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Interplay between normal forms and center manifold reduction for homoclinic predictors near Bogdanov-Takens bifurcation

M.M. Bosschaert^{*} Yu.A. Kuznetsov[†]

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Abstract: This paper provides for the first time correct third-order homoclinic predictors in *n*dimensional ODEs near a generic Bogdanov-Takens bifurcation point. To achieve this, higher-order time approximations to the nonlinear time transformation in the Lindstedt-Poincaré method are essential. Moreover, a correct transform between approximations to solutions in the normal form and approximations to solutions on the parameter-dependent center manifold is needed. A detailed comparison is done between applying different normal forms (smooth and orbital), different phase conditions, and different perturbation methods (regular and Lindstedt-Poincaré) to approximate the homoclinic solution near Bogdanov-Takens points. Examples demonstrating the correctness of the predictors are given. The new homoclinic predictors are implemented in the open-source MATLAB/GNU Octave continuation package MatCont.

Keywords: codimension 2 Bogdanov-Takens bifurcation, homoclinic solution, Lindstedt-Poincaré method, regular perturbation method, blow-up transformation, continuation software

2020 MSC: 37M20, 65P30, 34C37, 34B08, 34B15, 34B40, 34E10

1 Introduction

Let $f: \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}^n$ with $n \ge 2$, be smooth and suppose that the autonomous ordinary differential equation (ODE)

$$\dot{x}(t) = f(x(t), \alpha) \tag{1}$$

has equilibrium $x_0 = 0$ that undergoes a codimension two local bifurcation at the critical parameter value $\alpha_0 = 0$. Here the dot means the derivative with respect to the independent variable $t \in \mathbb{R}$. To understand the dynamics near the bifurcation point for nearby parameter values, one typically first restricts the ODE to the center manifold. By projecting the solutions on the center manifold onto the center subspace, one then obtains a n_c -dimensional ODE that locally governs the restricted dynamics. Using the normal form theory, one further tries to transform the restricted ODE into a simpler form, called the *critical normal form*.

If the canonical unfolding of the critical normal form is known and only qualitative behavior near the equilibrium is of interest, one can stop here. However, if one is interested in relating solutions of the unfolding to those of the original system (1) near the bifurcation point, one needs a relation between the *parameter-dependent normal form* and the restricted ODE, and also a relation between

^{*}Department of Mathematics, Hasselt University, Diepenbeek Campus, Agoralaan Gebouw D, 3590 Diepenbeek, Belgium (maikel.bosschaert@uhasselt.be).

[†]Department of Mathematics, Utrecht University, Budapestlaan 6, 3508 TA Utrecht, The Netherlands and

Department of Applied Mathematics, University of Twente, Zilverling Building, 7500AE Enschede, The Netherlands (I.A.Kouznetsov@uu.nl).

this restricted ODE on the parameter-dependent center manifold and the original system (1). These two relations can be found simultaneously utilizing the *homological equation* approach, see [5].

The solutions of interest here are the codimension one bifurcation curves emanating from a codimension two point and the corresponding orbits in phase space. In general, the bifurcation curves in the parameter-dependent normal form are not known exactly, but only by an approximation up to a certain order. Similarly, the transformation between of the normal form to the (parameter-dependent) center manifold is generally also only known up to a certain order. Then, by combining these two transformations, an approximation to the codimension one bifurcation curve and the corresponding phase orbits is obtained for the original system (1).

These approximations are particularly useful in numerical continuation software to start the continuation of the codimension one bifurcation curves emanating from the codimension two bifurcation points, where the defining systems for the orbits of interest become degenerate. A codimension two bifurcation that has attracted much attention is the *Bogdanov-Takens bifurcation* at which the critical equilibrium has a double zero eigenvalue. It is well known that under certain non-degeneracy and transversality conditions, three codimension one bifurcation curves emanate from the Bogdanov-Takens point: a saddle-node, an (Andronov-)Hopf, and a saddle-homoclinic bifurcation curve. Since the standard defining systems for the equilibrium bifurcations are non-degenerate at the Bogdanov-Takens point, one does not need an approximation to start continuation there. On the contrary, the standard defining system for the homoclinic solution does become degenerate, which is easily seen since the homoclinic orbit shrinks to the equilibrium point at the Bogdanov-Takens bifurcation.

Starting continuation of the homoclinic orbits from a Bogdanov-Takens point in ODEs attracted much attention. In planar systems, Melnikov's method was first applied to solve this problem in [43]. A first attempt to provide asymptotic approximations to the homoclinic bifurcation curve near a generic codimension two Bogdanov-Takens bifurcation point in general *n*-dimensional systems was made in [4]. By applying a singular rescaling to the (one of the equivalent) parameter-dependent normal form on the center manifold, a perturbed planar Hamiltonian system is obtained. The unperturbed Hamiltonian system contains an explicit homoclinic solution. A first-order correction in parameterspace can subsequently be obtained by setting up the problem as a branching problem in a suitable Banach space, see [4]. Then, by using the regular perturbation method, higher-order approximations to the homoclinic bifurcation curve can be obtained. Unfortunately in [4], even the first-order correction in the phase-space was not derived. Nonetheless, it was proven that the obtained homoclinic predictor converges to the true solution under the Newton iterations in the perturbed Hamiltonian systems.

In [40] the work continued by obtaining a second-order correction in parameter and phase-space to the homoclinic bifurcation curve for the perturbed Hamiltonian system. However, a new problem was overlooked. The normal form used in [40] is a normal form for C^{∞} -equivalence (also called *smooth orbital equivalence*), i.e., besides a C^{∞} -coordinate change, also a time reparametrization must be taken into account, which was not the case in [40]. In the subsequent paper [41], this problem was resolved by considering a smooth normal form for the Bogdanov-Takens bifurcation point, which is a normal form for C^{∞} -conjugacy (*smooth equivalence*).

In the follow-up paper [1], progress was made in obtaining a uniform approximation in the time along the homoclinic solution, using a generalization of the Lindstedt-Poincaré method. This removes the so-called parasitic turns near the saddle point, as observed in [40]. Although, as pointed out by [2], there were mistakes in the third-order approximation with the Lindstedt-Poincaré method, the homoclinic predictor from [40] for the smooth normal form improved significantly in the phase space.

Nonetheless, the problem of correctly lifting the asymptotics in the normal form to the parameterdependent center manifold remained unnoticed and unsolved. Effectively, only the zeroth-order approximation to the homoclinic solutions in the phase space, i.e., a transformed homoclinic solution of the unperturbed Hamiltonian system, was available for a general *n*-dimensional system.

In this paper, we will provide for the first time the third-order homoclinic predictor for the homoclinic solutions emanating from a generic Bogdanov-Takens point for a general *n*-dimensional system. For this, we need to consider several additional systems to be solved in the homological equation method that were previously not taken into account. During the derivation of the coefficients of the normal form and the transformations, we will show that there is no need to solve certain systems simultaneously, see the so-called 'big' system in [40, 41, 1]. Furthermore, by allowing a transformation of time between the normal form and the original system, we can use the parameter-dependent *smooth orbital* normal form of the codimension two Bogdanov-Takens bifurcation point when approximating the homoclinic solution up to order three. This normal form is considerably simpler than previously employed smooth normal forms. The derivation of the coefficients will be the subject of Section 2.

Having derived the parameter-dependent center manifold transformation suitable for lifting the third-order homoclinic asymptotics present in different generic Bogdanov-Takens normal forms, we turn our attention to obtain the asymptotics in Section 3. We will revisit both regular perturbation method and the generalized Lindstedt-Poincaré method considered previously.

The non-uniqueness of the homoclinic solution due to a time shift results in the non-uniqueness for the systems to be solved in the regular perturbation method. To obtain uniqueness, a so-called *phase condition* needs to be satisfied. The phase condition used in [40] originates from a theoretical setting in [4]. In Section 3.1 we use a geometrically motivated phase condition which slightly improves the regular perturbation solution. Furthermore, by modifying [4, Proposition 4.3], we use symmetry arguments to simplify the calculations.

In Section 3.2 the generalized Lindstedt-Poincaré method for the approximation of homoclinic orbits is improved by introducing an additional transformation of time after applying the usual nonlinear time transformation. The resulting algorithm solitary relies on polynomial division and does not involve any hyperbolic or trigonometric functions as in [2, 1]. We show that for the quadratic Bogdanov-Takens normal form, we can represent the homoclinic solution in phase-space with only one single parameter.

In Section 3.2.2 we provide an explicit third-order homoclinic approximation in the perturbed Hamiltonian system using the algorithm described in Section 3.2.1. Here we also provide a third-order approximation to the reparametrization of time. The profiles of the homoclinic solution will only then be approximated accurately, resulting in a robust initial predictor for starting continuation of the branch of homoclinic orbits. In [1] the importance of the time-reparametrization was not recognized, and the zeroth-order approximation was used. We will demonstrate in detail that it is essential to use the higher time reparametrization by comparing the Lindstedt-Poincaré method with and without the higher-order time-reparametrization. Effectively, using the Lindstedt-Poincaré method without the higher-order time-reparametrization is equivalent to the zeroth-order regular perturbation method.

The algorithm given in Section 3.2.1 is implemented in Appendix D in the programming language Julia [6] for the quadratic normal form for the Bogdanov-Takens codimension two bifurcation. Here we gain some insight about the finite convergence radius of the homoclinic asymptotics and the speed of the algorithm.

By combining the homoclinic asymptotics derived in Section 3 with the parameter-dependent center manifold transformation obtained in Section 2, we get the correct homoclinic predictor for a general n-dimensional system. It will be shown in Section 4 how to incorporate the time translation into the homoclinic predictor.

Then we compare the homoclinic predictor for the smooth normal form and the smooth orbital normal form. In Section 4.3, it will be shown that these two predictors are asymptotically equivalent, up to a phase shift. Then, by choosing the constants of integration in the time translation in a specific manner, we show equivalence between the predictors.

All the above methods are implemented in the open-source bifurcation and continuation software MatCont [18]. In Section 4.4 we describe the new implementation of the homoclinic predictor in the latest version of MatCont. We show how to use the obtained predictors to construct an initial prediction for the defining system of the homoclinic solutions. Besides an initial prediction also an initial tangent vector is necessary to start continuation. Our implementation resolves the issue of possible continuation in the wrong direction, i.e., towards the Bogdanov-Takens point.

The effectiveness of the new predictors is demonstrated on the topological normal form and on two four-dimensional models from Neuroscience and Quantum Field Theory in Section 5. A comparison

between the new homoclinic predictor near a generic codimension 2 Bogdanov-Takens bifurcation and the predictor from [1] is given. It will be shown that the order of the higher-order approximations to the homoclinic solutions in the normal form is preserved under the parameter-dependent center manifold transformation. Complementary to Section 5 an online Jupyter Notebook is provided in which ten different models are considered using the new homoclinic predictor and comparing different approximation methods in detail.

2 Center manifold reduction combined with normalization and time reparametrization

Consider system (1), which has a codimension two bifurcation at $\alpha = 0$ of equilibrium $x \equiv 0$. Let the normal form on the n_c -dimensional center manifold be given by

$$\frac{d}{d\eta}w(\eta) = G(w(\eta),\beta), \qquad G \colon \mathbb{R}^{n_c} \times \mathbb{R}^2 \to \mathbb{R}^{n_c}, \tag{2}$$

where G is assumed to be one of the (known) equivalent normal forms. Suppose that a parameterdependent approximation to an emanating codimension one bifurcation curve in (2) is given by

$$\epsilon \mapsto (w_{\epsilon}(\eta), \beta_{\epsilon}). \tag{3}$$

By the Shoshitaishvili reduction principle the solutions on the parameter-dependent center manifold can be parametrized by

$$x(t(\eta)) = H(w(\eta), \alpha), \qquad \qquad H: \mathbb{R}^{n_c} \times \mathbb{R}^2 \to \mathbb{R}^n, \qquad (4)$$

$$= K(\beta), \qquad \qquad K \colon \mathbb{R}^2 \to \mathbb{R}^2. \tag{5}$$

Now let the time η be defined through the time rescaling

 α

$$\frac{dt}{d\eta} = \theta(w,\beta), \qquad \theta \colon \mathbb{R}^{n_c} \times \mathbb{R}^2 \to \mathbb{R}^n.$$
(6)

Then the invariance of the center manifold implies the homological equation

$$f(H(w,\beta), K(\beta))\theta(w,\beta) = H_w(w,\beta)G(w,\beta).$$
(7)

The unknowns in (7) are H, K, θ , and the normal form coefficients in (9). By expanding the functions H, K, θ , and f and collecting terms of equal power in (w, β) , we obtain linear systems which can be solved at each order, potentially depending on non-uniqueness present in lower order systems.

To determine which coefficients are needed to include in the expansions of H, K, and θ , we need to consider which terms in the expansion of the reduced system on the center manifold of (1) affect the approximation (3). It is important here to not only take into account the terms that affect the approximation that are present in the normal form G but also terms that are *not in the normal form*, as long as the approximation (3) is affected by the terms. This has not been understood correctly and will be made explicit for the approximation of the homoclinic bifurcation curve emanating from the generic codimension two Bogdanov-Takens bifurcation point in the next section.

2.1 Parameter-dependent normal form

Suppose that the ODE (1) undergoes a generic codimension two Bogdanov-Takens bifurcation at $(x, \alpha) \equiv (x_0, \alpha_0)$. That is the linearization of (1) has a double, but not semisimple, zero eigenvalue,

while all other eigenvalues are away from the imaginary axis. The critical smooth normal form on the two-dimensional center manifold is given by [3, 23]

$$\begin{cases} \dot{w}_0 = w_1, \\ \dot{w}_1 = aw_0^2 + bw_0w_1 + \mathcal{O}(||w||^3), \end{cases}$$

where

 w_i is a shorthand notion for $w_i(\eta)$ for i = 0, 1, and the dot is the derivative with respect to η .

Under these non-degeneracy and certain transversality conditions, the *topological normal form* for the codimension two Bogdanov-Takens bifurcation is given by

 $ab \neq 0$,

$$\begin{cases} \dot{w}_0 = w_1, \\ \dot{w}_1 = \beta_1 + \beta_2 w_1 + a w_0^2 + b w_0 w_1, \end{cases}$$
(8)

see [7, 8, 44, 23, 38]. It is well known that in system (8) three codimension one bifurcation curves emanate from $(\beta_1, \beta_2) = (0, 0)$: a saddle-node, a Hopf, and a saddle-homoclinic bifurcation curve.

By using either the regular perturbation or the Lindstedt-Poincaré method, an approximation to the homoclinic bifurcation curve and the corresponding solution can, theoretically, be obtained up to any order in the singular-rescaling parameter ϵ , see [40, 41, 1, 2].

To obtain the second-order homoclinic approximation to the homoclinic solutions on the center manifold in (1), it is, in general, insufficient to only consider the topological normal form (8), see [10]. One way to deal with this problem is to consider the smooth parameter-dependent normal form

$$\begin{cases} \dot{w}_0 = w_1, \\ \dot{w}_1 = \beta_1 + \beta_2 w_1 + (a + a_1 \beta_2) w_0^2 + (b + b_1 \beta_2) w_0 w_1 + e w_0^2 w_1 + d w_0^3 + g(w, \beta), \end{cases}$$
(9)

with

$$g(w,\beta) = \mathcal{O}(|\beta_1| ||w||^2 + |\beta_2|w_1^2 + ||\beta||^2 ||w||^2 + ||\beta|| ||w||^3 + ||w||^4)$$

as in [41, 1]. Here $w_i = w_i(t)$ (i = 0, 1) now depends explicitly on t as in the original ODE (1).

However, in this paper, we will allow for a time-reparametrization and use the $C^\infty\text{-equivalent}$ normal form

$$\begin{cases} \dot{w}_0 = w_1, \\ \dot{w}_1 = \beta_1 + \beta_2 w_1 + a w_0^2 + b w_0 w_1 + w_0^2 w_1 h(w_0, \beta) + w_1^2 Q(w_0, w_1, \beta), \end{cases}$$
(10)

where h is C^{∞} and Q is N-flat, see [10]. Here the dot represents the derivative with respect to the new time η of $w_i(\eta)(i = 0, 1)$. Furthermore, we will show that we can assume h(0, 0) = 0. Note that we do not impose the coefficients to be a = 1 and $b = \pm 1$ as in [10]. This simplifies the systems to be solved in the next section without complicating the solutions for the homoclinic corrections. Indeed, we can scale-out the coefficients a and b in the singular-rescaling. Also note that the normal form (10) was used in [10] to study degenerate (codimension 3) Bogdanov-Takens bifurcations, while we found it to be essential for constructing homoclinic predictors in the case of generic codimension two Bogdanov-Takens bifurcations.

To approximate the homoclinic solutions emanating from the Bogdanov-Takens point we apply the singular rescaling

$$w_0 = \frac{a}{b^2}u\epsilon^2, \quad w_1 = \frac{a^2}{b^3}v\epsilon^3, \quad \beta_1 = -4\frac{a^3}{b^4}\epsilon^4, \quad \beta_2 = \frac{a}{b}\tau\epsilon^2, \quad s = \frac{a}{b}\epsilon\eta, \quad (\epsilon \neq 0), \tag{11}$$

to (10) with h(0,0) = 0 to obtain the second order nonlinear oscillator

$$\ddot{u} = -4 + u^2 + \dot{u} \left(u + \tau \right) \epsilon + \mathcal{O}(\epsilon^4).$$
(12)

Here the dot represents the derivative with respect to s.

order	terms
ϵ^{-2}	u_0, α_2
ϵ^{-1}	u_1
ϵ^0	$u_0^2, u_0 \alpha_2, \alpha_2^2, \alpha_1$
ϵ^1	u_0u_1, u_1lpha_2
ϵ^2	$u_1^2, u_0^3, u_0^2 \alpha_2, u_0 \alpha_2^2, \alpha_2^3, u_0 \alpha_1, \alpha_1 \alpha_2$
ϵ^3	$u_0^2 u_1, u_0 u_1 \alpha_2, u_1 \alpha_2^2, u_1 \alpha_1$

Table 1: Terms in the reduced system on the center manifold that affect the third-order predictor.

2.2 Center manifold reduction for smooth orbital normal form

We want to relate the third-order homoclinic approximation in the smooth orbital normal form (10) to the homoclinic solutions of (1) near (x_0, α_0) . The third-order approximation depends, by definition, on the coefficients in ϵ up to order three in the perturbed Hamiltonian system (12), see Section 3.2. By inspecting the blowup transformation (11) we can determine exactly which coefficients in (1) must be included in the expansion of H, K and which multilinear forms to include in the expansion of f. Indeed, we search for those terms in the expansion of the reduced system on the center manifold of (1) that affect the coefficients in ϵ up to order three in (12). These are determined by solving the linear Diophantine equation

$$4i + 2j + 2k + 3l - 4 = n, \qquad n \in \{-2, -1, 0, 1, 2, 3\},$$
(13)

for $i, j, k, l \in \mathbb{N}_0$. In Table 1 the solutions to (13) are listed. To transform away these terms (into higher order terms), one needs exactly the corresponding coefficients in the expansions of H and K. To be concrete, suppose the term $\alpha_1 \alpha_2$ is present in the reduced system on the center manifold. Applying the blowup transformation (11), the term $\alpha_1 \alpha_2$ will show up in the coefficient of ϵ^2 in (12). Since we will derive a third-order approximation for (12) in which the corresponding term $\beta_1\beta_2$ is not present, this term needs to be transformed away. It is not too difficult to show that the coefficients needed to perform this operation in phase and parameter-space are precisely H_{0011} and K_{11} . In Appendix A an explicit example is given to show that the transformation in [1] leads to an incorrect predictor for the parameters.

Thus, we expand the mappings H, K, and θ , including precisely those coefficients needed to transfer the homoclinic predictor in the normal form to the center manifold maintaining the accuracy. Using Table 1 we write:



Figure 1: Schematic overview is which order the coefficients in the expansion of H, K and θ are derived.

$$\begin{split} f(x,\alpha) &= Ax + J_{1}\alpha + \frac{1}{2}B(x,x) + A_{1}(x,\alpha) + \frac{1}{2}J_{2}(\alpha,\alpha) + \frac{1}{6}C(x,x,x) \\ &+ \frac{1}{2}B_{1}(x,x,\alpha) + \frac{1}{2}A_{2}(x,\alpha,\alpha) + \frac{1}{6}J_{3}(\alpha,\alpha,\alpha) + \mathcal{O}\left(\|x\|^{4} + \|\alpha\|^{3}\right), \end{split} \tag{14} \\ &+ \frac{1}{2}B_{1}(x,x,\alpha) + \frac{1}{2}A_{2}(x,\alpha,\alpha) + \frac{1}{6}J_{3}(\alpha,\alpha,\alpha) + \mathcal{O}\left(\|x\|^{4} + \|\alpha\|^{3}\right), \end{aligned} \\ H(w,\beta) &= q_{0}w_{0} + q_{1}w_{1} + H_{0010}\beta_{1} + H_{0001}\beta_{2} + \frac{1}{2}H_{2000}w_{0}^{2} + H_{1100}w_{0}w_{1} + \frac{1}{2}H_{0200}w_{1}^{2} \\ &+ H_{1010}w_{0}\beta_{1} + H_{1001}w_{0}\beta_{2} + H_{0110}w_{1}\beta_{1} + H_{0101}w_{1}\beta_{2} + \frac{1}{2}H_{0002}\beta_{2}^{2} \\ &+ H_{0011}\beta_{1}\beta_{2} + \frac{1}{6}H_{3000}w_{0}^{3} + \frac{1}{2}H_{2100}w_{0}^{2}w_{1} + H_{1101}w_{0}w_{1}\beta_{2} + \frac{1}{2}H_{2001}w_{0}^{2}\beta_{2} \\ &+ \frac{1}{6}H_{0003}\beta_{2}^{3} + \frac{1}{2}H_{1002}w_{0}\beta_{2}^{2} + \frac{1}{2}H_{0102}w_{1}\beta_{2}^{2} \\ &+ \mathcal{O}(|w_{1}|^{3} + |w_{0}w_{1}^{2}| + |\beta_{2}w_{1}^{2}| + |\beta_{1}|||w||^{2} + |\beta_{1}^{2}|||w|| + |\beta_{1}^{2}| + ||(w,\beta)||^{4}), \end{aligned} \\ K(\beta) &= K_{10}\beta_{1} + K_{01}\beta_{2} + \frac{1}{2}K_{02}\beta_{2}^{2} + K_{11}\beta_{1}\beta_{2} + K_{03}\frac{1}{6}\beta_{2}^{3} \\ &+ \mathcal{O}(|\beta_{1}|^{2} + |\beta_{1}||\beta_{2}|^{2} + |\beta_{1}|^{2}|\beta_{2}| + |||\beta||^{4}), \end{aligned} \tag{15}$$

where $A = f_x(x_0, \alpha_0)$, $J_1 = f_\alpha(x_0, \alpha_0)$, and B, J_2, J_3, C, A_1, A_2 and B_1 are the standard multilinear forms, introduced for readability.

