal{E}nd_{X/\!\!/G}(\pi_{s*}\mathcal{U})$ and $\pi_{s*}\mathcal{U}$ is reflexive (see e.g. [ŠVdB17, Lemma 4.1.3]). Then Λ is an NCCR by definition. $\square$

7.3. **The linear case.** We fix some notation that will be in use throughout this section. Let $W = W_0 \oplus W_1$, $W_0 = W^{G_e}$ and U be representations of G as in §5.3. Put $S = k[W]$, graded by giving the elements of $W_i^\vee$ degree i. We define $H \supset G_e$ as the pointwise stabilizer of W_0 (this is a normal subgroup of G). Let $\Lambda = \operatorname{End}_{G,S}(U \otimes S)$, graded with the grading induced from S.

For arbitrary representations V, U of G we denote $M_{G,V}(U) := (U \otimes k[V])^G$. We may omit V or G in the notation if they are clear from the context. The following lemma may be of independent interest.

Lemma 7.8. *$M(U)$ is Cohen-Macaulay S^G-module if and only if $M_H(U)$ is Cohen-Macaulay S^H-module.*

Proof. The if direction follows by applying $(-)^{G/H}$ so we concentrate on the only if direction and assume that $M(U)$ is a Cohen-Macaulay S^G-module.

We have as G/H-representations

$$\begin{aligned} M_H(U) &= k[W_0] \otimes M_{H,W_1}(U) \\ &= \bigoplus_{V,V' \in \operatorname{rep}(G/H)} (M_{W_0}(V^*) \otimes V) \otimes (M_{W_1}(U \otimes V') \otimes V'^*), \end{aligned} \tag{7.1}$$

where in the second line we use e.g.

$$\begin{aligned} M_{H,W_1}(U) &= (U \otimes k[W_1])^H \\ &= \bigoplus_{V' \in \operatorname{rep}(G/H)} ((U \otimes k[W_1])^H \otimes V')^{G/H} \otimes V'^* \\ &= \bigoplus_{V' \in \operatorname{rep}(G/H)} ((U \otimes V' \otimes k[W_1])^H)^{G/H} \otimes V'^* \\ &= \bigoplus_{V' \in \operatorname{rep}(G/H)} (U \otimes V' \otimes k[W_1])^G \otimes V'^*. \end{aligned}$$

Applying G/H to (7.1) we obtain as S^G-modules

$$M(U) = \bigoplus_{V \in \operatorname{rep}(G/H)} M_{W_0}(V^*) \otimes M_{W_1}(U \otimes V). \tag{7.2}$$

As $k[W_0]^{G/H} \otimes k[W_1]^G = k[W_0]^{G/H} \otimes (k[W_1]^H)^{G/H} \subset (k[W_0] \otimes k[W_1]^H)^{G/H} = k[W]^G$ is a finite extension of rings and by hypotheses $M(U)$ is a Cohen-Macaulay $k[W]^G$-module, $M(U)$ is also a Cohen-Macaulay $k[W_0]^{G/H} \otimes k[W_1]^G$-module. By (7.2), $M_{W_0}(V^*) \otimes M_{W_1}(U \otimes V)$ is then a Cohen-Macaulay $k[W_0]^{G/H} \otimes k[W_1]^G$-module.

Consequently, $M_{W_1}(U\otimes V)$ is a Cohen-Macaulay $k[W_1]^G$-module if $M_{W_0}(V^*)\neq 0$ by [GW78, Theorem (2.2.5)]. Since G/H acts faithfully on W_0 (this is the point where the definition of H is used) it follows (e.g by the proof of [Alp86, Theorem II.7.1]) that $M_{W_0}(V^*)=M_{G/H,W_0}(V^*)\neq 0$ for every $V\in\operatorname{rep}(G/H)$. Thus,

$$M_{H,W_1}(U)=\bigoplus_{V\in\operatorname{rep}(G/H)} M_{W_1}(U\otimes V)\otimes V^*$$

is a Cohen-Macaulay $k[W_1]^G$-module. Hence $M_H(U)=k[W_0]\otimes M_{H,W_1}(U)$ is a Cohen-Macaulay $k[W_0]\otimes k[W_1]^G$-module. Since $k[W_0]\otimes k[W_1]^G\subset k[W]^H$ is a finite ring extension, $M_H(U)$ is a Cohen-Macaulay $k[W]^H$-module. □

The following lemma gives a necessary condition for $\Lambda=\operatorname{End}_{G,S}(U\otimes S)$ to be homologically homogeneous.

Lemma 7.9. *Assume that $\Lambda=\operatorname{End}_{G,S}(U\otimes S)$ is homologically homogeneous and that $U\otimes\mathcal{O}_W$ is a generator of $\operatorname{Qch}(W/G)$ in codimension 1. Then U and $\operatorname{Ind}_H^G\operatorname{Res}_H^G U$ contain the same irreducible G-representations. Moreover, if G is finite then U contains all irreducible G-representations.*

Proof. Let $P=\operatorname{Ind}_K^G\operatorname{Res}_K^G U\otimes S$ for $K=H$, or alternatively K may be the trivial group if G is finite, and put $Q=U\otimes S$. Consider the evaluation map of (G,S)-modules

$$\phi:\operatorname{Hom}_{G,S}(Q,P)\otimes_\Lambda Q\to P.$$

We will prove below that ϕ is an isomorphism and hence in particular surjective. Assuming this is the case then by writing $\operatorname{Hom}_{G,S}(Q,P)$ as a quotient of $\Lambda^{\oplus N}$ as right Λ-module we find that P is a quotient of $Q^{\oplus N}$ as (G,S)-module. Tensoring with $S/S_{>0}$ we obtain that $\operatorname{Ind}_K^G\operatorname{Res}_K^G U$ is a quotient of $U^{\oplus N}$. Since U is a summand of $\operatorname{Ind}_K^G\operatorname{Res}_K^G U$ this proves that U and $\operatorname{Ind}_K^G\operatorname{Res}_K^G U$ contain the same irreducible G-representation.

Now we turn to proving that ϕ is an isomorphism. If G is finite then Q is a projective Λ-module by the Auslander-Buchsbaum formula [IR08, Proposition 2.3] and if $K=H$ then we show in the next paragraph that $\operatorname{Hom}_{G,S}(Q,P)$ is a projective Λ-module. Hence in both cases (G finite or $K=H$) ϕ is a map between reflexive (G,S)-modules. Now the kernel and the cokernel of the evaluation map are supported in codimension 2 as Q is by assumption a generator in codimension 1. Hence ϕ is an isomorphism.

Suppose now $K=H$. Let $V=\operatorname{Hom}(U,\operatorname{Ind}_H^G\operatorname{Res}_H^G U)$. Then $\operatorname{Hom}_{G,S}(Q,P)=M(V)$ and as promised we have to show that $M(V)$ is a projective Λ-module. Now note that $\operatorname{Res}_H^G\operatorname{Ind}_H^G\operatorname{Res}_H^G U$ and $\operatorname{Res}_H^G U$ are equal up to multiplicities (see e.g. the proof of [ŠVdB17, Lemma 4.5.1]). Using the definition of V it then follows easily that $\operatorname{Res}_H^G V$ and $\operatorname{Res}_H^G\operatorname{End}(U)$ are equal up to multiplicities. Since $\Lambda=M(\operatorname{End}(U))$ is assumed to be homologically homogeneous, it is in particular a Cohen-Macaulay S^G-module (e.g. [SVdB08, Theorem 2.3]) and hence it follows by Lemma 7.8 that $M_H(\operatorname{End}(U))$ is a Cohen-Macaulay S^H-module. Thus the same is true for $M_H(V)$. Using Lemma 7.8 again we obtain that $M(V)$ is Cohen-Macaulay S^G-module. Hence $M(V)$ is a projective Λ-module, using [IR08, Proposition 2.3] again. □

8. Semi-orthogonal decomposition

We assume that X is as §4 and that X in addition satisfies (H2). Assume we are in the setting of §5, in particular §5.1.

In §6 we embedded $D(\Lambda)$ for $\Lambda = \pi_{s*}\,\mathcal{E}nd_X(\mathcal{U})$ in $D(X^R/G)$ via $D(\Lambda')$ for $\Lambda' = \pi^R_{s*}\,\mathcal{E}nd_{X^R}(\xi^{R*}\mathcal{U})$. If Λ is Cohen-Macaulay then so is Λ' by Lemma 6.4, and this enabled us to embed $D(\Lambda)$ in $D(\mathbf{X}/G)$. However, a similar statement for finite global dimension is not true. The reader may consult §8.5 for an explicit counterexample. This hampers the inductive construction of semi-orthogonal decomposition of $D(\mathbf{X}/G)$ with $D(\Lambda)$ as a component. In order to remediate the situation we need to tweak the vector bundle $\xi^{R*}\mathcal{U}$ by adding suitable twists.

Let $\mathcal{U}$ be a vector bundle on X/G and let N be as in Proposition 5.2(6). Put

$$\mathcal{U}^R = \bigoplus_{i=0}^{N-1}(\xi^{R*}\mathcal{U})(i), \quad \Lambda^R = \pi^R_{s*}\,\mathcal{E}nd_{X^R}(\mathcal{U}^R). \tag{8.1}$$

Remark 8.1. The most important property of $\mathcal{U}^R$ is that $\mathcal{U}^R(1) \cong \mathcal{U}^R$ locally over $X^R/\!\!/G$. This follows by the definition of N.

The advantage of $\mathcal{U}^R, \Lambda^R$ (in contrast to $\mathcal{U}', \Lambda'$) is that they inherit more favorable properties from $\mathcal{U}$, Λ.

8.1. Some subcategories of $D(X^R/G)$.

8.1.1. *Local generators.* We slightly generalise some definitions and results from [ŠVdB16, §3.5]. Let Y, Y' be smooth G-varieties such that good quotients $Y \to Y/\!\!/G$, $Y' \to Y'/\!\!/G$, respectively, exist. Let $\phi : Y' \to Y$ be a G-equivariant map, denote by $\bar{\phi} : Y'/\!\!/G \to Y/\!\!/G$ the corresponding quotient map. (In our application below $\bar{\phi}$ will be proper.) For open $U \subset Y/\!\!/G$ we write $\tilde{U} = U \times_{Y/\!\!/G} Y' \subset Y'$.

Definition 8.2. Let $(E_i)_{i\in I}$ be a collection of objects in $D(Y'/G)$. The category $\mathcal{D}$ *locally generated over* $Y/\!\!/G$ by $(E_i)_{i\in I}$ is the full subcategory of $D(Y'/G)$ spanned by all objects $\mathcal{F}$ such that for every affine open $U \subset Y/\!\!/G$ the object $\mathcal{F}|\tilde{U}$ is in the subcategory of $D(\tilde{U}/G)$ generated[11] by $(E_i|\tilde{U})_i$. We use the notation $\mathcal{D} = \langle E_i \mid i \in I\rangle^{\mathrm{loc}}_{Y/\!\!/G}$.

The objects $F, H \in D(Y'/G)$ are *locally isomorphic over* $Y/\!\!/G$ if there exists a covering $Y/\!\!/G = \bigcup_{i\in I} U_i$ such that $F|\tilde{U}_i \cong H|\tilde{U}_i$ for all i.

