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Local Blaschke–Kakutani ellipsoid characterization and Banach's isometric subspaces problem

Sergei Ivanov [a,b], Daniil Mamaev [a,c,*], Anya Nordskova [a,d]

[a] *St. Petersburg Department of Steklov Mathematical Institute, Fontanka 27, St. Petersburg 191023, Russia*
[b] *St. Petersburg State University, Universitetskaya nab. 7-9, St. Petersburg 199034, Russia*
[c] *University College London, Gower Street, London WC1E 6BT, United Kingdom*
[d] *Universiteit Hasselt, Agoralaan, gebouw D, 3590 Diepenbeek, Belgium*

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ABSTRACT

We prove the following local version of Blaschke–Kakutani's characterization of ellipsoids: Let V be a finite-dimensional real vector space, $B \subset V$ a convex body with 0 in its interior, and $2 \leq k < \dim V$ an integer. Suppose that the body B is contained in a cylinder based on the cross-section $B \cap X$ for every k-plane X from a connected open set of linear k-planes in V. Then in the region of V swept by these k-planes B coincides with either an ellipsoid, or a cylinder over an ellipsoid, or a cylinder over a k-dimensional base.

For $k = 2$ and $k = 3$ we obtain as a corollary a local solution to Banach's isometric subspaces problem: If all cross-sections of B by k-planes from a connected open set are linearly equivalent, then the same conclusion as above holds.

* Corresponding author at: University College London, Gower Street, London WC1E 6BT, United Kingdom.

E-mail addresses: svivanov@pdmi.ras.ru (S. Ivanov), daniil.mamaev.21@ucl.ac.uk (D. Mamaev), anya.nordskova@uhasselt.be (A. Nordskova).

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1. Introduction

A classical result by Kakutani ([11], see also [8, Theorem 12.5]) characterizes Euclidean spaces among finite-dimensional normed spaces as follows:

Let $V = (V, \|\cdot\|)$ be an n-dimensional normed space and $2 \leq k < n$. Suppose that for every k-dimensional linear subspace X of V, there exists a linear projector $P_X : V \to X$ onto X with unit operator norm, i.e. such that $\|P_X(v)\| \leq \|v\|$ for all $v \in V$. Then $\|\cdot\|$ is a Euclidean (i.e., an inner product) norm.

This fact is also known as the Blaschke–Kakutani characterization; it can be seen as the dual form of Blaschke's characterization of ellipsoids via planarity of shadow boundaries. The projector property always holds for 1-dimensional subspaces (by the Hahn–Banach theorem), which explains the condition $k \geq 2$.

In this paper we characterize norms for which the same assumption is satisfied locally, that is for subspaces X ranging over an open subset of the respective Grassmannian. The answer is that the local structure of the norm near these subspaces is either Euclidean or cylindrical, see Theorem 1.2 below for the precise formulation.

Our main motivation and application is a local version of a low-dimensional solution of Banach's problem about normed spaces where all subspaces of a fixed dimension are isometric. Using the approach from [10] together with Theorem 1.2 we show that the answer to the local version of Banach's problem for $k = 2, 3$ is the same as in Theorem 1.2, see Theorem 1.4.

In the case when the norm is smooth and strictly convex, the local Blaschke–Kakutani characterization was obtained by Calvert [4]. The cylindrical case does not appear in [4] because of the strict convexity assumption. The local Banach's isometric subspaces problem for $k = 2$ and smooth strictly convex norms was solved in [9] as a part of the proof of a Finsler geometry result. Note that the Blaschke–Kakutani characterization easily reduces to the case $k = 2$ while Banach's problem does not.

Definitions and formulations

We state our results in convex geometry terms rather than in terms of norms. As usual, a norm on a vector space V is represented by its unit ball B, which is a convex body in V. Since a part of our motivation comes from Finsler geometry, we do not assume that B is symmetric and hence consider norms that are not necessarily symmetric ("Minkowski norms", see section 2). The existence of a norm non-increasing projector to a linear subspace $X \subset V$ is equivalent to the property that B is contained in a cylinder with base $B \cap X$, see Definition 1.1 and Lemma 2.2 below.

By $\mathrm{Gr}_k V$ we denote the Grassmannian of k-dimensional linear subspaces of a vector space V. Two linear subspaces $X, Y \subset V$ are called *complementary* if $V = X + Y$ and $X \cap Y = 0$. A *convex body* is a compact convex set with a nonempty interior.

Definition 1.1. Let V be a real n-dimensional vector space and $1 \le k < n$ an integer. A set $C \subset V$ is called a *k-cylinder* if it can be represented in the form

$$C = K + Y$$

where K is a k-dimensional convex body in a linear subspace $X \in \mathrm{Gr}_k V$, and Y is a linear subspace complementary to X.

The set K is referred to as a *base* and Y as a *generatrix* of C. If the value of k is clear from context, we omit it and call k-cylinders simply *cylinders*.

Note that the generatrix of a cylinder is unique but a base is not. In fact, if C is a k-cylinder then for every linear subspace $X' \in \mathrm{Gr}_k V$ such that the set $K' = C \cap X'$ is compact, K' is a base of C. (The compactness of K' is equivalent to $X' \cap Y = 0$ where Y is the generatrix).

Our first result is the following theorem.

Theorem 1.2. *Let V be a real n-dimensional vector space, $B \subset V$ a convex body containing 0 in its interior, $2 \le k < n$ an integer, and $\mathcal{U} \subset \mathrm{Gr}_k V$ a nonempty connected open set.*

Suppose that for every $X \in \mathcal{U}$ the body B is contained in a k-cylinder with base $B \cap X$. Then there exists a set $B' \subset V$ such that

$$B \cap X = B' \cap X \quad \text{for all } X \in \mathcal{U} \tag{1.1}$$

and at least one of the following holds:

(1) *B' is a k-cylinder;*
(2) *$B' = \{v \in V : Q(v) \le 1\}$ for some nonnegative definite quadratic form Q on V.*

The case (1) in Theorem 1.2 occurs if all k-cylinders from the assumption have the same generatrix. In (2), the set B' may be an ellipsoid or, if Q is degenerate, a cylinder over an m-dimensional ellipsoid for some $m \ge k$. The cylindrical and degenerate cases are unavoidable in the local setting. In fact, any B satisfying the conclusion of the theorem satisfies its assumption, see Lemma 2.5.

Remark 1.3. If $\mathcal{U}$ in Theorem 1.2 is the entire Grassmannian $\mathrm{Gr}_k V$ then (1.1) implies that $B' = B$. Then, since B is compact, the cylindrical and degenerate cases are ruled out and the only remaining option is a sublevel set of a positive definite quadratic form. Thus Theorem 1.2 implies the original Blaschke–Kakutani characterization and moreover generalizes it to non-symmetric norms.

We also note that the cylindrical and degenerate cases cannot occur if B is strictly convex or, more generally, if the union of the subspaces from $\mathcal{U}$ contains an extreme point of B. (Recall that an *extreme point* of B is a point $p \in B$ such that $B \setminus \{p\}$ is convex).

The proof of Theorem 1.2 is given in sections 5 and 6 after technical preparations in sections 2–4. In section 5 we handle the case $n = 3$ and in section 6 the proof is finished by induction on dimension.

The ellipsoid and cylinder cases in dimension 3 are separated by whether the correspondence between crossing planes and cylinders containing the body is 1-to-1 or not. In the latter case we deduce the result via convex geometry arguments, and in the former case the proof is based on a local version of the fundamental theorem of projective geometry. This is similar to proofs in [4] and [9].

Banach's isometric subspaces problem

In 1932, Banach [2] posed the following problem:

Let $V = (V, \|\cdot\|)$ be a normed vector space and $2 \le k < \dim V$ an integer. Suppose that all k-dimensional linear subspaces of V are isometric. Is $\|\cdot\|$ necessarily an inner product norm?

The problem translates into the language of convex geometry as follows: Consider a convex body $B \subset V$ (the unit ball of a norm) and suppose that all cross-sections of B by k-dimensional linear subspaces are *linearly equivalent*, that is, for every $X, Y \in \mathrm{Gr}_k V$ there exists a linear map $L \colon X \to Y$ such that $L(B \cap X) = B \cap Y$. The question is whether such a body is necessarily a centered ellipsoid.

It is answered affirmatively in some dimensions and remains open in others. For a long time the only known result was the solution for $k = 2$ by Auerbach, Mazur, and Ulam [1]. Then Dvoretzky [6] solved the problem for infinite-dimensional spaces and Gromov [7] settled the case of even k and the case $\dim V \ge k + 2$ for all k. Recently the problem was solved for $k \equiv 1 \bmod 4$ except $k = 133$ by Bor, Hernández Lamoneda, Jiménez-Desantiago, and Montejano [3] and for $k = 3$ by the authors [10].

The proofs in [1,7,3] rely on algebraic topology of Grassmannians to find obstructions to the existence of certain families of linear equivalences. In contrast, the proof for $k = 3$ in [10] is based on differential geometric analysis in a neighborhood of a single cross-section. This suggests that it makes sense to consider a local version of the problem where the linear equivalence is assumed only for a small open set of cross-sections. Such a local result was obtained in [9] for $k = 2$, $n = 3$ and a smooth strictly convex body; the conclusion is that the respective part of the body coincides with an ellipsoid.

In this paper we extend the results of [9] and [10] and solve the local version of Banach's problem for $k = 2$ and $k = 3$. Like in the case of Theorem 1.2, the problem admits locally cylindrical solutions in addition to locally ellipsoidal ones.

Theorem 1.4. *Let V be an n-dimensional real vector space, $B \subset V$ a convex body containing 0 in its interior, $k \in \{2, 3\}$, and $\mathcal{U} \subset \mathrm{Gr}_k V$ a nonempty connected open set.*

Suppose that for every $X_1, X_2 \subset \mathcal{U}$ the cross-sections $B \cap X_1$ and $B \cap X_2$ are linearly equivalent.

Then the same conclusion as in Theorem 1.2 holds, namely there exists $B' \subset V$ such that $B \cap X = B' \cap X$ for all $X \in \mathcal{U}$ and B is either a k-cylinder or a sublevel set of a nonnegative definite quadratic form (or both).

