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Inputs, Outputs, and Composition in the Logic of Information Flows

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The logic of information flows (LIF) is a general framework in which tasks of a procedural nature can be modeled in a declarative, logic-based fashion. The first contribution of this paper is to propose semantic and syntactic definitions of inputs and outputs of LIF expressions. We study how the two relate and show that our syntactic definition is optimal in a sense that is made precise. The second contribution is a systematic study of the expressive power of sequential composition in LIF. Our results on composition tie in the results on inputs and outputs, and relate LIF to first-order logic (FO) and bounded-variable LIF to bounded-variable FO.

This paper is the extended version of a paper presented at KR 2020 [2].

CCS Concepts: • **Computing methodologies** → **Knowledge representation and reasoning**; • **Theory of computation** → **Logic**; • **Software and its engineering** → *Software verification and validation.*

Additional Key Words and Phrases: dynamic logic, expressive power, binary relations on valuations

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1 INTRODUCTION

The Logic of Information Flows (LIF) [26, 27] is a knowledge representation framework designed to model and understand how information propagates in complex systems, and to find ways to navigate it efficiently. The basic idea is that modules, that can be given procedurally or declaratively, are the atoms of a logic whose syntax resembles first-order logic, but whose semantics produces new modules. In LIF, atomic modules are modeled as relations with designated input and output arguments. Computation is modeled as propagation of information from inputs to outputs, similarly to propagation of tokens in Petri nets. The specification of a complex system then amounts to *connecting* atomic modules together. For this purpose, LIF uses the classical logic connectives, i.e., the boolean operators, equality, and existential quantification. The goal is to start from constructs that are well understood, and to address the fundamental question of *what logical means are necessary*

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and sufficient to model computations declaratively. The eventual goal, which goes beyond the topic of this paper, is to come up with restrictions or extensions of LIF that make the computations efficient.

In its most general form, LIF is a rich family of logics with recursion and higher-order variables. Atomic modules are given by formulae in various logics, and may be viewed as solving the task of Model Expansion [22]: the input structure is expanded to satisfy the specification of a module thus producing an output. The semantics is given in terms of pairs of structures. We can, for example, give a graph (a relational structure) on the input of a module that returns a Hamiltonian cycle on the output, and compose it sequentially with a module that checks whether the produced cycle is of even length. One can vary both the expressiveness of logics for specifying atomic modules and the operations for combining modules, to achieve desirable complexity of the computation for the tasks of interest.

Many issues surrounding LIF, however, are already interesting in a first-order setting (see, e.g., [1]); and in fact such a setting is more generic than the higher-order setting, which can be obtained by considering relations as atomary data values. Thus, in this paper, we give a self-contained, first-order presentation of LIF. Syntactically, atomic modules here are relation atoms with designated input and output positions. Such atoms are combined using a set of algebraic operations into *LIF expressions.* The semantics is defined in terms of pairs of valuations of first-order variables; the first valuation represents a situation right before applying the module, while the second represents a possible situation immediately afterwards. The results in this paper are then also applicable to the case of higher-order variables.

Our contributions can be summarized as follows.

(1) While the input and output arguments of atomic modules are specified by the vocabulary, it is not clear how to designate the input and output variables of a complex LIF expression that represents a compound module. Actually, coming up with formal definitions of what it means for a variable to be an input or output is a technically and philosophically interesting undertaking. We propose semantic definitions, based on natural intuitions, which are, of course, open to further debate. The semantic notions of input and output turn out to be undecidable. This is not surprising, since LIF expressions subsume classical first-order logic formulas, for which most inference tasks in general are undecidable.
(2) We proceed to give an approximate, syntactic definition of the input and output variables of a formula, which is effectively computable. Indeed, our syntactic definition is *compositional*, meaning that the set of syntactic input (or output) variables of a formula depends only on the top-level operator of the formula, and the syntactic inputs and outputs of the operands. We prove our syntactic input–output notion to be *sound:* every semantic input or output is also a syntactic input or output, and the syntactic inputs and outputs are connected by a property that we call *input–output determinacy.* Moreover, we prove an optimality result: our definition provides the most precise approximation to semantic input and outputs among all compositional and sound definitions.
(3) We investigate the expressive power of sequential composition in the context of LIF. The sequential composition of two modules is fundamental to building complex systems. Hence, we are motivated to understand in detail whether or not this operation is expressible in terms of the basic LIF connectives. This question turns out to be approachable through the notion of inputs and outputs. Indeed, there turns out to be a simple expression for the composition of *io-disjoint* modules. Here, io-disjointness means that inputs and outputs do not overlap. For example, a module that computes a function of x and returns the result in y is io-disjoint; a module that stores the result back in x, thus overwriting the original input, is not.

(4) We then use the result on io-disjoint expressions to show that composition is indeed an expressible operator in the classical setting of LIF, where there is an infinite supply of fresh variables. (In contrast, the expression for io-disjoint modules does not need extra variables.)
(5) Finally, we complement the above findings with a result on LIF in a bounded-variable setting: in this setting, composition is necessarily a primitive operator.

Many of our notions and results are stated generally in terms of transition systems (binary relations) on first-order valuations. Consequently, we believe our work is also of value to settings other than LIF inasmuch as they involve dynamic semantics. Several such settings, where input–output specifications are important, are discussed in the related work section.

The rest of this paper is organized as follows. In Section 2, we formally introduce the Logic of Information Flows from a first-order perspective. Section 3 presents our study concerning the notion of inputs and outputs of complex expressions. Section 7 then presents our study on the expressibility of sequential composition. Section 8 discusses related work. We conclude in Section 9. In Sections 4, 5, and 6, we give extensive proofs of theorems we discuss in Section 3.

2 PRELIMINARIES

A *(module) vocabulary* $\mathcal{S}$ is a triple $(Names, ar, iar)$ where:

- *Names* is a nonempty set, the elements of which are called *module names*;
- *ar* assigns an arity to each module name in *Names*;
- *iar* assigns an input arity to each module name M in *Names*, where $iar(M) \leq ar(M)$.

We fix a countably infinite universe **dom** of data elements. An *interpretation* D of $\mathcal{S}$ assigns to each module name M in *Names* an $ar(M)$-ary relation $D(M)$ over **dom**.

Furthermore, we fix a universe of *variables* $\mathbb{V}$. This set may be finite or infinite; the size of $\mathbb{V}$ will influence the expressive power of our logic. A *valuation* is a function from $\mathbb{V}$ to **dom**. The set of all valuations is denoted by $\mathcal{V}$. We say that ν_1 and ν_2 *agree on* $Y \subseteq \mathbb{V}$ if $\nu_1(y) = \nu_2(y)$ for all $y \in Y$ and that they *agree outside* Y if they agree on $\mathbb{V} - Y$. Sometimes, we simply write $\nu_1 = \nu_2$ on (outside) Y to say that they agree on (outside) Y. A *partial valuation* on $Y \subseteq \mathbb{V}$ is a function from Y to **dom**; we will also call this a Y-valuation. If ν is a valuation, we use $\nu|_Y$ to denote its restriction to Y. Let ν be a valuation and let ν_1 be a partial valuation on $Y \subseteq \mathbb{V}$. Then the substitution of ν_1 into ν, denoted by $\nu[\nu_1]$, is defined as $\nu_1 \cup (\nu|_{\mathbb{V}-Y})$. In the special case where ν_1 is defined on a single variable x with $\nu_1(x) = d$, we also write $\nu[\nu_1]$ as $\nu[x : d]$.

We assume familiarity with the syntax and semantics of first-order logic (FO, relational calculus) over $\mathcal{S}$ [8] and use := to mean "is by definition".

2.1 Binary Relations on Valuations

The semantics of LIF will be defined in terms of binary relations on $\mathcal{V}$ (abbreviated BRV: Binary Relations on Valuations). Before formally introducing LIF, we define operations on BRVs corresponding to the classical logical connectives, adapted to a dynamic semantics. For boolean connectives, we simply use the standard set operations. For equality, we introduce selection operators. For existential quantification, we introduce cylindrification operators.

Let A and B be BRVs, let Z be a finite set of variables, and let x and y be variables.

- **Set Operations:** $A \cup B$, $A \cap B$, and $A - B$ are well known.
- **Composition**

$$A \,;\, B := \{(\nu_1, \nu_2) \mid \exists \nu_3 : (\nu_1, \nu_3) \in A \text{ and } (\nu_3, \nu_2) \in B\}.$$

- **Converse**

$$A^{\smile} := \{(\nu_1, \nu_2) \mid (\nu_2, \nu_1) \in A\}.$$

- **Left and Right Cylindrifications**

$$\mathrm{cyl}^l_Z(A) := \{(\nu_1, \nu_2) \mid \exists \nu_1' : (\nu_1', \nu_2) \in A \text{ and } \nu_1' \text{ and } \nu_1 \text{ agree outside } Z\};$$
$$\mathrm{cyl}^r_Z(A) := \{(\nu_1, \nu_2) \mid \exists \nu_2' : (\nu_1, \nu_2') \in A \text{ and } \nu_2' \text{ and } \nu_2 \text{ agree outside } Z\}.$$

- **Left and Right Selections**

$$\sigma^l_{x=y}(A) := \{(\nu_1, \nu_2) \in A \mid \nu_1(x) = \nu_1(y)\};$$
$$\sigma^r_{x=y}(A) := \{(\nu_1, \nu_2) \in A \mid \nu_2(x) = \nu_2(y)\}.$$

- **Left-to-Right Selection**

$$\sigma^{lr}_{x=y}(A) := \{(\nu_1, \nu_2) \in A \mid \nu_1(x) = \nu_2(y)\}.$$

If $\bar{x}$ and $\bar{y}$ are tuples of variables of length n, we write $\sigma^{lr}_{\bar{x}=\bar{y}}(A)$ for

$$\sigma^{lr}_{x_1=y_1} \sigma^{lr}_{x_2=y_2} \ldots \sigma^{lr}_{x_n=y_n}(A)$$

and if z is a variable we write cyl^l_z for $\mathrm{cyl}^l_{\{z\}}$. Intuitively, a BRV is a dynamic system that manipulates the interpretation of variables. A pair (ν_1, ν_2) in a BRV represents that a transition from ν_1 to ν_2 is possible, i.e., that when given ν_1 as input, the values of the variables can be updated to ν_2. The operations defined above correspond to manipulations/combinations of such dynamic systems. Union, for instance, represents a non-deterministic choice, while composition corresponds to composing two such systems. Cylindrification corresponds, in the dynamic view, to following the underlying BRV followed by ignoring (erasing) some value either in the inputs or the outputs. The selection operations correspond to performing checks, on the input, the output, or a combination of both, after performing what the underlying BRV does.

Some of the above operators are redundant, in the sense that they can be expressed in terms of others, for instance, $A \cap B = A - (A - B)$. We also have:

Lemma 2.1. *For any* BRV *A, and any variables x and y, the following hold:*

$$\mathrm{cyl}^r_x(A) = (\mathrm{cyl}^l_x(A^\smile))^\smile$$
$$\mathrm{cyl}^l_x(A) = (\mathrm{cyl}^r_x(A^\smile))^\smile$$
$$\sigma^r_{x=y}(A) = A \cap \mathrm{cyl}^l_x \sigma^{lr}_{(x,x)=(y,x)} \mathrm{cyl}^l_x(A)$$
$$\sigma^l_{x=y}(A) = A \cap \mathrm{cyl}^r_x \sigma^{lr}_{(y,x)=(x,x)} \mathrm{cyl}^r_x(A)$$
$$\sigma^l_{x=y}(A) = \sigma^r_{x=y}(A^\smile)^\smile$$

The expression for $\sigma^r_{x=y}$ can be explained as follows. First, we copy x from right to left by applying cyl^l_x followed by $\sigma^{lr}_{x=x}$. Selection $\sigma^r_{x=y}$ can now be simulated by $\sigma^{lr}_{x=y}$. The original x value on the left is restored by a final application of cyl^l_x and intersecting with the original A.

2.2 The Logic of Information Flows

The language of LIF expressions α over a vocabulary $\mathcal{S}$ is defined by the following grammar:

$$\alpha ::= id \mid M(\overline{z}) \mid \alpha \cup \alpha \mid \alpha \cap \alpha \mid \alpha - \alpha \mid \alpha \,;\alpha \mid \alpha^\smile \mid \mathrm{cyl}^l_Z(\alpha) \mid \mathrm{cyl}^r_Z(\alpha) \mid \sigma^{lr}_{x=y}(\alpha) \mid \sigma^l_{x=y}(\alpha) \mid \sigma^r_{x=y}(\alpha)$$

Here, M is any module name in $\mathcal{S}$; Z is a finite set of variables; $\overline{z}$ is a tuple of variables; and x, y are variables. For *atomic module expressions*, i.e., expressions of the form $M(\overline{z})$, the length of $\overline{z}$ must equal $ar(M)$. In practice, we will often write $M(\overline{x}; \overline{y})$ for atomic module expressions, where $\overline{x}$ is a tuple of variables of length $iar(M)$ and $\overline{y}$ is a tuple of variables of length $ar(M) - iar(M)$.

We will define the semantics of a LIF expression α, in the context of a given interpretation D, as a BRV which will be denoted by $[\![\alpha]\!]_D$. Thus, adapting Gurevich's terminology [13, 14], the semantics $[\![\alpha]\!]$ of a LIF expression α is a *global* BRV. Formally, we define a global BRV to be a function Q that maps interpretations D of $\mathcal{S}$ to BRVs. Thus, $[\![\alpha]\!]$ corresponds to the global BRV Q where $Q(D) = [\![\alpha]\!]_D$.

For atomic module expressions, we define

$$[\![M(\overline{x};\overline{y})]\!]_D := \{(\nu_1,\nu_2) \in \mathcal{V}\times\mathcal{V} \mid \nu_1(\overline{x})\cdot\nu_2(\overline{y}) \in D(M) \text{ and } \nu_1 \text{ and } \nu_2 \text{ agree outside } \overline{y}\}.$$

Here, $\nu_1(\overline{x})\cdot\nu_2(\overline{y})$ denotes the *concatenation* of tuples. Intuitively, the semantics of an expression $M(\overline{x};\overline{y})$ represents a transition from ν_1 to ν_2: the inputs of the module are "read" in ν_1 and the outputs are updated in ν_2. The value of every variable that is not an output is preserved; this important semantic principle is a realization of the commonsense law of *inertia* [19, 20].

We further define

$$[\![id]\!]_D := \{(\nu,\nu) \mid \nu \in \mathcal{V}\}.$$

The semantics of other operators is obtained directly by applying the corresponding operation on BRVs, e.g.,

$$[\![\alpha - \beta]\!]_D := [\![\alpha]\!]_D - [\![\beta]\!]_D.$$
$$[\![\sigma^{lr}_{x=y}(\alpha)]\!]_D := \sigma^{lr}_{x=y}([\![\alpha]\!]_D).$$

We say that α and β are *equivalent* if $[\![\alpha]\!]_D = [\![\beta]\!]_D$ for each interpretation D, i.e., if they denote the same global BRV.

2.3 Satisfiability of LIF Expressions

In this section, we will show that the problem of deciding whether a given LIF expression is satisfiable is undecidable. Thereto we begin by noting that first-order logic (FO) is naturally embedded in LIF in the following manner. When evaluating FO formulas on interpretations, we agree that the domain of quantification is always **dom**.

Lemma 2.2. *Let $\mathcal{S}$ be a vocabulary with $iar(R) = 0$ for every $R \in \mathcal{S}$. Then, for every FO formula φ over $\mathcal{S}$, there exists a LIF expression α_φ such that for every interpretation D the following holds:*

$$[\![\alpha_\varphi]\!]_D = \{(\nu,\nu) \mid D,\nu \models \varphi\}.$$

Proof. The proof is by structural induction on φ.

- If φ is $x = y$, take $\alpha_\varphi = \sigma^r_{x=y}(id)$.
- If φ is $R(\bar{x})$ for some $R \in \mathcal{S}$, take $\alpha_\varphi = id \cap R(;\bar{x})$.
- If φ is $\varphi_1 \vee \varphi_2$, take $\alpha_\varphi = \alpha_{\varphi_1} \cup \alpha_{\varphi_2}$.
- If φ is $\neg\varphi_1$, take $\alpha_\varphi = id - \alpha_{\varphi_1}$.
- If φ is $\exists x\, \varphi_1$, take $\alpha_\varphi = \sigma^{lr}_{x=x}(\mathrm{cyl}^l_x(\mathrm{cyl}^r_x(\alpha_{\varphi_1})))$. □

It is well known that satisfiability of FO formulas over a fixed countably infinite domain is undecidable. This leads to the following undecidability result.

Problem: **Satisfiability**
Given: a LIF expression α.
Decide: Is there an interpretation D such that $[\![\alpha]\!]_D \neq \emptyset$?

Proposition 2.3. *The satisfiability problem is undecidable.*

Proof. The proof is by reduction from the satisfiability of FO formulas. Let φ be an FO formula and let α_φ be the LIF expression obtained from Lemma 2.2. It is clear that α_φ is satisfiable if and only if φ is. □

3 INPUTS AND OUTPUTS

We are now ready to study inputs and outputs of LIF expressions, and, more generally, of global BRVs. We first investigate what inputs and outputs mean on the semantic level before introducing a syntactic definition for LIF expressions.

3.1 Semantic Inputs and Outputs for Global BRVs

Intuitively, an output is a variable whose value can be changed by the expression, i.e., a variable that is not subject to inertia.

Definition 3.1. A variable x is a *semantic output* for a global BRV Q if there exists an interpretation D and $(\nu_1, \nu_2) \in Q(D)$ such that $\nu_1(x) \neq \nu_2(x)$. We use $O^{\text{sem}}(Q)$ to denote the set of semantic output variables of Q. If α is a LIF expression, we call a variable a *semantic output* of α if it is a semantic output of $[\![\alpha]\!]$. We also write $O^{\text{sem}}(\alpha)$ for the semantic outputs of α. A variable that is not a semantic output is also called an *inertial variable*.

Defining semantic inputs is a bit more subtle. Intuitively, a variable is an input for a BRV if its value on the left-hand side matters for determining the right-hand side (i.e., that if the value of the input would have been different, so would have been the right-hand side; which is in fact a very coarse counterfactual definition of actual causality [18]). However, a naive formalization of this intuition would result in a situation in which all inertial variables (variables that are not outputs) are inputs since their value on the right-hand side always equals to the one on the left-hand side. A slight refinement of our intuition is that the inputs are those variables whose value matters for determining the possible values of the outputs. This is formalized in the following definitions.

Definition 3.2. Let Q be a global BRV and X, Y be sets of variables. We say that X *determines* Q on Y if for every interpretation D, every $(\nu_1, \nu_2) \in Q(D)$ and every ν_1' such that $\nu_1' = \nu_1$ on X, there exists a ν_2' such that $\nu_2' = \nu_2$ on Y and $(\nu_1', \nu_2') \in Q(D)$.

Definition 3.3. A variable x is a *semantic input* for a global BRV Q if $\mathbb{V} - \{x\}$ does not determine Q on $O^{\text{sem}}(Q)$. The set of input variables of Q is denoted by $I^{\text{sem}}(Q)$. A variable is a *semantic input* of a LIF expression α if it is a semantic input of $[\![\alpha]\!]$; the semantic inputs of α are denoted by $I^{\text{sem}}(\alpha)$.

From Definition 3.2, we can rephrase the definition for semantic inputs to:

Proposition 3.4. *A variable x is a semantic input for a global* BRV *Q if and only if there is an interpretation D, a value $d \in$* **dom***, and $(\nu_1, \nu_2) \in Q(D)$ such that there is no valuation ν_2' that agrees with ν_2 on $O^{\text{sem}}(Q)$ and $(\nu_1[x : d], \nu_2') \in Q(D)$.*

The following proposition shows that the semantic inputs of Q are indeed exactly the variables that determine Q.

Proposition 3.5. *If a set of variables X determines a global* BRV *Q on $O^{\text{sem}}(Q)$, then $I^{\text{sem}}(Q) \subseteq X$.*

Proof. Let v be any variable in $I^{\text{sem}}(Q)$. We know that $\mathbb{V} - \{v\}$ does not determine Q on $O^{\text{sem}}(Q)$. If $v \notin X$, then $X \subseteq \mathbb{V} - \{v\}$, so X would not determine Q on $O^{\text{sem}}(Q)$, which is impossible. Hence, v must be in X as desired. □

Under a mild assumption, also the converse to Proposition 3.5 holds:[1]

Proposition 3.6. *Assume there exists a finite set of variables that determines a global* BRV Q *on* $O^{\text{sem}}(Q)$*. Then,* $I^{\text{sem}}(Q)$ *determines* Q *on* $O^{\text{sem}}(Q)$*.*

Proof. Let $(\nu_1, \nu_2) \in Q(D)$ and $\nu_1' = \nu_1$ on $I^{\text{sem}}(Q)$ for some interpretation D and valuations ν_1, ν_2, and ν_1'. To show that $I^{\text{sem}}(Q)$ determines Q on $O^{\text{sem}}(Q)$, we need to find a valuation ν_2' such that $(\nu_1', \nu_2') \in Q(D)$ and $\nu_2' = \nu_2$ on $O^{\text{sem}}(Q)$. By assumption, let X be a finite set of variables that determines Q on $O^{\text{sem}}(Q)$.

Thereto, take ν_1'' to be the valuation $\nu_1'[\nu_1|_X]$ which is the valuation ν_1' after changing the values for the variables in X to be as in ν_1. Thus, $\nu_1'' = \nu_1$ on X, while $\nu_1'' = \nu_1'$ outside X. Since X determines Q on $O^{\text{sem}}(Q)$, we know that there is a valuation ν_2'' such that $(\nu_1'', \nu_2'') \in Q(D)$ and $\nu_2'' = \nu_2$ on $O^{\text{sem}}(Q)$. To reach our goal, we would like to do incremental changes to ν_1'' in order to be similar to ν_1' while showing that each of the intermediate valuations does satisfy the determinacy conditions.

From construction, we know that $\nu_1'' = \nu_1'$ on $X \cap I^{\text{sem}}(Q)$. Using the finiteness assumption for X, let $X - I^{\text{sem}}(Q)$ be the set of variables $\{x_1, \dots, x_n\}$. Define the sequence of valuations $\mu_0, \mu_1, \dots, \mu_n$ such that

- $\mu_0 := \nu_1''$; and
- $\mu_i := \mu_{i-1}[\{x_i \mapsto \nu_1'(x_i)\}]$ so μ_i is μ_{i-1} after changing the value of x_i to be as in ν_1'.

We **claim** that for $i \in \{0, \dots, n\}$, there exists a valuation κ_i such that $(\mu_i, \kappa_i) \in Q(D)$ and $\kappa_i = \nu_2$ on $O^{\text{sem}}(Q)$. Since μ_n is clearly the same valuation as ν_1', we can then take ν_2' to be κ_n which is the required.

We verify our **claim** by induction.

Base Case: for $i = 0$, we can see that $\kappa_0 = \nu_2''$.

Inductive Step: for $i > 0$, by assumption, we know that there is a valuation κ_{i-1} such that $(\mu_{i-1}, \kappa_{i-1}) \in Q(D)$ and $\kappa_{i-1} = \nu_2$ on $O^{\text{sem}}(Q)$. It is clear that $\mu_i = \mu_{i-1}$ outside $\{x_i\}$ which is $\mathbb{V} - \{x_i\}$. Since $x_i \notin I^{\text{sem}}(Q)$, we know that $\mathbb{V} - \{x_i\}$ determines Q on $O^{\text{sem}}(Q)$. Hence, there is a valuation κ_i such that $(\mu_i, \kappa_i) \in Q(D)$ and $\kappa_i = \kappa_{i-1}$ on $O^{\text{sem}}(Q)$. Since $\kappa_{i-1} = \nu_2$ on $O^{\text{sem}}(Q)$, we can see that $\kappa_i = \nu_2$ on $O^{\text{sem}}(Q)$. □

In the next remark, we show that without our assumption, we can find an example of a global BRV that is not determined on its semantic outputs by its semantic inputs.

Remark 3.7. Let Q be the global BRV that maps every D to the same BRV, namely:

$$Q(D) = \{(\nu_1, \nu_2) \in \mathcal{V} \times \mathcal{V} \mid \nu_1 \text{ and } \nu_2 \text{ differ on finitely many variables}\}.$$

Since the variables that can be changed by Q are not restricted, we see that $O^{\text{sem}}(Q) = \mathbb{V}$. We now verify that $I^{\text{sem}}(Q) = \emptyset$. Let v be any variable. We can see that $v \notin I^{\text{sem}}(Q)$. Thereto, we check that $\mathbb{V} - \{v\}$ determines Q on $O^{\text{sem}}(Q)$. Let D be an interpretation and ν_1, ν_2, and ν_1' valuations such that $(\nu_1, \nu_2) \in Q(D)$ and $\nu_1' = \nu_1$ outside $\{v\}$. Since ν_1 and ν_2 differ on finitely many variables, we can see that ν_1' and ν_2 also do. Hence, $(\nu_1', \nu_2) \in Q(D)$.

Finally, we verify that $I^{\text{sem}}(Q)$ does not determine Q on $O^{\text{sem}}(Q)$. To see a counterexample, let D be an interpretation and ν_1 be the valuation that assigns 1 to every variable. We can see that $(\nu_1, \nu_1) \in Q(D)$. Let ν_2 be the valuation that assigns 2 to every variable. It is clear that $\nu_2 = \nu_1$ on $I^{\text{sem}}(Q) = \emptyset$, however, clearly $(\nu_2, \nu_1) \notin Q(D)$. By Proposition 3.6, we know that there is no finite set of variables that does determine Q on $O^{\text{sem}}(Q)$. □

[1] Proposition 3.6 indeed provides a converse to Proposition 3.5: given that $I^{\text{sem}}(Q)$ determines Q on $O^{\text{sem}}(Q)$ and $I^{\text{sem}}(Q) \subseteq X$ for some set X, clearly also X determines Q on $O^{\text{sem}}(Q)$.