In loc. cit. we only considered the case $Y' = Y$, $\phi = \mathrm{id}$ (so there was no "over $Y/\!\!/G$"). The proofs of the following analogues of the results from loc. cit. remain valid in this slightly more general setting; note only that instead of π_{s*} for $\pi_s : Y'/G \to Y'/\!\!/G$ we use $R\bar{\phi}_*\pi_{s*}$ (taking into account that ϕ is now not the identity) and that in loc. cit. we used small categories, instead of the large, cocomplete, categories we are using here.

Lemma 8.3. [ŠVdB16, Lemma 3.5.3] *Let $(E_i)_{i\in I}$ be a collection of perfect objects in $D(Y'/G)$ and let $\mathcal{F} \in D(Y'/G)$. Let $Y/\!\!/G = \bigcup_{j=1}^n U_j$ be a finite open affine covering of $Y/\!\!/G$. If for all j one has that $\mathcal{F}|\tilde{U}_j$ is in the subcategory of $D(\tilde{U}_j/G)$*

[11] Assume $\mathcal{T}$ is a triangulated category closed under coproduct. Let $\mathcal{S} = (T_i)_{i\in I}$ be a set of objects in $\mathcal{T}$. Then the subcategory of $\mathcal{T}$ *generated* by $\mathcal{S}$ is the smallest triangulated subcategory of $\mathcal{T}$ closed under isomorphism and coproduct which contains $\mathcal{S}$.

generated by $(E_i|\tilde{U}_j)_i$ then $\mathcal{F}$ is in the subcategory of $D(Y'/G)$ locally generated over $Y/\!\!/G$ by $(E_i)_{i\in I}$.

The following result shows that semi-orthogonal decompositions can be constructed locally.

Proposition 8.4. [ŠVdB16, Proposition 3.5.8] *Let I be a totally ordered set. Assume $\mathcal{D} \subset D(Y'/G)$ is locally generated over $Y/\!\!/G$ by a collection of subcategories $\mathcal{D}_i$ closed under coproduct and local isomorphism over $Y/\!\!/G$. Assume that $R\bar{\phi}_*\pi_{s*}\,\mathrm{R}\mathcal{H}om_{Y'}(\mathcal{D}_i,\mathcal{D}_j) = 0$ for $i > j$. Then $\mathcal{D}$ is generated by $(\mathcal{D}_i)_i$ and in particular we have a semi-orthogonal decomposition $\mathcal{D} = \langle \mathcal{D}_i \mid i \in I\rangle$.*

It is convenient to pick for every $E \in D(Y'/G)$ a K-injective resolution (with injective terms) $E \to I_E$ and to represent $R\bar{\phi}_*\pi_{s*}\,\mathrm{R}\mathcal{H}om_{Y'}(E,F)$ on $Y/\!\!/G$ by the complex of sheaves $U \mapsto \bar{\phi}_* \operatorname{Hom}_{\tilde{U}}(I_E|\tilde{U}, I_F|\tilde{U})^G$.[12] With this representation

$$\Lambda := R\bar{\phi}_*\pi_{s*}\,\mathrm{R}\mathcal{E}nd_{Y'}(E) := R\bar{\phi}_*\pi_{s*}\,\mathrm{R}\mathcal{H}om_{Y'}(E,E)$$

is a sheaf of DG-algebras on $Y/\!\!/G$ and $R\bar{\phi}_*\pi_{s*}\,\mathrm{R}\mathcal{H}om_{Y'}(E,F)$ is a sheaf of right Λ-DG-modules.

Lemma 8.5. [ŠVdB16, Lemma 3.5.6] *Assume that $\mathcal{D} \subset D(Y'/G)$ is locally generated over $Y/\!\!/G$ by the perfect complex E. Let $\Lambda = R\bar{\phi}_*\pi_{s*}\,\mathrm{R}\mathcal{E}nd_{Y'}(E)$ be the sheaf of DG-algebras on $Y/\!\!/G$ as defined above. The functors*

$$\mathcal{D} \to D(\Lambda) : F \mapsto R\bar{\phi}_*\pi_{s*}\,\mathrm{R}\mathcal{H}om_{Y'}(E,F),$$
$$D(\Lambda) \to \mathcal{D} : H \mapsto \bar{\phi}^{-1}H \overset{L}{\otimes}_{\bar{\phi}^{-1}\Lambda} E$$

are well-defined (the second functor is computed starting from a K-flat resolution[13] *of H) and yield inverse equivalences between $\mathcal{D}$ and $D(\Lambda)$.*

8.1.2. *Locally generated subcategories.* We define some locally generated subcategories of $D(X^R/G)$ which we will need for the semi-orthogonal decomposition of $D(\Lambda^R)$.

Let

$$\mathcal{C}_{X^R} := \langle \mathcal{U}^R\rangle^{\mathrm{loc}}_{X^R/\!\!/G} \subset D(X^R/G).$$

Our aim will be to define a semi-orthogonal decomposition of $\mathcal{C}_{X^R}$.

Let

$$\tilde{\mathcal{C}}_X := \langle \xi^{R*}\mathcal{U}\rangle^{\mathrm{loc}}_{X/\!\!/G} \subset \mathcal{C}'_{X^R} := \langle \xi^{R*}\mathcal{U}\rangle^{\mathrm{loc}}_{X^R/\!\!/G} \subset \mathcal{C}_{X^R}. \tag{8.2}$$

Let Y be a connected component of Z, G_Y be the stabilizer of Y in Z (not pointwise), let $E^R_Y = (\xi^R)^{-1}(Y)$ and let $s_Y : E^R_Y/G_Y \to X^R/G$ be the inclusion. Let $\mathcal{U}_Y$, $\mathcal{U}_{E^R_Y}$ be the restrictions of $\mathcal{U}$, $\xi^{R*}\mathcal{U}$ to Y, E^R_Y, respectively. Let

[12] It is enough to show that each term in $\mathcal{H}om_{Y'}(I_E, I_F)$, and consequently in $\mathcal{H}om_{Y'}(I_E,I_F)^G$, is flabby, since flabby sheaves are acyclic for $\bar{\phi}_*$ and $\bar{\phi}_*$ has finite cohomological dimension. Let $\mathcal{F},\mathcal{I} \in \mathrm{Qch}(Y'/G)$ with $\mathcal{I}$ injective. Let $j : U \hookrightarrow Y'$ be an open immersion. As $j_!j^*\mathcal{F} \hookrightarrow \mathcal{F}$ and $\mathcal{I}$ is injective in $\mathrm{Mod}(Y'/G)$ (as Y' is noetherian), $\mathcal{H}om_{Y'}(\mathcal{F},\mathcal{I}) \twoheadrightarrow \mathcal{H}om_X(j_!j^*\mathcal{F},\mathcal{I}) = \mathcal{H}om_{U'}(j^*\mathcal{F}, j^*\mathcal{I}) = \mathcal{H}om_{U'}(\mathcal{F}\,|_{U'}, \mathcal{I}\,|_{U'})$.

[13] Such a K-flat resolution is constructed in the same way as for DG-algebras (see [Kel94, Theorem 3.1.b]). One starts from the observation that for every $M \in D(\Lambda)$ there is a morphism $\bigoplus_{i\in I} j_{i!}(\Lambda|U_i) \to M$ with open immersions $(j_i : U_i \to X/\!\!/G)_{i\in I}$, which is an epimorphism on the level of cohomology.

$\pi_{Y,s} : Y/G_Y \to Y/\!\!/G_Y$, $\pi_{E^R_Y,s} : E^R_Y/G_Y \to E^R_Y/\!\!/G_Y$, be the quotient maps. Let

$$\mathcal{C}_{Y,n} = \langle s_{Y*}\mathcal{U}_{E^R_Y}(n)\rangle^{\mathrm{loc}}_{X/\!\!/G}.$$

Lemma 8.6. *We have $\mathcal{C}_{Y,n} \subset \mathcal{C}_{X^R}$ for all $n \in \mathbb{Z}$.*

Proof. Recall the standard exact sequence (6.11) on X^R/G

$$0 \to \xi^{R*}\mathcal{U}(1) \xrightarrow{t} \xi^{R*}\mathcal{U} \to s_*\mathcal{U}_{E^R} \to 0. \tag{8.3}$$

Let $n \in \mathbb{Z}$. Since $\xi^{R*}\mathcal{U}(n)$ belongs to $\mathcal{C}_{X^R}$ by Remark 8.1 (and the definition of $\mathcal{U}^R$), it follows from (twisting) (8.3) that $s_*\mathcal{U}_{E^R}(n) \in \mathcal{C}_{X^R}$. As the (local) generator $s_{Y*}\mathcal{U}_{E^R_Y}(n)$ of $\mathcal{C}_{Y,n}$ is its direct summand (identifying $Y/G_Y = (\cup_{g\in G} gY)/G$, $E^R_Y/G_Y = E^R_{\cup_{g\in G} gY}/G$) the lemma follows. □

Proposition 8.7. *Assume that Λ is Cohen-Macaulay. Let $Z_1, \dots, Z_t$ be representatives for the orbits of the G-action on the connected components of Z. There is a semi-orthogonal decomposition*

$$\mathcal{C}_{X^R} = \langle \tilde{\mathcal{C}}_X, \mathcal{C}_{Z_1,0}, \dots, \mathcal{C}_{Z_1,c_1-2}, \dots, \mathcal{C}_{Z_t,0}, \dots, \mathcal{C}_{Z_t,c_t-2}\rangle,$$

where $c_i = \operatorname{codim}(Z_i, X)$. Moreover, the components corresponding to different Z_i are orthogonal.

Proof. By (8.2), Lemma 8.6, we have $\tilde{\mathcal{C}}_X, \mathcal{C}_{Z_i,n} \subset \mathcal{C}_{X^R}$, respectively. We apply Proposition 8.4.

By definition of $\mathcal{C}_{Z_i,n}$ it is clear that the components corresponding to different i are orthogonal. To obtain the orthogonality for $\mathcal{C}_{Z_i,n}$ for fixed i, let us first recall (6.13) for an easier reference

$$\mathrm{R}\mathcal{H}om_{X^R}(\xi^{R*}\mathcal{U}, s_*\mathcal{U}_{E^R}) = s_*\,\mathrm{R}\mathcal{H}om_{E^R}(\mathcal{U}_{E^R}, \mathcal{U}_{E^R}). \tag{8.4}$$

Applying $\mathrm{R}\mathcal{H}om_{X^R}(-, s_*\mathcal{U}_{E^R}(n))$ to (8.3)[14] and using (8.4) it is then enough to show that

$$R\overline{\xi^R_*}\pi_{s*}(s_{Z_i*}\,\mathrm{R}\mathcal{H}om_{E^R_{Z_i}}(\mathcal{U}_{E^R_{Z_i}}, \mathcal{U}_{E^R_{Z_i}}(l))) = 0$$

for $-(c_i - 2) - 1 \le l < 0$. This holds by Lemma 6.7.

To obtain the orthogonality of $\mathcal{C}_{Z_i,n}$ for $n = 0, \dots, c_i - 2$ and $\tilde{\mathcal{C}}_X$ we need

$$R\overline{\xi^R_*}\pi_{s*}\,\mathrm{R}\mathcal{H}om_{X^R}(s_{Z_i*}\mathcal{U}_{E^R_{Z_i}}(n), \xi^{R*}\mathcal{U}) = 0 \tag{8.5}$$

for l in the indicated range.

