Crepant resolutions of stratified varieties via gluing

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Abstract

Let X be a variety with a stratification $\mathcal{S}$ into locally closed subvarieties such that X is locally a product along each stratum (for example, a symplectic singularity). We prove that the assignment of an open subset $U \subset X$ to its set of isomorphism classes of (locally projective) crepant resolutions of U is an $\mathcal{S}$-constructible sheaf of sets, by giving a parallel transport type procedure to extend resolutions uniquely along paths within a stratum. For each $s \in S$ a stratum, parallel transport gives an action of the fundamental group $\pi_1(S, s)$ on the set of germs of crepant resolutions at s, which can be interpreted as an obstruction to extending a resolution in a neighborhood of s to a neighborhood of S. Consequently, we obtain an obstruction to extending compatible local resolutions around basepoints in the stratification to global resolutions of X. In particular, a variety with unique local projective crepant resolutions has a unique global projective crepant resolution. Our results generalize to partial crepant resolutions (where the source is not smooth).

We discuss many examples, coming from symplectic singularities (symmetric powers and Hilbert schemes of surfaces with du Val singularities, finite quotients of tori, multiplicative and Nakajima quiver varieties) as well as canonical threefold singularities.

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1 Introduction

Let X be a variety. Consider the classification of all isomorphism classes of pairs $(Y, \rho : Y \to X)$ where Y is a variety and ρ is a birational morphism. The category of such pairs (Y, ρ) is discrete: given (Y_1, ρ_1) and (Y_2, ρ_2), if there is an isomorphism $\varphi : Y_1 \to Y_2$ satisfying $\rho_1 = \rho_2 \circ \varphi$ then φ is unique. Consequently, the set of isomorphism classes forms a sheaf (whereas, a priori, one might only expect the categories of resolutions to form a stack).

In more detail, suppose we have an open covering $\bigsqcup U_i \to X$ of X and birational morphisms $V_i \to U_i$. We seek a variety Y with a map to X and an open covering $\bigsqcup V_i \to Y$ such that the diagram

$$\begin{array}{ccc} V_i & \longrightarrow & Y \\ \downarrow & & \downarrow \\ U_i & \longrightarrow & X \end{array}$$

commutes for all i. If such Y exists, it is unique up to unique isomorphism. Indeed, for each i, j, if $V_i \to U_i$ and $V_j \to U_j$ are compatible over a nonempty overlap $U_i \cap U_j$, then they uniquely glue, by the preceding paragraph. So the only condition for the existence of Y is that the maps are pairwise compatible, i.e., every pair $V_i \to U_i$ and $V_j \to U_j$ restrict to isomorphic maps over the intersection $U_i \cap U_j$.

We will be especially interested in the case where each $V_i \to U_i$ is (locally) projective and crepant. In this case, $Y \to X$ is locally projective and crepant. As a special case, suppose that each U_i admits a unique locally projective crepant resolution $V_i \to U_i$. Then the gluing condition is automatic, and we conclude that X admits a unique locally projective crepant resolution $Y \to X$. The most well-known example of this phenomenon is if X has only du Val surface singularities, in which case the resolution $Y \to X$ is projective and minimal. We give more interesting examples below.

In this paper, we classify locally projective (partial) crepant resolutions from local data, under certain hypotheses. Our primary setting of interest is that of symplectic resolutions. In order for X to admit a symplectic resolution, it must be a symplectic singularity in the sense of Beauville [Bea00]. Symplectic singularities have a finite stratification by symplectic leaves.

Our main theorem shows that isomorphism classes of locally projective symplectic resolutions of X are in bijection with compatible, monodromy-free choices of local resolutions at basepoints of the strata. Furthermore, (globally) projective resolutions with fixed relatively ample line bundle on X are in bijection with compatible systems of local resolutions with fixed relatively ample line bundle.

We prove a much more general statement for (partial) crepant resolutions, that does not require the symplectic structure, at the price of assuming the existence of a nice stratification.

We provide two examples (the first straightforward, the second less so) where the utility of this approach is visible without the need to develop the general obstruction theory.

Example 1.1. Let C_2 act on $(\mathbb{C}^*)^2$ by $(z, w) \mapsto (z^{-1}, w^{-1})$. The singularities of $(\mathbb{C}^*)^2/C_2$ are the four fixed points $(\pm 1, \pm 1)$. Each singularity is analytically locally the A_1-singularity $\mathbb{C}^2/C_2$, which has a unique crepant resolution given by blowing up the singularity. The set of unique analytic-local crepant resolutions of a neighborhood of each singularity glue to a unique global crepant resolution: compatibility is automatic since each overlap has a unique crepant resolution. Additionally, the invariance condition described below (see e.g., Corollary 1.8) is trivially satisfied as the fundamental group is acting on a singleton set. The resulting resolution is, of course, the blowup of the singular locus.

Example 1.2. Fix $n \in \mathbb{N}$. The group $B_n := (C_2)^n \rtimes \mathfrak{S}_n$ acts on $(\mathbb{C}^*)^{2n}$ where $\mathfrak{S}_n$ denotes the symmetric group permuting the n copies of $(\mathbb{C}^*)^2$ and $(C_2)^n$ acts diagonally with C_2 acting as in Example 1.1. The singular locus of the quotient $X := (\mathbb{C}^*)^{2n}/B_n$ is the union of the diagonal (where two pairs are equal) with the locus of n-tuples of pairs of complex numbers containing a pair of the form $(\pm 1, \pm 1)$. Along a point of a diagonal-type stratum—that is, a stratum where some pairs are equal, but none are $(\pm 1, \pm 1)$—the singularity is a product of singularities $\mathbb{C}^{2m}/\mathfrak{S}_m$, which admit a unique resolution given by $\mathrm{Hilb}^{[2m]}(\mathbb{C}^2)$. Singularities where some number of pairs are one of the four pairs $(\pm 1, \pm 1)$ are the same as in $\mathrm{Sym}^n(\mathbb{C}^2/C_2)$, with C_2 now acting by $\pm I$. Our results then imply the formula:

$$\begin{aligned} &\#\{\text{isomorphism classes of locally projective crepant resolutions of } X\} \\ &\qquad = (\#\{\text{isomorphism classes of projective crepant resolutions of } \mathrm{Sym}^n(\mathbb{C}^2/C_2)\})^4 = n^4, \end{aligned}$$

where the last equality is a special case of Bellamy's formula[1] in [Bel16, Proposition 1.2]. To construct the resolutions, independently pick a local crepant projective resolution around each of the most singular points (all pairs are equal and are one of the four pairs $(\pm 1, \pm 1)$), and these spread out uniquely to a

[1] Bellamy is counting symplectic resolutions but the notions of symplectic and crepant agree whenever a symplectic resolution exists, see e.g. [Kal03, Proposition 3.2].

global resolution. In fact, we can see that all resolutions are *globally* projective. See Section 4.2 for more details.

Our key condition on the singularities is the following. Let X be a complex analytic variety equipped with a finite stratification $\{S_i\}$ by locally closed, connected, smooth subvarieties.

Definition 1.3. X is said to be *locally a product along* S_i if, for every $s \in S_i$, there is a connected neighborhood $U \ni s$ in X, a pointed stratified variety $F \ni 0$, and a stratified isomorphism $\varphi : (U \cap S_i) \times F \to U$ which restricts to the inclusion on $(U \cap S_i) \times \{0\}$. We call the triple (U, F, φ) a *local product neighborhood* and refer to F as a *local slice* to S_i.

Note that the condition implies that the germs of X along $s \in S_i$ are locally constant in s. In particular, since S_i is connected, all germs of X along points of S_i are isomorphic.

Our main results concern varieties X with a finite stratification $\mathcal{S}$ by such strata, such that F is stratified and the isomorphism φ respects the stratification. We are interested in crepant resolutions of X, so we assume that X is normal and that its canonical divisor is $\mathbb{Q}$-Cartier. For technical purposes we will also need to assume that F admits a projective crepant resolution $\tilde{F} \to F$ with $H^1(\tilde{F}, \mathcal{O}) = 0$ (see Definition 3.2), in other words local resolutions can be obtained by resolving the slice. This holds in all our examples; for the general case, see Remark 3.4.

Since X is assumed to be normal, and our neighborhoods U are connected, they are irreducible.

Definition 1.4. For an open set $U \subset X$, let $\mathcal{C}_U$ denote the set of isomorphism classes of locally projective, crepant resolutions of U. Similarly, let $\widetilde{\mathcal{C}}_U$ denote the set of isomorphism classes of pairs of a projective crepant resolution of U with a relatively ample line bundle.

In the main body of the text, we will relax the condition that the source is smooth and study partial crepant resolutions dominated by a terminal crepant resolution.

The assignments $U \mapsto \mathcal{C}_U$ and $U \mapsto \widetilde{\mathcal{C}}_U$ are presheaves of sets. In fact, they are sheaves, thanks to the uniqueness of gluing. Our main result is that these sheaves are constructible along $\mathcal{S}$.

We first explain the case of a variety with just two strata:

Example 1.5. Suppose that X has two strata: the open stratum S_0 (the smooth locus), and one singular connected stratum S_1, along which X is locally a product. Pick a basepoint $s_1 \in S_1$. We will prove that the fundamental group $\pi_1(S_1, s_1)$ acts on the stalk of the sheaf $\mathcal{C}_{(-)}$ at s_1, denoted $\mathcal{C}_{s_1}$. The fixed points of this action are the resolutions that extend to a (locally projective, crepant) resolution on a neighborhood U_{S_1} of all of S_1. Since these resolutions are isomorphisms over $S_0 \cap U_{S_1}$, they glue to the identity resolution $S_0 \to S_0$, to give a global resolution of X. So $\mathcal{C}_X$ can be viewed as the subset of fundamental group invariant resolutions of $\mathcal{C}_{s_1}$. Similarly $\widetilde{\mathcal{C}_X}$ can be viewed as the subset of fundamental group invariant elements of $\widetilde{\mathcal{C}_{s_1}}$.

Example 1.6. Let X be singular with stratification $\mathcal{S} := \{S_0, S_1, S_2\}$, such that $\overline{S_i} \supset S_{i+1}$ for $i = 0, 1$. Again we have a minimum stratum, S_2, and so can pick a resolution in a neighborhood of some $s_2 \in S_2$ and then ask if it extends to neighborhoods U_{S_2}, U_{S_1}, and then U_{S_0}. In this way $\mathcal{C}_X$ is again the subset of $\mathcal{C}_{s_2}$ invariant under the fundamental groups of S_2 and S_1. In words, resolutions are determined by local resolutions around their "most singular" points.

These examples suggest the following theorem, our main general result:

Theorem 1.7. *For a variety X with stratification as above, the sheaves $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are $\mathcal{S}$-constructible. The same is true for the subsheaves of smooth crepant resolutions.*

In the course of the proof, which uses some birational geometry, we prove that a more fundamental object is a constructible sheaf, namely the presheaf $\mathcal{P}$ of relative Picard groups, $U \mapsto \mathrm{Pic}(\rho^{-1}(U)/U)$ (see Corollary 1.14 below). Restricted to the codimension two strata, we recover the local systems defined by Namikawa to understand the deformation theory of symplectic singularities in [Nam11]; these local systems determine the Namikawa–Weyl (or symplectic Galois) group [Nam10], with monodromy indicating the presence of non-simply laced types. This group was recently shown to act on cohomology of fibers of symplectic resolutions, defining a vast generalization of Springer theory [MN19].

The theorem implies the following description of the possible projective crepant resolutions:

Corollary 1.8. *Fix $(X,\ \mathcal{S})$ a stratified variety as above. Pick a basepoint $s_i \in S_i$ for each stratum $S_i \in \mathcal{S}$. Then isomorphism classes of resolutions in $\mathcal{C}_X$ and $\widetilde{\mathcal{C}}_X$ are in bijection, via restriction, with the set of compatible, $\pi(S_i, s_i)$-invariant local resolutions of neighborhoods of the s_i.*

The second goal of this paper is to express the aforementioned compatibility conditions in terms of linear algebra, thereby establishing a framework to build global resolutions from local ones.

It is well known that a local system of sets on a space is equivalent to a functor from the fundamental groupoid to sets; by choosing a basepoint this is then equivalent to an action of the fundamental group on the fiber at the basepoint. In the stratified setting, MacPherson explained that constructible sheaves can similarly be defined as functors from the exit path category (see [Tre09, Theorem 1.2]). Let us recall the statement.

Definition 1.9. Given a finitely stratified topological space $X = \sqcup_i S_i$ with S_i locally closed smooth manifolds an *exit path* is a path $\gamma : [0,1] \to X$ such that, for $0 \le t_1 < t_2 \le 1$, the dimension of the stratum containing $\gamma(t_1)$ is less than or equal to the dimension of the stratum containing $\gamma(t_2)$.

These are all concatenations of paths of a simpler form:

Definition 1.10. A *simplified exit path* is $\gamma : [0,1] \to X$ such that $\gamma((0,1])$ all lies in the same stratum.

So, a simplified exit path either lies in a single stratum, or immediately exits one stratum to another. All exit paths are finite concatenations of simplified exit paths (up to parameterization).

Definition 1.11. Let $\mathbf{Ex}(X, \mathcal{S})$ be the category whose objects are points $x \in X$ and whose morphisms are tame homotopy classes of exit paths, through exit paths.

Here, following [Tre09], a tame homotopy is a map $H : [0,1] \times [0,1] \to X$ such that the source can be continuously triangulated with faces mapping to the same stratum.

Theorem 1.12 (MacPherson; see [Tre09])**.** *There is an equivalence between $\mathcal{S}$-constructible sheaves and functors $F : \boldsymbol{Ex}(X, \mathcal{S}) \to \mathbf{Sets}$. This equivalence sends F to the sheaf $\mathcal{F}$ given by:*

$$\mathcal{F}(U) = \Gamma(U, F) := \left\{ (\rho_x)_{x \in U} \in \prod_{x \in U} F(x) \;\middle|\; \rho_y = F([\gamma])(\rho_x),\ \textit{for all exit paths } \gamma : x \to y \textit{ in } U \right\}.$$

Thus, Theorem 1.7 can be rephrased as the following:

Theorem 1.13. *There are functors C and $\widetilde{C} : \boldsymbol{Ex}(X, \mathcal{S}) \to \mathbf{Sets}$, sending $x \in X$ to the stalks $\mathcal{C}_x$ and $\widetilde{\mathcal{C}}_x$, such that $\mathcal{C}(U) = \Gamma(U, C)$ and $\widetilde{\mathcal{C}}(U) = \Gamma(U, \widetilde{C})$. The same is true for the functors giving only the smooth resolutions.*

To keep the prerequisites to a minimum, we will not rely on MacPherson's equivalence, and will instead prove Theorem 1.7 on each stratum separately. This only requires the equivalence between local systems and actions of the fundamental groupoid. We will then recall and use the following easier part of MacPherson's result: There is a unique parallel transport on exit paths, such that global sections of $\mathcal{F}$ are the same as exit path compatible choices of local sections. A proof is provided for the reader's convenience in Section 3.3.

If we label by S_i the strata of X and choose basepoints $s_i \in S_i$, it is clear that $f \in \Gamma(X, C)$ is uniquely determined by its values in the stalks $C(s_i)$: we take the ones compatible under exit paths with endpoints among the s_i. We deduce a more precise version of Corollary 1.8, where the compatibility between local resolutions is given by one being sent to another (up to isomorphism) via exit paths with endpoints the s_i. Note that everything is determined by the local resolutions on the closed strata (although to express compatibility we need to consider exit paths to non-closed strata).

To express the isomorphism classes of local projective crepant resolutions, we use that two such local resolutions are related by a birational transformation given by a line bundle. Fix $\tilde{X}$ a projective crepant resolution of X (if one exists). Then isomorphism classes of projective symplectic resolutions correspond to Mori chambers in the movable cone $\mathrm{Mov}(\tilde{X}/X) \subseteq \mathrm{Pic}(\tilde{X}/X)$. (Note that, when X is affine, it is a consequence of [BCHM10, Theorem 1.3.2] that there are only finitely many such chambers; see Remark 3.4 below for more details.)

Applying this to strata, we can express parallel transport along exit paths in terms of linear monodromy on the relative Picard group. We obtain the following, which is a consequence of the proof of Theorem 1.7:

Corollary 1.14. *Assume that X is equipped with a stratification satisfying our hypotheses. Then the presheaf $\mathcal{P}$ of local relative Picard groups is a constructible sheaf of abelian groups. The sheaf $\widetilde{\mathcal{C}}_X$ is the subsheaf of movable bundles (a subsheaf of sets). The sheaf $\mathcal{C}_X$ is the sheafification of the associated presheaf of sets of Mori cones.*

If a point $x \in X$ has a neighborhood isomorphic to a neighborhood of a Nakajima quiver variety, then, under mild conditions, [BCS22] implies that the Mori chamber structure of its resolution is given by a hyperplane arrangement coming from geometric invariant theory (GIT). Varieties that are formally locally quiver varieties include multiplicative quiver varieties and character varieties of Riemann surfaces [KS23, Theorem 5.4], and more generally moduli of objects in 2-Calabi–Yau categories by Davison [Dav21, Theorem 5.11], which also includes Higgs bundle moduli spaces and moduli of sheaves on K3 or abelian surfaces. In all of these cases, to classify resolutions, we are reduced to determining the linear monodromies on the aforementioned hyperplane arrangements given by a set of generating exit paths.

Specializing Theorem 1.7 allows us to prove classification results in the following cases, constituting our third goal:

(1) varieties with each singularity locally having a unique projective crepant resolution,

(2) symmetric powers of surfaces with du Val singularities,

(3) certain multiplicative quiver varieties, and

(4) finite symplectic quotients of symplectic tori.