The proof of Theorem 1.4 is a combination of Theorem 1.2 and the results of [10]. In fact, the key propositions in [10] show that the assumptions of Theorem 1.4 imply those of Theorem 1.2. See section 7 for details.

Remark 1.5. A related question is whether a convex body in $\mathbb{R}^n$ is uniquely determined, up to a symmetry or homothety, by the congruence or affine types of its k-dimensional cross-sections through the origin. For congruence types this question goes back to Nakajima [12] and Süss [14]. In general the answer is negative as shown by Zhang [15]. However in the symmetric case an application of the spherical Radon transform shows that the answer is affirmative, moreover a symmetric body is uniquely determined by the areas of its cross-sections.

This type of questions can be localized as well, for example, one may ask the following: *Let B_1 and B_2 be origin symmetric bodies in $\mathbb{R}^n$, $2 \le k < n$, and $\mathcal{U} \subset \mathrm{Gr}_k\mathbb{R}^n$ an open set. Suppose that for every $X \in \mathcal{U}$ the sections $B_1 \cap X$ and $B_2 \cap X$ are congruent (or, more generally, are linearly equivalent and have the same area). Is it true that $B_1 \cap X = B_2 \cap X$ for all $X \in \mathcal{U}$?*

Note that Proposition 3.1 implies an affirmative answer to this question when one of the bodies is an ellipsoid. Also see Purnaras–Saroglou [13] for a local problem of a similar flavor.

2. Preliminaries

2.1. Notation and conventions

In this paper, a "vector space" always means a finite-dimensional real vector space and a "subspace" means a linear subspace. For vector spaces X and Y, $\mathrm{Hom}(X, Y)$ denotes the space of linear maps from X to Y, and $X^* = \mathrm{Hom}(X, \mathbb{R})$ the dual space to X. For a vector space V, the Grassmann manifold $\mathrm{Gr}_k V$ consists of (unoriented) k-dimensional subspaces of V. For $X \in \mathrm{Gr}_k V$ and $m \ge k$ we denote by $\mathrm{Gr}_m(V, X)$ the set of subspaces from $\mathrm{Gr}_m V$ containing X:

$$\mathrm{Gr}_m(V, X) := \{W \in \mathrm{Gr}_m V : X \subset W\}. \tag{2.1}$$

For a subset $S \subset V$ we denote by $\mathrm{LinSpan}\, S$ the smallest linear subspace of V containing S. For $v \in V \setminus \{0\}$ the line through v is denoted by $\mathbb{R}v = \{\lambda v \mid \lambda \in \mathbb{R}\}$.

For complementary subspaces $X, Y \subset V$ we denote by pr_X^Y the projection to X along Y:

$$\mathrm{pr}_X^Y\colon V \to V, \quad v \mapsto \text{ the unique point in } (v+Y)\cap X \tag{2.2}$$

Note that any linear projector (i.e., idempotent linear map) $P \in \mathrm{Hom}(V,V)$ is uniquely of the above form, with $X = \operatorname{im} P$ and $Y = \ker P$.

For a convex set $K \subset V$ we denote by ∂K the relative boundary of K, that is the boundary in the topology of its affine span. A convex set $B \subset V$ is called a *convex body* if it is compact and has nonempty interior. By an *ellipsoid* we mean the unit ball of an inner product norm in a vector space. In other words, all ellipsoids are assumed to be centered at 0. The same terminology adjustment applies to ellipses in dimension 2.

A *Minkowski seminorm* on a vector space V is a function $\Phi\colon V \to \mathbb{R}_+$ which is positively 1-homogeneous and subadditive (and hence convex). A *Minkowski norm* is a Minkowski seminorm which is positive on $V \setminus \{0\}$. The difference from usual norms is that a Minkowski norm is not assumed symmetric. There is a standard bijection between Minkowski seminorms and convex sets with 0 in the interior. Namely, to each Minkowski seminorm Φ one associates its unit ball $B_\Phi = \{x \in V : \Phi(x) \le 1\}$, and for every convex set $B \subset V$ with 0 in the interior there is a corresponding Minkowski seminorm $\Phi^B(x) = \inf\{\lambda > 0 \mid x/\lambda \in B\}$. Note that Φ^B is a Minkowski norm if and only if B is compact.

Recall that a *supporting hyperplane* of a convex body B at a point $p \in \partial B$ is an affine hyperplane $H \ni p$ such that $H \cap \operatorname{Int} B = \varnothing$. We say that a point $p \in \partial B$ is a *smooth point* of ∂B if B has a unique supporting hyperplane at p. In this case the hyperplane is also called the tangent hyperplane of B at b. If B is the unit ball of a Minkowski norm Φ then the smoothness of a point $p \in \partial B$ is equivalent to the property that Φ is differentiable at p. Note that a Minkowski norm is differentiable almost everywhere (since it is a convex function) and hence almost all points of the boundary of a convex body are smooth points.

2.2. Assumptions and assertions of Theorem 1.2

Definition 2.1. Let Φ be a Minkowski seminorm on a vector space V. A linear subspace $X \subset V$ is called Φ-*contracting* if there exists a linear projector P from V onto X such that $\Phi(P(v)) \le \Phi(v)$ for all $v \in V$.

Recall that every such linear projector P is of the form pr_X^Y for some subspace Y complementary to X. We refer to Y as a *contracting direction* for X.

Being Φ-contracting is a closed condition: For every $k \le n$ the set of k-dimensional Φ-contracting subspaces is closed in $\mathrm{Gr}_k V$. Also note that if X is Φ-contracting then X is $(\Phi|_W)$-contracting for every subspace $W \subset V$ containing X (to prove this, just restrict P to W).

The following lemma provides various reformulations for the assumption of Theorem 1.2. It will be handy throughout the proof.

Lemma 2.2. *Let Φ be a Minkowski seminorm on a vector space V, B its unit ball, and $X, Y \subset V$ complementary subspaces. Then the following conditions are equivalent.*

(1) $B \subset (B \cap X) + Y$;
(2) $\mathrm{pr}_X^Y B = B \cap X$;
(3) X *is Φ-contracting with contracting direction* Y;
(4) $(p + Y) \cap \operatorname{Int} B = \varnothing$ *for all* $p \in \partial B \cap X$.

Proof. (1) $\Rightarrow$ (2). The inclusion $B \cap X \subset \mathrm{pr}_X^Y B$ is trivial. The reverse one follows from (1) and the identity $(\mathrm{pr}_X^Y)^{-1}(B \cap X) = (B \cap X) + Y$.

(2) $\Rightarrow$ (3). We have to show that $\Phi(\mathrm{pr}_X^Y(v)) \le \Phi(v)$ for every $v \in V$. If $\Phi(v) = 1$ then $v \in B$, therefore $\mathrm{pr}_X^Y(v) \in B$ by (2), hence $\Phi(\mathrm{pr}_X^Y(v)) \le 1$. If $\Phi(v) > 0$ then the desired inequality follows by homogeneity. If $\Phi(v) = 0$ then $tv \in B$ for all $t \ge 0$, this and (2) imply that $t\,\mathrm{pr}_X^Y(v) \in B$ for all $t \ge 0$, therefore $\Phi(\mathrm{pr}_X^Y(v)) = 0$.

(3) $\Rightarrow$ (4). If $q \in (p + Y) \cap \operatorname{Int} B$ for some $p \in \partial B \cap X$, then $\mathrm{pr}_X^Y q = p$ hence

$$1 > \Phi(q) \overset{(3)}{\ge} \Phi\left(\mathrm{pr}_X^Y q\right) = \Phi(p) = 1$$

and we obtain a contradiction.

(4) $\Rightarrow$ (1). Suppose that (1) is false and pick $b_0 \in B$ such that $b_0 \notin (B \cap X) + Y$. Let $p_0 = \mathrm{pr}_X^Y(b_0)$, then $p_0 \notin B$, hence $\Phi(p_0) > 1$. Let $p = p_0/\Phi(p_0)$, then $p \in \partial B \cap X$. Now observe that the set $p + Y$ contains a point $q = b_0/\Phi(p_0)$ which belongs to $\operatorname{Int} B$, contrary to (4). $\square$

The assumption of Theorem 1.2 says that for every $X \in \mathcal{U}$ the condition (1) from Lemma 2.2 is satisfied for some $Y = Y_X \in \mathrm{Gr}_{n-k}V$. In view of Lemma 2.2(3), this can be restated as follows: every $X \in \mathcal{U}$ is Φ-contracting, where Φ is the Minkowski norm associated to B.

Definition 2.3. Let V be a vector space, $B_1, B_2 \subset V$ two convex sets with zero in the interiors, and $X \in \mathrm{Gr}_k V$. We say that B_1 and B_2 *coincide near* X if there exists an open set $U \subset V$ such that $X \subset U$ and $B_1 \cap U = B_2 \cap U$.

A convex body $B \subset V$ is called *locally cylindrical near* X if there exists a k-cylinder C such that B and C coincide near X. Note that in this case $B \cap X$ is a base of C (see Definition 1.1) since $C \cap X = B \cap X$ and $B \cap X$ is compact.

The next lemma shows that the assumption of Theorem 1.2 is local.

Lemma 2.4. *Let V be an n-dimensional vector space, $X \in \mathrm{Gr}_k V$, and $B_1, B_2 \subset V$ two convex sets with zero in the interiors. Assume that B_1 and B_2 coincide near X and B_1 is contained in a k-cylinder C with base $B_1 \cap X$. Then $B_2 \subset C$ as well.*

Proof. Let $K = B_1 \cap X = B_2 \cap X$ and $C = K + Y$ where Y is a subspace complementary to X. Suppose to the contrary that $B_2 \not\subset C$. Then by Lemma 2.2 there exist $p \in \partial K$ and $q \in B_2$ such that

$$q \in (p + Y) \cap \operatorname{Int} B_2.$$

Since $p \in B_2$ and $q \in \operatorname{Int} B_2$, for every $\varepsilon \in (0, 1)$ the point $p_\varepsilon := p + \varepsilon(q - p)$ belongs to $\operatorname{Int} B_2$. Since B_1 and B_2 coincide near X, it follows that $p_\varepsilon \in \operatorname{Int} B_1$ for a sufficiently small ε. On the other hand, $p_\varepsilon \in p + Y \subset \partial C$. This contradicts the assumption that $B_1 \subset C$. $\square$

In the last lemma of this section we show that Theorem 1.2 is in fact an if-and-only-if statement.