The reader should not be lulled into believing that $I^{\text{sem}}(Q)$ determines a global BRV Q on the set $\mathbb{V}$ of *all* variables since $I^{\text{sem}}(Q)$ determines Q on $O^{\text{sem}}(Q)$ and no other variable outside $O^{\text{sem}}(Q)$ can have its value changed. In the following remark, we give a simple counterexample.

Remark 3.8. We show that $I^{\text{sem}}(Q)$ does not necessarily determine Q on $\mathbb{V}$ for every global BRV Q. Take Q to be defined by the LIF expression $\sigma^{l}_{x=y}(id)$, so, for every D, we have

$$Q(D) = \{(\nu, \nu) \mid \nu \in \mathcal{V} \text{ such that } \nu(x) = \nu(y)\}.$$

It is clear that $O^{\text{sem}}(Q) = \emptyset$ and $I^{\text{sem}}(Q) = \{x, y\}$.

Let ν_1 be the valuation that assigns 1 to every variable. Clearly, $(\nu_1, \nu_1) \in Q(D)$ for any interpretation D. Now take ν'_1 to be the valuation that assigns 1 to x and y, while it assigns 2 to every other variable. It is clear that $\nu'_1 = \nu_1$ on $I^{\text{sem}}(Q)$.

If $I^{\text{sem}}(Q)$ were to determine Q on $\mathbb{V}$, we should find a valuation ν_2 that agrees with ν_1 on $\mathbb{V}$ such that $(\nu'_1, \nu_2) \in Q(D)$. In other words, this means that $(\nu'_1, \nu_1) \in Q(D)$, which is clearly not possible. □

Following the above remark, we note that in order to determine a global BRV Q on the set $\mathbb{V}$ of *all* variables we need to know the values of the input variables of Q along with the values of the variables outside the outputs of Q. In the following proposition, we show that when the inputs determine Q on the outputs, also a variant of the notion of determinacy holds, compared to the one we defined in Definition 3.2.

Proposition 3.9. *Let Q be a global* BRV. *If $I^{\text{sem}}(Q)$ determines Q on $O^{\text{sem}}(Q)$, then for every interpretation D, every $(\nu_1, \nu_2) \in Q(D)$ and every ν'_1 that agrees with ν_1 on $I^{\text{sem}}(Q)$ and outside $O^{\text{sem}}(Q)$, we have $(\nu'_1, \nu_2) \in Q(D)$.*

Proof. Suppose that

$$(\nu_1, \nu_2) \in Q(D) \text{ and } \nu'_1 = \nu_1 \text{ on } I^{\text{sem}}(Q) \text{ and outside } O^{\text{sem}}(Q). \tag{1}$$

By definition of $O^{\text{sem}}(Q)$, all variables outside $O^{\text{sem}}(Q)$ are inertial, so

$$\nu_1 = \nu_2 \text{ outside } O^{\text{sem}}(Q). \tag{2}$$

Since $I^{\text{sem}}(Q)$ determines Q on $O^{\text{sem}}(Q)$, by Definition 3.2 there exists ν'_2 such that

(i) $\nu'_2 = \nu_2$ on $O^{\text{sem}}(Q)$;
(ii) $(\nu'_1, \nu'_2) \in Q(D)$.

From (ii) and the definition of semantic outputs, we have

$$\nu'_1 = \nu'_2 \text{ outside } O^{\text{sem}}(Q). \tag{3}$$

Chaining together Equations 3, 1 and 2, we obtain that $\nu'_2 = \nu_2$ outside $O^{\text{sem}}(Q)$. Combining this with (i), we obtain $\nu_2 = \nu'_2$. Whence, by (ii), $(\nu'_1, \nu_2) \in Q(D)$ as desired. □

Intuitively, the inputs and outputs are the only variables that matter for a given global BRV, similar to how in classical logic the free variables are the only ones that matter. All other variables can take arbitrary values, *but*, their values are preserved by inertia, i.e., remain unchanged by the dynamic system. We now formalize this intuition.

Definition 3.10. Let Q be a global BRV and X a set of variables. We say that Q is *inertially cylindrified* on X if:

(1) all variables in X are inertial; and
(2) for every interpretation D, every $(\nu_1, \nu_2) \in Q(D)$, and every X-valuation ν' also $(\nu_1[\nu'], \nu_2[\nu']) \in Q(D)$.

Proposition 3.11. *Every global* BRV Q *is inertially cylindrified outside the semantic inputs and outputs of* Q *assuming that* $I^{\text{sem}}(Q)$ *determines* Q *on* $O^{\text{sem}}(Q)$.

Proof. Let Q be a global BRV such that $I^{\text{sem}}(Q)$ determines Q on $O^{\text{sem}}(Q)$. Moreover, let X be the set of variables that are neither semantic inputs nor semantic outputs of Q. It is trivial to show that all the variables in X are inertial since none of the variables in X is a semantic output of Q. What remains to show is that for every interpretation D, every $(\nu_1, \nu_2) \in Q(D)$, and every X-valuation ν' also $(\nu_1[\nu'], \nu_2[\nu']) \in Q(D)$.

Let $(\nu_1, \nu_2) \in Q(D)$ for an arbitrary interpretation D and let $\nu_1' = \nu_1[\nu']$ be a valuation for some X-valuation ν'. Since $\nu_1' = \nu_1$ on $I^{\text{sem}}(Q)$, we know by determinacy that there is a valuation ν_2' such that $(\nu_1', \nu_2') \in Q(D)$ and $\nu_2' = \nu_2$ on $O^{\text{sem}}(Q)$. We now argue that $\nu_2' = \nu_2[\nu']$. On the variables of $O^{\text{sem}}(Q)$, we know that $\nu_2 = \nu_2[\nu']$, whence, $\nu_2' = \nu_2[\nu']$ on $O^{\text{sem}}(Q)$. Now we consider the variables that are not in $O^{\text{sem}}(Q)$. It is clear that $\nu_1 = \nu_2$ outside $O^{\text{sem}}(Q)$, whence, $\nu_1[\nu'] = \nu_1' = \nu_2' = \nu_2[\nu']$ outside $O^{\text{sem}}(Q)$. □

Remark 3.12. Without the assumption, we can give an example of a global BRV that is not inertially cylindrified outside its semantic inputs and outputs. Let Q be the global BRV that maps every D to the same BRV, namely:

$$Q(D) = \{(\nu, \nu) \mid \nu \in \mathcal{V} \text{ and no value in the domain occurs infinitely often in } \nu\}.$$

It is clear that $O^{\text{sem}}(Q) = \emptyset$ and $I^{\text{sem}}(Q) = \emptyset$.

We proceed to verify that Q is not inertially cylindrified on $\mathbb{V}$. Let D be any interpretation and ν be any valuation that maps every variable to a unique value from the domain. We can see that $(\nu, \nu) \in Q(D)$ since every value in ν appears only once. Now fix some $a \in \mathbf{dom}$ arbitrarily and consider the valuation ν' that maps every variable to a. We can see that $(\nu[\nu'], \nu[\nu']) \notin Q(D)$ since a appears infinitely often in $\nu' = \nu[\nu']$. □

We remark that the converse of Proposition 3.11 is not true:

Remark 3.13. Consider the same global BRV Q discussed in Remark 3.7 where we showed that $I^{\text{sem}}(Q)$ does not determine Q on $O^{\text{sem}}(Q)$. Recall that $O^{\text{sem}}(Q) = \mathbb{V}$, so the set of variables outside the semantic inputs and outputs is empty. Trivially, however, Q is inertially cylindrified on $\emptyset$.

3.2 Semantic Inputs and Outputs for LIF Expressions

As we have discussed in the previous section, if there is a finite set of variables that determines a global BRV on its semantic outputs, then the global BRV has the properties of determinacy and inertial cylindrification. Indeed, that is the assumption made in Proposition 3.6 from which results like Proposition 3.9 and Proposition 3.11 follow. For the correctness of our later arguments it is important to emphasize that this assumption is indeed satisfied for global BRVs that are the semantics of LIF expressions. Indeed, in Section 3.3, we will show that there does exist, for every LIF expression α, a finite set of variables, what we will call the "syntactic input variables" of α, that does determine $[\![\alpha]\!]$ on a set of "syntactic output variables", which will include $O^{\text{sem}}(\alpha)$ as desired.

For atomic LIF expressions, the semantic inputs and outputs are easy to determine, as we will show first. Unfortunately, we show next that the problem is undecidable for general expressions.

We show that semantic inputs and outputs are exactly what one expects for atomic modules:

Proposition 3.14. *If* α *is an atomic LIF expressions* $M(\bar{x}; \bar{y})$, *then*

- $I^{\text{sem}}(\alpha) = \{x_i \mid \bar{x} = x_1, \ldots, x_n \text{ for } i \in \{1, \ldots, n\}\}$; *and*
- $O^{\text{sem}}(\alpha) = \{y_i \mid \bar{y} = y_1, \ldots, y_m \text{ for } i \in \{1, \ldots, m\}\}$.

Example 3.15. A variable can be both input and output of a given expression. A very simple example is an atomic module $P_1(x; x)$. To illustrate where this can be useful, assume **dom** $= \mathbb{Z}$ and consider an interpretation D such that $D(P_1) = \{(n, n+1) \mid n \in \mathbb{Z}\}$. In that case, the expression $P_1(x; x)$ represents a dynamic system in which the value of x is incremented by 1; x is an output of the system since its value is changed; it is an input since its original value matters for determining its value in the output.

We will now show that the problem of deciding whether a given variable is a semantic input or output of a LIF expression is undecidable. Proposition 2.3 showed that satisfiability of LIF expressions is undecidable. This leads to the following undecidability results.

Problem: **Semantic Output Membership**
Given: a variable x and a LIF expression α.
Decide: $x \in O^{\text{sem}}(\alpha)$?

Proposition 3.16. *The semantic output membership problem is undecidable.*

Proof. The proof is by reduction from the satisfiability of LIF expressions. Let α be a LIF expression. Take β to be $\text{cyl}^l_x(\alpha)$. What remains to show is that $x \in O^{\text{sem}}(\beta) \Leftrightarrow \alpha$ is satisfiable.

($\Rightarrow$) Let $x \in O^{\text{sem}}(\beta)$. Then, there is certainly an interpretation D and valuations ν_1 and ν_2 such that $(\nu_1, \nu_2) \in [\![\text{cyl}^l_x(\alpha)]\!]_D$. Hence, there is also a valuation ν_1' such that $(\nu_1', \nu_2) \in [\![\alpha]\!]_D$. Certainly, α is satisfiable.

($\Leftarrow$) Let α be satisfiable. Then, there is an interpretation D and valuations ν_1 and ν_2 such that $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$. Also, let ν' be an $\{x\}$-valuation that maps x to a with $a \neq \nu_2(x)$. It is clear then that $(\nu_1[\nu'], \nu_2) \in [\![\text{cyl}^l_x(\alpha)]\!]_D$. We thus see that $x \in O^{\text{sem}}(\beta)$. □

Problem: **Semantic Input Membership**
Given: a variable x and a LIF expression α.
Decide: $x \in I^{\text{sem}}(\alpha)$?

Proposition 3.17. *The semantic input membership problem is undecidable.*

Proof. Let α be a LIF expression. Take β to be $\sigma^l_{x=z}(\alpha)$, where z is a variable that is not used in α and different from x. What remains to show is that $x \in I^{\text{sem}}(\beta) \Leftrightarrow \alpha$ is satisfiable.

($\Rightarrow$) Let $x \in I^{\text{sem}}(\beta)$. Then, certainly, there is an interpretation D and valuations ν_1 and ν_2 such that $(\nu_1, \nu_2) \in [\![\sigma^l_{x=z}(\alpha)]\!]_D \subseteq [\![\alpha]\!]_D$. Certainly, α is satisfiable.

($\Leftarrow$) Let α be satisfiable. Then, there is an interpretation D and valuations ν_1 and ν_2 such that $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$. Without loss of generality, we can assume that $\nu_1(z) = \nu_1(x)$ since z is a fresh variable. Hence, $(\nu_1, \nu_2) \in [\![\sigma^l_{x=z}(\alpha)]\!]_D$. Let ν_1' be a valuation that agrees with ν_1 outside x such that $\nu_1'(x) \neq \nu_1(x)$. Since x and z are different variables, also $\nu_1'(x) \neq \nu_1'(z)$, so clearly there is no valuation ν_2' such that $(\nu_1', \nu_2') \in [\![\sigma^l_{x=z}(\alpha)]\!]_D$. We then see that $x \in I^{\text{sem}}(\beta)$. □

3.3 Syntactic Inputs and Outputs

Since the membership problems for both semantic inputs and semantic outputs are undecidable, to determine inputs and outputs in practice, we will need decidable approximations of these concepts. Before giving our syntactic definition, we define some properties of candidate definitions.

Definition 3.18. Let I and O be functions from LIF expressions to sets of variables. We say that (I, O) is a *sound input–output definition* if the following hold:

- If $\alpha = M(\overline{x}; \overline{y})$, then $I(\alpha) = \overline{x}$ and $O(\alpha) = \overline{y}$,

- $O(\alpha) \supseteq O^{\text{sem}}(\alpha)$, and
- $I(\alpha)$ determines $[\![\alpha]\!]$ on $O(\alpha)$.

The first condition states that on atomic expressions (of which we know the inputs), I and O are defined correctly. The next condition states O approximates the semantic notion correctly. We only allow for overapproximations; that is, false positives are allowed while false negatives are not. The reason for this is that falsely marking a variable as non-output while it is actually an output would mean incorrectly assuming the variable cannot change value. The last condition establishes the relation between I and O, and is called *input–output determinacy*. It states that the inputs need to be large enough to determine the outputs, as such generalizing the defining condition of semantic inputs.

We first remark that a proposition similar to Proposition 3.5 can be made about sound output definitions.

Proposition 3.19. *Let (I, O) be a sound input-output definition, α be a LIF expression, and X a set of variables. If X determines $[\![\alpha]\!]$ on $O(\alpha)$, then $I^{\text{sem}}(\alpha) \subseteq X$.*

Proof. By soundness, we know that $O^{\text{sem}}(\alpha) \subseteq O(\alpha)$. It follows that X determines $[\![\alpha]\!]$ on $O^{\text{sem}}(\alpha)$. Indeed, in general, if X determines $[\![\alpha]\!]$ on some Y, then clearly also X determines $[\![\alpha]\!]$ on Z for any set Z such that $Z \subseteq Y$. By Proposition 3.5, then, we obtain $I^{\text{sem}}(\alpha) \subseteq X$. □

This proposition along with the input-output determinacy condition imply a condition similar to the second condition about the inputs:

Proposition 3.20. *Let (I, O) be a sound input-output definition and α be a LIF expression. Then, $I(\alpha) \supseteq I^{\text{sem}}(\alpha)$.*

Proof. The proof follows from Proposition 3.19 and knowing that $I(\alpha)$ determines $[\![\alpha]\!]$ on $O(\alpha)$. □

Besides requiring that our definitions to be sound, we will focus on definitions that are *compositional*, in the sense that definitions of inputs and outputs of compound expressions can be given in terms of their direct subexpressions essentially treating subexpressions as black boxes. This means that the definition nicely follows the inductive definition of the syntax. Formally,

Definition 3.21. Suppose I and O are functions from LIF expression to sets of variables. We say that (I, O) is *compositional* if for all LIF expressions α_1, α_2, β_1, and β_2 with $I(\alpha_1) = I(\alpha_2)$, $O(\alpha_1) = O(\alpha_2)$, $I(\beta_1) = I(\beta_2)$, and $O(\beta_1) = O(\beta_2)$ the following hold:

- For every unary operator $\square$: $I(\square\alpha_1) = I(\square\alpha_2)$, and $O(\square\alpha_1) = O(\square\alpha_2)$; and
- For every binary operator $\boxdot$: $I(\alpha_1 \boxdot \beta_1) = I(\alpha_2 \boxdot \beta_2)$, and $O(\alpha_1 \boxdot \beta_1) = O(\alpha_2 \boxdot \beta_2)$.

The previous definition essentially states that in order to be compositional, the inputs and outputs of $\alpha_1 \boxdot \beta_1$ and $\square\alpha_1$ should only depend on the inputs and outputs of α_1 and β_1, and not on their inner structure.

The following lemma rephrases input–output determinacy in terms of the inputs and outputs: in order to determine the output-value of an inertial variable, we need to know its input-value.

Lemma 3.22. *Let (I, O) be a sound input–output definition and let α be a LIF expression.*

(1) If α is satisfiable, then $O(\alpha) - O^{\text{sem}}(\alpha) \subseteq I(\alpha)$.
(2) Moreover, if (I, O) is compositional, then $O(\alpha) - O^{\text{sem}}(\alpha) \subseteq I(\alpha)$ holds even if α is not satisfiable.

Proof. Let $x \in O(\alpha) - O^{\text{sem}}(\alpha)$. For the sake of contradiction, assume that $x \notin I(\alpha)$, so Proposition 3.11 is applicable since $x \notin I^{\text{sem}}(\alpha)$ as we know by the soundness of (I, O). Hence, $[\![\alpha]\!]$ is inertially cylindrified on $\{x\}$. We claim that this contradicts the fact that (I, O) is a sound definition. In particular, we can verify that $I(\alpha)$ can not determine $[\![\alpha]\!]$ on $O(\alpha)$ in case α is satisfiable. Let D be an interpretation and ν_1 and ν_2 be valuations such that $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$. We also know that $\nu_1(x) = \nu_2(x)$ since $[\![\alpha]\!]$ is inertially cylindrified on $\{x\}$. Take ν_1' to be the valuation $\nu_1[\{x \mapsto a\}]$ where $a \neq \nu_1(x)$. By determinacy, we know that there is a valuation ν_2' such that $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$ and $\nu_2' = \nu_2$ on $O(\alpha)$. Thus, $\nu_2'(x) = \nu_2(x) \neq a$ since $x \in O(\alpha)$. On the other hand, $\nu_2'(x) = \nu_1'(x) = a$ since $[\![\alpha]\!]$ is inertially cylindrified on $\{x\}$. Hence, a contradiction.

For the compositional case, we can always replace subexpressions by atomic expressions with the same inputs and outputs to ensure satisfiability. It is clear that when α is an atomic module expression, it is always satisfiable. Now, consider any LIF expression α which is of the form $\square\alpha_1$ or $\alpha_1 \boxdot \alpha_2$, where $\square$ is any of the unary operators and $\boxdot$ is any of the binary operator. Construct two atomic expressions M_1 and M_2 such that $I(M_i) = I(\alpha_i)$ and $O(M_i) = O(\alpha_i)$ for $i = 1, 2$. By compositionality, we know that $I(\square\alpha_1) = I(\square M_1)$ and $O(\square\alpha_1) = O(\square M_1)$ for any unary operator, while $I(\alpha_1 \boxdot \alpha_2) = I(M_1 \boxdot M_2)$ and $O(\alpha_1 \boxdot \alpha_2) = O(M_1 \boxdot M_2)$ for any binary operator. Next, we give examples for an interpretation D in which any $[\![\square M_1]\!]_D$ and $[\![M_1 \boxdot M_2]\!]_D$ can be shown not be empty, so $\square M_1$ and $M_1 \boxdot M_2$ are satisfiable expressions.

In what follows, let ν_1 be the valuation that assigns 1 to every variable.

Case $\boxdot$ is $-$. Let D be the interpretation with

- $D(M_1) = \{(1, \ldots, 1; 1, \ldots, 1)\}$; and
- $D(M_2) = \emptyset$.

It is clear that $(\nu_1, \nu_1) \in [\![M_1]\!]_D$, and $(\nu_1, \nu_1) \notin [\![M_2]\!]_D$, whence, $(\nu_1, \nu_1) \in [\![M_1 - M_2]\!]_D$.

All other cases. Let D be the interpretation with

- $D(M_1) = \{(1, \ldots, 1; 1, \ldots, 1)\}$; and
- $D(M_2) = \{(1, \ldots, 1; 1, \ldots, 1)\}$.

We can see that $(\nu_1, \nu_1) \in [\![M_1]\!]_D$. Consequently, $(\nu_1, \nu_1) \in [\![\square M_1]\!]_D$ for any unary operator $\square$. We can also see that $(\nu_1, \nu_1) \in [\![M_2]\!]_D$, whence, $(\nu_1, \nu_1) \in [\![M_1 \boxdot M_2]\!]_D$ for any binary operator $\boxdot \in \{\cup, \cap, ;\}$. □

We now provide a sound and compositional input–output definition. While the definition might seem complex, there is a good reason for the different cases. Indeed, as we show below in Theorem 3.28, our definition is optimal among the sound and compositional definitions. In the definition, the condition $x =_{\text{syn}} y$ simply means that x and y are the same variable. Moreover, $\triangle$ denotes the symmetric difference of two sets, precisely, $A \triangle B := (A \cup B) - (A \cap B)$.

Definition 3.23. The syntactic inputs and outputs of a LIF expression α, denoted $I^{\text{syn}}(\alpha)$ and $O^{\text{syn}}(\alpha)$ respectively, are defined recursively as given in Table 1.

While it would be tedious to discuss the motivation for all the cases of Definition 3.23 (their motivation will be clarified in Theorem 3.28), we discuss here one of the most difficult parts, namely the case where $\alpha = \sigma^{lr}_{x=y}(\alpha_1)$. For a given interpretation D,

$$[\![\alpha]\!]_D = \{(\nu_1, \nu_2) \in [\![\alpha_1]\!]_D \mid \nu_1(x) = \nu_2(y)\}.$$

First, since $[\![\alpha]\!]_D \subseteq [\![\alpha_1]\!]_D$, it is clear that the outputs of α should be a subset of those of α_1 (if α_1 admits no pairs in its semantics that change the value of a variable, then neither does α). For the special case in which x and y are the same variable, this selection enforces x to be inertial, i.e., it should not be an output of α.

Secondly, all inputs of α_1 remain inputs of α. Since we select those pairs whose y-value on the right equals the x-value on the left, clearly x must be an input of α (the special case $x =_{\text{syn}} y$

α	$I^{\mathrm{syn}}(\alpha)$	$O^{\mathrm{syn}}(\alpha)$
id	$\emptyset$	$\emptyset$
$M(\overline{x};\overline{y})$	$\{x_1,\ldots,x_n\}$ where $\overline{x}=x_1,\ldots,x_n$	$\{y_1,\ldots,y_n\}$ where $\overline{y}=y_1,\ldots,y_n$
$\alpha_1\cup\alpha_2$	$I^{\mathrm{syn}}(\alpha_1)\cup I^{\mathrm{syn}}(\alpha_2)\cup(O^{\mathrm{syn}}(\alpha_1)\vartriangle O^{\mathrm{syn}}(\alpha_2))$	$O^{\mathrm{syn}}(\alpha_1)\cup O^{\mathrm{syn}}(\alpha_2)$
$\alpha_1\cap\alpha_2$	$I^{\mathrm{syn}}(\alpha_1)\cup I^{\mathrm{syn}}(\alpha_2)\cup(O^{\mathrm{syn}}(\alpha_1)\vartriangle O^{\mathrm{syn}}(\alpha_2))$	$O^{\mathrm{syn}}(\alpha_1)\cap O^{\mathrm{syn}}(\alpha_2)$
$\alpha_1-\alpha_2$	$I^{\mathrm{syn}}(\alpha_1)\cup I^{\mathrm{syn}}(\alpha_2)\cup(O^{\mathrm{syn}}(\alpha_1)\vartriangle O^{\mathrm{syn}}(\alpha_2))$	$O^{\mathrm{syn}}(\alpha_1)$
$\alpha_1\,;\alpha_2$	$I^{\mathrm{syn}}(\alpha_1)\cup(I^{\mathrm{syn}}(\alpha_2)-O^{\mathrm{syn}}(\alpha_1))$	$O^{\mathrm{syn}}(\alpha_1)\cup O^{\mathrm{syn}}(\alpha_2)$
$\alpha_1^{\smile}$	$O^{\mathrm{syn}}(\alpha_1)\cup I^{\mathrm{syn}}(\alpha_1)$	$O^{\mathrm{syn}}(\alpha_1)$
$\mathrm{cyl}^l_x(\alpha_1)$	$I^{\mathrm{syn}}(\alpha_1)-\{x\}$	$O^{\mathrm{syn}}(\alpha_1)\cup\{x\}$
$\mathrm{cyl}^r_x(\alpha_1)$	$I^{\mathrm{syn}}(\alpha_1)$	$O^{\mathrm{syn}}(\alpha_1)\cup\{x\}$
$\sigma^{lr}_{x=y}(\alpha_1)$	$\begin{cases} I^{\mathrm{syn}}(\alpha_1) & \text{if } x=_{\mathrm{syn}}y \text{ and } y\notin O^{\mathrm{syn}}(\alpha_1)\\ I^{\mathrm{syn}}(\alpha_1)\cup\{x,y\} & \text{if } x\neq_{\mathrm{syn}}y \text{ and } y\notin O^{\mathrm{syn}}(\alpha_1)\\ I^{\mathrm{syn}}(\alpha_1)\cup\{x\} & \text{otherwise}\end{cases}$	$\begin{cases} O^{\mathrm{syn}}(\alpha_1)-\{x\} & \text{if } x=_{\mathrm{syn}}y\\ O^{\mathrm{syn}}(\alpha_1) & \text{otherwise}\end{cases}$
$\sigma^{l}_{x=y}(\alpha_1)$	$\begin{cases} I^{\mathrm{syn}}(\alpha_1) & \text{if } x=_{\mathrm{syn}}y\\ I^{\mathrm{syn}}(\alpha_1)\cup\{x,y\} & \text{otherwise}\end{cases}$	$O^{\mathrm{syn}}(\alpha_1)$
$\sigma^{r}_{x=y}(\alpha_1)$	$\begin{cases} I^{\mathrm{syn}}(\alpha_1) & \text{if } x=_{\mathrm{syn}}y\\ I^{\mathrm{syn}}(\alpha_1)\cup(\{x,y\}-O^{\mathrm{syn}}(\alpha_1)) & \text{otherwise}\end{cases}$	$O^{\mathrm{syn}}(\alpha_1)$

Table 1. Syntactic inputs and outputs for LIF expressions.

and $y\notin O^{\mathrm{syn}}(\alpha_1)$ only covers cases where α_1 and α are actually equivalent). Whether y is an input depends on α_1: if $y\notin O^{\mathrm{syn}}(\alpha_1)$, y is inertial. Since we compare the input-value of x with the output-value of y, essentially this is the same as comparing the input-values of both variables, i.e., the value of y on the input-side matters. On the other hand, if $y\in O^{\mathrm{syn}}(\alpha_1)$, the value of y can be changed by α_1 and thus this selection does not force y to be an input.