We can relax (1) to also allow isolated singularities which may have multiple resolutions; it remains true that locally projective crepant resolutions are classified by arbitrary choices of local projective crepant resolutions at the isolated singularities. This includes the case where all singularities are either du Val or isolated, such as for canonical three-folds and symplectic singularities of dimension four.

Finally, in the main body of the paper, we work in the greater generality of partial crepant resolutions (not requiring that the source be smooth); note that these always exist, unlike smooth crepant resolutions.

The paper is structured as follows: Section 2 gives background results on stratified varieties and in particular establishes a large class of stratified spaces with the local product condition. Section 3 contains the parallel transport argument as well as the main results of the paper, and useful corollaries. Section 4 applies the main results to the examples listed above.

1.1 Conventions

We will be working with analytic varieties over the complex numbers. By a stratification, we mean a locally finite stratification by smooth, connected, locally closed subvarieties.

We denote the symmetric group on n letters by $\mathfrak{S}_n$ and the cyclic group of order n by C_n. For a finite set S we denote its cardinality by $\#S$. We use bold to indicate categories such as **Sets** for the category of sets. For a space X we use $\tilde{X}$ to denote (the source of) a resolution and $\overline{X}$ to denote its closure.

We refer to a not-necessarily isolated singularity as du Val if it is locally a product of a du Val singularity with a polydisc.

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2 Stratifications with product singularities

Let $\mathcal{S}$ denote a stratification of X by locally closed subvarieties. Let $S \in \mathcal{S}$ denote a stratum.

Definition 2.1. A stratification, $\mathcal{S}$, of X is *locally finite* if each $x \in X$ lies in the closure of only finitely many strata.

Throughout the paper we restrict to locally finite stratifications by smooth, connected, locally closed subvarieties, and simply refer to these as stratifications.

We will be interested in stratifications for which X decomposes as a product along each stratum (Definition 1.3). For such stratifications, X is homogeneous along strata:

Proposition 2.2. *Suppose that X is locally a product along a connected, locally closed, smooth subvariety S. Then for any $s, s' \in S$, the germs of X at s and at s' are isomorphic.*

Proof. Let $T \subseteq S$ be the set of all points whose germ is isomorphic to the one at s. This is open by the local product condition: germs at two points $(0, u_1), (0, u_2) \in X' \times U$ are isomorphic given that U is smooth. But now, the complement of T is also a union of open subsets, one for each isomorphism class of germs. Since S is connected, and T is nonempty, $T = S$. □

There are many nice examples of stratifications along which X is locally a product. One of the main cases we will consider is if X is Poisson. We briefly recall a few facts about this, referring to standard references for details. A variety X is *Poisson* if $\mathcal{O}_X$ has a Lie bracket $\{-,-\} : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$ which is a derivation in each component. To each local section $f \in \Gamma(U, \mathcal{O}_X)$ we obtain a vector field ξ_f given by $\xi_f(g) = \{f, g\}$. These vector fields form an integrable distribution (local analytic foliation) since $[\xi_f, \xi_g] = \xi_{\{f,g\}}$. A *symplectic leaf* S of X is a leaf of this foliation, i.e., a locally closed subvariety S of X such that $T_s S = \mathrm{Span}(\xi_f|_s : f \in \Gamma(U, \mathcal{O}_X),\ U$ a neighborhood of $x)$; we require that S be a maximal connected such subvariety. This ensures uniqueness of a symplectic leaf S through $s \in X$, if it exists.

By the Weinstein splitting theorem, if X is Poisson and $S \subseteq X$ is a symplectic leaf, then X decomposes locally as a product along S, see Appendix A.

As a consequence we have the following standard result:

Proposition 2.3. *If X is a Poisson variety with a finite stratification by symplectic leaves, then X is locally a product along this stratification.*

By [Kal06, Theorem 2.5], every symplectic singularity has a finite stratification by symplectic leaves. A symplectic singularity, following [Bea00], is a *normal* variety X such that:

- the smooth locus X^{sm} of X has a symplectic form ω;
- for some (equivalently, every) projective resolution of singularities $\rho : \tilde{X} \to X$, the pullback $\rho^*\omega$ extends to a regular two-form on $\tilde{X}$.

In fact, for[Kal06, Theorem 2.5], we can drop the assumption that X is normal: if X' is Poisson, then it satisfies the first two conditions if and only if the normalization $X \to X'$ satisfies all three conditions, since the normalization map induces a bijection between resolutions of singularities of X and of X'.

Proposition 2.4. *Let X be a smooth variety and G a finite group of automorphisms. Then X/G has a stratification along which X is locally a product.*

Proof. Stratify X by stabilizer subgroups. If S is a stratum corresponding to a subgroup $H < G$, then at a point $s \in S$, the action of H at a germ of s linearizes. The quotient germ is equivalent to a quotient V/K for V a vector space and $K < \mathrm{GL}(V)$ a finite group of linear automorphisms. We can write $V = V^K \oplus V'$ for some complementary K-representation V' of V^K. Then $V/K \cong V^K \times V'/K$. The stratum where the stabilizer is K is V/K, so we indeed get a product along this. □

Proposition 2.5. *Let X be a Poisson variety and G a finite group of Poisson automorphisms. Let S be a symplectic leaf of X. Then the image of S in X/G has a finite stratification by symplectic leaves.*

Proof. The proof follows in the same way as the previous one, noting now that in the case K acts symplectically on V, then the Hamiltonian vector fields at the origin of V/K span the subspace V^K/K. □

Corollary 2.6. *If X is a Poisson variety with a finite stratification by symplectic leaves, and G is a finite group of Poisson automorphisms, then X/G also has a finite stratification by symplectic leaves.*

In particular, this includes finite quotients of symplectic singularities, but actually these are themselves symplectic singularities by [Bea00, Proposition 2.4].

Proposition 2.7. *Let X be a smooth variety and $G \times X \to X$ an action of a reductive group G. Assume that X is equipped with a G-equivariant ample line bundle L (one can take L trivial if X is affine). Then $X/\!/_L G$ has a finite stratification along which it is locally a product.*

Proof. It suffices to prove the result for G connected, since by Proposition 2.4, we can take a further quotient by any finite group. Thanks to the GIT construction, every point in $Y := X/\!/_L G$ is the image of a closed G-orbit in X, whose stabilizer is a reductive subgroup. By Luna's slice theorem, the number of conjugacy classes of such reductive subgroups is finite, which gives a finite stratification of Y.

Let S be such a stratum corresponding to the conjugacy class $[H]$ for $H \leq G$ reductive. We claim that a local slice to S in Y is given by $V/\!/H$, for $V \subseteq X$ a slice guaranteed by Luna's theorem at a point $x \in X$ with closed G-orbit in the semi-stable locus of X which maps to s. The action of H fixes the origin $0 \in V$, and we can linearize the action so that $H < \mathrm{GL}(V)$. We then write $V = V^H \oplus V'$ for some H-subrepresentation V', with V^H/H locally giving the stratum S. This implies that the germ of Y at s is indeed a product along S, as desired. □

Proposition 2.8. *Let X be a Poisson variety and $G \times X \to X$ a Hamiltonian action of a reductive group G. Assume that X is equipped with a G-equivariant ample line bundle L (one can take L trivial if X is affine). Let $S \subseteq X$ be a symplectic leaf. Then the image of the reduced subvariety $S \cap \mu^{-1}(0)$ in $X/\!/\!/_L G$ has a finite stratification into symplectic leaves.*

Proof. The proof begins exactly as before: we reduce to the case that G is connected, and obtain a finite stratification of $X/\!/_L G$. We pull this back to a finite stratification of $\mu^{-1}(0) \subseteq X$. Now, since G is connected, it preserves the symplectic leaf S, and the image of $S \cap \mu^{-1}(0)$ is a locally closed Poisson subvariety which is locally the Hamiltonian reduction of S by G. Applying [Los17, Section 2.3] we see that the induced finite stratification of $S \cap \mu^{-1}(0)$ maps to symplectic leaves of Y. □

Corollary 2.9. *If X is Poisson with a finite stratification by symplectic leaves, G is a reductive group acting Hamiltonianly on X, and L is a G-equivariant ample line bundle on X, then the reduced quotient $X/\!/\!/_L G$ also has a finite stratification by symplectic leaves.*

In particular, we could take X to be a symplectic singularity in the corollary. We remark that, unlike in the case of finite groups, the quotient $X/\!/\!/_L G$ need not itself be a symplectic singularity.

Summarizing, the following classes of varieties have a finite stratification along which X is locally a product:

(1) any Poisson variety with finitely many symplectic leaves;

(2) any finite or GIT quotient of a smooth variety.

We also saw that the varieties in (1) are closed under finite or GIT Hamiltonian reductions, and that there the product decompositions are products of Poisson varieties.

Additionally, threefolds with canonical singularities X admit a stratification along which X is locally a product by [Rei80, Corollary 1.14].

3 Constructibility

The purpose of this section is to prove Theorem 1.7. That is, the restriction of each sheaf $\mathcal{C}$, $\widetilde{\mathcal{C}}$ to each stratum S is locally constant. The latter condition is well known to be equivalent to forming a functor from the fundamental groupoid of S to sets. In other words, we only need to show that $\mathcal{F}$ arises from a functor $F : \mathbf{Ex}'(X, \mathcal{S}) \to \mathbf{Sets}$, where $\mathbf{Ex}'(X, \mathcal{S})$ is the subcategory of $\mathbf{Ex}(X, \mathcal{S})$ whose morphisms are restricted to homotopy classes of paths on a single stratum, i.e., the invertible morphisms in $\mathbf{Ex}(X, \mathcal{S})$. This is what we will show here.

In this section we will relax the assumption that (the source of) a resolution is smooth.

Definition 3.1. For an open set $U \subset X$, let $\mathcal{C}_U$ denote the set of isomorphism classes of locally projective, *partial* crepant resolutions of U that are dominated by a locally projective, terminal crepant resolution. Similarly, let $\widetilde{\mathcal{C}}_U$ denote the set of isomorphism classes of pairs of such a resolution with a relatively ample line bundle.

Note that, under mild conditions on the singularities, [BCHM10] ensures that the domination condition is automatic, see Remark 3.4.

3.1 Picard groups and product decompositions for local resolutions

In order to construct parallel transportation for local resolutions and (relatively ample) line bundles, we need to study the Picard group of the local resolutions using a bit of birational geometry. An important consequence is that, given a single local projective crepant resolution is given by resolving the slice, then *all* local projective crepant resolutions are given this way.

Definition 3.2. A *projective (respectively locally projective) relative minimal model*[2] of U is a projective (resp. locally projective) crepant morphism $\tilde{U} \to U$ with $\tilde{U}$ having $\mathbb{Q}$-factorial, terminal singularities. If not specified, we assume projective.

Here by $\mathbb{Q}$-factorial we mean that every Weil divisor in $\tilde{U}$ has a multiple which is Cartier (hence defines a line bundle).

Recall that our assumption on each stratum S is that every point $s \in S$ has (a) a neighborhood U_s, (b) a pointed stratified variety $F \ni 0$, and (c) an stratified isomorphisms $\varphi : (U \cap S) \times F \to U$. We write this data as a triple (U, F, φ). We will restrict to a setting where the restriction map from (projective partial crepant) resolutions of U to F is an isomorphism. For this we will need the following key property of F:

$$F \textit{ admits a projective relative minimal model } \tilde{F} \to F \textit{ with } H^1(\tilde{F}, \mathcal{O}) = 0. \qquad (*)$$

If we restrict to F Stein, then this is a local condition, because relative minimal models are local, and if F is Stein then $H^1(\tilde{F}, \mathcal{O}) = 0$ is equivalent to $R^1\rho_*\mathcal{O}_{\tilde{F}} = 0$ for $\rho : \tilde{F} \to F$.

In many situations, including all of our main examples, we will be given $\tilde{F}$, so that $(*)$ holds. But let us take a moment to explain under what conditions $(*)$ is guaranteed to hold, according to the minimal model program.

Definition 3.3. If a variety is isomorphic to an analytic open subset of an (affine) algebraic variety, we call it "of (affine) algebraic origin". A variety of algebraic origin is covered by varieties of affine algebraic origin.

Remark 3.4. By [BCHM10, Corollary 1.4.3, Corollary 1.3.2], when F has affine algebraic origin, then condition $(*)$ holds whenever F has only canonical singularities (note that this is a biconditional, since the existence of a crepant resolution with terminal singularities shows that the original singularities were canonical). In fact, they also show that in this case $\tilde{F}$ is a relative Mori dream space over F, hence F has only finitely many partial projective crepant resolutions dominated by terminal ones. If we assume that X itself has canonical singularities (e.g., if X admits a smooth crepant resolution, or if X has symplectic singularities), then the same holds for F, and we then only need to check the algebraicity.

Moreover, applying their result to any partial projective crepant resolution of F, we get that it is always dominated by a terminal one (in fact a minimal model). So the condition of partial projective resolutions being dominated by terminal ones is in fact redundant in this case.

The condition of having algebraic origin appears to be unnecessary, by replacing results from the minimal model program used in [BCHM10] by those in [Fuj22, Theorem 1.6] in the analytic setting.[3]

[2]This is also called a projective (resp. locally projective) $\mathbb{Q}$-factorial terminalization.

[3]Thanks to Paolo Cascini for pointing out this reference, and thanks to Osamu Fujino for multiple clarifications and sending a brief note stating the needed assertions.

Given this, up to shrinking F, and using the definition of $\mathbb{Q}$-factorial in [Fuj22, Definition 2.38], $(*)$ holds precisely when X has canonical singularities (which includes all symplectic singularities). Note also that, if X has symplectic singularities, Kaledin conjectured in [Kal09, Conjecture 1.8] that F is always conical and hence of algebraic origin; this is true in all known examples.

Proposition 3.5. *Suppose that (U, F, φ) is a a local product neighborhood along S and F satisfies $(*)$. Assume that U_S is a polydisc (i.e., biholomorphic to a unit disc in $\mathbb{C}^n$). Then restriction yields a bijection $\mathcal{C}_U \to \mathcal{C}_F$. In particular, every such resolution $\tilde{U} \to U$ is isomorphic to $U_S \times \tilde{F}$, for some such resolution $\tilde{F} \to F$. Moreover, restriction induces an isomorphism $\mathrm{Pic}(\tilde{U}) \to \mathrm{Pic}(\tilde{F})$ which maps the ample cone onto the ample cone.*

Proof of Proposition 3.5. Note that, since U is assumed to be irreducible, so is F. Take a relative minimal model $\tilde{F}$ of F with $H^1(\tilde{F}, \mathcal{O}) = 0$. We then get a relative minimal model $\tilde{F} \times U_S$ of U. One can check that the projection map induces an isomorphism $\mathrm{Pic}(\tilde{F} \times U_S) \cong \mathrm{Pic}(\tilde{F})$ using the long exact sequence associated to the exponential sequence and Künneth formula together with: $H^1(\tilde{F}, \mathcal{O}) = 0 = H^1(U_S, \mathcal{O})$ and $H^{>0}_{\mathrm{dR}}(U_S) = 0$. This yields the final statement.

The isomorphism of Picard groups restricts to an identification of movable cones and Mori decompositions. Next, note that any two projective crepant birational morphisms $\tilde{U}_1, \tilde{U}_2 \to U$ from terminal irreducible varieties $\tilde{U}_1, \tilde{U}_2$ are isomorphic in codimension one [KM98, Theorem 3.52] (essentially, the crepancy and terminality imply the two have to extract the same divisors). Thus the Weil divisor class groups of $\tilde{U}_1$ and $\tilde{U}_2$ are identified. If $\tilde{U}_1$ is $\mathbb{Q}$-factorial, then this gives an identification $\mathrm{Pic}\,\tilde{U}_2 \subseteq \mathrm{Pic}\,\tilde{U}_1$. In this case, there is a birational map $\tilde{U}_1 \dashrightarrow \tilde{U}_2$ defined by a line bundle in $\tilde{U}_1$ (one which corresponds to a relatively ample bundle for $\tilde{U}_2 \to U$). If, instead, $\tilde{U}_2 \to U$ is not necessarily terminal, but is a partial crepant resolution which is dominated by a terminal one $V \to \tilde{U}_2 \to U$, then $\tilde{U}_2$ is obtained from V by a birational morphism defined by a line bundle in the nef cone of V, which is well known to lie in the closure of the movable cone. In view of the fact that all line bundles on our minimal model $\tilde{F} \times U_S \to U$ are pulled back from one on a minimal model $\tilde{F} \to F$, we see that all the other partial crepant resolutions are isomorphic to the product of U_S and the modification of $\tilde{F}$ defined by the appropriate line bundle. This yields the second statement. For the first, we use that the Mori decompositions of the movable cone are identical. ☐

Remark 3.6. In other words, germs of such partial resolutions at s are all isomorphic to local product germs along S: a local product neighborhood of s and a birational morphism $\tilde{F} \to F$.

3.2 Parallel transport within a stratum, and proof of Theorem 1.7

Here we show that a path $\gamma : I \to S$ induces a map (called "parallel transport"), $\mathcal{C}_{\gamma(0)} \to \mathcal{C}_{\gamma(1)}$, which only depends on the homotopy class of γ (fixing the endpoints). It will also follow from the construction that concatenation of paths corresponds to composition of parallel transport, with the constant path acting as the identity. This will imply the key constructibility property, Theorem 1.7.

Definition 3.7. Given a continuous map $f : B \to X$, a *locally compatible system of birational morphisms* is an open covering $\sqcup_i B_i \twoheadrightarrow B$, neighborhoods U_i of $f(B_i)$, and birational morphisms $\tilde{U}_i \to U_i$, such that for all i,j with $B_i \cap B_j \neq \emptyset$, there is a neighborhood of $f(B_i \cap B_j)$ on which the two birational morphisms restrict isomorphically.

