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Height-diameter models: a comprehensive review with new insights on relationships to generalized linear models and differential equations

BG. DA SILVA[a], CGB. DEMÉTRIO[a], RA. SERMARINI[a], G. MOLENBERGHS[b], G. VERBEKE[c], A. BEHLING[d], ERG. MARQUES[e], Y. ACCIOLY[e], MA. FIGURA[e]

[a] Luiz de Queiroz College of Agriculture – ESALQ, Av. Pádua Dias, 11, Piracicaba, Brazil

[b] Universiteit Hasselt, Agoralaan gebouw D, 3590 Diepenbeek, Bélgica

[c] Katholieke Universiteit Leuven, Oude Markt 13, 3000 Leuven, Bélgica

[d] Federal University of Paraná, Rua XV de Novembro, 1299, Centro, Curitiba, Brazil

[e] Fazenda Monte Alegre, S/Nº - Harmonia, Telêmaco Borba, Brazil

E-mail: brenogsilva@usp.br, clarice.demetrio@usp.br, ralcarde@usp.br, geert.molenberghs@uhasselt.be, geert.verbeke@kuleuven.be, alexandre.behling@yahoo.com.br, eduardo.girardi@klabin.com.br, yuri.accioly@klabin.com.br, mafigura@klabin.com.br

HIGHLIGHTS

- Technological advances expand height-diameter models with new variables.
- Literature review guides forest biometricians in choosing increment models.
- Classification of height models into four main groups.
- Data transformation unnecessary for generalized linear models.
- Model generation discussed for various species and formulated via differential equations.

SUMMARY

Forest scientists widely use regression models, particularly for height-diameter modeling. These models offer several benefits for estimating height in homogeneous or non-homogeneous stands. The original models for height estimation based on diameter at breast height have been extended to include other variables, thanks to technological advancements. The purpose of this article was to provide a literature review using the methodology outlined by Cervo and Bervian (2011), providing helpful information to forest biometricians in selecting a height-diameter model that utilizes historical data. The models could be classified into four main groups and extended to include other covariates besides diameter at breast height. Many of the models used data transformation but we showed that except for one group (nonlinear models), all other models can be considered a generalized linear model, with corresponding linear predictors and link functions. This paper also discusses the generation process of these models, the species to which they are commonly applied, and how they can be created using general ordinary differential equations.

Keywords: height-diameter relationship, height estimation, forestry data models, forest biometrics, sustainable forest practices.

1. Introduction

In the 17th century, wood was scarce throughout Europe, which led to the development of techniques to improve its utilization. One of the pioneers in sustainable forest practices was Von Carlowitz (1732), as cited by Ziello *et al.* (2012), who shared information about the development of trees over time. Paulsen (1795) suggested using yield tables for many species to study tree growth but the first growth curves for individual trees were created by Späth (1797), as mentioned by Scolforo (1998). Their goal was to increase the flexibility of analyzing measurement data in forest regions.

To ensure sustainable forest practices necessitates good forest measurement management systems that refer to techniques used to obtain accurate and reliable information about forest characteristics and attributes. Scott (1996) cites Beckmann's (1777) early proposal for a forest measurement system, which involved marking trees with nails of the same color based on their diameter class. By counting the nails of each color, estimates of the amount of wood in the region could be obtained. This contribution led to the development of estimation, prediction, and control methods associated with forest production and planning in the early 1900s (Batista, 2014).

One example of a forest measurement management system is the method of using permanent plots to study tree growth by setting aside specific areas for continuous monitoring of forest composition, structure, growth, and dynamics over time, collecting data on individual trees within a plot, without accounting for other potential influencing factors. This practice has its roots in the work of Hossfeld (1822), Hundenhagen (1826), and Huber (1828), as cited by Scolforo (1998). In North America, Spurr (1951) noted that between 1896 and 1898, Henry Solon Graves and Gifford Pinchot presented findings related to the growth ring method at cutting height.