Our syntactic definition is clearly compositional (since we only use the inputs and outputs of subexpressions). An important result is that our definition is also sound, i.e., our syntactic concepts are overapproximations of the semantic concepts.

Theorem 3.24 (Soundness Theorem). *$(I^{\mathrm{syn}}, O^{\mathrm{syn}})$ is a sound input–output definition.*

Proof. The proof is given in Section 4. □

Of course, since the semantic notions of inputs and outputs are undecidable and our syntactic notions clearly are decidable, expressions exist in which the semantic and syntactic notions do not coincide. We give some examples.

Example 3.25. Consider the LIF expression

$$\alpha := \sigma^l_{x=y}\sigma^r_{x=y}(R(x;y)).$$

In this case, $O^{\mathrm{sem}}(\alpha)=\emptyset$. However, it can be verified that $O^{\mathrm{syn}}(\alpha)=\{x,y\}$.

Example 3.26. Consider the LIF expression

$$\alpha := \sigma^{lr}_{x=x}\text{cyl}^r_x\text{cyl}^l_x(P(x;)).$$

While the expression $P(x;)$ may look erroneous at first sight, it is an allowed expression, where P denotes an atomic module with input arity one and output arity zero.

In this expression, we first cylindrify x on both sides and afterwards only select those pairs that have inertia, therefore, we reach an expression α that is equivalent to *id*. As such, x is inertially cylindrified in α where $x \notin O^{\text{sem}}(\alpha)$ and $x \notin I^{\text{sem}}(\alpha)$. However, $I^{\text{syn}}(\alpha) = \{x\}$.

These examples suggest that our definitions can be improved. Indeed, one can probably keep coming up with ad-hoc but more precise approximations of inputs and outputs for various specific patterns of expressions. Such improvements would not be compositional, as they would be based on inspecting the structure of subexpressions. In the following results, we show that $(I^{\text{syn}}, O^{\text{syn}})$ is actually the most precise sound and compositional input–output definition.

Theorem 3.27 (Precision Theorem). *Let α be a LIF expression that is either atomic, or a unary operator applied to an atomic module expression, or a binary operator applied to two atomic module expressions involving different module names. Then*

$$O^{\text{sem}}(\alpha) = O^{\text{syn}}(\alpha) \textit{ and } I^{\text{sem}}(\alpha) = I^{\text{syn}}(\alpha).$$

Proof. The proof is given in Section 5. □

Now, the precision theorem forms the basis for our main result on syntactic inputs and outputs, which states that Definition 3.23 yields the most precise sound and compositional input–output definition.

Theorem 3.28 (Optimality Theorem). *Suppose (I, O) is a sound and compositional input–output definition. Then for each LIF expression α,*

$$I^{\text{syn}}(\alpha) \subseteq I(\alpha) \textit{ and } O^{\text{syn}}(\alpha) \subseteq O(\alpha).$$

Proof. The proof is given in Section 6. □

4 SOUNDNESS THEOREM PROOF

In this section, we prove Theorem 3.24. Thereto, we need to verify its three conditions for every LIF expression α according to Definition 3.18:

Atomic Module Case: If $\alpha = M(\overline{x};\overline{y})$, then $I^{\text{syn}}(\alpha) = \overline{x}$ and $O^{\text{syn}}(\alpha) = \overline{y}$.
This is clear from the definitions.

Output Approximation: $O^{\text{syn}}(\alpha) \supseteq O^{\text{sem}}(\alpha)$.
The output approximation property is shown in Proposition 4.1, which is given in Section 4.1.

Syntactic Input-Output Determinacy: $I^{\text{syn}}(\alpha)$ determines $[\![\alpha]\!]$ on $O^{\text{syn}}(\alpha)$.
The syntactic input-output determinacy property is shown in Lemma 4.6, which is given in Section 4.3. First, however, in Section 4.2, we need to prove a syntactic version of Proposition 3.11, which will be used in the proof of the syntactic input-output determinacy property.

4.1 Proof of Output Approximation

In this section, we prove:

Proposition 4.1. *Let α be a LIF expression. Then, $O^{\text{sem}}(\alpha) \subseteq O^{\text{syn}}(\alpha)$.*

To prove this proposition, we introduce the following notion which is related to Definition 3.10.

Definition 4.2. A BRV B has inertia outside a set of variables Z if for every $(\nu_1, \nu_2) \in B$, we have $\nu_1 = \nu_2$ outside Z. A global BRV Q has inertia outside a set of variables Z if $Q(D)$ has inertia outside Z for every interpretation D.

Using this notion, Proposition 4.1 can be equivalently formulated as follows.

PROPOSITION 4.3 (INERTIA PROPERTY). *Let α be a LIF expression. Then, $[\![\alpha]\!]$ has inertia outside $O^{\text{syn}}(\alpha)$.*

In the remainder of this section we prove the inertia property by structural induction on the shape of LIF expressions. Also, we remove the superscript from O^{syn} and refer to it simply by O.

4.1.1 Atomic Modules. Let α be of the form $M(\bar{x}; \bar{y})$. Recall that $O(\alpha) = Y$ where Y is the set of variables in $\overline{y}$. The property directly follows from the definition of the semantics for atomic modules.

4.1.2 Identity. Let α be of the form *id*. Recall that $O(\alpha) = \emptyset$. The property directly follows from the definition of *id*.

4.1.3 Union. Let α be of the form $\alpha_1 \cup \alpha_2$. Recall that $O(\alpha) = O(\alpha_1) \cup O(\alpha_2)$. If $(\nu_1, \nu_2) \in [\![\alpha_1 \cup \alpha_2]\!]_D$, then at least one of the following holds:

(1) $(\nu_1, \nu_2) \in [\![\alpha_1]\!]_D$. Then, by induction we know that $\nu_1 = \nu_2$ outside $O(\alpha_1)$. Since $O(\alpha_1) \subseteq O(\alpha_1) \cup O(\alpha_2) = O(\alpha)$, we know that $\nu_1 = \nu_2$ outside $O(\alpha)$.
(2) $(\nu_1, \nu_2) \in [\![\alpha_2]\!]_D$. Similar.

4.1.4 Intersection. Let α be of the form $\alpha_1 \cap \alpha_2$. Recall that $O(\alpha) = O(\alpha_1) \cap O(\alpha_2)$. If $(\nu_1, \nu_2) \in [\![\alpha_1 \cap \alpha_2]\!]_D$, then $(\nu_1, \nu_2) \in [\![\alpha_1]\!]_D$ and $(\nu_1, \nu_2) \in [\![\alpha_2]\!]_D$. By induction, $\nu_1 = \nu_2$ outside $O(\alpha_1)$ and also $\nu_1 = \nu_2$ outside $O(\alpha_2)$. Hence, $\nu_1 = \nu_2$ outside $O(\alpha_1) \cap O(\alpha_2)$.

4.1.5 Composition. Let α be of the form $\alpha_1 \,;\, \alpha_2$. Recall that $O(\alpha) = O(\alpha_1) \cup O(\alpha_2)$. If $(\nu_1, \nu_2) \in [\![\alpha_1 \,;\, \alpha_2]\!]_D$, then there exists a valuation ν such that $(\nu_1, \nu) \in [\![\alpha_1]\!]_D$ and $(\nu, \nu_2) \in [\![\alpha_2]\!]_D$. By induction, $\nu_1 = \nu$ outside $O(\alpha_1)$ and also $\nu = \nu_2$ outside $O(\alpha_2)$. Hence, $\nu_1 = \nu_2 = \nu$ outside $O(\alpha_1) \cup O(\alpha_2)$.

4.1.6 Difference. Let α be of the form $\alpha_1 - \alpha_2$. Recall that $O(\alpha) = O(\alpha_1)$. The proof then follows immediately by induction.

4.1.7 Converse. Let α be of the form $\alpha_1^{\smile}$. Recall that $O(\alpha) = O(\alpha_1)$. The proof is immediate by induction.

4.1.8 Left and Right Selections. Let α be of the form $\sigma^{l}_{x=y}(\alpha_1)$ or $\sigma^{r}_{x=y}(\alpha_1)$. Recall that $O(\alpha) = O(\alpha_1)$. The proof is immediate by induction.

4.1.9 Left-to-Right Selection. Let α be of the form $\sigma^{lr}_{x=y}(\alpha_1)$. Recall the definition:

$$O(\alpha) = \begin{cases} O(\alpha_1) & \text{if } x \neq_{\text{syn}} y \\ O(\alpha_1) - \{x\} & \text{otherwise} \end{cases}$$

If $(\nu_1, \nu_2) \in [\![\sigma^{lr}_{x=y}(\alpha_1)]\!]_D$, then we know that

(1) $(\nu_1, \nu_2) \in [\![\alpha_1]\!]_D$;
(2) $\nu_1(x) = \nu_2(y)$.

By induction from (1), we know that $\nu_1 = \nu_2$ outside $O(\alpha_1)$. Hence, for $x \neq_{\text{syn}} y$ case we are done. In the other case, i.e., when x and y are the same variable, we must additionally show that $\nu_1(x) = \nu_2(x)$. This follows from (2) since now x and y are the same variable.

4.1.10 Right and Left Cylindrifications. Let α be of the form $\mathrm{cyl}^r_x(\alpha_1)$. The case for left cylindrification is analogous. Recall that $O(\alpha) = O(\alpha_1) \cup \{x\}$. If $(\nu_1, \nu_2) \in [\![\mathrm{cyl}^r_x(\alpha_1)]\!]_D$, then there exists ν such that

(1) $(\nu_1, \nu) \in [\![\alpha_1]\!]_D$;
(2) $\nu = \nu_2$ outside $\{x\}$.

By induction from (1), we know that $\nu_1 = \nu$ outside $O(\alpha_1)$. Combining this with (2), we know that $\nu_1 = \nu_2$ outside $O(\alpha_1) \cup \{x\}$ as desired.

4.2 Proof of Syntactic Free Variable Property

Lemma 4.4 (Syntactic Free Variable Property). *Let α be a LIF expression. Denote $I^{\mathrm{syn}}(\alpha) \cup O^{\mathrm{syn}}(\alpha)$ by $\mathit{fvars}(\alpha)$. Then, α is inertially cylindrified on $\mathbb{V} - \mathit{fvars}(\alpha)$.*

In the proof of this Lemma, we will often make use of Lemma 4.5. In what follows, for a set of variables X, we define $\overline{X}$ to be $\mathbb{V} - X$. In the rest of the section, we remove the superscript from O^{syn} and refer to it simply by O.

Lemma 4.5. *Let B be a* BRV *that has inertia on Y. Then, B is inertially cylindrified on Y if and only if B is inertially cylindrified on every $X \subseteq Y$.*

Proof. The 'if'-direction is immediate.[2] Let us now consider the 'only if'. To this end, suppose that $(\nu_1, \nu_2) \in B$ and that ν is a partial valuation on X. Extend ν to a valuation ν' by $\nu' = \nu_1$ on $Y - X$. Since B has inertia on Y, we know that $\nu_1 = \nu_2 = \nu'$ on $Y - X$. Thus, $\nu_1[\nu'] = \nu_1[\nu]$ and $\nu_2[\nu'] = \nu_2[\nu]$. The lemma now directly follows since B is inertially cylindrified on Y. □

This Lemma is always applicable for any LIF expression α and $Y = \mathbb{V} - \mathit{fvars}(\alpha)$. Indeed, for every interpretation D, we know by Proposition 4.3 that $[\![\alpha]\!]_D$ has inertia outside $O(\alpha) \subseteq \mathit{fvars}(\alpha)$.

We are now ready to prove Lemma 4.4.

4.2.1 Atomic Modules. Let α be of the form $M(\bar{x}; \bar{y})$. Recall that $\mathit{fvars}(\alpha) = X \cup Y$ where X and Y are the variables in $\overline{x}$ and $\overline{y}$, respectively. The property directly follows from the definition of the semantics for atomic modules.

4.2.2 Identity. Let α be of the form id. Recall that $\mathit{fvars}(\alpha) = \emptyset$. The property directly follows from the definition of id.

4.2.3 Union. Let α be of the form $\alpha_1 \cup \alpha_2$. Recall that $\mathit{fvars}(\alpha) = \mathit{fvars}(\alpha_1) \cup \mathit{fvars}(\alpha_2)$. If $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$, then $(\nu_1, \nu_2) \in [\![\alpha_1]\!]_D$ or $(\nu_1, \nu_2) \in [\![\alpha_2]\!]_D$. Let $Y = \mathbb{V} - \mathit{fvars}(\alpha)$ and let ν be a partial valuation on Y. Assume without loss of generality that $(\nu_1, \nu_2) \in [\![\alpha_1]\!]_D$. Clearly, $Y \subseteq \mathbb{V} - \mathit{fvars}(\alpha_1)$ since $\mathit{fvars}(\alpha_1) \subseteq \mathit{fvars}(\alpha)$. By induction and Lemma 4.5, we know that $(\nu_1[\nu], \nu_2[\nu]) \in [\![\alpha_1]\!]_D \subseteq [\![\alpha]\!]_D$ as desired.

4.2.4 Intersection. Let α be of the form $\alpha_1 \cap \alpha_2$. Recall that $\mathit{fvars}(\alpha) = \mathit{fvars}(\alpha_1) \cup \mathit{fvars}(\alpha_2)$. If $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$, then $(\nu_1, \nu_2) \in [\![\alpha_1]\!]_D$ and $(\nu_1, \nu_2) \in [\![\alpha_2]\!]_D$. Let $Y = \mathbb{V} - \mathit{fvars}(\alpha)$ and let ν be a partial valuation on Y. Clearly, $Y \subseteq \mathbb{V} - \mathit{fvars}(\alpha_i)$ with $i = 1, 2$ since $\mathit{fvars}(\alpha_i) \subseteq \mathit{fvars}(\alpha)$. By induction and Lemma 4.5, we know that $(\nu_1[\nu], \nu_2[\nu]) \in [\![\alpha_i]\!]_D$ with $i = 1, 2$, whence $(\nu_1[\nu], \nu_2[\nu]) \in [\![\alpha]\!]_D$ as desired.

[2] We remind the reader that a statement of the form 'Φ if and only if Ψ' states the equivalence $\Phi \Leftrightarrow \Psi$, with 'if' standing for the implication $\Leftarrow$ and 'only if' standing for the implication $\Rightarrow$.

4.2.5 Composition. Let α be of the form $\alpha_1 ; \alpha_2$. Recall that $fvars(\alpha) = fvars(\alpha_1) \cup fvars(\alpha_2)$. If $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$, then there exists a valuation ν_3 such that $(\nu_1, \nu_3) \in [\![\alpha_1]\!]_D$ and $(\nu_3, \nu_2) \in [\![\alpha_2]\!]_D$. Let $Y = \mathbb{V} - fvars(\alpha)$ and let ν be a partial valuation on Y. Clearly, $Y \subseteq \mathbb{V} - fvars(\alpha_i)$ with $i = 1, 2$ since $fvars(\alpha_i) \subseteq fvars(\alpha)$. By induction and Lemma 4.5, we know that $(\nu_1[\nu], \nu_3[\nu]) \in [\![\alpha_1]\!]_D$ and $(\nu_3[\nu], \nu_2[\nu]) \in [\![\alpha_2]\!]_D$. Therefore, we may conclude that $(\nu_1[\nu], \nu_2[\nu]) \in [\![\alpha_1 ; \alpha_2]\!]_D$.

4.2.6 Difference. Let α be of the form $\alpha_1 - \alpha_2$. Recall that $fvars(\alpha) = fvars(\alpha_1) \cup fvars(\alpha_2)$. If $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$, then we know that

(1) $(\nu_1, \nu_2) \in [\![\alpha_1]\!]_D$. By inertia, we know that $\nu_1 = \nu_2$ outside $O(\alpha_1) \subseteq fvars(\alpha_1) \subseteq fvars(\alpha)$.
(2) $(\nu_1, \nu_2) \notin [\![\alpha_2]\!]_D$.

Let $Y = \mathbb{V} - fvars(\alpha)$ and let ν be a partial valuation on Y. Clearly, $Y \subseteq \mathbb{V} - fvars(\alpha_i)$ with $i = 1, 2$ since $fvars(\alpha_i) \subseteq fvars(\alpha)$. By induction from (1) and Lemma 4.5, we know that $(\nu_1[\nu], \nu_2[\nu]) \in [\![\alpha_1]\!]_D$. All that remains is to show that $(\nu_1[\nu], \nu_2[\nu]) \notin [\![\alpha_2]\!]_D$ Now, suppose for the sake of contradiction that $(\nu_1[\nu], \nu_2[\nu]) \in [\![\alpha_2]\!]_D$. By induction and Lemma 4.5, we know that $((\nu_1[\nu])[\nu_1|_Y], (\nu_2[\nu])[\nu_1|_Y]) \in [\![\alpha_2]\!]_D$. Clearly, $(\nu_1[\nu])[\nu_1|_Y] = \nu_1$. Moreover, $(\nu_2[\nu])[\nu_1|_Y] = \nu_2$ since $\nu_1 = \nu_2$ outside $fvars(\alpha)$ by (1). We have thus obtained that $(\nu_1, \nu_2) \in [\![\alpha_2]\!]_D$, which contradicts (2).

4.2.7 Converse. Let α be of the form $\alpha_1^\smile$. Recall that $fvars(\alpha) = fvars(\alpha_1)$. The property follows directly by induction since $fvars(\alpha_1) = fvars(\alpha_1^\smile)$.

4.2.8 Left Selection. Let α be of the form $\sigma^l_{x=y}(\alpha_1)$. Recall the definition:

$$fvars(\alpha) = \begin{cases} fvars(\alpha_1) & \text{if } x =_{\text{syn}} y \\ fvars(\alpha_1) \cup \{x, y\} & \text{otherwise} \end{cases}$$

In case of $x =_{\text{syn}} y$, clearly $[\![\sigma^l_{x=y}(\alpha_1)]\!]_D = [\![\alpha_1]\!]_D$. The property holds trivially by induction. In the other case when $x \neq_{\text{syn}} y$, if $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$, then $(\nu_1, \nu_2) \in [\![\alpha_1]\!]_D$. Let $Y = \mathbb{V} - fvars(\alpha)$ and let ν be a partial valuation on Y. Clearly, $Y \subseteq \mathbb{V} - fvars(\alpha_1)$ since $fvars(\alpha_1) \subseteq fvars(\alpha)$. By induction and Lemma 4.5, we know that $(\nu_1[\nu], \nu_2[\nu]) \in [\![\alpha_1]\!]_D$. Moreover, since $\{x, y\} \cap Y = \emptyset$, we know that the selection does not look at ν, whence $(\nu_1[\nu], \nu_2[\nu]) \in [\![\sigma^l_{x=y}(\alpha_1)]\!]_D$ as desired.

4.2.9 Right Selection. Let α be of the form $\sigma^r_{x=y}(\alpha_1)$. Recall the definition:

$$fvars(\alpha) = \begin{cases} fvars(\alpha_1) & \text{if } x =_{\text{syn}} y \\ fvars(\alpha_1) \cup \{x, y\} & \text{otherwise} \end{cases}$$

In case of $x =_{\text{syn}} y$, clearly $[\![\sigma^r_{x=y}(\alpha_1)]\!]_D = [\![\alpha_1]\!]_D$. Hence, the property holds trivially by induction. In the other case, it is analogous to $\sigma^l_{x=y}(\alpha_1)$ since here also $\{x, y\} \cap (\mathbb{V} - fvars(\alpha)) = \emptyset$.

4.2.10 Left-to-Right Selection. Let α be of the form $\sigma^{lr}_{x=y}(\alpha_1)$. Recall the definition:

$$fvars(\alpha) = \begin{cases} fvars(\alpha_1) & \text{if } x =_{\text{syn}} y \text{ and } y \notin O(\alpha_1) \\ fvars(\alpha_1) \cup \{x, y\} & \text{otherwise} \end{cases}$$

In case of $x =_{\text{syn}} y$ and $y \notin O(\alpha_1)$, clearly $[\![\sigma^{lr}_{x=y}(\alpha_1)]\!]_D = [\![\alpha_1]\!]_D$. Hence, the property holds trivially by induction. In the other case, it is analogous to $\sigma^l_{x=y}(\alpha_1)$ since here also $\{x, y\} \cap (\mathbb{V} - fvars(\alpha)) = \emptyset$.

4.2.11 Right and Left Cylindrifications. Let α be of the form $\text{cyl}^r_x(\alpha_1)$. The case for left cylindrification is analogous. Recall that $fvars(\alpha) = fvars(\alpha_1) \cup \{y\}$. If $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$, then there exists a valuation ν_3 such that

(1) $(\nu_1, \nu_3) \in [\![\alpha_1]\!]_D$;

(2) $\nu_3 = \nu_2$ outside $\{x\}$.

Let $Y = \mathbb{V} - \mathit{fvars}(\alpha)$ and let ν be a partial valuation on Y. Clearly, $Y \subseteq \mathbb{V} - \mathit{fvars}(\alpha_1)$ since $\mathit{fvars}(\alpha_1) \subseteq \mathit{fvars}(\alpha)$. By induction from (1) and Lemma 4.5 we know that $(\nu_1[\nu], \nu_3[\nu]) \in [\![\alpha_1]\!]_D$. Since $x \notin Y$, we know from (2) that $\nu_3[\nu] = \nu_2[\nu]$ outside $\{x\}$. Hence, we can conclude that $(\nu_1[\nu], \nu_2[\nu]) \in [\![\alpha]\!]_D$.

4.3 Proof of Syntactic Input-Output Determinacy

Syntactic Input-Output determinacy follows from the proof of the following Lemma.

Lemma 4.6 (Syntactic Input-Output Determinacy). *Let α be a LIF expression. Then, for every interpretation D, every $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$ and every ν_1' that agrees with ν_1 on $I^{\mathrm{syn}}(\alpha)$, there exists a valuation ν_2' that agrees with ν_2 on $O^{\mathrm{syn}}(\alpha)$ and $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$.*

In the proof, we will make use of a useful alternative formulation of syntactic input-output determinacy which is defined next.

Definition 4.7 (Alternative Input-Output Determinacy). A LIF expression α is said to satisfy *alternative input-output determinacy* if for every interpretation D, every $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$ and every ν_1' that agrees with ν_1 on $I^{\mathrm{syn}}(\alpha)$ and outside $O^{\mathrm{syn}}(\alpha)$, we have $(\nu_1', \nu_2) \in [\![\alpha]\!]_D$.

The following Lemma shows that the two notions are equivalent. In what follows, will remove the superscript from I^{syn} and O^{syn} and refer to them as I and O, respectively.

Lemma 4.8. *Every LIF expression α satisfies* alternative input-output determinacy *if and only if it satisfies* syntactic input-output determinacy.

Proof. The proof of the 'if'-direction is similar to the proof of Proposition 3.9. Now, we proceed to verify the other direction. Suppose that $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$ and $\nu_1' = \nu_1$ on $I(\alpha)$. We now construct a new valuation ν_1'' such that it agrees with ν_1' on $\mathit{fvars}(\alpha)$ and it agrees with ν_1 elsewhere. We thus have the following properties for ν_1'':

(1) $\nu_1'' = \nu_1'$ on $\mathit{fvars}(\alpha)$, and
(2) $\nu_1'' = \nu_1$ on $\mathbb{V} - \mathit{fvars}(\alpha)$.

We know that $\nu_1' = \nu_1$ on $I(\alpha)$ by assumption, whence (1) implies that $\nu_1'' = \nu_1$ on $I(\alpha)$ since $I(\alpha) \subseteq \mathit{fvars}(\alpha)$. Combining this with (2), we know that $\nu_1'' = \nu_1$ on $I(\alpha)$ and outside $\mathit{fvars}(\alpha)$. Thus, alternative input-output determinacy implies that $(\nu_1'', \nu_2) \in [\![\alpha]\!]_D$. Since $\nu_1'' = \nu_1'$ on $\mathit{fvars}(\alpha)$, we know that there is a partial valuation ν on $\mathbb{V} - \mathit{fvars}(\alpha)$ such that $\nu_1''[\nu] = \nu_1'$. By syntactic free variable, we know that $(\nu_1''[\nu], \nu_2[\nu]) \in [\![\alpha]\!]_D$. Thus, $(\nu_1', \nu_2[\nu]) \in [\![\alpha]\!]_D$ as desired. □

We are now ready for the proof of Lemma 4.6. In this proof, we will use the notation $\overline{X}$ to mean $\mathbb{V} - X$. Moreover, since we established by Lemma 4.8 that the two definitions for input-output determinacy are equivalent, we will verify any of them for each LIF expression.

4.3.1 Atomic Modules. Let α be of the form $M(\bar{x}; \bar{y})$. Recall the definitions:

- $I(\alpha) = X$ where X are the variables in $\overline{x}$;
- $O(\alpha) = Y$ where Y are the variables in $\overline{y}$.

Syntactic input-output determinacy directly follows from the definition of the semantics for atomic modules.

4.3.2 Identity. Let α be of the form *id*. Recall that the definition for $I(\alpha) = O(\alpha) = \emptyset$. We proceed to verify that α satisfies alternative input-output determinacy. Indeed, this is true since $\overline{O(\alpha)} \cup I(\alpha) = \mathbb{V}$.