Remark 3.8. Another way to view this definition is to define a topology on the isomorphism classes of germs of birational morphisms to X[4], so that a locally compatible system defines a continuous lift of the map f to a map $B \to \{\text{isomorphism classes of germs of birational morphisms to } X\}$.

We will especially be interested in the following cases:

- $B = I$ is an interval, and $f =: \gamma$ is a path, and
- $B = I \times I$ is a product of intervals, and $f =: H$ is a homotopy.

We consider here the case where $\mathrm{im}(f) \subset S$ a single stratum of X.

Lemma 3.9. *Suppose that X is locally a product along the stratum S whose factors F satisfy $(*)$. Let $f : B \to S$ be a continuous map. Suppose that f has two locally compatible systems of projective partial crepant resolutions dominated by terminal ones (sections of $\mathcal{C}$). If $f(B)$ is connected, then the systems are isomorphic if and only if they are isomorphic over some nonempty open subset of $f(B)$.*

Proof. One direction is clear. Suppose we have two locally compatible systems over B, as stated. By refining the covers of B and taking intersections of the neighborhoods of their images under f, one can assume a single index set for each cover $\{B_i\}$ and each $\{U_i\}$. By assumption, each birational morphism is given by a product $\mathrm{id}_S \times \pi_F : (U_i \cap S) \times \tilde{F} \to (U_i \cap S) \times F$. Consider the set

$$\mathcal{T} := \{b \in B : \text{ there exists a neighborhood } U_{f(b)} \text{ such that the systems are isomorphic over } U_{f(b)}\}.$$

[4]We are ignoring the set-theoretic issue that the isomorphism classes of germs need not form a set.

We assume that $\mathcal{T}$ is nonempty and need to show that it is all of B. As B is connected, and $\mathcal{T}$ is open by definition, it suffices to show that $\mathcal{T}$ is closed. We argue the complement is open by picking $b \in B\backslash\mathcal{T}$ and finding B_0 an open neighborhood of b in $B\backslash\mathcal{T}$.

Since $b \in B\backslash\mathcal{T}$, there exists $U_{f(b)}$ open with the product decomposition $U_{f(b)} \cong U_S \times F$ with the systems non-isomorphic on all open subsets containing $f(b)$. Assume, by shrinking $U_{f(b)}$ if necessary, that U_S is a polydisc. By Proposition 3.5, the two compatible systems restrict to products of non-isomorphic partial resolutions of F with the identity along U_S. Now let $B_0 \subseteq B$ be an open neighborhood of b in the covering, with neighborhood U_0 of $f(B_0)$. By shrinking B_0 if necessary, we may assume $f(B_0) \subseteq U_S$. By shrinking F and B_0 if necessary, we can assume that the image of $f(B_0) \times F$ in $U_{f(b)}$ is contained in U_0.

The partial resolutions of $U_{f(b)}$ are pulled back from partial resolutions of F, which are non-isomorphic by assumption. Then, they restrict to non-isomorphic partial resolutions of the image of $f(B_0) \times F$ in $U_{f(b)}$. In particular, the two partial resolutions of U_0 are non-isomorphic. This implies that $B_0 \subseteq B \setminus \mathcal{T}$, so that $\mathcal{T}$ is closed. ☐

Lemma 3.10. *Let $\gamma : I \to S \subseteq X$ be a path in a stratum S of X, along which X is locally a product with factors F satisfying $(*)$. Given a local partial resolution in $\mathcal{C}$, there is a unique extension to a compatible system for γ (a section of $\mathcal{C}$ over γ).*

Lemma 3.10 implies that, given a path γ, we obtain by extension and restriction a map $\mathcal{C}_{\gamma(0)} \to \mathcal{C}_{\gamma(1)}$, which is inverted by the backwards path.

Proof. Clearly, the relative partial resolution at $\gamma(0)$ can be viewed as a compatible system over $[0, t_0)$ for some t_0. If the system does not extend to all of $[0, 1]$, let t_0 be maximal such that the system extends uniquely to $[0, t_0)$ (this exists by taking the union of all such half-open intervals on which we have a compatible extension).

As in the proof of Lemma 3.9, consider a local product neighborhood $U_{\gamma(t_0)} \cong U_S \times F$, for $U_S \subseteq S$ containing $\gamma(t_0)$. Let $t_1 < t_0$ be such that $\gamma([t_1, t_0]) \subseteq U_S$. Take a local partial resolution $\tilde{U}_{\gamma(t_1)} \to U_{\gamma(t_1)}$ of $\mathcal{C}$ at $\gamma(t_1)$. Let $V_S \subseteq U_S$ be an open polydisc neighborhood of $\gamma(t_1)$ and $F' \subseteq F$ be a pointed open subset such that the image of $V_S \times F'$ is contained in $U_{\gamma(t_1)}$. Then the partial resolution of the image of $V_S \times F'$ is pulled back from one for F'.

Pull this partial resolution for F' back to a new partial resolution $\tilde{V}_{\gamma(t_0)} \to V_{\gamma(t_0)}$, with $V_{\gamma(t_0)}$ the image of $U_S \times F'$. We define a new compatible system on $[0, t_0]$ as follows: let this new one be included as a local partial resolution for an open interval $J \supseteq [t_1, t_0]$. On $J' := J \cap [0, t_0)$, this local partial resolution defines an extension of the partial resolution of the image of $V_S \times F'$ in $U_{\gamma(t_1)}$ we considered before.

On the other hand, our original system of local partial resolutions is compatible with this same partial resolution. By Lemma 3.9, since J' is connected, it follows that $\tilde{V}_{\gamma(t_0)}$ and our original system define isomorphic systems of local resolutions for $\gamma|_{J'}$.

As a result, the partial resolution $\tilde{V}_{\gamma(t_0)}$ is compatible with all local partial resolutions for intervals intersecting J'. In particular, it defines a compatible extension. The uniqueness of the extension follows from Lemma 3.9.

This contradicts the definition of t_0. ☐

Now this parallel transport will give us the exit path constructibility for the single stratum. The key remaining observation is homotopy invariance of parallel transport.

Lemma 3.11. *Let $H : I \times I \to S \subseteq X$ be a homotopy in a stratum along which X is locally a product with factors F satisfying $(*)$. Suppose that we are given a partial resolution in $\mathcal{C}$ at $H(0, 0)$. Then this extends uniquely to a compatible system over $I \times I$.*

Proof. Order $I \times I$ lexicographically. Assume there is no compatible extension over all of $I \times I$. Let (s_0, t_0) be maximal such that compatible local partial resolutions of $\mathcal{C}$ are defined for γ restricted to the set $(I \times I)_{<_{\text{lex}}(s_0,t_0)} := \{(s, t) \mid (s, t) <_{\text{lex}} (s_0, t_0)\}$. To find this, take the union of all open sets of the form $[0, s) \times I$ over which we have a compatible extension, forming a set $[0, s_0) \times I$. Since these extensions are all unique by Lemma 3.9, they are compatible, and we obtain a compatible extension on $[0, s_0) \times I$. Then take the union of all sets $([0, s_0) \times I) \cup \{s_0\} \times [0, t))$ (for all t) over which we have a (necessarily unique) compatible extension. We end up with a compatible extension over $(I \times I)_{<_{\text{lex}}(s_0,t_0)}$.

Now, take a ball B_0 about (s_0, t_0) such that there is a local product neighborhood of $\gamma(B_0)$ isomorphic to $U_S \times F$ with F satisfying $(*)$. Let $(s_1, t_1) <_{\text{lex}} (s_0, t_0)$ be in B_0 and $(B_1, \tilde{U}_1 \to U_1)$ a local partial resolution in our system (for $\gamma(B_1) \subseteq U_1 \subseteq X$). By shrinking B_1 and U_1 we can assume that U_1 is the image of $V_S \times F'$ under the product neighborhood of $\gamma(B_0)$. Let us also assume that $B_1 \subseteq [0, s_0) \times I$.

The local partial resolution for B_1, by Proposition 3.5, is then pulled back from one of F'. Shrinking the product neighborhood for B_0 to the image of $U_S \times F'$, we can pull back the same local partial resolution of F' to $U_S \times F'$. Thus we get a partial resolution for a neighborhood of $\gamma(B_0)$, $B_0 \ni (s_0, t_0)$, which is compatible with the one we had for $B_1 \ni (s_1, t_1)$ (possibly shrinking B_1).

Next, restrict the partial resolution for B_0 to $B_0' := B_0 \cap ([0, s_0) \times I)$. Note that B_0' is connected, and it contains $B_1 \ni (s_1, t_1)$. By Lemma 3.9, it follows that the partial resolution for $\gamma|_{B_0'}$ is isomorphic to the restriction of our original system to B_0'. Therefore, our partial resolution for B_0 is compatible with the original system.

This contradicts the choice of (s_0, t_0). □

Put together, given a path $\gamma : I \to S$ and a partial resolution at $\gamma(0)$, we can extend it uniquely (by Lemma 3.10) to a compatible system of partial resolutions for γ. By restriction, we obtain a partial resolution at $\gamma(1)$. We define the parallel transport $\mathcal{C}_{\gamma(0)} \to \mathcal{C}_{\gamma(1)}$ to be the resulting map.

Given two homotopic paths from s_0 to s_1, by Lemma 3.11, we can uniquely extend a partial resolution from $\mathcal{C}$ at s_0 to one over the whole homotopy $I \times I \to S$. This means that the two parallel transport maps differ by parallel transport of a constant path at s_1, so they are the same.

Observe that by construction, the concatenation of two paths induces concatenation of parallel transport, and a constant path induces the identity map on stalks.

This completes the proof of Theorem 1.7 for the sheaf $\mathcal{C}$.

To extend to the sheaf $\widetilde{\mathcal{C}}$, simply note that Lemmas 3.9, 3.10, and 3.11 all follow in the case of compatible systems of local partial resolutions $\rho_i : \tilde{U}_i \to U_i$ and choices of ρ_i-ample line bundle, where now compatibility means that over an overlap, the local partial resolutions restrict to isomorphic ones such that the isomorphism $\tilde{U}_i \to \tilde{U}_j$ pulls the line bundle L_j back to one isomorphic to L_i. Lemma 3.9 then generalizes to say that two such systems are isomorphic if and only if they are isomorphic over a nonempty subset of $f(B)$; and Lemmas 3.10 and 3.11 generalize to show that a local partial resolution and relatively ample line bundle extend uniquely to paths and homotopies in S, respectively. In all cases the argument is precisely the same, using the last assertion of Proposition 3.5, that pullback induces an isomorphism $\mathrm{Pic}(\tilde{F}) \to \mathrm{Pic}(\tilde{U})$ and our ample line bundles are therefore pulled back from $\tilde{F}$.

Finally, given a compatible system of local partial crepant resolutions over $\gamma : I \to S$, the locus of points where the local resolution is smooth is clearly open. By Proposition 3.5, the same is true for the locus of singular partial resolutions. Since the interval is connected, we conclude that the sub-presheaves of smooth crepant resolutions form constructible subsheaves of $\mathcal{C}$ and $\tilde{\mathcal{C}}$.

This completes the proof that the given sheaves come from functors $\mathbf{Ex}'(X, \mathcal{S}) \to \mathbf{Sets}$, and hence by the comments in the preceding section, of Theorem 1.7 and hence Theorem 1.13.

Proof of Corollary 1.14. We prove this in the setting of a partial projective crepant resolution. The construction of the sheaves $\mathcal{C}, \widetilde{\mathcal{C}}$ generalize to form a more fundamental object, the presheaf of relative Picard groups:

$$\mathcal{P} : \mathbf{Op}(X) \to \mathbf{Vect}_{\mathbb{Q}} \qquad \mathcal{P}(U) = \mathrm{Pic}(\tilde{U}/U) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Recall, by assumption that each $s \in S$ has a neighborhood U_s with a decomposition

$$U_s \cong (U_s \cap S) \times F$$

as analytic open sets, for some pointed manifold F. A local minimal model for s is a birational morphism

$$U_s \times \tilde{F} \to U_s \times F.$$

Any two choices $\tilde{F}_1$ and $\tilde{F}_2$ of local models have canonically isomorphic Picard groups $\mathrm{Pic}(\tilde{F}_1) \cong \mathrm{Pic}(\tilde{F}_2)$. Therefore, $\mathcal{P}(U_s) = \mathrm{Pic}(\tilde{F}/F) \otimes_{\mathbb{Z}} \mathbb{Q}$.

The proof that $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ are constructible actually shows that $\mathcal{P}$ is an $\mathcal{S}$-constructible sheaf of $\mathbb{Q}$-vector spaces: this is because the constructibility comes from a local constancy of the relative Picard groups along strata. Then the sheaves $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are recovered from $\mathcal{P}$ together with its relative Mori fan (saying which bundles are ample for each choice of local resolution). So in a sense $\mathcal{P}$ is the more fundamental object of study. □

Remark 3.12. The proofs of Lemma 3.10 and 3.11 are similar and can be generalized: for any $\gamma : B \to S$ and local partial resolution at $\gamma(b_0)$, we can consider pairs of an open subset $B' \subseteq B$ containing b' and a compatible system of local partial resolutions over B' extending the given model. The above argument shows that, if there is a point $x \in \partial B'$ such that a sufficiently small ball B_x around x has connected intersection with B', then we can uniquely extend the system to $B' \cup B_x$. We then use an ordering on B to inductively extend to all of B.

Remark 3.13. [5] Given a path $\gamma : [0,1] \to S$, using local product neighborhoods about points of the path, we can find a non-canonical isomorphism of the germ of X at $\gamma(0)$ and the germ of X at $\gamma(1)$, as in Proposition 2.2. Then one can check that the monodromy about the path is given by composing resolutions by this automorphism of the germ. This gives a restriction on the monodromy action: it can only relate partial resolutions which differ by an automorphism of the germ of the base. In particular, the two local partial resolutions are isomorphic as abstract varieties (at least, after restricting to suitable open neighborhoods of the singularity).

In more detail, postcomposition with the automorphism group G_x of the germ of X at $s \in S$ gives an action on the set $\mathcal{C}_s$ of local partial crepant resolutions at s. This action is determined by the action of G_x on the discrete Picard group of the germ, which factors through $\pi_0(G_x)$. The above shows that every monodromy automorphism of $\mathcal{C}_s$ is obtainable from this action.

[5]Thanks to Richard Thomas for the question that led to this remark.

3.3 Exit paths

Given a constructible sheaf, it easily follows that parallel transport can be extended to exit paths, and that global sections are the same as exit-path compatible sections (a weak form of MacPherson's equivalence):

Proposition 3.14. *Let $\mathcal{F}$ be a constructible sheaf on a finitely stratified variety $(X, \mathcal{S})$ with S_i locally closed smooth manifolds. Given an exit path $\gamma : [0,1] \to X$, there is a unique parallel transport map $\gamma_* : \mathcal{F}_{\gamma(0)} \to \mathcal{F}_{\gamma(1)}$ such that, for each $s \in \mathcal{F}_{\gamma(0)}$, there is an open covering of $[0,1]$ with compatible local sections restricting at $\gamma(0), \gamma(1)$ to s and $\gamma_*(s)$, respectively. Moreover, a global section of $\mathcal{F}$ on X is the same as an element of $\mathcal{F}_x$ for all $x \in X$ compatible with this parallel transport for all exit paths.*

Proof. To define parallel transport along a simplified exit path $\gamma : [0,1] \to X$, we first note that for each local section $s \in \mathcal{F}_{\gamma(0)}$, there is some ε for which the section is defined on $\gamma[0,\varepsilon]$. We can then extend to the whole path using parallel transport along the stratum. Since all exit paths are compositions of simplified exit paths, this defines parallel transport for all exit paths. To see that the parallel transport is unique, note that the parallel transport along a given stratum is unique, as is restriction of a local section to a smaller section. Now if we have a section over $\gamma([0,\varepsilon])$ and restrict it to a section over $\gamma([0,\delta])$ with $0 < \delta < \varepsilon$, the parallel transport along $[\delta, \varepsilon]$ recovers the same section as the latter parallel transport is unique.

A global section of a constructible sheaf on $(X, \mathcal{S})$ is given by a collection of sections f_i on neighborhoods U_i of each stratum S_i which agree on overlaps. We can choose U_i so that U_i intersects only those strata S_j such that $S_i \subseteq \overline{S_j}$ (note that by the local product condition, this is equivalent to $S_i \cap \overline{S_j} \neq \emptyset$). The sections on each stratum are the same as choices of stalks at each point compatible with parallel transport along the stratum (by the equivalence between local systems and functors from the fundamental groupoid to sets). These sections are compatible if and only if they restrict to the same local section in some neighborhood of each point s in the overlap. The latter can be checked by compatibility with an exit path ending at s, in view of the uniqueness of the parallel transport. It follows that any choice of local sections compatible under exit paths glues to form a global section. The converse is clear from uniqueness of parallel transport. □

We can moreover restrict to simplified exit paths, since all exit paths are compositions of these. We can be more restrictive and check compatibility only with certain paths:

Proposition 3.15. *Let $(X, \mathcal{S})$ be a finitely stratified variety by strata satisfying Definition 1.3. Compatibility of a collection $(f_x \in \mathcal{F}_x)$ of elements of stalks with all exit paths can be checked on simplified exit paths beginning at a fixed basepoint s_i of each stratum S_i. Moreover we can restrict to exit paths of the following form, for each stratum S_i:*

1. *For each $q \in S_i$, a single path from s_i to q;*
2. *A set of closed paths generating the fundamental group $\pi_1(S_i, s_i)$;*
3. *For each stratum $S_j \neq S_i$ such that $\overline{S_j} \supseteq S_i$, any choice of neighborhood U_i of s_i in X, and each component of $U_i \cap S_j$, a single simplified exit path from s_i to the given component.*

Proof. First, using paths in (1) and (2) we can obtain all homotopy classes of paths in the stratum S_i. It is standard that parallel transport for a local system along homotopic paths is the same, so these are enough to guarantee we obtain a global section on S_i.