We first apply $\mathrm{R}\mathcal{H}om_{X^R/G}(-, \xi^{R*}\mathcal{U})$ to (8.3) and obtain the distinguished triangle

$$\begin{aligned}\mathrm{R}\mathcal{H}om_{X^R}(s_*\mathcal{U}_R, \xi^{R*}\mathcal{U}) &\to \mathrm{R}\mathcal{H}om_{X^R}(\xi^{R*}\mathcal{U}, \xi^{R*}\mathcal{U}) \\ &\xrightarrow{\mathrm{R}\mathcal{H}om_{X^R}(\xi^{R*}\mathcal{U}\otimes t, \xi^{R*}\mathcal{U})} \mathrm{R}\mathcal{H}om_{X^R}(\xi^{R*}\mathcal{U}(1), \xi^{R*}\mathcal{U}) \to\end{aligned}$$

where $t : \mathcal{O}_{X^R/G}(1) \to \mathcal{O}_{X^R/G}$ denotes the canonical map. By Lemma 8.8 below this may be rewritten as

$$\begin{aligned}\mathrm{R}\mathcal{H}om_{X^R}(s_*\mathcal{U}_R, \xi^{R*}\mathcal{U}) &\to \mathrm{R}\mathcal{H}om_{X^R}(\xi^{R*}\mathcal{U}, \xi^{R*}\mathcal{U}) \\ &\xrightarrow{\mathrm{R}\mathcal{H}om_{X^R}(\xi^{R*}\mathcal{U}, \xi^{R*}\mathcal{U}\otimes t)} \mathrm{R}\mathcal{H}om_{X^R}(\xi^{R*}\mathcal{U}, \xi^{R*}\mathcal{U}(-1)) \to .\end{aligned}$$

[14] We obtain an analogue of (6.12).

By applying $\mathrm{R}\mathcal{H}om_{X^R/G}(\xi^{R*}\mathcal{U},-(-1))$ to (8.3) we then deduce that

$$\mathrm{R}\mathcal{H}om_{X^R}(s_*\mathcal{U}_{E^R},\xi^{R*}\mathcal{U})\cong \mathrm{R}\mathcal{H}om_{X^R}(\xi^{R*}\mathcal{U},s_*\mathcal{U}_{E^R}(-1))[-1].$$

Twisting and applying (8.4) we moreover have

$$\mathrm{R}\mathcal{H}om_{X^R}(s_*\mathcal{U}_{E^R}(n),\xi^{R*}\mathcal{U})\cong s_*\,\mathrm{R}\mathcal{H}om_{E^R}(\mathcal{U}_{E^R},\mathcal{U}_{E^R}(-1-n))[-1].$$

Applying $R\overline{\xi^R}_*\pi_{s*}$ and using Lemma 6.7, we obtain (8.5).

We now prove the generation property. We reduce to the affine X containing one representative Z_j of connected components of Z by Proposition 8.4 (and Lemma 8.3). By (8.3), it follows that $\langle\tilde{\mathcal{C}}_X,\mathcal{C}_{Z_j,0},\dots,\mathcal{C}_{Z_j,c_j-2}\rangle$ contains $\xi^{R*}\mathcal{U}(i)$ for $0\le i\le c_j-1$. By the proof of [VdB04b, Lemma 3.2.2] (which applies in the G-equivariant setting) it follows that $\mathcal{C}_{X^R}$ contains $\xi^{R*}\mathcal{U}(i)$ for all $i\in\mathbb{Z}$. We now show that $(\xi^{R*}\mathcal{U}(i))_{i\in\mathbb{Z}}$ generate $\mathcal{C}_{X^R}$. Let $0\neq\mathcal{F}\in D(X^R/G)$. We need to show that

$$\begin{aligned}(\mathrm{RHom}_{X^R/G}(\xi^{R*}\mathcal{U}(i),\mathcal{F})=0 \text{ for all } i\in\mathbb{Z}) \implies&\\ \pi^R_{s*}\,\mathrm{R}\mathcal{H}om_{X^R}(\mathcal{U}^R,\mathcal{F})=\pi^R_{s*}((\mathcal{U}^R)^\vee\otimes\mathcal{F})=0&\end{aligned}$$

or equivalently

$$\begin{aligned}(\pi^R_{s*}\,\mathrm{R}\mathcal{H}om_{X^R}(\mathcal{U}^R,\mathcal{F})=\pi^R_{s*}((\mathcal{U}^R)^\vee\otimes\mathcal{F})\neq 0) \implies&\\ \mathrm{RHom}_{X^R/G}(\xi^{R*}\mathcal{U}(i),\mathcal{F})\neq 0 \text{ for some } i\in\mathbb{Z}.&\end{aligned}$$

We assume that $\pi^R_{s*}((\mathcal{U}^R)^\vee\otimes\mathcal{F})\neq 0$. Recall that $\mathcal{O}(N)=\pi_s^{R*}\mathcal{M}$ for an ample line bundle $\mathcal{M}$ on $X^R/\!\!/G$ by Proposition 5.2(6). Then $\mathrm{Hom}_{X^R/\!\!/G}(\mathcal{M}(m),\pi^R_{s*}((\mathcal{U}^R)^\vee\otimes\mathcal{F}))\neq 0$ for $m\ll 0$ (since $X^R/\!\!/G$ is proper over affine $X/\!\!/G$ by Proposition 5.2(2)). Thus

$$\begin{aligned}0\neq \mathrm{RHom}_{X^R/\!\!/G}(\mathcal{M}(m),\pi^R_{s*}((\mathcal{U}^R)^\vee\otimes\mathcal{F}))=\mathrm{RHom}_{X^R/G}(\mathcal{O}(mN),(\mathcal{U}^R)^\vee\otimes\mathcal{F})=&\\ \oplus_{0\le i<N}\mathrm{RHom}_{X^R/G}(\xi^{R*}\mathcal{U}(i+mN),\mathcal{F})&\end{aligned}$$

and the generation follows. □

We have used the following lemma.

Lemma 8.8. *Let $\mathcal{F}$, $\mathcal{G}\in\mathrm{Qch}(X^R/G)$ and $t:\mathcal{O}_{X^R/G}(1)\to\mathcal{O}_{X^R/G}$ the canonical map. Then the following diagram is commutative*

$$\begin{tikzcd}
\mathrm{R}\mathcal{H}om_{X^R}(\mathcal{F},\mathcal{G}) \arrow[r, equal] \arrow[d, "\mathrm{R}\mathcal{H}om_{X^R}(\mathcal{F},t)"'] & \mathrm{R}\mathcal{H}om_{X^R}(\mathcal{F},\mathcal{G}) \arrow[d, "\mathrm{R}\mathcal{H}om_{X^R}(t,\mathcal{G})"] \\
\mathrm{R}\mathcal{H}om_{X^R}(\mathcal{F},\mathcal{G}(-1)) \arrow[r, "\cong"'] & \mathrm{R}\mathcal{H}om_{X^R}(\mathcal{F}(1),\mathcal{G})
\end{tikzcd}$$

8.2. **Orlov's semi-orthogonal decomposition for the Reichstein transform.** We are now ready to formulate our next main result which is *an analogue for the Reichstein transform of Orlov's semi-orthogonal decomposition for a blowup* [Orl93].

Theorem 8.9. *Let X be a smooth G-scheme such that a good quotient $\pi:X\to X/\!\!/G$ exists. Assume furthermore that (X,G) satisfies (H2).*[15] *Let $Z\subset X$ be the locus of maximal stabilizer dimension and let $Z_1,\dots,Z_t$ be representatives for the*

[15](H2) was imposed at the beginning of §8 and has been used throughout §8.1.2 implicitly via results in §6.

orbits of the G-action on the connected components of Z. Let G_i be the stabilizer of Z_i.

Let $\mathcal{U}$ be a G-equivariant vector bundle on X such that $\pi_{s*}\,\mathcal{E}nd_X(\mathcal{U})$ is Cohen-Macaulay, and put

$$\begin{array}{rcll} \mathcal{C}_{X^R} & \coloneqq & \langle \mathcal{U}^R\rangle^{\mathrm{loc}}_{X^R/\!\!/G} & \subset D(X^R/G), \\ \mathcal{C}_X & \coloneqq & \langle \mathcal{U}\rangle^{\mathrm{loc}}_{X/\!\!/G} & \subset D(X/G), \\ \mathcal{C}_{Z_i} & \coloneqq & \langle \mathcal{U}_{Z_i}\rangle^{\mathrm{loc}}_{Z_i/\!\!/G_i} & \subset D(Z_i/G_i). \end{array}$$

Let $\xi^R_E : E^R \to Z$ denote the restriction/corestriction of $\xi^R : X^R \to X$.

The following holds.

1. *$L\xi^{R*} : D(X/G) \to D(X^R/G)$ is fully faithful when restricted to $\mathcal{C}_X$.*
2. *The composition*
$$F_i : D(Z_i/G_i) \hookrightarrow D(Z/G) \xrightarrow{L\xi^{R*}_E} D(E^R/G) \xrightarrow{Rs_*} D(X^R/G)$$
is fully faithful when restricted to $\mathcal{C}_{Z_i}$.
3. *There is a semi-orthogonal decomposition of $\mathcal{C}_{X^R}$*
$$\langle L\xi^{R*}\mathcal{C}_X, (F_1\mathcal{C}_{Z_1})(0), \ldots, (F_1\mathcal{C}_{Z_1})(c_1-2), \ldots, (F_t\mathcal{C}_{Z_t})(0), \ldots, (F_t\mathcal{C}_{Z_t})(c_t-2)\rangle$$
where $c_i = \operatorname{codim}(Z_i, X)$. Moreover, the components corresponding to different Z_i are orthogonal.

Proof. (1) This follows from Corollary 6.3.

(2) This follows from Corollary 6.8.

(3) In the notation of §8.1.2, $F_i\mathcal{C}_{Z_i}(n) = \mathcal{C}_{Z_i,n}$, $L\xi^{R*}\mathcal{C}_X = \tilde{\mathcal{C}}_X$. The claim then follows immediately from Proposition 8.7. □

Corollary 8.10. *Let the notations and assumptions be as in the previous theorem and define in addition sheaves of algebras on $X^R/\!\!/G$, $X/\!\!/G$, $Z_i/\!\!/G_i$ via:*

$$\Lambda^R \coloneqq \pi^R_{s*}\,\mathcal{E}nd_{X^R}(\mathcal{U}^R), \quad \Lambda \coloneqq \pi_{s*}\,\mathcal{E}nd_X(\mathcal{U}), \quad \Lambda_{Z_i} \coloneqq \pi_{Z_i,s,*}\,\mathcal{E}nd^R_X(\mathcal{U}_{Z_i})$$

where $\pi_{Z_i} : Z_i \to Z_i/\!\!/G_i$ is the good quotient.

There is a semi-orthogonal decomposition

$$D(\Lambda^R) \cong \langle D(\Lambda), \underbrace{D(\Lambda_{Z_1}), \ldots, D(\Lambda_{Z_1})}_{c_1-1}, \ldots, \underbrace{D(\Lambda_{Z_t}), \ldots, D(\Lambda_{Z_t})}_{c_t-1}\rangle$$

where $c_i = \operatorname{codim}(Z_i, X)$. Moreover, the components corresponding to different Z_i are orthogonal.

Proof. This is an immediate consequence of Theorem 8.9 using Lemma 8.5.

