The Hirsch Function and its Properties

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ABSTRACT

The Hirsch function, denoted as h_f, of a given continuous function f is a new function depending on a parameter. It exists provided some assumptions are satisfied. If this parameter takes the value one, we obtain the well-known *h*-index. We prove several properties of the Hirsch function and characterize the shape of general functions that are Hirsch functions. We, moreover, present a formula that enables the calculation of f, given its Hirsch function h_f.

Keywords: *h*-index, H-function, Hirsch function.

INTRODUCTION

Hirsch[1] introduced the well-known discrete h-index. This indicator was later followed by the *g*-index,[2] and many other variants. The idea of considering (discrete) *h*- and *g*-indices with a variable parameter, originates from van Eck and Waltman.[3] Recently, Lathabai[4] introduced the ψ-index as the indicator with the largest offset-ability. These indicators are discrete, which makes them easy to apply, but it is well-known that for theoretical investigations a continuous version is more feasible. Hence, from now on we will work in a continuous context.

Let $f:R^+ \to R^+$ be a function. Then we define for all $\theta \in R_0^+ = R^+ \setminus \{0\}$:

$$x = h_f(\theta) \Leftrightarrow f(x) = \theta x \quad (1)$$

We only consider those cases for which (1) has a unique solution. If f(0) = 0 then we exclude a possible extra solution of $x = h_f(\theta) = 0$ unless this is a unique solution. Figure 1 illustrates some special cases.

Case a). does not lead to a valid solution of (1) as $y = \theta x$ and f(x) intersect in more than one point.

Case b). Here f(0) = f'(0) = 0. Here we do not consider x =0, so that (1) has a unique solution for all $\theta \in R_0^+$.

Case c). Here f(0) = 0 and $f'(0) = \theta_0 > 0$. We do not consider x = 0 as a solution of (1) if $\theta > \theta_0$ and do consider x = 0 as a solution if $0 < \theta \le \theta_0$.

Case d). Here we have $x = h_f(\theta) = 0$, for all $\theta \in R_0^+$.

Although it is possible to solve such special cases differently, the main point is that we know unambiguously what we mean by the notation $h_f(\theta)$. As $h_f(\theta)$ is now clearly defined we obtain a well-defined function h_f.

Definition: The Hirsch function

The function $h_f: \theta \in R_0^+ \to h_f(\theta) \in R^+$ is called the Hirsch function.

For $\theta = 1$, we obtain the well-known h-index[1] of the continuous function f, explaining the naming of this function. We further note that h_f is not defined in point zero so we can say that for f = 0 (the null function) $h_f(\theta) = 0$.

The Hirsch function has been used implicitly by Egghe and Rousseau[5] (without naming it as such) and later by Egghe[6,7] while, as mentioned above, the idea of considering h-indices with a variable parameter, originates from van Eck and Waltman.[3]

The Hirsch function is not defined as an explicit function but implicitly through equation (1). We first provide a characterization of such functions.

Theorem 1

Let φ be a function defined on R^+, continuous in 0. Let further f be a function, continuous in the point $\varphi(0)$ then the following two statements are equivalent:

$$\text{(i) } h_f = \varphi \text{ on } R_0^+$$

$$\text{(ii) } \forall \theta \in R^+ : f(\varphi(\theta)) = \theta . \varphi(\theta) \quad (2)$$

Proof. (i) $\Rightarrow$ (ii)

From (i) and (1) we obtain (2) $\forall \theta \in R_0^+$. For $\theta = 0$, we find, using the assumed continuity:

$$f(\varphi(0)) = f(\lim_{\theta \to 0} \varphi(\theta)) = \lim_{\theta \to 0} (f(\varphi(\theta))) = \lim_{\theta \to 0} (\theta . \varphi(\theta)) = 0$$
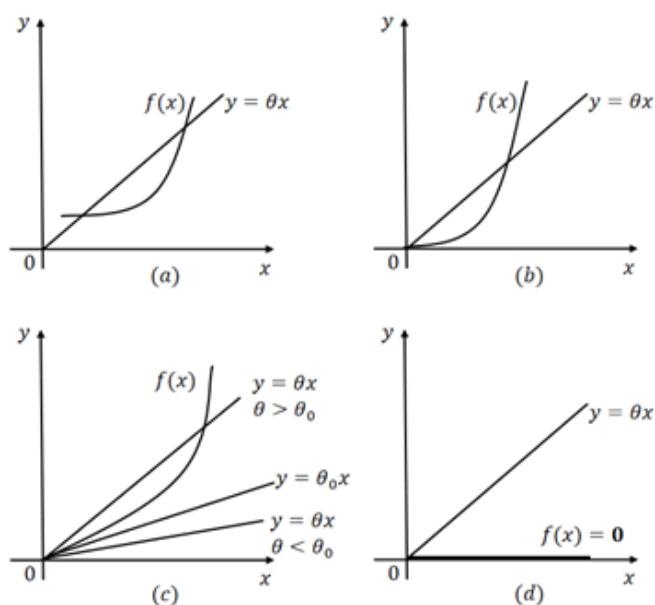
Figure 1: Some special cases.