It remains to show that the sections defined on each stratum are compatible using (3). Every simplified exit path can be written as a concatenation of such paths and exit paths which lie in an arbitrarily small ball about the initial point. So all we need to show is that we can take this initial point to be a fixed basepoint s_i of its stratum S_i. Let $q \in S$ be another point of the same stratum, and let γ be a path in S from s_i to q.

We can find an open covering $\mathcal{U}$ in X of the stratum S_i by open product neighborhoods such that $\mathcal{F}|_{U \cap S_i}$ is constant for each $U \in \mathcal{U}$ (since each point of S_i is contained in such a U). For each subinterval $[t_1, t_2] \subseteq [0,1]$ whose image under γ lies in a neighborhood $U \in \mathcal{U}$, we claim that compatibility with all simplified exit paths beginning on $\gamma([t_1, t_2])$ follows from compatibility with simplified exit paths beginning at any fixed point of $\gamma([t_1, t_2])$. For any $t_3 \in [t_1, t_2]$ and any simplified exit path β from $\gamma(t_3)$ to a point $x \in U \cap S_j$ for $j \neq i$, we can form a simplified exit path α from $\gamma(t_1)$ to a point $y \in U \cap S_j$ in the same stratum as x and a path γ' from x to y in $U \cap S_j$. Since $\mathcal{F}|_{U \cap S_i}$ is constant, the two composite paths from $\gamma(t_3)$ to $y \in U$ must have parallel transport yielding the unique extension of $f_{\gamma(t_3)}$ to a section of U. Since we already have sections defined on all strata, it follows that compatibility with β is equivalent to compatibility with α.

Since $[0,1]$ is covered by subintervals $[t_1, t_2]$ as in the preceding paragraph, we obtain that compatibility with exit paths departing $\gamma(0)$ is equivalent to compatibility at $\gamma(1)$. □

In the situation at hand, given a simplified exit path $[0,1] \to X$, the parallel transport of $s \in \mathcal{F}_{\gamma 0}$ along $[0, \varepsilon]$ is given by restricting a local partial resolution to neighborhoods of nearby points to $\gamma(0)$, followed by parallel transport in the stratum. The same argument works when including also the datum of a relatively ample line bundle.

We note that it is a consequence of the constructibility of the sheaves and [Tre09, Theorem 1.2] that the above actually defines functors $\mathbf{Ex}(X, \mathcal{S}) \to \mathbf{Sets}$ such that sections are given by stalks compatible by exit paths, although we do not need to use this. (This statement, in addition to the above, says that the above parallel transport for exit paths does not depend on any choices and is independent of tame homotopy of exit paths.)

3.4 Compatibility across strata

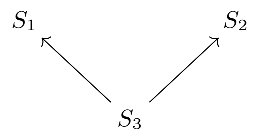
For a stratified variety $(X, \mathcal{S})$, the set of strata has a partial order given by $S \leq S'$ if $S \subseteq \overline{S'}$. We recall a few elementary notions for partially ordered sets.

Definition 3.16. The *Hasse diagram* for a partially ordered set $(X, \leq)$ is a directed[6] graph with vertex set X and an arrow $x \to x'$ if $x < x'$ and no x'' satisfies $x < x'' < x$.

Definition 3.17. Fix a partially ordered set $(X, \leq)$ and two elements $x_1, x_2 \in X$. The *join* $x_1 \vee x_2$ is the supremum (i.e., least upper bound) of x_1 and x_2. The *meet* $x_1 \wedge x_2$ is the infimum (i.e., greatest lower bound) of x_1 and x_2.

Thanks to our procedure, we can build resolutions up from the minimal strata. The minimal strata have no meet, and the compatibility condition can be checked in their join. If the join is an open stratum, the compatibility condition is automatically satisfied since that stratum has a unique (partial) crepant resolution. Similarly, the compatibility condition is satisfied for *actual* crepant resolutions if the join has codimension at most two, because there there is a unique crepant resolution (the minimal one).

For constructing resolutions, in the absence of monodromy, one can always extend a resolution up a chain $S_1 \leq S_2 \leq \cdots \leq S_n$. Notice in the following special case involving products, we can construct resolutions *down* the Hasse diagram (with no obstructions):



where the slice to S_3 is a product of slices to S_1 and S_2. (3.1)

Here, we mean that for $x \in S_3$, there is a local neighborhood of x of the form $U_3 \times F_1 \times F_2$ where U_3 is a neighborhood of x in S_3, and $U_3 \times F_1$ and $U_3 \times F_2$ map isomorphically to neighborhoods of x in $\overline{S_2}$ and $\overline{S_3}$, respectively (so the slice to x in S_1 is F_1 and the slice to x in S_2 is F_2).

Since a resolution for the stratum S_3 is uniquely determined by resolutions for S_1 and S_2. So we do not need to consider such strata when classifying resolutions.

Remark 3.18. Under a mild condition, such diagrams correspond to strata S_3 with decomposable slice. Namely, suppose that all the slices to the strata have exceptional basepoint (meaning that, in a neighborhood of the base point, all vector fields on the slice vanish at the basepoint). This includes the case that the finite stratification is by iterated singular loci (since by locally integrating vector fields, singular loci are preserved). In turn, this holds in the case of X Poisson with strata given by symplectic leaves.

Then such a diagram occurs whenever a stratum S_3 exists with slice decomposing as a product $F_1 \times F_2$: simply let S_1 be the generic stratum which near $s_3 \in S_3$ lies in $U_3 \times \{0\} \times F_2$, and similarly define S_2 as the generic stratum which near s_3 lies in $U_3 \times F_1 \times \{0\}$.

Without the assumption that basepoints of slices are exceptional, we can still define for a stratum S_3 with decomposable slice $F_1 \times F_2$ (with F_1, F_2 both singular) strata S_1, S_2 such that resolutions for S_i uniquely determine one for S_3. Indeed, by locally integrating vector fields, after shrinking the neighborhood we can further decompose F_i as $F_i' \times D_i$ for D_i some disc, and the basepoint of F_i' is exceptional. In this case, there will have to be strata S_i with slices $F_i' \times D_1 \times D_2$. In this case, though S_3 will note be a product of the slices to S_1 and S_2 (unless the D_i are zero-dimensional), it is still true that resolutions for S_1 and S_2 uniquely determine that of S_3, since local resolutions of a variety and of a variety times a disc are in bijection.

The upshot of this discussion is that:

(1) a choice of resolutions at the minimal strata can determine at most one resolution of X,

(2) we need not check compatibility for strata whose slices are products of two singular varieties (provided we determine their local resolutions from factor slices), and

(3) compatibility is automatic across strata whose join is an open stratum, or in the case of full crepant resolutions, for strata whose join is in codimension two.

[6]Note that most authors draw the Hasse diagram as an *undirected* graph, but place the vertex for x below the vertex for x' if $x < x'$ to indicate direction.

3.5 Corollaries of the main theorem

Let $(X, \mathcal{S})$ be as in Theorem 1.7. We will record some consequences of Theorem 1.7. For the first set of corollaries, we return to the setting of smooth projective crepant resolutions, as in the introduction.

Corollary 3.19. *Let $\{S_i\}_{i\in I}$ denote the* closed *strata in $\mathcal{S}$ and pick basepoints $s_i \in S_i$. If the germ of X at s_i has a unique local projective crepant resolution for all i, then they glue to form a global locally projective crepant resolution.*

More generally, if this is true for all strata except for isolated singularities, then locally projective crepant resolutions of X are in bijection with choices of local projective crepant resolutions of the isolated singularities.

Proof. It suffices to prove the assertion of the second paragraph. We use the subsheaf of $\mathcal{C}$ of smooth crepant resolutions (not the partial ones). All exit paths necessarily take the unique values everywhere except the isolated singularities, which can therefore take any value to get a global section. The result then follows from Theorem 1.7. ☐

Corollary 3.20. *Let X be a variety whose singularities are all either isolated, or have du Val singularities. Then crepant locally projective resolutions are in bijection with arbitrary choices of local crepant projective resolutions of the isolated singularities (when they exist).*

Here, by "having du Val singularities" we mean that some neighborhood is isomorphic to the product of a disc of some dimension with a du Val singularity. (Since du Val singularities have no moduli, this is equivalent to being a codimension two singularity whose generic two-dimensional slice is du Val, see [Rei80, Corollary 1.14].)

Proof. Let $Z \subseteq X$ be the singular locus. This is a Zariski subset which, by assumption, has only isolated singularities. At any smooth point, by assumption it is analytically locally the product of a du Val singularity and a smooth disc. So, X is stratified, with three types of strata: the open strata, the strata which locally have du Val singularities, and the isolated strata. Since du Val singularities all have unique projective crepant resolutions, the result now follows from Corollary 3.19. ☐

Corollary 3.21. *Let X be either a 4-dimensional variety with symplectic singularities, or a 3-dimensional canonical singularity. Then crepant locally projective resolutions of X are in bijection with arbitrary choices of local crepant projective resolutions of the isolated singularities (when they exist).*

Proof. Note that two-dimensional canonical singularities, two-dimensional symplectic singularities, and du Val singularities all coincide. Also, canonical and symplectic singularities are normal by definition, hence smooth in codimension one. Therefore, it suffices to show that canonical 3-dimensional and symplectic 4-dimensional singularities are stratified, with all non-isolated, non-open strata of codimension two. For symplectic singularities, this follows from the stratification into symplectic leaves. For canonical three-dimensional singularities, this was explained in [Rei80, Corollary 1.14] (as cited already). ☐

Corollary 3.22. *Let $(X, \mathcal{S})$ be a stratified variety satisfying Definition 1.3 and $\mathcal{F}$ a constructible sheaf. Let $U \subset X$ be an open subset of X such that for each stratum $S_i \in \mathcal{S}$, $U \cap S_i$ is connected and nonempty and the inclusion map $\iota : U \cap S_i \hookrightarrow S_i$ induces a surjection $\pi_1(U \cap S_i) \twoheadrightarrow \pi_1(S_i)$ on fundamental groups. Then the restriction map induces a bijection $\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F})$.*

Note that, under our assumptions, the subvariety U inherits a finite stratification by locally closed connected strata $U \cap S_i$ satisfying Definition 1.3.

Proof. Sections on U and on X can both be identified with choices of sections at every point compatible under the exit paths listed in Proposition 3.15. If we choose basepoints of X to be in U, then the list of exit paths can be chosen to be exactly the same (for item (2), it's because of the assumption on surjectivity of $\pi_1(S_i \cap U) \to \pi_1(S_i)$, together with the bijection between the connected strata, and for item (3), it's because an open neighborhood of a point of s_i is contained in both U and X, along with the bijection of connected strata), except we additionally need for sections on X to have compatibility with a single path from the basepoint of each stratum S_i to each point of $S_i \setminus U$ (1). The latter just says that the sections at $X \setminus U$ are uniquely determined from those on U. ☐

Corollary 3.23. *Let $(X, \mathcal{S})$ be a stratified variety with a unique minimal stratum given by a point $S_{\min} = \{s\}$, satisfying Definition 1.3. Then there is an injection $\mathcal{C}_X \hookrightarrow \mathcal{C}_{U_s}$ for U_s any neighborhood of s, and the same holds replacing $\mathcal{C}$ with $\widetilde{\mathcal{C}}$ and $\mathcal{P}$. If there exists a neighborhood U_s isomorphic to the neighborhood of 0 in a conical variety Y, then there is an injection $\mathcal{C}_X \hookrightarrow \mathcal{C}_Y$. Moreover, if $Y = \mathcal{M}_{0,0}(Q, d)$ is a Nakajima quiver variety (see Subsection 4.4 for the definition) for an unframed quiver Q with some $d_i = 1$ such that there exists a simple representation, then the set $\mathcal{C}_{\mathcal{M}_{0,0}(Q,d)}$ can be described by GIT chambers following [BCS22, Corollary 4.7].*

Proof. The first statement is an immediate consequence of the constructibility of the sheaves (or of the exit path description of global sections). For a conical variety X with cone point 0, one can choose a stratification $\mathcal{S}$ where each stratum S_i is conical (i.e., $0 \in \overline{S}_i$). Therefore, compatibility can be checked in any neighborhood of 0.

Next notice that if a local resolution has monodromy along the loop γ in a stratum S, then the non-triviality of the homotopy class $[\gamma]$ implies γ has a non-zero winding number about 0. Consequently the resolution does not arise as an extension of one in a neighborhood of the cone point.

Therefore, any resolution of any neighborhood of 0 extends uniquely to X. □

3.6 Reduction to combinatorics and linear algebra

Our main result says the assignment of $U \subset X$ open to the set $\mathcal{C}_U$ of isomorphism classes of projective crepant resolutions on U as an $\mathcal{S}$-constructible sheaf. Such an $\mathcal{S}$-constructible sheaf can in turn be described as a functor $F : \mathbf{Ex}(X, \mathcal{S}) \to \mathbf{Sets}$. By choosing the auxiliary data of a basepoint $s \in S$ for each stratum in $\mathcal{S}$ and a generating set of simplified exit paths $\{\gamma_{s,s'}\}$, such a functor is a system of sets $F(s) := \mathcal{C}_s$ with generalization maps $\mathrm{gen}_{s,s'} := F(\gamma_{s,s'}) : \mathcal{C}_s \to \mathcal{C}_{s'}$ if $\overline{S'} \cap S \neq \emptyset$.

In nice cases, for each pair of strata, it suffices to choose only one exit path between them (as a consequence, we only need to do this for neighboring strata), as we now explain. Suppose $s_i \in S_i$ for $i = 1, 2$ and $s_2 \in \overline{S}_1$. Suppose s_2 has a neighborhood basis $\{U_i\}$ such that $U_i \cap S_1$ is connected (hence path-connected, since S_1 is a manifold). Then any two simplified exit paths from s_2 into S_1 differ by an invertible morphism in the exit path category. A stronger, sufficient condition that ensures the need for only a single exit path from s_2 into S_1 is that $\overline{S}_1$ is *topologically unibranch* at s_2.

Definition 3.24. [Mum81, Definition (3.9)] Let X be a variety and fix $x \in X$. We say X is *topologically unibranch* at x if for all Zariski closed subsets $Y \subset X$, x has a neighborhood basis $\{U_n\}$ in the complex topology such that $U_n \backslash (U_n \cap Y)$ is connected.

This property is equivalent to the statement that the fiber of x in the the normalization of X is a single point (see the exposition in [MO15, Section 5.6] on Zariski's main theorem [Zar43], where this is the claim U3 $\iff$ U4).

Note that, by local triviality along a stratum and the assumption that strata are connected, these properties hold at s_i if and only if they hold at any other point of the same stratum. In this situation, we can refine Proposition 3.15 to only need a single simplified exit path at each s_i to each stratum S_j with $\overline{S_j} \ni s_i$. As we observed already, these are the same as the strata with $\overline{S_j} \supseteq S_i$. Finally, by composing these with an arbitrary path from the endpoint of the simplified exit path to the basepoint s_j, we obtain the following:

Corollary 3.25. *Consider a stratified variety satisfying Definition 1.3. Suppose that for every pair of strata S_i, S_j with $\overline{S_j} \supseteq S_i$, $\overline{S_j}$ is topologically unibranch at s_i (or just that there is a neighborhood basis $\{U_k\}$ of $\overline{S_j}$ at s_i such that $U_k \cap S_j$ is connected for all k). Then global sections of a constructible sheaf $\mathcal{F}$ are in bijection with choices of sections at basepoints s_i which are compatible with parallel transport along:*

1. *Any set of closed paths generating the fundamental groups $\pi_1(S_i, s_i)$;*
2. *For each pair of distinct strata S_i, S_j with $\overline{S_j} \supset S_i$, any single exit path from s_i to s_j.*

Note that in (2) in the corollary, we can take the path to be a simplified exit path. Moreover, since compositions of exit paths are exit paths, we can also restrict in (2) to pairs of *neighboring* strata, i.e., strata such that there does not exist an intermediate stratum $S_k \notin \{S_i, S_j\}$ with $\overline{S_j} \supset S_k, \overline{S_k} \supset S_i$. These are the strata that are adjacent in the poset of strata under inclusion of closures, known as the Hasse diagram.

In view of Corollary 1.14, for each of the paths above, we only need to compute the parallel transport as a linear map on relative Picard spaces. Then, the global locally projective resolutions are given by choices of Mori cones at each basepoint compatible under this transport, and the global projective resolutions with fixed relatively ample bundle are given by choices of movable line bundles compatible under this transport. In this sense, we have reduced the classification of crepant resolutions to combinatorics and linear algebra, provided we have computed the local classification and the parallel transport.

Remark 3.26. The unibranch condition is not always satisfied in examples. For example, if X is the nilpotent cone of a semisimple Lie algebra, stratified by the adjoint orbits, it can happen that an orbit closure is not unibranch. This is true even though X admits a (unique) projective crepant resolution, the Springer resolution.

For instance, let S_1 be the nilpotent orbit in $\mathfrak{so}_7$ labeled by the partition $(3, 2, 2)$, of dimension 12. The orbit closure $\overline{S_1}$ is not unibranch at a point of the orbit S_2 labeled by $(2, 1^4)$ of dimension 10 (see [KP82, p. 595]).

4 Examples

4.1 Monodromy and incompatibility

We show by example that we need both the monodromy-free and compatibility assumptions in our main result.