4.3.3 Union. Let α be of the form $\alpha_1 \cup \alpha_2$. Recall the definitions:

- $I(\alpha) = I(\alpha_1) \cup I(\alpha_2) \cup (O(\alpha_1) \triangle O(\alpha_2))$;
- $O(\alpha) = O(\alpha_1) \cup O(\alpha_2)$.

We proceed to verify that α satisfies alternative input-output determinacy. If $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$, then $(\nu_1, \nu_2) \in [\![\alpha_1]\!]_D$ or $(\nu_1, \nu_2) \in [\![\alpha_2]\!]_D$. Assume without loss of generality that $(\nu_1, \nu_2) \in [\![\alpha_1]\!]_D$. Now, let ν_1' be a valuation such that $\nu_1' = \nu_1$ on $\overline{O(\alpha)} \cup I(\alpha)$. Moreover, we have the following:

$$\begin{aligned}\overline{O(\alpha)} \cup I(\alpha) &= \overline{O(\alpha_1) \cup O(\alpha_2)} \cup I(\alpha_1) \cup I(\alpha_2) \cup (O(\alpha_1) \triangle O(\alpha_2)) \\ &= \overline{O(\alpha_1) \cap O(\alpha_2)} \cup I(\alpha_1) \cup I(\alpha_2) \\ &= \overline{O(\alpha_1)} \cup \overline{O(\alpha_2)} \cup I(\alpha_1) \cup I(\alpha_2)\end{aligned}$$

Therefore, certainly $\nu_1' = \nu_1$ on $\overline{O(\alpha_1)} \cup I(\alpha_1)$. Thus, $(\nu_1', \nu_2) \in [\![\alpha_1]\!]_D$ by induction and Lemma 4.8, whence $(\nu_1', \nu_2) \in [\![\alpha]\!]_D$ as desired.

4.3.4 Intersection. Let α be of the form $\alpha_1 \cap \alpha_2$. Recall the definitions:

- $I(\alpha) = I(\alpha_1) \cup I(\alpha_2) \cup (O(\alpha_1) \triangle O(\alpha_2))$;
- $O(\alpha) = O(\alpha_1) \cap O(\alpha_2)$.

We proceed to verify that α satisfies alternative input-output determinacy. If $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$, then $(\nu_1, \nu_2) \in [\![\alpha_1]\!]_D$ and $(\nu_1, \nu_2) \in [\![\alpha_2]\!]_D$. Now, let ν' be a valuation such that $\nu_1' = \nu_1$ on $\overline{O(\alpha)} \cup I(\alpha)$. Just as in the case for $\cup$, we have that $\overline{O(\alpha)} \cup I(\alpha) = \overline{O(\alpha_1)} \cup \overline{O(\alpha_2)} \cup I(\alpha_1) \cup I(\alpha_2)$. Therefore, certainly $\nu_1' = \nu_1$ on $\overline{O(\alpha_1)} \cup I(\alpha_1)$. Thus, $(\nu_1', \nu_2) \in [\![\alpha_1]\!]_D$ and $(\nu_1', \nu_2) \in [\![\alpha_2]\!]_D$ by induction and Lemma 4.8, whence $(\nu_1', \nu_2) \in [\![\alpha]\!]_D$ as desired.

4.3.5 Composition. Let α be of the form $\alpha_1 \,;\, \alpha_2$. Recall the definitions:

- $I(\alpha) = I(\alpha_1) \cup (I(\alpha_2) - O(\alpha_1))$;
- $O(\alpha) = O(\alpha_1) \cup O(\alpha_2)$.

We proceed to verify that α satisfies syntactic input-output determinacy. If $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$, then there exists a valuation ν such that

(i) $(\nu_1, \nu) \in [\![\alpha_1]\!]_D$;
(ii) $(\nu, \nu_2) \in [\![\alpha_2]\!]_D$.

Now, let ν_1' be a valuation such that

$$\nu_1' = \nu_1 \text{ on } I(\alpha) = I(\alpha_1) \cup (I(\alpha_2) - O(\alpha_1)). \tag{1}$$

Since $I(\alpha_1) \subseteq I(\alpha)$, then by induction there exists a valuation ν' such that

(iii) $(\nu_1', \nu') \in [\![\alpha_1]\!]_D$;
(iv) $\nu' = \nu$ on $O(\alpha_1)$.

By applying inertia to (i) and (iii) we get that $\nu_1 = \nu$ and $\nu_1' = \nu'$ outside $O(\alpha_1)$. Combining this with (1) we have that $\nu' = \nu_1' = \nu_1 = \nu$ on $I(\alpha) \cap \overline{O(\alpha_1)} = (I(\alpha_1) \cup I(\alpha_2)) - O(\alpha_1)$. Together with (iv), this implies that

$$\nu' = \nu \text{ on } I(\alpha_1) \cup I(\alpha_2) \cup O(\alpha_1). \tag{2}$$

By induction from (ii), there exists ν_2' such that

(v) $(\nu', \nu_2') \in [\![\alpha_2]\!]_D$;
(vi) $\nu_2' = \nu_2$ on $O(\alpha_2)$.

From (iii) and (vi) we get that $(\nu'_1, \nu'_2) \in [\![\alpha]\!]_D$. All that remains to be shown is that $\nu'_2 = \nu_2$ on $O(\alpha)$. By applying inertia to (ii) and (v) we get that

$$\nu = \nu_2 \text{ outside } O(\alpha_2);$$
$$\nu' = \nu'_2 \text{ outside } O(\alpha_2).$$

Combining this with 2 we have that $\nu'_2 = \nu_2$ on $(I(\alpha_1) \cup I(\alpha_2) \cup O(\alpha_1)) \cap \overline{O(\alpha_2)} = (I(\alpha_1) \cup I(\alpha_2) \cup O(\alpha_1)) - O(\alpha_2)$. Together with (vi) this implies that $\nu'_2 = \nu_2$ on $I(\alpha_1) \cup I(\alpha_2) \cup O(\alpha_1) \cup O(\alpha_2)$, whence $\nu'_2 = \nu_2$ on $O(\alpha)$ as desired.

4.3.6 Difference. Let α be of the form $\alpha_1 - \alpha_2$. Recall that

- $I(\alpha) = I(\alpha_1) \cup I(\alpha_2) \cup (O(\alpha_1) \triangle O(\alpha_2))$;
- $O(\alpha) = O(\alpha_1)$.

We proceed to verify that α satisfies alternative input-output determinacy. If $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$, then we know that

(1) $(\nu_1, \nu_2) \in [\![\alpha_1]\!]_D$;
(2) $(\nu_1, \nu_2) \notin [\![\alpha_2]\!]_D$.

Now, let ν'_1 be a valuation such that $\nu'_1 = \nu_1$ on $\overline{O(\alpha)} \cup I(\alpha)$. Since $\overline{O(\alpha_1)} \subseteq \overline{O(\alpha)}$ and $I(\alpha_1) \subseteq I(\alpha)$, then $\nu'_1 = \nu_1$ on $\overline{O(\alpha_1)} \cup I(\alpha_1)$. Thus, by induction from (1) and Lemma 4.8, we know that $(\nu'_1, \nu_2) \in [\![\alpha_1]\!]_D$.

To prove that $(\nu'_1, \nu_2) \in [\![\alpha]\!]_D$, all that remains is to show that $(\nu'_1, \nu_2) \notin [\![\alpha_2]\!]_D$. Assume for the sake of contradiction that $(\nu'_1, \nu_2) \in [\![\alpha_2]\!]_D$. Since $\nu'_1 = \nu_1$ on $\overline{O(\alpha)} \cup I(\alpha)$ and $\overline{O(\alpha)} \cup I(\alpha) = \overline{O(\alpha_1)} \cup \overline{O(\alpha_2)} \cup I(\alpha_1) \cup I(\alpha_2)$, we know that $\nu'_1 = \nu_1$ on $\overline{O(\alpha_2)} \cup I(\alpha_2)$. Hence, $(\nu_1, \nu_2) \in [\![\alpha_2]\!]_D$ by induction and Lemma 4.8, which contradicts (2).

4.3.7 Converse. Let α be of the form $\alpha_1^{\smile}$. Recall the definitions:

- $I(\alpha) = I(\alpha_1) \cup O(\alpha_1)$;
- $O(\alpha) = O(\alpha_1)$.

Alternative input-output determinacy holds since $\overline{O(\alpha)} \cup I(\alpha) = \mathbb{V}$.

4.3.8 Left Selection. Let α be of the form $\sigma^l_{x=y}(\alpha_1)$. Recall the definitions:

- $I(\alpha) = \begin{cases} I(\alpha_1) & x =_{\text{syn}} y \\ I(\alpha_1) \cup \{x, y\} & \text{otherwise} \end{cases}$
- $O(\alpha) = O(\alpha_1)$.

We proceed to verify that α satisfies alternative input-output determinacy. Clearly, the property holds trivially by induction in case of $x =_{\text{syn}} y$. Indeed, in this case, $[\![\alpha]\!]_D = [\![\alpha_1]\!]_D$. In the other case when $x \neq_{\text{syn}} y$, if $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$, then we know that

(1) $(\nu_1, \nu_2) \in [\![\alpha_1]\!]_D$;
(2) $\nu_1(x) = \nu_1(y)$;

Let ν'_1 be a valuation such that $\nu'_1 = \nu_1$ on $\overline{O(\alpha)} \cup I(\alpha)$. In all cases, $O(\alpha) \subseteq O(\alpha_1)$. Hence, $\overline{O(\alpha_1)} \subseteq \overline{O(\alpha)}$. Moreover, in all cases $I(\alpha_1) \subseteq I(\alpha)$. Thus, $\nu'_1 = \nu_1$ on $\overline{O(\alpha_1)} \cup I(\alpha_1)$. By induction from (1) and Lemma 4.8, we know that $(\nu'_1, \nu_2) \in [\![\alpha_1]\!]_D$.

All that remains to be shown is that $\nu'_1(x) = \nu'_1(y)$. Since $\{x, y\} \subseteq I(\alpha)$, we know that $\nu'_1 = \nu_1$ on $\{x, y\}$ by assumption. Hence, $\nu'_1(x) = \nu'_1(y)$ by (2).

4.3.9 Right Selection. Let α be of the form $\sigma^{r}_{x=y}(\alpha_1)$. Recall the definitions:

- $I(\alpha) = \begin{cases} I(\alpha_1) & x =_{\text{syn}} y \\ I(\alpha_1) \cup (\{x, y\} - O(\alpha_1)) & \text{otherwise} \end{cases}$
- $O(\alpha) = O(\alpha_1)$.

We proceed to verify that α satisfies alternative input-output determinacy. Clearly, the property holds trivially by induction in case of $x =_{\text{syn}} y$. Indeed, in this case, $[\![\alpha]\!]_D = [\![\alpha_1]\!]_D$. In the other case when $x \neq_{\text{syn}} y$, if $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$, then we know that

(1) $(\nu_1, \nu_2) \in [\![\alpha_1]\!]_D$;
(2) $\nu_2(x) = \nu_2(y)$.

Let ν_1' be a valuation such that $\nu_1' = \nu_1$ on $\overline{O(\alpha)} \cup I(\alpha)$. In all cases, $O(\alpha) \subseteq O(\alpha_1)$. Hence, $\overline{O(\alpha_1)} \subseteq \overline{O(\alpha)}$. Moreover, in all cases $I(\alpha_1) \subseteq I(\alpha)$. Thus, $\nu_1' = \nu_1$ on $\overline{O(\alpha_1)} \cup I(\alpha_1)$. By induction from (1) and Lemma 4.8, we know that $(\nu_1', \nu_2) \in [\![\alpha_1]\!]_D$. Together with (2), we know that $(\nu_1', \nu_2) \in [\![\alpha]\!]_D$.

4.3.10 Left-to-Right Selection. Let α be of the form $\sigma^{lr}_{x=y}(\alpha_1)$. Recall the definitions:

- $I(\alpha) = \begin{cases} I(\alpha_1) & x =_{\text{syn}} y \text{ and } y \notin O(\alpha_1) \\ I(\alpha_1) \cup \{x, y\} & x \neq_{\text{syn}} y \text{ and } y \notin O(\alpha_1) \\ I(\alpha_1) \cup \{x\} & \text{otherwise} \end{cases}$
- $O(\alpha) = \begin{cases} O(\alpha_1) - \{x\} & x =_{\text{syn}} y \\ O(\alpha_1) & \text{otherwise} \end{cases}$

We proceed to verify that α satisfies alternative input-output determinacy. Clearly, $[\![\alpha]\!]_D = [\![\alpha_1]\!]_D$ in case of $x =_{\text{syn}} y$ and $y \notin O(\alpha_1)$. Hence, the property holds trivially by induction. In the other cases, if $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$, then we know that

(1) $(\nu_1, \nu_2) \in [\![\alpha_1]\!]_D$;
(2) $\nu_1(x) = \nu_2(y)$.

Let ν_1' be a valuation such that $\nu_1' = \nu_1$ on $\overline{O(\alpha)} \cup I(\alpha)$. In all cases, $O(\alpha) \subseteq O(\alpha_1)$. Hence, $\overline{O(\alpha_1)} \subseteq \overline{O(\alpha)}$. Moreover, in all cases $I(\alpha_1) \subseteq I(\alpha)$. Thus, $\nu_1' = \nu_1$ on $\overline{O(\alpha_1)} \cup I(\alpha_1)$. By induction from (1) and Lemma 4.8, we know that $(\nu_1', \nu_2) \in [\![\alpha_1]\!]_D$. All that remains to be shown is that $\nu_1'(x) = \nu_2(y)$. Since $x \in I(\alpha)$ and $\nu_1' = \nu_1$ on $\overline{O(\alpha)} \cup I(\alpha)$, we have $\nu_1'(x) = \nu_1(x)$. Together with (2), we get that $\nu_1'(x) = \nu_2(y)$ as desired, whence $(\nu_1', \nu_2) \in [\![\alpha]\!]_D$.

4.3.11 Right Cylindrification. Let α be of the form $\text{cyl}^{r}_{x}(\alpha_1)$. Recall the definitions:

- $I(\alpha) = I(\alpha_1)$;
- $O(\alpha) = O(\alpha_1) \cup \{x\}$;
- $\mathit{fvars}(\alpha) = \mathit{fvars}(\alpha_1) \cup \{x\}$.

We proceed to verify that α satisfies alternative input-output determinacy. If $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$, then there exists a valuation ν_2' such that

(1) $(\nu_1, \nu_2') \in [\![\alpha_1]\!]_D$;
(2) $\nu_2' = \nu_2$ outside $\{x\}$.

Now, let ν_1' be a valuation such that $\nu_1' = \nu_1$ on $\overline{O(\alpha)} \cup I(\alpha)$. We now split the proof in two cases:

- Suppose that $x \in \mathit{fvars}(\alpha_1)$. Then, $\mathit{fvars}(\alpha) = \mathit{fvars}(\alpha_1)$. Thus, we know that $\nu_1' = \nu_1$ on $\overline{\mathit{fvars}(\alpha_1)} \cup I(\alpha_1)$, whence $\nu_1' = \nu_1$ on $\overline{O(\alpha_1)} \cup I(\alpha_1)$. Thus, by induction from (1) and Lemma 4.8, we know that $(\nu_1', \nu_2') \in [\![\alpha_1]\!]_D$. Hence, $(\nu_1', \nu_2) \in [\![\alpha]\!]_D$.

- Conversely, suppose that $x \notin fvars(\alpha_1)$. We have $\overline{O(\alpha)} \cup I(\alpha) = (\overline{O(\alpha_1)} \cup I(\alpha_1)) - \{x\}$. Thus, $\nu_1'[\nu_1|_{\{x\}}] = \nu_1$ on $\overline{O(\alpha_1)} \cup I(\alpha_1)$ since $\nu_1' = \nu_1$ on $\overline{O(\alpha)} \cup I(\alpha)$. By induction and Lemma 4.8, then $(\nu_1'[\nu_1|_{\{x\}}], \nu_2') \in [\![\alpha_1]\!]_D$. By syntactic free variable, we know that $(\nu_1'[\nu_1|_{\{x\}}][\nu_1'|_{\{x\}}], \nu_2'[\nu_1'|_{\{x\}}]) \in [\![\alpha_1]\!]_D$ since $x \notin fvars(\alpha_1)$. Clearly, $\nu_1'[\nu_1|_{\{x\}}][\nu_1'|_{\{x\}}] = \nu_1'$, whence $(\nu_1', \nu_2'[\nu_1'|_{\{x\}}]) \in [\![\alpha_1]\!]_D$. Consequently, $(\nu_1', \nu_2) \in [\![\alpha]\!]_D$ as desired.

4.3.12 Left Cylindrification. Let α be of the form $\mathrm{cyl}^l_x(\alpha_1)$. Recall the definitions:

- $I(\alpha) = I(\alpha_1) - \{x\}$;
- $O(\alpha) = O(\alpha_1) \cup \{x\}$.

We proceed to verify that α satisfies alternative input-output determinacy. If $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$, then there exists a valuation ν_1' such that

(i) $\nu_1' = \nu_1$ outside $\{x\}$;
(ii) $(\nu_1', \nu_2) \in [\![\alpha_1]\!]_D$.

Now, let ν be a valuation such that

$$\nu = \nu_1 \text{ on } \overline{O(\alpha)} \cup I(\alpha). \tag{1}$$

Clearly, $\nu[\nu_1'|_{\{x\}}] = \nu$ outside $\{x\}$. Since $x \notin \overline{O(\alpha)} \cup I(\alpha)$, we also know that $\nu[\nu_1'|_{\{x\}}] = \nu$ on $\overline{O(\alpha)} \cup I(\alpha)$. Combining this with (1), we get that $\nu[\nu_1'|_{\{x\}}] = \nu_1'$ on $\overline{O(\alpha)} \cup I(\alpha) \cup \{x\}$. Clearly, $\overline{O(\alpha)} \cup I(\alpha) \cup \{x\} \supseteq \overline{O(\alpha_1)} \cup I(\alpha_1)$, whence $\nu[\nu_1'|_{\{x\}}] = \nu_1'$ on $\overline{O(\alpha_1)} \cup I(\alpha_1)$. By induction from (ii) and Lemma 4.8, we get that $(\nu[\nu_1'|_{\{x\}}], \nu_2) \in [\![\alpha_1]\!]_D$, whence also $(\nu, \nu_2) \in [\![\alpha_1]\!]_D$.

5 PRECISION THEOREM PROOF

In this section, we prove Theorem 3.27. By soundness and Proposition 3.20, it suffices to prove $O^{\mathrm{syn}}(\alpha) \subseteq O^{\mathrm{sem}}(\alpha)$ and $I^{\mathrm{syn}}(\alpha) \subseteq I^{\mathrm{sem}}(\alpha)$ for every LIF expression α. For the latter inequality, it will be convenient to use the equivalent definition of semantic input variables introduced in Proposition 3.4. Moreover, in the proof of the Precision Theorem, we will often make use of the following two technical lemmas.

Lemma 5.1. *Let M be a nullary relation name and let D be an interpretation where $D(M)$ is nonempty. Then $[\![M()]\!]_D$ consists of all identical pairs of valuations.*

Proof. The proof follows directly from the semantics of atomic modules. □

Lemma 5.2. *Suppose $\alpha_1 = M_1(\bar{x_1}; \bar{y_1})$ and $\alpha_2 = M_2(\bar{x_2}; \bar{y_2})$ where $M_1 \neq M_2$. Let α be either $\alpha_1 \cup \alpha_2$ or $\alpha_1 - \alpha_2$. Assume that $O^{\mathrm{syn}}(\alpha_i) \subseteq O^{\mathrm{sem}}(\alpha_i)$ and $I^{\mathrm{syn}}(\alpha_i) \subseteq I^{\mathrm{sem}}(\alpha_i)$ for $i = 1, 2$. Let $j \neq k \in \{1, 2\}$. If $[\![\alpha]\!]_D = [\![\alpha_k]\!]_D$ for any interpretation D where $D(M_j)$ is empty, then $O^{\mathrm{syn}}(\alpha_k) \subseteq O^{\mathrm{sem}}(\alpha)$ and $I^{\mathrm{syn}}(\alpha_k) \subseteq I^{\mathrm{sem}}(\alpha)$.*

Proof. First, we verify that $O^{\mathrm{syn}}(\alpha_k) \subseteq O^{\mathrm{sem}}(\alpha)$. Let $v \in O^{\mathrm{syn}}(\alpha_k)$. Since $O^{\mathrm{syn}}(\alpha_k) \subseteq O^{\mathrm{sem}}(\alpha_k)$, then $v \in O^{\mathrm{sem}}(\alpha_k)$. By definition, we know that there is an interpretation D' and $(\nu_1, \nu_2) \in [\![\alpha_k]\!]_{D'}$ such that $\nu_1(v) \neq \nu_2(v)$. Take D'' to be the interpretation where $D''(M) = D'(M)$ for any $M \neq M_j$ while $D''(M_j)$ is empty. Clearly, $(\nu_1, \nu_2) \in [\![\alpha]\!]_{D''}$, whence, $M_j \neq M_k$ and $[\![\alpha]\!]_{D''} = [\![\alpha_k]\!]_{D'}$. It follows then that $v \in O^{\mathrm{sem}}(\alpha)$.

Similarly, we proceed to verify $I^{\mathrm{syn}}(\alpha_k) \subseteq I^{\mathrm{sem}}(\alpha)$. Let $v \in I^{\mathrm{syn}}(\alpha_k)$. Since $I^{\mathrm{syn}}(\alpha_k) \subseteq I^{\mathrm{sem}}(\alpha_k)$, then $v \in I^{\mathrm{sem}}(\alpha_k)$. By definition, we know that there is an interpretation D', $(\nu_1, \nu_2) \in [\![\alpha_k]\!]_{D'}$, and $\nu_1'(v) \neq \nu_1(v)$ such that $(\nu_1', \nu_2') \notin [\![\alpha_k]\!]_{D'}$ for every valuation ν_2' that agrees with ν_2 on $O^{\mathrm{sem}}(\alpha_k)$.

Take D'' to be the interpretation where $D''(M) = D'(M)$ for any $M \neq M_j$ while $D''(M_j)$ is empty. Clearly, $[\![\alpha]\!]_{D''} = [\![\alpha_k]\!]_{D'}$, whence, $M_j \neq M_k$. Therefore, $O^{\mathrm{sem}}(\alpha_k) \subseteq O^{\mathrm{sem}}(\alpha)$. Hence, $v \in I^{\mathrm{sem}}(\alpha)$.

Indeed, $(\nu_1, \nu_2) \in [\![\alpha]\!]_{D''}$ and for any valuation ν_2' if ν_2' agrees with ν_2 on $O^{\mathrm{sem}}(\alpha)$, then ν_2' agrees with ν_2 on $O^{\mathrm{sem}}(\alpha_k)$. □

The proof of Theorem 3.27 is done by extensive case analysis. Intuitively, for each of the different operations, and every variable $z \in O^{\mathrm{syn}}(\alpha)$, we construct an interpretation D such that z is not inertial in $[\![\alpha]\!]_D$ and thus $z \in O^{\mathrm{sem}}(\alpha)$. Similarly, for every variable $z \in I^{\mathrm{syn}}(\alpha)$, we construct an interpretation D as a witness of the fact that $\mathbb{V} - \{z\}$ does not determine $[\![\alpha]\!]$ on $O^{\mathrm{sem}}(\alpha)$ and thus that $z \in I^{\mathrm{sem}}(\alpha)$. In the proof, we often remove the superscript from I^{syn} and O^{syn} and refer to them by I and O, respectively.

5.1 Atomic Modules

Let α be of the form α_1, where α_1 is $M(\bar{x}; \bar{y})$. Recall the definition:

- $O^{\mathrm{syn}}(\alpha) = Y$, where Y are the variables in $\bar{y}$;
- $I^{\mathrm{syn}}(\alpha) = X$, where X are the variables in $\bar{x}$.

We first proceed to verify $O^{\mathrm{syn}}(\alpha) \subseteq O^{\mathrm{sem}}(\alpha)$. Let $v \in Y$. Consider an interpretation D where

$$D(M) = \{(1, \dots, 1; 2, \dots, 2)\}.$$

Let ν_1 be the valuation that is 1 everywhere. Also let ν_2 be the valuation that is 2 on Y and 1 everywhere else. Clearly, $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$ since $\nu_1(\bar{x}) \cdot \nu_2(\bar{y}) \in D(M)$ and ν_1 agrees with ν_2 outside Y. Hence, $v \in O^{\mathrm{sem}}(\alpha)$ since $\nu_1(v) \neq \nu_2(v)$.

Now we proceed to verify $I^{\mathrm{syn}}(\alpha) \subseteq I^{\mathrm{sem}}(\alpha)$. Let $v \in X$. Consider the same interpretation D and the same valuations ν_1 and ν_2 as discussed above. We already established that $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$. Take $\nu_1' := \nu_1[v : 2]$. We establish that $v \in I^{\mathrm{sem}}(\alpha)$ by arguing that there is no ν_2' for which $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$. Indeed, this is true since $v \in X$. Consequently, $v \in I^{\mathrm{sem}}(\alpha)$.

5.2 Identity

Let α be of the form *id*. We recall that $I^{\mathrm{syn}}(\alpha)$ and $O^{\mathrm{syn}}(\alpha)$ are both empty. Hence, $O^{\mathrm{syn}}(\alpha) \subseteq O^{\mathrm{sem}}(\alpha)$ and $I^{\mathrm{syn}}(\alpha) \subseteq I^{\mathrm{sem}}(\alpha)$ is trivial.

5.3 Union

Let α be of the form $\alpha_1 \cup \alpha_2$, where α_1 is $M_1(\bar{x_1}; \bar{y_1})$ and α_2 is $M_2(\bar{x_2}; \bar{y_2})$. We distinguish different cases based on whether M_1 or M_2 is nullary. If M_1 and M_2 are both nullary there is nothing to prove.