Where we have used that we already know (2) for $\theta > 0$. Hence, $f(\varphi(0)) = 0 = 0.\varphi(0)$, which is (2) for $\theta = 0$.

(ii) $\Rightarrow$ (i)

From (2), (1), and the assumed uniqueness we have that $\forall \theta \in R_0^+$: $h_f(\theta) = \varphi(\theta)$, by the definition of h_f. □

Next, we will study the following problems.

(a). Given f, determine h_f. This is the formalism shown in (1). We give one simple example: let f(x) = C > 0 (C fixed). Then (1) leads to the equation $C = \theta x$. Hence $\forall \theta \in R_0^+$ $h_f(\theta) = x = C/\theta$. We come to the same result using (2). Indeed, $\forall \theta \in R_0^+$ $h_f(\theta) = \varphi(\theta) = C/\theta$.

(b). Given φ, determine f such that $\varphi = h_f$. This problem already places some extra requirements on φ without which $\varphi = h_f$ is impossible. Consider e.g., the example above: with $\forall x \in R_0^+$ $\varphi(x) = C/x$. Then (2) leads to $f\left(\frac{C}{x}\right) = x.\frac{C}{x} = C$. As the range of C/x is R_0^+, f(x) = C on R_0^+ and thus also f(x) = C on R^+ by the continuity of f.

(c). Neither f nor φ is given, but a general relationship between f and φ. Here we consider two subcases.

φ is given via a relation with f

Example 1. $\varphi = f$ (the simplest possible relation). Using (2) we have $\forall \theta \in R^+$: $f(f(\theta)) = \theta.f(\theta)$. If f is continuous then this relation can only occur if f = 0 (the null function) or $f(x) = x^\alpha$, where α is the golden section, $\frac{\sqrt{5}+1}{2}$.[6] Its proof uses the Fibonacci sequence.

Example 2. $\varphi = f \cdot f$. Using (2) this leads to $f(f(f(\theta))) = \theta.f(f(\theta))$, $\forall \theta \in R^+$, see Egghe.[8] Again, for f continuous, this requirement leads to two possible solutions, namely f = 0 or $f(x) = x^\beta$, with $\beta \approx 1.4648493$ (smaller than the golden section). Its proof uses a variant of the Fibonacci sequence.

In the same vein, one can consider the case $\varphi = f°f°\ldots°f$ —n times.

f is given via a relationship with φ

Example 1. The function f = φ. Although this is essentially the same as the previous example 1, (2) leads to $\forall \theta \in R^+$: $\varphi(\varphi(\theta)) = \theta.\varphi(\theta)$, leading to φ (=f) = 0 or $\varphi(x)$ (=f(x)) = x^α.

Example 2. f = $\varphi \cdot \varphi$

This example is different. Via (2) we find: $\varphi(\varphi(\varphi(\theta))) = \theta.\varphi(\varphi(\theta))$, $\forall \theta \in R^+$, see Egghe.[6] For φ continuous, this leads to $\varphi = 0$ or $\varphi(x) = x^\alpha$, with $\alpha \approx 1.3247178$. hence $f(x) = x^{(\alpha 2)}$.

This ends the introduction. Next, we will study the basic properties of the Hirsch function.

Properties of the Hirsch function

Theorem 2

The function h_f is injective on the set $\{\theta \in R_0^+ \| h_f(\theta) \neq 0\}$.

Notation. We denote $\{\theta \in R_0^+ \| h_f(\theta) \neq 0\}$ as $\{h_f \neq 0\}$.

Proof. Let $x_1 = h_f(\theta_1) = h_f(\theta_2) = x_2$. Then (1) implies that $f(x_1) = \theta_1 x_1$ and $f(x_2) = \theta_2 x_2$. As $x_1 = x_2$ and f is a function this implies that $\theta_1 x_1 = \theta_2 x_2$, leading to $\theta 1 = \theta 2$ if $x1 = x2 \neq 0$. □

The next theorem provides a new characterization of h_f.

