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An Unbounded Number of Canard Limit Cycles in Linear Regularizations of Piecewise Linear Systems

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Abstract

The purpose of this paper is to study the number of limit cycles of canard type in linear regularizations of piecewise linear systems with non-monotonic transition functions. Using the notion of slow divergence integral and elementary breaking mechanisms, we construct systems with an arbitrary finite number of hyperbolic limit cycles. The Hopf breaking mechanism deals with transition functions with precisely one critical point in the interval $(-1, 1)$. On the other hand, the jump breaking mechanism produces any number of limit cycles using transition functions with precisely three critical points in $(-1, 1)$.

Keywords Canard cycles · Slow divergence integral · Slow-fast Hopf point · Jump point · Regularization

Mathematics Subject Classification 34E15 · 34E17 · 34C40

Contents

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1 Introduction

In this paper we consider planar piecewise linear (PWL) vector fields of the form

$$Z(z) = \begin{cases} X(z) \text{ for } h(z) > 0, \\ Y(z) \text{ for } h(z) < 0, \end{cases} \qquad z = (x, y) \in \mathbb{R}^2, \tag{1}$$

where the vector fields $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ and the function $h : \mathbb{R}^2 \to \mathbb{R}$, $\nabla h \neq 0$, are each affine. We assume that $h(x, y) = x$. The set $\Sigma := h^{-1}(0)$ is called the switching line. This class of piecewise smooth vector fields is the central topic of a wide number of papers, see for instance (Gasull et al. 2020; Llibre et al. 2013; Han and Zhang 2010; Llibre and Teruel 2014; Freire et al. 2012; Huan and Yang 2014; Li et al. 2021; Li and Llibre 2020; Medrado and Torregrosa 2015).

Limit cycles of (1) have attracted great attention from many mathematicians and interesting results have been proven. Lum and Chua (1991) conjectured that the number of limit cycles of (1) in the continuous case (i.e., $X(z) = Y(z)$ for all $z \in \Sigma$) is at most one. The conjecture was proven by Freire et al. (1998).

The determination of the maximum number of crossing limit cycles in discontinuous PWL systems (1) is more challenging (see e.g. Braga and Mello (2013); Huan and Yang (2012); Freire et al. (2013); Llibre et al. (2013); Llibre and Ponce (2012); Esteban et al. (2021); Li and Llibre (2021) and references therein). Using a case-independent approach based on integral representations of the Poincaré half-maps (Carmona and Fernández-Sánchez 2021), Carmona, Fernández-Sánchez and Novaes showed that the maximum number of crossing limit cycles is uniformly bounded by 8 (see (Carmona et al. 2023)) and gave the first case-independent proof of Lum and Chua's conjecture (see (Carmona et al. 2021)). To the best of our knowledge, 3 crossing limit cycles have been found (see (Huan and Yang 2012; Llibre and Ponce 2012)).

We point out that the interest in the number of crossing limit cycles for (1) is closely related to the second part of Hilbert's 16th problem (Smale 2000). The problem asks if there is a finite upper bound on the number of limit cycles for polynomial vector fields of a given degree n, and it is unsolved even for quadratic vector fields (Dumortier et al. 1994).

In this paper, we focus on the following natural question, also related to Hilbert's 16th problem: *Is there an upper bound on the number of limit cycles of regularized piecewise polynomial vector fields*? Even though regularized vector fields are not polynomial, this question is still interesting and its answer is non-trivial as we shall discuss.

Before we continue with the discussion, let us precisely define *linear* and *nonlinear* regularizations. Consider a piecewise polynomial vector field Z defined in an analogous way as in Equation (1). Then, one defines a smooth vector field depending on a parameter $\lambda \in \mathbb{R}$ as

$$\widetilde{Z}(\lambda, z) = \left(\widetilde{Z}_1(\lambda, z), \widetilde{Z}_2(\lambda, z)\right), \quad z = (x, y) \in \mathbb{R}^2, \tag{2}$$

satisfying $\widetilde{Z}(1, z) = X(z)$ and $\widetilde{Z}(-1, z) = Y(z)$. The vector field $\widetilde{Z}$ given in (2) is called *continuous combination* of Z, and if $\widetilde{Z}$ is linear with respect to λ then it is called *convex combination* of Z. Observe that, if we replace λ by $\operatorname{sgn}(h(x, y))$ in Equation (2), then we recover the expression of the piecewise polynomial vector field Z.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a function satisfying the following conditions: **(1)** φ is C^∞-smooth and **(2)** $\varphi(t) = -1$ if $t \leq -1$ and $\varphi(t) = 1$ if $t \geq 1$. We call φ a *transition function*. The transition function φ is *monotonic* if it satisfies **(3)** $\varphi'(t) > 0$ if $t \in (-1, 1)$.

If $\widetilde{Z}$ is not a convex combination, then the smooth vector field

$$Z_\varepsilon(x, y) := \widetilde{Z}\left(\varphi\left(\frac{h(x, y)}{\varepsilon}\right), x, y\right), \qquad 0 < \varepsilon \ll 1 \tag{3}$$

is called φ *non-linear regularization of* Z (or simply non-linear regularization). When $\widetilde{Z}$ is a convex combination, we say that Z_ε is a φ*-linear* regularization of Z (or simply linear regularization). We refer to (4). The classical *Sotomayor–Teixeira regularization* (Sotomayor and Teixeira 1996) concerns the case where $\widetilde{Z}$ is convex and φ is monotonic. In both linear and non-linear regularizations, the dynamics strongly depend on the transition function φ adopted.

In Huzak and Kristiansen (2023), it has been proved that the number of limit cycles of regularized (according to Sotomayor–Teixeira (Sotomayor and Teixeira 1996)) piecewise quadratic systems is unbounded. More precisely, there exists a piecewise quadratic vector field satisfying the following property: for a given integer $k > 0$, there is a monotonic transition function φ_k such that the regularized vector field has at least $k + 1$ hyperbolic limit cycles. Of course, one can also expect an unbounded number of limit cycles when considering Sotomayor–Teixeira regularization of piecewise polynomial systems of higher degree. We believe that the only case where one could expect a finite upper bound on the number of limit cycles is Sotomayor–Teixeira regularizations of PWL systems (see (Huzak and Kristiansen 2024)).

One can also state a similar problem for different regularization processes. In De Maesschalck et al. (2026) the authors considered non-linear regularizations and proved that non-linearly regularized PWL vector fields can also produce an unbounded number of limit cycles using monotonic transition functions. This result is true even for non-linear regularizations of quadratic degree with respect to λ, and the very same paper considers limit cycles of non-linearly regularized PWL vector fields of higher degree in λ. See (Jeffrey 2018; Novaes and Jeffrey 2015) for theoretical aspects and applications of such regularizations.

The goal of this paper is to prove that the number of limit cycles of *linear regularizations* of PWL systems (1) is *unbounded*, using *non-monotonic* transition functions. More precisely, we consider the linear regularization of PWL vector fields

$$Z_\varepsilon(x,y) := \frac{1+\varphi\left(\frac{h(x,y)}{\varepsilon}\right)}{2}X(x,y) + \frac{1-\varphi\left(\frac{h(x,y)}{\varepsilon}\right)}{2}Y(x,y), \tag{4}$$

where $\varepsilon > 0$ is a small parameter, and X, Y and h are introduced in (1). The main result of this paper is the following:

- There exist linear vector fields X and Y such that the following is true: for any integer $k > 0$, there is a non-monotonic transition function $\varphi : \mathbb{R} \to \mathbb{R}$ such that the φ-linear regularization (4) has at least $k+1$ hyperbolic limit cycles, for each $\varepsilon > 0$ small enough. Moreover, the number of critical points of φ is fixed (that is, independent of k).

For a precise statement of this result we refer to Theorems A and B in Sect. 2. Compared to Huzak and Kristiansen (2023), the main novelty of this paper is that elementary breaking mechanisms (Hopf and jump) can be generated using PWL vector fields-at the cost of introducing non-monotonic transition functions with a minimal number of critical points.

Therefore, it remains an open problem to find the maximum number of limit cycles in regularized PWL vector fields following the classical Sotomayor–Teixeira process.