Remark 1. Notice that compared with [1] there are four additional terms in the expansion of H, i.e. with coefficients $H_{0011}, H_{1002}, H_{0102}$, and H_{0003} , and two additional terms in the expansion of K, with coefficients K_{11} and K_{03} .

2.2.1 (Generalized) eigenvectors

We assume that the equilibrium (x_0, α_0) has a double (but not semisimple) zero eigenvalue, while all other eigenvalues are away from the imaginary axis. Thus, there exist two real linearly independent (generalized) eigenvectors, $q_0, q_1 \in \mathbb{R}^n$, of A, such that

$$Aq_0 = 0, \qquad Aq_1 = q_0, \tag{18}$$

and two left (generalized) eigenvectors $p_1^T, p_0^T \in \mathbb{R}^n$, of A, such that

$$p_1 A = 0, \qquad p_0 A = p_1$$

These vectors can be normalized to satisfy

$$p_i q_j = \delta_{ij}, \qquad i = 0, 1, \ j = 0, 1.$$

As in [39], we impose the condition

$$q_0^T q_0 = 1, \qquad q_1^T q_0 = 0,$$
 (19)

to uniquely define the vectors $\{q_0, q_1, p_1, p_0\}$ up to a plus or minus sign.

Note that collecting the coefficients of the linear terms in w in the homological equation are precisely the systems defining the (generalized) eigenvectors (18).

2.2.2Critical normal form coefficients

Collecting the quadratic coefficients in w from the homological (7) yields the systems

$$-AH_{2000} = B(q_0, q_0) - 2aq_1, \tag{20}$$

$$-AH_{1100} = B(q_0, q_1) - bq_1 + \theta_{1000}q_0 - H_{2000},$$
(21)

$$-AH_{0200} = B(q_1, q_1) - 2H_{1100}.$$
(22)

The Fredholm solvability condition for the first two systems yields the well known expressions

$$a = \frac{1}{2} p_1 B(q_0, q_0),$$

$$b = p_1 B(q_0, q_1) + p_0 B(q_0, q_0),$$

for the critical coefficients, see for example [37]. By the non-degeneracy conditions, we have that $ab \neq 0$. *Remark* 2. Since we assume that $p_1B(q_0, q_0) \neq 0$ we can use the freedom in the eigenvectors,

$$(q_0, q_1) \to c_1(q_0, q_1), \qquad (p_1, p_0) \to \frac{1}{c_1}(p_1, p_0),$$

to normalize the critical coefficient

$$a = p_1 B(q_0, q_0),$$

to one. Solving for c_1 then gives

$$c_1 = \frac{1}{p_1 B(q_0, q_0)}.$$

Alternatively, the freedom could have been used to set b = 1. To have the situation a = 1 and $b = \pm 1$, as in [10], the coefficient in front of the constant term in the expansion of θ , i.e., θ_{0000} , should be used. For convenience, we fixed this constant to 1.

Now that (20) and (21) are solvable, we can define ~

$$\begin{aligned} \dot{H}_{2000} &= -\mathbf{A}^{\mathrm{INV}} \left(B(q_0, q_0) - 2aq_1 \right), \\ \dot{H}_{1100} &= -\mathbf{A}^{\mathrm{INV}} \left(B(q_0, q_1) - bq_1 - \hat{H}_{2000} \right). \end{aligned}$$

The expression $x = A^{INV} y$ is defined by solving the non-singular bordered system

$$\begin{pmatrix} A & p_1^T \\ q_0^T & 0 \end{pmatrix} \begin{pmatrix} x \\ s \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix},$$

for the unknown $(x, s) \in \mathbb{R}^{n+1}$ that necessarily satisfies s = 0. The properties of bordered linear systems and their role in numerical bifurcation analysis are discussed in [36] and [22, Chapter 3].

It follows that the general solutions to the coefficients H_{2000} and H_{1100} are given by

$$H_{2000} = H_{2000} + \gamma_1 q_0,$$

$$H_{1100} = \hat{H}_{1100} + \gamma_1 q_1 - \theta_{1000} q_1 + \gamma_2 q_0.$$

The constant γ_1 is determined by the solvability condition from (22), which gives

$$\gamma_1 = p_0 \left(B(q_0, q_1) - \hat{H}_{2000} \right) + \frac{1}{2} p_1 B(q_1, q_1) + \theta_{1000}$$

To determine the constant γ_2 and the coefficient θ_{1000} we consider the w_0^3 and $w_0^2 w_1$ terms in the homological equation. After some simplification, we obtain

$$-AH_{3000} = 3B(H_{2000}, q_0) + C(q_0, q_0, q_0) + 6a\theta_{1000}q_1 - 6aH_{1100},$$
(23)

$$-AH_{2100} = -2aH_{0200} - 2bH_{1100} - H_{3000} + 2B(H_{1100}, q_0) + B(H_{2000}, q_1)$$
(24)
+ 2\theta_{1000}(bq_1 - \theta_{1000}q_0 + H_{2000}) + C(q_0, q_0, q_1).

The solvability condition of the first equation determines θ_{1000} as

$$\theta_{1000} = -\frac{1}{12a} p_1 \left\{ 3B(\hat{H}_{2000}, q_0) + C(q_0, q_0, q_0) \right\} + \frac{1}{2} p_1 \hat{H}_{1100}.$$
⁽²⁵⁾

After a rather lengthy calculation, we obtain that γ_2 is determined by

$$\gamma_{2} = \frac{1}{6a} \bigg[p_{1} \bigg\{ 2B(\hat{H}_{1100}, q_{0}) + B(\hat{H}_{2000}, q_{1}) + C(q_{0}, q_{0}, q_{1}) \bigg\}$$

$$+ 2ap_{0}B(q_{1}, q_{1}) + 2bp_{0} \bigg(B(q_{0}, q_{1}) - \hat{H}_{2000} \bigg)$$

$$+ p_{0} \bigg(3B(\hat{H}_{2000}, q_{0}) + C(q_{0}, q_{0}, q_{0}) \bigg)$$

$$+ \gamma_{1}b - 10ap_{0}\hat{H}_{1100} + 2b\theta_{1000} \bigg].$$

$$(26)$$

Since the systems in (22)–(24) are now all consistent, we are allowed to take the bordered inverses to obtain

$$H_{0200} = -A^{INV} \left[B(q_1, q_1) - 2H_{1100} \right],$$

$$H_{3000} = -A^{INV} \left[3B(H_{2000}, q_0) + C(q_0, q_0, q_0) + 6a\theta_{1000}q_1 - 6aH_{1100} \right],$$
(27)

$$H_{2100} = -\mathbf{A}^{\text{INV}} \left[-2aH_{0200} - 2bH_{1100} - H_{3000} + 2B(H_{1100}, q_0), \right]$$
(28)

$$+B(H_{2000},q_1)+2\theta_{1000}(bq_1-\theta_{1000}q_0+H_{2000})+C(q_0,q_0,q_1)].$$

2.2.3 Parameter-dependent linear terms

The coefficients of the linear terms in β give the systems

$$-AH_{0001} = J_1 K_{01}, -AH_{0010} = J_1 K_{10} - q_1.$$
⁽²⁹⁾

Since p_1 and J_1 are known, we can calculate

$$\nu = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} := (p_1 J_1)^T.$$

By the transversality condition, the vector ν is nonzero. It then follows from the Fredholm alternative that $V = \int \hat{V}$

$$K_{01} = \delta_1 K_{01},$$

$$H_{0001} = \delta_1 \left(\hat{H}_{0001} + \gamma_3 q_0 \right),$$

$$K_{10} = \hat{K}_{10} + \delta_2 K_{01},$$

$$H_{0010} = \hat{H}_{0010} + \delta_2 H_{0001} + \gamma_4 q_0.$$

where

$$\hat{K}_{10} = \frac{1}{\|\tau\|^2} \nu,$$

$$\hat{H}_{0010} = \mathbf{A}^{\mathrm{INV}} \left(q_1 - J_1 \hat{K}_{10} \right),$$

$$\hat{K}_{01} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \hat{K}_{10},$$

$$\hat{H}_{0001} = -\mathbf{A}^{\mathrm{INV}} J_1 \hat{K}_{01}.$$

and $\delta_{1,2}$, $\gamma_{3,4}$ are real constants determined by the solvability condition of the $w\beta$ terms in the homological equation. Collecting the corresponding systems in the homological equation yields

$$\begin{aligned} -AH_{1001} &= B(H_{0001}, q_0) + A_1(q_0, K_{01}), \\ -AH_{0101} &= B(H_{0001}, q_1) + A_1(q_1, K_{01}) - H_{1001} - q_1 + \theta_{0001}q_0, \\ -AH_{1010} &= B(H_{0010}, q_0) + A_1(q_0, K_{10}) - H_{1100} + \theta_{1000}q_1, \\ -AH_{0110} &= B(H_{0010}, q_1) + A_1(q_1, K_{10}) - H_{0200} - H_{1010}. \end{aligned}$$

The solvability condition for the first two systems yields

$$\begin{split} \gamma_{3} &= -\frac{p_{1}\left(B(\hat{H}_{0001},q_{0}) + A_{1}(q_{0},\hat{K}_{01})\right)}{2a}, \\ \delta_{1} &= \frac{1}{p_{1}\left(B(\hat{H}_{0001},q_{1}) + A_{1}(q_{1},\hat{K}_{01})\right) + p_{0}\left(B(\hat{H}_{0001},q_{0}) + A_{1}(q_{0},\hat{K}_{01})\right) + \gamma_{3}b}, \end{split}$$

while the solvability condition for the latter two systems yields

$$\gamma_{4} = \frac{p_{1}H_{1100} - \theta_{1000} - p_{1}\left(B(\hat{H}_{0010}, q_{0}) + A_{1}(q_{0}, \hat{K}_{10})\right)}{2a},$$

$$\delta_{2} = -p_{1}\left(B(\hat{H}_{0010}, q_{1}) + A_{1}(q_{1}, \hat{K}_{10})\right) - \gamma_{4}b + p_{1}H_{0200},$$

$$-p_{0}\left(B(\hat{H}_{0010}, q_{0}) + A_{1}(q_{0}, \hat{K}_{10}) - H_{1100}\right).$$

Note that the denominator in δ_1 is nonzero by the transversality condition.

2.2.4 Coefficients H_{1010} and H_{0110}

$$H_{1010} = -A^{INV} \left[B(H_{0010}, q_0) + A_1(q_0, K_{10}) - H_{1100} + \theta_{1000}q_1 \right],$$

$$H_{0110} = -A^{INV} \left[B(H_{0010}, q_1) + A_1(q_1, K_{10}) - H_{0200} - H_{1010} \right].$$

2.2.5 Coefficients $(\theta_{0001}, \gamma_5), H_{1001}, H_{0101}, H_{2001}, H_{1101}$

Define

$$\hat{H}_{1001} = -\mathbf{A}^{\text{INV}} \left[B(H_{0001}, q_0) + A_1(q_0, K_{01}) \right],$$

$$\hat{H}_{0101} = -\mathbf{A}^{\text{INV}} \left[B(H_{0001}, q_1) + A_1(q_1, K_{01}) - H_{1001} - q_1 \right],$$

so that

$$\begin{aligned} H_{1001} &= \hat{H}_{1001} + \gamma_5 q_0, \\ H_{0101} &= \hat{H}_{0101} + \gamma_5 q_1 - \theta_{0001} q_1 \end{aligned}$$

In order to determine γ_5 and θ_{0001} , we consider the systems corresponding to the $w_0^2\beta_2$ and $w_0w_1\beta_2$ terms in the homological equation. These are given by

$$-AH_{2001} = -2aH_{0101} + A_1(H_{2000}, K_{01}) + B(H_{0001}, H_{2000}) + 2B(H_{1001}, q_0) + 2a\theta_{0001}q_1 + B_1(q_0, q_0, K_{01}) + C(H_{0001}, q_0, q_0), -AH_{1101} = -bH_{0101} - H_{1100} - H_{2001} + A_1(H_{1100}, K_{01}) + \theta_{1000}(H_{1001} + q_1 - \theta_{0001}q_0) + B(H_{0001}, H_{1100}) + B(H_{0101}, q_0) + B(H_{1001}, q_1) + \theta_{0001}(H_{2000} + bq_1 - \theta_{1000}q_0) + B_1(q_0, q_1, K_{01}) + C(H_{0001}, q_0, q_1).$$
(30)

The Fredholm solvability condition leads to the following system to be solved

$$\begin{pmatrix} -2a & -4a \\ -b & -b \end{pmatrix} \begin{pmatrix} \gamma_5 \\ \theta_{0001} \end{pmatrix} = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix},$$
(31)

where

$$\begin{aligned} \zeta_{1} &= p_{1} \left[-2a\hat{H}_{0101} + A_{1}(H_{2000}, K_{01}) + B(H_{0001}, H_{2000}) \\ &+ 2B(\hat{H}_{1001}, q_{0}) + B_{1}(q_{0}, q_{0}, K_{01}) + C(H_{0001}, q_{0}, q_{0}) \right], \\ \zeta_{2} &= p_{1} \left[-b\hat{H}_{0101} - H_{1100} + A_{1}(H_{1100}, K_{01}) + \\ &\theta_{1000}(\hat{H}_{1001} + q_{1}) + B(H_{0001}, H_{1100}) + B(\hat{H}_{0101}, q_{0}) + \\ &B(\hat{H}_{1001}, q_{1}) + B_{1}(q_{0}, q_{1}, K_{01}) + C(H_{0001}, q_{0}, q_{1}) \right] \\ &+ p_{0} \left[-2a\hat{H}_{0101} + A_{1}(H_{2000}, K_{01}) + B(H_{0001}, H_{2000}) \\ &+ 2B(\hat{H}_{1001}, q_{0}) + B_{1}(q_{0}, q_{0}, K_{01}) + C(H_{0001}, q_{0}, q_{0}) \right]. \end{aligned}$$

Notice that the matrix has a non-vanishing determinant by the non-degeneracy condition. Now that the systems in (30) are solvable we obtain

$$H_{2001} = -A^{INV} \left[-2aH_{0101} + A_1(H_{2000}, K_{01}) + B(H_{0001}, H_{2000}) + 2B(H_{1001}, q_0) + 2a\theta_{0001}q_1 + B_1(q_0, q_0, K_{01}) + C(H_{0001}, q_0, q_0) \right],$$

$$H_{1101} = -A^{INV} \left[-bH_{0101} - H_{1100} - H_{2001} + A_1(H_{1100}, K_{01}) + \theta_{1000}(H_{1001} + q_1 - \theta_{0001}q_0) + B(H_{0001}, H_{1100}) + B(H_{0101}, q_0) + B(H_{1001}, q_1) + \theta_{0001}(H_{2000} + bq_1 - \theta_{1000}q_0) + B_1(q_0, q_1, K_{01}) + C(H_{0001}, q_0, q_1) \right].$$
(33)

2.2.6 Coefficients K_{11} and H_{0011}

Collecting the systems corresponding to the $\beta_1\beta_2$ term in the homological equation yields

$$-AH_{0011} = J_1K_{11} + A_1(H_{0001}, K_{10}) + A_1(H_{0010}, K_{01}) + B(H_{0001}, H_{0010}) + J_2(K_{01}, K_{10}) + \theta_{0001}q_1 - H_{0101}.$$
(34)

Using the identity

$$p_1 J_1 K_{10} = 1$$

from the second system in (29) combined with the solvability condition yields

$$K_{11} = -p_1 \left[A_1(H_{0001}, K_{10}) + A_1(H_{0010}, K_{01}) + B(H_{0010}, H_{0001}) + J_2(K_{10}, K_{01}) + \theta_{0001}q_1 - H_{0101} \right] K_{10}.$$

It follows that

$$H_{0011} = -A^{INV} [J_1 K_{11} + A_1(H_{0001}, K_{10}) + A_1(H_{0010}, K_{01}) + B(H_{0001}, H_{0010}) + J_2(K_{01}, K_{10}) + \theta_{0001}q_1 - H_{0101}].$$
(35)

2.2.7 Coefficients $K_{02}, H_{0002}, H_{1002}, H_{0102}$

The systems corresponding to the $\beta_2^2, w_0\beta_2^2$ and $w_1\beta_2^2$, terms in the homological equation yields

$$-AH_{0002} = J_1 K_{02} + 2A_1 (H_{0001}, K_{01}) + B(H_{0001}, H_{0001}) + J_2 (K_{01}, K_{01}),$$

$$-AH_{1002} = 2A_1 (H_{1001}, K_{01}) + A_1 (q_0, K_{02}) + A_2 (q_0, K_{01}, K_{01})$$

$$+ B(q_0, H_{0002}) + 2B(H_{0001}, H_{1001}) + 2B_1 (q_0, H_{0001}, K_{01})$$

$$+ C(q_0, H_{0001}, H_{0001}),$$

$$-AH_{0102} = 2A_1 (H_{0101}, K_{01}) + A_1 (q_1, K_{02}) + A_2 (q_1, K_{01}, K_{01})$$

$$+ B(q_1, H_{0002}) + 2B(H_{0001}, H_{0101}) + 2B_1 (q_1, H_{0001}, K_{01})$$

$$+ C(q_1, H_{0001}, H_{0001}) + 2\theta_{0001} (H_{1001} + q_1 - \theta_{0001} q_0)$$

$$- 2H_{0101} - H_{1002}.$$

(36)

The first system is solved similarly as (34). However, here we need to use hypernormalization in order to make the second and third systems consistent. Thus, we define

$$\hat{K}_{02} = -p_1 \left[2A_1(H_{0001}, K_{01}) + B(H_{0001}, H_{0001}) + J_2(K_{01}, K_{01}) \right] K_{10},$$

$$\hat{H}_{0002} = -A^{INV} \left[J_1 \hat{K}_{02} + 2A_1(H_{0001}, K_{01}) + B(H_{0001}, H_{0001}) + J_2(K_{01}, K_{01}) \right].$$

Then the general solutions can be written as

$$K_{02} = \hat{K}_{02} + \delta_3 K_{01},$$

$$H_{0002} = \hat{H}_{0002} + \delta_3 H_{0001} + \gamma_6 q_0.$$

Substituting these two expressions into (36) and using the solvability condition yields

$$\begin{split} \gamma_6 &= -\frac{1}{2a} p_1 \left[2A_1(H_{1001}, K_{01}) + A_1(q_0, \hat{K}_{02}) + A_2(q_0, K_{01}, K_{01}) \right. \\ &+ B(q_0, \hat{H}_{0002}) + 2B(H_{0001}, H_{1001}) + 2B_1(q_0, H_{0001}, K_{01}) \\ &+ C(q_0, H_{0001}, H_{0001}) \right], \\ \delta_3 &= -p_1 \left[2A_1(H_{0101}, K_{01}) + A_1(q_1, \hat{K}_{02}) + A_2(q_1, K_{01}, K_{01}) \right. \\ &+ B(q_1, \hat{H}_{0002}) + 2B(H_{0001}, H_{0101}) + 2B_1(q_1, H_{0001}, K_{01}) \\ &+ C(q_1, H_{0001}, H_{0001}) + 2\theta_{0001}(H_{1001} + q_1) - 2H_{0101} \right] \\ &- p_0 \left[2A_1(H_{1001}, K_{01}) + A_1(q_0, \hat{K}_{02}) + A_2(q_0, K_{01}, K_{01}) \right. \\ &+ B(q_0, \hat{H}_{0002}) + 2B(H_{0001}, H_{1001}) + 2B_1(q_0, H_{0001}, K_{01}) \\ &+ C(q_0, H_{0001}, H_{0001}) \right] - \gamma_6 b. \end{split}$$

Now that the last two systems in (36) are consistent, we obtain

$$\begin{split} H_{1002} &= - \mathbf{A}^{\mathrm{INV}} \left[2A_1(H_{1001}, K_{01}) + A_1(q_0, K_{02}) + A_2(q_0, K_{01}, K_{01}) \right. \\ &+ B(q_0, H_{0002}) + 2B(H_{0001}, H_{1001}) + 2B_1(q_0, H_{0001}, K_{01}) \\ &+ C(q_0, H_{0001}, H_{0001}) \right], \\ H_{0102} &= - \mathbf{A}^{\mathrm{INV}} \left[2A_1(H_{0101}, K_{01}) + A_1(q_1, K_{02}) + A_2(q_1, K_{01}, K_{01}) \right. \\ &+ B(q_1, H_{0002}) + 2B(H_{0001}, H_{0101}) + 2B_1(q_1, H_{0001}, K_{01}) \\ &+ C(q_1, H_{0001}, H_{0001}) + 2\theta_{0001}(H_{1001} + q_1 - \theta_{0001}q_0) \\ &- 2H_{0101} - H_{1002} \right]. \end{split}$$

2.2.8 Coefficients K_{03} and H_{0003}

Collecting the systems corresponding to the β_2^3 term in the homological equation yields

$$-AH_{0003} = J_1K_{03} + A_1(H_{0001}, K_{02}) + A_1(H_{0002}, K_{01}) + 2(A_1(H_{0001}, K_{02}) + A_1(H_{0002}, K_{01}) + 3B(H_{0001}, H_{0002}) + 3J_2(K_{01}, K_{02}) + 3A_2(H_{0001}, K_{01}, K_{01}) + 3B_1(H_{0001}, H_{0001}, K_{01}) + C(H_{0001}, H_{0001}, H_{0001}) + J_3(K_{01}, K_{01}, K_{01}).$$

This equation is solved similarly as equation (34). We obtain

$$\begin{split} K_{03} &= -p_1 \left[A_1(H_{0001}, K_{02}) + A_1(H_{0002}, K_{01}) + 2A_1(H_{0001}, K_{02}) \right. \\ &+ 2A_1(H_{0002}, K_{01}) + 3B(H_{0001}, H_{0002}) + 3J_2(K_{01}, K_{02}) \\ &+ 3A_2(H_{0001}, K_{01}, K_{01}) + 3B_1(H_{0001}, H_{0001}, K_{01}) \\ &+ C(H_{0001}, H_{0001}, H_{0001}) + J_3(K_{01}, K_{01}, K_{01}) \right] K_{10}, \\ H_{0003} &= - A^{INV} \left[J_1 K_{03} + A_1(H_{0001}, K_{02}) + A_1(H_{0002}, K_{01}) + 2A_1(H_{0001}, K_{02}) \right. \\ &+ 2A_1(H_{0002}, K_{01}) + 3B(H_{0001}, H_{0002}) + 3J_2(K_{01}, K_{02}) \\ &+ 3A_2(H_{0001}, K_{01}, K_{01}) + 3B_1(H_{0001}, H_{0001}, K_{01}) \\ &+ C(H_{0001}, H_{0001}, H_{0001}) + J_3(K_{01}, K_{01}, K_{01}) \right]. \end{split}$$

2.3 Center manifold reduction for smooth normal form

If we do not allow for a reparametrization of time, we can no longer consider the normal form (10). Instead, we need to use the smooth normal form as introduced in [41], i.e., equation (9). Applying the

blowup transformation

$$\beta_1 = -\frac{4}{a}\epsilon^4, \quad \beta_2 = \frac{b}{a}\tau\epsilon^2, \quad w_0 = \frac{1}{a}u\epsilon^2, \quad w_1 = \frac{1}{a}v\epsilon^3, \quad s = \epsilon t, \quad (\epsilon \neq 0), \tag{37}$$

to the smooth normal form, we obtain the second-order nonlinear differential equation

$$\ddot{u} = -4 + u^2 + \frac{b}{a}\dot{u}(u+\tau)\epsilon + \frac{1}{a^2}u^2(\tau ba_1 + du)\epsilon^2 + \frac{1}{a^2}u\dot{u}(\tau bb_1 + eu)\epsilon^3 + \mathcal{O}(\epsilon^4).$$
(38)

Here the dot represents the derivative with respect to s.