Theorem 3

Let m be a function of functions m: f → m(f), then the following statements are equivalent:

(i) m(f) = h_f

(ii) $\forall \theta \in R_0^+$: $m\theta(f) = \psi_f^{-1}(\theta) = x$, where ψ_f is injective, and defined as:

$$\psi_f(x) = \frac{f(x)}{x} = \theta \qquad (3)$$

Proof. (i) $\Rightarrow$ (ii)

From (i) and (1) it follows that $\forall \theta \in R_0^+$: $x = m_\theta(f) \Leftrightarrow f(x) = \theta x$ $\Leftrightarrow \theta = f(x)/x \Leftrightarrow x = \psi_f^{-1}(\theta)$. Moreover, from the fact that h_f is a function, it follows that ψ_f is injective.

(ii) $\Rightarrow$ (i)

It follows similarly from (ii) and (1) that m(f) = h_f.

Remark

As $h_f^{-1} = \psi_f$, with ψ_f defined in (3) it follows that h_f^{-1} is a function on R_0^+. This immediately leads to (see also Theorem 2):

the function f is continuous $\Leftrightarrow h_f^{-1}$ is continuous $\qquad (4)$

The two implications in (4) do not hold for h_f (see further). To study this, we recall two results (stated as lemmas) from real analysis.

Lemma 1

If f is continuous on an interval (possibly infinitely long) and injective then f is strictly monotonous.

Lemma 2

If f is injective, then the following two statements are equivalent:

(i) f is continuous on [a,b]

(ii) The function f^{-1} is continuous on [f(a), f(b)] (or [f(b), f(a)])

The proof can be found using Lemma 1 and,[9] Theorem 2.27.

Notation. The domain of a function f is denoted as D(f).

Theorem 4

If D(f) is an interval, then f is continuous implies that h_f is continuous.

Proof. $D(\psi_f) = D(f) \setminus \{0\}$, hence an interval. If f is continuous then also ψ_f is continuous and ψ_f is an injection (by Theorem 3). Applying now Lemma 2 on ψ_f shows that ψ_f^{-1} is a continuous function. It then follows from Theorem 3 that $h_f = \psi_f^{-1}$ is also continuous. □

Theorem 4 does not hold if one removes the requirement that D(f) is an interval. This is illustrated in Figure 2.

We know that f is continuous if and only if h_f^{-1} is continuous. Yet, we will show that the implication h_f continuous $\Rightarrow$ f continuous does not always hold. For this, we need some preliminary results.

Lemma 3. If $f:R^+ \rightarrow R^+$ is continuous and injective on {f≠0}, then one of the following three statements hold:
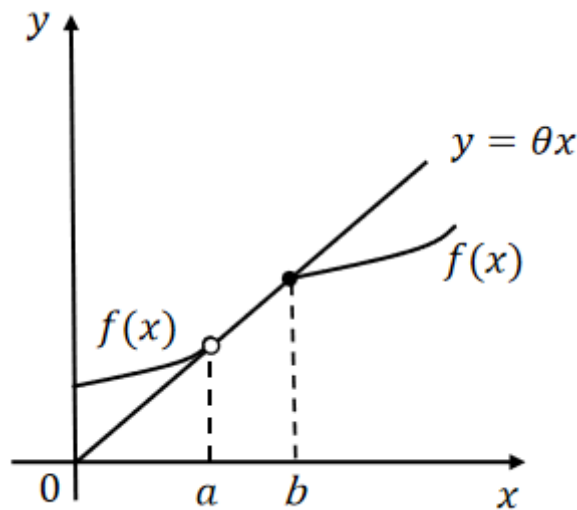
(i) f is injective

Figure 2: A function f, continuous on its domain D(f) = [0, a[∪[b, +∞[and a function $h_f(\theta)$ which is not continuous.

(ii) $\exists y_0 > 0$ such that $f|_{[0,y_0]} = 0$ and f (> 0) is injective, hence strictly increasing, on] y_0, +∞,

(iii) $\exists x_0 \geq 0$ such that $f|_{[x_0,+\infty]} = 0$ and f (> 0) is injective, hence strictly decreasing, on [0, x_0 [. Note that if x_0 = 0, this includes the case f = 0.