5.3.1 M_1 is nullary, M_2 is not. Clearly, $[\![\alpha]\!]_D = [\![\alpha_2]\!]_D$ for any interpretation D where $D(M_1)$ is empty. By induction and Lemma 5.2, we establish that $O^{\mathrm{syn}}(\alpha_2) \subseteq O^{\mathrm{sem}}(\alpha)$ and $I^{\mathrm{syn}}(\alpha_2) \subseteq I^{\mathrm{sem}}(\alpha)$. Since $I(\alpha_1)$ and $O(\alpha_1)$ are both empty, then we observe that

- $O^{\mathrm{syn}}(\alpha) = O(\alpha_2)$;
- $I^{\mathrm{syn}}(\alpha) = I(\alpha_2) \cup O(\alpha_2)$.

Thus, $O(\alpha_2) \subseteq O^{\mathrm{sem}}(\alpha)$ and $I(\alpha_2) \subseteq I^{\mathrm{sem}}(\alpha)$ is trivial.

We proceed to verify $O(\alpha_2) - I(\alpha_2) \subseteq I^{\mathrm{sem}}(\alpha)$. Let $v \in O(\alpha_2) - I(\alpha_2)$. Consider the interpretation D where $D(M_1)$ is not empty and

$$D(M_2) = \{(1, \dots, 1; 2, \dots, 2)\}.$$

Let ν_1 be the valuation that is 1 everywhere. Clearly, $(\nu_1, \nu_1) \in [\![\alpha]\!]_D$ since $(\nu_1, \nu_1) \in [\![\alpha_1]\!]_D$ by Lemma 5.1. Take $\nu_1' := \nu_1[v : 2]$. We establish that $v \in I^{\mathrm{sem}}(\alpha)$ by arguing that there for every valuation ν_2' for which $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$, we show that ν_2' and ν_1 disagrees on $O^{\mathrm{sem}}(\alpha)$. Thereto, suppose $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$. In particular, $(\nu_1', \nu_2') \in [\![\alpha_1]\!]_D$, whence $\nu_2' = \nu_1'$ by Lemma 5.1. Indeed, $v \in O(\alpha_2)$, $O(\alpha_2) \subseteq O^{\mathrm{sem}}(\alpha)$, and $\nu_2'(v) = 2$ but $\nu_1(v) = 1$. Otherwise, $(\nu_1', \nu_2') \in [\![\alpha_2]\!]_D$. However, since $v \in O(\alpha_2)$, then $\nu_2'(v) = 2$ as well. Therefore, there is no ν_2' that agrees with ν_1 on $O^{\mathrm{sem}}(\alpha)$ and $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$ at the same time. We conclude that $v \in I^{\mathrm{sem}}(\alpha)$.

5.3.2 M_2 is nullary, M_1 is not. This case is symmetric to the previous one.

5.3.3 Neither M_1 nor M_2 is nullary. Recall the definitions:

- $O^{\mathrm{syn}}(\alpha) = O(\alpha_1) \cup O(\alpha_2)$;
- $I^{\mathrm{syn}}(\alpha) = I(\alpha_1) \cup I(\alpha_2) \cup (O(\alpha_1) \triangle O(\alpha_2))$.

We first proceed to verify that $O^{\mathrm{syn}}(\alpha) \subseteq O^{\mathrm{sem}}(\alpha)$. By induction, $O^{\mathrm{syn}}(\alpha_i) = O^{\mathrm{sem}}(\alpha_i)$ for $i = 1$ or 2. Consequently, if $v \in O(\alpha_i)$, then there is an interpretation D_i, and $(\nu_1, \nu_2) \in [\![\alpha_i]\!]_D$ such that $\nu_1(v) \neq \nu_2(v)$. Indeed, $v \in O^{\mathrm{sem}}(\alpha)$ since $([\![\alpha_1]\!]_D \cup [\![\alpha_2]\!]_D) \subseteq [\![\alpha]\!]_D$ for any interpretation D.

We then proceed to verify that $I^{\mathrm{syn}}(\alpha) \subseteq I^{\mathrm{sem}}(\alpha)$. The proof has four possibilities. Each case is discussed separately below.

When $v \in I(\alpha_1)$. If $v \in I(\alpha_1)$ and $M_1 \neq M_2$, it is clear that $[\![\alpha]\!]_D = [\![\alpha_1]\!]_D$ for any interpretation D where $D(M_2)$ is empty. By Lemma 5.2 and by induction, we easily establish that $v \in I^{\mathrm{sem}}(\alpha)$.

When $v \in I(\alpha_2)$. This case is symmetric to the previous one.

When $v \in O(\alpha_1) - (O(\alpha_2) \cup I(\alpha_1) \cup I(\alpha_2))$. If $v \in O(\alpha_1) - (O(\alpha_2) \cup I(\alpha_1) \cup I(\alpha_2))$, then consider the interpretation D such that $D(M_1) = \{(1, \dots, 1; 1, \dots, 1)\}$ and $D(M_2) = \{(1, \dots, 1; 1, \dots, 1)\}$. Let ν_1 be the valuation that is 2 on v and 1 elsewhere. Clearly, $(\nu_1, \nu_1) \in [\![\alpha]\!]_D$, whence $(\nu_1, \nu_1) \in [\![\alpha_2]\!]_D$. Now take $\nu_1' := \nu_1[v : 1]$. If we can show that ν_2' does not agree with ν_1 on $O^{\mathrm{sem}}(\alpha)$ for any valuation ν_2' such that $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$, we are done. Thereto, suppose that there exists a valuation ν_2' such that $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$.

- In particular, if $(\nu_1', \nu_2') \in [\![\alpha_1]\!]_D$, then $\nu_2'(v) = 1$ since $v \in O(\alpha_1)$.
- Otherwise, if $(\nu_1', \nu_2') \in [\![\alpha_2]\!]_D$, then $\nu_2'(v) = \nu_1'(v) = 1$ since $v \notin (I(\alpha_2) \cup O(\alpha_2))$.

In both cases, ν_2' have to be 1 on v which disagrees with ν_1 on v. Since $v \in O(\alpha_1)$ and $O(\alpha_1) \subseteq O^{\mathrm{sem}}(\alpha)$, then ν_2' does not agree with ν_1 on $O^{\mathrm{sem}}(\alpha)$ as desired. We conclude that $v \in I^{\mathrm{sem}}(\alpha)$.

When $v \in O(\alpha_2) - (O(\alpha_1) \cup I(\alpha_1) \cup I(\alpha_2))$. This case is symmetric to the previous one.

5.4 Intersection

Let α be of the form $\alpha_1 \cap \alpha_2$, where α_1 is $M_1(\bar{x}_1; \bar{y}_1)$ and α_2 is $M_2(\bar{x}_2; \bar{y}_2)$. We distinguish different cases based on whether M_1 or M_2 is nullary. If M_1 and M_2 are both nullary there is nothing to prove.

5.4.1 M_1 is nullary, M_2 is not. In this case, $I(\alpha_1)$ and $O(\alpha_1)$ are both empty, then we observe that

- $O^{\mathrm{syn}}(\alpha) = \emptyset$;
- $I^{\mathrm{syn}}(\alpha) = I(\alpha_2) \cup O(\alpha_2)$.

It is trivial to verify $O^{\mathrm{syn}}(\alpha) \subseteq O^{\mathrm{sem}}(\alpha)$ since $O^{\mathrm{syn}}(\alpha)$ is empty.

We proceed to verify $I^{\mathrm{syn}}(\alpha) \subseteq I^{\mathrm{sem}}(\alpha)$. Let $v \in I(\alpha_2) \cup O(\alpha_2)$. Consider an interpretation D where $D(M_1)$ is not empty and $D(M_2) = \{(1, \dots, 1; 1, \dots, 1)\}$. Let ν_1 be the valuation that is 1 everywhere. Clearly, $(\nu_1, \nu_1) \in [\![\alpha]\!]_D$ since $(\nu_1, \nu_1) \in [\![\alpha_1]\!]_D$ and $(\nu_1, \nu_1) \in [\![\alpha_2]\!]_D$. Take $\nu_1' := \nu_1[v : 2]$. We establish that $v \in I^{\mathrm{sem}}(\alpha)$ by arguing that there is no valuation ν_2' for which $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$. Thereto, suppose $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$. In particular, when $v \in I(\alpha_2)$, it is clear that $(\nu_1', \nu_2') \notin [\![\alpha_1]\!]_D$. In the other case when $v \in O(\alpha_2) - I(\alpha_2)$, there is no ν_2' such that (ν_1', ν_2') belongs to both $[\![\alpha_1]\!]_D$ and $[\![\alpha_2]\!]_D$. Indeed, the value for $\nu_2'(v)$ will never be agreed upon by α_1 and α_2. Hence, $(\nu_1', \nu_2') \notin [\![\alpha]\!]_D$ as desired. We conclude that $v \in I^{\mathrm{sem}}(\alpha)$.

5.4.2 M_2 is nullary, M_1 is not. This case is symmetric to the previous one.

5.4.3 Neither M_1 nor M_2 is nullary. Recall the definitions:

- $O^{\text{syn}}(\alpha) = O(\alpha_1) \cap O(\alpha_2)$;
- $I^{\text{syn}}(\alpha) = I(\alpha_1) \cup I(\alpha_2) \cup (O(\alpha_1) \vartriangle O(\alpha_2))$.

We first proceed to verify $O^{\text{syn}}(\alpha) \subseteq O^{\text{sem}}(\alpha)$. Let $v \in O(\alpha_1) \cap O(\alpha_2)$. Consider an interpretation D such that $D(M_1) = \{(1, \ldots, 1; o_1, \ldots, o_m)\}$, where $o_1, \ldots, o_m$ are all the combinations of $\{1, 2\}$. Similarly,

$$D(M_2) = \{(1, \ldots, 1; o_1, \ldots, o_n)\},$$

where $o_1, \ldots, o_n$ are all the combinations of $\{1, 2\}$.

Let ν_1 be the valuation that is 1 everywhere. Also let ν_2 be the valuation that is 2 on $O(\alpha_1) \cap O(\alpha_2)$ and 1 elsewhere. Clearly, $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$, whence $(\nu_1, \nu_2) \in [\![\alpha_1]\!]_D$ and $(\nu_1, \nu_2) \in [\![\alpha_2]\!]_D$. Hence, $v \in O^{\text{sem}}(\alpha)$. Indeed, $\nu_2(v) \neq \nu_1(v)$ for $v \in O^{\text{syn}}(\alpha)$.

We then proceed to verify $I^{\text{syn}}(\alpha) \subseteq I^{\text{sem}}(\alpha)$. Let $v \in I(\alpha_1) \cup I(\alpha_2) \cup (O(\alpha_1) \vartriangle O(\alpha_2))$. Consider an interpretation D where $D(M_1) = \{(1, \ldots, 1; 1, \ldots, 1)\}$. Similarly, $D(M_2) = \{(1, \ldots, 1; 1, \ldots, 1)\}$. Let ν_1 be the valuation that is 1 everywhere. Clearly, $(\nu_1, \nu_1) \in [\![\alpha]\!]_D$, whence $(\nu_1, \nu_1) \in [\![\alpha_1]\!]_D$ and $(\nu_1, \nu_1) \in [\![\alpha_2]\!]_D$. Take $\nu_1' := \nu_1[v : 2]$. We establish that $v \in I^{\text{sem}}(\alpha)$ by arguing that there is no valuation ν_2' such that $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$. Indeed, this is clear when $v \in I(\alpha_1)$ or $v \in I(\alpha_2)$. On the other hand, when $v \in (O(\alpha_1) \vartriangle O(\alpha_2)) - (I(\alpha_1) \cup I(\alpha_2))$, we have ν_2' for which $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$ whence $(\nu_1', \nu_2') \in [\![\alpha_1]\!]_D$ and $(\nu_1', \nu_2') \in [\![\alpha_2]\!]_D$. This is not possible since v belongs to either $O(\alpha_1)$ or $O(\alpha_2)$, but not both. Hence, the value for $\nu_2'(v)$ will never be agreed upon by α_1 and α_2. We conclude that $v \in I^{\text{sem}}(\alpha)$.

5.5 Difference

Let α be of the form $\alpha_1 - \alpha_2$, where α_1 is $M_1(\bar{x}_1; \bar{y}_1)$ and α_2 is $M_2(\bar{x}_2; \bar{y}_2)$. We distinguish different cases based on whether M_1 or M_2 is nullary. If M_1 and M_2 are both nullary there is nothing to prove.

5.5.1 M_1 is nullary, M_2 is not. In this case, $I(\alpha_1)$ and $O(\alpha_1)$ are empty. In particular, $O^{\text{syn}}(\alpha)$ is empty, so $O^{\text{syn}}(\alpha) \subseteq O^{\text{sem}}(\alpha)$ is trivial.

We proceed to verify $I^{\text{syn}}(\alpha) \subseteq I^{\text{sem}}(\alpha)$. Observe that

$$I^{\text{syn}}(\alpha) = I(\alpha_2) \cup O(\alpha_2).$$

Let $v \in I^{\text{syn}}(\alpha)$. Consider the interpretation D where $D(M_1)$ is not empty and $D(M_2) = \{(1, \ldots, 1; 1, \ldots, 1)\}$. Let ν_1 be the valuation that is 2 on v and 1 elsewhere. Clearly, $(\nu_1, \nu_1) \in [\![\alpha]\!]_D$. Take $\nu_1' := \nu_1[v : 1]$. We establish that $v \in I^{\text{sem}}(\alpha)$ by arguing that there is no ν_2' for which $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$. Thereto, suppose $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$. In particular, $(\nu_1', \nu_2') \in [\![\alpha_1]\!]_D$, whence $\nu_2' = \nu_1'$ by Lemma 5.1. However, $(\nu_1', \nu_1') \in [\![\alpha_2]\!]_D$, so $(\nu_1', \nu_2') \notin [\![\alpha]\!]_D$ as desired.

5.5.2 M_2 is nullary, M_1 is not. Clearly, $[\![\alpha]\!]_D = [\![\alpha_1]\!]_D$ for any interpretation D where $D(M_2)$ is empty. By induction and Lemma 5.2, we establish that $O^{\text{syn}}(\alpha_1) \subseteq O^{\text{sem}}(\alpha)$ and $I^{\text{syn}}(\alpha_1) \subseteq I^{\text{sem}}(\alpha)$. Since $I(\alpha_2)$ and $O(\alpha_2)$ are both empty, then we observe that

- $O^{\text{syn}}(\alpha) = O(\alpha_1)$;
- $I^{\text{syn}}(\alpha) = I(\alpha_1) \cup O(\alpha_1)$.

Thus, $O(\alpha_1) \subseteq O^{\text{sem}}(\alpha)$ and $I(\alpha_1) \subseteq I^{\text{sem}}(\alpha)$ is trivial.

We proceed to verify $O(\alpha_1) - I(\alpha_1) \subseteq I^{\text{sem}}(\alpha)$. Let $v \in O(\alpha_1) - I(\alpha_1)$. Consider the interpretation D where $D(M_2)$ is not empty and $D(M_1) = \{(1, \ldots, 1; 1, \ldots, 1)\}$. Let ν_1 be the valuation that is 2 on v and 1 elsewhere and let ν_2 be the valuation that is 1 everywhere. Clearly, $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$. Take $\nu_1' := \nu_1[v : 1]$. We establish that $v \in I^{\text{sem}}(\alpha)$ by arguing that there is no ν_2' for which $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$. Thereto, suppose $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$. In particular, $(\nu_1', \nu_2') \in [\![\alpha_1]\!]_D$, whence $\nu_2' = \nu_1'$ from the structure of D. However, $(\nu_1', \nu_1') \in [\![\alpha_2]\!]_D$ by Lemma 5.1, so $(\nu_1', \nu_2') \notin [\![\alpha]\!]_D$ as desired.

5.5.3 Neither M_1 nor M_2 is nullary. Recall the definitions:

- $O^{\text{syn}}(\alpha) = O(\alpha_1)$;
- $I^{\text{syn}}(\alpha) = I(\alpha_1) \cup I(\alpha_2) \cup (O(\alpha_1) \triangle O(\alpha_2))$.

The proof of $O^{\text{syn}}(\alpha) \subseteq O^{\text{sem}}(\alpha)$ is done together with the proof that $v \in I^{\text{sem}}(\alpha)$ for every $v \in I(\alpha_1)$. Discussions for the other cases for $v \in I^{\text{syn}}(\alpha)$ follow afterwards. Since $M_1 \neq M_2$, it is clear that $[\![\alpha]\!]_D = [\![\alpha_1]\!]_D$ for any interpretation D where $D(M_2)$ is empty. By induction and Lemma 5.2, we establish that $O^{\text{syn}}(\alpha_1) \subseteq O^{\text{sem}}(\alpha)$ and $I^{\text{syn}}(\alpha_1) \subseteq I^{\text{sem}}(\alpha)$. Thus, $O(\alpha_1) \subseteq O^{\text{sem}}(\alpha)$ and $I(\alpha_1) \subseteq I^{\text{sem}}(\alpha)$ is trivial.

When $v \in I(\alpha_2) - I(\alpha_1)$. Let $v \in I(\alpha_2) - I(\alpha_1)$. Consider an interpretation D where $D(M_1) = \{(1, \dots, 1; 1, \dots, 1)\}$ and similarly $D(M_2) = \{(1, \dots, 1; 1, \dots, 1)\}$. Let ν_1 be the valuation that 2 on v and 1 elsewhere. Also, let ν_2 be the valuation that is 1 on $O(\alpha_1)$ and agrees with ν_1 everywhere else. Clearly, $(\nu_1, \nu_2) \in [\![\alpha_1]\!]_D$. Further, $(\nu_1, \nu_2) \notin [\![\alpha_2]\!]_D$. Indeed, since $v \in I(\alpha_2)$ then ν_1 should have the value of 1 on v for (ν_1, ν_2) to be in $[\![\alpha_2]\!]_D$. Take $\nu_1' := \nu_1[v : 1]$. We establish that $v \in I^{\text{sem}}(\alpha)$ by arguing that there is no ν_2' for which $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$. Thereto, suppose that $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$. Hence, $(\nu_1', \nu_2') \in [\![\alpha_1]\!]_D$ and $(\nu_1', \nu_2') \in [\![\alpha_2]\!]_D$. Indeed, $(\nu_1', \nu_2') \in [\![\alpha_1]\!]_D$ whence $\nu_1' = \nu_2'$. Clearly, $(\nu_1', \nu_1') \in [\![\alpha_2]\!]_D$ showing that $(\nu_1', \nu_1') \notin [\![\alpha]\!]_D$ as desired. Therefore, $v \in I^{\text{sem}}(\alpha)$.

When $v \in (O(\alpha_1) \triangle O(\alpha_2)) - (I(\alpha_1) \cup I(\alpha_2))$. Let $v \in (O(\alpha_1) \triangle O(\alpha_2)) - (I(\alpha_1) \cup I(\alpha_2))$. Consider an interpretation D where $D(M_1) = \{(1, \dots, 1; 1, \dots, 1)\}$ and $D(M_2) = \{(1, \dots, 1; 1, \dots, 1)\}$. Let ν_1 be the valuation that is 2 on v and 1 elsewhere. Also let ν_2 be the valuation that is 1 on $O(\alpha_1)$ and agrees with ν_1 everywhere else. Clearly, $(\nu_1, \nu_2) \in [\![\alpha_1]\!]_D$. Furthermore, $(\nu_1, \nu_2) \notin [\![\alpha_2]\!]_D$. In particular, when $v \in O(\alpha_1) - (I(\alpha_1) \cup I(\alpha_2) \cup O(\alpha_2))$, we know that $\nu_1(v) = 2$ and $\nu_2(v) = 1$. Since $v \notin O(\alpha_2)$, then $(\nu_1, \nu_2) \notin [\![\alpha_2]\!]_D$. In the other case, when $v \in O(\alpha_2) - (I(\alpha_1) \cup I(\alpha_2) \cup O(\alpha_1))$, we know that $\nu_1(v) = \nu_2(v) = 2$ since $(\nu_1, \nu_2) \in [\![\alpha_1]\!]_D$. Consequently, $(\nu_1, \nu_2) \notin [\![\alpha_2]\!]_D$ since $v \in O(\alpha_2)$ but $\nu_2(v) = 2$. We verify that $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$. Take $\nu_1' := \nu_1[v : 1]$. We establish that $v \in I^{\text{sem}}(\alpha)$ by arguing that there is no ν_2' for which $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$. Thereto, suppose that $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$. Hence, $(\nu_1', \nu_2') \in [\![\alpha_1]\!]_D$ and $(\nu_1', \nu_2') \in [\![\alpha_2]\!]_D$. Indeed, $(\nu_1', \nu_2') \in [\![\alpha_1]\!]_D$ whence $\nu_1' = \nu_2'$. Clearly, $(\nu_1', \nu_1') \in [\![\alpha_2]\!]_D$ showing that $(\nu_1', \nu_1') \notin [\![\alpha]\!]_D$ as desired. Therefore, $v \in I^{\text{sem}}(\alpha)$.

5.6 Composition

Let α be of the form $\alpha_1 ; \alpha_2$, where α_1 is $M_1(\bar{x_1}; \bar{y_1})$ and α_2 is $M_2(\bar{x_2}; \bar{y_2})$. We distinguish different cases based on whether M_1 or M_2 is nullary. If M_1 and M_2 are both nullary there is nothing to prove.

5.6.1 M_1 is nullary, M_2 is not. Clearly, $[\![\alpha]\!]_D = [\![\alpha_2]\!]_D$ for any interpretation D where $D(M_1)$ is not empty. In this case, $I(\alpha_1)$ and $O(\alpha_1)$ are both empty, then we observe that

- $O^{\text{syn}}(\alpha) = O(\alpha_2)$;
- $I^{\text{syn}}(\alpha) = I(\alpha_2)$.

First, we verify $O^{\text{syn}}(\alpha) \subseteq O^{\text{sem}}(\alpha)$. Let $v \in O(\alpha_2)$. We know that $O^{\text{syn}}(\alpha_2) \subseteq O^{\text{sem}}(\alpha_2)$ by induction, then $v \in O^{\text{sem}}(\alpha_2)$. By definition, we know that there is an interpretation D' and $(\nu_1, \nu_2) \in [\![\alpha_2]\!]_{D'}$ such that $\nu_1(v) \neq \nu_2(v)$. Take D'' to be the interpretation where $D''(M) = D'(M)$ for any $M \neq M_1$ while $D''(M_1)$ is not empty. Clearly, $(\nu_1, \nu_2) \in [\![\alpha]\!]_{D''}$, whence, $M_1 \neq M_2$, $(\nu_1, \nu_1) \in [\![\alpha_1]\!]_{D''}$ by Lemma 5.1, and $[\![\alpha]\!]_{D''} = [\![\alpha_2]\!]_{D'}$. It follows then that $v \in O^{\text{sem}}(\alpha)$.

Similarly, we proceed to verify $I^{\text{syn}}(\alpha) \subseteq I^{\text{sem}}(\alpha)$. Let $v \in I(\alpha_2)$. We know that $I^{\text{syn}}(\alpha_2) \subseteq I^{\text{sem}}(\alpha_2)$ by induction, then $v \in I^{\text{sem}}(\alpha_2)$. By definition, we know that there is an interpretation D', $(\nu_1, \nu_2) \in [\![\alpha_2]\!]_{D'}$, and $\nu_1'(v) \neq \nu_1(v)$ such that $(\nu_1', \nu_2') \notin [\![\alpha_2]\!]_{D'}$ for every valuation ν_2' that agrees with ν_2 on $O^{\text{sem}}(\alpha_2)$.

Take D'' to be the interpretation where $D''(M) = D'(M)$ for any $M \neq M_1$ while $D''(M_1)$ is not empty. Clearly, $[\![\alpha]\!]_{D''} = [\![\alpha_2]\!]_{D'}$, whence, $M_1 \neq M_2$. Therefore, $O^{\text{sem}}(\alpha_2) \subseteq O^{\text{sem}}(\alpha)$. Hence, $v \in I^{\text{sem}}(\alpha)$. Indeed, $(\nu_1, \nu_2) \in [\![\alpha]\!]_{D''}$ and for any valuation ν_2' if ν_2' agrees with ν_2 on $O^{\text{sem}}(\alpha)$, then ν_2' agrees with ν_2 on $O^{\text{sem}}(\alpha_2)$.

5.6.2 *M_2 is nullary, M_1 is not.*

Clearly, $[\![\alpha]\!]_D = [\![\alpha_1]\!]_D$ for any interpretation D where $D(M_2)$ is not empty. In this case, $I(\alpha_2)$ and $O(\alpha_2)$ are both empty, then we observe that

- $O^{\text{syn}}(\alpha) = O(\alpha_1)$;
- $I^{\text{syn}}(\alpha) = I(\alpha_1)$.

First, we verify $O^{\text{syn}}(\alpha) \subseteq O^{\text{sem}}(\alpha)$. Let $v \in O(\alpha_1)$. We know that $O^{\text{syn}}(\alpha_1) \subseteq O^{\text{sem}}(\alpha_1)$ by induction, then $v \in O^{\text{sem}}(\alpha_1)$. By definition, we know that there is an interpretation D' and $(\nu_1, \nu_2) \in [\![\alpha_1]\!]_{D'}$ such that $\nu_1(v) \neq \nu_2(v)$. Take D'' to be the interpretation where $D''(M) = D'(M)$ for any $M \neq M_2$ while $D''(M_2)$ is not empty. Clearly, $(\nu_1, \nu_2) \in [\![\alpha]\!]_{D''}$, whence, $M_1 \neq M_2$, $(\nu_2, \nu_2) \in [\![\alpha_2]\!]_{D''}$ by Lemma 5.1, and $[\![\alpha]\!]_{D''} = [\![\alpha_1]\!]_{D'}$. It follows then that $v \in O^{\text{sem}}(\alpha)$.