Note that, by using hypernormalization, we can still simplify the smooth normal form. Indeed, as already remarked in [39] the coefficient e can be set to zero. Furthermore, it can be seen from the system in (31) that either the coefficient a_1 or b_1 can also be removed. The natural choice here is for the coefficient b_1 to be set to zero in the normal form. The parameter-dependent center manifold transformation in this situation is obtained by first setting the coefficients θ_{1000} and θ_{0001} to zero in Section 2.2. Equation (23) becomes

$$-AH_{3000} = 3B(H_{2000}, q_0) + C(q_0, q_0, q_0) - 6dq_1 - 6aH_{1100},$$
(39)

and equation (25) is removed. After γ_2 in (26) has been calculated, the Fredholm solvability condition yields that

$$d = \frac{1}{6}p_1 \left[3B(H_{2000}, q_0) + C(q_0, q_0, q_0) - 6a\hat{H}_{1100} \right].$$

Now that (39) is consistent, we can replace (27) with

$$H_{3000} = -\mathbf{A}^{\mathrm{INV}} \left[3B(H_{2000}, q_0) + C(q_0, q_0, q_0) - 6dq_1 - 6aH_{1100} \right]$$

Next, we replace the first equation in (30) with

$$-AH_{2001} = -2a_1q_1 - 2aH_{0101} + A_1(H_{2000}, K_{01}) + B(H_{0001}, H_{2000}) + 2B(H_{1001}, q_0) + B_1(q_0, q_0, K_{01}) + C(H_{0001}, q_0, q_0)$$
(40)

and the system in (31) becomes the single equation

$$\gamma_5 = -\frac{\zeta_2}{b}.$$

Here ζ_2 is still given by the second equation in (32) (with θ_{1000} still set to zero), while ζ_1 is no longer needed. Applying the Fredholm solvability condition to (40) yields

$$a_{1} = \frac{1}{2} \left[-2aH_{0101} + A_{1}(H_{2000}, K_{01}) + B(H_{0001}, H_{2000}) + 2B(H_{1001}, q_{0}) \right.$$
$$\left. + B_{1}(q_{0}, q_{0}, K_{01}) + C(H_{0001}, q_{0}, q_{0}) \right].$$

Since (40) is now consistent, we can replace the first equation in (33) with

$$H_{2001} = -A^{INV} \left[-2aH_{0101} + A_1(H_{2000}, K_{01}) + B(H_{0001}, H_{2000}) \right. \\ \left. + 2B(H_{1001}, q_0) - 2a_1q_1 + B_1(q_0, q_0, K_{01}) + C(H_{0001}, q_0, q_0) \right].$$

The remaining systems and equations are unchanged.

To compare the homoclinic predictors under different transformations, we also provide the parameterdependent center manifold transformation for the smooth normal form (9) without transforming away the coefficients e and b_1 . In this case, in addition to the modification given above, we also set γ_2 and γ_5 to zero. Then the system in (24) becomes

$$-AH_{2100} = -2eq_1 - 2aH_{0200} - 2bH_{1100} - H_{3000} + 2B(H_{1100}, q_0) + B(H_{2000}, q_1) + C(q_0, q_0, q_1),$$
(41)

while the second systems in (30) should be replaced with

1

$$-AH_{1101} = -b_1q_1 - bH_{0101} - H_{1100} - H_{2001} + A_1(H_{1100}, K_{01}) + + B(H_{0001}, H_{1100}) + B(H_{0101}, q_0) + B(H_{1001}, q_1) + B_1(q_0, q_1, K_{01}) + C(H_{0001}, q_0, q_1).$$
(42)

Applying the Fredholm solvability condition to these equations gives

$$e = \frac{1}{2} p_1 \left[-2aH_{0200} - 2bH_{1100} - H_{3000} + 2B(H_{1100}, q_0) + B(H_{2000}, q_1) + C(q_0, q_0, q_1) \right],$$

$$b_1 = p_1 \left[-bH_{0101} - H_{1100} - H_{2001} + A_1(H_{1100}, K_{01}) + B(H_{0001}, H_{1100}) + B(H_{0101}, q_0) + B(H_{1001}, q_1) + B_1(q_0, q_1, K_{01}) + C(H_{0001}, q_0, q_1) \right].$$

Now that (41) and (42) are consistent, we can replace (28) and the second system in (33) with

$$H_{2100} = -\mathbf{A}^{\mathrm{INV}} \left[-2eq_1 - 2aH_{0200} - 2bH_{1100} - H_{3000} + 2B(H_{1100}, q_0) \right. \\ \left. + B(H_{2000}, q_1) + C(q_0, q_0, q_1) \right].$$

and

$$H_{1101} = -A^{INV} [-b_1q_1 - bH_{0101} - H_{1100} - H_{2001} + A_1(H_{1100}, K_{01}) + B(H_{0001}, H_{1100}) + B(H_{0101}, q_0) + B(H_{1001}, q_1) + B_1(q_0, q_1, K_{01}) + C(H_{0001}, q_0, q_1)],$$

respectively. The remaining systems and equations are unchanged.

Remark 3. The derivation in [1, 40] leads to a 'big' system in which equations need to be solved simultaneously. The derivation presented here does not involve a 'big' system, making the expressions more suitable to implement for infinitely-dimensional ODEs generated by partial and delay differential equations, to which the (parameter-dependent) center manifold theorem applies.

3 Homoclinic asymptotics

In this section, we derive third-order asymptotics to the homoclinic solution near the generic Bogdanov-Takens point. We revisited the standard regular perturbation method, but with a different phase condition. In Section 5 we will show that this improves the accuracy of the homoclinic approximation in the normal form. Then, in Section 3.2 we revisited the Lindstedt-Poincaré method. By an additional non-linear time transformation, we obtain a very simple algorithm to approximate the homoclinic solution.

3.1 The Regular Perturbation Method with norm minimizing phase condition

For $\epsilon = 0$, (12) is a Hamiltonian system with the first integral

$$H(u, \dot{u}) = \frac{1}{2}\dot{u}^2 + 4u - \frac{1}{3}u^3.$$

The Hamiltonian system has the well-known explicit homoclinic solution $(u_0(s), \dot{u}_0(s))$ given by

$$u_0(s) = 6 \tanh^2(s) - 4.$$

Thus, for $(u, \epsilon, \tau) = (u_0, 0, \tau)$ there exists a trivial branch of homoclinic orbits in (12). In [4] it is shown that there exists a bifurcation point at $\tau = \frac{10}{7}$ from which a smooth non-trivial branch emanates transversally. Parametrizing this branch by ϵ , we formally have

$$u(s,\epsilon) = \sum_{i\geq 0} u_i(s)\epsilon^i, \qquad \tau(\epsilon) = \sum_{i\geq 0} \tau_i\epsilon^i.$$
(43)

Substituting (43) into (12) and collecting equal terms in ϵ yields the following differential equations to be solved:

$$\ddot{u}_0 - u_0^2 + 4 = 0, (44)$$

$$\ddot{u}_i - 2u_0 u_i = z_i,\tag{45}$$

$$\dot{u}_i(\pm\infty) = \ddot{u}_i(\pm\infty) = 0, \qquad i \in \mathbb{N}.$$

Here z_i dependents on the sums and products of u_j, \dot{u}_j and τ_{j-1} for $0 \le j < i$. Multiplying equation (45) by \dot{u} and integrating from s_0 to s yields,

$$\int_{s_0}^s \dot{u}_0 \ddot{u}_i - 2\dot{u}_0 u_0 u_i \, dx = \int_{s_0}^s \dot{u}_0 z_i \, dx.$$

Using integration by parts twice then gives

$$\left(\dot{u}_{0}\dot{u}_{i}-u_{i}\ddot{u}_{0}\right)\Big|_{s_{0}}^{s}=\int_{s_{0}}^{s}\dot{u}_{0}z_{i}\ dx.$$
(46)

Notice that solutions $(\dot{u}_i(s), \ddot{u}_i(s))$ must vanish at plus and minus infinity. We obtain that τ_{i-1} is given by the condition

$$0 = \int_{-\infty}^{\infty} \dot{u}_0 z_i \, dx.$$

To simplify the equations that follow below we would like to use [4, Proposition 4.2].

However, we noticed that the proposition is not precise enough for the conclusion to hold. Indeed, the proof relies on the uniqueness of the non-trivial branch of homoclinic orbits. However, we see that the left-hand side (45) is invariant under the transformation

$$u_i \to u_i + \gamma \dot{u}_0, \qquad \gamma \in \mathbb{R}, \ i \in \mathbb{N}.$$

We, therefore, slightly modify the proposition with an additional assumption.

Proposition 4. Assume that the perturbed Hamiltonian system (12) is obtained from the normal form (10) by the singular rescaling (11). Then the non-trivial branch of homoclinic solutions

$$(u(s,\epsilon), \dot{u}(s,\epsilon), \epsilon, \tau(\epsilon)), \quad \epsilon < |\epsilon_0|,$$

for some $\epsilon_0 > 0$ satisfies

$$\tau(\epsilon) = \tau(-\epsilon). \tag{47}$$

Furthermore, if the solutions u_i are even functions for i even, then the solution u contains the additional symmetry

$$u(s, -\epsilon) = u(-s, \epsilon).$$

Proof. The proof follows almost entirely [4, Proposition 4.2]. Thus, it can be shown that the transformation (11) induces the symmetry

$$\varphi(s, D_0(u^0, \dot{u}^0)^T, -\epsilon, \tau) = D_0 \varphi(-s, (u^0, \dot{u}^0)^T, \epsilon, \tau)$$

on the flow φ of (12), where

$$D_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

From this relation one can then conclude that if $(u(s,\epsilon), \dot{u}(s,\epsilon), \epsilon, \tau(\epsilon))$ is a homoclinic solution to (12) then so is $(\tilde{u}(-s,\epsilon), -\dot{\tilde{u}}(-s,\epsilon), -\epsilon, \tau(\epsilon))$. The proof in [4] then finishes with the remark that these two homoclinic solutions must be equal by the uniqueness of the non-trivial branch. However, by the *non-uniqueness* of the non-trivial branch, we obtain the relation

$$(u(s,\epsilon),\epsilon,\tau(\epsilon)) = (u(-s,-\epsilon),\epsilon,\tau(-\epsilon)) + (\gamma(\epsilon)\dot{u}_0(s),\epsilon,0),$$

where

$$\gamma(\epsilon) = \sum_{i \ge 1} \gamma_i \epsilon^i, \qquad \gamma_i \in \mathbb{R}.$$
(48)

Thus, τ is indeed an even function of ϵ . Using the expansion for u from (43) we see that by inspecting the coefficients of equal powers in ϵ we only need to impose that $\gamma_i = 0$ for i even and then the assertion follows.

It, therefore, follows from Proposition 4 together with equality (46) that the condition for solving τ_{i-1} in (49) simplifies to

$$0 = \int_0^\infty \dot{u}_0 z_i \, dx. \tag{49}$$

for *i* even, whereas $\tau_i = 0$ for *i* odd.

From (46) we obtain the solution

$$u_i = \dot{u}_0 \int \frac{1}{\dot{u}_0^2} \int \dot{u}_0 z_i \, dx \, dx, \qquad i \in \mathbb{N},\tag{50}$$

or

$$u_{i} = \left(\dot{u}_{0} \int \frac{1}{\dot{u}_{0}^{2}} dx\right) \int \dot{u}_{0} z_{i}(u, \dot{u}, \tau) dx - \dot{u}_{0} \int \left(\int \frac{1}{\dot{u}_{0}^{2}} dx\right) \dot{u}_{0} z_{i}(u, \dot{u}, \tau) dx.$$

From (50) we see that there are two integration constants involved. The first integration constant, originating from the inner integral, is needed to ensure the boundedness of the solution. The second integration constant introduces precisely the freedom

$$u_i \to u_i + \gamma_i \dot{u}_0, \qquad i \in \mathbb{N},$$

with $\gamma_i \in \mathbb{R}$ constants. In [40] the condition

$$\dot{u}_i(0) = 0 \tag{51}$$

is imposed to ensure the uniqueness of the solution. This phase condition is also used in [4] a theoretical setting. However, a more natural phase condition would be to minimize the L^2 -distance between the current and previous solution obtained from the regular perturbation method. This phase condition is also used in [20, 11, 46, 21] for numerical continuation of heteroclinic and homoclinic orbits. Using the L^2 phase condition yields

$$\int_{-\infty}^{\infty} \langle (\dot{u}_0(s), \ddot{u}_0(s))(u_i(s) + \gamma_i \dot{u}_0(s), \dot{u}_i(s) + \gamma_i \ddot{u}_0(s)) \rangle \, ds = 0, \qquad i \in \mathbb{N}$$

By Proposition 4 this phase condition is equivalent to the condition that

$$\int_{0}^{\infty} \langle (\dot{u}_{0}(s), \ddot{u}_{0}(s))(u_{i}(s) + \gamma_{i}\dot{u}_{0}(s), \dot{u}_{i}(s) + \gamma_{i}\ddot{u}_{0}(s)) \rangle \, ds = 0, \qquad i \in \mathbb{N},$$
(52)

for *i* odd and $\gamma_i = 0$, for *i* even, if we ensure that u_i is even for *i* even. By using integration by parts together with (44) and subsequently solving (52) for γ_i we obtain

$$\gamma_i = -\frac{35}{2592} \int_0^\infty \dot{u}_0 (1 - 2u_0) u_i(s) \, ds, \qquad i \in \mathbb{N}.$$
(53)

In [40], the phase condition

$$\int_{-\infty}^{\infty} (u(s) - u_0(s)) \dot{u}_0(s) \, ds = 0, \qquad i \in \mathbb{N}, \tag{54}$$

was also tested. This phase condition only minimizes the L_2 -distance of the *u*-component between the current and the zeroth-order solution obtained from the regular perturbation method. It is reported in [40] that for (54) no substantial superiority over using phase condition (51) was found. Our findings show that, at least for (52), this is only partially true. Indeed, the numerical simulations in Section 5.1 show that, as one would expect, using the phase condition (52) does indeed improve the approximation to the homoclinic orbit. However, when the homoclinic approximations are lifted from the normal form to the center manifold, the phase conditions are, in general, not preserved, and the improvements are no longer observed.

As we will see below, the L_2 phase condition (52) is more difficult to solve. It is, therefore, more efficient to use the orbital normal form (8) instead of the smooth normal form (9).

3.1.1 Third-order homoclinic approximation

For i = 1 we obtain the equation

$$z_1(s) = (u_0(s) + \tau_0)\dot{u}_0(s).$$

Condition (49) yields

$$\tau_0 = \frac{10}{7}.$$

Then from (50) we obtain the solution

$$u_1(s) = -\frac{6}{7}\dot{u}_0(s)\log(\cosh(s)).$$

The L_2 phase condition then yields that

$$\gamma_1 = -\frac{3}{245}(70\log(2) - 59).$$

Note that the integral to be evaluated in (53) is labor-intensive and prone to error. Therefore, we used the (freely available) Wolfram Engine [28] (although not open source). Correcting the previous solution u_1 leads to the solution

$$u_1(s) = \frac{3}{245}(59 - 70\log(2\cosh(s)))\dot{u}_0(s).$$

Continuing with the second-order system we have the equation

$$z_2 = (u_0 + \tau_0)\dot{u}_1 + u_1\dot{u}_0 + u_1^2$$

Here we directly used that $\tau_1 = 0$ by the symmetry as explained above. From (50) we obtain

$$u_2(s) = \frac{1}{60025} 36 \operatorname{sech}^2 s \left[3 \operatorname{sech}^2 s \left\{ 70 \log(2 \cosh s) (105 \log(2 \cosh s) - 247) \right. \right. \\ \left. + 6289 \right\} - 2(3675s \tanh s + 210 \log(2 \cosh s) (35 \log(2 \cosh s) - 94) + 7129) \right].$$

Notice that, since u_2 is an even function, we automatically have that γ_2 vanishes.

For i = 3 we have

$$z_3 = (u_2 + \tau_2)\dot{u}_0 + (u_0 + \tau_0)\dot{u}_2 + u_1(\dot{u}_1 + 2u_2).$$

Condition (49) yields

$$\tau_2 = \frac{288}{2401}$$

Then from (50) we obtain the solution

$$\begin{aligned} u_3(s) &= \frac{216 \operatorname{sech}^2 s}{14706125} \left[\operatorname{sech}^2 s \left\{ 3675 s (210 \log(2 \cosh s) - 247) \right. \\ &+ \tanh s \left(-171500 (\cosh(2s) - 5) \log^3(2 \cosh s) + 7350 (129 \cosh(2s) - 470) \right. \\ &\left. \log^2(2 \cosh s) + 4456830 \log(2 \cosh s) - 966242 \right) \right\} \\ &\left. -70 \left\{ 210 s (35 \log(2 \cosh s) - 47) + 30673 \tanh s \log(\cosh s) \right\} \right]. \end{aligned}$$

Trying to solve the integral in (53) with the Wolfram Engine yields

$$\int_{0}^{\infty} \dot{u}_{0}(1-2u_{0})u_{3}(s) ds = \frac{16\left(-5234558923+331676100\pi^{2}+6260972760\log(2)\right)}{514714375}$$
(55)
$$-\frac{155520}{343} \int_{0}^{\infty} \log^{3}(2\cosh s) \operatorname{sech}^{6} s \tanh^{2} s \, ds$$
$$\frac{31104}{343} \int_{0}^{\infty} \log^{3}(2\cosh s) \operatorname{sech}^{6} s \tanh^{2} s \cosh(2s) \, ds$$
$$\frac{622080}{343} \int_{0}^{\infty} \log^{3}(2\cosh s) \operatorname{sech}^{8} s \tanh^{2} s \, ds$$
$$-\frac{124416}{343} \int_{0}^{\infty} \log^{3}(2\cosh s) \operatorname{sech}^{8} s \tanh^{2} s \cosh(2s) \, ds,$$

i.e., the Wolfram Engine was unable to solve the integral. We observe that, in order to solve (55), it is sufficient to solve integrals of the form

$$I_n := \int_0^\infty \log^3(2\cosh s) \operatorname{sech}^n s \, ds, \tag{56}$$

with n = 4, 6, 8 and n = 10. After a lengthy calculation, see Appendix B, we obtain the closed form expression

$$\begin{split} I_n &= 2^{n-3} 3 \sum_{k=0}^{\frac{n}{2}-1} \binom{\frac{n}{2}-1}{k} (-1)^k \\ & \left[\frac{1}{(\frac{n}{2}+k)^4} + \frac{8}{2k+n} \left(\frac{H_{\frac{n}{2}+k}}{(2k+n)^2} + \frac{H_{\frac{n}{2}+k}^{(2)} - \zeta(2)}{2(2k+n)} + \frac{H_{\frac{n}{2}+k}^{(3)} - \zeta(3)}{4} \right) \right]. \end{split}$$

where ζ is the Riemann zeta function and $H_n^{(m)}$ is the *n*th generalized harmonic number of order *m*. Explicitly we obtain

$$I_4 = \frac{82}{27} - \frac{5\pi^2}{36} - \zeta(3),$$

$$I_6 = \frac{38342}{16875} - \frac{47\pi^2}{450} - \frac{4\zeta(3)}{5},$$

$$I_8 = \frac{25545482}{13505625} - \frac{319\pi^2}{3675} - \frac{24\zeta(3)}{35},$$

$$I_{10} = \frac{5428830032}{3281866875} - \frac{7516\pi^2}{99225} - \frac{64\zeta(3)}{105}$$

From the above integrals, we deduce that

$$\gamma_3 = \frac{264\zeta(3)}{343} - \frac{884895199}{7147176750} - \frac{100\pi^2}{3087} - \frac{1104228\log 2}{420175}$$

Since, by the symmetry, $\tau_3 = 0$, we obtain the third-order predictor

$$\begin{cases} w_0(\eta) = \frac{a}{b^2} \left(\sum_{i=0}^3 u_i(\frac{a}{b}\epsilon\eta)\epsilon^i + \mathcal{O}(\epsilon^4) \right) \epsilon^2, \\ w_1(\eta) = \frac{a^2}{b^3} \left(\sum_{i=0}^3 \dot{u}_i(\frac{a}{b}\epsilon\eta)\epsilon^i + \mathcal{O}(\epsilon^4) \right) \epsilon^3, \\ \beta_1 = -4\frac{a^3}{b^4}\epsilon^4, \\ \beta_2 = \frac{a}{b}\epsilon^2 \left(\frac{10}{7} + \frac{288}{2401}\epsilon^2 + \mathcal{O}(\epsilon^4) \right), \end{cases}$$
(57)

for the smooth orbital normal form (10).