The hyperbolic limit cycles in Theorems A and B are produced by canard cycles associated with the slow-fast system

$$\begin{cases} \dot{x} = \dfrac{X_1(\varepsilon x, y) + Y_1(\varepsilon x, y)}{2} + \varphi(x)\dfrac{X_1(\varepsilon x, y) - Y_1(\varepsilon x, y)}{2}, \\ \dot{y} = \varepsilon\left(\dfrac{X_2(\varepsilon x, y) + Y_2(\varepsilon x, y)}{2} + \varphi(x)\dfrac{X_2(\varepsilon x, y) - Y_2(\varepsilon x, y)}{2}\right), \end{cases} \tag{5}$$

obtained after the performing $x = \varepsilon\tilde{x}$ to (4) and multiplication by $\varepsilon > 0$ (we drop the tildas in (5)). Canard cycles are limit periodic sets of (5), defined for $\varepsilon = 0$, that can produce limit cycles of (5) (or (4)) for $\varepsilon > 0$ small. They consist of fast (horizontal) orbits and at least one attracting and one repelling portion of the curve of singularities (for more details, see Sect. 3). Clearly, we have to find appropriate PWL vector field (X, Y) and transition function φ in (5) such that the canard cycles exist.

In this paper, we consider two important types of canard cycles that can occur in (5) when $\varepsilon \to 0$. The first type is related to the so-called *Hopf breaking mechanism* (De Maesschalck et al. 2021, Section 6.3) (see also Sect. 3.1). This mechanism contains a slow-fast Hopf/canard point near which the passage from an attracting branch to a repelling branch of the curve of singularities is possible, see Fig. 1(a). The Hopf breaking mechanism has been used to prove Theorem A (see Sect. 4.1).

The second type deals with the so-called *jump mechanism* (De Maesschalck et al. 2021, Section 6.2) (see also Sect. 3.2). In this case, we have two (generic) jump points that are connected by a fast orbit, see Fig. 1(b). In Theorem B, we generate limit cycles using the jump breaking mechanism (see Sect. 4.2).

If system (5) has a slow-fast Hopf point or a jump point at $(x, y) = (x_0, y_0)$, with $x_0 \in (-1, 1)$, then $\varphi'(x_0) = 0$ (that is, x_0 is a critical point of the transition function), see Sect. 3. This has already been observed in Perez et al. (2023) for jump points. We can therefore have a Hopf (or jump) breaking mechanism in the slow-fast system (5)

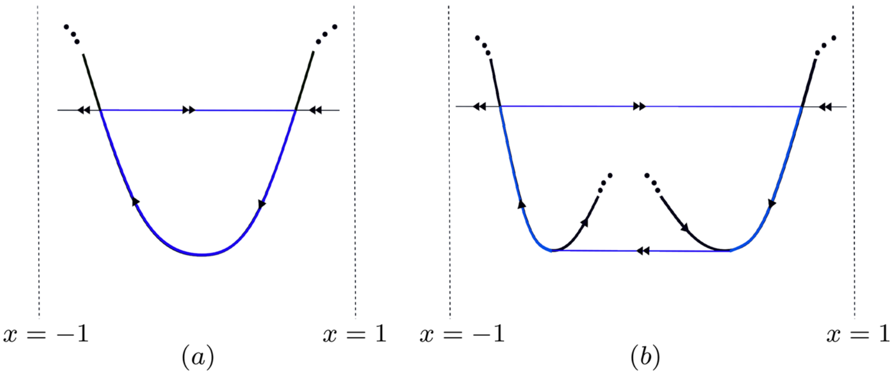


Fig. 1 Possible canard cycles in (5) for $\varepsilon = 0$. (a) Canard cycle with a slow-fast Hopf point. (b) Canard cycle containing a jump connection

only if we drop the monotonicity condition in the Sotomayor–Teixeira regularization. This paper can also be seen as a continuation of Perez et al. (2023), when we deal with limit cycles produced by canard cycles such as in Fig. 1(b). For a precise definition of slow-fast Hopf points and jump points, see Sect. 3.

In Theorems A and B, we choose X and Y in such a way that the slow-fast system (5) becomes a classical Liénard equation (see e.g. the system (10) in Sect. 3). Limit cycles of canard type in (slow-fast) classical Liénard equations can be studied using the notion of slow divergence integral (De Maesschalck et al. 2021, Chapter 5) (see (De Maesschalck and Dumortier 2011; De Maesschalck and Huzak 2015; Dumortier et al. 2007; Huzak and De Maesschalck 2014) and references therein). Simple zeros of the slow divergence integral correspond to hyperbolic limit cycles (see Theorems 3.1 and 3.2 in Sects. 3.1 and 3.2). Moreover, the main tool applied in the proof of the main results of De Maesschalck et al. (2026); Huzak and Kristiansen (2023) discussed previously was the slow divergence integral.

In this paper, we are only interested in the limit cycles produced by canard cycles located inside the regularization stripe. The study of canard cycles with portions located outside the stripe, in linear regularizations of PWL systems with non-monotonic transition functions, is left for future research. See also De Maesschalck et al. (2026).

This paper is organized as follows. In Sect. 2, we state our main results. In Sect. 3, we define Hopf and jump mechanisms in slow-fast systems, and then we discuss these notions for classical Liénard equations and regularizations of PWL systems. Finally, in Sect. 4, we prove Theorems A and B.

2 Statement of the Main Results

We consider $h(x, y) = x$ in Equation (4), and we obtain

$$Z_\varepsilon(x, y) := \frac{1 + \varphi\left(\frac{x}{\varepsilon}\right)}{2} X(x, y) + \frac{1 - \varphi\left(\frac{x}{\varepsilon}\right)}{2} Y(x, y), \tag{6}$$

where $\varepsilon > 0$ is a parameter kept small, $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$. Theorems A and B stated below deal with the number of limit cycles of (6), with linear vector fields X and Y and non-monotonic transition functions φ.

Theorem A (Hopf breaking mechanism). *There exists a PWL vector field*

$$Z(x, y, \alpha) = \begin{cases} X(x, y, \alpha) = (-3 + y, \alpha - x), \ x > 0, \\ Y(x, y, \alpha) = (-1 + y, \alpha - x), \ x < 0, \end{cases} \tag{7}$$

depending on a parameter $\alpha \in \mathbb{R}$, such that the following is true: for any integer $k > 0$, there exist a non-monotonic transition function $\varphi_k : \mathbb{R} \to \mathbb{R}$, with precisely 1 critical point in $(-1, 1)$, and a smooth function $a_k : [0, \varepsilon_k] \to \mathbb{R}$, with $\varepsilon_k > 0$ small enough and $a_k(0) = 0$, such that the φ_k-linear regularization of (7) with $\alpha = \varepsilon a_k(\varepsilon)$ has at least $k + 1$ hyperbolic limit cycles, for each $\varepsilon \in (0, \varepsilon_k]$.

We prove Theorem A in Sect. 4.1. The limit cycles in this theorem are produced using the Hopf breaking mechanism (see Figs. 1(a) and 2). On the other hand, Theorem B (proven in Sect. 4.2) deals with limit cycles generated by generic jump breaking mechanisms (see Figs. 1(b) and 4).

Theorem B (Jump breaking mechanism) *There exists a PWL vector field*

$$Z(x, y) = \begin{cases} X(x, y) = (-3 + y, -x), & x > 0, \\ Y(x, y) = (-1 + y, -x), & x < 0, \end{cases} \tag{8}$$

such that the following is true: for any integer $k > 0$, there exist a smooth b-family of non-monotonic transition functions $\varphi_{b,k} : \mathbb{R} \to \mathbb{R}$, with precisely 3 critical points in $(-1, 1)$, and a smooth function $b_k : [0, \varepsilon_k] \to \mathbb{R}$, with $\varepsilon_k > 0$ small enough and $b_k(0) = 0$, such that the $\varphi_{b,k}$-linear regularization in (6) with $b = b_k(\varepsilon)$ has at least $k + 1$ hyperbolic limit cycles, for each $\varepsilon \in (0, \varepsilon_k]$.

Observe that one can produce as many limit cycles as desired, either considering a one parameter family of PWL vector fields (Theorem A), or one parameter family of non-monotonic transition functions (Theorem B). In the former case, the transition function φ_k has precisely one critical point of Morse type. In the second case, the number of critical points of $\varphi_{b,k}$ in Theorem B is three, all of them also of Morse type. However, in both cases the number of critical points is fixed, that is, it does not increase as k increases. The reason a fixed number of critical points suffices lies in the geometry of the breaking mechanisms: the Hopf mechanism requires one critical point, while the jump mechanism requires three. For further details, see Sects. 3 and 4.