Lemma 2.5. *Let V be an n-dimensional vector space, $B \subset V$ a convex body with zero in the interior, and $X \in \mathrm{Gr}_k V$. Assume that at least one of the following holds:*

(1) *B is locally cylindrical near X;*
(2) *B coincides with $B' = \{v \in V : Q(v) \le 1\}$ near X, where Q is a nonnegative definite quadratic form on V.*

Then B is contained in a k-cylinder with base $B \cap X$.

Proof. First assume (1) and let C be the corresponding cylinder (see Definition 2.3). The desired property follows from Lemma 2.4 applied to C and B in place of B_1 and B_2, respectively.

Now assume (2) and let Y be the orthogonal complement to X with respect to the symmetric bilinear form associated to Q. Since $B' \cap X = B \cap X$ is compact, $Q|_X$ is positive definite and therefore Y is a complementary subspace to X. From the orthogonality we have $Q(\mathrm{pr}_X^Y v) \le Q(v)$ for all $v \in V$, therefore B' is contained in a cylinder $C = (B' \cap X) + Y$. Applying Lemma 2.4 to B' and B finishes the proof. $\square$

3. Quadratic forms

The goal of this section is to prove the following local version of the well-known fact that a normed space is Euclidean if all of its subspaces of fixed dimension $k \ge 2$ are Euclidean. Though the statement looks standard, we could not find it in the literature and the proof is not so immediate as one might expect.

Proposition 3.1. *Let Φ be a Minkowski norm on an n-dimensional vector space V, $2 \le k < n$ an integer, and $\mathcal{U} \subset \mathrm{Gr}_k V$ a connected nonempty open set. Suppose that for every $X \in \mathcal{U}$ the restriction $\Phi|_X$ is an inner product norm on X.*

Then there exists a unique quadratic form Q on X such that $(\Phi|_X)^2 = Q|_X$ for all $X \in \mathcal{U}$. Moreover Q is nonnegative definite.

The following notation and terminology will be handy throughout the proof. For a basis $\mathbf{v} = (v_1, \dots, v_n)$ of a vector space V we denote by $\Pi_{ij}^{\mathbf{v}}$ its coordinate planes:

$$\Pi_{ij}^{\mathbf{v}} = \operatorname{LinSpan}\{v_i, v_j\}, \qquad 1 \le i \ne j \le n.$$

If $\mathcal{U} \subset \mathrm{Gr}_2 V$ is an open set of planes, we say that a basis $\mathbf{v}$ is *$\mathcal{U}$-compatible* if all its coordinate planes $\Pi_{ij}^{\mathbf{v}}$ belong to $\mathcal{U}$. Clearly for any plane $\Pi \in \mathcal{U}$ every basis (v_1, v_2) of Π can be extended to a $\mathcal{U}$-compatible basis of V (just choose the remaining vectors sufficiently close to Π).

We precede the proof of Proposition 3.1 with a couple of lemmas.

Lemma 3.2. *Let $\mathbf{v} = (v_1, \dots, v_n)$ be a basis of a vector space V and $F\colon V \to \mathbb{R}$ a function whose restrictions to the coordinate planes $\Pi_{ij}^{\mathbf{v}}$ are quadratic forms on these planes. Then there exists a unique quadratic form Q on V such that $Q|_{\Pi_{ij}^{\mathbf{v}}} = F|_{\Pi_{ij}}$ for all $i \ne j$.*

Proof. Let $(x_1, \dots, x_n)$ be the coordinates on V with respect to the basis $\mathbf{v}$. We construct the (symmetric) matrix (c_{ij}) of Q in these coordinates from the values of F on the coordinate planes.

First define $c_{ii} = F(v_i)$ for all $1 \le i \le n$. Then for each pair i, j with $i \ne j$, consider the quadratic form $Q_{ij} := F|_{\Pi_{ij}^{\mathbf{v}}}$ on the plane $\Pi_{ij}^{\mathbf{v}}$. Since $Q_{ij}(v_i) = F(v_i) = c_{ii}$ and $Q_{ij}(v_j) = F(v_j) = c_{jj}$, the coordinate expression of Q_{ij} has the form

$$Q_{ij}(x_i, x_j) = c_{ii}x_i^2 + c_{jj}x_j^2 + 2c_{ij}x_i x_j$$

for some $c_{ij} \in \mathbb{R}$. We use this expression to define c_{ij}.

The resulting quadratic form $Q(x_1, \dots, x_n) = \sum_{i,j} c_{ij}x_i x_j$ satisfies $Q|_{\Pi_{ij}^{\mathbf{v}}} = Q_{ij}$ for all $i \ne j$. The uniqueness is obvious from the construction. $\square$

The next lemma essentially covers the case $n = 3$ of Proposition 3.1. Note that in this case we do not assume that $\mathcal{U}$ is connected.

Lemma 3.3. *Let X be a 3-dimensional vector space, $F\colon X \to \mathbb{R}$ a continuous function, and $\mathcal{U} \subset \mathrm{Gr}_2 V$ a nonempty open set. Suppose that for every $\Pi \in \mathcal{U}$ the restriction $F|_\Pi$ is a quadratic form on Π. Then there exists a unique quadratic form Q on X such that $F|_\Pi = Q|_\Pi$ for all $\Pi \in \mathcal{U}$.*

Proof. Fix a $\mathcal{U}$-compatible basis $\mathbf{v} = (v_1, v_2, v_3)$ of X. By Lemma 3.2, there exists a unique quadratic form Q on X that coincides with F on the coordinate planes $\Pi_{ij}^{\mathbf{v}}$. We show that this Q satisfies $F|_\Pi = Q|_\Pi$ for all $\Pi \in \mathcal{U}$.

First consider a plane $\Pi \in \mathcal{U}$ which is *generic* in the sense that it does not contain any of the vectors v_1, v_2, v_3. Since F and Q coincide on the coordinate planes, they coincide on the lines

$$\ell_{ij} := \Pi \cap \Pi_{ij}^{\mathbf{v}}, \qquad 1 \le i < j \le 3.$$

Both $F|_\Pi$ and $Q|_\Pi$ are quadratic forms on Π, and a quadratic form on Π is uniquely determined by its values on the three distinct lines ℓ_{ij}. Hence $F|_\Pi = Q|_\Pi$ if Π is generic. To finish the proof, observe that any non-generic plane Π can be approximated by generic ones and the identity $F|_\Pi = Q|_\Pi$ follows by continuity. $\square$

Proof of Proposition 3.1. Let V, Φ and $\mathcal{U} \subset \mathrm{Gr}_k V$ be as in Proposition 3.1. Define $F = \Phi^2$ and

$$\Omega = \bigcup_{X \in \mathcal{U}} X \setminus \{0\}.$$

It is easy to see that Ω is a connected open subset of V.

The assertion of the proposition can be rewritten as follows: there exists a unique quadratic form Q on V such that $F|_\Omega = Q|_\Omega$ and furthermore Q is nonnegative definite. First we verify the uniqueness and nonnegative definiteness of such Q. The uniqueness follows from the facts that Ω is open and a quadratic form is uniquely determined by its restriction to any open set.

Now suppose that Q is a quadratic form such that $F|_\Omega = Q|_\Omega$ and $Q(v) < 0$ for some $v \in V$. Fix $p \in \Omega$ and define $f(t) = F(p+tv)$ for all $t \in \mathbb{R}$. The function f is convex since $F = \Phi^2$ is a convex function on V. The identity $F|_\Omega = Q|_\Omega$ implies that $f(t) = Q(p+tv)$ for all t sufficiently close to 0. Therefore f is smooth near 0 and $f''(0) = 2Q(v) < 0$, contrary to the convexity of f. This contradiction shows that Q must be nonnegative definite.

It remains to prove the existence of a quadratic form Q such that $F|_\Omega = Q|_\Omega$. First we reduce this statement to the special case when $k = 2$. Consider the set

$$\mathcal{U}_2 = \{\Pi \in \mathrm{Gr}_2 V : \Pi \subset X \text{ for some } X \in \mathcal{U}\}$$

and observe that $\mathcal{U}_2$ is a connected open subset of $\mathrm{Gr}_2 V$, $\Phi|_\Pi$ is a quadratic form for every $\Pi \in \mathcal{U}_2$, and $\bigcup_{\Pi \in \mathcal{U}_2} \Pi \setminus \{0\} = \Omega$. Thus it suffices to prove the proposition for $k = 2$ and $\mathcal{U}_2$ in place of $\mathcal{U}$. We therefore assume $k = 2$ for the rest of the proof.

For a $\mathcal{U}$-compatible basis $\mathbf{v} = (v_1, \dots, v_n)$ of V, we denote by $Q^{\mathbf{v}}$ the quadratic form on V satisfying $F|_{\Pi_{ij}^{\mathbf{v}}} = Q^{\mathbf{v}}|_{\Pi_{ij}^{\mathbf{v}}}$ for all $1 \le i < j \le n$. Such a form exists and is unique by Lemma 3.2. Clearly $Q^{\mathbf{v}}$ does not change if the vectors of $\mathbf{v}$ are permuted or multiplied by nonzero scalars.

Claim. *Let $v_1, \dots, v_n \in V$ and $t \in \mathbb{R}$ be such that $\mathbf{v} = (v_1, v_2, \dots, v_n)$ and*

$$\mathbf{v}' = (v_1 + tv_2, v_2, \dots, v_n)$$

are $\mathcal{U}$-compatible bases. Then $Q^{\mathbf{v}} = Q^{\mathbf{v}'}$.

