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Topological formulation of termination properties of iterates of functions

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Abstract

We consider a number of decision problems, that appear in the dynamical systems and database literature, concerning the termination of iterates of real functions. These decision problems take a function $f:\mathbb{R}^n \to \mathbb{R}^n$ as input and ask, for example, whether this function is *mortal*, *nilpotent*, *terminating*, or *reaches a fixed point* on a given point in $\mathbb{R}^n$. We associate topologies to functions $f:\mathbb{R}^n \to \mathbb{R}^n$ and study some basic properties of these topologies. The contribution of this paper is a translation of the above mentioned decision problems into decision problems concerning well-known properties of topologies, e.g., connectivity. We also show that connectivity of topologies on $\mathbb{R}^n$ is undecidable for $n > 1$.

Keywords: Decision problems; Iterates of functions; Dynamical systems; Fixed points; Transitive closure; Topology; Databases

1. Introduction and summary of results

We consider properties, that originate from dynamical systems theory [1,2,5] but are also relevant to database theory [3], of iterates of functions $f:\mathbb{R}^n \to \mathbb{R}^n$ (by $\mathbb{R}$ we denote the real numbers). Here, we focus on four such properties. We abbreviate the origin $(0, 0, \ldots, 0)$ of $\mathbb{R}^n$ by $\mathbf{0}$. We call a function $f:\mathbb{R}^n \to \mathbb{R}^n$ *mortal* if $f(\mathbf{0}) = \mathbf{0}$ and if for each $\boldsymbol{x} \in \mathbb{R}^n$ there exists a natural number $k \geqslant 1$ such that $f^k(\boldsymbol{x}) = \mathbf{0}$ [2]. A function $f:\mathbb{R}^n \to \mathbb{R}^n$ is called *nilpotent* if $f(\mathbf{0}) = \mathbf{0}$ and if there exists a natural number $k \geqslant 1$ such that for all $\boldsymbol{x} \in \mathbb{R}^n$, $f^k(\boldsymbol{x}) = \mathbf{0}$ [2]. Clearly, nilpotency is a more restrictive property than mortality. The transitive closure of the graph of a function $f:\mathbb{R}^n \to \mathbb{R}^n$, viewed as a binary relation over $\mathbb{R}^n$, is traditionally computed by computing the $2n$-ary relations $TC_1(f), TC_2(f), TC_3(f), \ldots$, where $TC_1(f) = \text{graph}(f)$ and $TC_{i+1}(f) := TC_i(f) \cup \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{2n} \mid (\exists z)((\boldsymbol{x}, z) \in TC_i(f) \wedge f(z) = \boldsymbol{y})\}$. We call a function f *terminating* if this iterative computation of the transitive closure terminates after a finite number of iterations, i.e., if there exists a $k \geqslant 1$ such that $TC_{k+1}(f) = TC_k(f)$. Since these are Boolean properties of functions, we can associate to them a decision problem (i.e., the mapping that takes a function as input and returns whether the function has the property). Another decision problem is the *point-to-fixed-point problem*, which asks whether for a given algebraic number $\boldsymbol{x}$ and a given piecewise affine function

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$f:\mathbb{R}^n \to \mathbb{R}^n$, the sequence $\boldsymbol{x}, f(\boldsymbol{x}), f^2(\boldsymbol{x}), f^3(\boldsymbol{x}), \ldots$ reaches a fixed point, i.e., whether there exists a $k \geqslant 1$ such that $f^k(\boldsymbol{x}) = f^{k+1}(\boldsymbol{x})$ [1,5].

In the field of dynamical systems, it is often important that these decision problems are computable (or decidable), in the sense that there exists an algorithm that takes as input some finite representation of a function $f:\mathbb{R}^n \to \mathbb{R}^n$ and returns as output whether f has the property. About the above mentioned decision problems the following is known. Mortality and nilpotency are known to be undecidable for piecewise affine functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ and for functions from $\mathbb{R}$ to $\mathbb{R}$ the (un)decidability of these properties is open [2]. Termination of functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ is undecidable but termination of continuous semi-algebraic functions from $\mathbb{R}$ to $\mathbb{R}$ is decidable [3]. The decidability of the point-to-fixed-point problem is open for $n = 1$, even for piecewise linear functions with only two non-constant pieces [1,5].

The decidability of these decision problems has also implications in the area of database theory. For example, the decidability of termination of continuous semi-algebraic functions from $\mathbb{R}$ to $\mathbb{R}$ was used to obtain extensions of first-order logics with recursion, based on a transitive-closure operator [3], that are used as query languages for constraint databases [6]. These extensions of first-order logics are more expressive than these logics as such and they allow the expression of recursive queries whose computation is guaranteed to terminate. Decidability results concerning termination for wider classes of real functions may lead to even more powerful query languages.