3 Generic Breaking Mechanisms

In this section, we consider planar slow-fast systems of the form

$$\begin{cases} \dot{x} = f(x, y, \varepsilon), \\ \dot{y} = \varepsilon^l g(x, y, \varepsilon), \end{cases} \tag{9}$$

where $\varepsilon \geq 0$ is the singular perturbation parameter kept close to zero, l is a positive integer, and f and g are C^∞-smooth functions. The overdot denotes the derivative of $x(t)$ and $y(t)$ with respect to the *fast time* t. When $\varepsilon = 0$, the set $S = \{f(x, y, 0) = 0\}$ is a curve of singularities of (9), and it has horizontal intervals as fast regular orbits. A singularity $(x_0, y_0) \in S$ is *normally hyperbolic* if $\frac{\partial f}{\partial x}(x_0, y_0, 0) \neq 0$.

We also recall the notion of *classical Liénard equations*, because they will play an important role in the proofs of Theorems A and B. Consider a classical Liénard equation

$$\begin{cases} \dot{x} = y - F(x), \\ \dot{y} = -\varepsilon^2 x, \end{cases} \tag{10}$$

where F is a C^∞-smooth function, and we adopt $l = 2$ (we postpone the explanation of this choice to Sect. 3.3). When $\varepsilon = 0$, the set $S = \{y = F(x)\}$ is a curve of singularities of system (10), and a singularity $(x_0, F(x_0)) \in S$ is normally hyperbolic if $F'(x_0) \neq 0$ (the prime denotes the derivative with respect to x). The singularity is attracting when $F'(x_0) > 0$ and repelling when $F'(x_0) < 0$. Singularities $(x_0, F(x_0)) \in S$ that satisfy $F'(x_0) = 0$ are called *contact points*.

Limit cycles of slow-fast systems (9) (and in particular of classical Liénard systems (10)) can be produced by slow-fast cycles defined for $\varepsilon = 0$. A *slow-fast cycle* consists of fast orbits and compact portions of S. We say that a slow–fast cycle is a *canard cycle* if it contains at least one attracting and one repelling portion of S. When canard cycles appear, the following two mechanisms play an essential role: **(a)** Hopf mechanism (De Maesschalck et al. 2021, Section 6.3) (see Sect. 3.1) and **(b)** jump mechanism (De Maesschalck et al. 2021, Section 6.2) (see Sect. 3.2).

3.1 The Hopf Breaking Mechanism

A singularity $(x_0, y_0) \in S$ is a *slow-fast Hopf point of* (9) if (see also (De Maesschalck et al., 2021, Definition 2.4))

$$\begin{aligned} f(x_0, y_0, 0) = g(x_0, y_0, 0) = \frac{\partial f}{\partial x}(x_0, y_0, 0) = 0, \\ \frac{\partial^2 f}{\partial x^2}(x_0, y_0, 0) \neq 0 \text{ and } \frac{\partial g}{\partial x}(x_0, y_0, 0) \cdot \frac{\partial f}{\partial y}(x_0, y_0, 0) < 0, \end{aligned} \tag{11}$$

Assume that (9) has a slow-fast Hopf point at $(x, y) = (x_0, y_0)$. Since $\frac{\partial f}{\partial y}(x_0, y_0, 0) \neq 0$, the curve of singularities $S = \{f(x, y, 0) = 0\}$ of (9) for $\varepsilon = 0$, near $(x, y) = (x_0, y_0)$, can be represented as $y = \kappa(x)$ where κ is a smooth function satisfying $\kappa(x_0) = y_0$. Notice that S has a quadratic contact with fast horizontal orbits of (9) with $\varepsilon = 0$, at $(x, y) = (x_0, y_0)$. Using (11), it is clear that the singularities with x close to x_0 and $x \neq x_0$ are normally hyperbolic, that is,

$$\frac{\partial f}{\partial x}(x, \kappa(x), 0) \neq 0,$$

for $x \neq x_0$. Then we can define the notion of slow vector field (see (De Maesschalck et al., 2021, Chapter 3)) along normally hyperbolic portions of the curve of singularities near $x = x_0$ (its flow is often called the slow dynamics). More precisely, if we write $\tau = \varepsilon^l t$, then system (9) becomes

$$\begin{cases} \varepsilon^l x' = f(x, y, \varepsilon), \\ \quad y' = g(x, y, \varepsilon), \end{cases} \tag{12}$$

where the prime $'$ denotes the derivative of $x(\tau)$ and $y(\tau)$ with respect to the *slow time* τ. The systems (9) and (12) are equivalent for $\varepsilon > 0$. If we let $\varepsilon \to 0$ in (12), we obtain the slow system

$$\begin{cases} 0 = f(x, y, 0), \\ y' = g(x, y, 0). \end{cases} \tag{13}$$

Using (13) and $y' = \kappa'(x)x' = -\frac{\frac{\partial f}{\partial x}(x,\kappa(x),0)}{\frac{\partial f}{\partial y}(x,\kappa(x),0)}x'$, we obtain the slow vector field

$$x' = -\frac{\frac{\partial f}{\partial y}(x, \kappa(x), 0)}{\frac{\partial f}{\partial x}(x, \kappa(x), 0)} g(x, \kappa(x), 0). \tag{14}$$

If $(x, y) = (x_0, y_0)$ is a slow-fast Hopf point, then using (11) and L'Hospital's rule it follows that (14) can be regularly extended through $x = x_0$ and the slow dynamics points, locally near $x = x_0$, from the normally attracting branch to the normally repelling branch of $y = \kappa(x)$. This means that, for $\varepsilon > 0$, orbits follow the attracting branch, pass through the slow-fast Hopf point, and then follow the repelling branch. See Fig. 1(a).

For classical Liénard equations (10), we assume

$$F(0) = F'(0) = 0 \quad \text{and} \quad \frac{F'(x)}{x} > 0, \ \forall x \in [-\rho, \rho], \tag{15}$$

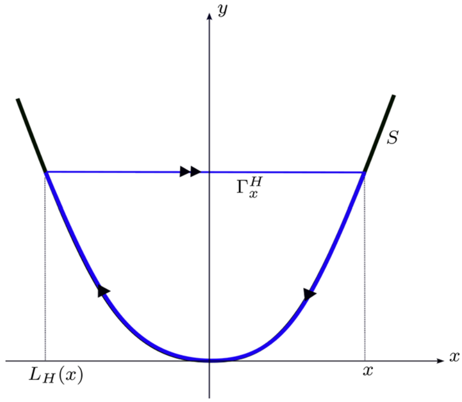
where ρ is a positive constant. The assumptions in (15) imply that the curve of singularities S contains a normally attracting branch ($x \in (0, \rho]$), a normally repelling branch ($x \in [-\rho, 0)$), and a slow-fast Hopf point at $(x, y) = (0, 0)$. In this case, the slow dynamics is given by

$$x' = \frac{dx}{d\tau} = -\frac{x}{F'(x)}, \tag{16}$$

where $\tau = \varepsilon^2 t$ is the slow time. Then the slow dynamics points from the attracting part to the repelling part of S near $x = 0$. See Fig. 2.

Denote the *fast relation function* by L_H (see Dumortier (2011)). For $x > 0$, a canard cycle Γ_x^H is the union of a fast orbit at height $y = F(x)$ and the part of the parabolic curve S between the α-limit point $(L_H(x), F(L_H(x)))$ and the ω-limit point $(x, F(x))$ of the fast orbit. We have $L_H(x) < 0$ and $F(x) = F(L_H(x))$. Such canard cycles are well-defined for $x \in (0, \min\{\rho, L_H^{-1}(-\rho)\}]$. See Fig. 2.

Fig. 2 A canard cycle Γ_x^H created by the Hopf mechanism



The slow divergence integral (see (De Maesschalck et al., 2021, Chapter 5) and De Maesschalck and Huzak (2015)) associated with Γ_x^H is given by

$$I_H(x) := \int_x^{L_H(x)} \frac{(F'(s))^2}{s} ds, \quad x \in (0, \min\{\rho, L_H^{-1}(-\rho)\}]. \tag{17}$$

This is the integral of the divergence of the vector field (10) for $\varepsilon = 0$ (which is equal to $-F'(x)$) with respect to the slow time τ (which is $d\tau = -\frac{F'(x)}{x} dx$).