Proof. By the definition of $Q^{\mathbf{v}'}$ it suffices to show that

$$Q^{\mathbf{v}}|_{\Pi^{\mathbf{v}'}_{ij}} = F|_{\Pi^{\mathbf{v}'}_{ij}} \tag{3.1}$$

for all $1 \le i < j \le n$. Observe that $\Pi^{\mathbf{v}'}_{ij} = \Pi^{\mathbf{v}}_{ij}$ if $i, j \ge 2$ or $\{i, j\} = \{1, 2\}$, so (3.1) trivially holds in these cases. It remains to verify (3.1) for $i = 1$ and $j > 2$. Fix $j > 2$ and apply Lemma 3.3 to the 3-dimensional subspace

$$X = \mathrm{LinSpan}\{v_1, v_2, v_j\},$$

the set $\mathcal{U} \cap \mathrm{Gr}_2 X$ in place of $\mathcal{U}$, and the function $F|_X$ in place of F. This yields a quadratic form Q on X such that $Q|_\Pi = F|_\Pi$ for all planes $\Pi \in \mathcal{U} \cap \mathrm{Gr}_2 X$. In particular Q and F coincide on the planes $\Pi^{\mathbf{v}}_{12}$, $\Pi^{\mathbf{v}}_{1j}$ and $\Pi^{\mathbf{v}}_{2j}$, therefore $Q = Q^{\mathbf{v}}|_X$ by the uniqueness part of Lemma 3.2. On the other hand, Q and F coincide on the plane $\Pi^{\mathbf{v}'}_{1j} = \mathrm{LinSpan}\{v_1 + tv_2, v_j\}$ since this plane also belongs to $\mathcal{U} \cap \mathrm{Gr}_2 X$. Therefore (3.1) holds for all $1 \le i < j \le n$ and Claim follows. □

Fix $p \in \Omega$ and choose a $\mathcal{U}$-compatible basis $\mathbf{v} = (v_1, \dots, v_n)$ such that $v_1 = p$. Fix $\varepsilon > 0$ so small that for every point $p' \in V$ of the form

$$p' = p + \sum_{i=1}^{n} t_i v_i \quad \text{where } t_i \in (-\varepsilon, \varepsilon) \text{ for all } 1 \le i \le n, \tag{3.2}$$

the collection $\mathbf{v}' = (p', v_2, \dots, v_n)$ is a $\mathcal{U}$-compatible basis. We are going to show that $Q^{\mathbf{v}'} = Q^{\mathbf{v}}$ for every such $\mathbf{v}'$.

Let $p' \in V$ be as in (3.2). Connect p to p' by a sequence $p_0 = p, p_1, \dots, p_n = p'$ where

$$p_m = p + \sum_{i=1}^{m} t_i v_i, \qquad m = 1, \dots, n,$$

and let $\mathbf{v}^m = (p_m, v_2, \dots, v_n)$ for each m. By the choice of ε, each $\mathbf{v}^m$ is a $\mathcal{U}$-compatible basis. Observe that $Q^{\mathbf{v}^1} = Q^{\mathbf{v}}$ since $\mathbf{v}^1$ is obtained from $\mathbf{v}$ by rescaling the first basis vector. For $2 \le m \le n$, the basis $\mathbf{v}^m$ is obtained from $\mathbf{v}^{m-1}$ by a transformation as in Claim (up to a permutation of indices), hence $Q^{\mathbf{v}^m} = Q^{\mathbf{v}^{m-1}}$. Thus

$$Q^{\mathbf{v}} = Q^{\mathbf{v}^1} = Q^{\mathbf{v}^2} = \cdots = Q^{\mathbf{v}^n} = Q^{\mathbf{v}'}.$$

In particular, since p' is one of the basis vectors in $\mathbf{v}'$, we have $Q^{\mathbf{v}}(p') = Q^{\mathbf{v}'}(p') = F(p')$.

Thus $Q^{\mathbf{v}}(p') = F(p')$ for any point p' of the form (3.2). The range of such points p' is an open neighborhood of p, therefore we have proven the following statement (where $Q^{\mathbf{v}}$ is renamed to Q_p): For every $p \in \Omega$ there exists a quadratic form Q_p on V such that Q_p and F coincide in a neighborhood of p. Since a quadratic form is uniquely determined by its restriction to any open set, such Q_p is unique for every $p \in \Omega$ and the map $p \mapsto Q_p$ is locally constant on $\mathcal{U}$. Since Ω is connected, it follows that all Q_p, $p \in \Omega$, are one and the same quadratic form. Denote this quadratic form by Q and observe that $Q(p) = Q_p(p) = F(p)$ for all $p \in \Omega$. Thus $Q|_\Omega = F|_\Omega$ and Proposition 3.1 follows. $\square$

4. Local cylinders

In this section we collect technical facts about locally cylindrical convex bodies, see Definition 2.3. Throughout this section V is an n-dimensional vector space, Φ is a Minkowski norm on V, and B is the unit ball of Φ.

Lemma 4.1. *Let $X_1, X_2 \in \mathrm{Gr}_k V$ and $Y \in \mathrm{Gr}_{n-k} V$ be such that*

$$B \subset (B \cap X_i) + Y \text{ for } i = 1, 2.$$

Then $(B \cap X_1) + Y = (B \cap X_2) + Y$.

Proof. The assumption of the lemma implies that

$$(B \cap X_1) + Y \subset B + Y \subset ((B \cap X_2) + Y) + Y = (B \cap X_2) + Y.$$

Swapping X_1 and X_2 yields the opposite inclusion, hence the result. $\square$

Lemma 4.2. *Let $\mathcal{U} \subset \mathrm{Gr}_k V$ be an open set and $X_0 \in \mathcal{U}$. Suppose that all subspaces from $\mathcal{U}$ are Φ-contracting with the same contracting direction $Y_0 \in \mathrm{Gr}_{n-k} V$ (see Definition 2.1). Then B is locally cylindrical near X_0.*

Proof. By Lemma 2.2 we have $B \subset (B \cap X) + Y_0$ for all $X \in \mathcal{U}$. Then Lemma 4.1 implies that all cylinders $(B \cap X) + Y_0$, $X \in \mathcal{U}$, are in fact one and the same cylinder, which we denote by C. Let $U \subset V$ be the union of $\operatorname{Int} B$ and all subspaces from $\mathcal{U}$. The set U is open, contains X_0, and satisfies $B \cap U = C \cap U$ since $C \cap X = ((B \cap X) + Y_0) \cap X = B \cap X$ for every $X \in \mathcal{U}$. Thus B and C coincide near X_0 hence B is locally cylindrical near X_0. $\square$

Lemma 4.3. *Let $\mathcal{U} \subset \mathrm{Gr}_k V$ be a nonempty connected open set. Suppose that B is locally cylindrical near X for every $X \in \mathcal{U}$. Then there exists a k-cylinder C such that $B \cap X = C \cap X$ for all $X \in \mathcal{U}$.*

Proof. First we show that for every $X \in \mathcal{U}$, a k-cylinder that coincides with B near X is unique. Indeed, suppose that for some $X \in \mathcal{U}$ there are two such cylinders, C_1 and C_2, and observe that C_1 and C_2 coincide near X. By Lemma 2.4 applied to C_2 in place of B_2 and C_1 in place of both B_1 and C, it follows that $C_2 \subset C_1$. Similarly $C_1 \subset C_2$, hence $C_1 = C_2$, showing the desired uniqueness.

Let C_X denote the above unique cylinder. Pick $X \in \mathcal{U}$ and let $U \subset V$ be a neighborhood of X such that $B \cap U = C_X \cap U$ (see Definition 2.3). Since B is compact, there exists a neighborhood $\mathcal{U}_X \subset \mathcal{U}$ of X such that $B \cap X' \subset U$ for all $X' \in \mathcal{U}_X$. Then C_X and B coincide near X' for every $X' \in \mathcal{U}_X$. Hence $C_{X'} = C_X$, by the above uniqueness applied to X' in place of X.

Thus the map $X \mapsto C_X$ is locally constant and hence constant on $\mathcal{U}$. Denote the constant cylinder C_X by C and observe that $B \cap X = C_X \cap X = C \cap X$ for every $X \in \mathcal{U}$. $\square$

The next lemma provides a convenient reformulation of the conclusion of Theorem 1.2.

Lemma 4.4. *Let $\mathcal{U} \subset \mathrm{Gr}_k V$ be a nonempty connected open set. Suppose that for every $X \in \mathcal{U}$ at least one of the following holds:*

(1) *B is locally cylindrical near X;*
(2) *$B \cap X$ is an ellipsoid.*

Then the conclusion of Theorem 1.2 holds for B and $\mathcal{U}$.

Proof. If all intersections $B \cap X$ are ellipsoids then $\Phi|_X$ is an inner product norm for every $X \in \mathcal{U}$ (recall that all ellipsoids in this paper are 0-centered). Then Proposition 3.1 implies that there exists a nonnegative definite quadratic form Q on V such that $(\Phi|_X)^2 = Q|_X$ or, equivalently, $B \cap X = B' \cap X$ for all $X \in \mathcal{U}$ where $B' = \{v \in V : Q(v) \le 1\}$. This is the second option in the conclusion of Theorem 1.2.

Now assume that some of the intersections $B \cap X$, $X \in \mathcal{U}$, are not ellipsoids. Let $\mathcal{U}_0$ be a connected component of the nonempty open set

$$\mathcal{U}' := \{X \in U : B \cap X \text{ is not an ellipsoid}\}.$$

By our assumptions B is locally cylindrical near X for every $X \in \mathcal{U}_0$. By Lemma 4.3 there exists a k-cylinder C_0 such that $B \cap X = C_0 \cap X$ for all $X \in \mathcal{U}_0$. If $\mathcal{U}_0 = \mathcal{U}$ then this is the first option in the conclusion of Theorem 1.2. It remains to rule out the case when $\mathcal{U}_0 \ne \mathcal{U}$.

Suppose that $\mathcal{U}_0 \ne \mathcal{U}$. Then, since $\mathcal{U}$ is connected, there exists $X_1 \in \mathcal{U} \setminus \mathcal{U}_0$ that belongs to the closure of $\mathcal{U}_0$. Clearly $X_1 \notin \mathcal{U}'$, hence $B \cap X_1$ is an ellipsoid. On the other hand, all intersections of the form $B \cap X$ where $X \in \mathcal{U}_0$, are linearly equivalent since they are compact cross-sections of the same cylinder C_0. Since the linear equivalence is a closed condition, it follows that $B \cap X_1$ is linearly equivalent to $B \cap X_0$ for any $X_0 \in \mathcal{U}_0$. However $B \cap X_0$ is not an ellipsoid, a contradiction. $\square$

The next lemma will allow us to reduce the theorems to the codimension 1 case.