Similarly, we proceed to verify $I^{\text{syn}}(\alpha) \subseteq I^{\text{sem}}(\alpha)$. Let $v \in I(\alpha_1)$. We know that $I^{\text{syn}}(\alpha_1) \subseteq I^{\text{sem}}(\alpha_1)$ by induction, then $v \in I^{\text{sem}}(\alpha_1)$. By definition, we know that there is an interpretation D', $(\nu_1, \nu_2) \in [\![\alpha_1]\!]_{D'}$, and $\nu_1'(v) \neq \nu_1(v)$ such that $(\nu_1', \nu_2') \notin [\![\alpha_1]\!]_{D'}$ for every valuation ν_2' that agrees with ν_2 on $O^{\text{sem}}(\alpha_1)$.

Take D'' to be the interpretation where $D''(M) = D'(M)$ for any $M \neq M_2$ while $D''(M_2)$ is not empty. Clearly, $[\![\alpha]\!]_{D''} = [\![\alpha_1]\!]_{D'}$, whence, $M_1 \neq M_2$. Therefore, $O^{\text{sem}}(\alpha_1) \subseteq O^{\text{sem}}(\alpha)$. Hence, $v \in I^{\text{sem}}(\alpha)$. Indeed, $(\nu_1, \nu_2) \in [\![\alpha]\!]_{D''}$ and for any valuation ν_2' if ν_2' agrees with ν_2 on $O^{\text{sem}}(\alpha)$, then ν_2' agrees with ν_2 on $O^{\text{sem}}(\alpha_1)$.

5.6.3 *Neither M_1 nor M_2 is nullary.*

Recall the definitions:

- $O^{\text{syn}}(\alpha) = O(\alpha_1) \cup O(\alpha_2)$;
- $I^{\text{syn}}(\alpha) = I(\alpha_1) \cup (I(\alpha_2) - O(\alpha_1))$.

We first proceed to verify $O^{\text{syn}}(\alpha) \subseteq O^{\text{sem}}(\alpha)$. Let $v \in O(\alpha_1) \cup O(\alpha_2)$. Consider an interpretation D such that

$$D(M_1) = \{(1, \ldots, 1; 2, \ldots, 2), (i_1, \ldots, i_m; 3, \ldots, 3)\},$$

where $i_1, \ldots, i_m$ are all the combinations of $\{1, 2\}$. Similarly,

$$D(M_2) = \{(1, \ldots, 1; 2, \ldots, 2), (i_1, \ldots, i_n; 3, \ldots, 3)\},$$

where $i_1, \ldots, i_n$ are all the combinations of $\{1, 2\}$.

Let ν_1 be the valuation that is 1 everywhere. Also, let ν be the valuation that is 2 on $O(\alpha_1)$ and 1 elsewhere. Clearly, $(\nu_1, \nu) \in [\![\alpha_1]\!]_D$. Let ν_2 be the valuation that is 3 on $O(\alpha_2)$, 2 on $O(\alpha_1) - O(\alpha_2)$, and 1 elsewhere. Clearly, $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$, whence $(\nu, \nu_2) \in [\![\alpha_2]\!]_D$. Hence, $v \in O^{\text{sem}}(\alpha)$. Indeed, $\nu_2(v) \neq \nu_1(v)$ for $v \in O^{\text{syn}}(\alpha)$.

Now we proceed to verify $I^{\text{syn}}(\alpha) \subseteq I^{\text{sem}}(\alpha)$. Let $v \in I(\alpha_1) \cup (I(\alpha_2) - O(\alpha_1))$. Consider an interpretation D where $D(M_1) = \{(1, \ldots, 1; 1, \ldots, 1)\}$ and similarly $D(M_2) = \{(1, \ldots, 1; 1, \ldots, 1)\}$. Let ν_1 be the valuation that is 1 everywhere. Clearly, $(\nu_1, \nu_1) \in [\![\alpha]\!]_D$, whence $(\nu_1, \nu_1) \in [\![\alpha_1]\!]_D$ and $(\nu_1, \nu_1) \in [\![\alpha_2]\!]_D$.

Take $\nu_1' := \nu_1[v : 2]$. We establish that $v \in I^{\text{sem}}(\alpha)$ by arguing that there is no valuation ν_2' for which $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$. In particular, when $v \in I(\alpha_1)$. Clearly, there is no ν_2' such that $(\nu_1', \nu_2') \in [\![\alpha_1]\!]_D$. On the other hand, when $v \in I(\alpha_2) - O(\alpha_1)$. Clearly, $(\nu_1', \nu) \in [\![\alpha_1]\!]_D$, whence $\nu = \nu_1'$. However, there is no ν_2' such that $(\nu_1', \nu_2') \in [\![\alpha_2]\!]_D$. Thus, there is no ν_2' such that $(\nu_1', \nu_2') \notin [\![\alpha]\!]_D$ as desired. We conclude that $v \in I^{\text{sem}}(\alpha)$.

5.7 Converse

Let α be of the form $\alpha_1^{\smile}$, where $\alpha_1 := M(\bar{x}; \bar{y})$. Recall the definitions:

- $O^{\text{syn}}(\alpha) = O(\alpha_1)$;
- $I^{\text{syn}}(\alpha) = I(\alpha_1) \cup O(\alpha_1)$.

We first proceed to verify $O^{\text{syn}}(\alpha) \subseteq O^{\text{sem}}(\alpha)$. Let $v \in O(\alpha_1)$. Consider an interpretation D where

$$D(M) = \{(1, \ldots, 1; 2, \ldots, 2)\}.$$

Let ν_1 be the valuation that is 2 on $O(\alpha_1)$ and 1 elsewhere. Also let ν_2 be the valuation that is 1 everywhere. Clearly, $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$ since $(\nu_2, \nu_1) \in [\![\alpha_1]\!]_D$. Therefore, $v \in O^{\text{sem}}(\alpha)$ since $\nu_1(v) \neq \nu_2(v)$.

Now we proceed to verify $I^{\text{syn}}(\alpha) \subseteq I^{\text{sem}}(\alpha)$. Let $v \in I(\alpha_1) \cup O(\alpha_1)$. Consider the same interpretation D and the same valuations ν_1 and ν_2. We established that $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$. Take $\nu_1' := \nu_1[v : 3]$. We establish that $v \in I^{\text{sem}}(\alpha)$ by arguing that there is no ν_2' for which $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$. Indeed, when $v \in O(\alpha_1)$, then ν_1 has to be 2 on v. In the other case, when $v \in I(\alpha_1) - O(\alpha_1)$, then ν_1 has to be 1 on v. Thus, there is no ν_2' for which $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$ as desired. Consequently, $v \in I^{\text{sem}}(\alpha)$.

5.8 Left Cylindrification

Let α be of the form $\text{cyl}^l_x(\alpha_1)$, where $\alpha_1 := M(\bar{x}; \bar{y})$. Recall the definitions:

- $O^{\text{syn}}(\alpha) = O(\alpha_1) \cup \{x\}$;
- $I^{\text{syn}}(\alpha) = I(\alpha_1) - \{x\}$.

We first proceed to verify $O^{\text{syn}}(\alpha) \subseteq O^{\text{sem}}(\alpha)$. Let $v \in O(\alpha_1) \cup \{x\}$. Consider an interpretation D where

$$D(M) = \{(1, \ldots, 1; 2, \ldots, 2)\}.$$

Let ν_1 be the valuation that is 3 on x and 1 elsewhere. Also let ν_2 be the valuation that is 2 on $O(\alpha_1)$ and 1 everywhere else. Clearly, $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$ since $(\nu_1[x : 1], \nu_2) \in [\![\alpha_1]\!]_D$. Therefore, $v \in O^{\text{sem}}(\alpha)$ since $\nu_1(v) \neq \nu_2(v)$.

Now we proceed to verify $I^{\text{syn}}(\alpha) \subseteq I^{\text{sem}}(\alpha)$. Let $v \in I(\alpha_1) - \{x\}$. Consider the same interpretation D and the same valuations ν_1 and ν_2. We established that $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$. Take $\nu_1' := \nu_1[v : 2]$. We establish that $v \in I^{\text{sem}}(\alpha)$ by arguing that there is no ν_2' for which $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$. Indeed, this is true since $v \in I(\alpha_1) - \{x\}$. Consequently, $v \in I^{\text{sem}}(\alpha)$.

5.9 Right Cylindrification

Let α be of the form $\text{cyl}^r_x(\alpha_1)$, where $\alpha_1 := M(\bar{x}; \bar{y})$. Recall the definitions:

- $O^{\text{syn}}(\alpha) = O(\alpha_1) \cup \{x\}$;
- $I^{\text{syn}}(\alpha) = I(\alpha_1)$.

We first proceed to verify $O^{\text{syn}}(\alpha) \subseteq O^{\text{sem}}(\alpha)$. Let $v \in O(\alpha_1) \cup \{x\}$. Consider an interpretation D where

$$D(M) = \{(1, \ldots, 1; 2, \ldots, 2)\}.$$

Let ν_1 be the valuation that is 1 everywhere. Also let ν_2 be the valuation that is 2 on $O(\alpha_1)$ and on x and 1 everywhere else. Clearly, $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$ since either $(\nu_1, \nu_2[x : 1]) \in [\![\alpha_1]\!]_D$ or $(\nu_1, \nu_2) \in [\![\alpha_1]\!]_D$. Therefore, $v \in O^{\text{sem}}(\alpha)$ since $\nu_1(v) \neq \nu_2(v)$.

Now we proceed to verify $I^{\text{syn}}(\alpha) \subseteq I^{\text{sem}}(\alpha)$. Let $v \in I(\alpha_1)$. Consider the same interpretation D and the same valuations ν_1 and ν_2. We established that $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$. Take $\nu_1' := \nu_1[v : 2]$. We establish that $v \in I^{\text{sem}}(\alpha)$ by arguing that there is no ν_2' for which $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$. Indeed, this is true since $v \in I(\alpha_1)$. Consequently, $v \in I^{\text{sem}}(\alpha)$.

5.10 Left Selection

Let α be of the form $\sigma^{l}_{x=y}(\alpha_1)$, where $\alpha_1 := M(\bar{u};\bar{w})$, $\bar{u} = u_1,\ldots,u_n$, and $\bar{w} = w_1,\ldots,w_m$. We distinguish different cases based on whether $x =_{\text{syn}} y$.

When x and y are the same variable ($x =_{\text{syn}} y$). Recall the definitions in this case:

- $O^{\text{syn}}(\alpha) = O(\alpha_1)$;
- $I^{\text{syn}}(\alpha) = I(\alpha_1)$.

We proceed to verify that $O^{\text{syn}}(\alpha) \subseteq O^{\text{sem}}(\alpha)$ and $I^{\text{syn}}(\alpha) \subseteq I^{\text{sem}}(\alpha)$. Indeed, this is true since $[\![\alpha]\!]_D = [\![\alpha_1]\!]_D$ for any interpretation D because of $x =_{\text{syn}} y$.

When x and y are different variables ($x \neq_{\text{syn}} y$). Recall the definitions in this case:

- $O^{\text{syn}}(\alpha) = O(\alpha_1)$;
- $I^{\text{syn}}(\alpha) = I(\alpha_1) \cup \{x, y\}$.

We first proceed to verify $O^{\text{syn}}(\alpha) \subseteq O^{\text{sem}}(\alpha)$. Let $v \in O(\alpha_1)$. Consider an interpretation D where

$$D(M) = \{(1,\ldots,1;2,\ldots,2)\}.$$

Let ν_1 be the valuation that is 1 everywhere. Also let ν_2 be the valuation that is 2 on $O(\alpha_1)$ and 1 everywhere else. Clearly, $(\nu_1,\nu_2) \in [\![\alpha]\!]_D$ since $(\nu_1,\nu_2) \in [\![\alpha_1]\!]_D$ and $\nu_1(x) = \nu_1(y)$. Therefore, $v \in O^{\text{sem}}(\alpha)$ since $\nu_1(v) \neq \nu_2(v)$.

Now we proceed to verify $I^{\text{syn}}(\alpha) \subseteq I^{\text{sem}}(\alpha)$. Let $v \in I(\alpha_1) \cup \{x, y\}$. Consider an interpretation D where

$$D(M_1) = \{(1,\ldots,1;1,\ldots,1)\}.$$

Let ν_1 be the valuation that is 1 everywhere. Clearly, $(\nu_1,\nu_1) \in [\![\alpha]\!]_D$ since $(\nu_1,\nu_1) \in [\![\alpha_1]\!]_D$ and $\nu_1(x) = \nu_1(y)$. Take $\nu_1' := \nu_1[v:2]$. We establish that $v \in I^{\text{sem}}(\alpha)$ by arguing that there is no ν_2' for which $(\nu_1',\nu_2') \in [\![\alpha]\!]_D$. In particular, when $v \in I(\alpha_1)$, it is clear that there is no ν_2' such that $(\nu_1',\nu_2') \in [\![\alpha_1]\!]_D$. In the other case, when v is either x or y, there is no ν_2' such that $(\nu_1',\nu_2') \in [\![\alpha]\!]_D$. Indeed, this is true since $x \neq_{\text{syn}} y$ and $\nu_1'(x) \neq \nu_1'(y)$. Consequently, $v \in I^{\text{sem}}(\alpha)$.

5.11 Right Selection

Let α be of the form $\sigma^{r}_{x=y}(\alpha_1)$, where $\alpha_1 := M(\bar{u};\bar{w})$, $\bar{u} = u_1,\ldots,u_n$, and $\bar{w} = w_1,\ldots,w_m$. We distinguish different cases based on whether $x =_{\text{syn}} y$.

When x and y are the same variable ($x =_{\text{syn}} y$). Recall the definitions in this case:

- $O^{\text{syn}}(\alpha) = O(\alpha_1)$;
- $I^{\text{syn}}(\alpha) = I(\alpha_1)$.

We proceed to verify that $O^{\text{syn}}(\alpha) \subseteq O^{\text{sem}}(\alpha)$ and $I^{\text{syn}}(\alpha) \subseteq I^{\text{sem}}(\alpha)$. Indeed, this is true since $[\![\alpha]\!]_D = [\![\alpha_1]\!]_D$ for any interpretation D because of $x =_{\text{syn}} y$.

When x and y are different variables ($x \neq_{\text{syn}} y$). Recall the definitions in this case:

- $O^{\text{syn}}(\alpha) = O(\alpha_1)$;
- $I^{\text{syn}}(\alpha) = I(\alpha_1) \cup (\{x, y\} - O(\alpha_1))$.

We first proceed to verify $O^{\text{syn}}(\alpha) \subseteq O^{\text{sem}}(\alpha)$. Let $v \in O(\alpha_1)$. Consider an interpretation D where

$$D(M) = \{(i_1,\ldots,i_n;2,\ldots,2)\},$$

such that $i_j = 2$ if u_j is either x or y and $u_j \notin O(\alpha_1)$, otherwise, $u_j = 1$. Let ν_1 be the valuation that is 2 on x if $x \notin O(\alpha_1)$, 2 on y if $y \notin O(\alpha_1)$, and 1 everywhere. Also let ν_2 be the valuation that is 2 on $O(\alpha_1)$ and agrees with ν_1 everywhere else. Clearly, $(\nu_1,\nu_2) \in [\![\alpha]\!]_D$ since $(\nu_1,\nu_2) \in [\![\alpha_1]\!]_D$ and $\nu_2(x) = \nu_2(y)$. Therefore, $v \in O^{\text{sem}}(\alpha)$ since $\nu_1(v) \neq \nu_2(v)$.

Now we proceed to verify $I^{\text{syn}}(\alpha) \subseteq I^{\text{sem}}(\alpha)$. Let $v \in I(\alpha_1) \cup \{x, y\}$. Consider an interpretation D where $D(M_1) = \{(1, \ldots, 1; 1, \ldots, 1)\}$. Let ν_1 be the valuation that is 1 everywhere. Clearly, $(\nu_1, \nu_1) \in [\![\alpha]\!]_D$ since $(\nu_1, \nu_1) \in [\![\alpha_1]\!]_D$ and $\nu_1(x) = \nu_1(y)$. Take $\nu_1' := \nu_1[v : 2]$. We establish that $v \in I^{\text{sem}}(\alpha)$ by arguing that there is no ν_2' for which $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$. In particular, when $v \in I(\alpha_1)$, it is clear that there is no ν_2' such that $(\nu_1', \nu_2') \in [\![\alpha_1]\!]_D$. Now we need to verify the same when v is x or y and $v \notin I(\alpha_1)$. Thereto, suppose $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$. In the case of v is x and $x \notin I(\alpha_1)$, this is only possible when $x \notin O(\alpha_1)$. Therefore, $\nu_2'(x) = \nu_1'(x) = 2$ but $\nu_2'(y) = 1$ whether $y \in O(\alpha_1)$ or not. Hence, $(\nu_1', \nu_2') \notin [\![\alpha]\!]_D$ since $x \neq_{\text{syn}} y$ and $\nu_2'(x) \neq \nu_2'(y)$. The case when v is y and $y \notin I(\alpha_1)$ is symmetric. Consequently, $v \in I^{\text{sem}}(\alpha)$.

5.12 Left-to-Right Selection

Let α be of the form $\sigma^{lr}_{x=y}(\alpha_1)$, where $\alpha_1 := M(\bar{u}; \bar{w})$, $\bar{u} = u_1, \ldots, u_n$, and $\bar{w} = w_1, \ldots, w_m$. We distinguish different cases based on whether $x =_{\text{syn}} y$ and $y \in O(\alpha_1)$.

When $x =_{\text{syn}} y$ and $y \in O(\alpha_1)$. Recall the definitions in this case:

- $O^{\text{syn}}(\alpha) = O(\alpha_1) - \{x\}$;
- $I^{\text{syn}}(\alpha) = I(\alpha_1) \cup \{x\}$.

In what follows, since $x =_{\text{syn}} y$ we will refer to both of them with x. We first proceed to verify $O^{\text{syn}}(\alpha) \subseteq O^{\text{sem}}(\alpha)$. Let $v \in O(\alpha_1) - \{x\}$. Consider an interpretation D such that $D(M) = \{(1, \ldots, 1; o_1, \ldots, o_m)\}$ where $o_j = 1$ if $w_j = y$, otherwise $o_j = 2$. Let ν_1 be the valuation that is 1 everywhere. Also let ν_2 be the valuation that is 2 on $O(\alpha_1) - \{x\}$ and 1 everywhere else. Clearly, $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$ since $(\nu_1, \nu_2) \in [\![\alpha_1]\!]_D$ and $\nu_1(x) = \nu_2(x)$. Therefore, $v \in O^{\text{sem}}(\alpha)$ since $\nu_1(v) \neq \nu_2(v)$.

Now we proceed to verify $I^{\text{syn}}(\alpha) \subseteq I^{\text{sem}}(\alpha)$. Let $v \in I(\alpha_1) \cup \{x\}$. Consider an interpretation D where

$$D(M) = \{(1, \ldots, 1; 1, \ldots, 1)\}.$$

Let ν_1 be the valuation that is 1 everywhere. Clearly, $(\nu_1, \nu_1) \in [\![\alpha]\!]_D$ since $(\nu_1, \nu_1) \in [\![\alpha_1]\!]_D$. Take $\nu_1' := \nu_1[v : 2]$. We establish that $v \in I^{\text{sem}}(\alpha)$ by arguing that there is no ν_2' for which $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$. Thereto, suppose that $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$. In particular, when $v \in I(\alpha_1)$ it is clear that $(\nu_1', \nu_2') \notin [\![\alpha_1]\!]_D$. On the other hand, when $v = x$ and $x \in O(\alpha_1) - I(\alpha_1)$, clearly $\nu_1'(x) = 2 \neq 1 = \nu_2'(x)$. Consequently, $v \in I^{\text{sem}}(\alpha)$.

When $x =_{\text{syn}} y$ and $y \notin O(\alpha_1)$. Recall the definitions in this case:

- $O^{\text{syn}}(\alpha) = O(\alpha_1)$;
- $I^{\text{syn}}(\alpha) = I(\alpha_1)$.

In what follows, since $x =_{\text{syn}} y$ we will refer to both of them with x. We proceed to verify $O^{\text{syn}}(\alpha) \subseteq O^{\text{sem}}(\alpha)$ and $I^{\text{syn}}(\alpha) \subseteq I^{\text{sem}}(\alpha)$. Indeed, this is true since $[\![\alpha]\!]_D = [\![\alpha_1]\!]_D$ for any interpretation D because of $x =_{\text{syn}} y$ and $x \notin O(\alpha_1)$.

When $x \neq_{\text{syn}} y$ and $y \in O(\alpha_1)$. Recall the definitions in this case:

- $O^{\text{syn}}(\alpha) = O(\alpha_1)$;
- $I^{\text{syn}}(\alpha) = I(\alpha_1) \cup \{x\}$.

We first proceed to verify $O^{\text{syn}}(\alpha) \subseteq O^{\text{sem}}(\alpha)$. Let $v \in O(\alpha_1)$. Consider an interpretation D such that

$$D(M) = \{(i_1, \ldots, i_n; o_1, \ldots, o_m)\},$$

where $i_j = 2$ if $u_j = x$, otherwise $i_j = 1$. Also, $o_j = 3$ if $w_j = x$, otherwise $o_j = 2$. Let ν_1 be the valuation that is 2 on x and 1 everywhere else. Also let ν_2 be the valuation that is 2 on $O(\alpha_1) - \{x\}$, 3 on x if $x \in O(\alpha_1)$ and agrees with ν_1 everywhere else. Clearly, $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$

since $(\nu_1, \nu_2) \in [\![\alpha_1]\!]_D$ and $\nu_1(x) = \nu_2(y)$. Therefore, $v \in O^{\text{sem}}(\alpha)$. Indeed, in both cases whether $x \in O(\alpha_1)$ or not, $\nu_1(v) \neq \nu_2(v)$.

Now we proceed to verify $I^{\text{syn}}(\alpha) \subseteq I^{\text{sem}}(\alpha)$. Let $v \in I(\alpha_1) \cup \{x\}$. Consider an interpretation D where

$$D(M) = \{(1, \ldots, 1; 1, \ldots, 1)\}.$$

Let ν_1 be the valuation that is 1 everywhere. Clearly, $(\nu_1, \nu_1) \in [\![\alpha]\!]_D$ since $(\nu_1, \nu_1) \in [\![\alpha_1]\!]_D$. Take $\nu_1' := \nu_1[v : 2]$. We establish that $v \in I^{\text{sem}}(\alpha)$ by arguing that there is no ν_2' for which $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$. Thereto, suppose that $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$. In particular, when $v \in I(\alpha_1)$ it is clear that $(\nu_1', \nu_2') \notin [\![\alpha_1]\!]_D$. On the other hand, when $v = x$ and $y \in O(\alpha_1)$, clearly $\nu_1'(x) = 2 \neq 1 = \nu_2'(y)$. Consequently, $v \in I^{\text{sem}}(\alpha)$.

When $x \neq_{\text{syn}} y$ and $y \notin O(\alpha_1)$. Recall the definitions in this case:

- $O^{\text{syn}}(\alpha) = O(\alpha_1)$;
- $I^{\text{syn}}(\alpha) = I(\alpha_1) \cup \{x, y\}$.

We first proceed to verify $O^{\text{syn}}(\alpha) \subseteq O^{\text{sem}}(\alpha)$. Let $v \in O(\alpha_1)$. Consider an interpretation D such that

$$D(M) = \{(1, \ldots, 1; 2, \ldots, 2)\}.$$

Let ν_1 be the valuation that is 1 everywhere. Also let ν_2 be the valuation that is 2 on $O(\alpha_1)$ and 1 everywhere else. Clearly, $(\nu_1, \nu_2) \in [\![\alpha]\!]_D$ since $(\nu_1, \nu_2) \in [\![\alpha_1]\!]_D$ and $\nu_1(x) = \nu_2(y)$. Indeed, this is true since $y \notin O(\alpha_1)$, then $\nu_1(y) = \nu_2(y)$. Therefore, $v \in O^{\text{sem}}(\alpha)$ since $\nu_1(v) \neq \nu_2(v)$.

Now we proceed to verify $I^{\text{syn}}(\alpha) \subseteq I^{\text{sem}}(\alpha)$. Let $v \in I(\alpha_1) \cup \{x, y\}$. Consider an interpretation D where

$$D(M) = \{(1, \ldots, 1; 1, \ldots, 1)\}.$$

Let ν_1 be the valuation that is 1 everywhere. Clearly, $(\nu_1, \nu_1) \in [\![\alpha]\!]_D$ since $(\nu_1, \nu_1) \in [\![\alpha_1]\!]_D$. Take $\nu_1' := \nu_1[v : 2]$. We establish that $v \in I^{\text{sem}}(\alpha)$ by arguing that there is no ν_2' for which $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$. Thereto, suppose that $(\nu_1', \nu_2') \in [\![\alpha]\!]_D$. In particular, when $v \in I(\alpha_1)$ it is clear that $(\nu_1', \nu_2') \notin [\![\alpha_1]\!]_D$. On the other hand, when $v = x$ or $v = y$, clearly $\nu_1'(x) \neq (\nu_1'(y) = \nu_2'(y))$ since $y \notin O(\alpha_1)$ and $x \neq_{\text{syn}} y$. Consequently, $v \in I^{\text{sem}}(\alpha)$.

6 OPTIMALITY THEOREM PROOF

In this section, we prove Theorem 3.28. Thus, we would like to show that

$$I^{\text{syn}}(\alpha) \subseteq I(\alpha) \text{ and } O^{\text{syn}}(\alpha) \subseteq O(\alpha).$$

for any LIF expression α, assuming that (I, O) is a sound and compositional input–output definition. The proof is by induction on the structure of α.

Atomic Modules. For atomic module expressions α, this follows directly from Theorem 3.27.

Identity. For $\alpha = id$, this is immediate since $I^{\text{syn}}(id) = O^{\text{syn}}(id) = \emptyset$.