Proof. Assume (i) is not the case, i.e., f is not injective. Yet, we know that f is injective on {f ≠ 0}. Hence, there exist x, y, 0 ≤ x < y such that f(x) = f(y) = 0.

$$\text{We then show that } f|_{[x,y]} = 0. \qquad (*)$$

Indeed, otherwise, there would exist z ∈]x,y[such that f(z) ≠ 0. Because f is continuous it assumes all values between f(x) = 0 and f(z) > 0 on]x,z[and similarly on the interval]z,y[. Consequently, there exist points x' and y', x'∈]x,z[and y' ∈]z,y[, (hence x' ≠ y') such that f(x') = f(y') = f(z)/2 ≠ 0, which contradicts the fact that f is injective on {f ≠ 0}.

Next, we show that

$$\text{either } f|_{[0,x]} = 0 \text{ or } f|_{[y,+\infty[} = 0. \qquad (**)$$

Assume this is not the case. Then there exists u ∈ [0,x[such that f(u) > 0 and v ∈]y, +∞[such that f(v) > 0. As f is continuous it takes all values between f(x) = 0 en f(u) > 0 on]u,x[and between f(y) = 0 and f(v) > 0 on]y,v[. Put a = min(f(u), f(v)) > 0. Then there exist x' in]u,x[and y' in]y,v[such that f(y') = f(x') = a ≠ 0 (and x' ≠ y'). This is in contradiction with the fact that f is injective on {f ≠ 0}.

From (*) and (**) it follows that $f|_{[0,y]} = 0$ or $f|_{[x, +\infty[} = 0$, with 0 ≤ x < y. In the first case we set y_0 = sup {y > 0 such that $f|_{[0,y]} = 0$}. Then we know that $f|_{[0,y_0]} = 0$ and not equal to zero (hence strictly positive) on the complement of [0,y_0]. From this, it follows that f is injective. Then f is strictly increasing on] y_0, +∞ [and (ii) has been proved.

In the second case we set x_0 = inf {x ≥ 0 such that $f|_{[x,+\infty[} = 0$}. Then $f|_{[x_0,+\infty]} = 0$ and on the complement of [x_0, +∞ [f ≠ 0 and hence injective. In this case, f decreases strictly on [0,x_0[and (ii) is proved. ¥

Corollary

A continuous Hirsch function h_f on R_0^+ is of one of the following three types:

(i) h_f is injective on R_0^+;

(ii) h_f = 0 on an interval [0,y_0], y_0 > 0 and strictly increasing on the compliment;

(iii) h_f = 0 on an interval [x_0, + ∞ [, x_0 > 0 and strictly decreasing on the complement; including the case h_f = 0.

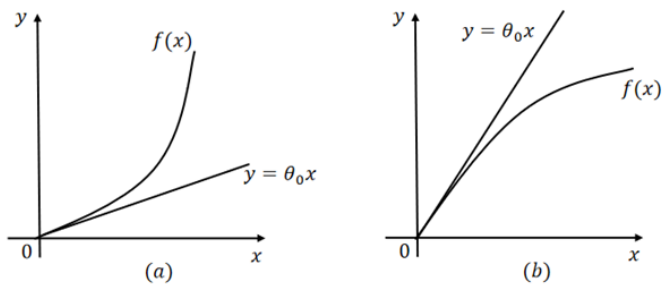Proof. This follows from Lemma 3 and Theorem 2, with f (in Lemma 3) replaced by h_f, defined on R_0^+.¥

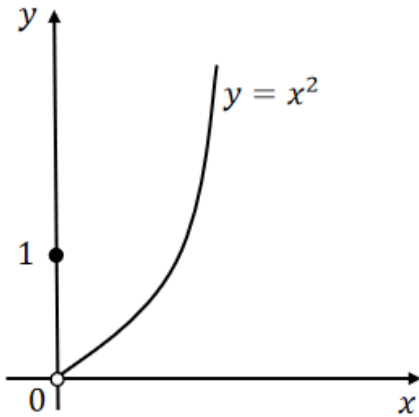**Figure 3:** Illustration of the Corrolary.

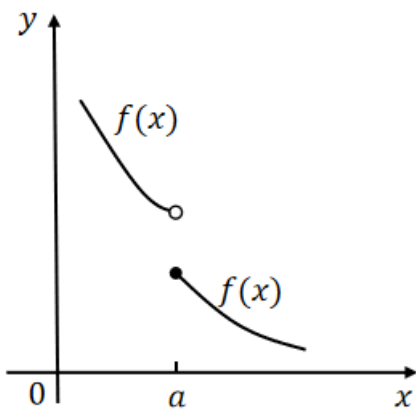**Figure 4:** The parabola y=x² with shifted origin.