The main contribution of this paper is a translation of these decision problems into decision problems about topologies. Hereto, we define the following topologies[1] associated to a function $f:\mathbb{R}^n \to \mathbb{R}^n$. We call a subset G of $\mathbb{R}^n$ *f-closed* if for every $\boldsymbol{x}$ in G, also $f(\boldsymbol{x})$ belongs to G, i.e., if $f(G) \subseteq G$. We denote the set of all f-closed subsets of $\mathbb{R}^n$ by C_f. We call a subset O of $\mathbb{R}^n$ *f-open* if for every $\boldsymbol{x}$ in O and for every $\boldsymbol{y} \in \mathbb{R}^n$ for which $f(\boldsymbol{y}) = \boldsymbol{x}$, also $\boldsymbol{y}$ belongs to O, i.e., if $f^{-1}(O) \subseteq O$. We denote the set of all f-open subsets of $\mathbb{R}^n$ by O_f.

[1] The notions from topology that we use can be found in most introductory topology books, e.g., [4].

We remark that the definitions and results presented here hold for arbitrary sets, rather than just for $\mathbb{R}^n$, but we stick to $\mathbb{R}^n$ since the mentioned decision problems are stated for $\mathbb{R}^n$.

The proofs of the following properties and theorems are postponed to the next section.

Property 1. *Both the structures $(\mathbb{R}^n, O_f)$ and $(\mathbb{R}^n, C_f)$ are topologies. Furthermore, C_f is the set of closed sets of $(\mathbb{R}^n, O_f)$ and O_f is the set of closed sets of $(\mathbb{R}^n, C_f)$.*

So, $(\mathbb{R}^n, O_f)$ and $(\mathbb{R}^n, C_f)$ are topologies in which both the open and the closed sets form a topology. We remark that these topologies $(\mathbb{R}^n, O_f)$ and $(\mathbb{R}^n, C_f)$ have no interesting separation properties [4] in the sense that both $(\mathbb{R}^n, O_f)$ and $(\mathbb{R}^n, C_f)$ are T_i, $i = 0, 1, 2, \ldots$ (among which Hausdorff) if and only if f is the identity. These topologies are also incomparable to the natural topology of $\mathbb{R}^n$ in the sense that none is finer than the other.

A basic property is the following.

Property 2. *For any function $f:\mathbb{R}^n \to \mathbb{R}^n$, $f:(\mathbb{R}^n, O_f) \to (\mathbb{R}^n, O_f)$ and $f:(\mathbb{R}^n, C_f) \to (\mathbb{R}^n, C_f)$ are continuous mappings.*

In dynamical systems, when looking at iterates of a function $f:\mathbb{R}^n \to \mathbb{R}^n$, the notion of orbit is widely used. For $\boldsymbol{x} \in \mathbb{R}^n$, the *orbit of $\boldsymbol{x}$ (with respect to f)* is defined as the set $\{\boldsymbol{x}, f(\boldsymbol{x}), f^2(\boldsymbol{x}), f^3(\boldsymbol{x}), \ldots\}$ and we denote it by $\mathrm{Orb}(\boldsymbol{x}, f)$. It is clear from the definition that the set $\mathrm{Orb}(\boldsymbol{x}, f)$ is the smallest f-closed set that contains $\boldsymbol{x}$. The set of orbits $\{\mathrm{Orb}(\boldsymbol{x}, f) \mid \boldsymbol{x} \in \mathbb{R}^n\}$ therefore forms a basis of $(\mathbb{R}^n, C_f)$. This basis is also minimal, in the sense that any other basis of $(\mathbb{R}^n, C_f)$ must contain $\{\mathrm{Orb}(\boldsymbol{x}, f) \mid \boldsymbol{x} \in \mathbb{R}^n\}$. Also, the closure in $(\mathbb{R}^n, C_f)$ of a subset A of $\mathbb{R}^n$ is the set $\bigcup_{\boldsymbol{x} \in A} \mathrm{Orb}(\boldsymbol{x}, f)$.

Since the open sets of the topology C_f are closed under iteration of f, this topology captures the essential elements one is interested in when looking at the iteration of function f. Also, the orbits, which play a central role in studying the iterates of functions in the dynamical systems literature (see, e.g., [7]), turn out to play a central role in the topology C_f. We remark that Monks discusses a related topology [8].