Binary Operators. For $\alpha = \alpha_1 \boxdot \alpha_2$, where $\boxdot$ is a binary operator, we define two atomic module expressions $\alpha_1' = M_1(\bar{x}; \bar{y})$ and $\alpha_2' = M_2(\bar{u}, \bar{v})$ where $\bar{x} = I(\alpha_1)$, $\bar{y} = O(\alpha_1)$, $\bar{u} = I(\alpha_2)$, and $\bar{v} = O(\alpha_2)$ with M_i distinct module names of the right arity.

Since (I, O) is sound, we know that the following holds for $i \in \{1, 2\}$:

$$I(\alpha_i) = I(\alpha_i') = I^{\text{syn}}(\alpha_i') \text{ and } O(\alpha_i) = O(\alpha_i') = O^{\text{syn}}(\alpha_i'). \tag{1}$$

Moreover by soundness and Proposition 3.20, we know that

$$I^{\text{sem}}(\alpha_1' \boxdot \alpha_2') \subseteq I(\alpha_1' \boxdot \alpha_2') \text{ and } O^{\text{sem}}(\alpha_1' \boxdot \alpha_2') \subseteq O(\alpha_1' \boxdot \alpha_2'). \tag{2}$$

From the Precision Theorem, we know that

$$I^{\text{syn}}(\alpha_1' \boxdot \alpha_2') = I^{\text{sem}}(\alpha_1' \boxdot \alpha_2') \text{ and } O^{\text{syn}}(\alpha_1' \boxdot \alpha_2') = O^{\text{sem}}(\alpha_1' \boxdot \alpha_2'). \tag{3}$$

From the compositionality of (I, O), we know that

$$I(\alpha_1' \boxdot \alpha_2') = I(\alpha_1 \boxdot \alpha_2) \text{ and } O(\alpha_1' \boxdot \alpha_2') = O(\alpha_1 \boxdot \alpha_2). \tag{4}$$

By combining Equations (2–4), we find that

$$I^{\text{syn}}(\alpha_1' \boxdot \alpha_2') \subseteq I(\alpha_1 \boxdot \alpha_2) \text{ and } O^{\text{syn}}(\alpha_1' \boxdot \alpha_2') \subseteq O(\alpha_1 \boxdot \alpha_2). \tag{5}$$

We now **claim** the following

$$I^{\text{syn}}(\alpha_1 \boxdot \alpha_2) \subseteq I^{\text{syn}}(\alpha_1' \boxdot \alpha_2') \text{ and } O^{\text{syn}}(\alpha_1 \boxdot \alpha_2) \subseteq O^{\text{syn}}(\alpha_1' \boxdot \alpha_2'). \tag{6}$$

If we prove our **claim**, then combining Equations (5–6) establishes our theorem for binary operators.

First, we prove our claim for the inductive cases for outputs of the different binary operators. From the inductive hypothesis and Equation (1), we know that for $i \in \{1, 2\}$:

$$O^{\text{syn}}(\alpha_i) \subseteq O(\alpha_i) = O^{\text{syn}}(\alpha_i').$$

Hence, it is clear that

- $O^{\text{syn}}(\alpha_1) \cup O^{\text{syn}}(\alpha_2)$ is a subset of $O^{\text{syn}}(\alpha_1') \cup O^{\text{syn}}(\alpha_2')$, which settles the cases when $\boxdot \in \{\cup, ;\}$ since $O^{\text{syn}}(\beta \boxdot \gamma) = O^{\text{syn}}(\beta) \cup O^{\text{syn}}(\gamma)$ for any LIF expressions β and γ;
- $O^{\text{syn}}(\alpha_1) \cap O^{\text{syn}}(\alpha_2)$ is a subset of $O^{\text{syn}}(\alpha_1') \cap O^{\text{syn}}(\alpha_2')$, which settles the case when $\boxdot$ is $\cap$ since $O^{\text{syn}}(\beta \boxdot \gamma) = O^{\text{syn}}(\beta) \cap O^{\text{syn}}(\gamma)$ for any LIF expressions β and γ;
- $O^{\text{syn}}(\alpha_1)$ is a subset of $O^{\text{syn}}(\alpha_1')$, which settles the case when $\boxdot$ is $-$ since $O^{\text{syn}}(\beta \boxdot \gamma) = O^{\text{syn}}(\beta)$ for any LIF expressions β and γ.

Now, we consider the inductive cases for the inputs of the different binary operators. Similar to the outputs, we know that for $i \in \{1, 2\}$:

$$I^{\text{syn}}(\alpha_i) \subseteq I(\alpha_i) = I^{\text{syn}}(\alpha_i').$$

Consequently,

- when $x \in I^{\text{syn}}(\alpha_1) \cup (I^{\text{syn}}(\alpha_2) - O^{\text{syn}}(\alpha_1))$, we consider the following cases:
  - if $x \in I^{\text{syn}}(\alpha_1)$, then it is clear that $x \in I^{\text{syn}}(\alpha_1')$;
  - if $x \in (I^{\text{syn}}(\alpha_2) - O^{\text{syn}}(\alpha_1))$, then we know that $x \in I^{\text{syn}}(\alpha_2')$. Moreover, since $x \notin O^{\text{syn}}(\alpha_1)$, we know by soundness of $(I^{\text{syn}}, O^{\text{syn}})$ that $x \notin O^{\text{sem}}(\alpha_1)$. Now, we differentiate between two cases
    - when $x \notin O^{\text{syn}}(\alpha_1')$, it is clear that $x \in (I^{\text{syn}}(\alpha_2') - O^{\text{syn}}(\alpha_1'))$;
    - when $x \in O^{\text{syn}}(\alpha_1')$, we know from Equation (1) that $x \in O(\alpha_1)$. From Lemma 3.22 and Equation (1), it follows that $x \in I(\alpha_1)$ and $x \in I^{\text{syn}}(\alpha_1')$.

  In all cases, we verify that $x \in I^{\text{syn}}(\alpha_1') \cup (I^{\text{syn}}(\alpha_2') - O^{\text{syn}}(\alpha_1'))$. This settles the case when $\boxdot$ is ; since $I^{\text{syn}}(\beta \boxdot \gamma) = I^{\text{syn}}(\beta) \cup (I^{\text{syn}}(\gamma) - O^{\text{syn}}(\beta))$ for any LIF expressions β and γ.
- when $x \in I^{\text{syn}}(\alpha_1) \cup I^{\text{syn}}(\alpha_2) \cup (O^{\text{syn}}(\alpha_1) \triangle O^{\text{syn}}(\alpha_2))$, we consider the following cases:
  - if $x \in I^{\text{syn}}(\alpha_i)$ for some i, then it is clear that $x \in I^{\text{syn}}(\alpha_i')$;
  - if $x \in O^{\text{syn}}(\alpha_i) - O^{\text{syn}}(\alpha_j)$ for $i \neq j$, we know that $x \in O^{\text{syn}}(\alpha_i')$. Since $x \notin O^{\text{syn}}(\alpha_j)$, we know by soundness that $x \notin O^{\text{sem}}(\alpha_j)$. Now, we differentiate between two cases
    - when $x \notin O^{\text{syn}}(\alpha_j')$, it is clear that $x \in (O^{\text{syn}}(\alpha_i') \triangle O^{\text{syn}}(\alpha_j'))$;
    - when $x \in O^{\text{syn}}(\alpha_j')$, we know from Equation (1) that $x \in O(\alpha_j)$. From Lemma 3.22 and Equation (1), it follows that $x \in I(\alpha_j)$ and $x \in I^{\text{syn}}(\alpha_j')$.

  In all cases, we verify that $x \in I^{\text{syn}}(\alpha_1') \cup I^{\text{syn}}(\alpha_2') \cup (O^{\text{syn}}(\alpha_1') \triangle O^{\text{syn}}(\alpha_2'))$. This settles the cases when $\boxdot \in \{\cup, \cap, -\}$ since $I^{\text{syn}}(\beta \boxdot \gamma) = I^{\text{syn}}(\beta) \cup I^{\text{syn}}(\gamma) \cup (O^{\text{syn}}(\beta) \triangle O^{\text{syn}}(\gamma))$ for any LIF expressions β and γ.

Unary Operators. We follow a similar approach for unary operators. For $\alpha = \square\alpha_1$, where $\square$ is a unary operator, we define one atomic module expression $\alpha_1' = M_1(\bar{x};\bar{y})$ where $\bar{x} = I(\alpha_1)$, and $\bar{y} = O(\alpha_1)$.

Since (I, O) is sound, we know that the following holds:

$$I(\alpha_1) = I(\alpha_1') = I^{\text{syn}}(\alpha_1') \text{ and } O(\alpha_1) = O(\alpha_1') = O^{\text{syn}}(\alpha_1'). \tag{7}$$

Moreover, we know that

$$I^{\text{sem}}(\square\alpha_1') \subseteq I(\square\alpha_1') \text{ and } O^{\text{sem}}(\square\alpha_1') \subseteq O(\square\alpha_1'). \tag{8}$$

From the precision theorem, we know that

$$I^{\text{syn}}(\square\alpha_1') = I^{\text{sem}}(\square\alpha_1') \text{ and } O^{\text{syn}}(\square\alpha_1') = O^{\text{sem}}(\square\alpha_1'). \tag{9}$$

From the compositionality of (I, O), we know that

$$I(\square\alpha_1') = I(\square\alpha_1) \text{ and } O(\square\alpha_1') = O(\square\alpha_1). \tag{10}$$

By combining Equations (8–10), we find that

$$I^{\text{syn}}(\square\alpha_1') \subseteq I(\square\alpha_1) \text{ and } O^{\text{syn}}(\square\alpha_1') \subseteq O(\square\alpha_1). \tag{11}$$

We now **claim** the following

$$I^{\text{syn}}(\square\alpha_1) \subseteq I^{\text{syn}}(\square\alpha_1') \text{ and } O^{\text{syn}}(\square\alpha_1) \subseteq O^{\text{syn}}(\square\alpha_1'). \tag{12}$$

If we prove our **claim**, then combining Equations (11–12) establishes our theorem for unary operators.

Proving our claim for the inductive cases for outputs of the different unary operators follows directly from the inductive hypothesis and Equation (7), which states that

$$O^{\text{syn}}(\alpha_1) \subseteq O(\alpha_1) = O^{\text{syn}}(\alpha_1').$$

Indeed, $O^{\text{syn}}(\square\alpha_1)$ and $O^{\text{syn}}(\square\alpha_1')$, respectively, simply equal $O^{\text{syn}}(\alpha_1)$ and $O^{\text{syn}}(\alpha_1')$, except for the possible addition or removal of some fixed variable that depends only on $\square$.

Now, we consider the inductive cases for inputs. Similar to the outputs, we know that

$$I^{\text{syn}}(\alpha_1) \subseteq I(\alpha_1) = I^{\text{syn}}(\alpha_1').$$

Here, we only discuss the cases for $\sigma^{lr}_{x=y}$ and $\sigma^{r}_{x=y}$ as all the other cases again follow directly from the above inclusion and the definition of I^{syn}.

We begin by the cases for $\sigma^{lr}_{x=y}$. The cases are:

- when $y \in O^{\text{syn}}(\alpha_1)$, we have
$$I^{\text{syn}}(\sigma^{lr}_{x=y}(\alpha_1)) = I^{\text{syn}}(\alpha_1) \cup \{x\} \subseteq I^{\text{syn}}(\alpha_1') \cup \{x\} = I^{\text{syn}}(\sigma^{lr}_{x=y}(\alpha_1')).$$
- when $y \notin O^{\text{syn}}(\alpha_1)$ and $x =_{\text{syn}} y$, we have
$$I^{\text{syn}}(\sigma^{lr}_{x=y}(\alpha_1)) = I^{\text{syn}}(\alpha_1) \subseteq I^{\text{syn}}(\alpha_1') \subseteq I^{\text{syn}}(\sigma^{lr}_{x=y}(\alpha_1')).$$
- when $y \notin O^{\text{syn}}(\alpha_1)$ and $x \neq_{\text{syn}} y$, by definition
$$I^{\text{syn}}(\sigma^{lr}_{x=y}(\alpha_1)) = I^{\text{syn}}(\alpha_1) \cup \{x, y\}.$$
In case $y \notin O^{\text{syn}}(\alpha_1')$, we are done since $I^{\text{syn}}(\alpha_1) \cup \{x, y\} \subseteq I^{\text{syn}}(\alpha_1') \cup \{x, y\} = I^{\text{syn}}(\sigma^{lr}_{x=y}(\alpha_1'))$. Otherwise, $y \in O^{\text{syn}}(\alpha_1')$ in which case $I^{\text{syn}}(\sigma^{lr}_{x=y}(\alpha_1')) = I^{\text{syn}}(\alpha_1') \cup \{x\}$. What remains to show is that $y \in I^{\text{syn}}(\alpha_1')$. By Equation 7, we have $y \in O(\alpha_1)$. Moreover, $y \notin O^{\text{sem}}(\alpha_1)$ since $y \notin O^{\text{syn}}(\alpha_1)$. By Lemma 3.22 and Equation 7, we obtain $y \in I(\alpha_1) = I^{\text{syn}}(\alpha_1')$ as desired.

Finally, we consider the case for $\sigma^r_{x=y}$ when $x \neq_{\text{syn}} y$. The case when $x =_{\text{syn}} y$ follows directly. By definition,

$$I^{\text{syn}}(\sigma^r_{x=y}(\alpha_1)) = I^{\text{syn}}(\alpha_1) \cup (\{x, y\} - O^{\text{syn}}(\alpha_1)).$$

We can focus on $z \in \{x, y\}$. If $z \in O^{\text{syn}}(\alpha_1)$ or $z \notin O^{\text{syn}}(\alpha_1')$, we are done. Now, consider the case when $z \notin O^{\text{syn}}(\alpha_1)$, but $z \in O^{\text{syn}}(\alpha_1')$. Similar to our reasoning for the last case in $\sigma^{lr}_{x=y}$, we can show that $z \in I^{\text{syn}}(\alpha_1')$, whence, $z \in I^{\text{syn}}(\sigma^{lr}_{x=y}(\alpha_1'))$ by definition.

7 PRIMITIVITY OF COMPOSITION

We now turn our attention to the study of composition in LIF. Indeed, LIF has a rich set of logical operators already, plus an explicit operator (;) for sequential composition. This begs the question whether composition is not already definable in terms of the other operators.

We begin by showing that for "well-behaved" expressions (all subexpressions have disjoint inputs and outputs) composition is redundant in LIF: every well-behaved LIF expression is equivalent to a LIF expression that does not use composition. As a corollary, we will obtain that composition is generally redundant if there is an infinite supply of variables. In contrast, in the bounded variable case, we will show that composition is primitive in LIF. Here, we use LIFnc to denote the fragment of LIF without composition.

7.1 When Inputs and Outputs are Disjoint, Composition is Non-Primitive

Our first non-primitivity result is based on inputs and outputs. We say that a LIF expression β is *io-disjoint* if $O^{\text{sem}}(\beta) \cap I^{\text{sem}}(\beta) = \emptyset$. The following theorem implies that if α, β, and all their subexpressions are io-disjoint, we can rewrite $\alpha ; \beta$ into a LIFnc expression. Of course, this property also holds in case $O^{\text{syn}}(\beta) \cap I^{\text{syn}}(\beta) = \emptyset$ since the syntactic inputs and outputs overapproximate the semantic ones.

Theorem 7.1. *Let α and β be LIF expressions such that β is io-disjoint. Then, $\alpha ; \beta$ is equivalent to*

$$\gamma := \text{cyl}^r_{O^{\text{sem}}(\beta)}(\alpha) \cap \text{cyl}^l_{O^{\text{sem}}(\alpha)}(\beta).$$

Intuitively, the reason why this expression works is as follows: we cylindrify α on the right. In general, this might result in a loss of information, but since we are only cylindrifying outputs of β, this means we only forget the information that would be overwritten by β anyway. Since the inputs and outputs of β are disjoint, β does not need to know what α did to those variables in order to determine its own outputs. We also cylindrify β on the left on the outputs of α, since these values will be set by α. One then still needs to be careful in showing that the intersection indeed removes all artificial pairs, by exploiting the fact that expressions are inertial outside their output.

Proof of Theorem 7.1. Let D be an interpretation. First, we show that $[\![\alpha ; \beta]\!]_D \subseteq [\![\gamma]\!]_D$. If $(\nu_1, \nu_2) \in [\![\alpha ; \beta]\!]_D$, then there is a ν_3 such that $(\nu_1, \nu_3) \in [\![\alpha]\!]_D$ and $(\nu_3, \nu_2) \in [\![\beta]\!]_D$. By definition of the outputs of β, ν_3 and ν_2 agree outside $O^{\text{sem}}(\beta)$. Hence, $(\nu_1, \nu_2) \in [\![\text{cyl}^r_{O^{\text{sem}}(\beta)}(\alpha)]\!]_D$. Similarly, we can show that $(\nu_1, \nu_2) \in [\![\text{cyl}^l_{O^{\text{sem}}(\alpha)}(\beta)]\!]_D$.

For the other inclusion, assume that $(\nu_1, \nu_2) \in [\![\gamma]\!]_D$. Using the definition of the semantics of cylindrification, we find ν_2' such that $(\nu_1, \nu_2') \in [\![\alpha]\!]_D$ and ν_2 agrees with ν_2' outside $O^{\text{sem}}(\beta)$ and we find a ν_1' such that ν_1' agrees with ν_1 outside $O^{\text{sem}}(\alpha)$ and $(\nu_1', \nu_2) \in [\![\beta]\!]_D$. Using the definition of output of β, we know that also ν_1' agrees with ν_2 outside the outputs of β, thus ν_1' and ν_2' agree outside the outputs of β, and hence definitely on the inputs of β. We can apply Proposition 3.6 thanks to the $(I^{\text{syn}}, O^{\text{syn}})$ soundness, $I^{\text{syn}}(\alpha)$ is finite and determines $O^{\text{syn}}(\alpha)$, which contains $O^{\text{sem}}(\alpha)$. So we guarantee that β is determined by its inputs, whence, there exists a ν_2'' such that $(\nu_2', \nu_2'') \in [\![\beta]\!]_D$ where $\nu_2'' = \nu_2$ on the outputs of β and, since β is inertial outside its outputs, $\nu_2'' = \nu_2'$ outside the outputs of β. But we previously established that ν_2' agrees with ν_2 outside the outputs of β, whence $\nu_2'' = \nu_2$. Summarized we now found that $(\nu_1, \nu_2') \in [\![\alpha]\!]_D$ and $(\nu_2', \nu_2) \in [\![\beta]\!]_D$, whence, $(\nu_1, \nu_2) \in [\![\alpha \,;\, \beta]\!]_D$ as desired. □

Given the undecidability results of Section 3, Theorem 7.1 is not effective. We can however give the following syntactic variant.

THEOREM 7.2. *Let α and β be LIF expressions such that $O^{\text{syn}}(\beta) \cap I^{\text{syn}}(\beta) = \emptyset$. Then, $\alpha \,;\, \beta$ is equivalent to*

$$\text{cyl}^r_{O^{\text{syn}}(\beta)}(\alpha) \cap \text{cyl}^l_{O^{\text{syn}}(\alpha)}(\beta).$$

PROOF. Since $I^{\text{syn}}(\beta) \cap O^{\text{syn}}(\beta) = \emptyset$, we obtain by Lemma 3.22 that $O^{\text{sem}}(\beta) = O^{\text{syn}}(\beta)$. Thus, we alternatively show that $\alpha \,;\, \beta$ is equivalent to the expression

$$\text{cyl}^r_{O^{\text{sem}}(\beta)}(\alpha) \cap \text{cyl}^l_{O^{\text{syn}}(\alpha)}(\beta).$$

We can also see that β is io-disjoint, since $I^{\text{syn}}(\beta) \cap O^{\text{syn}}(\beta) = \emptyset$ and $(I^{\text{syn}}, O^{\text{syn}})$ is sound. Thus, if we show that

$$[\![\text{cyl}^r_{O^{\text{sem}}(\beta)}(\alpha) \cap \text{cyl}^l_{O^{\text{syn}}(\alpha)}(\beta)]\!]_D = [\![\text{cyl}^r_{O^{\text{sem}}(\beta)}(\alpha) \cap \text{cyl}^l_{O^{\text{sem}}(\alpha)}(\beta)]\!]_D$$

for any interpretation D, we can apply Theorem 7.1 and we are done.

Thereto, let D be an interpretation. By soundness, it is clear that

$[\![\text{cyl}^l_{O^{\text{sem}}(\alpha)}(\beta)]\!]_D \subseteq [\![\text{cyl}^l_{O^{\text{syn}}(\alpha)}(\beta)]\!]_D$, so $[\![\text{cyl}^r_{O^{\text{sem}}(\beta)}(\alpha) \cap \text{cyl}^l_{O^{\text{sem}}(\alpha)}(\beta)]\!]_D \subseteq [\![\text{cyl}^r_{O^{\text{sem}}(\beta)}(\alpha) \cap \text{cyl}^l_{O^{\text{syn}}(\alpha)}(\beta)]\!]_D$.

What remains to show is that the other inclusion also holds. Thereto, let $(\nu_1, \nu_2) \in [\![\text{cyl}^r_{O^{\text{sem}}(\beta)}(\alpha) \cap \text{cyl}^l_{O^{\text{syn}}(\alpha)}(\beta)]\!]_D$. Clearly, $(\nu_1, \nu_2) \in [\![\text{cyl}^r_{O^{\text{sem}}(\beta)}(\alpha)]\!]_D$ and $(\nu_1, \nu_2) \in [\![\text{cyl}^l_{O^{\text{syn}}(\alpha)}(\beta)]\!]_D$. From $(\nu_1, \nu_2) \in [\![\text{cyl}^r_{O^{\text{sem}}(\beta)}(\alpha)]\!]_D$, we can see that $\nu_1 = \nu_2$ outside $O^{\text{sem}}(\alpha) \cup O^{\text{sem}}(\beta)$. From $(\nu_1, \nu_2) \in [\![\text{cyl}^l_{O^{\text{syn}}(\alpha)}(\beta)]\!]_D$, we can see that there is a valuation ν_1' such that $(\nu_1', \nu_2) \in [\![\beta]\!]_D$ and $\nu_1' = \nu_1$ outside $O^{\text{syn}}(\alpha)$. Define ν_1'' to be the valuation $\nu_1'[\nu_1|_{O^{\text{sem}}(\beta)}]$. By construction and io-disjointness of β, we see that $\nu_1'' = \nu_1'$ on $I^{\text{sem}}(\beta)$ and outside $O^{\text{sem}}(\beta)$. By Proposition 3.9, we obtain that $(\nu_1'', \nu_2) \in [\![\beta]\!]_D$. Define ν to be the valuation $\nu_1''[\nu_1|_{O^{\text{sem}}(\alpha)}]$. By the semantics of cylindrification, we see that $(\nu, \nu_2) \in [\![\text{cyl}^l_{O^{\text{sem}}(\alpha)}(\beta)]\!]_D$. Consequently, $\nu = \nu_2$ outside $O^{\text{sem}}(\alpha) \cup O^{\text{sem}}(\beta)$. Before, we established that ν_1 and ν_2 agree outside the same set of variables. So we obtain that $\nu = \nu_2 = \nu_1$ outside $O^{\text{sem}}(\alpha) \cup O^{\text{sem}}(\beta)$. Moreover, we know by construction that $\nu = \nu_1'' = \nu_1$ on $O^{\text{sem}}(\beta) \cup O^{\text{sem}}(\alpha)$. Then, ν is the same valuation as ν_1. So we obtain that $(\nu_1, \nu_2) \in [\![\text{cyl}^r_{O^{\text{sem}}(\beta)}(\alpha) \cap \text{cyl}^l_{O^{\text{sem}}(\alpha)}(\beta)]\!]_D$ as desired. □

Example 7.3 (Example 3.15 continued). Consider the expression

$$\alpha = P_1(x; x) \,;\, P_1(x; y).$$

with the interpretation D in Example 3.15. In that case, α first increments x by one and subsequently sets the value of y to one higher than x. Stated differently,

$$[\![\alpha]\!]_D = \{(\nu_1, \nu_2) \mid \nu_2(x) = \nu_1(x) + 1 \wedge \nu_2(y) = \nu_2(x) + 1 \text{ and } \nu_1(z) = \nu_2(z) \text{ for } z \notin \{x, y\}\}$$

Theorem 7.1 tells us that α is equivalent to

$$\text{cyl}^r_y(P_1(x; x)) \cap \text{cyl}^l_x(P_1(x; y)).$$

We see that

$$[\![\text{cyl}^r_y(P_1(x; x))]\!]_D = \{(\nu_1, \nu_2) \mid \nu_2(x) = \nu_1(x) + 1 \text{ and } \nu_1(z) = \nu_2(z) \text{ for } z \notin \{x, y\}\},$$
$$[\![\text{cyl}^l_x(P_1(x; y))]\!]_D = \{(\nu_1, \nu_2) \mid \nu_2(y) = \nu_2(x) + 1 \text{ and } \nu_1(z) = \nu_2(z) \text{ for } z \notin \{x, y\}\}.$$

The intersection of these indeed equals $[\![\alpha]\!]_D$.

Theorem 7.1 no longer holds in general if β can have overlapping inputs and outputs, as the following example illustrates.

Example 7.4. Consider the expression

$$\alpha := P_1(x;x)\ ;\ P_1(x;x).$$

with the interpretation D as in the example above. In this case, α increments the value of x by two. However, $[\![\mathrm{cyl}^r_x(P_1(x;x))]\!]_D$ and $[\![\mathrm{cyl}^l_x(P_1(x;x))]\!]_D$ are both equal to

$$\{(\nu_1,\nu_2) \mid \nu_1(z) = \nu_2(z) \text{ for all } z \neq x\}.$$

Hence, indeed, in this case α is not equivalent to

$$\mathrm{cyl}^r_x(P_1(x;x)) \cap \mathrm{cyl}^l_x(P_1(x;x)).$$

7.2 If $\mathbb{V}$ is Infinite, Composition is Non-Primitive

We know from Theorem 7.1 that if β is io-disjoint, α and β can be composed without using the composition operator. If $\mathbb{V}$ is sufficiently large, we can force any expression β to be io-disjoint by having β write its outputs onto unused variables instead of its actual outputs. The composition can then be eliminated following Theorem 7.1, after which we move the variables back so that the "correct" outputs are used. What we need to show is that "moving the variables around", as described above, is expressible without composition. As before, we define the operators on BRVs but their definition is lifted to LIF expressions in a straightforward way.