Lemma 4.5. *Let $X \in \mathrm{Gr}_k V$ be such that for every $W \in \mathrm{Gr}_{k+1}(V, X)$ the intersection $B \cap W$ is locally cylindrical near X. Then B is locally cylindrical near X.*

Proof. Let $K = B \cap X$. Choose

$$W_1, \dots, W_{n-k} \in \mathrm{Gr}_{k+1}(V, X) \text{ such that } \mathrm{LinSpan}\left(\bigcup_{i=1}^{n-k} W_i\right) = V.$$

For each $i = 1, \dots, n-k$, since $B \cap W_i$ is locally cylindrical near X, there exists $v_i \in W_i \setminus X$ such that $K + [-v_i, v_i] = B \cap (X + [-v_i, v_i])$ where $[-v_i, v_i]$ denotes the straight line segment between $-v_i$ and v_i. Define $Y = \mathrm{LinSpan}\{v_i\}_{i=1}^{n-k}$ and $C = K + Y$, then Y is a subspace of V complementary to X and C is a k-cylinder with base K.

We show that $B \subset C$. Lemma 2.2 implies that it is enough to check that $(p + Y) \cap \mathrm{Int}\, B = \varnothing$ for all $p \in \partial K$. Fix $p \in \partial K$ and let H be a supporting hyperplane to B at p. Then for any $i = 1, \dots, n-k$ the line $l_i = H \cap \mathrm{LinSpan}\{p, v_i\}$ is a supporting line to $B_i = B \cap \mathrm{LinSpan}\{p, v_i\}$ at p. As $p + [-v_i, v_i] \subset \partial B_i$, we have $p + v_i \in l_i$. Therefore $p + v_i \in H$ for all $i = 1, \dots, n-k$ hence $p + Y \subset H$ thus $(p + Y) \cap \mathrm{Int}\, B = \varnothing$.

Define a neighborhood U_0 of 0 in Y by

$$U_0 = \left\{ \sum_{i=1}^{n-k} t_i v_i : t_1, \dots, t_{n-k} \in \mathbb{R},\ \sum |t_i| < 1 \right\}$$

and let $U = X + U_0$. The choice of v_i and the convexity of B imply that $K + U_0 \subset B$. Combining this with $B \subset C$ we obtain $B \cap U \subset C \cap U = K + U_0 \subset B \cap U$ hence B is locally cylindrical near X. $\square$

5. Proof of Theorem 1.2 in dimension 3

In this section we prove Theorem 1.2 for $n = 3$ and $k = 2$. Let V, B, and $\mathcal{U}$ satisfy the assumptions of Theorem 1.2 with $n = 3$ and $k = 2$. That is, V is a 3-dimensional vector space, $B \subset V$ is a convex body with zero in the interior, $\mathcal{U} \subset \mathrm{Gr}_2 V$ is a nonempty connected open set, and for every $X \in \mathcal{U}$ there exists a line $L_X \subset \mathrm{Gr}_1 V$ such that B is contained in the 2-cylinder with base $B \cap X$ and generatrix L_X:

$$B \subset (B \cap X) + L_X. \tag{5.1}$$

Note that (5.1) implies that L_X is complementary to X, otherwise the set on the right-hand side would be two-dimensional and could not contain B.

We fix the above notation and assumptions for the rest of this section and denote by Φ the Minkowski norm associated to B.

To facilitate understanding, we first sketch the proof in the case of the classical Kakutani criterion (that is, $\mathcal{U} = \mathrm{Gr}_2 V$) for a smooth, strictly convex, 0-symmetric body B. In this case one can define a continuous bijection $\psi\colon \mathrm{Gr}_1 V \to \mathrm{Gr}_2 V$ as follows: for $L \in \mathrm{Gr}_1 V$, $\psi(L)$ is the plane from $\mathrm{Gr}_2 V$ parallel to the tangent planes of ∂B at the points of $L \cap \partial B$. The Grassmannians $\mathrm{Gr}_1 V$ and $\mathrm{Gr}_2 V$ can be regarded as real projective planes, which are dual to each other in the sense that points in one projective plane correspond to lines in the other. Indeed, a plane $X \in \mathrm{Gr}_2 V$ corresponds to the set $\{L \in \mathrm{Gr}_1 V : L \subset X\}$, which is a line of the projective structure of $\mathrm{Gr}_1 V$, and a line $L \in \mathrm{Gr}_1 V$ corresponds to the set $\{X \in \mathrm{Gr}_2 V : L \subset X\}$, which is a line of the projective structure of $\mathrm{Gr}_2 V$.

The assumption of the Kakutani criterion implies that the above map ψ sends projective lines of $\mathrm{Gr}_1 V$ to projective lines of $\mathrm{Gr}_2 V$. Indeed, for every $X \in \mathrm{Gr}_2 V$ and $L \in \mathrm{Gr}_1 X$ the plane $\psi(L)$ must contain the line L_X from (5.1), thus ψ sends the projective line of $\mathrm{Gr}_1 X$ corresponding to X to the projective line of $\mathrm{Gr}_2 X$ corresponding to L_X.

By the fundamental theorem of projective geometry it follows that ψ is a projective map, that is, ψ is induced by a linear bijection $F\colon V \to V^*$ between the 3-dimensional vector spaces with projectivizations $\mathbb{P}(V) = \mathrm{Gr}_1 V$ and $\mathbb{P}(V^*) = \mathrm{Gr}_2 V$. In the latter case the projectivization is given by the map $V^* \setminus \{0\} \to \mathrm{Gr}_2 V$, $f \mapsto \ker f$. To summarize, we obtain a linear bijection $F\colon V \to V^*$ such that $\psi(\mathbb{R}v) = \ker(F(v))$ for all $v \in V \setminus \{0\}$. It is not hard to show that such a linear parametrization of tangent directions of ∂B is possible only if B is an ellipsoid. (This last step is detailed in Lemma 5.6 below).

We now turn to the general case of Theorem 1.2 for $n = 3$ and $k = 2$. The difference of the proof from the sketch above is that we have to tackle the cylindrical case (see Lemma 5.2 below) and in the non-cylindrical case it is more natural to construct the projective dual of ψ. The proof is composed of several lemmas.

Lemma 5.1. *For every $X \in \mathcal{U}$ there is a unique line $L_X \in \mathrm{Gr}_1(V)$ satisfying (5.1).*

Proof. Suppose to the contrary that for some $X \in \mathcal{U}$ there exist two distinct lines $L_1 \neq L_2$ such that $B \subset K + L_i$, $i = 1, 2$, where $K = B \cap X$. Note that both L_1 and L_2 are complementary to X.

Pick a smooth point p of ∂K such that the supporting line l of K at p is not contained in the plane $p + L_1 + L_2$. Then $l + L_1$ and $l + L_2$ are two distinct supporting planes of B at p. Hence $B \subset H_1 \cap H_2$ where H_1 and H_2 are closed half-spaces of V bounded by $l + L_1$ and $l + L_2$ respectively.

Pick a plane $X' \in \mathcal{U}$ such that $p \in X'$ and $X' \neq X$. Let $L = L_{X'} \in \mathrm{Gr}_1 V$ be a line satisfying (5.1) for X'. By Lemma 2.2 we have

$$B \cap X' = \mathrm{pr}^L_{X'}(B) \supset \mathrm{pr}^L_{X'}(K). \tag{5.2}$$

Assume for a moment that $L \not\subset X$. Then the restriction of $\mathrm{pr}^L_{X'}$ to X is a linear isomorphism between X and X'. Since $\mathrm{pr}^L_{X'}(p) = p$, it follows that p is a smooth point of $\mathrm{pr}^L_{X'}(K)$. This and (5.2) imply that p is a smooth point of $B \cap X'$. On the other hand, $B \cap X'$ is contained in the set $X' \cap H_1 \cap H_2$, which is a non-straight solid angle with vertex at p. Hence p is not a smooth point of $B \cap X'$, a contradiction.

This contradiction shows that the assumption $L \not\subset X$ is false. Thus for every plane $X' \in \mathcal{U}$ such that $p \in X'$ and $X' \neq X$, one has $L_{X'} \subset X$ (for any choice of $L_{X'}$). Pick a sequence $\{X_i\}$ of such planes converging to X. For every i we have a line $L_{X_i} \subset X$ satisfying (5.1) for X_i. Passing to a subsequence if necessary we may assume that the lines L_{X_i} converge to some line $L_0 \subset X$, then $B \subset (B \cap X) + L_0$ by continuity. However, the set $(B \cap X) + L_0$ is two-dimensional, a contradiction. This finishes the proof of Lemma 5.1. $\square$

With the help of Lemma 5.1, we can now define a map $\varphi\colon \mathcal{U} \to \mathrm{Gr}_1(V)$ by $\varphi(X) = L_X$ where L_X satisfies (5.1). Since (5.1) is a closed condition on a pair $(X, L) \in \mathrm{Gr}_2 V \times \mathrm{Gr}_1 V$, the uniqueness of L_X implies that φ is continuous. We fix the notation φ for the rest of this section.

In the next lemma we handle the degenerate case when φ is not injective.

Lemma 5.2. *Suppose that the above map φ is not injective. Then there exists a 2-cylinder $C \subset V$ such that $B \cap X = C \cap X$ for all $X \in \mathcal{U}$.*

Proof. Fix a line $L_0 \in \mathrm{Gr}_1 V$ having more than one φ-preimage. For a point $p \in V$ define

$$\mathcal{U}_p = \{X \in \mathcal{U} \colon p \in \mathcal{U}\}$$

Claim. *Let $X_0, X_1 \in \mathcal{U}$ be such that $X_0 \neq X_1$ and $\varphi(X_0) = \varphi(X_1) = L_0$. Then there exists a point $p \in (\partial B \cap X_0) \setminus X_1$ such that $\varphi(X) = L_0$ for all $X \in \mathcal{U}_p$.*

Proof. Recall that every compact convex set in a finite-dimensional vector space is a convex hull of its extreme points. This implies that there exists an extreme point of $B \cap X_0$ outside the line $X_0 \cap X_1$. Let p be such a point.