In these permanent plot studies, dendrometric variables such as total height (h) and diameter at breast height (D), measured at 1.30 m, are essential. These variables allow researchers to quantify and describe the growth, volume, biomass, and other characteristics of trees. Moreover, they provide a basis for classifying forest stands by productivity, supporting the interpretation of historical data and enabling long-term comparisons within the study area (Jesus *et al*. 2015). However, it can be difficult to accurately measure certain variables in the field. For instance, when measuring tree height, visualizing the tree crowns can be time-consuming and challenging. This can affect the precision of measurements and increase the cost of forest inventories (Buford 1986, Dantas *et al.* 2020). Alternatively, tree height can be estimated based on diameters, which are easier and quicker to measure (Schmidt 1977, Moreira *et al.* 2015). The use of height-diameter models was first introduced by Trorey (1932) and was later applied in studies by Huxley and Teissier (1936), as well as by Prodan (1944). One common method for estimating tree heights involves measuring the heights of a sample of the trees in the area and the diameters of all trees. Using the height-diameter pairs obtained from these measurements, it is possible to estimate a mathematical equation to calculate the heights of the remaining trees (Moreira *et al.* 2015).

Chapman and Meyer (1949) and Bruce and Schumacher (1950) noted that tree height can be affected by more than just its diameter. Factors such as age, species, canopy size, site quality, density, and sociological position (which involves the classification and distribution of tree species based on their ecological, behavioral, and structural characteristics) also play a role. Additionally, silvicultural practices, which involve a range of forest management activities and techniques aimed at promoting healthy and sustainable forest growth while maximizing their economic, social, and environmental benefits, can have a significant impact.

As discussed, different factors can cause significant variations in tree heights with the same diameter, affecting the intercept and slope of growth curves. This variation in growth curves reflects the stand's growth and development stage: a steep curve indicates a young population still in development, while a flat curve suggests a decrease in growth rate associated with older populations (Bartoszeck *et al*. 2004, Figueiredo Filho *et al*. 2015).

Understanding and evaluating the characteristics of forest stand structure and developmental stages is essential for assessing their ecological and economic potential. This involves conducting a comprehensive statistical analysis by selecting an appropriate model to examine growth patterns in height and diameter over time. Additionally, it is important to consider the average height ratio across diameter classes at a specific age, the mortality ratio, and the changes in dominant height over time (Fontes *et al.* 2003, Sanquetta *et al.* 2014). Dominant height (Hd) refers to the average height of the tallest trees in a specific forest area. Another variable that has been used for characterizing the diameter distribution is the quadratic mean diameter (dg), obtained by averaging the squares of the individual diameters of all trees in the plot, The main purpose of this paper is to conduct a comprehensive review of articles on height-diameter modeling, with a focus on the properties of different functional forms. As a result, it will be easier for researchers to build a model that best suits their particular data and purposes. Several models, such as linear and non-linear ones, have been created to consider various factors that affect height-diameter curves. Generic equations have been examined alongside existing models to establish a reliable framework. (Mendes *et al.* 2006, de Oliveira *et al.* 2011, Sáez-Cano *et al.* 2021).

In the development of forest models, ordinary differential equations (ODE) have great potential for examining how different curves of a function of height, , as a function of diameter, , behave. They offer biological interpretability and at the same time enable comparison between models (Garcia 1983, Hamlin and Leary 1987, Narmontas *et al.* 2020). It has been observed that there is a lack of research connecting an ODE to a generalization that can lead to various equations commonly used for different species. Additionally, we highlight the importance and interpretability of the first and second derivatives of the ODEs in this study to further clarify the differences and similarities between the models that have been used and to place them in a broad, elegant, and flexible modeling framework.

A further key aim of this manuscript is to establish a connection between most height-diameter models and their equivalent representations in the framework of Generalized Linear Models (GLM). This helps to overcome the limitations often encountered in traditional statistical analyses, posed by the typical assumptions of linearity, independence, homogeneity of variances, and normality, which are not always met in forestry measurements and therefore should be relaxed. Traditional approaches often resort to data transformation to meet these assumptions, but such an approach is not always effective or appropriate and may complicate interpretation.

Fortunately, GLM presents itself as a more elegant and unified alternative, but it is underused in forestry research. GLMs provide a flexible framework that can handle various types of data and distributions, making them more suitable for the complexity of forestry data. In a GLM, the mean is transformed by a link function, instead of transforming the response itself. The two opposing methods of transformation can lead to quite different results. Transforming the mean often allows for easier interpretation, especially because mean parameters remain on the same scale as the measured responses. While Harrison *et al.* (1986), Krumland and Wensel (1988) and Cimini and Salvati (2011) make use of it, to our knowledge there is no literature presenting its relationships with height-diameter models.