We are ready to summarize our main translation results.

Theorem 1. *For any function $f : \mathbb{R}^n \to \mathbb{R}^n$ we have the following equivalences.*

(i) *f is mortal if and only if $f(\mathbf{0}) = \mathbf{0}$ and $(\mathbb{R}^n, C_f)$ is connected*;
(ii) *f reaches a fixed point on $\boldsymbol{x} \in \mathbb{R}^n$ if and only if the smallest f-closed set containing $\boldsymbol{x}$ is finite and contains a singleton closed subset*;
(iii) *f is terminating if and only if there is a uniform bound on the size of the elements of the minimal basis of the topology $(\mathbb{R}^n, C_f)$*;
(iv) *f is nilpotent if and only if f is terminating, $\{\mathbf{0}\}$ is f-closed, and the only f-open set containing $\mathbf{0}$ is $\mathbb{R}^n$.*

Theorem 1 gives a translation of decision problems from dynamical systems theory and database theory into decision problems about topologies (and vice versa). Progress on decision problems about topologies could therefore contribute to both these areas. However, to the best of our knowledge, there is no literature on results concerning decidable properties of (even finitely-presented) topologies.

This result has a corollary concerning the undecidability of testing connectivity of topologies on $\mathbb{R}^2$. There are obviously uncountably many topologies on $\mathbb{R}^2$, but if we restrict our attention to those topologies that allow some finite representation and if we agree that the topology $(\mathbb{R}^2, C_f)$ can be represented suitably by some finite description of f, Theorem 1 and the earlier stated result that says that mortality is undecidable for piecewise affine functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ [2], imply that connectivity of topologies on $\mathbb{R}^n$ is *undecidable*, for $n \geqslant 2$.

Corollary 1. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a piecewise continuous linear function. The connectivity of topologies of the form $(\mathbb{R}^n, C_f)$ is undecidable for $n > 1$.*

The following sections are organized as follows. In Section 2, we prove the results that were stated in this section. We end the paper with a section where we discuss the topologies of congruent functions.

2. Proofs of the results

In this section we prove the results from Section 1.

Proof of Property 1. First we show that $(\mathbb{R}^n, O_f)$ is a topology. It is immediately clear from the definition that $\emptyset$ and $\mathbb{R}^n$ are f-open. Let O_1 and O_2 belong to O_f. If $\boldsymbol{x} \in O_1 \cap O_2$ and $\boldsymbol{y} \in \mathbb{R}^n$ such that $f(\boldsymbol{y}) = \boldsymbol{x}$, then also $\boldsymbol{y} \in O_1$ and $\boldsymbol{y} \in O_2$ and hence $\boldsymbol{y} \in O_1 \cap O_2$. Therefore, also $O_1 \cap O_2$ is f-open. Finally, let O_i ($i \in I$) belong to O_f (I is an arbitrary index set, such that $\{O_i \mid i \in I\}$ is an arbitrary subset of O_f). We have to show that $\bigcup_{i \in I} O_i$ belongs to O_f. Let $\boldsymbol{x}$ belong to $\bigcup_{i \in I} O_i$. Then there exists a $k \in I$ such that $\boldsymbol{x} \in O_k$. For all $\boldsymbol{y} \in \mathbb{R}^n$ with $f(\boldsymbol{y}) = \boldsymbol{x}$, we have that $\boldsymbol{y} \in O_k$ since O_k is f-open and thus $\boldsymbol{y} \in \bigcup_{i \in I} O_i$. This shows that O_f is a topology on $\mathbb{R}^n$.

Next, we show (1) that for any $G \in C_f$, $\mathbb{R}^n \setminus G$ is in O_f and (2) that for any $O \in O_f$, $\mathbb{R}^n \setminus O$ is in C_f. If G is in C_f, then abbreviate by O the set $\mathbb{R}^n \setminus G$. Let $\boldsymbol{x}$ be in O and suppose $\boldsymbol{y}$ is such that $f(\boldsymbol{y}) = \boldsymbol{x}$. Suppose that $\boldsymbol{y} \notin O$, then $\boldsymbol{y} \in G$ and thus $\boldsymbol{x} = f(\boldsymbol{y})$ in G (by definition of C_f). This contradicts the assumption and (1) is proved. For (2), if O is in O_f, then abbreviate by G the set $\mathbb{R}^n \setminus O$. Let $\boldsymbol{x}$ be in G and suppose $\boldsymbol{y} = f(\boldsymbol{x})$ and suppose that $\boldsymbol{y} \notin G$. Then $\boldsymbol{y} \in O$ and thus $\boldsymbol{x}$ is in O (by definition of O_f). This contradicts the assumption and proves (2).