Let $q = \mathrm{pr}^{L_0}_{X_1} p$. Then $q \neq p$ as $p \notin X_1$, and $q \in B$ by Lemma 2.2. Consider the set

$$\mathcal{U}_{pq} = \{X \in \mathcal{U} : X \cap (p, q) \text{ is a single point}\}$$

where (p, q) denotes the open line segment between p and q. We claim that $\varphi(X) = L_0$ for all $X \in \mathcal{U}_{pq}$. Suppose to the contrary that $\varphi(X) = L \neq L_0$ for some $X \in \mathcal{U}_{pq}$. Consider the points $p' = \mathrm{pr}^L_X(p)$ and $q' = \mathrm{pr}^L_X(q)$. Since $p, q \in B$, Lemma 2.2 implies that $p', q' \in B \cap X$. The assumption that $L \neq L_0$ implies that $p' \neq q'$. Thus (p', q') is a nontrivial open line segment in $B \cap X$. Moreover (p', q') contains the intersection point of X and (p, q) since this point is preserved by pr^L_X. Now consider points $p'' = \mathrm{pr}^{L_0}_{X_0}(p')$ and $q'' = \mathrm{pr}^{L_0}_{X_0}(q')$. They belong to $B \cap X_0$ by Lemma 2.2, they are distinct since $p' \neq q'$

and L_0 is complementary to X, and we have $p \in (p'', q'')$ since (p', q') contains a point from $(p, q) \subset L_0$. This contradicts the choice of p as an extreme point of $B \cap X_0$. This contradiction shows that $\varphi(X) = L_0$ for all $X \in \mathcal{U}_{pq}$. Then Claim follows by continuity as every plane from $\mathcal{U}_p$ can be approximated by planes from $\mathcal{U}_{pq}$. $\square$

Fix $X_0, X_1 \in \mathcal{U}$ such that $X_0 \neq X_1$ and $\varphi(X_0) = \varphi(X_1) = L_0$. Let $p \in (\partial B \cap X_0) \setminus X_1$ be a point provided by Claim. Applying Claim to X_0 and any plane from $\mathcal{U}_p \setminus \{X_0\}$ we obtain another point $p' \in (\partial B \cap X_0) \setminus \mathbb{R}p$ such that $\varphi(X) = L_0$ for all $X \in \mathcal{U}_{p'}$.

Let $C = (B \cap X_0) + L_0$. For every $X \in \mathcal{U}_p \cup \mathcal{U}_{p'}$ we have $\varphi(X) = L_0$ and hence $B \subset (B \cap X) + L_0$ by the definition of φ. This and Lemma 4.1 imply that $(B \cap X) + L_0 = C$ and hence $B \cap X = C \cap X$ for all $X \in \mathcal{U}_p \cup \mathcal{U}_{p'}$. Thus $B \cap U = C \cap U$ where $U \subset V$ is the union of all planes from $\mathcal{U}_p \cup \mathcal{U}_{p'}$.

Since $\mathbb{R}p \neq \mathbb{R}p'$, U contains an open neighborhood of $X_0 \setminus 0$ and every plane $X \in \mathcal{U}$ sufficiently close to X_0 is contained in U. For every such plane $X \subset U$ we have

$$B \cap X = B \cap U \cap X = C \cap U \cap X = C \cap X,$$

therefore $(B \cap X) + L_0 = C$ and hence $\varphi(X) = L_0$. Thus X_0 has a neighborhood in $\mathcal{U}$ where φ is constant. Since X_0 is an arbitrary element of $\varphi^{-1}(L_0)$, it follows that $\varphi^{-1}(L_0)$ is an open set. Since $\mathcal{U}$ is connected, this implies that φ is constant, thus $\varphi(X) = L_0$ for all $X \in \mathcal{U}$.

Now Lemma 4.1 applied to X_0 and any $X \in \mathcal{U}$ implies that $(B \cap X) + L_0 = C$ and hence $B \cap X = C \cap X$, finishing the proof of Lemma 5.2. $\square$

Lemma 5.2 implies Theorem 1.2 in the case when φ is not injective. Now we consider the case when φ is injective.

Lemma 5.3. *Let $X_1, X_2, X_3 \in \mathcal{U}$ be distinct planes containing a common line $\ell \in \mathrm{Gr}_1 V$. Then the lines $\varphi(X_1), \varphi(X_2), \varphi(X_3)$ are contained in one plane from $\mathrm{Gr}_2 V$.*

Proof. First consider the case when $\ell \cap \partial B$ contains a smooth point p of ∂B. Let T be the unique supporting plane of B at p. By Lemma 2.2, for every $j \in \{1, 2, 3\}$ the straight line $p + \varphi(X_j)$ does not intersect $\operatorname{Int} B$ and hence, by the smoothness of B at p, this line is contained in T. Therefore the lines $\varphi(X_1), \varphi(X_2), \varphi(X_3)$ are contained in the plane from $\mathrm{Gr}_2 V$ parallel to T. This proves the lemma in the case when $\ell \cap \partial B$ contains a smooth point of ∂B.

The general case follows by continuity, since smooth points are dense in ∂B and any triple of planes $X_1, X_2, X_3 \in \mathrm{Gr}_2 V$ with $\ell = X_1 \cap X_2 \cap X_3 \in \mathrm{Gr}_1 V$ can be approximated by similar configurations where intersection lines contain smooth points of ∂B. $\square$

Each of the Grassmannians $\mathrm{Gr}_1 V$ and $\mathrm{Gr}_2 V$ carries a natural structure of a real projective plane as explained in the beginning of this section.

Lemma 5.3 says that φ preserves collinearity with respect to these projective structures: it sends any three collinear points of $\mathcal{U} \subset \mathrm{Gr}_2 V$ to three collinear points of $\mathrm{Gr}_1 V$. We use the following generalization of the fundamental theorem of projective geometry.

Proposition 5.4 *([5, Theorem 3.2]). Let $U \subset \mathbb{RP}^2$ be a connected open set and $\varphi\colon U \to \mathbb{RP}^2$ an injective map such that for any three collinear points of $x, y, z \in U$ their images $\varphi(x), \varphi(y), \varphi(z)$ are also collinear. Assume that the image $\varphi(U)$ contains three non-collinear points. Then φ is the restriction of a projective map.*

If the map $\varphi\colon \mathcal{U} \to \mathrm{Gr}_1 V$ defined above is injective, then it satisfies the assumptions of Proposition 5.4. Indeed, φ preserves collinearity by Lemma 5.3, and its continuity and injectivity imply that the image $\varphi(U)$ is not contained in one projective line of $\mathrm{Gr}_1 V$, hence $\varphi(U)$ contains three non-collinear points. Thus there exists a projective map $\widetilde{\varphi}\colon \mathrm{Gr}_2 V \to \mathrm{Gr}_1 V$ such that $\widetilde{\varphi}|_{\mathcal{U}} = \varphi$.

We now construct a dual projective map $\psi\colon \mathrm{Gr}_1 V \to \mathrm{Gr}_2 V$. Pick $L \in \mathrm{Gr}_1 V$ and consider the set $P_L = \{X \in \mathrm{Gr}_2 V : L \subset X\}$. It is a projective line in $\mathrm{Gr}_2 V$, hence its image $\widetilde{\varphi}(P_L)$ is a projective line in $\mathrm{Gr}_1 V$. This means that $\widetilde{\varphi}(P_L) = \mathrm{Gr}_1(P'_L)$ for some plane $P'_L \in \mathrm{Gr}_2 V$. We define $\psi(L) = P'_L$. This yields an injective map $\psi\colon \mathrm{Gr}_1 V \to \mathrm{Gr}_2 V$ uniquely characterized by the property

$$L \subset X \iff \widetilde{\varphi}(X) \subset \psi(L) \qquad \text{for all } L \in \mathrm{Gr}_1 V \text{ and } X \in \mathrm{Gr}_2 V. \tag{5.3}$$

In particular (5.3) implies that ψ preserves collinearity, therefore it is a projective map. Hence ψ is the projectivization of some linear bijection $F\colon V \to V^*$ (recall that $\mathrm{Gr}_1 V = \mathbb{P}(V)$ and $\mathrm{Gr}_2 V = \mathbb{P}(V^*)$). This means that

$$\ker(F(p)) = \psi(\mathbb{R}p) \quad \text{for all } p \in V \setminus \{0\} \tag{5.4}$$

Lemma 5.5. *Assume that φ is injective and let $F\colon V \to V^*$ be as above. Let $p \in \partial B$ be a point contained in at least one plane from $\mathcal{U}$. Then $p + \ker(F(p))$ is a supporting plane of B at p.*

Proof. First assume that p is a smooth point of ∂B. Let T be the unique supporting plane of B at p and $T_0 \in \mathrm{Gr}_2 V$ the plane parallel to T through the origin. Pick distinct $X_1, X_2 \in \mathcal{U}$ such that $p \in X_1 \cap X_2$. Similarly to the proof of Lemma 5.3, we have $p + \varphi(X_i) \subset T$ and hence $\varphi(X_i) \subset T_0$ for $i = 1, 2$. On the other hand, (5.3) implies that $\varphi(X_i) \subset \psi(X_1 \cap X_2)$ for $i = 1, 2$. Since there is only one plane containing $\varphi(X_1)$ and $\varphi(X_2)$, it follows that $\psi(X_1 \cap X_2) = T_0$. This and (5.4) imply that $\ker(F(p)) = T_0$. Thus $p + \ker(F(p)) = p + T_0 = T$.

We have shown that the assertion of the lemma holds in the case when p is a smooth point of ∂B. The general case follows by continuity. $\square$

The next lemma, together with Proposition 3.1, proves Theorem 1.2 for $n = 3, k = 2$ in the case when φ is injective.

Lemma 5.6. *Suppose that φ is injective. Then $B \cap X$ is an ellipse for every $X \in \mathcal{U}$ (recall that all ellipses in this paper are 0-centered).*

Proof. Fix $X \in \mathcal{U}$ and let $K = B \cap X$. For each $p \in X$, define $f(p) \in X^*$ by $f(p) = F(p)|_X$, where F is the map from Lemma 5.5. Then $f\colon X \to X^*$ is a linear map. Applying Lemma 5.5 to $p \in \partial K$ we obtain that $f(p) \neq 0$ and $p + \ker(f(p))$ is a supporting line of K at p.