Example 4.1. (Monodromy can occur) Let Q be the quiver with a single vertex and $g \geq 3$ loops. Pick the dimension vector $\alpha = (2)$. By Bellamy–Schedler, the quiver variety $\mathcal{M}_{0,0}(Q,(2))$ is locally factorial, terminal, and singular and hence does not admit a symplectic resolution. The same holds for the variety $\mathcal{M}_{0,0}(Q,(2))\backslash\{0\}$, which we claim has a local resolution around every point. To see this notice that the stratification of $\mathcal{M}_{0,0}(Q,(2))$ into its symplectic leaves is:

$$\{0\} \qquad \{\rho = \rho_1 \oplus \rho_2 \mid \rho_1 \ncong \rho_2\} \qquad \{\rho \text{ simple}\}$$

For the minimal stratum of $\mathcal{M}_{0,0}(Q,(2))\backslash\{0\}$ we have the local quiver $\bullet \to \bullet$ with dimension vector $(1,1)$. This is not smooth but admits a symplectic resolution since the dimension vector $(1,1)$ is indivisible, i.e., the greatest common divisor of its entries is one. The local quiver for the dense open leaf of simple representations is $\bullet \circlearrowleft g'$ (i.e., the quiver with a single vertex and g' arrows). Therefore, the local quiver variety at a smooth point is the affine space $\mathbb{C}^{2g'}$. Hence $\mathcal{M}_{0,0}(Q,(2))\backslash\{0\}$ admits a local symplectic resolution in some neighborhood of every point, yet these do not glue to a global resolution. As there is a single minimum stratum, there is no incompatibility obstruction, so the obstruction must be monodromy.

Example 4.2. (Incompatibility can occur) Let $\overline{\mathcal{O}}$ be the minimal nilpotent orbit closure of $\mathfrak{sl}_3(\mathbb{C})$, defined explicitly as $\{A \in \mathfrak{sl}_3(\mathbb{C}) : A^2 = 0\}$. Note that $\overline{\mathcal{O}}$ has a stratification into $\{0\}$ and $\mathcal{O}$, both contractible. Further, $\overline{\mathcal{O}}$ admits two symplectic resolutions of the form $T^*(\mathrm{SL}_3(\mathbb{C})/P) \to \overline{\mathcal{O}}$ corresponding to the two parabolic subgroups of $\mathrm{SL}_3(\mathbb{C})$. We can also write these resolutions as $T^*\mathrm{Gr}(1,3)$ and $T^*\mathrm{Gr}(2,3)$, or the cotangent bundle of $\mathbb{P}^1$ and its dual. So the product $X \times X \times X$ has contractible strata and admits 2^3 resolutions.

Let $Y = \overline{\mathcal{O}} \times \overline{\mathcal{O}} \times \overline{\mathcal{O}}\backslash\{(0,0,0)\}$. There are three minimal leaves: $\overline{\mathcal{O}} \times 0 \times 0$, $\times\overline{\mathcal{O}} \times 0$, and $0 \times 0 \times \overline{\mathcal{O}}$. Each has slice isomorphic to $\mathcal{O} \times \mathcal{O}$ and hence has 4 resolutions. Of the $4^3 = 64$ possible choices, only 8 can result in an actual resolution. Indeed, the choices can only be compatible when they come from a triple of choices by restriction, i.e., from a choice of resolution at the cone point we threw out. This can be seen by restricting from each pair of minimal leaves to their product. Moreover, since the fundamental group of each stratum is trivial, there is no monodromy for resolutions along the strata.

Example 4.3. (Constructing monodromy) Let X be a symplectic singularity with two symplectic resolutions X_1 and X_2 that differ by composing with an automorphism $\phi : X \to X$ of the singularity (in particular, X_1 is isomorphic to X_2 as varieties). Let S be symplectic with an automorphism ψ whose non-identity powers $\{\psi^n \mid \psi^n \neq \mathrm{id}_S\}_{n\in\mathbb{N}}$ have no fixed points. Assume further that the order of ϕ divides the order of ψ (or the order of ψ is infinity). The quotient $X \times S/G$ by the group $G := \langle \phi \times \psi \rangle$ has non-trivial monodromy along a path from (x,s) to $(\phi(x),\psi(s))$ in $X \times S$.

For example, for $n, m \in \mathbb{N}$ with $m < n/2$ define

$$X_{n,m} = \{A \in \mathfrak{sl}_n(\mathbb{C}) \mid A^2 = 0,\ \mathrm{rank}_{\mathbb{C}}(A) \leq m\}.$$

There are non-isomorphic Springer resolutions $T^*(\mathrm{Gr}(m,n)), T^*(\mathrm{Gr}(n-m,n) \to X_{n,m}$ whose sources are isomorphic as varieties (via an inner product $\langle -,- \rangle$ on $\mathbb{C}^n$ that identifies each m-plane with its orthogonal complement $(n-m)$-plane), and this isomorphism induces a nontrivial automorphism of $X_{n,m}$, namely the transpose. Note that ϕ squares to the identity. Let $S = (\mathbb{C})^2\backslash\{(0,0)\}$ with action of $C_2 = \langle \psi \rangle$ taking $(x,y) \mapsto (-x,-y)$. Then $X_{n,m} \times S/((A,x,y) \sim (A^\vee,-x,-y))$ has monodromy taking $T^*(\mathrm{Gr}(m,n))$ to $T^*(\mathrm{Gr}(n-m,n))$ along a path from (A,x,y) to $(A^\vee,-x,-y)$ in $X \times S$.

4.2 Symmetric powers of surfaces with du Val singularities

Let X be a surface with du Val singularities and a symplectic form on the smooth locus. Consider $\mathrm{Sym}^n(X) := X^{\times n}/\mathfrak{S}_n$ where $\mathfrak{S}_n$ denotes the symmetric group on n-letters acting by permuting the factors. Write $Y := \mathrm{Sym}^n(X)$ and consider a stratification of Y as

$$Y = \bigsqcup_{\substack{f\,:\,X^{\mathrm{sing}} \to \mathbb{N} \\ \sum f(z) \leq n}} Y_f \qquad Y_f := \bigsqcup_{\substack{f\,:\,X^{\mathrm{sing}} \to \mathbb{N} \\ \sum f(z) \leq n}} \mathrm{Sym}^{n-\sum f}(X\backslash X^{\mathrm{sing}}) \times \prod_{z \in X_{\mathrm{sing}}} (z,\ldots,z)\ (f(z) \text{ times}).$$

Fix $\{U_z^j\}_{j\in\mathbb{N}}$ a contractible neighborhood basis for the isolated singularity in X. Then each Y_f has an neighborhood basis $\{U_f^j\}_{j\in\mathbb{N}}$ given by

$$U_f^j := \mathrm{Sym}^{n-\sum f(z)}\left(X\backslash \bigsqcup_{z\in X^{\mathrm{sing}}} U_z^j\right) \times \prod_{z\in X^{\mathrm{sing}}} \mathrm{Sym}^{p(z)}(U_z^j)$$

Since z is du Val, its formal neighborhood in X, denoted $\hat{X}_z$, is of the form $\hat{X}_z = \widehat{\mathbb{C}^2/\Gamma_z}$ for some $\Gamma_z \subset \mathrm{SL}_2(\mathbb{C})$. It follows that resolutions of U_f^N for $N \gg 0$ correspond to a choice of resolution of $\mathrm{Sym}^{f(z)}(\mathbb{C}^2/\Gamma_z)$ for all $z \in X^{\mathrm{sing}}$.

Proposition 4.4. *Every projective, crepant resolution of Y is uniquely determined by its restriction to open neighborhoods of the points $(z, \ldots, z)$ (n times) for $z \in X^{\mathrm{sing}}$. Consequently,*

$$\mathcal{C}_Y \cong \prod_{z \in X^{\mathrm{sing}}} \mathcal{C}_{\mathrm{Sym}^n(\hat{X}_z)} \cong \prod_{z \in X^{\mathrm{sing}}} \mathcal{C}_{\mathrm{Sym}^n(\widehat{\mathbb{C}^2/\Gamma_z})}$$

for some $\Gamma_z \subset SL_2(\mathbb{C})$. Hence each projective, crepant resolution of Y is locally given by variation of stability parameter.

By [BC20, Corollary 1.3], $\mathcal{C}_{\mathrm{Sym}^n(\widehat{\mathbb{C}^2/\Gamma_z})}$ can be identified with chambers in the GIT fan modulo the Weyl group action. The later can be computed explicitly, for W with Coxeter number h and exponents $e_1, \ldots, e_\ell$

$$\#\mathcal{C}_{\mathrm{Sym}^n(\widehat{\mathbb{C}^2/\Gamma_z})} = \prod_{i=1}^{\ell} \left(\frac{(n-1)h}{e_i + 1} + 1 \right)$$

see [Bel16, Proposition 1.2]. For example, in type E, writing T, O, I for the binary tetrahedral, binary octahedral, and binary icosahedral groups respectively, we have

$$\begin{aligned}
\#\mathcal{C}_{\mathrm{Sym}^n(\widehat{\mathbb{C}^2/T})} &= \frac{n}{30}(1728n^5 - 4320n^4 + 4140n^3 - 1900n^2 + 417n - 35) \\
\#\mathcal{C}_{\mathrm{Sym}^n(\widehat{\mathbb{C}^2/O})} &= \frac{n}{280}(59049n^6 - 183708n^5 + 229635n^4 - 147420n^3 + 51156n^2 - 9072n + 640) \\
\#\mathcal{C}_{\mathrm{Sym}^n(\widehat{\mathbb{C}^2/I})} &= \frac{n}{1344}(1265625n^7 - 4725000n^6 + 7323750n^5 - 6100500n^4 + 2943325n^3 \\
&\qquad - 820260n^2 + 121796n - 7392).
\end{aligned}$$

Proof. To prove the claim, for each $x_i \in X$ pick U_i a contractible neighborhood, such that the collection $\{U_i\}$ is pairwise disjoint.

Let $V = X \setminus \left(\sqcup_i \overline{U_i}\right)$ where $\overline{U_i}$ denotes the closure of U_i in S. The inclusion map $\iota_V : V \to X$ induces a surjection on the fundamental group of the smooth locus X^{sm} and the inclusion $\iota_U : (\sqcup_i U_i) \to X$ induce surjections on the fundamental groups of the singular strata, $X^{\mathrm{sing}} = \{x_i\}$. So their union $\iota : V \sqcup (\sqcup_i U_i) \to X$ induces surjections on the fundamental group of each stratum. Therefore, the induced inclusion on the nth symmetric power $\mathrm{Sym}^n(\iota) : \mathrm{Sym}^n(V \sqcup (\sqcup_i U_i)) \to \mathrm{Sym}^n(X)$ induces surjections on the fundamental groups of each stratum. By Corollary 3.22, we can detect monodromy in the open subset $\mathrm{Sym}^n(V \sqcup (\sqcup_i U_i))$. This variety consists of products with each piece (1) contractible or (2) having a unique crepant resolution of singularities. So in either case, the fundamental group action on the set of crepant resolutions is trivial. Consequently, there is no obstruction extending local resolutions to entire stratum.

Let x_1 and x_2 be du Val singularities in the surface X. Fix $0 < m < n$. Define the strata in $Y = \mathrm{Sym}^n(X)$

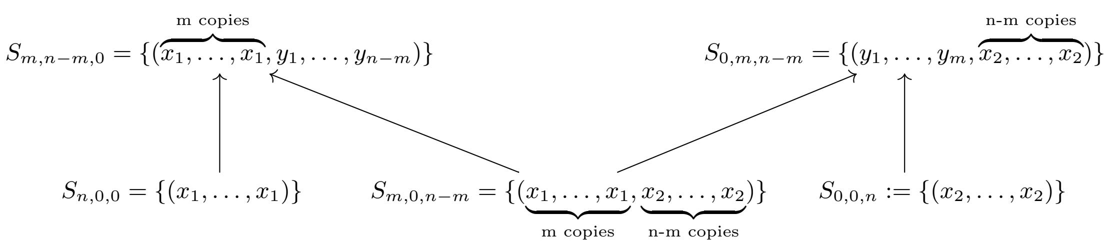
$$S_{a,b,c} := \{(\underbrace{x_1, \ldots, x_1}_{\text{a copies}}, y_1, y_2, \ldots, y_b, \underbrace{x_2, \ldots, x_2}_{\text{c copies}}) \mid y_i \in S^{\mathrm{sm}}, y_i \neq y_j \text{ for } i \neq j\} \qquad a + b + c = n.$$

The four chains:

$$S_{n,0,0} \to S_{n-1,1,0} \to \cdots \to S_{m,n-m,0} \qquad S_{m,0,n-m} \to S_{m,1,n-m-1} \to \cdots \to S_{m,n-m,0}$$

$$S_{0,0,n} \to S_{0,1,n} \to \cdots \to S_{0,m,n-m} \qquad S_{m,0,n-m} \to S_{m-1,1,n-m} \to \cdots \to S_{0,m,n-m}$$

form an M-configuration in the Hasse diagram



So as explained in Subsection 3.4 in general, a choice of resolutions in neighborhoods of the points $(x_1, \ldots, x_1)$ and $(x_2, \ldots, x_2)$ determine resolutions in neighborhoods of the point in $S_{m,0,n-m}$ for all $0 < m < n$. Note that we can check compatibility across $S_{n,0,0}$ and $S_{0,0,n}$ in the diagonal stratum $\{(x, x, \ldots, x) : x \in S\}$ as it is the join $S_{n,0,0} \vee S_{0,0,n}$. The diagonal stratum has a unique resolution given by the Hilbert scheme, so all choices of resolutions glue compatibility in the diagonal.

Finally, to conclude that the locally projective resolutions are indeed projective, we run the argument again, this time extending local ample line bundles to a global ample line bundle. Unlike for resolutions, we are not checking that *all* possible choices of bundles extend, but rather that *some* choice extends. Since ampleness can be checked locally on the singularity, we only need that the line bundles glue to a global line bundle.

Let $Z := \mathbb{C}^2/G$ be a du Val singularity, and let $\widetilde{Z}$ be its minimal resolution (which is crepant). Recall that the Hilbert-Chow morphism $\mathrm{Hilb}^{[n]}(\widetilde{Z}) \to \mathrm{Sym}^n(\widetilde{Z})$ is a symplectic resolution of singularities. There is a map $\mathrm{exc} : \mathrm{Pic}(\mathrm{Hilb}^{[n]}(Z)) \to \mathbb{Z}$ projecting to the exceptional divisor. For different choices of relatively ample bundle at the most singular points to glue compatibly (and to fit into an M-configuration) we need their images under exc to agree. This can be arranged e.g., by rescaling an arbitrary collection.

Since line bundles on a contractible neighborhood of a point are trivial, the argument above implies there is no monodromy, except possibly on the diagonal stratum.

□

4.3 Hilbert schemes of a surface with du Val singularities

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ be a non-trivial, finite subgroup and let $n > 1$. The nth symmetric power $\mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$ has multiple non-isomorphic symplectic resolutions of singularities. One such resolution is given by

$$n\Gamma\text{-}\mathrm{Hilb}(\mathbb{C}^2) := \{\Gamma\text{-invariant ideals } I \subset \mathbb{C}[x,y] \mid \mathbb{C}[x,y]/I \cong \mathbb{C}[\Gamma]^{\oplus n} \text{ as } \Gamma\text{-representations}\}.$$

where $\mathbb{C}[\Gamma]$ denotes the regular representation. Additionally, $\mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$ has a *partial* resolution of singularities given by

$$\mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma) := \{\text{ideals } I \subset \mathbb{C}[x,y]^\Gamma \mid \dim_{\mathbb{C}}(\mathbb{C}[x,y]/I) = n\}.$$

Craw–Gammelgaard–Gyenge–Szendrői prove that the invariants map

$$n\Gamma\text{-}\mathrm{Hilb}(\mathbb{C}^2) \to \mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma) \qquad I \mapsto I \cap \mathbb{C}[x,y]^\Gamma$$

is the unique symplectic resolution of singularities for $\mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$ [CGGS21, Theorem 1.1].

Now let Z be a surface with du Val singularities and let $X := \mathrm{Hilb}^{[n]}(Z)$ be its Hilbert scheme. By assumption, for each $z \in Z$ the formal neighborhood U_z, is either smooth or du Val. In each case, $\mathrm{Hilb}^{[n]}(U_z)$, which is smooth if U_z is smooth by [Fog68, Theorem 2.4], has a unique symplectic resolution of singularities. Hence by Corollary 3.19, these local resolutions glue uniquely to a locally projective symplectic resolution.

This symplectic resolution is locally an nΓ-Hilbert scheme and if Z is a global quotient, then this description holds globally, establishing (global) projectivity.

In the general case, the resolution is also (globally) projective, because we can take relatively ample line bundles for local resolutions at each singularity, and they all restrict to some power of the ample generator of $\mathrm{Hilb}^{[n]}(Z^\circ)$ where Z° is the smooth locus. Then, up to replacing the relatively ample bundles at the singularities by suitable powers, these bundles are compatible and hence glue to a global ample line bundle.

Remark 4.5. If $X = \mathrm{Hilb}^{[n]}(Z)$ for some surface Z, then X admits a crepant resolution precisely when Z has at worst du Val singularities. The preceding gives the existence of a unique resolution when X has at worst du Val singularities. Conversely, note that all of the singularities of Z appear in X, and already in the set of distinct n-tuples of points of Z. But it is well known that the canonical surface singularities are precisely the du Val singularities.

4.4 Multiplicative quiver varieties

Let Q be a quiver, k a field, and A a k-algebra with a kQ_0-bimodule structure. Notice that an A-module M has a dimension vector $d := (d_i) := (\dim(e_i \cdot M)) \in \mathbb{N}^{Q_0}$. And $G_d := \prod_i \mathrm{GL}_{d_i}(k)$ acts on the space of A-modules by conjugation. Following King [Kin94], define $\theta \in \mathrm{Hom}_{\mathrm{grp}}(\prod_i \mathrm{GL}_{d_i}(k), \mathbb{C}^*) = \mathbb{Z}^{Q_0}$ to be a character of G_d. The quiver variety for A is the moduli space $\mathcal{M}_{d,\theta}(A)$ of d-dimensional, θ-semistable representations of A.