The paper is structured as follows. The Materials and Methods section outlines the process of selecting height-diameter models from the literature for comparison and grouping using ODEs and GLMs. It also includes a description of a case study from a black wattle experiment. The Results section focuses on the application of these models to the black wattle data and resulting inferences, offering a detailed analysis of their effectiveness. Finally, the Discussion and Conclusion section offers a broader perspective, presenting general observations associated with the theory of GLMs and its advantages associated with the results, and outlining the implications for future research in the area.

2. Materials and Methods

In the initial phase of the study, we followed the methodology proposed by Cervo and Bervian (2011) to choose scientific articles published between January 1990 and May 2022. The articles were sourced from online search engines like Google Scholar and Web of Science, as well as from scientific journals such as Southern Forests, Journal of Forest Science, Forest Science, Annals of Forest Science, Forest Ecology and Management, Forestry, Silva Fennica, Canadian Journal of Forest Research, Ecological Indicators, and Ecological Modeling. These journals were selected because of their high impact factor and their inclusion in a list of important papers presented by Gregoire (2012). In addition, to comprehend the reasoning behind certain models, texts referenced by the same author in journals, dissertations, and/or theses were also reviewed.

To identify height-diameter models, we searched for keywords such as "modelos hipsométricos", "relação hipsométrica", "curva hipsométrica", "height-diameter model", "height-diameter equation", "height-diameter relationship", "height-diameter models", "hypsometric models", "hypsometric relationship", and "hypsometric curve". We included some Portuguese terms, as Brazil is a leading country in forest plantation, with high annual wood production per area and a short cycle, according to the 2021 Annual Report of the Brazilian Tree Industry.

The models were then grouped according to some properties of the different functional forms using ODE, when possible. The first derivative of with respect to represents the height-diameter model given a current diameter . It provides information about how tree height is changing as increases, becoming equally interpretable as other explanatory variables are incorporated into the model. It allows forest managers to obtain information about competition between trees, consequently optimizing wood production.

The second derivative of with respect to is useful for evaluating the concavity or convexity of the curve generated by the equation. A positive second derivative indicates that the height increment is increasing as increases, while a negative second derivative indicates that the height increment is decreasing.

Additionally, the models were linked to the equivalent GLM. A GLM, as formulated by Nelder and Wedderburn (1972), allows the response variable to assume a distribution from an exponential family of distributions, with mean and variance , where is a constant dispersion parameter and is called the variance function. The variance function will allow for a specific type of heterogeneity of variance proportional to given by, for example, , for normal, gamma, and inverse Gaussian distributions, respectively (Cordeiro *et al.* 2024). The mean is related to the covariates by a linear predictor, , where is a design matrix and is a parameter vector, through a link function, . The parameter vector is estimated by the maximum likelihood method using an iteratively weighted least squares algorithm (IWLS).

For a classical linear model, the (transformed) response variable has a normal distribution with mean and constant variance, that is, with , and identity link, . With Gaussian outcomes, the mean and variance are functionally independent, and the mean is equal to the linear predictor. In other generalized linear models, a transformation of the mean equals the linear prediction, by a link function, while the variance follows as a specific function of the mean. The parameter vector is estimated using the ordinary least squares algorithm, a particular case of IWLS.

To illustrate the fitting of the different models, we use a data set from an experiment conducted in stands of black wattle (*Acacia mearnsii* De Wild.), 10.75 years old, in the municipality of Encruzilhada do Sul, state of Rio Grande do Sul, Brazil, in June and July 2014. This region is characterized by climate, relief type, and soil in Mochiutti (2007). One circular plot with 400 was randomly allocated in the stand, where refers to a specific area or plot of forest that is relatively uniform in species composition, age structure, and other characteristics, typically used in forestry management (Batista, 2014). All trees in the plot were measured, using a dendrometric tape for diameter at breast height and a Vertex hypsometry for total height.