Among other things this implies that there is a way to continuously assign a supporting line to each point of ∂K. This is possible only if ∂K is a C^1 curve and the Minkowski norm Φ is C^1 away from 0. Now for every $p \in \partial K$, the line $p + \ker(f(p))$ is the tangent line of ∂K at p.

We turn this family of supporting lines into a linear vector field W as follows. Fix a nonzero skew-symmetric bilinear form ω on X, and for each $p \in X$ let $W(p) \in X$ be the unique vector satisfying

$$\omega(W(p), q) = f(p)(q) \text{ for all } q \in X.$$

Then $W\colon X \to X$ is a non-degenerate linear map, and for every $p \in \partial K$ we have $W(p) \in \ker(f(p))$, hence the direction of $W(p)$ is the tangent direction of ∂K at p. We have therefore obtained a linear vector field W on X that is tangent to ∂K. It is well-known that the existence of such a vector field implies that K is an ellipse, see e.g. [10, Lemma 3.4]. $\square$

We now compose the proof of Theorem 1.2 for $n = 3$ and $k = 2$ from the results of this section. For V, B, $\mathcal{U}$ as in the theorem, define a continuous map $\varphi\colon \mathcal{U} \to \mathrm{Gr}_1 V$ as explained after Lemma 5.1. Then there are two cases: either φ is injective or not. If φ is not injective then Lemma 5.2 implies that the alternative (1) of the conclusion of Theorem 1.2 takes place. If φ is injective then Lemma 5.6 implies that $B \cap X$ is an ellipse and hence $\Phi|_X$ is a Euclidean norm for every $X \in \mathcal{U}$. This and Proposition 3.1 imply that the alternative (2) of the conclusion of Theorem 1.2 takes place. Thus Theorem 1.2 holds for $n = 3$ and $k = 2$.

6. Proof of Theorem 1.2 in higher dimensions

In this section we finish the proof of Theorem 1.2. First we observe that for every fixed n and k the statement of Theorem 1.2 is equivalent to the following proposition.

Proposition 6.1. *Let $k \geq 2$ and $n \geq k + 1$ be integers. Let V be an n-dimensional vector space, Φ a Minkowski norm on V, and B the unit ball of Φ. Let $X_0 \in \mathrm{Gr}_k V$ be such that* *all k-dimensional subspaces X from a neighborhood of X_0 in $\mathrm{Gr}_k V$ are Φ-contracting (see Definition 2.1). Then at least one of the following holds.*

(1) *B is locally cylindrical near X_0 (see Definition 2.3).*
(2) *$B \cap X_0$ is an ellipsoid (recall that all ellipsoids in this paper are 0-centered).*

To show that Theorem 1.2 is equivalent to Proposition 6.1, first observe that the assumptions on X in Theorem 1.2 and Proposition 6.1 are equivalent by Lemma 2.2. The conclusion of Theorem 1.2 trivially implies that of Proposition 6.1. Conversely, by Lemma 4.4 the conclusion of Proposition 6.1 implies that of Theorem 1.2.

The proof of Proposition 6.1 occupies the rest of this section. We argue by induction with base $n = 3$ and $k = 2$ established in section 5. The induction step is based on the following lemma.

Lemma 6.2. *Let $X_1, X_2 \in \mathrm{Gr}_{n-1}V$ be two hyperplanes and $L_1, L_2 \in \mathrm{Gr}_1 V$ two lines, $X_1 \neq X_2$ and $L_1 \neq L_2$. Suppose that X_1 and X_2 are Φ-contracting with contracting directions L_1 and L_2, respectively (see Definition 2.1).*

Then the subspace $X_1 \cap X_2 \in \mathrm{Gr}_{n-2}V$ is Φ-contracting.

Proof. Define $W = X_1 \cap X_2$ and $Z = L_1 + L_2$. We are going to show that $W \cap Z = 0$ and the projector $\mathrm{pr}_W^Z : V \to W$ does not increase Φ.

Consider the map $T = \mathrm{pr}_{X_1}^{L_1} \circ \mathrm{pr}_{X_2}^{L_2}$. Note that $\Phi(T(v)) \leq \Phi(v)$ for all $v \in V$ as $\mathrm{pr}_{X_1}^{L_1}$ and $\mathrm{pr}_{X_2}^{L_2}$ do not increase Φ. We also have $T(V) \subset X_1$ and $T|_W = \mathrm{id}_W$. Since $L_1 \neq L_2$, T has no fixed points outside W. Define $L = Z \cap X_1$ and note that $\dim L = 1$. By construction we have $T(v) - v \in Z$ for all $v \in V$, therefore $T(L) \subset L$ and moreover $T(p + L) \subset p + L$ for every $p \in X_1$. Pick $p \in X_1 \setminus W$ and consider the affine map $T|_{p+L}$ from the line $p + L$ to itself. This map cannot be a nontrivial translation of $p + L$ since T does not increase Φ and sublevel sets of $\Phi|_{p+L}$ are bounded. Therefore $T|_{p+L}$ has a fixed point, hence $(p + L) \cap W \neq \varnothing$. Thus $L \not\subset W$, $X_1 = W \oplus L$, and $W \cap Z = 0$.

Now we have a projector pr_W^Z and it remains to show that it does not increase Φ. Let p be as above and $q = \mathrm{pr}_W^Z(p)$. Note that q is the unique intersection point of $p + L$ and W hence the unique fixed point of $T|_{p+L}$. Since $T(L) \subset L$ and T does not increase Φ, the restriction $T|_L$ is a multiplication by some $\lambda \in [-1, 1]$, therefore $T(p) = T(p - q) + T(q) = \lambda(p - q) + q$. If $\lambda = 1$ then $T(p) = p$ which contradicts our choice of p as $p \notin W$ and T has no fixed points outside of W. If $\lambda = -1$ then $q = \frac{p+T(p)}{2}$ and then $\Phi(q) \leq \Phi(p)$ since Φ is convex and $\Phi(T(p)) \leq \Phi(p)$. If $|\lambda| < 1$ then $q = \lim_{m\to\infty} T^m(p)$, hence $\Phi(q) \leq \Phi(p)$ since T does not increase Φ.

We have shown that $\Phi(\mathrm{pr}_W^Z(p)) \leq \Phi(p)$ for an arbitrary $p \in X_1 \setminus W$. Thus pr_W^Z does not increase Φ on X_1. Since $\mathrm{pr}_W^Z = \mathrm{pr}_W^Z \circ \mathrm{pr}_{X_1}^{L_1}$, it follows that pr_W^Z does not increase Φ everywhere. $\square$

Proof of Proposition 6.1. Recall that Proposition 6.1 and Theorem 1.2 are equivalent for every fixed n and k. The case $n = 3$ is covered in section 5, so we assume that $n \geq 4$. Arguing by induction, we assume that Proposition 6.1 and Theorem 1.2 are proven for all $3 \leq n' < n$ in place of n.

Let V, Φ, B, X_0 be as in Proposition 6.1. If $B \cap X_0$ is an ellipsoid then the second alternative of Proposition 6.1 takes place, so we assume that $B \cap X_0$ is not an ellipsoid.

First assume that $n > k+1$. For every $W \in \mathrm{Gr}_{k+1}(V, X_0)$ the assumptions of Proposition 6.1 are satisfied for W in place of V, $\Phi|_W$ in place of Φ, and $B \cap W$ in place of B. Since $B \cap X_0$ is not an ellipsoid, the $(k+1)$-dimensional case of Proposition 6.1 implies that $B \cap W$ is locally cylindrical near X_0 for every $W \in \mathrm{Gr}_{k+1}(V, X_0)$. By Lemma 4.5 it follows that B is locally cylindrical near X_0. This finishes the proof of Proposition 6.1 for $n > k + 1$.

Now assume that $n = k + 1$. Let $\mathcal{U}$ be a neighborhood of X_0 in $\mathrm{Gr}_{n-1}V$ such that all subspaces from $\mathcal{U}$ are Φ-contracting. We consider two cases.

Case 1: All subspaces from $\mathrm{Gr}_{n-2}X_0$ are Φ-contracting. Then the $(n-1)$-dimensional case of Theorem 1.2 applies to X_0, $B \cap X_0$, and $\mathrm{Gr}_{n-2}X_0$ in place of V, B, and $\mathcal{U}$ respectively, and we conclude that $B \cap X_0$ is an ellipsoid. (Other possibilities for B' in Theorem 1.2 are excluded as explained in Remark 1.3).

Case 2: There exists $W_0 \in \mathrm{Gr}_{n-2}X_0$ that is not Φ-contracting. Define

$$\Sigma = \{W \in \mathrm{Gr}_{n-2}V : W \text{ is not } \Phi\text{-contracting}\}$$

Since being Φ-contracting is a closed condition, Σ is an open subset of $\mathrm{Gr}_{n-2}V$.

Let $L_0 \in \mathrm{Gr}_1 V$ be a contracting direction for X_0 (see Definition 2.1). Pick $X_1 \in \mathcal{U}$ such that $X_1 \neq X_0$ and $X_1 \cap X_0 = W_0$. Applying Lemma 6.2 to the hyperplanes X_0 and X_1 we conclude that L_0 is the unique contracting direction for X_1, otherwise W_0 would be Φ-contracting.

Now consider the set

$$\mathcal{U}_0 = \{X \in \mathcal{U} : \ X \neq X_1 \text{ and } X \cap X_1 \in \Sigma\}.$$

It is an open subset of $\mathcal{U}$ containing X_0. For every $X \in \mathcal{U}_0$ we apply Lemma 6.2 to X and X_1 and conclude that contracting directions for X and X_1 coincide (since $X \cap X_1$ is not Φ-contracting). Thus all hyperplanes from $\mathcal{U}_0$ have the same contracting direction L_0, and an application of Lemma 4.2 shows that B is locally cylindrical near X_0.

Thus we have shown that in all cases one of the alternatives from the conclusion of Proposition 6.1 holds for an arbitrary $X_0 \in \mathcal{U}$. This finishes the proof of Proposition 6.1 and Theorem 1.2. $\square$

7. Proof of Theorem 1.4

The proof of Theorem 1.4 is essentially the same as that of the main result of [10] except that the use of the global Kakutani criterion is replaced by an application of

Theorem 1.2. Below we go through the steps of the proof from [10] for $k = 3$ and fill out missing bits in the case $k = 2$ (which was not considered in [10]).