Note that we could also deduce this corollary from Proposition 8.7 and Lemma 8.5 together with results from §6; i.e. for $\mathcal{C}_{X^R}$, $\mathcal{C}_{Z_i,n}$ we could apply the lemma with $Y' = Y = X^R, \phi = \mathrm{id}$, and then use Corollary 6.8 to further describe the latter, and for $\tilde{\mathcal{C}}_X$ with $Y' = X^R, Y = X, \phi = \xi^R$ followed by Lemma 6.1. □

8.3. **Properties of $\mathcal{U}^R, \Lambda^R$ inherited from $\mathcal{U}, \Lambda$.** For use below we recall that $\mathcal{U}_{E^R}$ was defined as the restrictions of $\xi^{R*}\mathcal{U}$ to the exceptional divisor E^R/G for the morphism $\xi^R : X^R/G \to X/G$. We similarly let $\mathcal{U}^R_{E^R}$ be the restriction of $\mathcal{U}^R$ to E^R/G.

Lemma 8.11. *The sheaf of rings on $E^R /\!\!/ G$*

$$\bar{\Lambda} := \oplus_{n=-\infty}^{\infty} \pi^R_{E,s,*} \mathcal{H}om_{E^R}(\mathcal{U}^R_{E^R}, \mathcal{U}^R_{E^R}(n))$$

is strongly graded. If in the linear case as in Lemmas 5.4, 4.6 then on $E^R /\!\!/ G$

$$\bar{\theta}_* \Lambda^R \cong \bar{\Lambda}_{\geq 0}.$$

Proof. We start by proving that $\bar{\Lambda}$ is strongly graded. Since $\mathcal{U}^R_{E^R}(1) \cong \mathcal{U}^R_{E^R}$ locally over $E^R /\!\!/ G$ (restricting the local isomorphism $\mathcal{U}^R(1) \cong \mathcal{U}^R$ over $X^R /\!\!/ G$ in Remark 8.1 to E^R/G), $\bar{\Lambda}$ has a unit in degree 1 and it is thus strongly graded.

We now prove the second statement. We have $\mathcal{U}^R = \theta^* \mathcal{U}^R_{E^R}$ (using the linearity assumption $\mathcal{U} = U \otimes \mathcal{O}_W$). We compute

$$\begin{aligned}
\bar{\theta}_* \Lambda^R &= \bar{\theta}_* \pi^R_{s*} \mathcal{E}nd_{X^R}(\theta^* \mathcal{U}^R_{E^R}) \\
&\cong \pi^R_{E,s,*} \mathcal{H}om_{E^R}(\mathcal{U}^R_{E^R}, \theta_* \theta^* \mathcal{U}^R_{E^R}) \\
&\cong \oplus_{n=0}^{\infty} \pi^R_{E,s,*} \mathcal{H}om_{E^R}(\mathcal{U}^R_{E^R}, \mathcal{U}^R_{E^R}(n)).
\end{aligned}$$ □

Lemma 8.12. *Let X be a scheme and let A be a strongly graded sheaf of algebras on X. If A is homologically homogeneous then A_0 and $A_{\geq 0}$ are homologically homogeneous on X.*

Proof. This is a local statement so we may assume that A is a strongly graded ring. It is clear that $A[t]$ for $|t| = -1$ is also strongly graded. Since $A_{\geq 0} \cong A[t]_0$ we need two facts:

(1) If A is homologically homogeneous then so is $A[t]$.
(2) If A is strongly graded and homologically homogeneous then so is A_0.

The first fact is [BH84, Theorem 7.3]. The second follows since the categories of A_0-modules and graded A-modules are in this case equivalent [NvO82, Theorem I.3.4], and by [SVdB08, Proposition 2.9]. □

The next proposition exhibits some properties of the pair $(X, \mathcal{U})$ which lift to the pair $(X^R, \mathcal{U}^R)$.

Proposition 8.13. (1) *If $\Lambda = \pi_{s*} \mathcal{E}nd_X(\mathcal{U})$ is homologically homogeneous on $X /\!\!/ G$ then the same is true for $\Lambda^R = \pi^R_{s*} \mathcal{E}nd_{X^R}(\mathcal{U}^R)$.*
(2) *If $\mathcal{U}$ is generator in codimension one then the same is true for $\mathcal{U}^R$.*

Proof. (1) We reduce to the linear case by Lemmas 5.3, 4.6. Let $\check{\Lambda}$ be the sheaf of rings on $X^R /\!\!/ G$ defined by

$$\bar{\theta}_* \check{\Lambda} = \bar{\Lambda} = \oplus_{n=-\infty}^{\infty} \pi^R_{E,s,*} \mathcal{H}om_{E^R/G}(\mathcal{U}^R_{E^R}, \mathcal{U}^R_{E^R}(n))$$

where the right-hand side is to be viewed as a sheaf of $\bar{\theta}_* \mathcal{O}_{X^R /\!\!/ G}$ algebras. Let

$$\Gamma = \oplus_{n=-\infty}^{\infty} \pi^R_{E,s,*} \mathcal{H}om_{E^R/G}(\mathcal{U}_{E^R}, \mathcal{U}_{E^R}(n)).$$

Using the definition of N we get

$$\bar{\theta}_*\check{\Lambda} \cong \begin{pmatrix} \Gamma & \Gamma(1) & \cdots & \Gamma(N-1) \\ \Gamma(-1) & \ddots & & \vdots \\ \vdots & & & \\ \Gamma(-N+1) & \cdots & & \Gamma \end{pmatrix}$$

as sheaves of $\mathbb{Z}$-graded algebras on $E^R /\!\!/ G$, where $?(i)$ denotes the grading shift.

Let $R = k[W]^G$ with W graded as in Lemma 5.5. Note that $\operatorname{Proj} R \cong E^R /\!\!/ G$. Let $\hat{\Lambda}$ be the sheaf of graded $\mathcal{O}_{E^R/\!\!/G}$-algebras associated to the $k[W]^G$-algebra Λ, defined by $\hat{\Lambda}(U_f) = \Lambda_f$, where $U_f = \operatorname{Spec}((R_f)_0)$ for $f \in R_{>0}$.

We claim that $\hat{\Lambda} \cong \Gamma$. We will now confuse quasi-coherent sheaves on affine schemes with their global sections. First note that $\Lambda = (\operatorname{End}(U) \otimes k[W])^G$. Hence $\hat{\Lambda}(U_f) = \Lambda_f = (\operatorname{End}(U) \otimes k[W]_f)^G$. On the other hand,

$$\begin{aligned} \Gamma(U_f) &= \oplus_{n=-\infty}^{\infty} \pi^R_{E,s,*} \mathcal{H}om_{E^R/G}(U \otimes \mathcal{O}_{E^R}, U \otimes \mathcal{O}_{E^R}(n))(U_f) \\ &= (\operatorname{End}(U) \otimes \oplus_{n=-\infty}^{\infty} \Gamma(E^R_f, \mathcal{O}(E^R)(n)))^G \\ &= (\operatorname{End}(U) \otimes \oplus_{n=-\infty}^{\infty} (k[W]_f)_n)^G \\ &= (\operatorname{End}(U) \otimes k[W]_f)^G. \end{aligned}$$

Since Λ is homologically homogeneous, so is $\hat{\Lambda}$ and therefore Γ. Thus, $\bar{\theta}_*\check{\Lambda}$ is homologically homogeneous. Since $\bar{\theta}_*\check{\Lambda}$ is strongly graded by Lemma 8.11 and $\bar{\theta}_*\Lambda^R = (\bar{\theta}_*\check{\Lambda})_{\geq 0}$ by Lemma 8.11, Λ^R is homologically homogeneous by Lemma 8.12.

(2) Also (2) can be checked étale locally, so we can reduce to the linear case by Lemmas 5.3, 4.6. Since X and X^R differ in codimension 1 by the exceptional divisor E^R, we need to show that, generically on E^R, $\mathcal{U}^R$ generates $D(E^R/G)$. It is enough to check that $\mathcal{U}^R_y = \mathcal{U}^R \otimes k(y)$ contains all the irreducible representations of G_y for a generic point $y \in E^R$. Denote $H = G_y$ and let x a generic point in W such that $y = [x]$. Then H (not necessarily pointwise) stabilizes the line ℓ passing through 0, x, and the action of H on ℓ is then given by a character $\alpha \in X(H)$. Let $K = \ker \alpha$, which is the (finite) stabilizer of x. Thus, H/K can be considered as a subgroup of G_m, and it is therefore cyclic or G_m. However, if the H/K were G_m then 0 would be in the closure of the orbit of x. This is a contradiction since x is generic in W and W satisfies $(H2)$. Thus, H/K is a finite cyclic group and it acts on the line ℓ by a generator of $X(H/K)$. Since $\mathcal{U}^R_y = \oplus_{i=0}^{N-1} U \otimes (\ell^*)^{\otimes i}$ as H-representations and $(\ell^*)^{\otimes N}$ is trivial by the definition of N (see Proposition 5.2(6)), and by the assumption U contains all irreducible representations of K, Lemma 8.14 below implies that all irreducible representations of $H = G_y$ are contained in $\mathcal{U}^R_y$. □

The following lemma was used in the proof of Proposition 8.13(2), which might also be of independent interest when viewed as a recognition criterion for induced representations.

Lemma 8.14 (Recognition criterion)**.** *Let K be a normal subgroup of H such that H/K is finite and let V be a representation of H. If $V \otimes V' :=: V$ for every representation V' of H/K, then $V :=: \operatorname{Ind}^H_K \operatorname{Res}^H_K V$ (see §3).*

Proof. $\tilde{V} := \operatorname{Ind}_K^H \operatorname{Res}_K^H V \cong k[H/K] \otimes V$ as H-representations, where the action on the right-hand side is diagonal. Since by the assumption $k[H/K] \otimes V :=: V$, we obtain $\tilde{V} :=: V$ as desired. □

8.4. **Semi-orthogonal decomposition of the Kirwan resolution.** In the next theorem we collect the results we have obtained.

Theorem 8.15. *Let X be a smooth G-scheme such that a good quotient $\pi : X \to X/\!\!/G$ exists. Assume furthermore that (X, G) satisfies (H2).*[16] *Let $\mathcal{U}$ be a G-equivariant vector bundle on X. Assume that $\Lambda := \pi_{s*}\,\mathcal{E}nd_X(\mathcal{U})$ is homologically homogeneous on $X/\!\!/G$ and that $\mathcal{U}$ is a generator in codimension 1 (see Definitions 7.3, 7.2).*

Let us assume that the Kirwan resolution $\mathbf{X}/G$ is obtained by performing n successive Reichstein transforms and Z_j is blown-up at the j-th step in X_j. Let $Z_{j1}, \dots, Z_{jt_j}$ be representatives for the orbits of the G-action on the connected components of Z and let $G_{Z_{ji}}$ be the stabilizer of Z_{ji} (as a connected component). Denote by $\pi_{Z_{ji}} : Z_{ji} \to Z_{ji}/\!\!/G_{ji}$ the quotient map. Let $\mathcal{U}_0 = \mathcal{U}$ and let $\mathcal{U}_i = \mathcal{U}_{i-1}^R$, $1 \le i \le n$ where $(-)^R$ is as in (8.1). Let $\mathcal{U}_{j,Z_{ji}}$ be the restriction of $\mathcal{U}_j$ to Z_{ji} and set $\Lambda_{Z_{ji}} = \pi_{Z_{ji},s,*}\,\mathcal{E}nd_{Z_{ji}}(\mathcal{U}_{j,Z_{ji}})$.