Definition 7.5. Let B be a BRV and let $\bar{x}$ and $\bar{y}$ be disjoint tuples of distinct variables of the same length. The *right move* is defined as follows:

$\mathrm{mv}^r_{\bar{x}\to\bar{y}}(B) := \{(\nu_1,\nu_2') \mid \nu_2'(\bar{x}) = \nu_1(\bar{x}) \text{ and } \exists \nu_2 : (\nu_1,\nu_2) \in B \text{ and } \nu_2'(\bar{y}) = \nu_2(\bar{x}) \text{ and } \nu_2 = \nu_2' \text{ outside } \bar{x}\cup\bar{y}\}.$

This operation can be expressed without composition, which we show in the following lemma:

Lemma 7.6. *Let $\bar{x}$ and $\bar{y}$ be disjoint tuples of distinct variables of the same length. Then, for any* BRV *B, we have*

$$\mathrm{mv}^r_{\bar{x}\to\bar{y}}(B) = \sigma^{lr}_{\bar{x}=\bar{x}}\mathrm{cyl}^r_{\bar{x}}\sigma^r_{\bar{x}=\bar{y}}\mathrm{cyl}^r_{\bar{y}}(B).$$

Proof. We give a "proof by picture". Consider an arbitrary $(\nu_1,\nu_2) \in B$:

$\bar{x}$	$\bar{y}$	*rest*	$\bar{x}$	$\bar{y}$	*rest*
$\bar{a}$	$\bar{b}$	$\bar{c}$	$\bar{d}$	$\bar{e}$	$\bar{f}$

We will verify that when we apply the LHS and the RHS on this pair of valuations, we obtain identical results.

For the LHS, we see that $\mathrm{mv}^r_{\bar{x}\to\bar{y}}(B)$ yields the following pair of valuations when applied on (ν_1,ν_2):

$\bar{x}$	$\bar{y}$	*rest*	$\bar{x}$	$\bar{y}$	*rest*
$\bar{a}$	$\bar{b}$	$\bar{c}$	$\bar{a}$	$\bar{d}$	$\bar{f}$

Now, we check the RHS. We see that the following set of pairs of valuations is the result of $\mathrm{cyl}^r_{\bar{y}}(B)$ when applied on (ν_1,ν_2):

$\bar{x}$	$\bar{y}$	*rest*	$\bar{x}$	$\bar{y}$	*rest*
$\bar{a}$	$\bar{b}$	$\bar{c}$	$\bar{d}$	$*$	$\bar{f}$

Here the asterisk denotes a "wildcard", i.e., any valuation on $\bar{y}$ is allowed.

Then, we see that $\sigma^r_{\bar{x}=\bar{y}}\text{cyl}^r_{\bar{y}}(B)$ yields:

$\bar{x}$	$\bar{y}$	*rest*	$\bar{x}$	$\bar{y}$	*rest*
$\bar{a}$	$\bar{b}$	$\bar{c}$	$\bar{d}$	$\bar{d}$	$\bar{f}$

Next, we see that $\text{cyl}^r_{\bar{x}}\sigma^r_{\bar{x}=\bar{y}}\text{cyl}^r_{\bar{y}}(B)$ yields:

$\bar{x}$	$\bar{y}$	*rest*	$\bar{x}$	$\bar{y}$	*rest*
$\bar{a}$	$\bar{b}$	$\bar{c}$	$*$	$\bar{d}$	$\bar{f}$

Finally, we see that $\sigma^{lr}_{\bar{x}=\bar{x}}\text{cyl}^r_{\bar{x}}\sigma^r_{\bar{x}=\bar{y}}\text{cyl}^r_{\bar{y}}(B)$ yields the following pair of valuations which is the same as the result of the LHS.

$\bar{x}$	$\bar{y}$	*rest*	$\bar{x}$	$\bar{y}$	*rest*
$\bar{a}$	$\bar{b}$	$\bar{c}$	$\bar{a}$	$\bar{d}$	$\bar{f}$

□

Lemma 7.7. *Let A and B be* BRV*s and let $\bar{x}$ and $\bar{y}$ be disjoint tuples of distinct variables of the same length such that all variables in $\bar{y}$ are inertially cylindrified in A and B. In that case:*

$$A\,;B = \text{mv}^r_{\bar{y}\to\bar{x}}(A\,;\text{mv}^r_{\bar{x}\to\bar{y}}(B))$$

What this lemma shows is that we can temporarily move certain variables (the $\bar{x}$) to unused variables (the $\bar{y}$) and then move them back. The proof of this lemma is:

Proof of Lemma 7.7. Again we give a proof by picture. Let the left be a generic pair of valuations that belongs to A, while the one on the right be a generic one that belongs to B. The "–" here represents inertial cylindrification.

$\bar{x}$	$\bar{y}$	*rest*	$\bar{x}$	$\bar{y}$	*rest*
$\bar{a}$	–	$\bar{b}$	$\bar{c}$	–	$\bar{d}$

$\bar{x}$	$\bar{y}$	*rest*	$\bar{x}$	$\bar{y}$	*rest*
$\bar{e}$	–	$\bar{f}$	$\bar{g}$	–	$\bar{h}$

For the LHS, we see that composition can only be applied if $\bar{c} = \bar{e}$ and $\bar{d} = \bar{f}$. Under this assumption, we get that $A\,;B$ yields the following:

$\bar{x}$	$\bar{y}$	*rest*	$\bar{x}$	$\bar{y}$	*rest*
$\bar{a}$	–	$\bar{b}$	$\bar{g}$	–	$\bar{h}$

Now, we check the RHS. We see that $\text{mv}^r_{\bar{x}\to\bar{y}}(B)$ yields the following when applied on the generic pair belonging to B:

$\bar{x}$	$\bar{y}$	*rest*	$\bar{x}$	$\bar{y}$	*rest*
$\bar{e}$	$*$	$\bar{f}$	$\bar{e}$	$\bar{g}$	$\bar{h}$

To apply the composition in the RHS, we must have $\bar{c} = \bar{e}$ and $\bar{d} = \bar{f}$, which are the same restrictions we had in applying the composition in the LHS, so the expression $A\,;\text{mv}^r_{\bar{x}\to\bar{y}}(B)$ yields:

$\bar{x}$	$\bar{y}$	*rest*	$\bar{x}$	$\bar{y}$	*rest*
$\bar{a}$	$*$	$\bar{b}$	$\bar{e}$	$\bar{g}$	$\bar{h}$

Finally, applying the last move operation, $\mathrm{mv}^r_{\bar{y}\to\bar{x}}(A\,;\mathrm{mv}^r_{\bar{x}\to\bar{y}}(B))$ yields:

$\bar{x}$	$\bar{y}$	*rest*	$\bar{x}$	$\bar{y}$	*rest*
$\bar{a}$	–	$\bar{b}$	$\bar{g}$	–	$\bar{h}$

which is clearly identical to what we had from the LHS. □

This finally brings us to the main result of the current subsection.

Theorem 7.8. *If $\mathbb{V}$ is infinite, then every LIF expression is equivalent to a LIFnc expression.*

Proof. We prove this theorem by induction on the number of compositions operators in a LIF expression γ. The base case (no composition operators), is trivial. For the inductive case, consider an expression η containing at least one composition operator. We show how to rewrite η equivalently with one composition operator less. Thereto, take any subexpression $\alpha\,;\beta$ such that α and β are LIFnc expressions. We eliminate this composition as follows. Choose a tuple of variables $\bar{y}$ of the same length as $O^{\mathrm{syn}}(\beta)$, such that $\bar{y}$ does not occur in γ. In that case, $\bar{y}$ is inertially cylindrified in α and in β, and hence, Lemma 7.7 yields that $\alpha\,;\beta$ is equivalent to

$$\mathrm{mv}^r_{\bar{y}\to O^{\mathrm{syn}}(\beta)}(\alpha\,;\mathrm{mv}^r_{O^{\mathrm{syn}}(\beta)\to\bar{y}}(\beta)).$$

We will next show that $\mathrm{mv}^r_{O^{\mathrm{syn}}(\beta)\to\bar{y}}(\beta)$ is io-disjoint. Indeed, from the equivalence in Lemma 7.6 and the soundness of our definitions, we can see that

$$O^{\mathrm{sem}}(\mathrm{mv}^r_{O^{\mathrm{syn}}(\beta)\to\bar{y}}(\beta)) = O^{\mathrm{sem}}(\sigma^{lr}_{\bar{x}=\bar{x}}\mathrm{cyl}^r_{\bar{x}}\sigma^r_{\bar{x}=\bar{y}}\mathrm{cyl}^r_{\bar{y}}(\beta)) \subseteq O^{\mathrm{syn}}(\sigma^{lr}_{\bar{x}=\bar{x}}\mathrm{cyl}^r_{\bar{x}}\sigma^r_{\bar{x}=\bar{y}}\mathrm{cyl}^r_{\bar{y}}(\beta)) = O^{\mathrm{syn}}(\beta)\cup\bar{y}.$$

Moreover, we generally have $O^{\mathrm{sem}}(\mathrm{mv}^r_{\bar{x}\to\bar{y}}(\gamma))\cap\bar{x}=\emptyset$ for any $\bar{x}$ and any LIF expression γ in which $\bar{y}$ is inertially cylindrified. As a consequence, $O^{\mathrm{sem}}(\mathrm{mv}^r_{O^{\mathrm{syn}}(\beta)\to\bar{y}}(\beta))\subseteq\bar{y}$.

Also, we can see that

$$I^{\mathrm{sem}}(\mathrm{mv}^r_{O^{\mathrm{syn}}(\beta)\to\bar{y}}(\beta)) = I^{\mathrm{sem}}(\sigma^{lr}_{\bar{x}=\bar{x}}\mathrm{cyl}^r_{\bar{x}}\sigma^r_{\bar{x}=\bar{y}}\mathrm{cyl}^r_{\bar{y}}(\beta)) \subseteq I^{\mathrm{syn}}(\sigma^{lr}_{\bar{x}=\bar{x}}\mathrm{cyl}^r_{\bar{x}}\sigma^r_{\bar{x}=\bar{y}}\mathrm{cyl}^r_{\bar{y}}(\beta)) = I^{\mathrm{syn}}(\beta)\cup O^{\mathrm{syn}}(\beta).$$

Since $\bar{y}$ does not occur in β, we indeed obtain that is $\mathrm{mv}^r_{O^{\mathrm{syn}}(\beta)\to\bar{y}}(\beta)$ io-disjoint.

$$O^{\mathrm{sem}}(\mathrm{mv}^r_{O^{\mathrm{syn}}(\beta)\to\bar{y}}(\beta)) \subseteq \bar{y}\cap(I^{\mathrm{syn}}(\beta)\cup O^{\mathrm{syn}}(\beta))=\emptyset.$$

We can now apply Theorem 7.2 to eliminate the composition yielding the LIFnc expression

$$\mathrm{mv}^r_{\bar{y}\to O^{\mathrm{syn}}(\beta)}(\mathrm{cyl}^r_{O^{\mathrm{syn}}(\mathrm{mv}^r_{O^{\mathrm{syn}}(\beta)\to\bar{y}}(\beta))}(\alpha)\cap\mathrm{cyl}^l_{O^{\mathrm{syn}}(\alpha)}(\mathrm{mv}^r_{O^{\mathrm{syn}}(\beta)\to\bar{y}}(\beta))).$$

□

7.3 If $\mathbb{V}$ is Finite, Composition is Primitive

The case that remains is when $\mathbb{V}$ is finite. We will show that in this case, composition is indeed primitive by relating bounded-variable LIF to bounded-variable first-order logic.

Assume $\mathbb{V}=\{x_1,\dots,x_n\}$. Since BRVs involve pairs of $\mathbb{V}$-valuations, we introduce a copy $\mathbb{V}_y=\{y_1,\dots,y_n\}$ disjoint from $\mathbb{V}$. For clarity, we also write $\mathbb{V}_x$ for $\mathbb{V}$. As usual, by FO$[k]$ we denote the fragment of first-order logic that uses only k distinct variables. We observe the following:

Proposition 7.9. *For every LIF expression α, there exists an* FO$[3n]$ *formula φ_α with free variables in $\mathbb{V}_x\cup\mathbb{V}_y$ such that*

$$(\nu_1,\nu_2)\in[\![\alpha]\!]_D \textit{ if and only if } D,(\nu_1\cup\nu_2')\models\varphi_\alpha,$$

where ν_2' is the $\mathbb{V}_y$-valuation such that $\nu_2'(y_i)=\nu_2(x_i)$ for each i. Furthermore, if α is a LIFnc expression, φ_α can be taken to be a FO$[2n]$ *formula.*

Proof. The proof is by induction on the structure of α (using Lemma 2.1, we omit redundant operators).

We introduce a third copy $\mathbb{V}_z = \{z_1, \ldots, z_n\}$ of $\mathbb{V}$. For every $u, v \in \{x, y, z\}$ we define ρ_{uv} as follows:

$$\rho_{uv} : \mathbb{V}_u \to \mathbb{V}_v : u_i \mapsto v_i$$

Using these functions, we can translate a valuation ν on $\mathbb{V} = \mathbb{V}_x$ to a corresponding valuation on $\mathbb{V}_u$ with $u \in \{y, z\}$. Clearly, $\nu \circ \rho_{ux}$ does this job.

In the proof, we actually show a stronger statement by induction, namely that for each α and for every $u \neq v \in \{x, y, z\}$ there is a formula φ_α^{uv} with free variables in $\mathbb{V}_u \cup \mathbb{V}_v$ in $\mathrm{FO}[\mathbb{V}_x \cup \mathbb{V}_y \cup \mathbb{V}_z]$ such that for every D,

$$(\nu_1, \nu_2) \in [\![\alpha]\!]_D \text{ if and only if } D, (\nu_1 \circ \rho_{ux} \cup \nu_2 \circ \rho_{vx}) \models \varphi_\alpha^{uv}.$$

Since the notations x, y, z, u and v are taken, we use notations a, b and c for variables.

- $\alpha = id$. Take φ_α^{uv} to be $\bigwedge_{i=1}^n u_i = v_i$.
- $\alpha = M(\overline{a}; \overline{b})$. Take φ_α^{uv} to be $M(\rho_{xu}(\overline{a}), \rho_{xv}(\overline{b})) \wedge \bigwedge_{c \notin \overline{b}} \rho_{xu}(c) = \rho_{xv}(c)$.
- $\alpha = \alpha_1 \cup \alpha_2$. Take φ_α^{uv} to be $\varphi_{\alpha_1}^{uv} \vee \varphi_{\alpha_2}^{uv}$.
- $\alpha = \alpha_1 - \alpha_2$. Take φ_α^{uv} to be $\varphi_{\alpha_1}^{uv} \wedge \neg\varphi_{\alpha_2}^{uv}$.
- $\alpha = \alpha_1 \,;\, \alpha_2$. Let $w \in \{x, y, z\} - \{u, v\}$. Take φ_α^{uv} to be $\exists w_1 \ldots \exists w_n\ (\varphi_{\alpha_1}^{uw} \wedge \varphi_{\alpha_2}^{wv})$.
- $\alpha = \alpha_1^{\smile}$. By induction, $\varphi_{\alpha_1}^{vu}$ exists. This formula can serve as φ_α^{uv}.
- $\alpha = \sigma_{a=b}^{lr}(\alpha_1)$. Take φ_α^{uv} to be $\varphi_{\alpha_1}^{uv} \wedge \rho_{xu}(a) = \rho_{xv}(b)$.
- $\alpha = \mathrm{cyl}_a^l(\alpha_1)$. Take φ_α^{uv} to be $\exists \rho_{xu}(a)\ \varphi_{\alpha_1}^{uv}$. □

Now that we have established that LIFnc can be translated into FO[$2n$], all that is left to do is find a Boolean query that can be expressed in LIF with n variables, but not in FO[$2n$]. We find such a query in the existence of a $3n$-clique. We will first show that we can construct a LIFnc expression α_{2n} such that, given an interpretation D interpreting a binary relation R, $[\![\alpha_{2n}]\!]_D$ consists of all $2n$-cliques of R. Next, we show how α_{2n} can be used (with composition) to construct an expression $\alpha_{\exists 3n}$ such that $[\![\alpha_{\exists 3n}]\!]_D$ is non-empty if and only if R has a $3n$-clique. Since this property cannot be expressed in FO[$2n$], we can conclude that composition must be primitive.

To avoid confusion, we recall that a set L of k data elements is a k-clique in a binary relation R, if any two distinct a and b in L, we have $(a, b) \in R$ (and also $(b, a) \in R$).

Proposition 7.10. *Suppose that $|\mathbb{V}| = n$ with $n \geq 2$ and let $\mathcal{S} = \{R\}$ with $ar(R) = iar(R) = 2$. There exists a LIF expression α_{2n} such that*

$$[\![\alpha_{2n}]\!]_D = \{(\nu_1, \nu_2) \mid \nu_1(\mathbb{V}) \cup \nu_2(\mathbb{V}) \text{ is a } 2n\text{-clique in } D(R)\}.$$

Proof. Throughout this proof, we identify a pair (ν_1, ν_2) of two valuations with the $2n$ tuple of data elements

$$\nu_1(x_1, \ldots, x_n) \cdot \nu_2(x_1, \ldots, x_n).$$

Before coming to the actual expression for α_{2n}, we introduce some auxiliary concepts. First, we define

$$all := \mathrm{cyl}_{\mathbb{V}}^l\, \mathrm{cyl}_{\mathbb{V}}^r(id).$$

It is clear that

$$[\![all]\!]_D = \{(\nu_1, \nu_2) \in \mathcal{V} \times \mathcal{V}\}.$$

A first condition for being a $2n$-clique is that all data elements are different. It is clear that the expression

$$\alpha_{=} := \bigcup_{x \neq y \in \mathbb{V}} \left(\sigma^{l}_{x=y}(all) \cup \sigma^{r}_{x=y}(all)\right) \cup \bigcup_{x,y \in \mathbb{V}} \sigma^{lr}_{x=y}(all)$$

has the property that $[\![\alpha_{=}]\!]_D$ consists of all $2n$-tuples where at least one data element is repeated. Hence, $[\![\alpha_{\neq}]\!]_D$ consists of all $2n$-tuples of distinct data elements, where

$$\alpha_{\neq} := all - \alpha_{=}.$$

The second condition for being a $2n$-clique is that each two distinct elements are connected by R. In order to check this, we define the following expressions for each two variables x and y:

$$R^{l}_{x,y} := \mathrm{cyl}^{l}_{\mathbb{V}-\{x,y\}}\mathrm{cyl}^{r}_{\mathbb{V}}(R(x,y;) \cap R(y,x;))$$
$$R^{r}_{x,y} := \mathrm{cyl}^{l}_{\mathbb{V}}\mathrm{cyl}^{r}_{\mathbb{V}-\{x,y\}}(R(x,y;) \cap R(y,x;))$$
$$R^{lr}_{x,y} := \mathrm{cyl}^{l}_{\mathbb{V}-\{x\}}\mathrm{cyl}^{r}_{\mathbb{V}-\{y\}}(R(x,y;) \cap R(y,x;))$$

With these definitions, for instance $[\![R^{lr}_{x_i,x_j}]\!]_D$ consists of all $2n$-tuples such that the i^{th} and the $n+j^{\text{th}}$ element are connected (in two directions) in R, and similar properties hold for R^l and R^r. From this, it follows that the expression

$$\alpha_{2n} = \alpha_{\neq} \cap \bigcap_{x \neq y \in \mathbb{V}} (R^{l}_{x,y} \cap R^{r}_{x,y}) \cap \bigcap_{x,y \in \mathbb{V}} R^{lr}_{x,y}$$

satisfies the proposition; it intersects $\alpha_{\neq}$ with all the expressions stating that each two data elements must be (bidirectionally) connected by R. □

Notice that α_{2n} can be used to compute *all* the $2n$-cliques of the input interpretation. We now use α_{2n} to check for existence of $3n$-cliques.

Proposition 7.11. *Suppose that* $|\mathbb{V}| = n$ *with* $n \geq 2$ *and let* $\mathcal{S} = \{R\}$ *with* $ar(R) = iar(R) = 2$. *Define*

$$\alpha_{\exists 3n} := (\alpha_{2n} \,;\, \alpha_{2n}) \cap \alpha_{2n}.$$

Then, for every interpretation D, $[\![\alpha_{\exists 3n}]\!]_D$ *is non-empty if and only if* $D(R)$ *has a* $3n$*-clique.*

It is well known that existence of a $3n$-clique is not expressible in FO$[2n]$ [7]. The above proposition thus immediately implies primitivity of composition.

Theorem 7.12. *Suppose that* $|\mathbb{V}| = n \geq 2$. *Then, composition is primitive in LIF. Specifically, no LIFnc expression is equivalent to the LIF expression* $\alpha_{\exists 3n}$.

8 RELATED WORK

LIF grew out of the **Algebra of Modular Systems** [25], which was developed to provide foundations for programming from available components. That paper mentions information flows, in connection with input–output behavior in classical logic, for the first time. The paper also surveys earlier work from the author's group, as well as other closely related work.

In a companion paper [1], we report on an application of LIF to **querying under limited access patterns**, as for instance offered by web services [21]. That work also involves inputs and outputs, but only of a syntactic nature, and for a restricted variant of LIF (called "forward" LIF) only. The property of io-disjointness turned also to be important in that work, albeit for a quite different purpose.

Our results also relate to **the evaluation problem for LIF**, which takes as input a LIF expression α, an interpretation D, and a valuation ν_1, and where the task is to find all ν_2 such that $(\nu_1, \nu_2) \in$

$[\![\alpha]\!]_D$. From our results, it follows that only the value of ν_1 on the input variables is important, and similarly we are only interested in the values of each ν_2 on the output variables. A subtle point, however, is that D may be infinite, and moreover, even if D itself is not infinite, the output of the evaluation problem may still be. In many cases, it is still possible to obtain a finite representation, for instance by using quantifier elimination techniques as done in Constraint Databases [17].

We have defined the semantics of LIF algebraically, in the style of **cylindric set algebra** [15, 16]. An important difference is the dynamic nature of BRVs which are sets of *pairs* of valuations, as opposed to sets of valuations which are the basic objects in cylindric set algebra.

Our optimality theorem was inspired by work on **controlled FO** [9], which had as aim to infer boundedness properties of the outputs of first-order queries, given boundedness properties of the input relations. Since this inference task is undecidable, the authors defined syntactic inferences similar in spirit to our syntactic definition of inputs and outputs. They show (their Proposition 4.3) that their definitions are, in a sense, sharp. Note that our optimality theorem is stronger in that it shows that no other compositional and sound definition can be better than ours. Of course, the comparison between the two results is only superficial as the inference tasks at hand are very different.

The Logic of Information Flows is similar to **dynamic predicate logic** (DPL) [12], in the sense that formulas are also evaluated with respect to pairs of valuations. There is, however a key difference in philosophy between the two logics. LIF starts from the idea that well-known operators from first-order logic can be used to describe combinations and manipulations of dynamic systems, and as such provides a means for procedural knowledge in a declarative language. The dynamics in LIF are dynamics of the described system. Dynamic predicate logic, on the other hand starts from the observation that, in natural language, operators such as conjunction and existential quantification are dynamic, where the dynamics are in the process of parsing a sentence, often related to coreference analysis. To the best of our knowledge, inputs and outputs of expressions have not been studied in DPL.

Since we developed a large part of our work in the general setting of BRVs, and thus of **transition systems**, we expect several of our results to be applicable in the context of other formalisms where specifying inputs and outputs is important, such as API-based programming [5] and synthesis [3, 6], privacy and security, business process modeling [4], and model combinators in Constraint Programming [11].

9 CONCLUSION AND FUTURE WORK

Declarative modeling is of central importance in the area of Knowledge Representation and Reasoning. The Logic of Information Flows provides a framework to investigate how, and to what degree, dynamic or imperative features can be modeled declaratively. In this paper we have focused on inputs, outputs, and sequential composition, as these three concepts are fundamental to modeling dynamic systems. There are many directions for further research.

Inputs and outputs are not just relevant from a theoretic perspective, but can also have ramifications on computation. Indeed, they form a first handle to parallelize computation of complex LIF expressions, or to decompose problems.

In this paper, we have worked with a basic set of operations motivated by the classical logic connectives. In order to provide a fine control of computational complexity, or to increase expressiveness, it makes sense to consider other operations.

The semantic notions developed in this paper (inputs, outputs, soundness) apply to global BRVs in general, and hence are robust under varying the set of operations. Moreover, our work delineates and demonstrates a methodology for adapting syntactic input–output definitions to other operations.

A specific operation that is natural to investigate its primitivity is *converse*. The converse of a BRV A is defined to be $\{(v_2, v_1) \mid (v_1, v_2) \in A\}$. In the context of LIF [27] it can model constraint solving by searching for an input to a module that produces a desired outcome. When we add converse to LIF with only a single variable ($|\mathbb{V}| = 1$), and the vocabulary has only binary relations of input arity one, then we obtain the classical calculus of relations [24]. There, converse is known to be primitive [10]. When the number of variables is strictly more than half of the maximum arity of relations in the vocabulary, converse is redundant in LIF, as can be shown using similar techniques as used in this paper to show redundancy of composition. Investigating the exact number of variables needed for non-primitivity is an interesting question for further research.

Another direction for further research is to examine fragments of LIF for which the semantic input or output problem may be decidable, or even for which the syntactic definitions coincide with the semantic definitions.

Finally, an operation that often occurs in dynamic systems is the fixed point construct used by [27]. It remains to be seen how our work, and the further research directions mentioned above, can be extended to include the fixpoint operation.

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