Remark 5. In [40] there is the remark that the author [4] was unable to find a tangent predictor due to the (normalized) form. The system in [4, Equation (4.5)] to be solved is given by

$$\begin{cases} \dot{x} - y = \frac{1}{2}a_1\hat{x}^2 + a_2\tau_0\hat{x} + a_3\tau_0^2, \\ \dot{y} - 2\hat{x}x = b_1\hat{x}\hat{y} + b_2\tau_0\hat{y}, \end{cases}$$
(58)

here (\hat{x}, \hat{y}) is the zeroth-order solution (u_0, \dot{u}_0) . The coefficients a_1, a_2, b_1, b_2 are different normal form coefficients then used in this paper, but τ_0 is identical to the τ_0 used in this paper. Using the same technique as above it is easy to derive that

$$\begin{aligned} x(s) &= \left(a - \frac{1}{16}a_3s\tau_0^2 - \frac{1}{96}a_3\tau_0^2\sinh(2s) - \frac{1}{48}a_2\tau_0\sinh(2s) + \frac{1}{24}a_3\tau_0^2\coth(s) \right. \\ &\quad \left. - \frac{1}{24}a_2\tau_0\coth(s) + \frac{a_1s}{8} - \frac{1}{48}a_1\sinh(2s) + \frac{1}{12}a_1\coth(s) + \frac{1}{15}b_2\tau_0\cosh(2s) \right. \\ &\quad \left. + \frac{1}{240}b_2\tau_0\cosh(4s) - \frac{2}{21}b_1\cosh(2s) - \frac{1}{168}b_1\cosh(4s) \right. \\ &\quad \left. - \frac{6}{7}b_1\log(\cosh(s)) \right) \hat{y}(s) \end{aligned}$$

is a solution to (58). Therefore, by [35, 31] the convergence of the homoclinic solution in the perturbed Hamiltonian system follows. Although the author in [4] was unable to provide a tangent approximation in phase-space, they did prove, by refining the convergence cones from [32], the convergence of the zeroth-order approximation in phase-space in the perturbed Hamiltonian system.

3.2 A polynomial Lindstedt-Poincaré method

The Lindstedt-Poincaré method considers a nonlinear time transformation defined implicitly through the relation

$$\frac{d\xi}{ds} = \omega(\xi),\tag{59}$$

which can be used to remove the so-called *secular terms*, i.e., terms growing without bound, appearing in the process of approximating periodic orbits in weakly nonlinear oscillators using the regular perturbation approach.

The Lindstedt-Poincaré method is also used to approximate homoclinic solutions in nonlinear oscillators, referred to as the generalized Lindstedt-Poincaré method, see [12, 14, 15, 13, 16]. In this case, there are no terms growing without bound when applying the regular perturbation approach. Instead, there are so-called *parasitic turns*, see [40, Figure 1]. The nonlinear transformation (59) can then be used to remove the parasitic turns. In fact, using the nonlinear transformation, one can obtain a very simple form for the solution of the homoclinic orbit in phase-space, see [12, Equation 35] and [2].

In both cases, i.e., when approximating periodic orbits or homoclinic orbits, we do the same: a nonlinear time transformation is used to obtain a uniform approximation of the orbit in time.

3.2.1 General method

Substituting the parameterization of time ω (59) into (12) yields

$$\omega \frac{d}{d\xi} \left(\omega \hat{u}' \right) - \hat{u}^2 + 4 = \epsilon \omega \hat{u}' (\hat{u} + \tau) + \mathcal{O}(\epsilon^4).$$
(60)

were $\hat{u}(\xi(s)) = u(s)$.

We now perform one additional transformation of time

$$\frac{d\zeta}{d\xi} = 1 - \zeta^2 \tag{61}$$

to simplify the solutions obtained below. Note that this transformation implies that $\zeta = \tanh(\xi + c_1)$, where c_1 is some constant. Without loss of generality, we can assume that $c_1 = 0$ since c_1 just shifts the homoclinic solution in time. Substituting (61) into (60) yields

$$(1-\zeta^2)\tilde{\omega}\frac{d}{d\zeta}\left((1-\zeta^2)\tilde{\omega}\tilde{u}'\right) - \tilde{u}^2 + 4 = \epsilon\tilde{\omega}(1-\zeta^2)\tilde{u}'(\tilde{u}+\tau) + \mathcal{O}(\epsilon^4),\tag{62}$$

where the prime ' now represents the derivative with respect to the variable ζ , $\tilde{u}(\zeta(\xi(s))) = u(s)$, and $\tilde{\omega}(\zeta) = \omega(\xi(\zeta))$.

Expanding \tilde{u} and τ in ϵ

$$\tilde{u}(\zeta) = \sum_{i=0} \tilde{u}_i(\zeta)\epsilon^i, \qquad \tau = \sum_{i=0} \tau_i \epsilon^i, \tag{63}$$

substituting into (60), and collecting terms of equal power in ϵ , we obtain the following systems to be solved:

$$(1 - \zeta^2)\tilde{\omega}_0 \left((1 - \zeta^2)\tilde{\omega}_0 \tilde{u}'_0 \right)' - \tilde{u}_0^2 + 4 = 0, \tag{64}$$

$$(1 - \zeta^{2}) \left((1 - \zeta^{2}) \tilde{u}_{i}' \right)' - 2 \tilde{u}_{0} \tilde{u}_{i} + 2(1 - \zeta^{2}) \tilde{\omega}_{i} \left((1 - \zeta^{2}) \tilde{u}_{0}' \right)' + (1 - \zeta^{2})^{2} \tilde{u}_{0}' \tilde{\omega}_{i}'$$

$$= \tau_{i-1} \left(1 - \zeta^{2} \right) \tilde{u}_{0}' + z_{i}, \qquad i \in \mathbb{N}.$$
(65)

Here z_i contains the sums and products of terms in $\tilde{u}_j, \tilde{\omega}_j$ and τ_{j-1} with $0 \leq j \leq i-1$, with τ_{-1} is defined to be zero.

Theorem 6. Equations (64) and (65) are solvable for every $i \in \mathbb{N}_0$, with

$$\tilde{u}_i(\zeta) = \sigma_i \zeta^2 + \delta_i, \tag{66}$$

where σ_i and δ_i are constants to be determined.

Proof. It is easy to see that equation (64) is solvable with $\sigma_0 = 6$, $\delta_0 = -4$, and $\tilde{\omega}_0(\zeta) = 1$.

Assume that for i = 1, ..., n-1, the systems given by (65) are solvable for $\tilde{\omega}_i$ and \tilde{u}_i . Furthermore, also assume that for i = 1, ..., n, \tilde{u}_i is of the form (66). We will show that the system (65) with i = n is solvable for $\tilde{\omega}_n$.

First notice that (65) is just a first order ordinary differential equation in $\tilde{\omega}_i$:

$$\tilde{\omega}_{i}' + \frac{2\left((1-\zeta^{2})\tilde{u}_{0}'\right)'}{(1-\zeta^{2})\tilde{u}_{0}'}\tilde{\omega}_{i} = \frac{2\tilde{u}_{0}\tilde{u}_{i} - (1-\zeta^{2})\left((1-\zeta^{2})\tilde{u}_{i}'\right)' + \tau_{i-1}\left(1-\zeta^{2}\right)\tilde{u}_{0}' + z_{i}}{(1-\zeta^{2})^{2}\tilde{u}_{0}'}.$$
(67)

Multiplying by the integrating factor

$$(1 - \zeta^2)^2 \left(\tilde{u}_0'\right)^2 \tag{68}$$

and subsequently integrating with respect to ζ yields the identity

$$\tilde{\omega}_{i} = \frac{(1-\zeta^{2})\left((1-\zeta^{2})\tilde{u}_{0}'\right)'\tilde{u}_{i} - (1-\zeta^{2})^{2}\tilde{u}_{0}'\tilde{u}_{i}' + (g_{i}(\zeta) - g_{i}(1))}{((1-\zeta^{2})\tilde{u}_{0}')^{2}} \\ = -\frac{\sigma_{i}}{12} - \frac{(1-\zeta^{2})\left((1-\zeta^{2})\tilde{u}_{0}'\right)'\tilde{u}_{i} + (g_{i}(\zeta) - g_{i}(1))}{((1-\zeta^{2})\tilde{u}_{0}')^{2}}$$
(69)

where

$$g_i(\zeta) = \tau_{i-1} \int \left(1 - \zeta^2\right) \left(\tilde{u}_0'\right)^2 \, d\zeta + \int \tilde{u}_0' z_i \, d\zeta.$$

Here we used identity

$$\left((1-\zeta^2)\left((1-\zeta^2)\tilde{u}_0'\right)'\right)' = 2\tilde{u}_0\tilde{u}_0',$$

obtained from differentiating equation (64) and then using integrating by parts. Furthermore, we have chosen the integration constant $g_i(1)$ such that numerator in (69) vanishes for $\zeta = 1$. Indeed, for $\tilde{\omega}_i$ to be well-defined, the numerator in (69) must have roots of at least multiplicity two at $\zeta = 0$ and $\zeta = \pm 1$. By setting $\zeta = -1, 0$ in the numerator of (69), we obtain the equations

$$0 = g_i(-1) - g_i(1), \tag{70}$$

$$0 = 12\delta_i - (g_i(0) - g_i(1)), \tag{71}$$

respectively. The first equation can be solved explicitly for τ_i . Since $g_i(0) = 0$, it follows that $\delta_i = \frac{g_i(1)}{12}$. To show that the roots $\zeta = 0$ and $\zeta = \pm 1$ have multiplicity two, we notice that differentiation of the numerator in (69) with respect to ζ is equal to multiplying the right-hand side of (67) with the integrating factor (68), i.e.,

$$\tilde{u}_{0}^{\prime}\left(2\tilde{u}_{0}\tilde{u}_{i}-(1-\zeta^{2})\left((1-\zeta^{2})\tilde{u}_{i}^{\prime}\right)^{\prime}+\tau_{i-1}\left(1-\zeta^{2}\right)\tilde{u}_{0}^{\prime}-z_{i}\right).$$
(72)

Since $\tilde{u}'_0 = 12\zeta$, we can factor out $\zeta = 0$. Then substituting $\zeta = \pm 1$ into (72), the following equation needs to be satisfied

$$2\tilde{u}_0(\pm 1)\tilde{u}_i(\pm 1) + z_i(\pm 1) = 0.$$

Notice that this condition is equivalent to the condition obtained by substituting $\zeta = \pm 1$ into (65). Therefore, by solving the above equation for either ± 1 , yields

$$\sigma_i = -\delta_i - \frac{z_i(1)}{4}.\tag{73}$$

Lastly, notice that for i = 1 we have the solution

$$\tau_0 = \frac{10}{7}, \quad \sigma_1 = 0, \quad \delta_1 = 0, \quad \tilde{\omega}_1(\zeta) = \frac{6}{7}\zeta.$$

Corollary 7. For $i \in \mathbb{N}_0$ the polynomials $\tilde{\omega}_i$ (69) have rational coefficients. Also, the τ_i, σ_i and δ_i are rational.

Proof. The proof follows from a simple induction argument taking into account to structure of z_i , $i \in \mathbb{N}$, in (65).

Corollary 8. The following relation holds

$$\sigma_i = \delta_i = \tau_i = 0, \quad for \ i \ odd$$

Proof. From Proposition 4 we have that the branch of non-trivial homoclinic orbits has the following symmetry

$$u(-s,\epsilon) = u(s,-\epsilon) + \gamma \dot{u}_0(s), \quad \tau(\epsilon) = \tau(-\epsilon),$$

for $s \in \mathbb{R}$ and some open neighborhood of $\epsilon = 0$. Since $u(s, \epsilon) = \tilde{u}(\zeta(\xi(s))), \epsilon)$ and

$$\tilde{u}(\zeta(\xi(s)),\epsilon) = \sigma(\epsilon)\zeta^2(\xi(s)) + \delta(\epsilon),$$

where $\sigma(\epsilon) = \sum_i \sigma_i \epsilon^i$ and $\delta(\epsilon) = \sum_i \delta_i \epsilon^i$, it follows that

$$\begin{aligned} \sigma(\epsilon)\zeta^2(\xi(-s)) + \delta(\epsilon) &= u(-s,\epsilon) = u(s,-\epsilon) + \gamma \dot{u}_0(s) \\ &= \sigma(-\epsilon)\zeta^2(\xi(s)) + \delta(-\epsilon) + \gamma(\epsilon) \left[1 - \zeta^2(\xi(s))\right] 12\zeta(\xi(s)). \end{aligned}$$

Therefore, σ , δ , and τ are even functions in ϵ , from which the assertion follows.

Corollary 9. For the quadratic Bogdanov-Takens normal form (8) we have the relation that

$$\sigma_i = -\delta_i, \qquad \text{for } i \ge 1. \tag{74}$$

Proof. Applying the singular rescaling (11) to the normal form (8), and consecutive applying the nonlinear time transformations (59) and (61), we obtain (62) without the higher-order terms in ϵ . After some calculations we obtain the explicit expression for z_i with $i \ge 1$ in (65), namely

$$z_{i}(\zeta) = \sum_{k=1}^{i-1} u_{k} u_{i-k} + (1-\zeta^{2}) \left\{ \sum_{l=1}^{i-1} u_{l}' \tau_{i-1-l} + \sum_{k=1}^{i-1} \sum_{l=0}^{i-1-k} \omega_{k} u_{l}' \tau_{i-1-l-k} + \sum_{k=0}^{i-1} \sum_{l=0}^{i-1-k} \omega_{k} u_{l}' u_{i-1-l-k} - \sum_{l=1}^{i-1} \omega_{l} \left((1-\zeta^{2}) u_{i-l}' \right)' - \sum_{k=1}^{i-1} \sum_{l=0}^{i-k} \omega_{l} \left((1-\zeta^{2}) \omega_{k} u_{i-l-k}' \right)' \right\}.$$

From Corollary 8, we have that $z_1(1) = 0$. By assuming that the relation (74) holds for i = 1, 2, ..., n - 1, $n \in \mathbb{N}$, we see directly that $z_i(1) = 0$. The assertion now follows by (73), with i = n.

Remark 10. From Corollary 8 it follows that the solution \tilde{u} for the quadratic Bogdanov-Takens normal form (8) can be represented by the single parameter σ

$$\tilde{u}(\zeta) = 2 - (1 - \zeta^2) \sum_{i \ge 0} \sigma_i \epsilon^i.$$

Consequently, \hat{u} becomes

$$\hat{u}(\xi) = 2 - \operatorname{sech}^2(\xi) \sum_{i \ge 0} \sigma_i \epsilon^i.$$

3.2.2 Third-order orbital homoclinic approximation

For the third-order homoclinic predictor we obtain

$$\sigma = 6 + \frac{18}{49}\epsilon^{2} + \mathcal{O}(\epsilon^{4}),$$

$$\delta = -4 - \frac{18}{49}\epsilon^{2} + \mathcal{O}(\epsilon^{4}),$$

$$\tau = \frac{10}{7} + \frac{288}{2401}\epsilon^{2} + \mathcal{O}(\epsilon^{4}),$$

$$\tilde{\omega}(\zeta) = 1 - \frac{6}{7}\zeta\epsilon + \left(\frac{9}{98} + \frac{27}{98}\zeta^{2}\right)\epsilon^{2} + \left(-\frac{198}{2401}\zeta + \frac{18}{343}\zeta^{3}\right)\epsilon^{3} + \mathcal{O}(\epsilon^{4}).$$
(75)

From which it follows that

$$\tilde{u}(\zeta) = 2 - \left(1 - \zeta^2\right) \left(6 + \frac{18}{49}\epsilon^2\right) + \mathcal{O}(\epsilon^4),\tag{76}$$

$$\tilde{v}(\zeta) = -2\tilde{\omega}(\zeta)\sigma(1-\zeta^2)\zeta = -\left[-12 + \frac{72}{7}\zeta\epsilon - \left(\frac{90}{49} + \frac{162}{49}\zeta^2\right)\epsilon^2 + \left(\frac{3888}{2401}\zeta - \frac{216}{343}\zeta^3\right)\epsilon^3\right](1-\zeta^2)\zeta + \mathcal{O}(\epsilon^4).$$
(77)

The relation $\xi(s)$ is obtained by solving the ODE

$$\frac{d\xi}{ds}(s) = \tilde{\omega}(\tanh(\xi(s))). \tag{78}$$

Thus, we substitute

$$\xi(s) = s + \xi_1(s)\epsilon + \xi_2(s)\epsilon^2 + \xi_3(s)\epsilon^3 + \mathcal{O}(\epsilon^4),$$

into (78) and expand the resulting equation in ϵ to obtain

$$\frac{d\xi_1}{ds}(s) = -\frac{6\tanh(s)}{7},\tag{79}$$

$$\frac{d\xi_2}{ds}(s) = \frac{18 + 54 \tanh^2(s) - 168\xi_1(s) + 168 \tanh^2(s)\xi_1(s)}{196},$$
(80)
$$\frac{d\xi_3}{ds}(s) = -\frac{198 \tanh(s)}{2401} + \frac{18 \tanh^3(s)}{343} - \frac{27}{49}(-\tanh(s)\xi_1(s) + \tanh^3(s)\xi_1(s)) \\
- \frac{6}{7}(-\tanh(s)\xi_1^2(s) + \tanh^3(s)\xi_1^2(s) + \xi_2(s) - \tanh^2(s)\xi_2(s)).$$

Here we directly used that $\xi_0(s) = s$. By solving these equations recursively we obtain

$$\begin{aligned} \xi_1(s) &= c_1 - \frac{6}{7} \log(\cosh(s)), \\ \xi_2(s) &= c_2 - \frac{18s}{49} + \frac{45 \tanh(s)}{98} - \frac{6}{7} c_1 \tanh(s) + \frac{36}{49} \tanh(s) \log(\cosh(s)), \\ \xi_3(s) &= c_3 + \frac{1}{4802} \left(3 \operatorname{sech}^2(s) \left(-504 \log^2(\cosh(s)) - 276 \cosh(2s) \log(\cosh(s)) \right) \right. \\ &+ 102 \log(\cosh(s)) + 14(18s - 49c_2) \sinh(2s) + 1176c_1 \log(\cosh(s)) \\ &+ 546 - 686c_1^2 - 441c_1) \right). \end{aligned}$$

The constants $c_i(i = 1, 2, 3)$ lead to different phase conditions. A computationally simple phase condition is given by

$$\xi_i(0) = 0, \qquad \text{for } i = 1, 2, 3.$$
 (81)

These results in the constraint v(0) = 0, i.e., the phase condition used in [40]. Solving (81) leads to the solution

$$c_1 = 0, \qquad c_2 = 0, \qquad c_3 = -\frac{117}{343}.$$
 (82)

Substituting the above expression for ξ into (11) we obtain the third-order predictor

$$\begin{cases}
w_0(\eta) = \frac{a}{b^2} \tilde{u} \left(\tanh\left(\xi\left(\frac{a}{b}\epsilon\eta\right)\right) \right) \epsilon^2, \\
w_1(\eta) = \frac{a^2}{b^3} \tilde{v} \left(\tanh\left(\xi\left(\frac{a}{b}\epsilon\eta\right)\right) \right) \epsilon^3, \\
\beta_1 = -4\frac{a^3}{b^4} \epsilon^4, \\
\beta_2 = \frac{a}{b} \epsilon^2 \tau,
\end{cases}$$
(83)

where τ, \tilde{u} and \tilde{v} are given by (75)–(77), respectively.

Remark 11. By expanding $\tilde{u}(\tanh(\xi(s)))$ in ϵ up to third-order we obtain

$$u(s) = u_0(s) + u_1(s)\epsilon + u_2(s)\epsilon^2 + u_3(s)\epsilon^3,$$

where

$$\begin{split} u_{0}(s) &= 6 \tanh^{2}(s) - 4, \quad u_{1}(s) = -\frac{72b \tanh(s) \operatorname{sech}^{2}(s) \log(\cosh(s))}{7a}, \\ u_{2}(s) &= \frac{18}{49} \operatorname{sech}^{2}(s) \left(-12s \tanh(s) - 24(\log(\cosh(s)) - 1) \log(\cosh(s)) + 3 \operatorname{sech}^{2}(s)(32 \log(\operatorname{sech}(s)) + 12 \log(\cosh(s))(\log(\cosh(s)) + 2) - 5) + 14\right), \\ u_{3}(s) &= -\frac{27 \operatorname{sech}^{5}(s)}{2401} \left(-273 \sinh(s) + 91 \sinh(3s) + 84s \cosh(3s)(2 \log(\cosh(s)) - 1)\right) \end{split}$$
(84)
$$&- 84s \cosh(s)(6 \log(\cosh(s)) - 1) - 1232 \sinh(s) \log^{3}(\cosh(s)) + 112 \sinh(3s) \log^{3}(\cosh(s)) + 112 \sinh(3s) \log^{3}(\cosh(s)) + 2016 \sinh(s) \log^{2}(\cosh(s)) - 336 \sinh(3s) \log^{2}(\cosh(s)) + 904 \sinh(s) \log(\cosh(s)) - 104 \sinh(3s) \log(\cosh(s))) . \end{split}$$

Together with (75), this is precisely the solution obtained by using the regular perturbation method to (12) with phase condition $\dot{u}(0) = 0$.

Note that for the conjecture in [1, Section 7] to hold, the phase condition (82) must be satisfied.