There exists a semi-orthogonal decomposition

$$\langle D(\Lambda), D(\Lambda_{Z_{ji}})_{1\le j\le n, 1\le i\le t_j, 0\le k\le c_{ji}-2}\rangle \tag{8.6}$$

of $D(\mathbf{X}/G)$, where $c_{ji} := \operatorname{codim}(Z_{ji}, X_j)$, and the terms appear in the lexicographic order (according to the label (j, i, k)).

Remark 8.16. The assumptions on $\mathcal{U}$ and Λ are satisfied if we assume that Λ is an NCCR of $X/\!\!/G$ and X is "generic". See Proposition 7.7.

Proof of Theorem 8.15. The theorem follows from Corollary 8.10, once we prove that when we perform the last Reichstein transform we get $D(\Lambda^R) \cong D(\mathbf{X}/G)$.

Assume thus that we are at the last step of the Kirwan resolution. We have $\Lambda^R = \pi_{s*}^R\,\mathcal{E}nd_{\mathbf{X}}(\mathcal{U}_n)$. Moreover, $\mathbf{X}/G$ is a smooth Deligne-Mumford stack, $\mathcal{U}_n$ is generator in codimension 1 by Proposition 8.13(2) (and the assumption on $\mathcal{U}$), and Λ^R is homologically homogeneous by Proposition 8.13(1) (and the assumption on Λ). Hence, by Theorem 7.6, $\mathcal{U}_n$ is full. Consequently, Lemma 7.2 implies that $\operatorname{Qch}(\mathbf{X}/G) \cong \operatorname{Qch}(\Lambda^R)$. Then, $D(\Lambda^R) \cong D(\mathbf{X}/G)$ as $D_{\operatorname{Qch}}(-) = D(\operatorname{Qch}(-))$ in our case by (the proof of) [HNR19, Theorem 1.2]. □

Remark 8.17. The embedding $D(\Lambda) \hookrightarrow D(\mathbf{X}/G)$ obtained from (8.6) is the same one as the one obtained from the diagonal in (6.8). Indeed tracing through the various constructions we find that both embeddings are obtained as the composition of $D(\Lambda) \cong \langle\mathcal{U}\rangle_{X/\!\!/G}^{\mathrm{loc}} \subset D(X/G)$ with the pullback $D(X/G) \to D(\mathbf{X}/G)$.

Remark 8.18. Theorem 8.15 will not be the end of our story as we will show in §9 that the components $D(\Lambda_{Z_{ji}})$ of the semi-orthogonal decomposition (8.6) can be decomposed further as sums of derived categories of Azumaya algebras on smooth Deligne-Mumford stacks. In other words the "extra" components to be added to the noncommutative resolution to obtain the Kirwan resolution are very close to commutative (they are "gerby"). The precise statement, which is given in Corollary

[16](H2) was imposed at the beginning of §8 and has been used throughout the section explicitly or implicitly via results in §6.

9.9, is a bit technical but it becomes very easy in the case that G is abelian. In that case we have

$$D(\Lambda_{Z_{ji}}) \cong D(Z_{ji}/(G/H_{ji}))^{\oplus N_{ji}}$$

where H_{ji} is the stabilizer of Z_{ji} and N_{ji} is the number of distinct H_{ji}-characters occurring in $\mathcal{U}_{j,Z_{ji},x}$ for some $x \in Z_{ji}$. Thus in the abelian case the extra components are truly commutative.

8.5. **A counterexample.** Here we give an explicit example of a Cohen-Macaulay Λ such that $\operatorname{gl}\dim \Lambda < \infty$, $\operatorname{gl}\dim \Lambda' = \infty$. This was announced in the beginning of §8, from where we borrow the notations.

Example 8.19. Assume that Λ is homologically homogeneous graded algebra and let R be the center of Λ. We note that $(\Lambda_f)_{\geq 0}$ and $(\Lambda_f)_0$ for a homogeneous $f \in R_{>0}$ need not have finite global dimension. As explained in the first paragraph of the proof of Lemma 6.4, Λ' is locally of the form $(\Lambda_f)_{\geq 0}$.

For example, let $G = G_m$ act on a 4-dimensional vector space W with weights $-2, -1, 1, 2$. Let U be another G-representation with weights $0, 1, 2$. Let $S = \operatorname{Sym} W^\vee$, $R = S^G$, $\Lambda = (\operatorname{End}(U) \otimes S)^G$. Then Λ is an NCCR of R [VdB04a, Theorem 8.9], and thus in particular homologically homogeneous. We let f be the product of the weight vectors in $W^\vee \subset S$ with weights $-2, 2$ (which is G-invariant and thus belongs to R), and claim that $\operatorname{gl}\dim(\Lambda_f)_0 = \infty$, which implies $\operatorname{gl}\dim(\Lambda_f)_{\geq 0} = \infty$ by [ŠVdB17, Lemma 4.3.2].

Note that $B := (\Lambda_f)_0 = (\operatorname{End}(U) \otimes (S_f)_0)^G = M_{G,(S_f)_0}(\operatorname{End} U)$. By [ŠVdB17, Lemma 4.2.1], it is enough to show that the global dimension of $(\operatorname{End}(U) \otimes k[N_x])^{G_x}$ is infinite for some closed point $x \in \operatorname{Spec}((S_f)_0)$ with closed G_m-orbit and with (linear) slice N_x. To compute the slice N_x we observe that $\operatorname{Spec}((S_f)_0)$ is an open subset in $\mathbb{P}(W) = (W \setminus \{0\})/G_m$. Hence we may compute the slice in $\mathbb{P}(W)$. Let x^* in W be a lift of x and let N_{x^*} be the slice in W of x^* for the $G \times G_m$-action. Then it is easy to see that $N_x /\!\!/ G_x$ is the same as $N_{x^*} /\!\!/ (G \times G_m)_{x^*}$. The weights of W, U as $G \times G_m$-representation are respectively $(-2, 1), (-1, 1), (1, 1), (2, 1)$ and $(0, 0), (1, 0), (2, 0)$. We take the point $x = [a : 0 : 0 : b] \in \operatorname{Spec}(S_f)_0 \subset \mathbb{P}(W)$. The stabilizer of x^* is $\mathbb{Z}_4$, embedded in $G \times G_m$ via (ϵ, ϵ^2), where ϵ is a primitive 4-th root of unity. The actions of $\mathbb{Z}_4$ on N_{x^*}, U have weights $1/4, 3/4$ and $0, 1/4, 2/4$, respectively. Thus, $N_x /\!\!/ G_x \cong N_{x^*} /\!\!/ \mathbb{Z}_4$ is a Gorenstein singularity. We moreover have $(\operatorname{End}(U) \otimes k[N_x])^{G_x} = (\operatorname{End}(U) \otimes k[N_{x^*}])^{\mathbb{Z}_4} = \operatorname{End}_{k[N_{x^*}]^{\mathbb{Z}_4}}((U \otimes k[N_{x^*}])^{\mathbb{Z}_4})$, where the last equality follows by [ŠVdB17, Lemma 4.1.3] because N_{x^*} is a generic $\mathbb{Z}_4$-representation (see [ŠVdB17, Definition 1.3.4]). If B would have finite global dimension it would be an NCCR. However this is impossible by [IW14, Proposition 4.5] since $(U \otimes k[N_{x^*}])^{\mathbb{Z}_4}$ is not an "MM-module" (see loc.cit.) as $U \subset k\mathbb{Z}_4$ (as $\mathbb{Z}_4$-representation) and $\operatorname{End}_{k[N_{x^*}]^{\mathbb{Z}_4}}(k\mathbb{Z}_4 \otimes k[N_{x^*}])^{\mathbb{Z}_4})$ is a Cohen-Macaulay $k[N_{x^*}]^{\mathbb{Z}_4}$-module.

9. Endomorphism sheaves in the case of constant stabilizer dimension

In this section, on smooth quotients stacks with constant stabilizer dimension, we give a geometric ("gerby") interpretation of sheaves of endomorphism algebras of vector bundles. In particular, this applies to $\Lambda_{Z_{ji}}$ appearing in Theorem 8.15.

9.1. **Normalizer of a representation.** We discuss some technical results we need later on. Let $H \subset G$ be an inclusion of reductive groups. We recall the following result for further reference.

Lemma 9.1. [LR79, Lemma 1.1] *$N(H)$ is reductive.*

Let V be an irreducible representation of H. Let $g \in N(H)$. Denote by $\sigma_g = g^{-1} \cdot g : H \to H$ and by ${}_{\sigma_g}V$ the corresponding twisted H-representation (i.e. the action of $h \in H$ on ${}_{\sigma_g}V_i$ is $h.v := (g^{-1}hg)v$). We set

$$N_V(H) := \{u \in N(H) \mid {}_{\sigma_u}V \underset{H}{\cong} V\}$$

so that we have inclusions

$$H \subset N_V(H) \subset N(H).$$

Lemma 9.2. *The index of $N_V(H)$ in $N(H)$ is finite.*

Proof. We claim that if a reductive group H is a normal subgroup in a reductive group K, then the image of the map

$$K \to \operatorname{Out}(H) = \operatorname{Aut}(H)/\operatorname{Inn}(H), \quad k \mapsto (h \mapsto khk^{-1}) \tag{9.1}$$

is finite. We apply this with $K = N(H)$, which is reductive by Lemma 9.1. If $u \in N(H)$ is in the kernel of (9.1) then σ_u is an inner automorphism of H and then ${}_{\sigma_u}V \cong V$. Hence $u \in N_V(H)$. So the kernel of (9.1) is contained in $N_V(H)$, which is therefore of finite index.

We now prove the claim. Note that we can assume that H is connected. Indeed H_e is a normal subgroup of K and furthermore $\operatorname{Out}(H) \to \operatorname{Aut}(H_e)/\operatorname{Inn}(H)$ has finite kernel (since the kernel is a subquotient of $\operatorname{Aut}(H/H_e)$ which is finite as H/H_e is finite), and $\operatorname{Aut}(H_e)/\operatorname{Inn}(H)$ is a quotient of $\operatorname{Out}(H_e)$.

Assuming H connected we have $H \subset K_e$. As K/K_e is finite we may then also assume that K is connected. Then $K = HQ$ for a subgroup Q of K such that Q and H commute [Spr98, Theorem 8.1.5, Corollary 8.1.6]. Thus, the image of K is trivial in this case. □

For use below we write $V \sim V'$ for $V, V' \in \operatorname{rep}(H)$ if there is some $g \in N(H)$ such that $V' \overset{H}{\cong} {}_{\sigma_g}V$. This defines an equivalence relation on $\operatorname{rep}(H)$ and the equivalence classes are in bijection with $N(H)/N_V(H)$. In particular by Lemma 9.2 they are finite.

9.2. **Actions with stabilizers of constant dimension.** Now we assume that Z is a G-equivariant connected[17] smooth k-scheme with a good quotient $\pi : Z \to Z/\!\!/G$. Moreover we assume that the stabilizers $(G_x)_{x\in Z}$ have dimension independent of x. As explained in §7.1 all orbits in Z are closed and all G_x are reductive.