The motivating data were analyzed using R software version 4.1.2 (R Core Team 2021). The GLMs were fitted using the `glm` function, while the nonlinear models were fitted with the `nls` function. The model checking was performed using half-normal plots with simulated envelopes, generated with the `hnp` package (Moral *et al.* 2017).

3. Results

Out of the 347 scientific texts that were found, 94 (27.08%) were chosen for the Interpretive Reading and Textual Commentary stage. The analysis showed that 41 height-diameter models were commonly used as shown in Table 1, with an adapted notation for easy comparison. All the models can be expressed as

(1)

where is a function (transformation) of , is a function of and other covariates, is a vector of parameters, and . To ensure biological accuracy, they are valid only on the first quadrant of the cartesian plane, where values for height and diameter are non-negative. According to their properties, they can be classified into four main groups of models that can be extended to include other covariates besides , as shown in the following.

3.1 Group 1 – Linear models

Studies in forest science to describe the relationship between height and diameter began with the pioneering work of Trorey (1932) who proposed a quadratic model [M1] of on , as shown in Table 1, to represent data from observational studies of several tree species located in different areas of British Columbia. He separated the data by site and forest age to obtain empirical curves using the mathematical interpretation of the quadratic equation's constants. Note that for this parabolic model with a maximum to have a biological interpretation. The maximum of the curve is attained at a diameter given by and a height of . Trorey assumed a constant maximum height from this point forward, as diameters larger than would result in decreasing height values. This could be a motivation for asymptotic models.

Assmann (1936), as cited by Scolforo (1993), proposed models [M2] and [M3], which are linear functions of and , respectively, while Henriksen (1950), as cited by Scolforo (1998), proposed model [M6], as an alternative model to [M1]. The models [M2] and [M3] have an asymptote equal to while model [M6] is not limited.

Models [M1], [M2], [M3], and [M6] from Table 1 is represented by equation (1) assuming that and is a linear function on , , , and , respectively. Differentiating with respect to , we get the height-diameter model as

where and are parameters representing the intercept and slope of a straight line and , respectively. This shows that the increment in can be proportional to a linear function of the diameter, the inverse of the squared diameter, the inverse of cubed diameter, or the inverse of diameter. Equivalent GLMs for these four models can be identified with response variable having a normal distribution, different linear predictors, and an identity link function.

The empirical models [M40] and [M41], proposed by de Azevedo *et al.* (2011), have (square root transformation) with that are linear functions in and , respectively, by equation (1). Alternatively, equivalent GLMs for these two models can be identified with response variable having a normal distribution, a square root link function, and linear predictors as linear functions of and , respectively.

Models [M1], [M2], [M3], [M6], [M40] and [M41] were fitted to the black wattle data to illustrate the elements of the curves (Figure 1). We can see that [M3] is not adequate for estimating the asymptote (it is smaller than the extreme values), arguably because the linear component is not included in the predictor. Combining [M2] and [M3], which is equivalent to including the linear term in the predictor for [M3], the estimate of the asymptote is more plausible. To further explore the difference between these models, although extrapolation should be done with caution, we expanded the range of values. As shown in Figure 2, except for the model whose linear predictor is the combination of models [M2] and [M3], the other models do not present curve differences.

3.2 Group 2 – Inverse polynomials

As alternative models to [M1], asymptotic models were proposed: [M4] by Näslund (1937), [M8] and [M9] by Petterson (1955), [M16] by Prodan (1965), and [M27] and [M28] by Azevedo *et al.* (1999). Näslund (1937) pioneered using the method of least squares to estimate the parameters, which, until then, was done approximately. Note that [M4] and [M16] are non-linear models while [M8], [M9], [M27], and [M28] use data transformation for as shown in Table 1.

Models [M4], [M8], [M9], [M16], [M27], and [M28] can be considered particular cases of the so-called inverse polynomials by Nelder (1966), which have better properties than standard polynomials, such as being generally nonnegative, bounded, and have a second-order form that has no built-in symmetry. From Table 1, we can get a general model as

or, equivalently,

Considering equation (3), for we have models [M4] and [M8], for we have model [M9] and for , we have models [M16], [M27] and [M28]. In this notation, the models will have asymptotes equal to , while, using Table 1 notation, the asymptotes take values , , , , , and , respectively, for models [M4], [M8], [M9], [M16], [M27], and [M28]. Equivalent GLMs for these six models can be identified with response variable having a normal distribution, different linear predictors, and an inverse link function.