3.2.3 Non-uniqueness homoclinic solution

Note that in Theorem 6 we could have assumed the solutions of (65) to be of the form

$$\tilde{u}(\zeta) = \sum_{i \ge 0} \left(\sigma_i \zeta^2 + \delta_i + \gamma_i (1 - \zeta^2) \tilde{u}_0'(\zeta) \right) \epsilon^i,$$
(85)

where $\gamma_0 = 0$ and $\gamma_i \in \mathbb{R}$ are constants to be determined by some phase condition. Thus, we have freedom in (81) and in (85) both originating from the non-uniqueness of the homoclinic orbit. The solutions (79) and (80) together with

$$\xi_{3}(s) = \frac{1}{4802} \left(18 \left[49\gamma_{1}(7\gamma_{1}+4) + 84s \tanh(s) + 92 \log(\operatorname{sech}(s)) - 105 \right] - 7 \operatorname{sech}^{2}(s) \left[-7\gamma_{1}(7\gamma_{1}-3)(35\gamma_{1}+9) + 18(7\gamma_{1}(7\gamma_{1}+4)+9) \log(\operatorname{sech}(s)) + 216 \log^{2}(\cosh(s)) - 234 \right] \right)$$

and

$$\gamma_1 = \frac{1}{35} \left(-4 - \frac{59}{\sqrt[3]{836 + 15\sqrt{4019}}} + \sqrt[3]{836 + 15\sqrt{4019}} \right)$$

also leads to the phase condition $\xi_i(0) = 0$ for i = 1, 2, 3. However, $v_3(0) = 0$ no longer holds. The solutions \tilde{u} and \tilde{v} are now given by

$$\begin{split} \tilde{u}(\zeta) &= 2 + \left(1 - \zeta^2\right) \left(-6 + 12\gamma_1 \zeta \epsilon + \left(6\gamma_1^2 - \frac{18}{49}\right) \epsilon^2 + \mathcal{O}(\epsilon^4)\right), \\ \tilde{v}(\zeta) &= (1 - \zeta^2) \tilde{\omega}(\zeta) u'(\zeta) = (1 - \zeta^2) \tilde{\omega}(\zeta) \sum_{i=0} \left(\sigma_i \zeta + 12\gamma_i \left(1 - 3\zeta^2\right)\right) \epsilon^i \\ &= (1 - \zeta^2) \left[12\zeta + \left(12\gamma_1 - 36\frac{2 + 7\gamma_1 \zeta^2}{7}\right) \epsilon + \\ & 6\zeta \frac{15 - 168\gamma_1 - 245\gamma_1^2 + 3(9 + 7\gamma_1(16 + 7\gamma_1))\zeta^2}{49} \epsilon^2 + \\ & \left(216\zeta^2 \frac{-18 + 7\zeta^2}{2401} + 6\gamma_1^3(-3 + 2\zeta^2 + \zeta^4) - 72\gamma_1^2 \frac{1 - 6\zeta^2 + 4\zeta^4}{7} - 54\gamma_1 \frac{-1 - 6\zeta^2 + 15\zeta^4}{49}\right) \epsilon^3 + \mathcal{O}(\epsilon^4) \right]. \end{split}$$

The numerical simulations in Section 5.1 show that for the normal form (8) these asymptotics are more accurate than the asymptotics derived in Section 3.2.2.

3.2.4 Comparison with the nonlinear periodic time-reparametrization

In [2] a different approach is used to approximate the homoclinic solution near the quadratic normal form of a generic codimension 2 Bogdanov-Takens bifurcation. The approach there is an application of the so-called Perturbation-Incremental Method described in [47]. Consider strongly nonlinear oscillators of the form

$$\ddot{x} + g(x) = \lambda f(x, \dot{x}, \mu) \dot{x}, \tag{86}$$

where g and f are arbitrary nonlinear functions, and λ and μ are parameters.

The authors in [47] perform a nonlinear periodic time reparametrization of the form

$$\frac{d\phi}{dt} = \Phi(\phi), \qquad \Phi(\phi + 2\pi) = \Phi(\phi) \tag{87}$$

to the system (86). Then it is assumed that there is a homoclinic orbit present which can be approximated by the solution

$$u(\phi) = p\cos(2\phi) + q,$$

where p and q are constants to be approximated. In [2, Theorem 1] it is shown that the solutions $u(\phi)$ and u(s) are related to each other through

$$\Phi(\phi) = \frac{\sqrt{2}}{2}\omega(\xi)\sin\phi.$$

It follows that one should be able to factor out the term $\sin \phi$ in (87). This is indeed precisely what we see in the transformation Φ in [2, (41)]. Thus, although the nonlinear periodic time reparametrization is analytically equivalent to the polynomial generalized Lindstedt-Poincaré method, it is geometrically *less intuitive* than using hyperbolic functions and computationally *more expensive* than using polynomials.

We also would like to point out that the singular rescaling

$$w_0 = u\epsilon^2$$
, $w_1 = v\epsilon^3$, $\beta_1 = -\epsilon^4$, $\beta_2 = \tau\epsilon^2$, $s = \epsilon\eta$, $(\epsilon \neq 0)$,

used in [2] applied to the quadratic Bogdanov-Takens normal form with coefficient a = 1 and b = 1 results in the $\sqrt{2}$ turning up in the calculations of homoclinic approximation. From a computational point of view, this is less ideal to work with.

Remark 12. Note that the polynomial generalized Lindstedt-Poincaré method described above only depends on the existence of the zeroth-order solution. Therefore, this method can be applied in similar situations where homoclinic orbits emanate from codimension two Bogdanov-Takens bifurcation points. For example, the transcritical codimension two bifurcation treated in [25], see also [9, Appendix C.2]. Furthermore, under certain symmetry present in the ODE heteroclinic solutions can also emanate from codimension two Bogdanov-Takens points which can be approximated using the polynomial Lindstedt-Poincaré method as described above.

4 Homoclinic asymptotic expansion in *n*-dimensional systems

In this section, we will provide third-order approximations to the homoclinic solution for (1) emanating from a generic codimension two Bogdanov-Takens bifurcation assumed to be at $x_0 \equiv 0$ and $\alpha_0 = 0$. A distinction between asymptotics derived with the smooth orbital and the smooth normal form is necessary. In case of the smooth orbital normal form, a further subdivision is made between the perturbation method used. This is a price we have to pay for using this simpler normal form. Using the obtained transformations for lifting the homoclinic orbits from the normal form to the parameterdependent center manifold in *n*-dimensional systems, we show the homoclinic asymptotics arising from the smooth orbital and smooth normal form are (asymptotically) equivalent in Section 4.3. We finish this section with our implementation in MatCont.

4.1 Homoclinic approximation using the smooth orbital normal form

First, we consider the situation where we have obtained a homoclinic predictor for the smooth orbital normal form (10). Depending on the used method to approximate the homoclinic solution, i.e., the regular perturbation or the Lindstedt-Poincaré method, we substitute either (57) or (83), into the parameter-dependent center manifold transformation H and K defined in (15) and (16). By truncating the higher-order terms in w and β we obtain the following approximation $(\bar{x}^o, \bar{\alpha}^o)$ to the homoclinic

solution

$$\bar{x}^{o}(\eta,\epsilon) = q_{0}w_{0}(\eta) + q_{1}w_{1}(\eta) + H_{0010}\beta_{1} + H_{0001}\beta_{2} + \frac{1}{2}H_{2000}w_{0}^{2}(\eta) + H_{1100}w_{0}w_{1}(\eta) \qquad (88)$$

$$+ \frac{1}{2}H_{0200}w_{1}^{2}(\eta) + H_{1010}w_{0}(\eta)\beta_{1} + H_{1001}w_{0}(\eta)\beta_{2} + H_{0110}w_{1}(\eta)\beta_{1}$$

$$+ H_{0101}w_{1}(\eta)\beta_{2} + \frac{1}{2}H_{0002}\beta_{2}^{2} + H_{0011}\beta_{1}\beta_{2} + \frac{1}{6}H_{3000}w_{0}^{3}(\eta)$$

$$+ \frac{1}{2}H_{2100}w_{0}^{2}(\eta)w_{1}(\eta) + H_{1101}w_{0}(\eta)w_{1}(\eta)\beta_{2} + \frac{1}{2}H_{2001}w_{0}^{2}(\eta)\beta_{2}$$

$$+ \frac{1}{6}H_{0003}\beta_{2}^{3} + \frac{1}{2}H_{1002}w_{0}(\eta)\beta_{2}^{2} + \frac{1}{2}H_{0102}w_{1}(\eta)\beta_{2}^{2},$$

$$\bar{\alpha}^{o}(\epsilon) = K_{10}\beta_{1} + K_{01}\beta_{2} + \frac{1}{2}K_{02}\beta_{2}^{2} + K_{11}\beta_{1}\beta_{2} + K_{03}\frac{1}{6}\beta_{2}^{3}.$$

$$(89)$$

Next, we use (17) to approximate $\eta(t)$ from the relation

$$\frac{dt}{d\eta} = 1 + \theta_{1000} w_0(\eta) + \theta_{0001} \beta_2, \tag{90}$$

where w_0 and β_2 are defined in (57) when using the predictor obtained by the regular perturbation method and defined in (83) when using the predictor obtained by the Lindstedt-Poincaré method. We will consider these two cases separately below.

Regular perturbation method

Integrating (90) with respect to η yields

$$t(\eta) = \int 1 + \theta_{1000} \frac{a}{b^2} u\left(\frac{a}{b}\epsilon\eta\right) \epsilon^2 + \theta_{0001} \frac{a}{b}\epsilon^2 \left(\frac{10}{7} + \frac{288}{2401}\epsilon^2 + \mathcal{O}(\epsilon^4)\right) d\eta$$

$$= \eta \left(1 + \theta_{0001} \frac{a}{b}\epsilon^2 \left(\frac{10}{7} + \frac{288}{2401}\epsilon^2 + \mathcal{O}(\epsilon^4)\right)\right) + \theta_{1000} \frac{a}{b^2}\epsilon^2 \int u\left(\frac{a}{b}\eta\epsilon\right) d\eta,$$
(91)

where u is the third-order approximation given in (57). To approximate the integral in the equation above uniformly in η , we make the substitution $s = \frac{a}{b}\eta\epsilon$. Then

$$\int u\left(\frac{a}{b}\eta\epsilon\right)\,d\eta = \frac{b}{a}\frac{1}{\epsilon}\int u(s)\,ds$$

where the integral on the right-hand side can be calculated to be

$$\int u(s) \, ds = 2(s - 3 \tanh s) - \frac{9}{7} \operatorname{sech}^2 s(\cosh(2s) - 4\log(\cosh s) - 1)\epsilon - \frac{9}{49} \operatorname{sech}^3 s\left[2\sinh s\left(\cosh(2s) - 12\log^2(\cosh s) + 6\right) - 12s\cosh s\right]\epsilon^2 - \frac{27\operatorname{sech}^4 s}{2401} \left[\cosh(4s) + \cosh(2s)\left(-112\log^3(\cosh s) + 168\log^2(\cosh s) + 188\log(\cosh s) + 7\right) + 8\left(28\log^3(\cosh s) - 21\log^2(\cosh s) - 21\log^2(\cosh s) - 29\log(\cosh s) - 21s\sinh(2s)\log(\cosh s) - 1\right)\right]\epsilon^3 + \mathcal{O}(\epsilon^4).$$
(92)

Here the constants of integration are calculated such that t(0) = 0. Thus, we obtain a third-order approximation for $t(\eta)$.

Lindstedt-Poincaré

The method is similar as for the regular perturbation method. We first integrate (90) with respect to η , so that

$$t(\eta) = \eta \left(1 + \theta_{0001} \frac{a}{b} \epsilon^2 \left(\frac{10}{7} + \frac{288}{2401} \epsilon^2 + \mathcal{O}(\epsilon^4) \right) \right) + \theta_{1000} \frac{a}{b^2} \epsilon^2 \int \hat{u} \left(\xi \left(\frac{a}{b} \eta \epsilon \right) \right) d\eta, \tag{93}$$

To approximate the integral in the above equation uniformly in η , we make the substitution $\tilde{\xi}(\eta) = \xi\left(\frac{a}{b}\epsilon\eta\right)$. Then

$$\int \hat{u}\left(\xi\left(\frac{a}{b}\eta\epsilon\right)\right)\,d\eta = \frac{b}{a}\frac{1}{\epsilon}\int \hat{u}(\tilde{\xi})/\omega(\tilde{\xi})\,d\tilde{\xi}.$$

Expanding the integrand $\hat{u}(\tilde{\xi})/\omega(\tilde{\xi})$, up to order three in ϵ and integrating with respect to $\tilde{\xi}$ yields

$$\int \frac{1}{\omega(\tilde{\xi})} \hat{u}\left(\tilde{\xi}\right) d\tilde{\xi} = 2\tilde{\xi} - 6\tanh(\tilde{\xi}) + \left(\frac{18\operatorname{sech}^2(\tilde{\xi})}{7} + \frac{12}{7}\log(\cosh(\tilde{\xi}))\right) \epsilon + \frac{9}{49}\left(4\tilde{\xi} - 9\tanh(\tilde{\xi}) + 5\tanh(\tilde{\xi})\operatorname{sech}^2(\tilde{\xi})\right) \epsilon^2 + \frac{18\left(-21\operatorname{sech}^4(\tilde{\xi}) + 47\operatorname{sech}^2(\tilde{\xi}) + 8\log(\cosh(\tilde{\xi}))\right)}{2401} \epsilon^3 + \mathcal{O}(\epsilon^4).$$
(94)

Substituting $\tilde{\xi}$ with $\xi(\frac{a}{b}\eta\epsilon)$ gives the relation $t(\eta)$ up to order three in ϵ .

Since we are interested in the inverse relation, i.e., $\eta(t)$, we numerically solve the equation

$$t(\eta) - t = 0$$

for η . This can easily be done within machine precision.

4.2 Homoclinic approximation using the smooth normal form

To lift the homoclinic approximation obtained for the smooth normal form to the parameter-dependent center manifold, we simply substitute either (118) and (120) into H and K. Thus, we obtain (88) and (89), where η is replaced by t, i.e.,

$$\begin{split} \bar{x}^{s}(t,\epsilon) &= q_{0}w_{0}(t) + q_{1}w_{1}(t) + H_{0010}\beta_{1} + H_{0001}\beta_{2} + \frac{1}{2}H_{2000}w_{0}^{2}(t) + H_{1100}w_{0}w_{1}(t) \\ &+ \frac{1}{2}H_{0200}w_{1}^{2}(t) + H_{1010}w_{0}(t)\beta_{1} + H_{1001}w_{0}(t)\beta_{2} + H_{0110}w_{1}(t)\beta_{1} \\ &+ H_{0101}w_{1}(t)\beta_{2} + \frac{1}{2}H_{0002}\beta_{2}^{2} + H_{0011}\beta_{1}\beta_{2} + \frac{1}{6}H_{3000}w_{0}^{3}(t) \\ &+ \frac{1}{2}H_{2100}w_{0}^{2}(t)w_{1}(t) + H_{1101}w_{0}(t)w_{1}(t)\beta_{2} + \frac{1}{2}H_{2001}w_{0}^{2}(t)\beta_{2} \\ &+ \frac{1}{6}H_{0003}\beta_{2}^{3} + \frac{1}{2}H_{1002}w_{0}(t)\beta_{2}^{2} + \frac{1}{2}H_{0102}w_{1}(t)\beta_{2}^{2}, \\ \bar{\alpha}^{s}(\epsilon) &= K_{10}\beta_{1} + K_{01}\beta_{2} + \frac{1}{2}K_{02}\beta_{2}^{2} + K_{11}\beta_{1}\beta_{2} + K_{03}\frac{1}{6}\beta_{2}^{3}. \end{split}$$

Note however that the coefficients of the mappings H and K are calculated as outlined in Section 2.3.

4.3 Comparison between smooth and orbital homoclinic predictors

Using the above transformations, we show that the homoclinic predictor for the smooth normal form (9), see Appendix C, is asymptotically equivalent to the orbital predictor derived in Section 3.2.2. Thus, we assume that (1) is given by

$$f(x_1(t), x_2(t), \alpha_1, \alpha_2) := \begin{pmatrix} x_1(t), \\ \alpha_1 + \alpha_2 x_1(t) + a x_0^2(t) + b x_0(t) x_1(t) \end{pmatrix},$$
(95)

where

$$g(x_1(t), x_2(t), \alpha_1, \alpha_2) = a_1 \alpha_2 x_0^2(t) + b_1 \alpha_2 x_0(t) x_1(t) + e x_0^2(t) x_1(t) + d x_0^3(t).$$

First we will focus on the predictors for the parameters. Using the procedure outlined in Section 2.2, we obtain that the coefficients for the parameter transformation K are given by

$$K_{10} = \begin{pmatrix} 1\\ \frac{ae-bd}{a^2} \end{pmatrix}, \quad K_{01} = \begin{pmatrix} 0\\ 1 \end{pmatrix}, \quad K_{11} = \frac{3a_1b - 4ab_1 + 2d}{ab} \begin{pmatrix} 1\\ \frac{(ae-bd)}{a^2} \end{pmatrix},$$
$$K_{02} = \begin{pmatrix} 0\\ \frac{2a_1b - 2ab_1 + d}{ab} \end{pmatrix}, \quad K_{03} = \begin{pmatrix} 0\\ 0 \end{pmatrix}.$$

From equation (89) we obtain the following approximation

$$\begin{pmatrix} \bar{\alpha}_1^o \\ \bar{\alpha}_2^o \end{pmatrix} = \begin{pmatrix} -\frac{4a^3}{b^4} \epsilon^4 + \frac{40a^3(-3a_1b + 4ab_1 - 2d)}{7b^6} \epsilon^6 + \frac{1152a^3(-3a_1b + 4ab_1 - 2d)}{2401b^6} \epsilon^8 + \mathcal{O}(\epsilon^{10}) \\ \frac{10a}{7b} \epsilon^2 + \left(\frac{288b}{2401a} + \frac{50a(2a_1b - 2ab_1 + d)}{49b^3} + \frac{4a(bd - ae)}{b^4} \right) \epsilon^4 + \mathcal{O}(\epsilon^6) \end{pmatrix}.$$
(96)

Since the equation (95) is the smooth normal form, we can directly use (118) to obtain the approximation

$$\begin{pmatrix} \bar{\alpha}_1^s \\ \bar{\alpha}_2^s \end{pmatrix} = \begin{pmatrix} -\frac{4}{a}\epsilon^4 \\ \frac{10b}{7a}\epsilon^2 + \frac{98b(50ab_1+73d)-9604ae-2450a_1b^2+288b^3}{2401a^2b}\epsilon^4 + \mathcal{O}(\epsilon^6) \end{pmatrix}.$$
 (97)

To compare these two predictors, we eliminate the parameter ϵ from both equations. It would be tempting to first make a substitution for ϵ^2 in the equations. However, since for the orbits we need odd powers in ϵ , we will continue without this substitution. We assume α_1 to be positive, which implies that the coefficient *a* is negative. The case that α_1 is negative is treated similarly and has been verified as well.

To eliminate ϵ from (96), we expand ϵ as a function of $\sqrt[4]{\alpha_1}$. Solving the resulting equation for real positive ϵ we obtain

$$\epsilon^{o}(\alpha_{1}) = \frac{\sqrt[4]{-ab}}{\sqrt{2a}} \sqrt[4]{\alpha_{1}} - \frac{5b(-4ab_{1}+3a_{1}b+2d)}{28\sqrt{2}a^{2}\sqrt[4]{-a}} \sqrt[4]{\alpha_{1}}^{3} + \mathcal{O}(\sqrt[4]{\alpha_{1}}^{5}).$$
(98)

Obviously, for the smooth predictor we obtain

$$\epsilon^s(\alpha_1) = \frac{\sqrt[4]{-a}}{\sqrt{2}}\sqrt[4]{\alpha_1}.$$

Here we added superscripts o and s to distinguish the different ϵ 's in the orbital and smooth homoclinic predictors, respectively.

Substituting $\epsilon = \epsilon^{o}(\alpha_1)$ into the second equation of (96) yields

$$\alpha_2(\alpha_1) = \frac{5b}{7\sqrt{-a}}\sqrt{\alpha_1} + \frac{-49b(50ab_1 + 73d) + 4802ae + 1225a_1b^2 - 144b^3}{4802a^2}\alpha_1 + \mathcal{O}(\alpha_1^{3/2}).$$

It can readably be seen that by eliminating ϵ from (97) using that a < 0 we obtain the same expression, i.e., the predictors agree up to the desired order.

Next, we turn our attention to the approximation of the homoclinic orbits. Due to the various time transformations involved in the predictors, we compare the asymptotic expansions of the orbital with the smooth homoclinic predictor. Therefore, we can directly use the asymptotic obtained from the regular perturbation method for both predictors. Thus, for the orbital predictor, we will use

$$\begin{cases} w_0(\eta) = \frac{a}{b^2} \left(\sum_{i=0}^3 u_i(\frac{a}{b}\epsilon\eta)\epsilon^i + \mathcal{O}(\epsilon^4) \right) \epsilon^2, \\ w_1(\eta) = \frac{a^2}{b^3} \left(\sum_{i=0}^3 \dot{u}_i(\frac{a}{b}\epsilon\eta)\epsilon^i + \mathcal{O}(\epsilon^4) \right) \epsilon^3, \end{cases}$$
(99)

with $u_i(i = 1, 2, 3)$ is given by (84), while for the smooth predictor we will use (120) instead.