Let H be the stabilizer of a point in the open ("principal") stratum of the Luna stratification [Lun73] (we call H a *generic stabilizer*). By the properties of the Luna stratification, H is uniquely determined up to conjugacy.

[17] The connectedness assumption is purely to simplify the notation. It is not a serious restriction as in general $X/G \cong \coprod_i X_i/G_i$ where the X_i are representatives of the orbits of the connected components of X and the G_i are their stabilizers.

Proposition 9.3. *Let $Z^{\langle H\rangle}$ be the union of connected components of Z^H which contain a point whose stabilizer is exactly H. Then $Z^{\langle H\rangle}$ is smooth and the canonical map*

$$\phi : G \times^{N(H)} Z^{\langle H\rangle} \overset{\cong}{\to} Z$$

is an isomorphism.

Proof. Since Z^H is smooth [CGP15, Proposition A.8.10(2)], $Z^{\langle H\rangle}$ is smooth. By [LR79], [PV94, Theorem 7.14], $Z^{\langle H\rangle}/\!\!/N(H) \cong Z/\!\!/G$. It thus follows that ϕ is surjective since $Z \to Z/\!\!/G$, $Z^{\langle H\rangle} \to Z^{\langle H\rangle}/\!\!/N(H)$ separate orbits (as all orbits are closed as mentioned in the beginning of this subsection). Moreover ϕ defines an isomorphism between the principal strata for the Luna stratification. Globally ϕ is quasi-finite since G acts with constant stabilizer dimension. As Z is normal, by Zariski's main theorem ϕ is an isomorphism. □

Remark 9.4. Assume that $(Z, \mathcal{L})$ is a linearized connected smooth G-scheme. A point $x \in Z^{\mathrm{ss}} := X^{\mathrm{ss},\mathcal{L}}$ is *stable in Mumford's sense* [MFK94] if Gx has maximal dimension and is closed in Z^{ss}. Let $Z^{\mathrm{ms}} \subset Z^{\mathrm{ss}}$ be the set of Mumford stable points. Proposition 9.3 applies to Z^{ms} and so gives a structure theorem for Z^{ms}. We have not been able to find this result in the literature.

For use below we introduce some associated notations. For $V \in \mathrm{rep}(H)$ we put $\mathcal{Z}_V := Z^{\langle H\rangle}/(N_V(H)/H)$. For convenience we list some easily verified properties of $\mathcal{Z}_V$.

- $\mathcal{Z}_V$ is a smooth Deligne-Mumford stack.
- The natural quotient map $\mathcal{Z}_V \to Z^{\langle H\rangle}/\!\!/(N(H)/H) \cong Z/\!\!/G$ is finite.
- $\mathcal{Z}_V$ may however be non-connected.
- If G is abelian then $\mathcal{Z}_V = Z/(G/H)$, independently of V.

9.3. **Equivariant vector bundles and Azumaya algebras.** In this section we assume as in §9.2 that Z has constant stabilizer dimension. We discuss some properties of equivariant vector bundles on Z. For simplicity we will phrase them for a fixed choice of H (within its conjugacy class) but it is easy to see that they are in fact independent of this choice.

If U is an $N(H)$-representation (possibly infinite dimensional) and V is an irreducible H-representation then we let $U(V)$ be the V-isotypical part of U; i.e. if $U_V := \mathrm{Hom}(V, U)^H$ then $U(V)$ is the image of the evaluation map

$$V \otimes U_V \to U.$$

The evaluation map is injective so it yields in particular an isomorphism as H-representations

$$V \otimes U_V \cong U(V) \tag{9.2}$$

where the H-action on U_H is trivial. Moreover there is an internal direct sum decomposition

$$U = \bigoplus_{V \in \mathrm{rep}(H)} U(V). \tag{9.3}$$

One checks that if $g \in N(H)$ then inside U

$$g(U(V)) = U({}_{\sigma_g}V). \tag{9.4}$$

It follows that $U(V)$ is in fact a $N_V(H)$-subrepresentation of U.

Let $\mathcal{U}$ be a G-equivariant vector bundle on Z. We write

$$\mathcal{U}^{\langle H\rangle} := \mathcal{U} \mid Z^{\langle H\rangle}.$$

Since $\mathcal{U}^{\langle H\rangle}$, locally over $Z^{\langle H\rangle}/\!\!/N(H)$, is in particular an $N(H)$-representation it makes sense to use the notation $\mathcal{U}^{\langle H\rangle}(V)$.

We will also put

$$\mathcal{U}^{\langle H\rangle}_V := \operatorname{Hom}(V,\mathcal{U}^{\langle H\rangle})^H.$$

From (9.2) (checking locally over $Z^{\langle H\rangle}/\!\!/N(H)$) we get

$$\mathcal{U}^{\langle H\rangle}(V) \cong V \otimes \mathcal{U}^{\langle H\rangle}_V \tag{9.5}$$

as H-equivariant coherent sheaves on $Z^{\langle H\rangle}$.

Lemma 9.5. *$\mathcal{U}^{\langle H\rangle}(V)$ and $\mathcal{U}^{\langle H\rangle}_V$ are vector bundles on $Z^{\langle H\rangle}$. Moreover $\mathcal{U}^{\langle H\rangle}(V)$ is in fact a $N_V(H)/H$-equivariant subbundle of $\mathcal{U}^{\langle H\rangle}$.*

Proof. By (9.5) it suffices to consider $\mathcal{U}^{\langle H\rangle}(V)$ for the first claim. We have a decomposition of coherent sheaves on $Z^{\langle H\rangle}$:

$$\mathcal{U}^{\langle H\rangle} = \bigoplus_{V\in \operatorname{rep}(H)} \mathcal{U}^{\langle H\rangle}(V). \tag{9.6}$$

This can be checked locally over $Z^{\langle H\rangle}/\!\!/N(H)$ so that we may assume that $Z^{\langle H\rangle}$ is affine. Then it follows from (9.3). In particular $\mathcal{U}^{\langle H\rangle}(V)$ is a direct summand of $\mathcal{U}^{\langle H\rangle}$. So it is a vector bundle.

The fact that $\mathcal{U}^{\langle H\rangle}(V)$ is $N_V(H)/H$-equivariant may again be checked in the case that Z is affine where it follows from the above discussed fact that $U(V)$ is $N_V(H)/H$-invariant. $\square$

Lemma 9.6. *Let $x \in Z^{\langle H\rangle}$. Then $\mathcal{U}^{\langle H\rangle}(V) \neq 0$ (which is equivalent to $\mathcal{U}^{\langle H\rangle}_V \neq 0$) if and only if there exists $V' \sim V$ such that V' appears in $\mathcal{U}^{\langle H\rangle}_x$.*

Proof. We first collect some easy facts. From (9.4) one may deduce that if $y \in Z^{\langle H\rangle}$ and $g \in N(H)$ then

$$\mathcal{U}^{\langle H\rangle}(V)_{g^{-1}y} \overset{H}{\cong} \mathcal{U}^{\langle H\rangle}({}_{\sigma_g}V)_y. \tag{9.7}$$

Moreover if y, y' are in the same connected component of $Z^{\langle H\rangle}$ then by semi-continuity

$$\mathcal{U}^{\langle H\rangle}(V)_{y'} \overset{H}{\cong} \mathcal{U}^{\langle H\rangle}(V)_y. \tag{9.8}$$

Now we prove the lemma.

($\Rightarrow$) If $\mathcal{U}^{\langle H\rangle}(V) \neq 0$ then there is some $y \in Z^{\langle H\rangle}$ such that $\mathcal{U}^{\langle H\rangle}(V)_y \neq 0$. y may be in a different component than x, but by combining (9.7)(9.8) we find that there exists $V' \sim V$ such that $\mathcal{U}^{\langle H\rangle}(V')_x \neq 0$ (as $N(H)$ acts transitively on the connected components by using the assumption that Z is connected).

($\Leftarrow$) This is proved by reversing the argument in ($\Rightarrow$). $\square$

For use below we will write $\langle \mathcal{U}\rangle \subset \operatorname{rep}(H)/\sim$ for the set of equivalence classes that contain a representation that appears in $\mathcal{U}^{\langle H\rangle}_x$ for some $x \in Z^{\langle H\rangle}$. It follows from (9.7) that $\langle \mathcal{U}\rangle$ is well-defined.

It follows from Lemma 9.5 that

$$\begin{aligned}\mathcal{A}_V &\coloneqq \mathcal{E}nd_{Z^{\langle H\rangle}}(\mathcal{U}^{\langle H\rangle}(V))^H\\ &\overset{(9.5)}{\cong} \mathcal{E}nd_{Z^{\langle H\rangle}}(\mathcal{U}_V^{\langle H\rangle})\end{aligned}$$

is a $N_V(H)/H$-equivariant sheaf of Azumaya algebras on $Z^{\langle H\rangle}$ which is trivial if we forget the $N_V(H)/H$-action. Below we consider $\mathcal{A}_V$ as living on the quotient stack $\mathcal{Z}_V = Z^{\langle H\rangle}/(N_V(H)/H)$ which was introduced in §9.2. Our main result in this section is the following.

Proposition 9.7. *Assume that $\mathcal{U}$ is a saturated G-equivariant vector bundle on Z. Put*

$$\Lambda = \pi_{s*}\,\mathcal{E}nd_Z(\mathcal{U})$$

where $\pi_s : Z/G \to Z/\!\!/G$ is the quotient map. Then we have an equivalence of abelian categories

$$\operatorname{Qch}(Z/\!\!/G,\Lambda) \cong \bigoplus_{V\in\langle\mathcal{U}\rangle/\sim} \operatorname{Qch}(\mathcal{Z}_V,\mathcal{A}_V). \tag{9.9}$$

If G is abelian then each class in $\operatorname{rep}(H)/\sim$ *is a singleton and* $\operatorname{Qch}(\mathcal{Z}_V,\mathcal{A}_V) \cong \operatorname{Qch}(Z/(G/H))$ *for $\{V\}\in\langle\mathcal{U}\rangle$.*

Proof. The part about general G follows by combining Lemmas 9.10 and 9.12 below where we use Lemma 9.6 to restrict the sum.

Let us now assume that G is abelian. Then $Z^{\langle H\rangle} = Z$ and we may drop $(-)^{\langle H\rangle}$ superscripts. It is obvious that every class in $\operatorname{rep}(H)/\sim$ is a singleton. Furthermore we may extend the H-action on V to a G-action (non-canonically). It follows that $\mathcal{U}_V$ is G/H-equivariant and $\mathcal{A}_V = \mathcal{E}nd_Z(\mathcal{U}_V)$ as G/H-equivariant sheaves of algebras. In other words $\mathcal{A}_V$ is a trivial Azumaya algebra on $Z/(G/H)$ and the result follows. □

Corollary 9.8. *With notations and hypotheses as in Proposition 9.7 we have a decomposition*

$$D(\Lambda) \cong \bigoplus_{V\in\langle\mathcal{U}\rangle/\sim} D(\mathcal{A}_V). \tag{9.10}$$

Proof. We only need to note that in $D_{\operatorname{Qch}}(-) = D(\operatorname{Qch}(-))$ in our case by (the proof of) [HNR19, Theorem 1.2]. □

9.4. **Geometric interpretation of $\Lambda_{Z_{ji}}$.** In particular, Proposition 9.7 and Corollary 9.8 apply to the setting of Theorem 8.15, and thus allow us to give a more geometric description of $\Lambda_{Z_{ji}}$ appearing there. For the convenience of the reader we repeat the statements in that setting.