If $A = \Pi^\lambda(Q)$ is the deformed preprojective algebra of the quiver Q, then $\mathcal{M}_{d,\theta}(A)$ is the Nakajima quiver variety. In this case we write $\mathcal{M}_{\lambda,\theta}(Q,d) := \mathcal{M}_{d,\theta}(\Pi^\lambda(Q))$ in order to conform to the more standard notation. If $A = \Lambda^q(Q)$ is the *multiplicative preprojective algebra* of Crawley-Boevey and Shaw [CBS06], then $\mathcal{M}_{d,\theta}(A)$ is the *multiplicative quiver variety.*

Multiplicative quiver varieties includes character varieties of Riemann surfaces with monodromy conditions (as open subsets, which are the entire variety in genus zero), and modifications of du Val singularities. We showed in [KS23, Theorem 5.4] that the formal local structure of multiplicative quiver varieties agrees with that of Nakajima quiver varieties. Consequently, for each multiplicative quiver variety $\mathcal{M}_{d,\theta}(\Lambda^q(Q))$ and each module M one can classify symplectic resolutions for a sufficiently small open neighborhood U_M of M. The main results of this paper give a theoretical technique to classify global symplectic resolutions, provided one can compute the symplectic leaves and calculate the monodromy and compatibility constraints. Without monodromy computations, we can still prove partial results.

For dimension vectors $d, d' \in \mathbb{N}^{Q_0}$ we write $d' < d$ if $d - d' \in \mathbb{N}^{Q_0}$. The decomposition type of a semisimple representation is the unordered collection of dimension vectors of simple summands with multiplicities. Note that every point of a moduli space $\mathcal{M}_{d,0}(A)$ of a quiver algebra A is represented by a unique semisimple representation up to isomorphism. Taking connected components of the loci with fixed decomposition type, we obtain a finite stratification of $\mathcal{M}_{d,0}(A)$ by connected strata (although these strata need not be smooth).

Proposition 4.6. *Let $A = kQ/I$ be an augmented algebra, i.e., $I \subseteq (kQ_1)$. Equip $\mathcal{M}_{d,0}(A)$ with the stratification above. Let the zero representation of a given dimension vector denote the one factoring through the quotient $A/(kQ_1) \cong kQ_0$.*

(1) The zero representation lies in the closure of every stratum (i.e., symplectic leaf) of $\mathcal{M}_{d,0}(A)$.

(2) The variety $\mathcal{M}_{d',0}(A)$ is connected for all $d' < d$.

Proof. (1) $\implies$ (2): By definition, every stratum is connected. Since the closure of a connected set is connected, and the union of intersecting connected sets is connected, we establish that $\mathcal{M}_{d,0}(A)$ is connected. Since $\mathcal{M}_{d',0}(A)$ appears as a union of strata in $\mathcal{M}_{d,0}(\Lambda^1(Q))$ for all $d' < d$, we have that all such multiplicative quiver varieties are connected.
(2) $\implies$ (1): By contrapositive, assume that the closure of some stratum of does not contain the zero representation. This stratum is a product of (Zariski) open strata in quiver varieties of dimension vectors adding to d, corresponding to the decomposition type of the representations in the stratum. Thus for some $d' < d$, there is an open stratum whose closure does not contain the zero representation. Assume d' is minimal with this property. Take the union X of all such closures, and let Y be the union of closures of open strata containing the zero representation. We claim that X and Y are open, so the moduli space is disconnected. Since these are closed sets whose union is the whole variety, it suffices to show that $X \cap Y = \emptyset$. If not then the intersection contains a non-open stratum whose closure does not contain the zero representation. This contradicts minimality of d'. □

We apply the preceding result in the case of $A = \Lambda^1(Q)$. Note that, in general, multiplicative quiver varieties may be reducible or even disconnected. There is no trouble applying our theory to this case (and indeed we did not assume irreducibility), as birational morphisms make sense in the reducible setting.

Corollary 4.7. *Suppose that d is a dimension vector such that $\mathcal{M}_{d',0}(\Lambda^1(Q))$ is connected for all $d' < d$. Then $\mathcal{C}_{\mathcal{M}_{d,0}(\Lambda^1(Q))} \hookrightarrow \mathcal{C}_{\mathcal{M}_{d,0}(\Pi(Q))}$, and the same is true replacing $\mathcal{C}$ by $\widetilde{\mathcal{C}}$ and $\mathcal{P}$.*

More explicitly, the map is given by restricting to a neighborhood of the zero representation, identifying with a neighborhood of the zero representation in an additive quiver variety and extending to the entire variety.

Proof. The inclusion $\mathcal{C}_{\mathcal{M}_{d,0}(\Lambda^1(Q))} \hookrightarrow \mathcal{C}_{\mathcal{M}_{0,0}(Q,d)}$ is an application of Corollary 3.23 since $\mathcal{M}_{d,0}(\Lambda^1(Q))$ has a unique minimal stratum given by the zero representation by Proposition 4.6. There exists a neighborhood of zero in $\mathcal{M}_{d,0}(\Lambda^1(Q))$ that is isomorphic to a neighborhood of zero in $\mathcal{M}_{0,0}(Q,d)$ by [KS23, Corollary 5.22], completing the proof. The same holds for the sheaves $\widetilde{\mathcal{C}}$ and $\mathcal{P}$. □

The original motivation for this paper was to extend the classification of symplectic resolutions of quiver varieties [BS16, BCS22] to multiplicative quiver varieties. For quiver varieties the symplectic leaves are given by representation type. Multiplicative quiver varieties have a stratification by representation type but these strata need not be connected and the symplectic leaves may be a finer stratification. In examples below, we establish that the stratifications into symplectic leaves and representation types agree. Hence the classification of symplectic resolutions of the multiplicative quiver variety $\mathcal{M}_{d,0}(\Lambda^1(Q))$ agree with the known classification of symplectic resolutions of the quiver variety $\mathcal{M}_{0,0}(Q,d)$.

Type $\tilde{A}$ For $Q = \widetilde{A}_{n-1}$, the cycle with n-vertices, $q = 1$, $d = 1$, and $\theta = 0$, our previous work [KS23, Corollary 6.14, Proposition 6.19] building on Shaw [Sha05, Theorem 4.1.1] describes $\mathcal{M}_{1,0}(\Lambda^1(Q))$ as the spectrum of the commutative ring $R_n := k[X,Y,Z]/(Z^n + XY + XYZ)$. Note that $(1+Z)$ is invertible in R_n as the relation can be rewritten:

$$\begin{aligned} Z^n + XY + XYZ = Z^n + XY(1+Z) = Z^n + 1 - 1 + XY(1+Z) = (Z^n+1) + XY(1+Z) - 1 \\ = (1+Z)(Z^{n-1} - Z^{n-2} + \cdots + (-1)^{n-1} + XY) - 1. \end{aligned}$$

Hence there is an isomorphism of rings

$$k[X,Y,Z][(1+Z)^{-1}]/(Z^n + XY) \to k[X,Y,Z]/(Z^n + XY + XYZ) \quad X \mapsto X(1+Z),\ Y \mapsto Y,\ Z \mapsto Z.$$

Since the vanishing of $Z^n + XY$ is the du Val singularity for $\widetilde{A}_{n-1}$, we conclude that this multiplicative quiver variety is a Zariski-open subset of the (additive) quiver variety. In particular, $\mathcal{M}_{1,0}(\Lambda^1(Q))$ has a unique symplectic resolution of singularities.

Example 4.8. Let $Q = \tilde{A_2}$, $d = (2, 3, 5)$, and $\lambda = 0 = \theta$. Then the canonical decomposition of d is

$$(2, 3, 5) = 2(1, 1, 1) + (0, 1, 0) + 3(0, 0, 1).$$

The decomposition on the level of quiver varieties, is

$$\begin{aligned}\mathcal{M}_{0,0}(Q, d) &\cong \mathrm{Sym}^2(\mathcal{M}_{0,0}(Q, (1, 1, 1))) \times \mathcal{M}_{0,0}(Q, (0, 1, 0)) \times \mathrm{Sym}^3(\mathcal{M}_{0,0}(Q, (0, 0, 1))) \\ &\cong \mathrm{Sym}^2(\mathcal{M}_{0,0}(Q, (1, 1, 1)))\end{aligned}$$

since $\mathcal{M}_{0,0}(Q, e_i)$ is a single point, for any elementary vector $e_i = (0, \dots, 0, 1, 0, \dots, 0)$. This realizes

$$\mathcal{M}_{0,0}(Q, d) = \mathrm{Sym}^2(\mathcal{M}_{0,0}(Q, (1, 1, 1))) = \mathrm{Sym}^2(\mathbb{C}^2/C_2)$$

as a symmetric product of a surface with a du Val (A_1) singularity. Hence, $\mathcal{M}_{d,0}(\Pi(Q))$ has two symplectic resolutions of singularities given by $\mathrm{Hilb}^{[2]}(\widetilde{\mathbb{C}^2/C_2})$ and $2C_2$-$\mathrm{Hilb}(\mathbb{C}^2)$.

We claim that the multiplicative quiver variety $\mathcal{M}_{d,0}(\Lambda^1(Q))$ has the same classification of symplectic resolutions. In fact, it has the same stratification into symplectic leaves by representation type and hence is a Zariski-open subset

$$\mathcal{M}_{d,0}(\Lambda^1(Q)) = \mathrm{Sym}^2(\mathcal{M}_{1,0}(\Lambda^1(Q)) \subset \mathrm{Sym}^2(\mathbb{C}^2/C_2) \cong \mathcal{M}_{d,0}(\Pi(Q)).$$

We conclude that all symplectic resolutions of this multiplicative quiver variety are given by variation of geometric invariant theory quotient (VGIT), since the same holds in the additive case by Bellamy–Craw [BC20, Corollary 1.3].

Hyperpolygon spaces Let Q_n be the star-shaped quiver with a central vertex v, n external vertices $w_1, \dots, w_n$, and an arrow from each external vertex to the central vertex. Let d be the dimension vector $(2, 1, 1, \dots, 1)$ where the vertices are ordered $(v, w_1, \dots, w_n)$. In [BCR$^+$21, Corollary 1.4], the authors show that all crepant resolutions of $\mathcal{M}_{0,0}(Q_n, d)$ are given by variation of GIT and consequently can be counted explicitly in terms of GIT chambers. Further [BCR$^+$21, Proposition 3.7] counts the resolutions to be 1, 81, and 1684 when $n = 4$, 5 and 6 respectively. We claim that these results are valid in the multiplicative setting.

Proposition 4.9. *There is a bijection $\mathcal{C}_{\mathcal{M}_{d,0}(\Lambda^1(Q_n))} \cong \mathcal{C}_{\mathcal{M}_{0,0}(Q_n,d)}$ given by the inclusion map from Corollary 4.7. In particular, when $n = 4, 5, 6$, $\mathcal{M}_{d,0}(\Lambda^1(Q_n))$ has 1, 81, and 1684 resolutions respectively. These are all globally projective.*

Lemma 4.10. *Fix $n, m \in \mathbb{N}$ with $m, n > 1$ and monic polynomials $\chi_i \in \mathbb{C}[x]$ of degree m. Let $X \subseteq \mathrm{GL}_m(\mathbb{C})^n$ the subset of matrices such that the i-th matrix has characteristic polynomial χ_i. Let $Y_c \subseteq \mathrm{GL}_m(\mathbb{C})$ denote the subset of matrices of trace c. Define the locus of n-tuples of $m \times m$ matrices*

$$Z := \{(A_1, A_2, \dots, A_n, B) \in X \times Y_c \mid A_1 A_2 \cdots A_n B = I\}.$$

Then, Z is irreducible.

Proof. First, note that the variety embeds into the subset of X of n-tuples such that the inverse of the product has a fixed value c of the trace.

Consider matrices $A_1, A_2, \dots, A_n$ with Jordan decompositions of the inverses $A_i^{-1} := S_i + N_i$, where S_i is semisimple and N_i is nilpotent. For each i and $\lambda \in \mathbb{C}$ consider

$$X_{i,\lambda} := (S_n + N_n)^{-1} \cdots (S_{i+1} + N_{i+1})^{-1}(S_i + \lambda N_i)^{-1}(S_{i+1} + N_{i-1})^{-1} \cdots (S_1 + N_1)^{-1}$$

rescaling the nilpotent part of the A_i^{-1}. Now $\mathrm{tr}(X_{i,\lambda})$ is a linear function of λ, so either it obtains all values once, or it is constant. The condition to be constant, $\mathrm{tr}(X_{i,\lambda}) = \alpha$ a fixed value, is closed on the n-tuples of matrices, hence also on X. It is not the entire space since we can take all $S_1, \dots, S_n$ to be diagonal, take two matrices N_i, N_j to have $\mathrm{tr}(N_i N_j) \neq 0$, and all other N_k with $k \notin \{i, j\}$ to be zero.

So, we get that on a dense open subset $U \subset X$, the locus with $\mathrm{tr}(X_{i,\lambda}) = \alpha$ is given by a unique choice of λ. So the map

$$\mathrm{tr}(X_{i,\lambda}) : U \times \mathbb{C}^* \to \mathbb{C}$$

has fibers which project isomorphically to U. In particular, the fibers are irreducible.

Applying the multiplication map $U \times \mathbb{C}^* \to U$, the image of this fiber gives an open dense irreducible set of n-tuples which have the given value of trace. The whole locus of desired tuples is thus irreducible. □

Proof. (Of Proposition 4.9) To obtain the inclusion map of Corollary 4.7, we need to establish that $\mathcal{M}_{d',0}(\Lambda^1(Q_n))$ is connected for each $d' < d$. This is a consequence of Lemma 4.10 for $m = 2$ and characteristic polynomials equal to $(x - 1)^2$, since a product of unipotent 2×2 matrices is unipotent if and only if it has trace two.

For the injection to be surjective, we need to show that every local resolution near zero extends to a global one. We will show these are in fact given by VGIT and hence globally projective, proving simultaneously the last statement.

In [KS23, Theorem 5.4], we establish that $\mathcal{M}_{d,0}(\Lambda^1(Q_n))$ is locally a quiver variety. Note that the hypotheses of [ST19, Theorem 1.5] are satisfied, so $\mathcal{M}_{d,0}(\Lambda^1(Q_n))$ has a resolution given by variation of stability parameter (for a generic choice of parameter). The varieties $\mathcal{M}_{d,0}(\Lambda^1(Q_n))$ and $\mathcal{M}_{0,0}(Q,d)$ have the same choices of stability parameter and the same symplectic leaves since the stratification by representation type is connected. It follows that the injection of resolutions $\mathcal{C}_{\mathcal{M}_{0,0}(Q,d)} \hookrightarrow \mathcal{C}_{\mathcal{M}_{d,0}(\Lambda^1(Q_n))}$ is surjective and we have the same classification of symplectic resolutions. □

The proof of the previous proposition holds in the following more general case:

Corollary 4.11. *Suppose Q is a quiver and d is a dimension vector satisfying:*

- *$\mathcal{M}_{d',0}(\Lambda^1(Q))$ is connected for all $d' < d$,*
- *The hypothesis of [BCS22, Theorem 1.2] holds for $\mathcal{M}_{0,0}(Q,d)$ showing it is a Mori dream space with all crepant projective resolutions given by VGIT.*

Then $\mathcal{C}_{\mathcal{M}_{d,0}(\Lambda^1(Q))} \cong \mathcal{C}_{\mathcal{M}_{0,0}(Q,d)}$ and all resolutions are given by VGIT (hence globally projective).

Note here that the hypotheses of [ST19, Theorem 1.5] for the multiplicative variety $\mathcal{M}_{d,0}(\Lambda^1(Q))$ to admit a symplectic resolution by VGIT are automatically satisfied when the conditions of [BCS22, Theorem 1.2] hold for $\mathcal{M}_{0,0}(Q,d)$, since the latter include the facts that the dimension vector is indivisible and contained in $\Sigma_{0,0}$.

4.5 Moduli spaces of objects in 2-Calabi–Yau categories

A category $\mathbf{C}$ is 2-Calabi–Yau if it has a Serre functor given by shift by 2. Observe that 2-Calabi–Yau categories arise naturally in geometry including any full subcategory of:

(1) the category of finite-dimensional modules for a 2-Calabi–Yau dg-algebra, which includes dg-preprojective algebras and multiplicative dg-preprojective algebras (see e.g., [BCS23]),

(2) the category of coherent sheaves for a 2-dimensional compact

(3) the wrapped Fukaya category of a real 4-dimensional Weinstein manifold [Gan12],

(4) the cluster category of a finite quiver (see e.g., [Kel08]), and

(5) the category of semistable Higgs bundles of fixed slope on a closed Riemann surface M_g.

The Higgs bundle example follows from realizing this category is a subcategory of coherent sheaves on the symplectic manifold $T^*(M_g)$. Note that, analogously, a full subcategory of (1) for the multiplicative preprojective algebra includes categories of local systems on Riemann surfaces with punctures, fixing monodromy conjugacy classes at the punctures (see [ST19, Theorem 3.6] and the preceding discussion).

The key idea in this section is to build crepant resolutions for a variety X by realizing X as the moduli space of objects for $\mathbf{C}$, a 2-Calabi–Yau category, $X \cong \mathcal{M}(\mathrm{ob}(\mathbf{C}))$. Davison proved that such moduli spaces are étale-locally quiver varieties [Dav21, Theorem 5.11, Theorem 1.2], or more precisely there is an étale neighborhood of a closed point $x \in X$ is isomorphic to an étale neighborhood of the zero representation in a conical quiver variety.