If we add a breaking parameter a to the slow-fast Hopf point at $(x, y) = (0, 0)$, the canard cycles Γ_x^H can produce limit cycles for $\varepsilon > 0$ small. More precisely, a criterion for the existence of limit cycles produced by Γ_x^H is given in Theorem 3.1, and its proof can be found in (De Maesschalck and Huzak, 2015, Theorem 2).

Theorem 3.1 *Suppose that $I_H(x)$ has exactly k simple zeros $x_1 < \cdots < x_k$ in $(0, \min\{\rho, L_H^{-1}(-\rho)\})$. Let $x_{k+1} \in (0, \min\{\rho, L_H^{-1}(-\rho)\}]$ satisfy $x_k < x_{k+1}$. Then there is a smooth function $a = a(\varepsilon)$ with $a(0) = 0$, so that the perturbed system*

$$\begin{cases} \dot{x} = y - F(x), \\ \dot{y} = \varepsilon^2 (a(\varepsilon) - x), \end{cases}$$

has $k + 1$ periodic orbits $\mathcal{O}_\varepsilon^{x_i}$, with $i = 1, \ldots, k + 1$, for each $\varepsilon > 0$ small enough. The periodic orbit $\mathcal{O}_\varepsilon^{x_i}$ is isolated, hyperbolic and Hausdorff close to the canard cycle $\Gamma_{x_i}^H$ as $\varepsilon \to 0$.

3.2 The Jump Breaking Mechanism

We say that (9) has a (generic) *jump point* (see (De Maesschalck et al., 2021, Definition 2.3)) at $(x, y) = (x_0, y_0)$ if

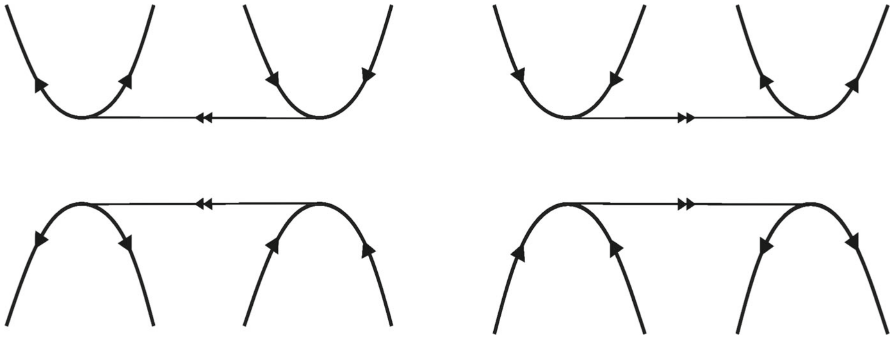


Fig. 3 Jump connections

$$f(x_0, y_0, 0) = \frac{\partial f}{\partial x}(x_0, y_0, 0) = 0, \quad \frac{\partial^2 f}{\partial x^2}(x_0, y_0, 0) \neq 0,$$
$$\frac{\partial f}{\partial y}(x_0, y_0, 0) \neq 0 \text{ and } g(x_0, y_0, 0) \neq 0. \tag{18}$$

If $(x, y) = (x_0, y_0)$ is a jump point, we can define the slow dynamics (14) in the same way as in Sect. 3.1. However, in this case (18) implies that the vector field in (14) is unbounded near $(x, y) = (x_0, y_0)$ and the orbits must jump. The slow dynamics is directed towards the jump point on both branches of S or away from the jump point on both branches.

We say that (9) has a *jump connection* if there are two jump points of (9) that are connected by a fast orbit and such that the curve of singularities S, locally near both jump points, is either concave up or concave down and the directions of the fast and slow dynamics are compatible (see Fig. 3). The slow dynamics is therefore directed towards one jump point and away from the other one, and the function g must have different sign near the jump points. Such a jump connection can be contained in canard cycles (see e.g. Figure 1(b)). Of course, system (9) can have other types of jump connections, for instance, one jump point is concave up and the other one is concave down (see (De Maesschalck et al., 2021, Section 6.2)). They are not considered in this paper.

In the context of slow-fast Liénard equations (10), we assume that F depends on a parameter $b \in \mathbb{R}$ kept close to 0. We write

$$\begin{cases} \dot{x} = y - F_b(x), \\ \dot{y} = -\varepsilon^2 x. \end{cases} \tag{19}$$

A contact point $(x, F_b(x)) \in S$ with $x \neq 0$ is a (generic) jump point if $F_b''(x) \neq 0$ (see (18)). For $b = 0$, we assume that system (19) has a jump connection, that is, it has two jump points, which will be denoted by $p_- = (x_-, F_0(x_-))$ and $p_+ = (x_+, F_0(x_+))$, and they are connected by a fast orbit γ. More precisely, the function F_0 has two minima of Morse type at $x = x_- < 0$ and $x = x_+ > 0$ such that $F_0(x_-) = F_0(x_+)$. We also suppose that

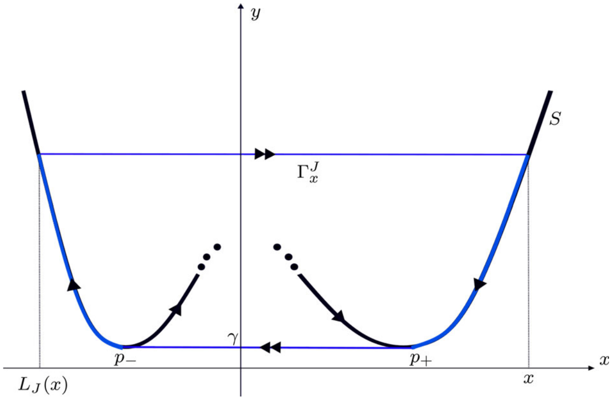


Fig. 4 A canard cycle Γ_x^J created by the jump mechanism, for $b = 0$

$$xF_0'(x) > 0,\ \forall x \in [x_- - \rho, x_-) \cup (x_+, x_+ + \rho], \tag{20}$$

for a positive constant ρ.

The slow dynamics associated to (19) is also given by Equation (16). However, in this case, using the assumption (20), the slow dynamics goes towards p_+ and moves away from p_-. Moreover, p_- (resp. p_+) is the ω-limit (resp. the α-limit) of the fast orbit γ. See Fig. 4.

We now define a canard cycle Γ_x^J for $x > x_+$ and $b = 0$. It is the union of a fast orbit at height $y = F_0(x)$, the fast orbit γ, the attracting portion of S between $(x, F_0(x)) \in S$ and $p_+ \in S$, and the repelling portion of S between $p_- \in S$ and $(L_J(x), F_0(L_J(x))) \in S$, where $L_J(x)$ is the fast relation function (see Fig. 4). We have $L_J(x) < x_-$ and $F_0(x) = F_0(L_J(x))$. We assume that there is a constant $\rho_0 > x_+$ such that the canard cycle Γ_x^J is well-defined for $x \in (\rho_0, \min\{x_+ + \rho, L_J^{-1}(x_- - \rho)\}]$, for some $\rho > 0$.

The jump points $p_\pm$ persist for all b close to 0, and we denote them by $p_\pm(b)$ with $p_\pm(0) = p_\pm$. We define

$$h(b) := F_b(x_+(b)) - F_b(x_-(b)).$$

Clearly, $h(0) = 0$. We assume that b is a regular parameter for the jump breaking mechanism, that is, $h'(0) \neq 0$.

The slow divergence integral related to Γ_x^J is given by

$$I_J(x) := \int_x^{x_+} \frac{(F_0'(s))^2}{s} ds + \int_{x_-}^{L_J(x)} \frac{(F_0'(s))^2}{s} ds, \tag{21}$$

for $x \in (\rho_0, \min\{x_+ + \rho, L_J^{-1}(x_- - \rho)\}]$.

A proof of the following result (similar to Theorem 3.1) can be found in Dumortier (2011).