We use the notation introduced in Theorem 8.15, moreover we set H_{ji} for the principal stabilizer of the action of G_{ji} on Z_{ji}, $\mathcal{A}_{Z_{ji},V} = \mathcal{E}nd_{Z^{\langle H_{ji}\rangle}}(\mathcal{U}_{j,Z_{ji},V}^{\langle H_{ji}\rangle})$ $\mathcal{Z}_{ji,V} = Z_{ji}^{\langle H_{ji}\rangle}/(N_V(H_{ji})/H_{ji})$.

Corollary 9.9. *Let the setting be as in Theorem 8.15. Then*

$$\mathrm{Qch}(Z_{ji}/\!\!/G_{ji}, \Lambda_{Z_{ji}}) \cong \bigoplus_{V\in\langle\mathcal{U}_{j,Z_{ji}}\rangle/\sim} \mathrm{Qch}(\mathcal{Z}_{ji,V}, \mathcal{A}_{Z_{ji},V}),$$

$$D(\Lambda_{Z_{ji}}) \cong \bigoplus_{V\in\langle\mathcal{U}_{j,Z_{ji}}\rangle/\sim} D(\mathcal{A}_{Z_{ji},V}).$$

Proof. We need to check that the hypotheses for Proposition 9.7 with $(Z, G, \mathcal{U}) = (Z_{ji}, G_{ji}, \mathcal{U}_{j,Z_{ji}})$ apply.

Recall that Z_{ji} is G_{ji}-equivariant smooth connected k-scheme, and by definition Z_{ji} has stabilizers of constant dimension. Let us denote $\mathcal{U}_{ji} := \mathcal{U}_{j,Z_{ji}}$. We only need to observe that the hypothesis on $\mathcal{U}$ imply that $\mathcal{U}_{ji}$ is saturated. This follows by Theorem 7.6 as its hypotheses are satisfied by Proposition 8.13 (and the initial hypothesis on Λ, $\mathcal{U}$). □

9.5. **A decomposition result.** In this section we assume as in §9.2 that Z has constant stabilizer dimension. We keep the notations introduced in the previous sections.

Lemma 9.10. *For $V \in \mathrm{rep}(H)$ consider the morphism*

$$\psi_V : Z^{\langle H\rangle}/\!\!/(N_V(H)/H) \to Z^{\langle H\rangle}/\!\!/(N(H)/H) \cong Z/\!\!/G.$$

Let $\mathcal{U}$ be a G-equivariant vector bundle on Z. Then we have

$$\pi_{s*}\,\mathcal{E}nd_Z(\mathcal{U}) \cong \bigoplus_{V\in\mathrm{rep}(H)/\sim} \psi_{V,*}\pi_{V,s,*}\mathcal{A}_V$$

where $\pi_s : Z/G \to Z/\!\!/G$, $\pi_{V,s} : \mathcal{Z}_V = Z^{\langle H\rangle}/(N_V(H)/H) \to Z^{\langle H\rangle}/\!\!/(N_V(H)/H)$ are the quotient maps.

Proof. We consider the corresponding quotient map

$$\pi^{\langle H\rangle}_{s*} : Z^{\langle H\rangle}/N(H) \to Z^{\langle H\rangle}/\!\!/N(H) \cong Z/\!\!/G.$$

Using $Z/\!\!/G = Z^{\langle H\rangle}/N(H)$ we obtain

$$\pi_{s*}\,\mathcal{E}nd_Z(\mathcal{U}) = \pi^{\langle H\rangle}_{s*}\,\mathcal{E}nd_{Z^{\langle H\rangle}}(\mathcal{U}^{\langle H\rangle}).$$

We may now restrict to the case[18] $Z = Z^{\langle H\rangle}$, $G = N(H)$. We drop all superscripts $(-)^{\langle H\rangle}$ from the notation.

Using (9.6) and Lemma 9.5 we obtain a G-equivariant decomposition of $\mathcal{U}$,

$$\mathcal{U} = \bigoplus_{V\in\mathrm{rep}(H)/\sim}\ \bigoplus_{V'\sim V} \mathcal{U}(V')$$

so that

$$\pi_{s*}\,\mathcal{E}nd_Z(\mathcal{U}) = \bigoplus_{V\in\mathrm{rep}(H)/\sim} \pi'_{s*}\left(\bigoplus_{V'\sim V} \mathcal{E}nd_Z(\mathcal{U}(V'))^H\right)$$

where π'_{s*} is the modified quotient map

$$Z/(G/H) \to Z/\!\!/(G/H).$$

[18]Note that $Z^{\langle H\rangle}$ may be nonconnected. So we are stepping out of our original context. However we will be careful not to use any results depending on connectedness.

Using the definition of $\mathcal{A}_V$ it is now sufficient to prove that the projection

$$\pi'_{s*}\left(\bigoplus_{V'\sim V}\mathcal{E}nd_Z(\mathcal{U}(V'))^H\right)\to\psi_{V,*}\pi_{V,s,*}\,\mathcal{E}nd_Z(\mathcal{U}(V))^H$$

is an isomorphism. This can be checked locally over $Z/\!\!/(G/H)$ and hence we may assume that Z is affine. Then it reduces to the algebraic statement in Lemma 9.11 below (with $G = G/H$, $K = N_V(H)/H$). □

Lemma 9.11. *Let $K \subset G$ be groups. Let $A = \bigoplus_{u\in G/K} A_u$ be an algebra equipped with a G-action such that $g(A_k) = A_{gk}$. Then projection induces an algebra isomorphism*

$$\Big(\bigoplus_{u\in G/K} A_u\Big)^G \to A_e^K.$$

Proof. Left to the reader. □

9.6. **Morita theory of $\mathcal{A}_V$.** In this section we assume as in §9.2 that Z has constant stabilizer dimension. We keep the notations introduced in the previous section. Consider the quotient map

$$\pi_{V,s} : \mathcal{Z}_V \to Z^{\langle H\rangle}/\!\!/(N_V(H)/H) = Z^{\langle H\rangle}/\!\!/N_V(H)$$

as well as the associated morphism of ringed stacks

$$\bar\pi_{V,s} : (\mathcal{Z}_V, \mathcal{A}_V) \to (Z^{\langle H\rangle}/\!\!/N_V(H), \pi_{V,s,*}\mathcal{A}_V).$$

Lemma 9.12. *Assume that $\mathcal{U}$ is a G-equivariant saturated vector bundle on Z. Then for every $V \in \mathrm{rep}(H)$ there is an equivalence of categories*

$$\bar\pi_{V,s,*} : \mathrm{Qch}(\mathcal{Z}_V, \mathcal{A}_V) \to \mathrm{Qch}(Z^{\langle H\rangle}/\!\!/N_V(H), \pi_{V,s,*}\mathcal{A}_V).$$

Proof. To simplify the notation we first replace Z by $Z^{\langle H\rangle}$ and G by $N(H)$ and drop all $(-)^{\langle H\rangle}$ superscripts. Since $X/G \cong X^{\langle H\rangle}/N(H)$ it is easy to see that this does not affect the saturation property of $\mathcal{U}$.

Next we further replace G by $N_V(H)$ which by Lemma 9.13 below also does not affect the saturation property.

As $\bar\pi_{V,s,*}\bar\pi^*_{V,s}$ is easily seen to be the identity, we have to prove that $\bar\pi^*_{V,s}\bar\pi_{V,s,*}$ is the identity.

This may be checked strongly étale locally on Z. Hence we may replace Z by $G\times^{G_x} S$ for S a smooth connected affine slice at $x \in Z$. Using $(G\times^{G_x} S)/(G/H) \cong S/(G_x/H)$ we may reduce do $Z = S$, $G = G_x$; i.e. $x \in Z$ is now a fixed point for G and we have to show that $\bar\pi^*_{V,s}\bar\pi_{V,s,*}$ is the identity on a neighborhood of x. Note that G/H is now a finite group.

Since $\bar\pi_{V,s,*}$ is exact and $\bar\pi^*_{V,s}$ is right exact it is sufficient to prove that for every $\mathcal{M} \in \mathrm{coh}(\mathcal{A}_V)$ there is a map $\mathcal{A}_V^{\oplus N} \to \mathcal{M}$ in $\mathrm{coh}(\mathcal{A}_V)$, whose cokernel is zero on a neighborhood of x. By lifting generators of $\mathcal{M}$ we may reduce to the case $Z = x$ and we have to show that $\mathcal{A}_V$ is a projective generator for $\mathrm{coh}(\mathcal{A}_V)$. As $\mathcal{U}$ is saturated this follows from Lemma 9.14 below (using that $H_x = H$). □

Lemma 9.13. *Assume that $\mathcal{U}$ is a saturated G-equivariant vector bundle on Z and K is a subgroup of G of finite index which contains H_x for all $x \in H$. Then the pullback of $\mathcal{U}$ to Z/K is also saturated.*

Proof. Let $x \in Z$. As G/K is finite we have $T_x(X)/T_x(Kx) = T_x(X)/T_x(Gx) = H_x$. As G_x/K_x is finite, the lemma follows from Mackey's restriction formula. $\square$

Lemma 9.14. *Let H be a normal subgroup of finite index in G. Assume that U is a finite dimensional G-representation which is up to nonzero multiplicities induced from H. If for $V \in \operatorname{rep}(H)$, ${}_{\sigma_g}V \cong V$ for all $g \in G$, then $\operatorname{End}_H(U(V))$ is a projective generator for $\operatorname{mod}(G/H, \operatorname{End}_H(U(V))$.*

Proof. Note that $\operatorname{Res}^G_H \operatorname{Ind}^G_H W :=: \bigoplus_{W' \sim W} W'$ for $W \in \operatorname{rep}(H)$ (see e.g. the proof of [ŠVdB17, Lemma 4.5.1]). Hence up to Morita equivalence we may assume $U = \bigoplus_{i=1}^n \operatorname{Ind}^G_H V_i$ with $V_i \in \operatorname{rep}(H)$ and $V_1 = V, V_2, \dots, V_n \not\sim V$ (as $\operatorname{Ind}^G_H W \cong \operatorname{Ind}^G_H W'$ if $W \sim W'$), so that $\operatorname{Ind}^G_H V_1$ and $\bigoplus_{i=2}^n \operatorname{Ind}^G_H V_i$ have no common H-summands. Thus $U(V) = \operatorname{Ind}^G_H V$ (as ${}_{\sigma_g}V \cong V$). We now put $U = U(V)$.

As G-representations we have

$$U \otimes k[G/H] \cong \operatorname{Ind}^G_H V \otimes k[G/H] = \operatorname{Ind}^G_H(V \otimes k[G/H]) :=: \operatorname{Ind}^G_H V = U \tag{9.11}$$

where we used the projection formula (i.e. the tensor identity, [Jan87, Proposition I.3.6]) for the second equality, and the fact $V :=: V \otimes k[G/H]$ as H-representations (since $k[G/H]$ is the trivial H-representation) for the :=:-relation.