Bellamy and the second-named author determine which quiver varieties admit projective, crepant (or equivalently symplectic) resolutions in [BS21]. Recently, Bellamy, Craw, and the second-named author give a criterion for a variety X presented as a GIT quotient to have all projective crepant resolutions given by variation of GIT [BCS22, Theorem 1.1, Condition 3.4]. Moreover, they prove that, under mild hypotheses, Nakajima quiver varieties satisfy this criterion, thus giving a classification of projective crepant resolutions by VGIT [BCS22, Theorem 1.2].

Consequently, for $\mathcal{M}(\mathrm{ob}(\mathbf{C}))$ the moduli of objects in a 2-Calabi–Yau category, one can compute the local classification of projective, crepant resolutions in the étale neighborhood of any point, by first identifying the neighborhood with that of a quiver variety, where every such resolution can be described by VGIT.

In more detail, consider the open subset of objects M with $\mathrm{End}(M)$ semisimple. Then there exists a quiver $Q = (Q_0, Q_1)$ with (1) $\mathrm{End}(M)$ Morita equivalent to kQ_0 and (2) the Morita equivalence taking $\mathrm{Ext}^1(M,M)$ to kQ_1. The quiver Q is called the Ext-quiver for M. And in the 2-Calabi–Yau case the entire Ext-algebra $\mathrm{Ext}^*(M,M)$ can be recovered (as a graded vector space) from Q since $\mathrm{Ext}^2(M,M) \cong \mathrm{Ext}^0(M,M)^*$ and $\mathrm{Ext}^n(M,M) = 0$ for $n > 2$. Note further that $\mathrm{Ext}^1(E_i,E_j) \cong \mathrm{Ext}^1(E_j,E_1)^*$ so $\dim(\mathrm{Ext}^1(E_i,E_j)) = \dim(\mathrm{Ext}^1(E_i,E_j))$ and hence Q is a doubled quiver. The multiplication on $\mathrm{Ext}^*(M,M)$ is determined by the pairing on $\mathrm{Ext}^1(M,M)$ (together with vanishing in degree ≥ 3). While, in general one would need to describe the entire A_∞-structure on $\mathrm{Ext}^*(M,M)$, Davison [Dav21, Theorem 1.2] proved that this algebra is formal.

By computing monodromy and compatibility, one could apply our local-to-global analysis, where the stratification $\mathcal{S}$ is determined by Ext-quiver type (passing to connected components), to classify global projective symplectic resolutions of X.

Even without computing monodromy and compatibility we can still apply Corollary 3.23 to obtain:

Corollary 4.12. *Let $\mathbf{C}$ be a 2-Calabi–Yau category and let $\mathcal{M}(ob(\mathbf{C}))$ denote its moduli space of objects. Assume $\mathcal{M}(ob(\mathbf{C}))$ has a unique minimal stratum with basepoint M. Let Q be the Ext-quiver for M and $d = (d_i := \dim(\mathrm{Ext}^1(e_iM, e_iM)))$. Then*

$$\mathcal{C}_{\mathcal{M}(ob(\mathbf{C}))} \hookrightarrow \mathcal{C}_{\mathcal{M}_{0,0}(Q,d)}.$$

More generally, we always have an embedding $\mathcal{C}_{\mathcal{M}(ob(\mathbf{C}))} \to \prod_i \mathcal{C}_{\mathcal{M}_{0,0}(Q_i,d_i)}$ into the product over all minimal strata of the set of local resolutions near a fixed basepoint of the stratum. The same holds replacing $\mathcal{C}$ by $\widetilde{\mathcal{C}}$ and $\mathcal{P}$.

4.6 Finite symplectic quotients of symplectic tori

Let $\mathbb{T} = (\mathbb{C}^*)^n$ be the n-dimensional complex torus, and let $\mathfrak{t} \cong \mathbb{C}^n$ be its Lie algebra. To this we can associate two natural symplectic varieties: $T^*(\mathbb{T}) \cong \mathbb{T} \times \mathfrak{t}$ and $\mathbb{T} \times \mathbb{T}$.

As explained in Section 2, any finite symplectic quotient has a finite stratification by symplectic leaves, with a local product decomposition. In this section we will look at the case where the actions are moreover group automorphisms of the torus.

Recall that the group of automorphisms of the torus $\mathbb{T}$ is the group of automorphisms of the lattice $\mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \cong \mathbb{Z}^n$ of characters, i.e., $\mathrm{GL}_n(\mathbb{Z})$. These induce symplectic automorphisms of $T^*(\mathbb{T})$ and $\mathbb{T} \times \mathbb{T}$. We will also consider the larger group $\mathrm{Sp}_{2n}(\mathbb{Z})$ of symplectic group automorphisms of $\mathbb{T} \times \mathbb{T}$.

4.6.1 Weyl group quotients

Let G be a reductive group with maximal torus $\mathbb{T}$ and Weyl group $W := N_G(\mathbb{T})/\mathbb{T}$. Here we consider the quotients $\mathbb{T} \times \mathbb{T}/W$ and $T^*(\mathbb{T})/W$.

Remark 4.13. The three classes of quotients (1) $\mathbb{T} \times \mathbb{T}/W$, (2) $T^*(\mathbb{T})/W$, and (3) $T^*(\mathfrak{t})/\!/\!/G$ are of interest in part because of their descriptions as the connected component of the identity of the quasi-Hamiltonian reductions: (1) $G \times G/\!/\!/G$, (2) $T^*(G)/\!/\!/G$, and (3) $T^*(\mathfrak{g})/\!/\!/G$. These statements can be found in [LNY23, Theorem 1.1.2, Theorem 1.1.4] but note Ansatz 1.2.5 and see Remark 1.1.5 for previous partial results including [Jos97] and [EG02] who obtain the result only after quotienting out the nilradical, i.e., for reduced rings. One could use the formula for $G \times G/\!/\!/G$ in [LNY23, Theorem 1.1.4] to analyze the other connected components, which are quotients of smaller tori.

Let us first consider the case where G is simply-connected and semisimple. Let $\mathfrak{S}_n$ denote the symmetric group on n letters and C_n denote the cyclic group of order n.

Type A (SL_{n+1}): The quotient $\mathbb{T} \times \mathbb{T}/W := (\mathbb{C}^*)^n \times (\mathbb{C}^*)^n/\mathfrak{S}_{n+1}$ can be viewed as the subset of $\mathrm{Sym}^{n+1}(\mathbb{C}^2)$ given by

$$\left\{ (\lambda_1, \mu_1), (\lambda_2, \mu_2), \ldots, (\lambda_{n+1}, \mu_{n+1}) \in \mathrm{Sym}^{n+1}(\mathbb{C}^2) \;\middle|\; \prod_{i=1}^{n+1} \lambda_i = 1 = \prod_{i=1}^{n+1} \mu_i \right\}.$$

Explicitly, $\lambda_{n+1} = 1/\prod_{i=1}^n \lambda_i$ and $\mu_{n+1} = 1/\prod_{i=1}^n \mu_i$ and in particular each $\lambda_i, \mu_i \neq 0$. Hence each singularity in $\mathbb{T} \times \mathbb{T}/W$ can be identified with a singularity in $\mathrm{Sym}^{n+1}(\mathbb{C}^2)$. Consequently, each singular point x has a neighborhood U_x with a unique resolution given by the Hilbert–Chow map $\mathrm{Hilb}^{[n+1]}(U_x) \to \mathrm{Sym}^{n+1}(U_x)$. By Corollary 3.19, these unique local resolutions glue to a unique global, crepant resolution. This resolution is a priori only locally projective but in this case is given by the global Hilbert scheme

$$\mathrm{Hilb}_1^{[n+1]}(\mathbb{C}^* \times \mathbb{C}^*) := \left\{ I \in \mathrm{Hilb}^{[n+1]}(\mathbb{C}^* \times \mathbb{C}^*) \;\middle|\; \prod_{i=1}^{n+1} \lambda_i = 1 = \prod_{i=1}^{n+1} \mu_i \right\}.$$

Type B (SO_{2n+1}) and type C (Sp_{2n}): Note that the reduced quotients in these cases are the same torus quotients, given as follows:

$$\mathbb{T} \times \mathbb{T}/W = (\mathbb{C}^*)^n \times (\mathbb{C}^*)^n/(C_2)^n \rtimes \mathfrak{S}_n = \mathrm{Sym}^n(\mathbb{C}^* \times \mathbb{C}^*/C_2)$$

where C_2 acts on $\mathbb{C}^* \times \mathbb{C}^*$ by $(z, w) \mapsto (z^{-1}, w^{-1})$ and hence fixes the four points $(\pm 1, \pm 1)$. These are the four singularities of $\mathbb{C}^* \times \mathbb{C}^*/C_2$ and each singularity is an A_1 singularity $\mathbb{C}^2/C_2$, where C_2 acts on $\mathbb{C}^2$ by $(z, w) \mapsto (-z, -w)$.

Denote by $\mathcal{C}_X$ the set of isomorphism classes of symplectic resolutions of the variety X. It follows that we obtain the following formula:

$$\#\mathcal{C}_{\mathbb{T}\times\mathbb{T}/W} = (\#\mathcal{C}_{\mathrm{Sym}^n(\mathbb{C}^2/C_2)})^4 = n^4.$$

Similarly, the number of symplectic resolutions for $T^*(\mathbb{C}^*)^n/((C_2)^n \rtimes \mathfrak{S}_n)$ is the square of the number of symplectic resolutions of $\mathrm{Sym}^n(\mathbb{C}^2)/C_2$.

In all other types, the formal completion of the quotient at the identity will recover the quotient $T^*\mathfrak{t}/\!/\!/W$, which does not admit a symplectic resolution by [Gor03, Bel09].

Next consider the general type A case. Then the existence of a symplectic resolution was classified in [BS23, Theorem 1.10]. In particular, for type $A_n, n \geq 2$, and G (almost) simple, then a resolution only exists if $G = \mathrm{SL}_{n+1}(\mathbb{C})$ as above.

Example 4.14. For example, for the case $G = \mathrm{PGL}_3(\mathbb{C})$, we obtain a quotient $\mathbb{T} \times \mathbb{T}/\mathfrak{S}_3$, for $\mathbb{T}$ a two-dimensional torus, which does not admit a resolution. One explicit way to think about the action is to consider $\mathbb{T} = (\mathbb{C}^*)^3/\mathbb{C}^* \subset \mathrm{PGL}_3(\mathbb{C})$ with the quotient action being the diagonal action, then to act by usual permutations. Then the element (123) fixes the nine points $Y \times Y$ for $Y = \{(1, \zeta, \zeta^2) \mid \zeta^3 = 1\}$. But the reflections (12) and (23) only fix the subset $Y \times \{1\}$ and $\{1\} \times Y$, respectively. This leaves $9 - (3 + 3 - 1) = 4$ nonresolvable C_3-singularities.

4.6.2 $\mathbb{T} \times \mathbb{T}/\Gamma$

In this section again $\mathbb{T} \cong (\mathbb{C}^*)^n$ but now W is replaced by $\Gamma \subset \mathrm{Sp}_{2n}(\mathbb{Z})$ a finite group acting on $\mathbb{T} \times \mathbb{T}$. We first restrict to the case $n = 1$, and $\Gamma \cong C_i$ for $i = 2$, 3, 4 or 6.

Define $\Gamma_i := \langle \gamma_i \rangle \cong C_i$ where γ_i is given by

$$\gamma_6 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \qquad \gamma_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \gamma_3 = \gamma_6^2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \qquad \gamma_2 = \gamma_4^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Γ_i acts on $\mathbb{C}^* \times \mathbb{C}^*$ via the weights matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (x, y) = (x^a y^b, x^c y^d)$$

so

$$\gamma_6 \cdot (x, y) = (y, x^{-1} y) \qquad \gamma_4 \cdot (x, y) = (y, x^{-1}) \qquad \gamma_3 \cdot (x, y) = (x^{-1} y, x^{-1}) \qquad \gamma_2 \cdot (x, y) = (x^{-1}, y^{-1}).$$

Each action preserves the symplectic form $\omega = dx/x \wedge dy/y$, as each weight matrix has determinant 1. The fixed points of each action are

$$\begin{aligned} (x, y) = (y, x^{-1} y) &\implies x = y,\ y = x^{-1} y \implies x = y = 1 \\ (x, y) = (y, x^{-1}) &\implies x = y,\ y = x^{-1} \implies x = y = \pm 1 \\ (x, y) = (x^{-1} y, x^{-1}) &\implies x^2 = y,\ y = x^{-1} \implies x = y^{-1} = \zeta_3 \\ (x, y) = (x^{-1}, y^{-1}) &\implies x = x^{-1},\ y = y^{-1} \implies x = \pm 1,\ y = \pm 1. \end{aligned}$$

Each fixed point gives a singularity in the orbit space that is locally du Val and hence has a unique crepant resolution. These glue to a unique crepant resolution of the entire variety. Moreover, this resolution is projective and so the set of projective crepant resolutions of $(\mathbb{C}^*)^2/\Gamma_i$ is a singleton.

These are the only finite group actions on $(\mathbb{C}^*)^2$ preserving $dx/x \wedge dy/y$. To see this, the following lemma is useful:

Lemma 4.15. *Suppose that $G < \mathrm{Sp}_{2m}(\mathbb{R})$ with complexified symplectic representation $\mathbb{C}^{2m}$ symplectically irreducible. Then $\mathbb{C}^{2m}$ is reducible as a complex representation.*

Proof. Observe that if a complex irreducible representation V of a finite group is defined over $\mathbb{R}$, it preserves a nondegenerate symmetric bilinear form (e.g., by averaging the standard inner product over the group), so it cannot preserve a symplectic form. □

Proposition 4.16. *Let Γ be a finite subgroup acting on $(\mathbb{C}^*)^2$ by group automorphisms preserving the invariant symplectic form $dx/x \wedge dy/y$. Then Γ is cyclic of order dividing $1, 2, 3, 4$, or 6 and the quotient $(\mathbb{C}^*)^2/\Gamma$ admits a unique projective crepant resolution of singularities.*

Proof. Such an action is given by a finite subgroup of $\mathrm{SL}_2(\mathbb{Z})$. These are cyclic of orders $1, 2, 3, 4$, or 6. An algebraic argument for this follows because $\mathrm{SL}_2(\mathbb{Z})$ is isomorphic to an amalgamated product $C_4 *_{C_2} C_6$.

For a geometric argument, first by Lemma 4.15, the group must act reducibly, hence it is conjugate to a subgroup of diagonal matrices therefore abelian. Next, the abelian finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ are cyclic. Finally, the cyclic subgroups with integral traces are precisely the ones of the given orders.

For the final statement, all singularities of the quotient are du Val and hence admit unique local projective crepant resolutions of singularities. So the quotient $(\mathbb{C}^*)^2/\Gamma$ admits a unique locally projective crepant resolution of singularities. But also, this is globally projective, since it is well known that the resolutions can be obtained by iterated blowups of the (isolated) singularities. □

Using these examples, we can extend the type B/C Weyl group quotients as follows:

Example 4.17. Let $W = \mathfrak{S}_n \ltimes \Gamma$ be a wreath product with $\Gamma < \mathrm{Sp}_2(\mathbb{Z})$ and $\mathfrak{S}_n$ permuting the pairs $(x_1, y_1), \dots, (x_n, y_n)$ and hence preserving the symplectic forms $\sum_{i=1}^n dx_i/x_i \wedge dy_i/y_i$ and $\sum_{i=1}^n dx_i \wedge dy_i$. So $W < \mathrm{Sp}_{2n}(\mathbb{Z})$. In particular we can let $\Gamma = C_m$ for $m \in \{1, 2, 3, 4, 6\}$. Then the quotient $\mathbb{T} \times \mathbb{T} /\!/\!/ W$ is identified with $\mathrm{Sym}^n(\mathbb{C}^\times \times \mathbb{C}^\times/\Gamma)$. In particular there is a symplectic resolution given by $\mathrm{Hilb}^{[n]}(\widetilde{\mathbb{C}^\times \times \mathbb{C}^\times/\Gamma})$.