It is also possible to see that model [M39], proposed by de Azevedo *et al.* (2011) to estimate the height of trees using a completely randomized design with four repetitions, is a type of inverse polynomial:

Models [M4], [M8], [M9], [M16], [M27], and [M28] were fitted to the black wattle data to illustrate the aspects of the curves (Figure 3), showing small differences between the curves for the transformed data model and the equivalent GLM in the observed data range, but the estimated asymptotes can be very different.

Like before, to get a feel for differences between the models, we showed their behavior over an expanded range (Figure 4). Observe that the transformed data model and the equivalent GLMs can be very different, justifying the differences in the estimation for the asymptotes.

3.3 Group 3 – Nonlinear linearizable models

The models [M7] proposed by Stoffels and Van Soest (1953), cited by Jesus *et al.* (2015), [M18] by Curtis (1967), and [M31] by de Barros *et al.* (2002) are different parametrizations of the same model and they can be grouped with models [M17] by Curtis (1967), [M22] by Silva (1980), and [M29], [M30] by de Barros *et al.* (2002), using a logarithmic transformation of . Model [M22] was used by Silva (1980) to analyze data from trees of different species in the Tapajós National Forest – Brazil, while models [M29] and [M31] were used by de Barros (2002) to estimate the height of trees of the species *Pinus oocarpa*. These can be viewed as particular cases of a general nonlinear linearizable regression model, as follows.

A nonlinear regression model is one in which the parameters appear nonlinearly (Ratkowsky, 1983). However, some of them can be linearized as, for example, the following model

used by Ali and Schaeffer (1987), thereby extending the Wood (1967) model for lactation curves.

Taking the logarithm of (4) we get

(5)

Considering (5) and assuming , and we get [M7] for and , [M17] for and , [M18] for and , [M22] for , and , [M29] for , and , and [M31] for and . Assuming , and we obtain [M30] for , and .

Note that the general expression of the ODE for the models represented by (4) is

giving the ODE for the models [M7], [M18] and [M31] for , [M22] for , [M29] for and [M30] for .

Equivalent GLMs for these six models can be identified with response variable having a normal distribution, different linear predictors, and a logarithmic link function.

Models [M7], [M17], [M18], [M22], [M29], [M30], and [M31] were fitted to the black wattle data to illustrate the elements of the curves (Figure 5), showing almost no difference between the transformed data model and a GLM with log-link, over the observed data range.

Also here, we examine the behavior of the models over an extended range. The transformed data model and the equivalent GLMs can be slightly different and [M30] may not represent the data generating process. Also, for the asymptotic ones, the estimated asymptotes can be smaller than some of the larger extreme values.

3.4 Group 4 – Nonlinear models

Other types of nonlinear models are the growth models that may or may not be sigmoidal. In the context of height-diameter modeling, the works of Richards (1959) and Turnbull (1963), cited by Crechi (1996) and Brito *et al.* (2007), pioneered the use of the growth models to explain the relation between height and diameter.

Given that there is an upperlimit (asymptote) for the average height of a tree as a function of D, we can assume that the height-diameter model is proportional to (complement to reach the theoretical maximum height), meaning that the velocity of growth decreases as (Law of Diminishing Increments) (Mitscherlich, 1909; Sorensen, 1983), that is

where is the average growth rate.

Solving the differential equation (7), we get

where C is an integration constant. Given that and . Therefore, we have the mathematical equation of [M12], used by Mitscherlich (1909) to study the effect of fertilization on crop productivity.

Noting that as the measurements are made at m, a minor reparameterization of [M12] gives model [M5] proposed by Meyer (1940) to analyze a data set of hemlock trees from a virgin forest in northern Pennsylvania.