Figure 5: Example of a case for which f is not continnous but h_f is.

We show that these three types occur.

(i) This class is best known as it includes the functions $f(x) = x^c$, $c > 1$. Then, $\forall\theta\in R_0^+ : h_f(\theta) = \theta^{\left(\frac{1}{(c-1)}\right)}$, which is a strictly increasing injection. Note that for $1/(c-1) = c$ we find $c = \alpha$ and hence $f = h_f$ on R_0^+.

(ii). See Figure 3a. The function f is strictly convex, $\theta_0 = f'(0) > 0$. By definition, h_f is zero on $[0,\theta_0]$ and strictly increasing on $[\theta_0,+\infty]$.

The consequence of Lemma 3 also provides conditions for an equation such as (2), Theorem 1, to have or not to have a solution,

These cases are discussed in the next theorem.

Theorem 5

If φ is a continuous function $R^+\to R^+$ which is not of the form (i), (ii), or (iii) of the above corollary then a function f such that $h_f = φ$ does not exist. If φ is of the form (i), (ii), or (iii) then the solution of (2), namely $h_f = φ$ is given by

$$\forall\, x \in R_0^+ : f(x) = x\, φ^{-1}(x) \qquad (5)$$

with $φ^{-1}$ the inverse function of the injective part of φ (abuse of notation). This function $φ^{-1}$ always exists, except when $x_0 = 0$ in (iii), in which case φ = 0 and φ = h_f with f = 0. We further note that in cases (ii) and (iii) f(0) = 0.

Proof

Case (i). In this case, φ is injective and (2) gives:

$$\forall\, \theta \in R_0^+ :\ f(φ(\theta)) = \theta\, φ(\theta)$$

Denoting φ(θ) by x we find that $φ^{-1}(x) = \theta$, which yields (5).

Case (ii). Now we know that there exists $y_0 > 0$ such that $φ|_{[0,y_0]} = 0$ and φ is strictly increasing (hence injective) on $]y_0, +\infty[$. Next, we set $f(x) = xφ^{-1}(x)$ on $φ([y_0,+\infty[)$. As φ is continuous and $φ|_{[0,y_0]} = 0$, $R_0^+\subset φ]y_0,+\infty[$. In this way, f is defined on R_0^+ with (2) holding on $[y_0, +\infty]$. Now define f(0) = 0, then we have, $\forall\, \theta \in [0,y_0]$:

$$f(φ(\theta)) = f(0) = 0 = \theta\, φ(\theta)$$

showing that (2) holds on R^+ and thus, by Theorem 1, $h_f = φ$ on R_0^+.

Case (iii) Now we know that there exists $x_0 \ge 0$ such that $φ|_{[x_0,+\infty[} = 0$ with φ strictly decreasing (and hence injective) on $[0,x_0]$. For $x_0 = 0$, φ = 0 on R^+ and we take f = 0 on R^+, leading to $h_f = φ$ on R_0^+ (by (1)).

Assume now that $x_0 > 0$. Define $f(x) = x\, φ^{-1}(x)$ on $φ([0,x_0[)\neq\varnothing$. As φ is continuous and $φ|_{[x_0,+\infty[} = 0$, $R_0^+\subset φ([0,x_0[)$. So far, we defined f on R_0^+, with (2) holding on $[0, x_0[$. Now, put f(0) = 0, then we have $\forall\theta\in[x_0,+\infty[$:

$$f(φ(\theta)) = f(0) = 0 = \theta\, φ(\theta)$$

showing again that (2) holds on R^+ and thus, by Theorem 1, $h_f = φ$ on R_0^+. □

Practical conclusion

Leaving x = 0 aside we see that the solution of $h_f = φ$ is given by equation (5) with $φ^{-1}$ the inverse of φ on the injective part of φ (and f = 0 for φ = 0).

Examples

(i) For $\varphi(x) = C/x$, $C>0$ constant, we see that φ is injective and $\varphi^{-1} = \varphi$. Then (5) yields: $f(x) = x.C/x = C$ and $h_f = \varphi$.

For $\varphi(x) = x^c$, $\varphi^{-1}(x) = x^{1/c}$ and, by (5), $f(x) = x^{1+(\frac{1}{c})}$; $h_f = \varphi$.