The above example actually produces all of the groups whose quotient admits a projective crepant resolution:

Proposition 4.18. *Let $\Gamma \subset Sp_{2n}(\mathbb{Z})$ be a finite subgroup acting on $(\mathbb{C}^*)^{2n}$ by group automorphisms preserving the invariant symplectic form $\sum_i dx_i/x_i \wedge dy_i/y_i$. The quotient $(\mathbb{C}^*)^{2n}/\Gamma$ admits a locally projective symplectic (equivalently crepant) resolution of singularities only if $\Gamma \cong \prod_i \Gamma_i$ with each of the Γ_i of the form $\mathfrak{S}_n \ltimes H$ with $H \in \{C_1, C_2, C_3, C_4, C_6\}$, where the derivative of the action near $1 \in (\mathbb{C}^*)^{2n}$ is given by the product of the usual reflection representations of each Γ_i.*

Proof. Suppose that $(\mathbb{C}^*)^{2n}/\Gamma$ admits a crepant resolution for $\Gamma < \mathrm{Sp}_{2n}(\mathbb{Z})$. Then the local model at the identity is $\mathbb{C}^{2n}/\Gamma$ for the same group. Let us assume the representation $\mathbb{C}^{2n}$ is symplectically irreducible, otherwise the pair $(\mathbb{C}^{2n}, \Gamma)$ decomposes as a product of pairs. Thus, $\mathbb{C}^{2n}$ is isomorphic to a sum of G-invariant Lagrangian subspaces $L \oplus L'$. If $\mathbb{C}^{2n}/G$ admits a locally projective symplectic resolution, then G must be generated by sympletic reflections [Ver00]. In turn, if G preserves a Lagrangian L, it must be a complex reflection group. Thanks to [Bel09] building on Ginzburg–Kaledin [GK04, Corollary 1.2.1] (and see also Etingof–Ginzburg [EG02, Corollary 1.1.4 (i)] and Namikawa [Nam11, Corollary 2.1]) we have a complete list of complex reflection groups Γ such that the symplectic quotient singularity $\mathbb{C}^{2n}/\Gamma$ admits a projective symplectic resolution. This includes the wreath products $\mathfrak{S}_n \ltimes \Gamma^n$ for $\Gamma < \mathrm{SL}_2(\mathbb{C})$ cyclic. In turn, as in Proposition 4.16, to have integral traces this means Γ is cyclic of order $1, 2, 3, 4$, or 6, since for $\gamma \in \Gamma$, a corresponding reflection in the wreath product has trace $2n - 2 + \mathrm{tr}(\gamma)$. Other than the wreath products and $\mathfrak{S}_{n+1}$, the only other complex reflection group with symplectic quotient admitting a symplectic resolution is the binary tetrahedral group. In Lemma 4.21 below, we show that the quotient of $(\mathbb{C}^*)^4$ by the binary tetrahedral group does not admit a symplectic resolution of singularities, completing the classification. ☐

For each group $\Gamma \subset \mathrm{Sp}_{2n}(\mathbb{Z})$ appearing in Proposition 4.18, there exists an action of Γ on $(\mathbb{C}^*)^{2n}$ such that the quotient $(\mathbb{C}^*)^{2n}/\Gamma$ admits a symplectic resolution of singularities. Thus, Proposition 4.18 completes the classification of the rationalizations of integral representations such that the corresponding torus quotient admits a symplectic resolution of singularities. But a rationalization of an integral representation may admit multiple inequivalent integral forms. Moreover, for many fixed Γ, we can find an action on $(\mathbb{C}^*)^{2n}$ such that the resulting quotient does *not* admit a symplectic resolution of singularities:

Example 4.19. If we view $\Gamma = \mathfrak{S}_3$ as the Weyl group W acting on the cotangent bundle $T^*((\mathbb{C}^*)^2)$, then [Gor03], shows that the quotient $(\mathbb{C}^*)^4/W$ obtained by restricting the action $(\mathbb{C}^*)^4 \subset (\mathbb{C}^*)^2 \times \mathbb{C}^2 \cong T^*((\mathbb{C}^*)^2)$ does not admit a symplectic resolution of singularities (see Subsection 4.6.1 on Weyl group quotients above).

There can even be nontrivial ways to extend integral representations:

Example 4.20. Let $C_4 = \langle i \rangle \subset \mathbb{C}$ act on the lattice $\mathbb{Z}^2 \cong \mathbb{Z}_1 \oplus \mathbb{Z}_i \subset \mathbb{C}$ by multiplication in $\mathbb{C}$. The group $C_4 \times C_4$ acts on $\mathbb{Z}^4 \cong \mathbb{Z}^2 \oplus \mathbb{Z}^2$ componentwise. It also acts on the equivalent lattice $\frac{1}{2}\mathbb{Z}^4$, with each action giving the same integral representation $C_4 \times C_4 \to \mathrm{GL}_4(\mathbb{Z})$. Next we consider the intermediate lattice

$$\mathbb{Z}^4 \subsetneq L := \langle v_1 := (1,0,0,0),\ v_2 := (0,1,0,0),\ v_3 := (0,0,1,0),\ v_4 := \frac{1}{2}(1,1,1,1) \rangle \subsetneq \frac{1}{2}\mathbb{Z}^4.$$

Notice that L contains $(0,0,0,1) = 2v_4 - (v_1 + v_2 + v_3)$ and therefore contains $(a/2, b/2, c/2, d/2)$ for $a, b, c, d \in 2\mathbb{Z} + 1$. The action of $C_4 \times C_4$ on $\mathbb{Z}^4$ restricts to L, giving a representation

$$C_4 \times C_4 \to \mathrm{GL}(L) \cong \mathrm{GL}(\mathbb{Z}^4)$$

that is rationally (but not integrally) equivalent to to the original representation. That it is not integrally equivalent follows because L is not spanned by elements which are fixed by one of the two C_4 factors, whereas $\mathbb{Z}^4$ is so spanned. This is true even though there is a short exact sequence

$$0 \to \mathbb{Z}^2 \to L \to \mathbb{Z}^2 \to 0,$$

with the first and third terms giving the standard rotation representations of the two factors of C_4. So there are nontrivial ways to extend integral representations.

We complete the prove of Proposition 4.18 by ruling out the binary tetrahedral group.

Lemma 4.21. *The quotient $(\mathbb{C}^*)^4/T$, where T is the binary tetrahedral group acting symplectically on $(\mathbb{C}^*)^4$ does not admit a symplectic resolution of singularities.*

Proof. First, observe that there may be many ways for T to act symplectically on $(\mathbb{C}^*)^4$. We are fixing only the complex symplectic representation of T, so the embedding $T \subseteq \mathrm{Sp}_4(\mathbb{C})$. This is conjugate to a real representation, $T \subseteq \mathrm{Sp}_4(\mathbb{R})$, and this real representation is uniquely determined up to isomorphism. One explicit way to understand it is to realize T inside the quaternions $\mathbb{H}$, as the group generated by $i, j, k, \frac{1}{2}(1+i+j+k)$. Then, every integral representation, up to isomorphism, can be constructed by choosing a full (rank four) T-invariant lattice $\Lambda \subseteq \mathbb{H}$.

We now consider the fixed points of T on $(\mathbb{C}^*)^4$. First consider the element $i \in T$. Now i acts on $\mathbb{Q}^4$ without eigenvalues ± 1, so this determines an action of the Gaussian integers, $\mathbb{Z}[i] \cong \mathbb{Z}[x]/(x^2+1)$, on Λ. This action is torsion free as Λ is torsion free. Since $\mathbb{Z}[i]$ is a PID, Λ is a free $\mathbb{Z}[i]$-module. So we can write the action in the standard way. The number of fixed points is four, the same as for block diagonal 90-degree rotation matrices, which is the γ_4 case above.

Next, in order to admit a symplectic resolution, each such fixed point must also be fixed by a reflection. Since i and any reflection generate T, we can assume that each of the above fixed points are fixed by all of T, at least to investigate the existence of a symplectic resolution.

We next turn to elements of order six (i.e., -1 times a reflection). For such an element γ there is a saturated sublattice of rank 2, Λ_0, on which γ acts by -1, and quotient lattice Λ/Λ_0 on which γ acts as a $\mathbb{Z}[\zeta]$-module, for ζ a primitive cube root of unity. The latter is also a PID and hence Λ/Λ_0 is a free $\mathbb{Z}[\zeta]$-module. So we can write the action of γ in the standard way for an order six element. Overall the action of γ is block-upper triangular with diagonal blocks $-I$ and the standard order-six action. There are still 4 fixed points, since the order six action has no fixed points (the equations to be fixed give a unique solution, regardless of the choice of four fixed points corresponding to the sublattice, i.e., quotient torus). By the preceding argument, each fixed point has stabilizer all of T.

However, the element $-I$ fixed sixteen points on the four-torus: all involutions. So there are twelve points with stabilizer C_2; they form one T-orbit. Thus there is a nonresolvable C_2-singularity of $(\mathbb{C}^*)^4/T$. $\square$

Remark 4.22. There is an obvious choice of lattice invariant under T in the proof: the lattice of Hurwitz quaternions. For this choice one can see that the fixed points are exactly as in the proof. Note that there could exist a different lattice with different fixed points: this would happen only if the elements i, j, k do not all share the same four fixed points. In this case, one of the non-identity fixed points of i would also have to be fixed for j and k, hence for all of Q_8, hence for all of T since Q_8 is normal. So we would get two fixed points for T, and two fixed points each for the groups $\langle i\rangle, \langle j\rangle, \langle k\rangle \cong C_4$, which all form a single orbit of T. Aside from these eight points we would have eight more fixed points for $-I$, breaking up into subsets of two fixed points for minus each reflection and its inverse. These sets of size two have stabilizers C_6, also nonresolvable. They form two orbits of size four. So in this model we would get one nonresolvable C_4 singularity and two nonresolvable C_6 singularities (instead of one nonresolvable C_2 singularity). It would be interesting to see if an appropriate lattice can be found to construct such a quotient.

A The Weinstein splitting theorem in the singular case

The celebrated Weinstein splitting theorem [Wei83, Theorem 2.1] gives a local decomposition of a smooth Poisson manifold into a symplectic part and a part with zero rank at the base point. This result goes through exactly the same in the case of analytic varieties, but for sake of completeness, we provide the statement and repeat the proof. Note that a version of this for formal neighborhoods was proved in [Kal06], as a consequence of Proposition 3.3 therein. By passing from analytic to formal neighborhoods, the result below also recovers such a decomposition.

Theorem A.1. *Let (X, σ) be a (real or complex) analytic Poisson variety and $x \in X$. Then there is a neighborhood $U \ni x$ of X and a pointed Poisson isomorphism $(U, x) \cong (S, s) \times (Z, z)$ where S is symplectic and the Poisson structure of Z vanishes at z.*

Proof. Let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$ depending on whether we work in the real analytic or complex analytic setting. The proof is by induction on the rank of σ. If this rank is zero, then S must be a point, and the statement is obvious. Assume the rank is nonzero. Restricting to a small enough analytic neighborhood U, we can find a Hamiltonian $f \in \mathcal{O}(U)$ such that $\xi_f|_x \neq 0$. Up to further restriction of U, we may assume it is a Zariski closed subset of a ball $B \subseteq \mathbb{K}^n$ for some n centered at $x = 0$. Then there exists a vector field $\widetilde{\xi_f}$ on B which is tangent to U and restricts there to ξ_f. Up to shrinking B and U, we may pick coordinates $x_1, \ldots, x_n$ on B centered at 0 with $\widetilde{\xi_f}(x_1) = 1$. Let $g := x_1|_U$. Then $\{f, g\} = 1$, and hence $[\xi_f, \xi_g] = \xi_1 = 0$. Since $\xi_f(f) = 0 = \xi_g(g)$, whereas $\xi_f(g) = 1 = -\xi_g(f)$, ξ_f and ξ_g are linearly independent at 0.

Since $\widetilde{\xi_f}$ is nonvanishing on B, by the Frobenius integrability theorem (for the holomorphic case, see [Voi02, 2.26]), we can locally integrate the distribution spanned by $\widetilde{\xi_f}$ to a submersion $B \to \mathbb{K}^{n-1}$ with $\widetilde{\xi_f}$ tangent to the fibers, up to shrinking B and U. This gives local coordinates $x_2, \ldots, x_n$ annihilated by $\widetilde{\xi_f}$. As a result, $\widetilde{\xi_f} = h\partial_1$ for some function h, which by assumption is nonvanishing in B, and hence invertible. Then the coordinates $\int h^{-1} \mathrm{d}x_1, x_2, \ldots, x_n$ have the property that $\widetilde{\xi_f} = \partial_1$.

Now, the ideal I_X of X is locally generated at $x = 0$ by some functions $g_1, \ldots, g_k$ independent of x_1. Shrinking U and B if necessary, we can assume these generate the ideal I_U of $U \subseteq B$, globally on B. Extend ξ_g to a vector field $\widetilde{\xi_g}$ on B tangent to U. Note that $[\partial_1, \xi]$ vanishes on U if and only if ξ is a sum of a vector field constant in x_1 (i.e., commuting with ∂_1) and a vector field itself vanishing on U (as I_U is generated by functions independent of x_1). So up to changing the extension $\widetilde{\xi_2}$ of ξ_2, we can assume that actually $[\widetilde{\xi_f}, \widetilde{\xi_g}] = 0$.

Then by the Frobenius integrability theorem again, we can find coordinates $x_3, \ldots, x_n$ with $\widetilde{\xi_f}(x_i) = 0 = \widetilde{\xi_g}(x_i)$ for $3 \leq i \leq n$ (again up to shrinking B and U). Since $\widetilde{\xi_f}, \widetilde{\xi_g}$ are tangent to U we get that I_U is generated by functions in $x_3, \ldots, x_n$. Moreover, we can further change the coordinates to $g, f, x_3, \ldots, x_n$, since $\widetilde{\xi_f}(g) = 1 = -\widetilde{\xi_g}(f)$, whereas $\widetilde{\xi_f}(f) = 0 = \widetilde{\xi_g}(g)$, so that the derivatives of f, g are linearly independent at x to those of $x_3, \ldots, x_n$.

In this new coordinate system we get a Poisson isomorphism $(U, x) \cong (S, s) \times (U', x)$, with (S, s) symplectic of dimension two and (U', x) Poisson, since $\{f, g\} = 1$ and $\{f, x_i\} = 0 = \{g, x_i\}, i \geq 3$.

Iterating the procedure, we will eventually obtain the theorem. □

A.1 A non-Poisson version of the splitting theorem

Rewriting the essential part of the argument of the Weinstein splitting theorem in the non-Poisson setting we arrive at the following, which we would imagine is known:

Lemma A.2. *Let X be a (real or complex) analytic variety and $\xi_1, \ldots, \xi_m$ vector fields on X which commute. Suppose that at some $x \in X$, $\xi_1|_x, \ldots, \xi_m|_x \in T_x X$ are linearly independent. Then in some local analytic neighborhood U of x, the ξ_i integrate to give a decomposition $U \cong D \times Z$ for D a polydisc of dimension m, with coordinates $x_1, \ldots, x_m$, such that $\xi_i = \partial_i$.*

Proof. We can take an analytic neighborhood U of x which is isomorphic to a Zariski closed subset of an open ball $B \subseteq \mathbb{K}^n$ centered at 0, for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We will think of U as a subset of B with $x \in U$ equal to $0 \in B$. We can pick U small enough that $\xi_1, \ldots, \xi_m$ are linearly independent on U. Let $\widetilde{\xi_1}, \ldots, \widetilde{\xi_m}$ be vector fields on B which are tangent to U and restrict there to $\xi_1, \ldots, \xi_m$. Applying the Frobenius integrability theorem (for the holomorphic case, again see [Voi02, 2.26]) to $\widetilde{\xi_1}$, we get a local submersion at $x = 0$, $B \to \mathbb{K}^{n-1}$ (up to shrinking B and U) with $\widetilde{\xi_1}$ tangent to the fibers. This defines coordinates $x_1, x_2, \ldots, x_n$ on B centered at 0 with $\widetilde{\xi_1} = h\partial_1$ for some function h nonvanishing on B. Changing coordinates $x_1 \mapsto \int h^{-1} \mathrm{d}x_1$ (if necessary after shrinking U), we get $\widetilde{\xi_1} = \partial_1$, as desired.

Next, since $\widetilde{\xi_1}$ is tangent to X, the ideal I_U is generated near $x \in \mathbb{K}^n$ by some functions in $x_2, \ldots, x_n$ (i.e., independent of x_1). We can assume these globally generate up to shrinking U and B. Then because $[\widetilde{\xi_1}, \widetilde{\xi_2}]$ vanishes on U, we see that $\widetilde{\xi_2}$ must be a sum of a vector field commuting with $\widetilde{\xi_1} = \partial_1$ and one vanishing on U. So we can find another choice of $\widetilde{\xi_2}$ (still restricting to ξ_2 on X) such that $[\widetilde{\xi_1}, \widetilde{\xi_2}] = 0$. Now applying the Frobenius integrability theorem again, we can assume that $\widetilde{\xi_1}, \widetilde{\xi_2}$ both annihilate $x_3, \ldots, x_n$. As before we can assume $\widetilde{\xi_1} = \partial_1$. Since they are linearly independent, $\widetilde{\xi_2} = f\partial_1 + g\partial_2$ with f, g independent of x_1 and g nonvanishing on B. Up to change of coordinates $x_1 \mapsto x_1 - \int fg^{-1} \mathrm{d}x_2$, we can assume $f = 0$, and up to change of coordinates $x_2 \mapsto \int g^{-1} \mathrm{d}x_2$, we can assume $g = 1$.

Inductively, if we have coordinates such that $\widetilde{\xi_1} = \partial_1, \ldots, \widetilde{\xi_k} = \partial_k$, then since these are tangent to U, the ideal I_U is locally generated at x by functions of $x_{k+1}, \ldots, x_n$. We can make another choice of $\widetilde{\xi_{k+1}}$ (restricting to ξ_{k+1} on X) which commutes with $\widetilde{\xi_1}, \ldots, \widetilde{\xi_k}$. Applying the Frobenius integrability theorem we have new coordinates $x_{k+2}, \ldots, x_n$ annihilated by $\widetilde{\xi_1}, \ldots, \widetilde{\xi_{k+1}}$. We can arrange that the latter vector fields are $\partial_1, \ldots, \partial_k$, together with something of the form $f_1\partial_1 + \cdots + f_{k+1}\partial_{k+1}$, with $f_{k+1}(0) \neq 0$. Now changes of coordinates as before replace $x_1, \ldots, x_{k+1}$ by new coordinates so that $\widetilde{\xi_i} = \partial_i$ for $1 \leq i \leq k+1$, completing the inductive step.

By induction we thus find coordinates on a ball about $x = 0$ in $\mathbb{K}^n$ so that $\widetilde{\xi_i} = \partial_i$ for all $1 \leq i \leq m$. The ideal I_X is generated by functions in the x_i for $i > m$. We obtain an analytic neighborhood of $x \in X$ isomorphic to a product of an open polydisc $D \subseteq \mathbb{K}^m$ and another variety Z (the vanishing locus of the generators of I_X in a polydisc with coordinates $x_{m+1}, \ldots, x_n$), with ξ_i the coordinate vector fields on D. □

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