Theorem 3.2 *Denote system* (19) *by* $X_{\varepsilon,b}$ *and suppose that* $I_J(x)$ *defined by* (21) *has exactly* k *simple zeros. Then there is a smooth function* $b = b(\varepsilon)$ *with* $b(0) = 0$*, so that* $X_{\varepsilon,b(\varepsilon)}$ *has* $k+1$ *periodic orbits for each* $\varepsilon > 0$ *sufficiently small. All of them are isolated, hyperbolic and Hausdorff close to canard cycles* Γ_x^J *as* $\varepsilon \to 0$.

3.3 Regularized PWL Systems and Generic Breaking Mechanisms

In this section, we discuss how the breaking mechanisms studied in Sects. 3.1 and 3.2 appear in regularizations of PWL vector fields (or more general piecewise smooth vector fields). The regularized system (5) is a special case of (9) with $l = 1$ and

$$f(x, y, \varepsilon) = \frac{X_1(\varepsilon x, y) + Y_1(\varepsilon x, y)}{2} + \varphi(x)\frac{X_1(\varepsilon x, y) - Y_1(\varepsilon x, y)}{2}, \tag{22}$$

and

$$g(x, y, \varepsilon) = \frac{X_2(\varepsilon x, y) + Y_2(\varepsilon x, y)}{2} + \varphi(x)\frac{X_2(\varepsilon x, y) - Y_2(\varepsilon x, y)}{2}. \tag{23}$$

Suppose that $(x, y) = (x_0, y_0)$ is a slow-fast Hopf point or a jump point of (5). Then Equation (22) implies that $\varphi'(x_0) = 0$ and $\varphi''(x_0) \neq 0$. Indeed, conditions $\frac{\partial f}{\partial x}(x_0, y_0, 0) = 0$ and $\frac{\partial^2 f}{\partial x^2}(x_0, y_0, 0) \neq 0$ can be written as

$$\varphi'(x_0)\,(X_1(0, y_0) - Y_1(0, y_0)) = 0 \text{ and } \varphi''(x_0)\,(X_1(0, y_0) - Y_1(0, y_0)) \neq 0.$$

This is equivalent to

$$\varphi'(x_0) = 0,\ \varphi''(x_0) \neq 0 \text{ and } X_1(0, y_0) - Y_1(0, y_0) \neq 0.$$

For the chosen $l = 1$ and functions f and g, we get the following result.

Proposition 3.3 *In Equation* (9)*, assume* $l = 1$ *and* f, g *defined in* (22) *and* (23)*, respectively. Then the following statements are true.*

(1) For $x_0 \in (-1, 1)$*, the point* (x_0, y_0) *is not a slow-fast Hopf point.*
(2) If (x_0, y_0) *and* (x_1, y_0) *are generic jump points with* $x_{0,1} \in (-1, 1)$*, then a jump connection is not possible.*

Proof (1). Suppose that (x_0, y_0) with $x_0 \in (-1, 1)$ is a slow-fast Hopf point, that is, the assumption (11) is satisfied. Conditions $\frac{\partial f}{\partial x}(x_0, y_0, 0) = 0$ and $\frac{\partial^2 f}{\partial x^2}(x_0, y_0, 0) \neq 0$ imply $\varphi'(x_0) = 0$, and $\frac{\partial g}{\partial x}(x_0, y_0, 0) \neq 0$ and (23) imply $\varphi'(x_0) \neq 0$, which is a contradiction.

(2). Suppose that there is a jump connection between (x_0, y_0) and (x_1, y_0), that is, the assumption (18) is satisfied for both points and the orientation of the slow dynamics is compatible. Since $f(x_i, y_0, 0) = \frac{\partial f}{\partial x}(x_i, y_0, 0) = 0$ and $\frac{\partial^2 f}{\partial x^2}(x_i, y_0, 0) \neq 0$, for $i = 0, 1$, it follows that

$$X_1(0, y_0) - Y_1(0, y_0) \neq 0, \quad \text{and} \quad \varphi(x_0) = \varphi(x_1) = -\frac{X_1(0, y_0) + Y_1(0, y_0)}{X_1(0, y_0) - Y_1(0, y_0)}.$$

However, $\varphi(x_0) = \varphi(x_1)$ and (23) imply $g(x_0, y_0, 0) = g(x_1, y_0, 0)$, therefore the orientation of the slow dynamics is not compatible, because the function g has the same sign when evaluated in such points. Therefore we do not have a jump connection. □

Remark 1 We cannot exclude the possibility of having slow-fast Hopf points or jump connections in (5). Proposition 3.3 only suggests that $l = 1$ is not a good choice. As we shall see in the following, they can occur when $g(x, y, \varepsilon) = \varepsilon \tilde{g}(x, y, \varepsilon)$ for some smooth function $\tilde{g}$ (recall that g is defined in (23)).

Consider a PWL vector field of the form

$$Z(x, y) = \begin{cases} X(x, y) = (a_1 + b_1 x + y, \varepsilon a + b_2 x), & x > 0, \\ Y(x, y) = (\alpha_1 + \beta_1 x + y, \varepsilon a + \beta_2 x), & x < 0. \end{cases} \tag{24}$$

A linear regularization (5) of Z leads to the slow-fast Liénard equation

$$\begin{cases} \dot{x} = y + \frac{1}{2}\Big(a_1 + \alpha_1 + (b_1 + \beta_1)\varepsilon x + \big(a_1 - \alpha_1 + (b_1 - \beta_1)\varepsilon x\big)\varphi(x)\Big), \\ \dot{y} = \varepsilon^2 \left(a + \frac{x}{2}\big(b_2 + \beta_2 + (b_2 - \beta_2)\varphi(x)\big)\right). \end{cases} \tag{25}$$

The system (25) is a special case of (9) with $l = 2$. For $\varepsilon = 0$, system (25) has the curve of singularities $\{y = F(x) := -\frac{1}{2}\big(a_1 + \alpha_1 + (a_1 - \alpha_1)\varphi(x)\big)\}$.

Observe that the systems (7), with $\alpha = \varepsilon a$, and (8) are of the form (24), and then we can explicitly write their regularizations in the form (25). In particular, these regularizations reduce to classical Liénard equations (see Sect. 4). The main advantage of working with the system (25) is that slow-fast Hopf points and jump connections can occur in (25).

4 Proof of the Main Results

In this section, we prove Theorems A and B. Theorem 3.1 (resp. Theorem 3.2), stated for the classical Liénard equations, will play a crucial role in the proof of Theorem A (resp. Theorem B).

In this paper, we do not focus on the description of the phase portraits for different values of the coefficients of X and Y in Equation (24) because we do not use such a description in the proof of Theorems A and B. Recall that we are only interested in the limit cycles inside the regularization stripe.

For the sake of readability, we rewrite system (5):

$$\begin{cases} \dot{x} = \dfrac{X_1(\varepsilon x, y) + Y_1(\varepsilon x, y)}{2} + \varphi(x)\dfrac{X_1(\varepsilon x, y) - Y_1(\varepsilon x, y)}{2}, \\ \dot{y} = \varepsilon\left(\dfrac{X_2(\varepsilon x, y) + Y_2(\varepsilon x, y)}{2} + \varphi(x)\dfrac{X_2(\varepsilon x, y) - Y_2(\varepsilon x, y)}{2}\right). \end{cases} \tag{26}$$

4.1 Proof of Theorem A

Define the function

$$F(x) := \frac{1}{2}x^2 + \delta F_o(x), \tag{27}$$

where F_o is an odd function satisfying $F_o'(0) = 0$ and δ is a parameter kept close to zero. Notice that assumption (15) is satisfied in any fixed segment $[-\rho, \rho]$ by taking δ small enough. By (De Maesschalck and Huzak, 2015, Proposition 1), the slow divergence integral (17) can be written as

$$I_H(x) = -2\delta\left(F_o(x) + O(\delta)\right).$$

Indeed, following the notation of (De Maesschalck and Huzak, 2015, Proposition 1), in our case we have $F_e(x) = \frac{1}{2}x^2$ and $f_e = 1$. This implies that simple zeros of F_o persist as simple zeros of I_H, for small $\delta \neq 0$.