Model [M13] was proposed by Von Bertalanffy (1957) for the study of allometric relationships. Taking into account the allocation of metabolic energy between the growth and the sustenance of an animal, he assumed that the growth of animals follows a process of synthesis (anabolism -) and degradation (catabolism -) and that the anabolic ratio is proportional to the body surface area, while the catabolic ratio is proportional to the biomass volume. Thus, based on this model the tree height-diameter model is given by

with a solution given by the mathematical equation of [M13] where . In addition, the inflection point of the curve is

Model [M21] was proposed by Bailey and Clutter (1974), using (logarithmic transformation), for the estimation of height curves of trees of the species *Pinus radiata* with as a covariate. It has, as particular cases, the models [M17] and [M18] for = 1 and → 0, respectively.

When studying individual growth, it is typical to observe sigmoidal curves. Such a curve starts with a low slope, increases the slope to an inflection point, and then levels off as it approaches an asymptote. Growth is fastest around the inflection point, and depending on the location of this point, sigmoidal curves will be either symmetric or asymmetric. Examples of models with sigmoidal curves are logistic [M10], Gompertz [M11], Mitcherlich [M12], Richards [M14], Chapman and Richards [M15], among others (Ratkowsky, 1983).

The logistic model [M10] was formulated by Verhulst (1838) as a population growth model with a more realistic biological interpretation than the exponential model of Malthus (1872). To obtain the mathematical equation for this model, it is assumed that the height-diameter model is proportional to (actual height) and (complement to reach the maximum theoretical height), that is,

with solution

where C is an integration constant, and . It can be seen that is the asymptote, is related to the model's intercept and is the average increment height. In addition, the inflection point of the curve is , showing that the logistic model is symmetric, which is not always realistic.

The asymmetric sigmoidal model proposed by Gompertz (1825) is obtained by assuming that the height-diameter model is proportional to and , that is,

with solution after reparameterization of the model [M11] given by

where C is an integration constant, = > 0, = > 0 and = > 0, the average increment height. It is noted that is the asymptote, is related to the model's intercept. In addition, the inflection point of the curve is .

Models [M14] and [M15] were proposed by Richards (1959) and Chapman (1961), respectively, as extensions of Von Bertalanffy's model [M13], for the analysis of fish and plant growth data. Pienaar and Turnbull (1973) generalized the Chapman-Richards model, with a height-diameter model:

wherein (8), is the allometric constant, with values = , respectively, for Von Bertalanffy, Mitscherlich, Gompertz, and Logistic models.

Another non-linear growth model proposed by Weibull (1951), [M20], was used by Bailey and Dell (1973), cited by Scolforo (1998), for the estimation of tree heights with different diametric classes in equi- and inequietric forests. In this model, the parameter represents the asymptote, and and are the shape and scale parameters of the curve, respectively. Models [M5], [M10], [M11], [M12], [M13], [M14], [M15], [M20] and [M21] were fitted to the black wattle data to illustrate the aspects of the curves (Figure 7). Here too we see that the estimated asymptotes can be smaller than some of the larger extreme values, except for models [M5], [M12] and [M21].

3.5 Models incorporating other variables

Other factors besides the diameter can affect the height of a tree, such as age, environmental variables, etc. The site quality in a forest is associated with the potential of wood production. The age of a tree is used in evaluating the growth and productivity of a site. Age is also used as a tool for silvicultural practices, in determining the present and future growth of the forest and in management plan decisions.

To capture the heterogeneous effect of ages, Curtis (1967), using (logarithmic transformation), was the first to include other independent variables, like model [M19], extending models [M17] and [M18], while Blanco (1983), cited by Figueiredo Filho *et al.* (2015), proposed model [M23], extending model [M16], for analysis of *Pinus elliotti* data from the National Forest of Três Barras – Brazil.

Model [35], proposed by de Barros *et al.* (2002), has the response transformed variable as a quadratic function of added by an interaction of by .

The most common index used for the site classification is the mean of the dominant height (Hd) per stand for a given age. Campos *et al.* (1986), cited by de Miranda *et al.* (2014), used as a covariate in model [M24] as an extension of model [M17], to estimate the total height of trees of *Eucalyptus grandis*. Models [M35], [M36] and [M37] used by de Barros *et al.* (2002) are extensions of [M1] by adding as a covariate and also a quadratic function of for the later. Model [M38] of Nogueira (2003), cited by de Miranda *et al.* (2014), was fitted to

Eucalyptus sp. and *Tectona grandis* data, using (logarithmic transformation), as an extension of model [M17] taking into account and .