(ii) and (iii). These cases are similar so we give just one example. For

$$\varphi(x) = \begin{cases} a^{x \cdot b} \text{-}1 (a>1, b>0), & x \geq b \\ 0 & 0 \leq x < b \end{cases}$$

we see that φ is strictly increasing on $[b, +\infty]$, and hence injective. On this set the function $\varphi^{-1}(x) = b + \log_a(x+1)$ and hence, using (5) we have:

$\forall x > 0$; $f(x) = x (b + \log_a(x+1))$ and $f(0) = 0$, showing that $h_f = \varphi$. Note that $f'(0) = b$ and that h_f is zero on $[0, b]$.

Finally, we come to the case "h_f continuous implies f continuous", the inverse statement of Theorem 4.

Theorem 6

If the range of f, denoted as $R(f)$ is an interval, then h_f continuous implies f continuous on $D(f) \cap R_0^+$.

Proof

As h_f is continuous everywhere, it is also continuous on $\{h_f \neq 0\}$, which is an interval inside $D(h_f) = D(\psi_f^{-1})$, by the corollary to Lemma 3. By Theorem 3 $D(\psi_f^{-1}) = R(\psi_f)$, which too is an interval because $R(f)$ is an interval. By Theorem 2 we know that h_f is injective on $\{h_f \neq 0\}$. Then it follows from Lemma 2 that $h_f^{-1} = \psi_f$ (by Theorem 3) is continuous on $\psi_f^{-1}(\{\psi_f^{-1} \neq 0\}) = D(f) \cap R_0^+$. Finally, as $f(x) = x \, \psi_f(x)$ on R_0^+ (by Theorem 3), this shows that f is continuous on $D(f) \cap R_0^+$.Y

The next example shows that f is not necessarily continuous in zero. Take

$$f(x) = \begin{cases} x^2 \text{ for } x>0 \\ 1 \text{ for } x=0 \end{cases}$$

Then $R(f) = R_0^+$, which is an interval, h_f is continuous on R_0^+ but f is not continuous in 0, see Figure 4.

We finish this article by remarking that the condition "$R(f)$ is an interval" is necessary for Theorem 6. Figure 5 provides an example of a function f which is not continuous on R_0^+, but h_f is continuous because $D(h_f)$ is not an interval, (because $R(f)$ is not an interval).

CONCLUSION

This article illustrates how a very practical tool, here the h-index, can inspire further developments, enriching theoretical informetric. The originality of this contribution lies in the finding that - essentially - the *h*-function is injective and that the original function can be recovered from the *h*-function by multiplying the inverse of this function with the independent variable x. Our results show the simplicity of the *h*-index tool.

Returning to the introduction, it seems natural to try to construct a similar theory based on the *g*-index or the ψ – index. Yet we have not been able to realize this, mainly because sums (or integrals) occur in these indices, while this is not the case for the h-index. This makes studying inverses (which we need), not a sinecure. It can be stated though that a theory like we developed for the *h*-index is possible for other indices, such as Kosmulski's.[10] We did not include this because it brings nothing new and is just more difficult.

The author hopes that his theoretical study will inspire colleagues to further useful developments or interesting applications.

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CONFLICT OF INTEREST

The authors declare no conflict of interest.

REFERENCES

1. Hirsch J.E. An index to quantify an individual's scientific research output. Proceedings of the National Academy of Sciences of the United States of America. 2005;102(46):16569-72.
2. Egghe L. An improvement of the h-index: The g-index. ISSI Newsletter. 2006;2(1): 8-9.
3. van Eck NJ, Waltman L. Generalizing the h-and g-indices. Journal of Informetrics. 2008;2(4);263-71.
4. Lathabai HH. ψ – index: a new overall productivity index for actors of science and technology. Journal of Informetrics. 2020;14: article 101096.
5. Egghe L, Rousseau R. Solution by step functions of a minimum problem in L2[0, T], using generalized h- and g-indices. Journal of Informetrics 2019;13(3):785-92.
6. Egghe L. A theory of pointwise defined impact measures. Journal of Informetrics 2021;15. article 101169.
7. Egghe L. Convergence of impact measures and impact bundles. Journal of Data and Information Science. 2022;7(3):5-19.
8. Egghe L. Hirsch meets Fibonacci and Narayana type variants. arXiv:2303.12083. (2023).
9. De Lillo NJ. Advanced Calculus with Applications. 1982. New York, Macmillan.
10. Kosmulski M. A new Hirsch-type index saves time and works equally well as the original index. ISSI Newsletter. 2006;2(3):4-6.