Following the procedure as outlined in Section 2.2, we obtain that the coefficients of the transformations H and θ for the smooth orbital normal form (10) are given by

$$\begin{aligned} q_0 &= \begin{pmatrix} 1\\0 \end{pmatrix}, \quad q_1 = \begin{pmatrix} 0\\1 \end{pmatrix}, \quad H_{2000} = \begin{pmatrix} -\frac{d}{2a}\\0 \end{pmatrix}, \quad H_{1100} = \begin{pmatrix} -\frac{3bd+4ae}{12a^2}\\0 \end{pmatrix}, \\ H_{0200} &= \begin{pmatrix} 0\\-\frac{3bd+4ae}{6a^2} \end{pmatrix}, \quad H_{3000} = \begin{pmatrix} 0\\-\frac{3bd}{2a}+2e \end{pmatrix}, \quad H_{2100} = \begin{pmatrix} 0\\b(-\frac{3bd+4ae}{6a^2} \end{pmatrix}, \\ H_{0010} &= \begin{pmatrix} \frac{d}{4a^2}\\0 \end{pmatrix}, \quad H_{1001} = \begin{pmatrix} -\frac{2ab_1+a_1b+d}{ab}\\0 \end{pmatrix}, \quad H_{0101} = \begin{pmatrix} 0\\-\frac{-3(6ab_1-4a_1b+b^2-3d)d+4abe}{12a^2b} \end{pmatrix}, \quad H_{0102} = \begin{pmatrix} 0\\(\frac{6ab_1-4a_1b-3d)(2ab_1-2a_1b-d)}{2a^2b^2} \end{pmatrix}, \\ H_{1010} &= \begin{pmatrix} 0\\-\frac{-3bd+4ae}{12a^2} \end{pmatrix}, \quad \theta_{1000} = -\frac{d}{2a}, \quad \theta_{0001} = -\frac{-2ab_1+2a_1b+d}{2ab}, \end{aligned}$$

while

$$H_{0001} = H_{2001} = H_{0002} = H_{1002} = H_{0003} = H_{0011} = H_{0110} = \begin{pmatrix} 0\\0 \end{pmatrix}$$

Thus, the third-order homoclinic predictor using the smooth orbital normal form in η is given by

$$\begin{split} \bar{x}_{\epsilon}^{o}(\eta) &= \begin{pmatrix} 1\\ 0 \end{pmatrix} w_{0}(\eta) + \begin{pmatrix} 0\\ 1 \end{pmatrix} w_{1}(\eta) + \begin{pmatrix} -\frac{d}{2a}\\ 0 \end{pmatrix} w_{0}^{2}(\eta) + \begin{pmatrix} -\frac{3bd+4ae}{12a^{2}}\\ 0 \end{pmatrix} w_{0}(\eta)w_{1}(\eta) + \\ & \begin{pmatrix} 0\\ -\frac{3bd+4ae}{6a^{2}} \end{pmatrix} w_{1}^{2}(\eta) + \begin{pmatrix} 0\\ -\frac{3bd}{2a} + 2e \end{pmatrix} w_{0}^{3}(\eta) + \begin{pmatrix} 0\\ \frac{b(-3bd+4ae)}{6a^{2}} \end{pmatrix} w_{0}^{2}(\eta)w_{1}(\eta) + \\ & \begin{pmatrix} \frac{d}{4a^{2}}\\ 0 \end{pmatrix} \beta_{1} + \begin{pmatrix} -\frac{2ab_{1}+a_{1}b+d}{ab}\\ 0 \end{pmatrix} w_{0}(\eta)\beta_{2} + \begin{pmatrix} 0\\ -\frac{6ab_{1}+4a_{1}b+3d}{2ab} \end{pmatrix} w_{1}(\eta)\beta_{2} + \\ & \begin{pmatrix} 0\\ -\frac{3(6ab_{1}-4a_{1}b+b^{2}-3d)d+4abe}{12a^{2}b} \end{pmatrix} w_{0}(\eta)w_{1}(\eta)\beta_{2} + \\ & \begin{pmatrix} 0\\ (\frac{6ab_{1}-4a_{1}b-3d)(2ab_{1}-2a_{1}b-d)}{2a^{2}b^{2}} \end{pmatrix} w_{1}(\eta)\beta_{2}^{2} + \begin{pmatrix} 0\\ -\frac{3bd+4ae}{12a^{2}} \end{pmatrix} w_{0}(\eta)\beta_{1}, \end{split}$$

where $w_{0,1}$ are given by (99) and $\beta_{1,2}$ by (57). Since (95) is the smooth normal form (9) we obtain from (120) the third-order homoclinic approximation in t

$$\bar{x}_{\epsilon}^{s}(t) = \frac{1}{a} \sum_{i=0}^{3} \begin{pmatrix} \left(u_{i}(\epsilon t)\epsilon^{i} + \mathcal{O}(\epsilon^{4})\right)\epsilon^{2} \\ \left(\dot{u}_{i}(\epsilon t)\epsilon^{i} + \mathcal{O}(\epsilon^{4})\right)\epsilon^{3} \end{pmatrix}.$$

To relate the smooth orbital predictor $\bar{x}^{o}_{\epsilon}(\eta)$ with the smooth predictor $\bar{x}^{s}_{\epsilon}(t)$, we need to consider the time transformation

$$t_{\epsilon}(\eta) = \eta \left(1 + \theta_{0001} \frac{a}{b} \epsilon^2 \left(\tau_0 + \tau_2 \epsilon^2 \right) \right) + \theta_{1000} \frac{1}{b} \epsilon \int u(s) \, ds,$$

where

$$\theta_{1000} = -\frac{d}{2a}, \quad \theta_{0001} = -\frac{-2ab_1 + 2a_1b + d}{2ab}, \quad s = \frac{a}{b}\eta\epsilon$$

and the integral is given by (94). We eliminate ϵ from $t_{\epsilon}(\eta)$ by substituting ϵ by $\epsilon^{o}(\alpha_{1})$ defined in (98). Subsequently, we substitute $t_{\epsilon^{o}(\alpha_{1})}(\eta)$ into $\bar{x}_{\epsilon^{s}(\alpha_{1})}^{s}(t)$. Thus, we now have two approximations, both parametrized by τ . To compare these approximations, we first rescale η by $\frac{b}{a}\frac{\eta}{\epsilon}$, otherwise the expansions in α_{1} become polynomial. We thus arrive at the following equation which should be satisfied

$$\bar{x}^{s}_{\epsilon^{s}(\alpha_{1})}\left(t_{\epsilon^{o}(\alpha_{1})}\left(\frac{b}{a}\frac{\eta}{\epsilon}\right)\right) = \bar{x}^{o}_{\epsilon^{o}(\alpha_{1})}\left(\frac{b}{a}\frac{\eta}{\epsilon}\right) + \begin{pmatrix}\mathcal{O}(\alpha_{1}^{3/2})\\\mathcal{O}(\alpha_{1}^{7/4})\end{pmatrix}$$

Expanding and simplifying the first component of $\bar{x}^o_{\epsilon^o(\alpha_1)}(\frac{b}{a}\eta/\epsilon)$ in α_1 gives

$$\begin{split} \left(\bar{x}^{o}_{\epsilon^{o}(\alpha_{1})}\left(\frac{b}{a}\frac{\eta}{\epsilon}\right)\right)_{1} &= \frac{3\operatorname{sech}^{2}\eta - 1}{\sqrt{-a}}\sqrt{\alpha_{1}} + \frac{18\sqrt{2}b\tanh\eta\operatorname{sech}^{2}\eta\log(\cosh\eta)}{7(-a)^{5/4}}\alpha_{1}^{3/4} \\ & \frac{1}{196a^{2}}\left[-6\operatorname{sech}^{2}\eta\left(7\left(5a_{1}b + 6b^{2} + 7d\right) - 72b^{2}(\log(\cosh\eta) - 1)\log(\cosh\eta)\right)\right. \\ & -36b^{2}\eta\tanh\eta\right) + 70a_{1}b + 9\operatorname{sech}^{4}\eta\left(24b^{2}(2 - 3\log(\cosh\eta))\log(\cosh\eta)\right) \\ & + 30b^{2} + 49d\right) + 98d\right]\alpha_{1} + \left[\tanh\eta\left\{-18b\log(\cosh\eta)\left(-980ab_{1} + 1225a_{1}b\right)\right. \\ & -336b^{2}(\log(\cosh\eta) - 3)\log(\cosh\eta) + 312b^{2} + 1176d\right) - 21\operatorname{sech}^{2}\eta\left(-1372ae\right) \\ & + 36b\log(\cosh\eta)\left(6b^{2}\log(\cosh\eta)(4\log(\cosh\eta) - 7) - 18b^{2} - 49d\right) + 234b^{3} \\ & + 1029bd\right) - 9604ae + 4914b^{3} + 7203bd\right\} - 4536b^{3}\eta\left(-2\log(\cosh\eta)\right) \\ & + \operatorname{sech}^{2}\eta(3\log(\cosh\eta) - 1) + 1\right)\right]\frac{\operatorname{sech}^{2}\eta}{4802\sqrt{2}(-a)^{11/4}}\alpha_{1}^{5/4} + \mathcal{O}(\alpha_{1}^{3/2}). \end{split}$$

Similarly, for the second component of $\bar{x}^o_{\epsilon^o(\alpha_1)}(\eta/\epsilon)$ we obtain

$$\begin{split} \left(\bar{x}^{o}_{\epsilon^{o}(\alpha_{1})}\left(\frac{b}{a}\frac{\eta}{\epsilon}\right)\right)_{2} &= \frac{3\sqrt{2}\tanh\eta\operatorname{sech}^{2}\eta}{\sqrt[4]{-a}}\alpha_{1}^{3/4} + \frac{9b\operatorname{sech}^{4}\eta}{7a}(\cosh(2\eta) - 2(\cosh(2\eta) - 2))\\ \log(\cosh\eta) - 1)\alpha_{1} + -\frac{3b\operatorname{sech}^{4}\eta}{196\sqrt{2}(-a)^{7/4}}(\sinh(2\eta)(35a_{1} - 144b(\log(\cosh\eta) - 2))\\ \log(\cosh\eta) + 48b) + 72b(2\eta - \eta\cosh(2\eta) + \tanh\eta(2\log(\cosh\eta)(6\log(\cosh\eta) - 7))\\ &- 3)))\alpha_{1}^{5/4} + \left(2\left(36b\log(\cosh\eta)\left(490ab_{1} - 490a_{1}b + 84b^{2}\log(\cosh\eta)(2\log(\cosh\eta) - 7)\right)\right)\right)\\ &+ (2\left(36b\log(\cosh\eta)\right)\left(490ab_{1} - 490a_{1}b + 84b^{2}\log(\cosh\eta)(2\log(\cosh\eta) - 7)\right)\\ &+ (2\left(36b\log(\cosh\eta)\right)\left(490ab_{1} - 490a_{1}b + 84b^{2}\log(\cosh\eta)(2\log(\cosh\eta) - 7)\right)\right)\\ &+ (2\left(36b\log(\cosh\eta)\right)\left(490ab_{1} - 490a_{1}b + 84b^{2}\log(\cosh\eta)(2\log(\cosh\eta) - 7)\right)\\ &+ (2\left(36b\log(\cosh\eta)\right)\left(490ab_{1} - 490a_{1}b + 84b^{2}\log(\cosh\eta)(2\log(\cosh\eta) - 7)\right)\\ &+ (2\left(36b\log(\cosh\eta)\right) - 138b^{2}\right) - 3b\left(-1960ab_{1} + 1960a_{1}b + 6462b^{2}\right)\\ &+ (12985d) + 7\operatorname{sech}^{2}\eta\left(-6860ae + 216b^{3}\log(\cosh\eta)(\log(\cosh\eta)(20\log(\cosh\eta) - 47) - 1)\right) + 1818b^{3} + 5145bd\right) + 48020ae\right) - 4536b^{3}\eta\tanh\eta\left(-4\log(\cosh\eta)\right)\\ &+ \operatorname{sech}^{2}\eta(12\log(\cosh\eta) - 7) + 4)\right)\frac{\operatorname{sech}^{2}\eta}{9604(-a)^{5/2}}\alpha_{1}^{3/2} + \mathcal{O}(\alpha_{1}^{7/4}). \end{split}$$

By expanding $\bar{x}_{\epsilon^s(\alpha_1)}^s\left(t_{\epsilon^o(\alpha_1)}\left(\frac{b}{a}\frac{\eta}{\epsilon}\right)\right)$ and simplifying we obtain

$$\begin{split} \bar{x}^{s}_{\epsilon^{s}(\alpha_{1})}\left(t_{\epsilon^{o}(\alpha_{1})}\left(\frac{b}{a}\frac{\eta}{\epsilon}\right)\right) &= \bar{x}^{o}_{\epsilon^{o}(\alpha_{1})}\left(\frac{b}{a}\frac{\eta}{\epsilon}\right) + \begin{pmatrix}\frac{(3bd-4ae)\operatorname{sech}^{2}(\eta)\tanh(\eta)}{\sqrt{2}(-a)^{11/4}}\alpha_{1}^{5/4}\\ \frac{(3bd-4ae)(\cosh(2\eta)-2)\operatorname{sech}^{4}(\eta)}{2(-a)^{5/2}}\alpha_{1}^{3/2}\end{pmatrix} \\ &= \bar{x}^{o}_{\epsilon^{o}(\alpha_{1})}\left(\frac{b}{a}\frac{\eta}{\epsilon}\right) + \begin{pmatrix}\frac{(3bd-4ae)u'_{0}(\eta)}{24\sqrt{2}(-a)^{11/4}}\alpha_{1}^{5/4}\\ \frac{(3bd-4ae)u'_{0}(\eta)}{24(-a)^{5/2}}\alpha_{1}^{3/2}\end{pmatrix}, \end{split}$$

i.e., the predictors differ by a phase shift. In fact, by using the freedom in the constants of integration in (92), we can let

$$t(\eta) \to t(\eta) + \theta_{1000} \frac{1}{b} \frac{2}{3} \left(\frac{4ae}{bd} - 3\right) \epsilon^2.$$

In this case, we will have equivalence between the predictors up to the desired order.

Remark 13. It is important to note here the phase condition used in the orbital predictor isn't preserved under the transformation H. Therefore, any improvements obtained in the approximation to the homoclinic solutions in a normal form due to a different phase condition is, in general, not preserved when lifting the approximations to the center manifold.

4.4 Implementation

In this section, we first briefly review the method used in MatCont to continue homoclinic solutions in autonomous ordinary differential equations (1) in two parameters as described in [46]. Then we describe our implementation in MatCont of the algorithm to start the continuation of the homoclinic solutions emanating from a codimension two Bogdanov-Takens point using the derived above homoclinic predictors and parameter-dependent center manifold.

4.4.1 Continuation of homoclinic solutions in MatCont

MatCont uses a correction-prediction continuation method applied to a defining system [17]. The defining system for the continuation of homoclinic solutions in ordinary differential equations of the form (1) in two parameters in MatCont are given by

$$\dot{x}(t) - 2Tf(x(t), \alpha) = 0,$$

$$f(s_0, \alpha) = 0,$$

$$\int_0^1 \tilde{x}(t)[x(t) - \tilde{x}(t)]dt = 0,$$

$$Q^{U^{\perp}, \mathrm{T}}(x(0) - s_0) = 0,$$

$$Q^{S^{\perp}, \mathrm{T}}(x(1) - s_0) = 0,$$

$$T_{22U}Y_U - Y_U T_{11U} + T_{21U} - Y_U T_{12U}Y_U = 0,$$

$$T_{22S}Y_S - Y_S T_{11S} + T_{21S} - Y_S T_{12S}Y_S = 0,$$

$$\|x(0) - s_0\| - \epsilon_0 = 0,$$

$$\|x(1) - s_0\| - \epsilon_1 = 0,$$

(100)

see [46]. Here the infinite time interval $]-\infty,\infty[$ of the homoclinic orbit is truncated to a finite interval [-T,T], where T > 0 is called the half-return time. The truncated interval is rescaled to the interval [0,1] and divided into **ntst** mesh-intervals. Each mesh interval is further subdivided by equidistant fine mesh points where the solution is approximated by a vector polynomial. Each mesh interval

contains a number of ncol collocation points where the first equation in (100) must be satisfied. The second equation in (100) locates the saddle point s_0 of the homoclinic orbit. The last two equations in (100) define the distance ϵ_0 and ϵ_1 between the saddle and the homoclinic solution at t = 0 and t = 1, respectively. The half-return time T, ϵ_0 , ϵ_1 are referred to as the homoclinic parameters. Either one or two of the homoclinic parameters must be allowed to vary. If two homoclinic parameters are selected to vary, the third equation (the phase condition) in (100) is added. The fourth and fifth equations in (100) place the solution at the endpoints in the unstable and stable eigenspace of linearization of the saddle point, respectively. The matrices $Q^{U^{\perp}} \in \mathbb{R}^{n \times n_S}$ and $Q^{S^{\perp}} \in \mathbb{R}^{n \times n_U}$ are not recalculated each continuation step, but constructed from the lower dimensional matrices $Y_U \in \mathbb{R}^{n_S \times n_U}$ and $Y_S \in \mathbb{R}^{n_U \times n_S}$. The fifth and sixth equations in (100), referred to as algebraic Ricatti equations, keep track of the lower dimensional matrix Y_U and Y_S , see [19]. The matrices Y_U and Y_S are initially set to zero.

Thus, in order to start continuation of homoclinic solutions near a codimension two Bogdanov-Takens point in (1), one needs to provide an initial approximation to

- the discretized orbit on the rescaled and truncated interval [0, 1],
- the parameter values α ,
- the saddle point s_0 ,
- the half-return time T,
- the initial distances ϵ_0 and ϵ_1 ,
- and an initial tangent vector for the next prediction.

Since the homoclinic predictors depend on the coefficients of the normal form, we first calculate the parameter-dependent center manifold transformation.

4.4.2 Multilinear forms

Unfortunately, not all multilinear forms in the expansion (14) needed for the derivation of the coefficients for the transformations H, K, and θ , were previously implemented in MatCont. We, therefore, developed new scripts which generate the necessary multilinear forms symbolically if the symbolic toolbox for MATLAB or for GNU Octave is installed. The multilinear forms can be generated with the graphical user interface of MatCont, or via the command-line interface. Examples are given in the Supplementary Materials. If the symbolic toolbox is not available, finite differences in combination with polarization identities are used instead. Note that symbolical derivatives generated with either GNU Octave or MatCont can be used interchangeably.

4.4.3 Coefficients of the parameter-dependent center manifold

Next, we compute the coefficients in the expansion G, H, K, and θ as derived in Section 2. Since we allow using finite differences, the results may become inaccurate. We, therefore, provide a warning message if one or more of the systems are not satisfied with a prescribed accuracy. The code for calculating the coefficients of the orbital, smooth, and hyper normal form can be found in the scripts BT_nmfm_orbital.m, BT_nmfm.m, and BT_nmfm_without_e_b1.m, respectively. These scripts can be called independently once a Bogdanov-Takens point is located.

4.4.4 The perturbation parameter and the half-return time

To provide the data listed at the end of Section 4.4.1, one first needs to select a suitable perturbation parameter ϵ in the homoclinic predictors in Section 4. In [1] a geometrically motived approach is

used to determine the perturbation parameter ϵ and half-return time T. The user first provides the amplitude A_0 of the homoclinic orbit, which can be approximated by

$$A_0 = \|\bar{x}(0,\epsilon) - s_0\|.$$

From the amplitude A_0 the initial perturbation parameter ϵ can then be estimated by truncating the homoclinic predictors in Section 4 up to second order in ϵ . Then, by using that the norm of the eigenvector q_0 is of unit length, see (19), we obtain

$$\epsilon^o = |b| \sqrt{\frac{A_0}{6|a|}}, \text{ and } \epsilon^s = \sqrt{\frac{A_0|a|}{6}},$$

for the orbital and smooth homoclinic predictor, respectively.

Secondly, the user provides the distance k, also referred to as **TTolerance** in MatCont, between the endpoints of the truncated homoclinic orbit and the saddle point

$$k = \|\bar{x}(\pm T, \epsilon) - s_0\|.$$
(101)

From this equation the half-return time T is solved by again truncating the homoclinic predictors given in Section 4 up to second order in ϵ . We obtain

$$\eta(T) = \left| \frac{b}{a} \frac{1}{\epsilon} \operatorname{arcsech} \left(\frac{|b|}{\epsilon} \sqrt{\frac{k}{6|a|}} \right) \right|, \quad \text{and} \quad T = \frac{1}{\epsilon} \operatorname{arcsech} \left(\sqrt{\frac{k}{A_0}} \right),$$

for the orbital and smooth homoclinic predictor, respectively. Thus, for the orbital predictor, the halfreturn time T has to be computed by numerical inverting $\eta(T)$, either using (91) or (93), depending on the perturbation method used.

Note that by truncating the homoclinic orbit up to second order, only the zeroth-order solution of u is used. Furthermore, since only the eigenvector q_0 is used in the approximation of the amplitude A_0 and the half-return time T, one should not expect to obtain an accurate approximation of the amplitude in general *n*-dimensional systems. For the half-return time T, there is an additional problem in the approximation. Namely, the homoclinic predictors are not even functions in t, i.e., we have the inequality

$$\|\bar{x}(T,\epsilon) - s_0\| \neq \|\bar{x}(-T,\epsilon) - s_0\|.$$

Thus, the more accurate interpretation of the amplitude A_0 and half-return time T is that they represent approximations for the amplitude and half-return time of u_0 on the center subspace and not for the homoclinic orbit on the center manifold. This, however, does not influence the convergence of the initial prediction for the homoclinic orbit, as long as the derived perturbation parameter ϵ is within the radius of convergence.

Instead of requiring the user to provide the amplitude A_0 and distance k, we determine the perturbation parameter ϵ automatically. Motivated by Appendix D an initial good guess for the orbital homoclinic predictor would be obtained with $\epsilon = 0.1$. The higher-order terms, not taken into account for the predictor, would have to behave very badly to not lead to convergence. The distance k we set to $k = \epsilon 10^{-4}$. In case that Newton does not converge, the perturbation ϵ is halved and k is updated. This process is then repeated until convergence is obtained, or by the maximum prescribed number of tries. In fact, for all buy one model in Section 5 and the Supplementary Materials, setting the perturbation parameter $\epsilon = 0.1$ and the distance $k = 10^{-4}$ while using the orbital predictor with either the Lindstedt-Poincaré or the regular perturbation method lead to convergence to the true homoclinic solution.

4.4.5 Homoclinic solution

After the perturbation parameter ϵ and the half-return time T have been determined and using the coefficients of H, K, and possibly θ , the homoclinic approximation is obtained by evaluating one of the homoclinic approximations given in Section 4 on the fine mesh points f_i for $i = 0, \ldots$ ntst \times ncol on the rescaled discretized interval [0, 1]. Note that we do need to translate the approximation to the Bogdanov-Takens phase point x_0 under consideration.

4.4.6 Saddle point

To obtain an approximation for the saddle point of the homoclinic orbit, we simply let t go to infinity in the homoclinic approximation obtained in the previous step. Note that τ goes to infinity as t does. Thus, using (88) we obtain

$$s_{0} = x_{0} + q_{0}w_{0}^{\infty} + \bar{H}_{0010}\beta_{1} + \bar{H}_{0001}\beta_{2} + \frac{1}{2}\bar{H}_{2000}(w_{0}^{\infty})^{2} + \bar{H}_{1010}w_{0}^{\infty}\beta_{1} + \bar{H}_{1001}w_{0}^{\infty}\beta_{2} + \frac{1}{2}\bar{H}_{0002}\beta_{2}^{2} + \bar{H}_{0011}\beta_{1}\beta_{2} + \frac{1}{6}\bar{H}_{3000}(w_{0}^{\infty})^{3} + \frac{1}{2}\bar{H}_{2001}(w_{0}^{\infty})^{2}\beta_{2} + \frac{1}{6}\bar{H}_{0003}\beta_{2}^{3} + \frac{1}{2}\bar{H}_{1002}w_{0}^{\infty}\beta_{2}^{2}$$

where

$$w_0^\infty = 2\frac{a}{b^2}\epsilon^2,$$

for the orbital predictor the approximation and

$$w_0^{\infty} = \frac{1}{a} \left(2 - 2\frac{5a_1b + 7d}{7a^2} \epsilon^2 \right) \epsilon^2,$$

for the smooth predictor.