Recall the PWL vector field (7), which is given by

$$Z(x, y, \alpha) = \begin{cases} X(x, y, \alpha) = (-3 + y, \alpha - x), \ x > 0, \\ Y(x, y, \alpha) = (-1 + y, \alpha - x), \ x < 0, \end{cases}$$

where α is a breaking parameter kept close to zero. The points $(\alpha, 3)$ and $(\alpha, 1)$ are linear centers of X and Y, respectively. Define $\psi(x) := F(x) - 2$, where F is given in (27). For $\alpha = \varepsilon a$, with a close to zero, consider the slow-fast system

$$\begin{cases} \dot{x} = y - (\psi(x) + 2), \\ \dot{y} = \varepsilon^2(a - x). \end{cases} \tag{28}$$

Recall that (28) is obtained from (26), with $\varphi = \psi$, but it is not yet a linear regularization of the PWL system (7) because ψ is not a transition function. However, ψ will be important for the construction of the transition function.

Let $k > 0$ be an integer. Take

$$F_o(x) = x^3(x^2 - \tilde{x}_1^2) \cdot \cdots \cdot (x^2 - \tilde{x}_k^2),$$

with $0 < \tilde{x}_1 < \cdots < \tilde{x}_k$. We can assume that the simple zeros $\tilde{x}_1, \ldots, \tilde{x}_k$ of F_o are small enough and fixed so that the canard cycle $\Gamma_{\tilde{x}_i}^H$, associated with (28) for $a = 0$, is contained in the stripe $\{-1 < x < 1\}$ (or, equivalently, $\tilde{x}_i < 1$ and $L_H(\tilde{x}_i) > -1$),

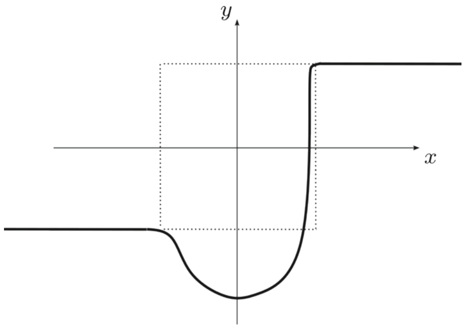


Fig. 5 A transition function φ_k with one critical point in $(-1, 1)$, located at $x = 0$

for each $i = 1, \ldots, k$, and for each δ small. It is clear from the construction of ψ and F_o that

$$\psi(0) = -2, \quad \psi'(0) = 0, \quad \psi''(0) > 0, \quad \psi(-1) < -1, \quad \psi(1) < 1,$$

and $\psi'(x) > 0$ for $x \in (0, 1]$ and $\psi'(x) < 0$ for $x \in [-1, 0)$, for each δ sufficiently small. Now, we fix a $\delta \neq 0$ sufficiently small so that the above properties are satisfied and such that the slow divergence integral $I_H(x)$ has k simple zeros $0 < x_1 \cdots < x_k$, with x_i close to $\tilde{x}_i$, $x_i < 1$ and $L_H(x_i) > -1$, $i = 1, \ldots, k$.

We choose $x_{k+1} > x_k$ satisfying $x_{k+1} < 1$ and $L_H(x_{k+1}) > -1$. From Theorem 3.1 it follows that there exists a smooth function $a = a_k(\varepsilon)$ with $a_k(0) = 0$, so that system (28), with $a = a_k(\varepsilon)$, has $k+1$ hyperbolic limit cycles $\mathcal{O}_\varepsilon^{x_i}$, with $i = 1, \ldots, k+1$, for each $\varepsilon > 0$ small enough. Moreover, the limit cycle $\mathcal{O}_\varepsilon^{x_i}$ converges in the Hausdorff sense to the canard cycle $\Gamma_{x_i}^H$ as $\varepsilon \to 0$. This implies that there is a constant $\rho \in (0, 1)$, with ρ close to 1, such that the limit cycle $\mathcal{O}_\varepsilon^{x_i}$ is contained in the stripe $\{-\rho < x < \rho\}$, for each $i = 1, \ldots, k+1$ and for each $\varepsilon > 0$ sufficiently small.

4.1.1 Construction of the Transition Function

Notice that the polynomial $\psi : \mathbb{R} \to \mathbb{R}$, defined above, is not a transition function. Our goal now is to construct a transition function φ_k satisfying $\varphi_k(x) \equiv \psi(x)$, for all $x \in [-\rho, \rho]$, and such that φ_k has no additional critical points, besides the one at $x = 0$ (see Fig. 5). Indeed, consider the *cut-off functions* $A, B : \mathbb{R} \to \mathbb{R}$ (see Lovett (2010)) satisfying

1. Both A, B are smooth;
2. $A(x) = 0$ for $x \leq \rho$, $A(x) = 1$ for $x \geq 1$, and $A'(x) > 0$ for $x \in (\rho, 1)$;
3. $B(x) = 0$ for $x \geq -\rho$, $B(x) = 1$ for $x \leq -1$, and $B'(x) < 0$ for $x \in (-1, -\rho)$.

Define

$$\varphi_k(x) := A(x)\,(1 - B(x)) - B(x) + \psi(x)\,(1 - A(x))\,(1 - B(x))\,.$$

We claim that $\varphi_k(x)$ is a transition function. It can be checked that

$$\varphi_k(x) = \begin{cases} -1, & x \leq -1, \\ -B(x) + \psi(x)\,(1 - B(x))\,, & x \in (-1, -\rho), \\ \psi(x), & x \in [-\rho, \rho], \\ A(x) + \psi(x)\,(1 - A(x))\,, & x \in (\rho, 1), \\ 1, & x \geq 1. \end{cases}$$

It remains to prove that $\varphi_k'(x) < 0$, for $x \in (-1, -\rho)$, and $\varphi_k'(x) > 0$, for $x \in (\rho, 1)$. Let us first consider the interval $x \in (-1, -\rho)$. Since $0 < B(x) < 1$, $B'(x) < 0$, $\psi(x) < -1$ and $\psi'(x) < 0$, for all $x \in (-1, -\rho)$, we have

$$\varphi_k'(x) = -B'(x)\,(1 + \psi(x)) + \psi'(x)\,(1 - B(x)) < 0,$$

for all $x \in (-1, -\rho)$. Similarly, since $0 < A(x) < 1$, $A'(x) > 0$, $\psi(x) < 1$ and $\psi'(x) > 0$, for all $x \in (\rho, 1)$, we have

$$\varphi_k'(x) = A'(x)\,(1 - \psi(x)) + \psi'(x)\,(1 - A(x)) > 0,$$

for each $x \in (\rho, 1)$. Thus, φ_k is a transition function with precisely one critical point in the interval $(-1, 1)$, which is $x = 0$.

Consider the slow-fast system

$$\begin{cases} \dot{x} = y - (\varphi_k(x) + 2), \\ \dot{y} = \varepsilon^2 (a - x). \end{cases} \tag{29}$$

Notice that the vector fields in (28) and (29) are equal in the stripe $\{-\rho < x < \rho\}$, and $\mathcal{O}_\varepsilon^{x_i}$ is therefore a (hyperbolic) limit cycle of (29) with $a = a_k(\varepsilon)$, for each $i = 1, \ldots, k + 1$ and for each $\varepsilon > 0$ small enough. We conclude that the φ_k-linear regularization (6), with X, Y defined above, and $\alpha = \varepsilon a_k(\varepsilon)$, has at least $k + 1$ hyperbolic limit cycles, for each $\varepsilon > 0$ small enough. This completes the proof of Theorem A.