From the above it follows that to prove that $(\mathbb{R}^n, C_f)$ is a topology it suffices to show that O_f is closed under arbitrary intersections. Let O_i, $i \in I$ belong to O_f. We show that $\bigcap_{i \in I} O_i$ belongs to O_f. Let $\boldsymbol{x}$ belong to $\bigcap_{i \in I} O_i$. Then $\boldsymbol{x} \in O_k$ for all $k \in I$. Hence, for all $\boldsymbol{y} \in \mathbb{R}^n$ with $f(\boldsymbol{y}) = \boldsymbol{x}$, we have that $\boldsymbol{y} \in O_k$ (since O_k is f-open) for all $k \in I$ and thus $\boldsymbol{y} \in \bigcap_{i \in I} O_i$. This completes the proof. □

Proof of Property 2. It suffices to show that for any f-closed set G, $f^{-1}(G)$ is also closed. Let $\boldsymbol{x}$ be an element of $f^{-1}(G)$. Then $f(\boldsymbol{x}) \in f(f^{-1}(G)) \subseteq G$. From the given fact that G is closed it therefore follows that $f^2(\boldsymbol{x}) \in G$, and thus $f(\boldsymbol{x}) \in f^{-1}(G)$. Therefore, $f^{-1}(G)$ is closed. □

For a function $f : \mathbb{R}^n \to \mathbb{R}^n$ and a point $\boldsymbol{p} \in \mathbb{R}^n$, the set of $\boldsymbol{x} \in \mathbb{R}^n$ for which $\boldsymbol{p}$ is the fixed point reached by $\boldsymbol{x}$, $f(\boldsymbol{x})$, $f^2(\boldsymbol{x})$, ..., will be denoted by $\mathrm{Fix}(f, \boldsymbol{p})$.

Lemma 1. *For any* $\boldsymbol{p} \in \mathbb{R}^n$, *the set* $\mathrm{Fix}(f, \boldsymbol{p})$ *is open and closed both in* $(\mathbb{R}^n, O_f)$ *and* $(\mathbb{R}^n, C_f)$.

Proof. If $\mathrm{Fix}(f, \boldsymbol{p})$ is empty then the lemma trivially holds. Otherwise, to show that $\mathrm{Fix}(f, \boldsymbol{p})$ is closed it suffices to remark that if $\boldsymbol{x}, f(\boldsymbol{x}), f^2(\boldsymbol{x}), \ldots$ reaches $\boldsymbol{p}$ as fixed point, then also $f(\boldsymbol{x}), f^2(\boldsymbol{x}), f^3(\boldsymbol{x}), \ldots$ reaches $\boldsymbol{p}$ as fixed point. To show that $\mathrm{Fix}(f, \boldsymbol{p})$ is open it suffices to remark that if $\boldsymbol{x}, f(\boldsymbol{x}), f^2(\boldsymbol{x}), \ldots$ reaches $\boldsymbol{p}$ as fixed point and if $\boldsymbol{x} = f(\boldsymbol{y})$, then also $\boldsymbol{y}, \boldsymbol{x}, f(\boldsymbol{x}), f^2(\boldsymbol{x}), f^3(\boldsymbol{x}), \ldots$ reaches $\boldsymbol{p}$ as fixed point. $\square$

We are now ready to give the proof of Theorem 1.

Proof of Theorem 1. For the only-if direction of item (i), assume that f is mortal. Then $f(\mathbf{0}) = \mathbf{0}$ follows from the definition of mortality. Assume that there exists a non-empty subset O of $\mathbb{R}^n$ such that O is open and closed in $(\mathbb{R}^n, C_f)$. Since O is non-empty there exists an $\boldsymbol{x}$ in O. From the fact that f is mortal, it follows that there is a natural number k such that $f^k(\boldsymbol{x}) = \mathbf{0}$. Since O is open in $(\mathbb{R}^n, C_f)$ (so, f-closed), therefore also $\mathbf{0} \in O$. Since O is closed in $(\mathbb{R}^n, C_f)$ (so, f-open), therefore also $\mathbb{R}^n \subseteq O$. In other words, $O = \mathbb{R}^n$ and $(\mathbb{R}^n, C_f)$ is connected since every open and closed subset of $\mathbb{R}^n$ is either empty or $\mathbb{R}^n$.