4.4.7 Tangent

Once a good initial prediction for the homoclinic solution to (100) has been obtained, a normalized tangent vector is needed to start the continuation process. The simplest method to obtain the normalized tangent vector is by calculating the one-dimensional null space of the sparse rectangular Jacobian of the defining system (100). Although this is easy to implement numerically via a QR-decomposition of the transpose, the drawback is that we do not have any control of the orientation of the tangent vector. Thus, we do not know in which direction the continuation starts. Obviously, we want to avoid continuing the homoclinic curve towards the Bogdanov-Takens point. One way to obtain the correct direction is by obtaining two approximations with close, but different perturbation parameters, from which the normalized tangent can be approximated. Computationally cheaper and more accurate is to first compute the null space V of the Jacobian as described above and subsequently inspect the sign of the α_1 component in the vector V. This sign should be equal to the sign of the derivative of the first component of $\bar{\alpha}$ with respect to ϵ , see (89). A simple calculation yields

$$\bar{\alpha}'(\epsilon) = K_{10}\beta_1'(\epsilon) + K_{01}\beta_2'(\epsilon) + K_{02}\beta_2\beta_2'(\epsilon) + K_{11}(\beta_1'(\epsilon)\beta_2 + \beta_1\beta_2'(\epsilon)) + \frac{1}{2}K_{03}\beta_2^2\beta_2'(\epsilon),$$

where $\beta'_1(\epsilon)$ and $\beta'_2(\epsilon)$ are given by

$$\begin{cases} \beta_1'(\epsilon) = -16\frac{a^3}{b^4}\epsilon^3, \\ \beta_2'(\epsilon) = \frac{a}{b}(2\tau_0 + 4\tau_2\epsilon^2)\epsilon \end{cases}$$

with $\tau_{0,1}$ are given in (75) for the orbital predictor, and by

$$\begin{cases} \beta_1'(\epsilon) = -\frac{16}{a}\epsilon^3, \\ \beta_2'(\epsilon) = \frac{b}{a}(2\tau_0 + 4\tau_2\epsilon^2)\epsilon, \end{cases}$$

where $\tau_{0,1}$ are given in (114) for the smooth homoclinic predictor. Thus, we simply change the sign of the vector V if the product of the α_1 component of V with the first component of $\bar{\alpha}'(\epsilon)$ is negative, otherwise, we leave the vector V unchanged.

5 Examples

In this section, we will compare the different methods at different orders to approximate the homoclinic solution in (1) near a generic codimension two Bogdanov-Takens bifurcation point. In the first example, we consider the topological normal form (8). By using convergence plots we will show that by considering different phase conditions in both the regular perturbation method and the Lindstedt-Poincaré method influences the accuracy of the approximation, see Figure 2 and Figure 5, respectively. In this example, we will also compare the regular perturbation method against the Lindstedt-Poincaré method with a higher-order approximation of the non-linear time transformation as derived in this paper, or without, as done in [1]. We do this twofold: using convergence plots, see Figure 3, and by inspecting the predicted homoclinic profiles with the Newton corrected homoclinic profiles, see Figure 4.

Next, we will consider two four-dimensional models in which generic codimension two Bogdanov-Takens bifurcations are present. Here we will show that the approximation order of the homoclinic asymptotic lifts correctly to the parameter-dependent center manifold, see Figures 6 and 7. In the second example, we also compare the predicted with the Newton corrected homoclinic orbits in a projection onto a three-dimensional slice of the full system. It is shown that with an amplitude of 0.1 the approximation is still very accurate. This should be compared with [1, section 6.2] in which the amplitude needed to be set to 4×10^{-5} to obtain convergence.

Here we will not demonstrate how to actually start the continuation of the homoclinic orbits with MatCont. For this we refer to the online Jupyter Notebook. In the Jupyter Book, a total of nine different models are considered demonstrating in detail how to start continuation from either an explicitly derived, or encountered during continuation, codimension two Bogdanov-Takens bifurcation point. Each model is treated in a separate Jupyter Notebook, which can be executed to reproduce the results obtained. We do like to note that in all cases the curve of homoclinic solutions could be started with the default settings without the need to adjust any parameters, see Section 4.4. This shows that the asymptotics obtained in this paper are very robust.

5.1 Topological normal form

In this example we compare five different methods to approximate the homoclinic solution present in the universal unfolding (8):

- the regular perturbation method,
- the regular perturbation method with L_2 phase condition,
- the Lindstedt-Poincaré method without a higher-order time approximation as in [1],
- the Lindstedt-Poincaré method with a higher-order time approximation as derived here,
- and the Lindstedt-Poincaré method with a different phase condition.



Figure 2: Log-log convergence plot comparing the different phase conditions when using the regular perturbation method (RP) for approximation the homoclinic solutions. The subscript in RP_i , $0 \le i \le 3$, refers to the approximation order. The superscript L_2 refers to the phase condition (52).

In Figure 2 a log-log convergence plot is shown comparing the asymptotics derived in [1] using the regular perturbation method with phase condition $\dot{u} = 0$ against the asymptotics derived here with the phase condition given in (52). On the abscissa is the amplitude A_0 and on the ordinate is the relative error δ between the components w_0 and w_1 of the predicted solution and the Newton corrected solution. We see that the L_2 phase condition is slightly, but noticeably, more accurate at each order, confirming the geometric intuition.

Next, we compare the regular perturbation method with the Lindstedt-Poincaré method to approximate the homoclinic solution in log-log plot in Figure 3. It is seen that the first order regular perturbation method slightly outperforms the Lindstedt-Poincaré method, while for the second and third-order the Lindstedt-Poincaré method are clearly better approximations than the regular perturbation method. The third-order approximation by the Lindstedt-Poincaré method without including a higher-order approximation of the non-linear time transformation results in the same order of accuracy as the zeroth-order regular perturbation method.

It is thus essential to include a higher-order approximation of the non-linear time transformation. To make it even more clear we plotted the profiles of the third-order approximations using the Lindstedt-Poincaré method as in [1], the regular perturbation method, and the Lindstedt-Poincaré, together with the Newton corrected solutions in Figure 4. We see that the Lindstedt-Poincaré method as in [1] approximates the solution rather poorly, whereas the approximation derived in Section 3.2.2 is very accurate. Note that when plotting these homoclinic approximations and corrections in (w_0, w_1) phase-space, this difference is not visible at all. This explains why this has been unnoticed in [1].

Lastly, in Figure 5, we compare the two different phase conditions when using the Lindstedt-Poincaré method in a log-log plot. It is clearly seen that the phase condition used in Section 3.2.3 improves, rather significantly, the accuracy of the third-order predictor. However, in contrast with the different phase conditions used in the regular perturbation method, we do not have any geometric (or analytical) explanation for this improvement.

Notice that here we do not compare the homoclinic predictors derived with different normal forms. Indeed, when considering the universal unfolding (8) the normal forms coincide, resulting in identical



Figure 3: Log-log convergence plot comparing the relative errors of the computed homoclinic w_0 and w_1 component with the predicted solution in the topological normal form using four different methods: Regular Perturbation (RP, yellow), Lindstedt-Poincaré without higher-order time approximation (LP_3^{2016} , red), and Lindstedt-Poincaré combined with higher-order time approximation (LP, blue).

predictors.

5.2 Hodgking-Huxley equations

The Hodgkin-Huxley equations [27] relate the difference in electric potential across the cell membrane V and gating variables m, n and h for ion channels to the stimulus intensity I and temperature T, as follows:

$$\begin{cases}
V = -G(V, m, n, h) + I, \\
\dot{m} = \Phi(T) \left[(1 - m) \alpha_m(V) - m \beta_m(V) \right], \\
\dot{n} = \Phi(T) \left[(1 - n) \alpha_n(V) - n \beta_n(V) \right], \\
\dot{h} = \Phi(T) \left[(1 - h) \alpha_h(V) - h \beta_h(V) \right],
\end{cases}$$
(102)

where

$$\Phi(T) = 3^{(T-6.3)/10},$$

$$G(V,m,n,h) = \bar{g}_{\rm Na}m^3h\left(V - \bar{V}_{\rm Na}\right) + \bar{g}_{\rm K}n^4\left(V - \bar{V}_{\rm K}\right) + \bar{g}_{\rm L}\left(V - \bar{V}_{\rm L}\right).$$

The equations modeling the variation of membrane permeability are:

$$\begin{aligned} \alpha_m(V) &= \Psi\left(\frac{V+25}{10}\right), & \beta_m(V) &= 4e^{V/18}, \\ \alpha_n(V) &= 0.1\Psi\left(\frac{V+10}{10}\right), & \beta_n(V) &= 0.125e^{V/80}, \\ \alpha_h(V) &= 0.07e^{V/20}, & \beta_h(V) &= \left(1+e^{(V+30)/10}\right)^{-1}, \end{aligned}$$



Figure 4: Comparison of the profiles of the predicted and corrected homoclinic orbit for (8) using different approximation methods, see Section 5.1 for a full description.



Figure 5: Log-log convergence plot comparing the relative errors of the computed homoclinic w_0 and w_1 component with the predicted solution in the topological normal form using Lindstedt-Poincaré with two different phase-condition, see Section 3.2.3.

with

$$\Psi(x) = \begin{cases} x/(e^x - 1), & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

The parameters \bar{g}_{ion} and \bar{V}_{ion} representing maximum conductance and equilibrium potential for the ion were obtained from experimental data by Hodgkin and Huxley, with the values given below:

$$\bar{g}_{Na} = 120 \text{mS/cm}^2, \quad \bar{g}_{K} = 36 \text{mS/cm}^2, \quad \bar{g}_{L} = 0.3 \text{mS/cm}^2, \\ \bar{V}_{Na} = -115 \text{mV}, \quad \bar{V}_{K} = 12 \text{mV}, \quad \bar{V}_{L} = 10.599 \text{mV}.$$

The values of \bar{V}_{Na} and \bar{V}_{K} can be controlled experimentally [26, 29]. The temperature is set to $T = 6.3^{\circ}$. It is easy to see that the equilibria of (102) can be parametrized by V

$$I(V) = G(V, m(V), n(V), h(V))$$

$$y(V) = \alpha_u(V) / (\alpha_u(V) + \beta_u(V)),$$

where $y \in \{m, n, h\}$, see also [24]. By calculating the Jacobian A of (102) at the equilibrium, we can derive the characteristic polynomial $\rho_A(\lambda)$. The equation $\rho_A(0) = 0$ can be solved analytically for \bar{V}_K . Using this solution for \bar{V}_k and plotting the curve $\rho'(0)$ reveals two potential candidates for Bogdanov-Takens points. Inspecting the geometric multiplicity of these two points narrows the possibilities down to the point

$$\begin{pmatrix} V\\m\\n\\h\\\bar{V}_k\\I \end{pmatrix} \approx \begin{pmatrix} -2.835463618170097\\0.07351498630356315\\0.361877602925177\\0.494859128785482\\-4.977020454108788\\-0.06185214966177632 \end{pmatrix}.$$
 (103)



Figure 6: Convergence plot for the homoclinic predictors near the Bogdanov-Takens bifurcation (103) in the Hodgkin-Huxley equations (102).

Inspecting the coefficients of the normal form shows that

$$a = 2.5515 \cdot 10^{-5}, \qquad b = -0.0075.$$

Thus, provided the transversality conditions are satisfied, we can use MatCont to start continuation of the homoclinic orbits emanating from this point.

In Figure 6 there are two log-log convergence plots shown. Note that in this and the next example, we show the relative error $\delta(X)$ between the predicted and corrected Newton solution to the defining system (100). In the left plot (a) we compare the regular perturbation method with the Lindstedt-Poincaré method. We see that compared with the previous example the Lindstedt-Poincaré method is slightly less accurate than the regular perturbation method and the second order. Nevertheless, we clearly see that the order of convergence lifts from the normal form to the two-dimensional center manifold in \mathbb{R}^4 . In the plot right (b) we compare four different third approximations to the homoclinic orbit

- the Lindstedt-Poincaré method using the smooth orbital normal form (the blue diamond),
- the Lindstedt-Poincaré method using the smooth normal form (the dashed light gray line),
- the regular perturbation method using the smooth normal form (the pink square),
- the Lindstedt-Poincaré method using the hyper-normal form (the green plus).

We see that both the Lindstedt-Poincaré method and the regular perturbation method using the smooth orbital normal form are in perfect agreement with the Lindstedt-Poincaré method using the smooth normal form. Only the homoclinic predictor using the hyper-normal form is slightly less accurate.

5.3 Homoclinic RG flows

In [30] an $\mathcal{N} = 1$ supersymmetric model of interacting scalar superfields Φ_{ab}^i that is invariant under the action of an $O(N) \times O(M)$ group in $d = 3 - \epsilon$ dimensions is considered. The coupling constants $g_i(i = 1, \dots, 4)$ satisfy the following differential equations

$$\dot{g} = -\epsilon g + \beta^{(2)}(g, M, N) + \mathcal{O}(g^5), \qquad g \in \mathbb{R}^4, \tag{104}$$

where the two-loop contributions $\beta_i^{(2)}(i=1,\ldots,4)$ are cubic in the coupling and the parameter ϵ is scaled to 1. The exact expression for $\beta_i^{(2)}$ are quite long can be found in [30, Appendix B] or in the Supplementary Materials.

In [30] a Bogdanov-Takens point near the parameter values M = 0.2945 and N = 4.036 is located. Using these parameter values we locate an equilibrium at

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix} = \begin{pmatrix} 0.0701457361241472 \\ -0.06520883770451065 \\ 0.001823543197553845 \\ 0.22874527306411319 \end{pmatrix}.$$

By continuing the equilibrium in the parameter M we detect several limit points and two Hopf points. We continue the second Hopf point at $M \approx 0.2958$ in parameters M and N. Several Bogdanov-Takens points are detected. The first Bogdanov-Takens point is located at

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix} = \begin{pmatrix} -0.715157316845187 \\ -0.250968103603174 \\ 0.510051114588271 \\ -0.391935453715783 \end{pmatrix},$$

with parameter values

$$(M, N) = (0.294477255737036, 4.035536108506390).$$

In Figure 7 we have created similar log-log convergence plots as in the previous example. The plots look very alike, only the homoclinic predictor using the hyper-normal form is in this model slightly more accurate than the other homoclinic approximations.

Lastly, in Figure 8, there are two additional plots. The left plot (a) compares the predicted (dashed, red) with the corrected (solid, blue) homoclinic orbits using the Lindstedt-Poincaré method with the smooth orbital normal form for amplitudes $A_0 = 10^{-3}$ to $A_0 = 0.1$. We see that they are in excellent agreement. In the right plot (b) we compared the computed homoclinic bifurcation curve (solid, blue) with the predicted values in parameter-space (M, N). Most important to notice here is that the predictor given in [1] (the yellow crosses) is less accurate than the second order predictor (blue plus signs) obtained in this paper.

6 Discussion

We have derived third-order predictors for the homoclinic curve emanating from the generic codimension two Bogdanov-Takens bifurcation in general n-dimensional autonomous ODEs. By considering the smooth orbital normal form (10) and incorporating the time-reparametrization in the homological equation (7) we were able to derive the third-order asymptotic of the homoclinic curve independent of any coefficients. However, for this simplification, there is a price to pay. Firstly, the systems to be solved to obtain the coefficients for the parameter-dependent center manifold becomes more difficult, see Section 2.2. Ideally, there should be an automatic algorithm in line with [42]. However, to the best



Figure 7: Convergence plot for the homoclinic predictors near one of the two Bogdanov-Takens bifurcation in the Homoclinic RG flows model (104).

of our knowledge, such algorithms do not exist yet. Secondly, the translation of time in the homological equation needs to be inverted numerically. This, however, can be done relatively cheap and is very accurate as shown by the examples.

We have explained how to obtain the correct transformation to the parameter-dependent center manifold by carefully inspecting which terms are in, *and are not in*, the normal form that affects the homoclinic asymptotic up to certain order. The comparison in Section 4.3 and the examples in Section 5 together with the examples in the supplementary materials show that we indeed have obtained the correct transformation.

The additional non-linear transformation (12) greatly simplifies the computation of the coefficients in the Lindstedt-Poincaré method since all calculations become essentially polynomial, which is ideal for computers to work with. Nonetheless, there the algorithm complexity grows exponentially as the order increases linearly. Also, the radius of convergence is clearly finite as shown in Appendix D. One way to increase the convergence radius is by using transformations as in [45]. However, we didn't include any results in this direction since it would distract too much from our main objectives.

Using different phase conditions can improve the accuracy of the homoclinic approximation. However, this only holds true when applied directly to the system considered. Indeed, the phase condition isn't invariant under the parameter-dependent center manifold transformation. Thus, its applicability is very limited. Furthermore, using a different phase condition may, somewhat unexpectedly, result in difficult integrals to be solved, see Section 3.1.

The higher-order approximation to the non-linear time transformation in the Lindstedt-Poincaré method turns out to be essential to obtain higher-order approximations to the homoclinic solutions. This is clearly seen by inspecting the profiles of the homoclinic solution in Figure 4 and in the convergence plot in Figure 3. Without the higher-order approximation the same convergence order as the unperturbed Hamiltonian solution, i.e., the zeroth-order solution. It should be noted that the higher-order approximation of the non-linear time transformation is more difficult to obtain. Therefore, we conclude that there seem to be *no benefits* of the Lindstedt-Poincaré method over the regular perturbation method for starting continuation of homoclinic orbits. Indeed, the numerical comparisons in Section 5 show similar accuracy of convergence at each order.

By comparing the convergence order of the regular perturbation method with the Lindstedt-Poincaré method we see that contrary to what one might expect, the regular perturbation method



Figure 8: Plot (a) compares the predicted (dashed, red) with the corrected (solid, blue) homoclinic orbits using the Lindstedt-Poincaré method with the smooth orbital normal form for amplitudes $A_0 = 10^{-3}$ to $A_0 = 1$. Plot (b) compares the computed homoclinic bifurcation curve (solid, blue) with the predicted values in parameter-space (M, N). The yellow crosses are the second order predictor obtained with the transformation as given in [1]. The blue plus signs are the second order predictor obtained in this paper. Lastly, the red boxes are the third-order predictor obtained in this paper.

may result in better accuracy at the same order. A possible explanation for this might be that although the Lindstedt-Poincaré method provides a uniform approximation in time along the homoclinic orbit of the numerical solution is truncated to a finite interval in which the 'parasitic turn' doesn't give a significant contribution. After all it then simply depends on the higher-order non-linear terms in the system which favor one method over the other.

A Explicit example demonstrating incorrect predictor

Although the second-order homoclinic approximation derived in [1] for the smooth normal form (9) is correct, the parameter and center manifold transformation are incorrect. To see this, we suppose (1) is given by

$$\dot{x} = f(x,\alpha) = \begin{pmatrix} x_1 \\ \alpha_1 + \alpha_2 x_1 + x_0^2 + x_0 x_1 + c_1 \alpha_2^3 \end{pmatrix},$$
(105)

for some arbitrary nonzero constant $c_1 \in \mathbb{R}$. We will now compare two different methods for obtaining a second-order approximation to the homoclinic solution in (105). To keep the exposition as clear as possible, we focus solely on the parameters. For the first method we directly apply the singular rescaling

$$\alpha_1 = -4\epsilon^4, \quad \alpha_2 = \eta\epsilon^2, \quad x_0 = \epsilon^2, \quad x_1 = \epsilon^3, \quad s = \epsilon t,$$

to (105). This yields the system

$$\begin{cases} \dot{u} = v, \\ \dot{v} = -4 + u^2 + v \left(u + \tau \right) \epsilon + c_1 \tau^3 \epsilon^2, \end{cases}$$

where the dot $\dot{}$ now represents the derivative with respect to s. Then, using the generalized Lindstedt-Poincaré method we obtain the approximation

$$(\alpha_1, \alpha_2) = \left(-4\epsilon^4, \frac{10}{7}\epsilon^2 + \frac{288 - 1250c_1}{2401}\epsilon^4 + \mathcal{O}(\epsilon^5)\right)$$
(106)

for the parameters. For the second method we use the predictor from [1]. That is, we use the secondorder homoclinic predictor derived for the smooth normal form (9). Then calculate the center manifold transformation, which for the two-dimensional systems reduces to a near-identity transformation, and parameter transformation to transfer the predictor the original system. We obtain that the nearidentity and parameter transformation are just the identities. Thus, we obtain the predictor

$$(\alpha_1, \alpha_2) = \left(-4\epsilon^4, \frac{10}{7}\epsilon^2 + \frac{288}{2401}\epsilon^4 + \mathcal{O}(\epsilon^5)\right).$$
(107)

Obviously, this result is wrong. To see why the latter second predictor doesn't contain the term c_1 we consider the near-identity transformation

$$\begin{cases} x = w, \\ \alpha = \beta + \begin{pmatrix} -c_1 \\ 0 \end{pmatrix} \beta_2^3. \end{cases}$$
(108)

Then system (105) becomes

$$\begin{cases} \dot{w}_0 = w_1, \\ \dot{w}_1 = \beta_1 + \beta_2 w_1 + w_0^2 + w_0 w_1. \end{cases}$$

Using the second-order predictor from [1] for the smooth normal form we obtain

$$(\beta_1, \beta_2) = \left(-4\epsilon^4, \frac{10}{7}\epsilon^2 + \frac{288}{2401}\epsilon^4 + \mathcal{O}(\epsilon^5)\right).$$

Then using the near-identity transformation (108) yields the predictor

$$(\alpha_1, \alpha_2) = \left(-4\epsilon^4 - c_1 \left(\frac{10}{7} \epsilon^2 + \frac{288}{2401} \epsilon^4 + \mathcal{O}(\epsilon^5) \right)^3, \frac{10}{7} \epsilon^2 + \frac{288}{2401} \epsilon^4 + \mathcal{O}(\epsilon^5) \right).$$

To compare this predictor with (106) we eliminate ϵ in both equations. This yields

$$\alpha_2(\alpha_1) = -\frac{5}{7}\sqrt{-\alpha_1} + \frac{288 - 1250c_1}{2401}\alpha_1 + \mathcal{O}(\alpha_1^{\frac{3}{2}}).$$

We conclude, as expected, that if the correct transformation is used between the normal form and the original system we keep the correct order of accuracy for the approximation. In [1] the coefficients H_{0003} and K_{03} (among other coefficients) are not incorporated into the parameter-dependent center manifold transformation (15) and (16) leading to the incorrect predictor (107).