We restate the key intermediate results from [10] in the following two propositions. The first one works in all dimensions and provides a special algebraic family of tangent directions to ∂B.

Proposition 7.1 *([10, Proposition 2.4] and [10, Remark 4.6]). Let V be a vector space, $\dim V = k+1 \geq 3$, $B \subset V$ a convex body containing 0 in its interior, and $\mathcal{U} \subset \mathrm{Gr}_k V$ an open set. Suppose that for every $X_1, X_2 \subset \mathcal{U}$ the cross-sections $B \cap X_1$ and $B \cap X_2$ are linearly equivalent.*

Then for almost every $X \in \mathcal{U}$ there exist a vector $\nu \in V \setminus X$ and a linear map

$$R\colon X^* \to \mathrm{Hom}(X, X)$$

such that for every $\lambda \in X^*$ the linear operator $R_\lambda = R(\lambda)\colon X \to X$ satisfies:

(1) $\mathrm{Trace}\, R_\lambda = 0$.
(2) *For every $p \in \partial B \cap X$, the vector $R_\lambda(p) + \lambda(p)\nu$ is tangent to ∂B at p.*

The notion of tangency to ∂B in Proposition 7.1(2) is defined as follows: A vector $v \in V$ is said to be tangent to ∂B at a point $p \in \partial B$ if for the Minkowski norm Φ associated to B the function $t \mapsto \Phi(x + tv)$ has zero derivative at $t = 0$. One can see that this is equivalent to the property that the tangent cone of B at p contains $\mathrm{LinSpan}\{v\}$.

The most important case in Proposition 7.1(2) is when $p \in \ker \lambda$. In this case the term $\lambda(p)\nu$ vanishes and hence $R_\lambda(p)$ is tangent to ∂K at p where $K = B \cap X$ is the respective cross-section. This property is a strong restriction on the pair (K, R) and at least in dimensions $k = 2, 3$ we have the following.

Proposition 7.2 *(cf. [10, Proposition 2.5]). Let X be a vector space, $\dim X = k \in \{2, 3\}$, and let $K \subset X$ be a convex body with 0 in its interior. Let $R\colon X^* \to \mathrm{Hom}(X, X)$ be a linear map such that for every $\lambda \in X^*$ the map $R_\lambda = R(\lambda)$ satisfies $\mathrm{Trace}\, R_\lambda = 0$ and*

$$\text{for every } p \in \partial K \cap \ker \lambda, \text{ the vector } R_\lambda(p) \text{ is tangent to } \partial K \text{ at } p. \tag{7.1}$$

Then $R_\lambda(p)$ is tangent to ∂K at p for **all** *$p \in \partial K$ and $\lambda \in X^*$.*

Proof. The case $k = 3$ is covered by [10, Proposition 2.5]. The proof for $k = 2$ can be assembled from arguments in [10] as follows.

Fix a basis (e_1, e_2) of X, and let (e_1^*, e_2^*) be the dual basis of X^*. For a point $p = xe_1 + ye_2 \in X$ define $\lambda_p = ye_1^* - xe_2^*$ and observe that $p \in \ker \lambda_p$. Hence by (7.1) the vector $W(p) := R_{\lambda_p}(p)$ is tangent to ∂K at p. Denote $R_{ij} = R_{e_i^*}(e_j)$ and rewrite $W(p)$ using the linearity of R:

$$W(p) = R_{\lambda_p}(p) = R_{ye_1^*-xe_2^*}(xe_1+ye_2) = xy(R_{11}-R_{22}) - x^2R_{21} + y^2R_{12}. \tag{7.2}$$

Thus W is a quadratic vector field on X and it is tangent to ∂K everywhere. This implies (see [10, Lemma 3.4]) that K is a 0-centered ellipse or $W = 0$. In the case of an ellipse the result follows from [10, Lemma 2.6], which is independent of the dimension.

It remains to consider the case $W = 0$ (cf. [10, Lemma 5.2]). In this case (7.2) vanishes as a function of x and y, therefore $R_{12} = R_{21} = 0$ and $R_{11} = R_{22}$. Since $\operatorname{Trace} R_{e_1^*} = \operatorname{Trace} R_{e_2^*} = 0$, we have $R_{11}^1 = -R_{12}^2 = 0$ and $R_{22}^2 = -R_{21}^1 = 0$ where R_{ij}^m, $m = 1, 2$, denotes the mth coordinate of R_{ij} with respect to the basis (e_1, e_2). Now the identity $R_{11} = R_{12}$ implies that $R_{11}^2 = R_{22}^2 = 0$ and $R_{22}^1 = R_{11}^1 = 0$. Thus the tensor R is zero, hence $R_\lambda(p) = 0$ for all $\lambda \in X^*$ and $p \in X$, and the assertion of the proposition follows. □

Now we deduce Theorem 1.4 from Propositions 7.1 and 7.2; the argument essentially repeats the one from [10, §2.3].

Let n, k, V, B and $\mathcal{U}$ be as in Theorem 1.4. First we assume that $n = k + 1$ and apply Proposition 7.1. Let X, R and ν be as in the assertion of Proposition 7.1 and $K = B \cap X$. Then K and R satisfy the assumptions of Proposition 7.2 and we conclude that for all $p \in \partial K$ and $\lambda \in X^*$, the vector $R_\lambda(p)$ is tangent to ∂K and hence to ∂B at p. Pick $p \in \partial K$ and choose $\lambda \in X^*$ such that $\lambda(p) \neq 0$. Now we have two vectors, $R_\lambda(p) + \lambda(p)\nu$ from Proposition 7.1 and $R_\lambda(p)$ from Proposition 7.2, such that they are both tangent to ∂B at p and ν is their linear combination. These properties imply that ν is also tangent to ∂B at p (a detailed proof of this implication can be found in [10, Lemma 2.3]).

Let $Y = \operatorname{LinSpan}\{\nu\}$. The above tangency and convexity of B imply that $(p + Y) \cap \operatorname{Int} B = \varnothing$ for all $p \in \partial K$. By Lemma 2.2 it follows that B is contained in the cylinder $(B \cap X) + Y$. Thus we have shown that almost every $X \in \mathcal{U}$ satisfies the assumption of Theorem 1.2 (the "almost every" is inherited from Proposition 7.1). This assumption is a closed condition, therefore it is satisfied for all $X \in \mathcal{U}$. Now we apply Theorem 1.2 and conclude that Theorem 1.4 holds for $n = k + 1$.

It remains to handle the case $n \geq k + 2$. By Lemma 4.4 it suffices to verify that for every $X \in \mathcal{U}$, B is locally cylindrical near X or $B \cap X$ is an ellipsoid. Pick $X \in \mathcal{U}$. For every $W \in \operatorname{Gr}_{k+1}(V, X)$, the assumption of Theorem 1.4 is satisfied for W in place of V, $B \cap W$ in place of B, and the connected component of $\mathcal{U} \cap \operatorname{Gr}_k(W)$ containing X in place of $\mathcal{U}$. If $B \cap X$ is not an ellipsoid then by the codimension 1 case of Theorem 1.4 proven above, $B \cap W$ is locally cylindrical near X for every $W \in \operatorname{Gr}_{k+1}(V, X)$. By Lemma 4.5 this implies that B is locally cylindrical near X. This finishes the proof of Theorem 1.4.

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References

[1] H. Auerbach, S. Mazur, S. Ulam, Sur une propriété caractéristique de l'ellipsoïde, Monatshefte Math. Phys. 42 (1) (1935) 45–48 (in French), MR 1550413.

[2] S. Banach, Théorie des opérations linéaires, Monografie Matematyczne, vol. 1, PWN, Warszawa, 1932 (in French); reprinted in Éditions Jacques Gabay, Sceaux, 1993, MR 1357166.

[3] G. Bor, L. Hernández Lamoneda, V. Jiménez-Desantiago, L. Montejano, On the isometric conjecture of Banach, Geom. Topol. 25 (5) (2021) 2621–2642, MR 4310896.

[4] B.D. Calvert, Characterizing a space which is Euclidean near a two-dimensional subspace, Houst. J. Math. 13 (3) (1987) 319–335, MR 0916140.

[5] A. Čap, M.G. Cowling, F. de Mari, M. Eastwood, R. McCallum, The Heisenberg group, $SL(3, \mathbb{R})$, and rigidity, in: Harmonic Analysis, Group Representations, Automorphic Forms and Invariant Theory, in: Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., vol. 12, World Sci. Publ., Hackensack, NJ, 2007, pp. 41–52, MR 2401809.

[6] A. Dvoretzky, A theorem on convex bodies and applications to Banach spaces, Proc. Natl. Acad. Sci. USA 45 (1959) 223–226, erratum, 1554. MR 105652.

[7] M.L. Gromov, On a geometric hypothesis of Banach, Izv. Akad. Nauk SSSR, Ser. Mat. 31 (1967) 1105–1114 (in Russian); English transl., Math. USSR, Izv. 1 (5) (1967) 1055–1064, MR 0217566.

[8] P.M. Gruber, Convex and Discrete Geometry, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 336, Springer, Berlin, 2007, MR 2335496.

[9] S. Ivanov, Monochromatic Finsler surfaces and a local ellipsoid characterization, Proc. Am. Math. Soc. 146 (4) (2018) 1741–1755, MR 3754357.

[10] S. Ivanov, D. Mamaev, A. Nordskova, Banach's isometric subspace problem in dimension four, Invent. Math. 233 (3) (2023) 1393–1425, MR 4623545.

[11] S. Kakutani, Some characterizations of Euclidean space, Jpn. J. Math. 16 (1939) 93–97, MR 895.

[12] S. Nakajima, Eine Kennzeichnung homothetischer Eiflächen, Tohoku Math. J. (1) 35 (1932) 285–286.

[13] I. Purnaras, C. Saroglou, Functions with isotropic sections, Trans. Am. Math. Soc. 374 (4) (2021) 3007–3024, MR 4223041.

[14] W. Süss, Zusammensetzung von Eikörpen und homothetische Eiflächen, Tohoku Math. J. (1) 35 (1932) 47–50.

[15] N. Zhang, On bodies with congruent sections or projections, J. Differ. Equ. 265 (5) (2018) 2064–2075, MR 3800112.