Remark 2 In the proof of Theorem A, we used the PWL system (7). Consider the following generalization of (7):

$$Z(x, y, \alpha) = \begin{cases} X(x, y, \alpha) = (-c_+ + y, \alpha - x),\ x > 0, \\ Y(x, y, \alpha) = (-c_- + y, \alpha - x),\ x < 0, \end{cases} \tag{30}$$

where α is a breaking parameter kept close to zero and $c_\pm \in \mathbb{R}$. The points (α, c_+) and (α, c_-) are linear centers of X and Y, respectively. The PWL system (7) is a special case of (30) with $c_+ = 3$ and $c_- = 1$. One can reproduce a completely analogous proof of Theorem A using (30), but for this we make the following assumption on $c_\pm$:

$$1 < 2c_- < 2c_+.$$

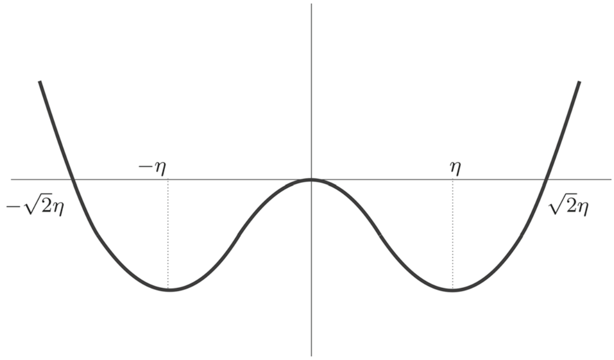


Fig. 6 Graph of the function F_b defined in (31) for $b = \delta = 0$. The points $x_\pm = \pm\eta$ are critical points and $z_\pm = \pm\sqrt{2}\eta$ are simple zeros. All these points are inside the open interval $(-1, 1)$

Define $\psi(x) := \frac{F(x)-C_+}{C_-}$, with $F(x)$ given by (27) and $C_+ = \frac{c_+ + c_-}{2}$ and $C_- = \frac{c_+ - c_-}{2}$. This assumption implies that $0 < C_- < C_+$, and that $\psi(-1) < -1$ and $\psi(1) < 1$, for δ sufficiently small. Moreover, using the definition of F, we have $\psi(0) < -1$, $\psi'(0) = 0$, $\psi''(0) > 0$, $\psi'(x) > 0$ for $x \in (0, 1]$ and $\psi'(x) < 0$ for $x \in [-1, 0)$, for each δ sufficiently small.

Now, with the above assumption in mind, one can prove Theorem A in a similar way using (30). However, the proof is given for $c_+ = 3$ and $c_- = 1$ for the sake of readability.

4.2 Proof of Theorem B

Consider the function

$$F_b(x) := P_e(x) + bx + \delta P_o(x), \tag{31}$$

in which b is a breaking parameter kept close to 0, δ is a perturbation parameter close to 0, P_e is given by

$$P_e(x) = \frac{x^4}{4} - \frac{\eta^2}{2}x^2,$$

where η is a positive and fixed constant satisfying $0 < \eta < \frac{\sqrt{2}}{2}$, and P_o is C^∞-smooth in $\mathbb{R}$ and odd in the symmetric set $D := (-\infty, -\eta] \cup [\eta, \infty)$ (that is, $P_o(-x) = -P_o(x)$ for all $x \in D$). Furthermore, we assume that $P_o(\eta) = P_o'(\eta) = 0$. Since P_o is odd in D, we have $P_o(-\eta) = P_o'(-\eta) = 0$.

The even polynomial P_e has a zero of multiplicity two positioned at the origin, and two simple zeros $z_\pm = \pm\sqrt{2}\eta \in (-1, 1)$. It is important to note that, for $b = 0$, the points $x_\pm = \pm\eta$ are critical points of Morse type (both minima) of F_0 and $F_0(x_-) = F_0(x_+)$, for every δ small enough ($x_\pm$ do not depend on δ). We refer to Fig. 6.

Using the properties of P_e and P_o, we can directly see that $F_0'(x) > 0$ for $x \in (\eta, 1]$ and $F_0'(x) < 0$ for $x \in [-1, -\eta)$, for every δ sufficiently small (see assumption (20)).

Consider the PWL vector field given in (8), which has the form

$$Z(x, y) = \begin{cases} X(x, y) = (y - 3, -x), \\ Y(x, y) = (y - 1, -x). \end{cases}$$

The singular points $(0, 3), (0, 1) \in \Sigma$ are linear centers of X, Y, respectively. In Equation (26), we set the function $\varphi(x) = \psi_b(x)$ satisfying

$$\psi_b(x) := F_b(x) - 2. \tag{32}$$

Then, using (8), we obtain the slow-fast system

$$\begin{cases} \dot{x} = y - F_b(x), \\ \dot{y} = -\varepsilon^2 x. \end{cases} \tag{33}$$

System (33) is not yet a regularization for (8), because ψ_b in (32) is not a transition function. Straightforward calculations lead to $F_b'(x) = \psi_b'(x)$, where the prime $'$ denotes the derivative with respect to the variable x.

The slow divergence integral (21) has the form

$$I_J(x) = \int_x^{\eta} \frac{\left(F_0'(s)\right)^2}{s} ds + \int_{-\eta}^{L_J(x)} \frac{\left(F_0'(s)\right)^2}{s} ds. \tag{34}$$

We keep x in (34) in a compact interval $D_1 \subset (\sqrt{2}\eta, 1)$. Then the fast relation function $L_J(x)$ and the canard cycle Γ_x^J are well defined, and $L_J(x) \in (-1, -\sqrt{2}\eta)$, for all $x \in D_1$ and every δ sufficiently small (see Figs. 4 and 6).

4.2.1 Regularity of the Breaking Parameter

We must verify that b is a regular breaking parameter, and it is enough to prove this for $\delta = 0$. We know that the jump points $x_\pm$ persist for all b close to 0. Using asymptotic expansions in b and the notation introduced in Sect. 3.2, we can write the solutions $x_\pm(b)$ of $F_b'(x) = 0$ as

$$x_+(b) = \eta - \frac{b}{2\eta^2} - \frac{3b^2}{8\eta^5} + O(b^3), \qquad x_-(b) = -\eta - \frac{b}{2\eta^2} + \frac{3b^2}{8\eta^5} + O(b^3).$$

It easily follows that

$$h(b) = F_b\left(x_+(b)\right) - F_b\left(x_-(b)\right) = 2\eta b + O(b^2).$$

Therefore $h'(0) = 2\eta \neq 0$ and it is a submersion. In other words, b is a regular breaking parameter.

4.2.2 Asymptotics of the Fast Relation Function

The next step is to find the asymptotics of the fast relation function $L_J(x)$. Recall that $b = 0$. Assuming $\delta = 0$, it follows that F_0 in (31) is an even function. Therefore, we can write $L_J(x) = -x + \delta L_1(x) + O(\delta^2)$. In addition, since P_o is odd in D and P_e is even, it is true that, for $x \in D_1 \subset D$,

$$P_o(L_J(x)) = -P_o(x) + O(\delta), \qquad P_e(L_J(x)) = P_e(x) - \delta P_e'(x)L_1(x) + O(\delta^2).$$

Therefore, from $F_0(x) = F_0(L_J(x))$ one obtains

$$P_e(x) + \delta P_o(x) = P_e(x) - \delta P_e'(x)L_1(x) - \delta P_o(x) + O(\delta^2),$$

and finally one obtains

$$L_1(x) = \frac{-2P_o(x)}{P_e'(x)} = \frac{-2P_o(x)}{x(x^2-\eta^2)}, \ x \in D_1. \tag{35}$$

4.2.3 Asymptotics of the Slow Divergence Integral

Now, we will study asymptotics of the slow divergence integral $I_J(x)$ given by (34). Recall that $x \in D_1$ and δ is close to zero.

We have

$$F_0'(s) = P_e'(s) + \delta P_o'(s), \qquad \left(F_0'(s)\right)^2 = \left(P_e'(s)\right)^2 + 2\delta P_e'(s)P_o'(s) + O(\delta^2).$$

Using the expression for P_e, we can write

$$\begin{aligned}\int_x^\eta \frac{(F_0'(s))^2}{s}ds &= \int_x^\eta s(s^2-\eta^2)^2 ds + 2\delta\int_x^\eta (s^2-\eta^2)P_o'(s)ds + O(\delta^2),\\ \int_{-\eta}^{L_J(x)} \frac{(F_0'(s))^2}{s}ds &= \int_{-\eta}^{L_J(x)} s(s^2-\eta^2)^2 ds + 2\delta\int_{-\eta}^{L_J(x)} (s^2-\eta^2)P_o'(s)ds\\ &\quad + O(\delta^2).\end{aligned} \tag{36}$$

Now, we will handle the integrals of Equation (36). Firstly, using straightforward computations and (35), one can show that

$$\begin{aligned}\int_x^\eta s(s^2-\eta^2)^2 ds + \int_{-\eta}^{L_J(x)} s(s^2-\eta^2)^2 ds &= -\delta x(x^2-\eta^2)^2 L_1(x) + O(\delta^2)\\ &= 2\delta(x^2-\eta^2)P_o(x) + O(\delta^2).\end{aligned} \tag{37}$$