B Integrals from Section 3.1.1

Making the substitution $s = \log(u)$ in (56) yields

$$I_n = 2^n \int_1^\infty \log^3\left(\frac{u^2+1}{u}\right) \frac{u^{n-1}}{(u^2+1)^n} \, du.$$

Then, by making the reciprocal substitution $u \to \frac{1}{u}$, we can show that

$$I_n = 2^{n-1} \int_0^\infty \log^3\left(\frac{u^2+1}{u}\right) \frac{u^{n-1}}{(u^2+1)^n} \, du.$$

The last integral can be separated into the four following integrals

$$I_n^{(1)} = \int_0^\infty \log^3(u^2 + 1) \frac{u^{n-1}}{(u^2 + 1)^n} du,$$
(109)

$$I_n^{(2)} = \int_0^\infty \log^2(u^2 + 1) \log u \frac{u^{n-1}}{(u^2 + 1)^n} du,$$
(110)

$$I_n^{(3)} = \int_0^\infty \log(u^2 + 1) \log^2 u \frac{u^{n-1}}{(u^2 + 1)^n} du,$$
(111)

$$I_n^{(4)} = \int_0^\infty \log^3 u \frac{u^{n-1}}{(u^2+1)^n} du.$$
 (112)

The integral (109) can easily be solved by first applying the substitutions $u \to u^2 + 1$ and $u \to \frac{1}{u}$ consecutively to obtain

$$I_n^{(1)} = -\frac{1}{2} \int_0^1 \log^3(u) \left(1-u\right)^{\frac{n}{2}-1} u^{\frac{n}{2}-1} du.$$

Then using the binomial theorem yields

$$I_n^{(1)} = -\frac{1}{2} \sum_{k=0}^{\frac{n}{2}-1} {\binom{n}{2}-1 \choose k} (-1)^k \int_0^1 \log^3(u) u^{\frac{n}{2}-1+k} \, du = \frac{1}{2} \sum_{k=0}^{\frac{n}{2}-1} {\binom{n}{2}-1 \choose k} \frac{(-1)^k 3!}{(\frac{n}{2}+k)^4},$$

where in the last equality we used the well-know identity

$$\int_0^1 u^m \log^n(u) \, du = (-1)^n \frac{n!}{(m+1)^{n+1}},\tag{113}$$

for n and m natural numbers.

To solve the integral (110) we make three consecutive substitutions: $u \to u^2$, $u \to u-1$, and $u \to \frac{1}{u}$. This gives

$$I_n^{(2)} = -\frac{1}{4} \int_0^1 \left(\log^3(v) - \log^2(v) \log(1-v) \right) (1-v)^{\frac{n}{2}-2} v^{\frac{n}{2}-1} dv.$$

Then, by using the binomial theorem and expanding the logarithm we obtain

$$I_n^{(2)} = \frac{1}{4} \sum_{k=0}^{\frac{n}{2}-1} {\binom{n}{2}-1 \choose k} (-1)^k \left[\int_0^1 \log^3(v)(1-v)^{\frac{n}{2}-2}v^{\frac{n}{2}-1}dv - \sum_{l=1}^{\infty} \frac{1}{l} \int_0^1 \log^2(v)v^{\frac{n}{2}-1+k+l}dv \right].$$

Using equality (113) once more yields

$$I_n^{(2)} = \frac{1}{4} \sum_{k=1}^{\frac{n}{2}-1} \binom{\frac{n}{2}-1}{k} (-1)^k \left[\frac{3!}{(\frac{n}{2}+k)^4} - 2\sum_{l=1}^{\infty} \frac{1}{l(\frac{n}{2}+k+l)^3} \right].$$

Fractional decomposition shows that the innermost summation in the last equation is equal to

$$\sum_{l=1}^{\infty} \frac{1}{l(\frac{n}{2}+k+l)^3} = \frac{8}{2k+n} \left(\frac{H_{\frac{n}{2}+k}}{(2k+n)^2} + \frac{H_{k+\frac{n}{2}}^{(2)} - \zeta(2)}{2(2k+n)} + \frac{H_{k+\frac{n}{2}}^{(3)} - \zeta(3)}{4} \right).$$

Thus, the integral in (110) is equal to

$$\begin{split} I_n^{(2)} &= \frac{1}{4} \sum_{k=1}^{\frac{n}{2}-1} \binom{\frac{n}{2}-1}{k} (-1)^k \left[\frac{3!}{(\frac{n}{2}+k)^4} - \frac{2}{2k+n} \left(\frac{4}{(2k+n)^2} H_{\frac{n}{2}+k} \right. \\ &+ \frac{2}{(2k+n)} H_{\frac{n}{2}+k}^{(2)} + H_{\frac{n}{2}+k}^{(3)} - \frac{2\zeta(2)}{2k+n} - 4\zeta(3) \right) \right]. \end{split}$$

Now most work is done, since subtracting two times (110) from (111) is equal to

$$\int_0^\infty \log^2(u^2 + 1) \log(u) \log\left(1 + \frac{1}{u^2}\right) \frac{u^{n-1}}{(u^2 + 1)^n} du.$$

The reportorial substitution $u \to \frac{1}{u}$ shows that this integral vanishes. The same substitution also shows that the integral in (112) vanishes. We thus obtain the closed-form expression

$$\begin{split} I_n &= 2^{n-3} 3 \sum_{k=0}^{\frac{n}{2}-1} \binom{\frac{n}{2}-1}{k} (-1)^k \\ & \left[\frac{1}{(\frac{n}{2}+k)^4} + \frac{8}{2k+n} \left(\frac{H_{\frac{n}{2}+k}}{(2k+n)^2} + \frac{H_{\frac{n}{2}+k}^{(2)} - \zeta(2)}{2(2k+n)} + \frac{H_{\frac{n}{2}+k}^{(3)} - \zeta(3)}{4} \right) \right], \end{split}$$

where ζ is the Riemann zeta function and $H_n^{(m)}$ is the *n*-th generalized harmonic number of order *m*.

C Asymptotics for homoclinic solution in the smooth normal form

Following the procedure outlined in Section 3.2 to the second-order nonlinear differential equation (38) obtained from the smooth normal form (9). For the third-order homoclinic predictor we obtain

$$\sigma = 6 + \frac{3\left(-70a_{1}b + 6b^{2} + 49d\right)}{49a^{2}}\epsilon^{2} + \mathcal{O}(\epsilon^{4}),$$

$$\delta = -4 + \frac{140a_{1}b - 18b^{2} - 245d}{49a^{2}}\epsilon^{2} + \mathcal{O}(\epsilon^{4}),$$

$$\tau = \frac{10}{7} + \frac{98b(50ab_{1} + 73d) - 9604ae - 2450a_{1}b^{2} + 288b^{3}}{2401a^{2}b}\epsilon^{2} + \mathcal{O}(\epsilon^{4}),$$

$$\tilde{\omega}(\zeta) = 1 - \frac{6b}{7a}\zeta\epsilon + \frac{70a_{1}b + 18b^{2}\left(3\zeta^{2} + 1\right) + 49d\left(9\zeta^{2} - 5\right)}{196a^{2}}\epsilon^{2} + \frac{\zeta}{2401a^{3}}\left(\left(-147b\left(20ab_{1} - 7d\zeta^{2} + 11d\right) - 9604ae\left(\zeta^{2} - 1\right)\right) + 1470a_{1}b^{2} + 18b^{3}\left(7\zeta^{2} - 11\right)\right)\epsilon^{3} + \mathcal{O}(\epsilon^{4}).$$

$$(114)$$

It follows that

$$\tilde{u}(\zeta) = 6\zeta^2 - 4 + \frac{-70a_1b\left(3\zeta^2 - 2\right) + 18b^2\left(\zeta^2 - 1\right) + 49d\left(3\zeta^2 - 5\right)}{49a^2}\epsilon^2 + \mathcal{O}(\epsilon^4),$$
(115)

$$\tilde{v}(\zeta) = -2\tilde{\omega}(\zeta)\sigma(1-\zeta^2)\zeta = -\left[-12 + \frac{72b}{7a}\zeta\epsilon - \frac{3}{49a^2}\left(70a_1b - 6b^2\left(9\zeta^2 + 5\right)\right)\right]$$
(116)

$$-147d (3\zeta^{2} - 1)) \epsilon^{2} + \frac{12\zeta}{2401a^{3}} (147b (20ab_{1} - 7d\zeta^{2} + 18d) +9604ae (\zeta^{2} - 1) - 2940a_{1}b^{2} - 18b^{3} (7\zeta^{2} - 18)) \epsilon^{3} (1 - \zeta^{2})\zeta + \mathcal{O}(\epsilon^{4}).$$

The relation $\xi(s)$ can be obtained by solving the ODE

$$\frac{d\xi}{ds}(s) = \tilde{\omega}(\tanh(\xi(s))). \tag{117}$$

Thus, we substitute

$$\xi(s) = s + \xi_1(s)\epsilon + \xi_2(s)\epsilon^2 + \xi_3(s)\epsilon^3 + \mathcal{O}(\epsilon^4),$$

into (117) and expand the resulting equation in ϵ to obtain

$$\begin{aligned} \frac{d\xi_1}{ds}(s) &= -\frac{6b\tanh(s)}{7a},\\ \frac{d\xi_2}{ds}(s) &= \frac{-168ab\operatorname{sech}^2(s)\xi_1(s) + 70a_1b + 9\left(6b^2 + 49d\right)\tanh^2(s) + 18b^2 - 245d}{196a^2},\\ \frac{d\xi_3}{ds}(s) &= \frac{\operatorname{sech}^3(s)}{4802a^3} \left(4116a^2b\sinh(s)(\xi_1(s))^2 - 4116a^2b\cosh(s)\xi_2(s)\right) \\ &+ 441a\left(6b^2 + 49d\right)\sinh(s)\xi_1(s) + 2\sinh(s)\left(-3b\cosh(2s)\right) \\ &\left(98(5ab_1 + d) - 245a_1b + 12b^2\right) - 1470abb_1 + 9604ae + 735a_1b^2 \\ &-162b^3 - 1323bd\right)\right). \end{aligned}$$

Here we directly used that $\xi_0(s) = s$. By solving these equations recursively we obtain

$$\begin{split} \xi_1(s) &= -\frac{6b}{7a} \log(\cosh(s)), \\ \xi_2(s) &= \frac{2s \left(35a_1b - 36b^2 + 98d\right) + 9 \tanh(s) \left(16b^2 \log(\cosh(s)) + 10b^2 - 49d\right)}{196a^2}, \\ \xi_3(s) &= \frac{1}{4802a^3} \left(-7 \operatorname{sech}^2(s) \left(1372ae - 27b \left(6b^2 + 49d\right) \log(\cosh(s))\right) \right. \\ &\quad + 216b^3 \log^2(\cosh(s)) - 234b^3 - 147bd\right) - 5880abb_1 \log(\cosh(s)) \\ &\quad + 9604ae + 42bs \tanh(s) \left(-35a_1b + 36b^2 - 98d\right) + 4410a_1b^2 \log(\cosh(s)) \\ &\quad - 1656b^3 \log(\cosh(s)) - 1638b^3 + 2940bd \log(\cosh(s)) - 1029bd\right). \end{split}$$

Here we used the phase condition that $\xi_i(0) = 0, i = 1, 2, 3$. This results in the constraint v(0) = 0. Substituting the above expression for ξ into (37) we obtain the third-order predictor

$$\begin{cases} w_0(t) = \frac{1}{a} \tilde{u} \left(\tanh\left(\xi(\epsilon t)\right) \right) \epsilon^2, \\ w_1(t) = \frac{1}{a} \tilde{v} \left(\tanh\left(\xi(\epsilon t)\right) \right) \epsilon^3, \\ \beta_1 = -\frac{4}{a} \epsilon^4, \\ \beta_2 = \frac{b}{a} \tau \epsilon^2, \end{cases}$$
(118)

where τ, \tilde{u} and \tilde{v} are given by (114)–(116), respectively.

Note that by expanding $\tilde{u}(\tanh(\xi(s)))$ in (118) up to order three in ϵ we obtain

$$\begin{aligned} u_0(s) &= 6 \tanh^2(s) - 4, \\ u_1(s) &= -\frac{72b \tanh(s) \operatorname{sech}^2(s) \log(\cosh(s))}{7a}, \\ u_2(s) &= (12s \sinh(2s) (35a_1b - 36b^2 + 98d) + 8 \cosh(2s) (7 (5a_1b + 9b^2 - 56d) \\ &-108b^2 \log^2(\cosh(s)) + 108b^2 \log(\cosh(s))) + 9 (35a_1b + 192b^2 \log^2(\cosh(s)) \\ &-96b^2 \log(\cosh(s)) - 64b^2 + 245d) - 7 \cosh(4s)(5a_1b + 7d)) \frac{\operatorname{sech}^4(s)}{196a^2}, \\ u_3(s) &= (-2 \sinh(s) (\cosh(2s) (-6b \log(\cosh(s)) (-980(ab_1 + 3d) + 1225a_1b + 312b^2) \\ &+7 (-1372ae + 234b^3 + 147bd) + 2016b^3 \log^3(\cosh(s)) - 6048b^3 \log^2(\cosh(s))) \\ &+ 6b \log(\cosh(s)) (980ab_1 - 1225a_1b + 1200b^2 - 9408d) + 7 (1372ae - 234b^3 \\ &-147bd) - 10080b^3 \log^3(\cosh(s)) + 15120b^3 \log^2(\cosh(s))) \\ &+ 42bs \cosh(3s) (35a_1b - 36b^2 + 98d) (2\log(\cosh(s)) - 1) \\ &+ 42bs \cosh(s) (-35a_1b + 36b^2 - 98d) (6\log(\cosh(s)) - 1)) \frac{3 \operatorname{sech}^5(s)}{4802a^3}. \end{aligned}$$

Together with (114), this is the solution obtained by using the regular perturbation method to the second-order nonlinear oscillator (38) obtained from the smooth normal form with phase condition $\dot{u}(0) = 0$. This gives us the third order homoclinic predictor

$$\begin{cases} w_0(t) = \frac{1}{a} \left(\sum_{i=0}^3 u_i(\epsilon t) \epsilon^i + \mathcal{O}(\epsilon^4) \right) \epsilon^2, \\ w_1(t) = \frac{1}{a} \left(\sum_{i=0}^3 \dot{u}_i(\epsilon t) \epsilon^i + \mathcal{O}(\epsilon^4) \right) \epsilon^3, \\ \beta_1 = -\frac{4}{a} \epsilon^4, \\ \beta_2 = \frac{b}{a} \epsilon^2 \tau, \end{cases}$$
(120)

where τ is given by (114) and $u_i(i = 0, ..., 3)$ are given by (119).

D Case study of the quadratic codim 2 Bogdanov-Takens normal form

In this section, we will numerically study the algorithm outlined in Section 3.2 for the quadratic codimension 2 Bogdanov-Takens normal form (8). Since the algorithm only relies on arithmetic and calculus on polynomials over the field \mathbb{Q} , see Corollary 7 there is no need to use propriety software for the implementation. We choose the relative new programming language Julia [6]. Julia natively supports arbitrary precision rational numbers. We use the package Polynomials.jl [34] to handle the differentiation and integration of polynomials, as well as polynomial division. Since the programming language Julia starts indexing arrays at 1, we use the package OffsetArrays [33] to lower the index to 0 to keep the indexing identical. The code is given in the listing below.

module BTQuadraticHomoclinic

```
using Polynomials, OffsetArrays
function z(i, \tau, \sigma, \omega)
    if i == 1
         p = 24*Polynomial([0, -2, 0, 3])
    else
         p = Polynomial([0,2]) * sum(\sigma[1] * \tau[i-1-1] for l in 1:i-1)
         p += Polynomial([0,2])*sum(\sigma[1]*\omega[k]*\tau[i-1-l-k]
                  for k in 1:i-1 for l in 0:i-1-k)
         p -= Polynomial([2,0,-6])*sum(\sigma[i-1]*\omega[1] for l in 1:i-1)
         p = 2*sum(\sigma[i-l-k]*\omega[l]*derivative(
                  Polynomial([0,1,0,-1])*\omega[k]) for k in 1:i-1 for l in 0:i-k)
         p \neq Polynomial([-1,0,1])*Polynomial([0,2])*sum(\sigma[1]*
                 \sigma[i-1-l-k]*\omega[k] for k in 0:i-1 for l in 0:i-2-k)
         p \neq Polynomial([-4,0,6]) * Polynomial([0,2]) * sum(\sigma[i-1-k] * \omega[k])
                  for k in 0:i-1)
         p \neq Polynomial([1,0,-1]) * sum(\sigma[k] * \sigma[i-k] for k in 1:i-1)
    end
    Polynomial([1,0,-1])*p
end
function solve(;order = n)
    \sigma = OffsetArray(zeros(Rational{BigInt}), order), 0:order-2)
    \tau = OffsetArray(zeros(Rational{BigInt}), order-1), 0:order-1)
    \omega = OffsetArray(Array{Polynomial}(undef, order), 0:order-1)
    \sigma[0], \omega[0] = 6, 1
    for i=1:order-1
         gi = integrate(Polynomial([0, 12//1])*z(i, \tau, \sigma, \omega))
         if i%2 == 1
             \tau[i-1] = -10//192*gi(1)
             \omega[i] = (\tau[i-1]*144*Polynomial([2//15, 0, 0, 1//3, 0, -1//5]))
                         + gi) ÷ (Polynomial([1,0,-1])*Polynomial([0,12]))^2
         else
              \sigma[i] = -gi(-1)//12
             \omega[i] = -\sigma[i]//6 + (\sigma[i]*Polynomial([-1,0,1])*
                   (Polynomial([-4,0,6])^2-4) + gi + 12*\sigma[i]) \div
                            (Polynomial([1,0,-1])*Polynomial([0,12]))^2
         end
    end
    τ, σ, ω
end
end
```

In [2] the impression is given that one can obtain higher-order approximations very fast using their algorithm. However, our algorithm, which should be superior, shows that the order of approximation and the computational cost is not linear related. We performed a benchmark to obtain a 20th-order approximation to τ , see Table 2. These results were obtained on the mobile CPU Intel i5-6200U (4) @ 2.800GHz with 11407MiB of memory. The coefficients for τ are shown in Table 3. We see that the length of the numerator and denominator increases at each (even) order. This also holds true for the coefficients of σ and the coefficients of the polynomials ω . Performing operations on these rational numbers become increasingly more difficult for the computer to deal with. In Figure 9 a log-linear plot is shown, plotting the time in seconds to solve the *ith* order equation (65). It took nearly 10 hours to

```
julia> @benchmark BTQuadraticHomoclinic.solve(order=20)
BenchmarkTools.Trial:
  memory estimate: 107.36 MiB
  allocs estimate: 5384379
  _____
  minimum time:
                   382.111 ms (21.56% GC)
  median time:
                   414.436 ms (24.41% GC)
  mean time:
                   459.831 ms (26.81% GC)
  maximum time:
                   577.254 ms (32.91% GC)
  _____
  samples:
                   11
  evals/sample:
                   1
```

-

Table 2: Benchmark to obtain a 20th-order approximation to τ in the quadratic normal form (8) using the algorithm outlined in Section 3.2. See in particular Corollary 9 and the algorithm above.

obtain the 200th order approximation of τ . A linear regression on the last 50 data points indicates the time increases exponentially. Extrapolation yields that it would take more than 17 years to solve the first 500 terms if memory doesn't become an issue. Since the algorithm is embarrassingly parallelizable we could speed up the process. However, exponential growth cannot be escaped.

i	$ au_i$
0	$\frac{10}{7}$
2	$\frac{288}{2401}$
4	$-\frac{240192}{45294865}$
6	$-\frac{37647386496}{11108339166925}$
$\overline{7}$	$rac{5154131788676352}{2724264638992521625}$
8	$-\frac{1828854022199737517568}{1622558397660750917241875}$
10	$\frac{4788476595594706542555091968}{6764713681983284597882721784375}$
12	$-\frac{333207890299772418358595124320937984}{750205319977359363432143692632890996875}$
14	$\frac{384324328969886017955349568884165743978164224}{1366817906371309055843841557575267655935039046875}$
16	$-\frac{2241496149333347510622026401379280877412634073959456768}{12451199287907137102778709466943415347212749443284171484375}$
18	$\frac{2640541575683401164949776492089908615592018826498686833538777088}{22685152569996135996229496037325376035529199986794205676507927734375}$
20	$-\frac{90999911407271629445902884163495991850622440695336663926865318101431386112}{1198588820362905667811216420985132469928184621078674921926335765852723935546875}$

Table 3: First 20 coefficients of τ

Next, we would like to make some comments on the radius of convergence of the asymptotic approximation to the homoclinic orbit in the quadratic Bogdanov-Takens bifurcation. In [2] there is the remark that the higher-order approximation can greatly improve the accuracy of the approximation for large parameter values. However, this fully depends on the radius of convergence of the series. To obtain a first impression, we compare the parameters computed from the numerical continued solution to the homoclinic solution in (8) using MatCont with the predicted parameters. Figure 10 reveals a typical situation in perturbation series. Increasing the order of the approximation parameter improves the approximation for small parameters, but for larger parameters, the approximation becomes much worse. Reproducing [2, Fig 3c], but increasing the order, shows that this solution is outside the radius of convergence.



Figure 9: Log-linear plot of the order i and the time in seconds it took to compute the coefficients.



Figure 10: Comparison between the numerical obtained continued homoclinic bifurcation curve emanating from the Bogdanov-Takens point in (8) with a = b = 1 and the parameter approximations with orders ranging from 10 to 200. For $\beta_1 \leq -8$ the approximation starts diverging. This indicates that the radius of convergence in the perturbation parameter ϵ is less than or equal to $\sqrt[4]{2}$. Note that for $\beta_1 > -8$ the tenth order is already indistinguishable from the numerical obtained parameters.

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