Scolforo (1998) proposed, as extensions of model [M24], model [M25] by incorporating as a covariate in the model and model [M26] with also an interaction of by (inverse of age).

Note that models [M19], [M24], [M25] and [M26] can be fitted using a GLM with response variable having a normal distribution, different linear predictors, and a logarithmic link function.

Another way to take into account proposed by de Barros *et al.* (2002) to estimate the height of trees of the species *Pinus oocarpa*, was to use as the response variable in models [M32], [M33] and [M34] as extensions of models [M6], [M1] and, once again [M1], respectively.

4. Discussion and Conclusion

It is noteworthy that there are several other regression models in the literature. However, as the objective of this study was to present the most commonly used models to describe the height-diameter relationship, many studies were excluded from the research. From a literature review, 41 height-diameter models were considered (Table 1). They could be classified into four main groups and extended to include other covariates besides . Many of the models used data transformations but we showed that, except for group 4 (nonlinear models), all other models have an equivalent GLM, with adequate linear predictors and link functions (Table 1).

These models have been widely used, as illustrated in Table 2, for different species under varying conditions to capture the effects associated with factors that interfere with tree development, showing positive and negative aspects. For the simple case study presented here, all models fitted well, as verified by half-normal plots (Supp Figure 1, Supp Figure 2, Supp Figure 3, Supp Figure 4, Supp Figure 5, Supp Figure 6, Supp Figure 7), and showing small differences between the curves for a transformed data model and a GLM in the observed data range, but the estimated asymptotes can be very different. Other types of GLM, involving different distributions are under development with applications in more complicated data sets.

The generalization of ordinary differential equations provides a deeper understanding of the principles associated with a series of equations studied, since by identifying common patterns and relationships between different models, it is possible to extract general information and insights, allowing the formulation of more comprehensive theories (Simmons, 2016).

Observing the models brought forward here, it is noted that some of them can be obtained through a generic ODE, assuming that the derivative / is proportional to a function of the diameter, like equation (2) for group 1, equation (6) for group 3 and equation (8) for group 4. At this point it is fundamental to highlight the importance of the biological knowledge about the factors that affect the development of a tree and how this happens, i.e, the models were grouped according to some properties of the different functional forms using ODE, when possible.

It was demonstrated that with some generic ODEs, it is possible to obtain a wide range of existing models, that is, it saves time and resources, as it avoids the need to derive new equations specific to each case. The interpretation of the first and second derivatives of the height-diameter model equations (ODE of first and order) help to understand the relation between and and plays important roles in evaluating forest settlement, growth, and management.

The GLM approach brings a new perspective to the analysis of forest data, offering not only greater flexibility in modeling, but also significant advantages in terms of computational efficiency. When comparing GLMs with classical linear models, one of the main advantages that emerges is the superior speed of the GLM estimation algorithm. The speed of analysis provided by GLMs allows for a more agile and efficient interpretation of data, a crucial aspect for contemporary research and management practices. Furthermore, GLMs stand out for their ability to accommodate a greater variety of data structures and variability patterns. This feature is especially valuable in the forestry field, where tree growth patterns and environmental interactions create scenarios that often defy the assumptions of conventional linear models. Rather than forcing a transformation of data to fit a model, GLMs adapt the model to more faithfully reflect the nature of the data, preserving its integrity and improving the interpretation of results.

The incorporation of GLMs into height-diameter modeling represents a significant advance in the way we approach data analysis in forestry studies. With this approach, we have the opportunity to more deeply explore tree growth dynamics and the environmental factors that influence them, providing valuable insights for forest management and conservation. This study therefore not only reviews existing modeling techniques but also demonstrates the potential of GLMs as a powerful and versatile tool in forestry research. With their efficiency and flexibility, GLMs are positioned as a preferred choice for future analyses, helping researchers draw more accurate and meaningful conclusions from their data.

Finally, with this research, we identified the factors that led the authors to develop the models proposed to express the height-diameter relationship and why a particular model can be employed. It is recommended that before choosing a model, one should verify for which species that model had already been well fitted, which are its mathematical properties and relationships with other models that already exist.

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Conflict of interest statement

None declared.

Data availability statement

The data underlying this article will be shared on reasonable request to the corresponding author.

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