Recall that the function P_o is odd in D. This, $L_J(x) = -x + O(\delta)$ and partial integration imply

$$\begin{aligned}\int_x^{\eta}(s^2-\eta^2)P_o'(s)ds &= -(x^2-\eta^2)P_o(x) - \int_x^{\eta} 2sP_o(s)ds,\\ \int_{-\eta}^{L_J(x)}(s^2-\eta^2)P_o'(s)ds &= -(x^2-\eta^2)P_o(x) - \int_{-\eta}^{L_J(x)} 2sP_o(s)ds + O(\delta).\end{aligned} \tag{38}$$

We also have

$$-\int_{-\eta}^{L_J(x)} 2\,sP_o(s)ds = -\int_x^{\eta} 2\,sP_o(s)ds + O(\delta).$$

Therefore, combining Equations (36), (37) and (38), one obtains the following asymptotics for the slow divergence integral I_J:

$$\begin{aligned}I_J(x) &= \int_x^{\eta}\frac{\left(F_0'(s)\right)^2}{s}ds + \int_{-\eta}^{L_J(x)}\frac{\left(F_0'(s)\right)^2}{s}ds\\ &= 2\delta(x^2-\eta^2)P_o(x) - 4\delta\left((x^2-\eta^2)P_o(x) + \int_x^{\eta}2sP_o(s)ds\right) + O(\delta^2)\\ &= -2\delta\left((x^2-\eta^2)P_o(x) + 2\int_x^{\eta}2sP_o(s)ds\right) + O(\delta^2)\\ &= 2\delta\int_{\eta}^{x}\Big(2sP_o(s) - (s^2-\eta^2)P_o'(s)\Big)ds + O(\delta^2),\end{aligned}$$

with $x \in D_1$. In the last step, we used partial integration.

Therefore, simple zeros of

$$I_1(x) = \int_{\eta}^{x}\Big(2sP_o(s) - (s^2-\eta^2)P_o'(s)\Big)ds \tag{39}$$

persist as simple zeros of I_J, for small $\delta \neq 0$.

4.2.4 Constructing Zeros of the Slow Divergence Integral

First, we show that there is a smooth function $P_o : \mathbb{R} \to \mathbb{R}$ (P_o is odd in D and $P_o(\eta) = P_o'(\eta) = 0$) such that the associated integral function I_1, defined by (39), has k simple zeros in the interior of D_1. Recall that $D_1 \subset (\sqrt{2}\eta, 1)$.

Consider a polynomial

$$\widetilde{P}(x) = (x^2-\eta^2)^3(x-\tilde{x}_1)\cdot\cdots\cdot(x-\tilde{x}_k),$$

with $\tilde{x}_1 < \cdots < \tilde{x}_k$ contained in the interior of D_1. For $x \geq \eta$, P_o is defined as follows:

$$P_o(x) := -(x^2 - \eta^2) \int_\eta^x \frac{\widetilde{P}'(s)}{(s^2 - \eta^2)^2} ds. \tag{40}$$

From (40) it follows that $P_o(\eta) = P_o'(\eta) = 0$. Since P_o has to be odd in D, we define $P_o(x) := -P_o(-x)$, for $x \leq -\eta$. Using cut-off functions (see Lovett (2010)), it is not difficult to see that P_o can be smoothly extended to $\mathbb{R}$ (notice that the behavior of P_o for $x \in (-\eta, \eta)$ is not relevant when we study zeros of I_J). We fix this P_o.

Now, it suffices to prove that $I_1(x) = \widetilde{P}(x)$, for $x \geq \eta$. This will imply that I_1 has k simple zeros $\tilde{x}_1, \ldots, \tilde{x}_k$. Indeed, we have

$$2sP_o(s) - (s^2 - \eta^2)P_o'(s) = \widetilde{P}'(s),\ s \geq \eta.$$

It follows from (40). Using (39) and $\widetilde{P}(\eta) = 0$, we conclude that $I_1(x) = \widetilde{P}(x)$, for $x \geq \eta$.

We know that the simple zeros of I_1 persist as simple zeros of I_J, for $\delta \neq 0$ sufficiently small. More precisely, we can fix a small $\delta \neq 0$ so that the slow divergence integral $I_J(x)$ has k simple zeros $x_1 < \cdots < x_k$ contained in the interior of D_1, with x_i close to $\tilde{x}_i$, $i = 1, \ldots, k$.

Now, as well as in the Hopf case, take $x_{k+1} > x_k$ satisfying $x_{k+1} \in D_1$. It follows from Theorem 3.2 that there exists a smooth function $b = b_k(\varepsilon)$ with $b_k(0) = 0$ such that system (33), with $F_b = F_{b_k(\varepsilon)}$, has $k+1$ hyperbolic limit cycles $\mathcal{O}_\varepsilon^{x_i}$, with $i = 1, \ldots, k+1$, for each $\varepsilon > 0$ small enough. Moreover, the limit cycle $\mathcal{O}_\varepsilon^{x_i}$ is Hausdorff close to the canard cycle $\Gamma_{x_i}^J$ as $\varepsilon \to 0$. Notice that the $k+1$ limit cycles are contained in the stripe $\{-1 < x < 1\}$.

4.2.5 Constructing the Transition Function

The construction of a suitable transition function $\varphi_{b,k}$ from ψ_b, defined in (32), can be done in the same fashion as in Sect. 4.1. Observe that $\psi_b(-1) < -1$ and $\psi_b(1) < 1$, for b and δ close to zero. We also have $\psi_0'(x) < 0$ in $[-1, -\rho]$ and $\psi_0'(x) > 0$ in $[\rho, 1]$, where $\rho \in (0, 1)$, with ρ close to 1 and $x_{k+1}, L_J(x_{k+1}) \in (-\rho, \rho)$, and δ is close to zero. It follows from the property of $F_0'(x)$ given in the beginning of Sect. 4.2. Notice that ψ_b has precisely three critical points in $(-\rho, \rho)$ and they are of Morse type, for b and δ close to zero.

Now, following the steps of Sect. 4.1, we replace ψ_b with a non-monotonic transition function $\varphi_{b,k}$ having precisely 3 critical points, see Fig. 7. The $\varphi_{b,k}$-linear regularization of the PWL system (8) has the form

$$\begin{cases} \dot{x} = y - (\varphi_{b(\varepsilon),k}(x) + 2), \\ \dot{y} = -\varepsilon^2 x, \end{cases} \tag{41}$$

and this system has $k+1$ hyperbolic limit cycles. This completes the proof of Theorem B.

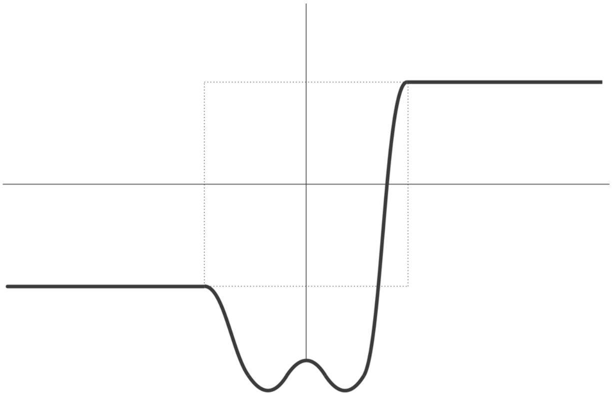


Fig. 7 Transition function $\varphi_{b,k}$ with three critical points in $(-1, 1)$

Remark 3 Consider

$$Z(x, y) = \begin{cases} X(x, y) = (-c_+ + y, -x), \ x > 0, \\ Y(x, y) = (-c_- + y, -x), \ x < 0, \end{cases} \tag{42}$$

The points $(0, c_+)$ and $(0, c_-)$ are linear centers of X and Y, respectively (see Remark 2). The PWL system (8) is a special case of (42) with $c_+ = 3$ and $c_- = 1$. One can reproduce a completely analogous proof of Theorem B using (42), under the following assumption on $c_\pm$:

$$\frac{1}{4} - \frac{\eta^2}{2} < c_- < c_+.$$

If we define $\psi_b(x) := \frac{F_b(x) - C_+}{C_-}$, with $F_b(x)$ given in (31) and $C_\pm$ introduced in Remark 2, then it is not difficult to see that such a ψ_b has similar properties as the function ψ_b defined in (32).

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Declarations

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