Consider the functor

$$F : \operatorname{mod}(G) \to \operatorname{mod}(G/H, \operatorname{End}_H(U)) : M \mapsto \operatorname{Hom}_H(U, M).$$

One checks that if $T \in \operatorname{mod}(G)$, $W \in \operatorname{mod}(G/H)$ then

$$F(T \otimes W) = F(T) \otimes_k W \tag{9.12}$$

in $\operatorname{mod}(G/H, \operatorname{End}_H(U))$. Applying F to (9.11) and using (9.12) with $W = k[G/H]$ we obtain

$$\operatorname{End}_H(U) \otimes k[G/H] :=: \operatorname{End}_H(U) \tag{9.13}$$

in $\operatorname{mod}(G/H, \operatorname{End}_H(U))$. As $\operatorname{End}_H(U) \otimes k[G/H]$ is tautologically a generator for $\operatorname{mod}(G/H, \operatorname{End}_H(U))$ it follows from (9.13) that $\operatorname{End}_H(U)$ is a generator for $\operatorname{mod}(G/H, \operatorname{End}_H(U))$. $\square$

10. Example

We demonstrate the above results on a simple example of the conifold singularity.

Assume that $X = W$ is a 4-dimensional vector space on which $G = G_m$ acts with weights $-1, -1, 1, 1$. Then $X/\!\!/G$ is a conifold singularity. In this case Z is the origin, and the Kirwan resolution $\mathbf{X}/G$ is obtained by one Reichstein transform.

A noncommutative crepant resolution Λ of $X/\!\!/G$, is given by a vector bundle $\mathcal{U} = \mathcal{O}_X \oplus \chi_1 \otimes \mathcal{O}_X$, where χ_i denotes 1-dimensional G-representation with weight i, i.e. $\Lambda = (\operatorname{End}(\chi_0 \oplus \chi_1) \otimes k[W^\vee])^G \cong \operatorname{End}_{k[W^\vee]^G}(k[W^\vee]^G \oplus (\chi_1 \otimes k[W^\vee])^G)$ [VdB04a, Theorem 8.9].

Note that $\Lambda_Z = \operatorname{End}(\chi_0 \oplus \chi_1)^G = k^{\oplus 2}$ and $\operatorname{codim}(Z, X) = 4$. By Theorem 8.15 we then obtain

$$D(\mathbf{X}/G) = \langle D(\Lambda), D(k), D(k), D(k), D(k), D(k), D(k) \rangle. \tag{10.1}$$

Remark 10.1. Note that $X/\!\!/G$ is as a toric variety given by a fan with a single cone σ generated by $(0,0,1), (1,0,1), (1,1,1), (0,1,1)$. Let $\Sigma = \sigma \cup \mathbb{R}_{\geq 0}(1,1,2)$. Then $\mathbf{X}/G$ is a toric stack given by the stacky fan

$$\mathbf{\Sigma} = (\Sigma, \{(0,0,1), (1,0,1), (1,1,1), (0,1,1), (2,2,4)\})$$

(representing the stacky blow up of $X /\!\!/ G$ in the origin) and $\mathbf{X} /\!\!/ G$ is a toric variety given by Σ (which is a blow-up of $X /\!\!/ G$ in the origin) [EM12, Theorem 4.7]. Using [BCS05, Proposition 4.5] one can (alternatively) check that $\operatorname{rk} K_0(\mathbf{X}/G) = 8$, which agrees with (10.1).

Appendix A. Local duality for graded rings

Let Λ be an $\mathbb{N}$-graded ring which is finitely generated as a module over its center R which in turn is a $\mathbb{N}$-graded k-algebra such that R_n is a finitely generated R_0-module. For convenience reasons we use *left* modules in this appendix. This allows us to literally use some results from [VdB97]. Needless to say this is only a notational issue and moreover in the rest of the paper we only use Corollary A.3 which is left right agnostic.

Below we write $D(\Lambda)$ for $D(\operatorname{Gr} \Lambda)$ and this convention extends to all related notations. Let D_R, D_{R_0} be the Grothendieck dualizing complexes of R, R_0 respectively and let D_Λ, D_{Λ_0} be the corresponding dualizing complexes of Λ, Λ_0; i.e. we have

$$\begin{aligned} D_\Lambda &= \operatorname{RHom}_R(\Lambda, D_R), \\ D_{\Lambda_0} &= \operatorname{RHom}_{R_0}(\Lambda_0, D_{R_0}). \end{aligned}$$

Let C be an arbitrary graded k-algebra. For M a complex of left graded $C \otimes \Sigma$-modules, where $\Sigma \in \{\Lambda, \Lambda_0\}$, we put

$$M^\vee = \operatorname{RHom}_{R_0}(M, D_{R_0}) \in D(C^\circ \otimes \Sigma^\circ).$$

Let $D_f(\Lambda_0)$ be the full subcategory of $D(\Lambda_0)$ consisting of complexes which have finitely generated cohomology. Then $(-)^\vee$ defines a duality between $D_f(\Lambda_0)$ and $D_f(\Lambda_0^\circ)$ (recall that D_{R_0} is bounded and has finite injective dimension).

Similarly, let $D^b_f(\Lambda)$ be the full subcategory of $D^b(\Lambda)$ consisting of complexes with cohomology which is finitely generated as Λ_0-module (or equivalently as R_0-module) in every degree. Then $(-)^\vee$ defines a duality between $D^b_f(\Lambda)$ and $D^b_f(\Lambda^\circ)$ (recall that D_{R_0} has finite injective dimension).

Remark A.1. If $M \in D(\Lambda_0)$ then by change of rings we have

$$M^\vee = \operatorname{RHom}_{\Lambda_0}(M, D_{\Lambda_0}).$$

The same formula holds if $M \in D(\Lambda)$, but unfortunately in that case the formula obscures the Λ-action on $M^\vee$.

Let $R\Gamma_{\Lambda_{\geq 1}}$ be the right derived functor of $\varinjlim_n \operatorname{RHom}_\Lambda(\Lambda/\Lambda_{\geq n}, -)$.

Proposition A.2. *Let $M \in D(C \otimes \Lambda)$. Then there is a local duality formula in $D(C^\circ \otimes \Lambda^\circ)$*

$$R\Gamma_{\Lambda_{>0}}(M)^\vee \cong \operatorname{RHom}_\Lambda(M, R\Gamma_{\Lambda_{>0}}(\Lambda)^\vee).$$

Proof. If C is absent then the formula is true on the level of complexes for $M = \Lambda$. One then proceeds as in the proof of [VdB97, Proposition 5.1] by replacing M by a free $C \otimes \Lambda$-resolution. □

Corollary A.3. *We have $D_\Lambda \cong R\Gamma_{\Lambda_{>0}}(\Lambda)^\vee$ in $D(\Lambda^e)$.*

Proof. We first check that the right-hand side is a dualizing complex. Note that $R\Gamma_{\Lambda_{>0}}(\Lambda)^\vee = R\Gamma_{R_{>0}}(\Lambda)^\vee$ so that we do not need to worry about the distinction between left and right. Following the proof of [VdB97, Theorem 6.3] we only need to check that $H^i(R\Gamma_{\Lambda_{>0}}(\Lambda)^\vee)$ is finitely generated. Following Lemma A.4 below it is enough to verify that $\Lambda_0 \overset{L}{\otimes}_\Lambda H^i(R\Gamma_{\Lambda_{>0}}(\Lambda)^\vee)$ has finitely generated cohomology as Λ_0-modules. We have the following formula as right Λ_0-modules

$$\begin{aligned}\operatorname{RHom}_{\Lambda_0^\circ}(\Lambda_0 \overset{L}{\otimes}_\Lambda R\Gamma_{\Lambda_{>0}}(\Lambda)^\vee, D_{\Lambda_0}) &= \operatorname{RHom}_{\Lambda^\circ}(\Lambda_0, \operatorname{RHom}_{\Lambda_0}(R\Gamma_{\Lambda_{>0}}(\Lambda)^\vee, D_{\Lambda_0}))\\ &= \operatorname{RHom}_{\Lambda^\circ}(\Lambda_0, R\Gamma_{\Lambda_{>0}}(\Lambda))\\ &= \operatorname{RHom}_{\Lambda^\circ}(\Lambda_0, \Lambda)\end{aligned}$$

where in the first line we have considered $\Lambda_0 \overset{L}{\otimes}_\Lambda R\Gamma_{\Lambda_{>0}}(\Lambda)^\vee$ as the complex of (Λ, Λ_0)-bimodules, and the third line follows by replacing Λ as a right Λ-module by an injective resolution. It follows that as left Λ_0-modules

$$\Lambda_0 \overset{L}{\otimes}_\Lambda R\Gamma_{\Lambda_{>0}}(\Lambda)^\vee = \operatorname{RHom}_{\Lambda^\circ}(\Lambda_0, \Lambda)^\vee,$$

which implies that $\Lambda_0 \overset{L}{\otimes}_\Lambda H^i(R\Gamma_{\Lambda_{>0}}(\Lambda)^\vee)$ indeed has finitely generated cohomology as Λ_0-modules.

The isomorphism $D_\Lambda \cong R\Gamma_{\Lambda_{>0}}(\Lambda)^\vee$ is a consequence of the uniqueness of "rigid" dualizing complexes [VdB97, Definition 6.1, Proposition 8.2(1)]. The fact that the right-hand side is rigid follows as in the proof of [VdB97, Proposition 8.2(2)], as for D_Λ this follows from the proof of [Yek99, Proposition 5.7]. □

Lemma A.4. *Let Λ be a left noetherian $\mathbb{N}$-graded ring. Assume that M is a right bounded complex of graded left Λ-modules with left bounded cohomology. Then the cohomology modules of M are finitely generated Λ-modules if and only if the cohomology modules of $\Lambda_0 \overset{L}{\otimes}_\Lambda M$ are finitely generated Λ_0-modules.*

Proof. We concentrate on the nonobvious direction.

Step 1. Assume that $M \in \operatorname{Gr}(\Lambda)$ has left bounded grading. By the graded Nakayama lemma, M is finitely generated if and only if $\Lambda_0 \otimes_\Lambda M$ is finitely generated (see e.g. [ATVdB90, Proposition 2.2]).

Step 2. Let now M be as in the statement of the lemma and assume that the cohomology modules of $\Lambda_0 \overset{L}{\otimes}_\Lambda M$ are finitely generated Λ_0-modules. Let m be maximal such that $H^m(M) \neq 0$. Then $H^m(\Lambda_0 \overset{L}{\otimes}_\Lambda M) = \Lambda_0 \otimes_\Lambda H^m(M)$. This follows by the appropriate hypercohomology spectral sequence. Hence by Step 1, $H^m(M)$ is finitely generated.

Step 3. Tensoring the distinguished triangle

$$\tau_{\leq m-1} M \to M \to H^m(M)[-m] \to$$

with Λ_0 yields the distinguished triangle

$$\Lambda_0 \overset{L}{\otimes}_\Lambda \tau_{\leq m-1} M \to \Lambda_0 \overset{L}{\otimes}_\Lambda M \to \Lambda_0 \overset{L}{\otimes}_\Lambda H^m(M)[-m] \to .$$

It now follows by Step 2 that $\Lambda_0 \overset{L}{\otimes}_\Lambda \tau_{\leq m-1} M$ has finitely generated cohomology. Now we repeat Steps 2,3 with $\tau_{\leq m-1} M$ replacing M. □

Remark A.5. If Λ is homologically homogeneous (c.f. Definition 7.3) of dimension d then $D_\Lambda = \omega_\Lambda[d]$, where $\omega_\Lambda := \mathrm{Hom}_R(\Lambda, \omega_R)$, by [SVdB08, Proposition 2.9].

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