For the if-direction of item (i), assume that $f(\mathbf{0}) = \mathbf{0}$ and $(\mathbb{R}^n, C_f)$ is connected. Since $\mathbf{0} \in \mathrm{Fix}(f, \mathbf{0})$ and since, by Lemma 1, $\mathrm{Fix}(f, \mathbf{0})$ is open and closed, $\mathrm{Fix}(f, \mathbf{0}) = \mathbb{R}^n$, or equivalently, f is mortal.

Item (ii) follows directly from the observation that the smallest f-closed set containing $\boldsymbol{x}$ is exactly $\mathrm{Orb}(\boldsymbol{x}, f)$. The condition that it should contain a singleton closed subset expresses that there is a fixed point rather than a cycle of length larger than one.

Item (iii) is straightforward since $\{\mathrm{Orb}(\boldsymbol{x}, f) \mid \boldsymbol{x} \in \mathbb{R}^n\}$ is the minimal basis for $(\mathbb{R}^n, C_f)$. A uniform bound k on the orbits $\mathrm{Orb}(\boldsymbol{x}, f)$ guarantees that the transitive closure of the graph of f terminates after at most $2k$ iterations and vice versa.

Finally, for the only-if direction of item (iv), assume that f is nilpotent. By the definition, there is a uniform bound k on the number of elements in all orbits. Therefore, f is terminating. Since all orbits contain $\mathbf{0}$, all open sets containing $\mathbf{0}$ equal $\mathbb{R}^n$. Since $f(\mathbf{0}) = \mathbf{0}$, clearly $\{\mathbf{0}\}$ is f-closed.

For the if-direction of item (iv), we assume the three given facts. From the fact that f is terminating, we know that there exists a uniform bound k on the size of the orbits of f. From the fact that $\{\mathbf{0}\}$ is f-closed, $f(\mathbf{0}) = \mathbf{0}$ follows. It remains to be shown that for all $\boldsymbol{x} \in \mathbb{R}^n$, $\mathbf{0} \in \mathrm{Orb}(\boldsymbol{x}, f)$. Suppose, there is an $\boldsymbol{x} \in \mathbb{R}^n$ such that $\mathbf{0} \notin \mathrm{Orb}(\boldsymbol{x}, f)$. Then $\mathbb{R}^n \setminus \mathrm{Orb}(\boldsymbol{x}, f)$ is an f-open set that contains $\mathbf{0}$ and is not equal to $\mathbb{R}^n$. This contradicts the second given fact. $\square$

3. The topologies of congruent functions

We call two functions $f, g : \mathbb{R}^n \to \mathbb{R}^n$ *congruent* if there exists a bijection $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h \circ f = g \circ h$. Intuitively it is clear that congruent functions share the same termination properties (such as mortality, nilpotency, termination and point-to-fixed-point). We can formally prove this by showing that congruent functions give rise to homeomorphic topologies.

Lemma 2. *Let* f *and* g *be two functions from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$. *A mapping* $h : \mathbb{R}^n \to \mathbb{R}^n$ *is an homeomorphism between the topological spaces* $(\mathbb{R}^n, C_f)$ *and* $(\mathbb{R}^n, C_g)$ *if and only if* h *is a bijection and for all* $\boldsymbol{x} \in \mathbb{R}^n$, $h(\mathrm{Orb}(\boldsymbol{x}, f)) = \mathrm{Orb}(h(\boldsymbol{x}), g)$.

Proof. First, we prove the if-direction. It suffices to prove that both h and h^{-1} are continuous. Let G' be any g-closed set, and let $G = h^{-1}(G')$. We show that G is f-closed. Indeed, let $\boldsymbol{x}$ be an element of G, and let $\boldsymbol{y} \in G'$ be such that $\boldsymbol{x} = h^{-1}(\boldsymbol{y})$. Since G' is g-closed, $\mathrm{Orb}(\boldsymbol{y}, g) \subseteq G'$ and $h^{-1}(\mathrm{Orb}(\boldsymbol{y}, g)) \subseteq G$. By the fact that for all $\boldsymbol{x} \in \mathbb{R}^n$, $h(\mathrm{Orb}(\boldsymbol{x}, f)) = \mathrm{Orb}(h(\boldsymbol{x}), g)$, $\mathrm{Orb}(\boldsymbol{x}, f) = h^{-1}(\mathrm{Orb}(\boldsymbol{y}, g)) \subseteq G$, and hence $f(\boldsymbol{x})$ is also in G. So, G is f-closed and h is continuous. Similarly, one can show that also h^{-1} is continuous.

For the only-if direction, we proceed as follows. Clearly, $\mathrm{Orb}(h(\boldsymbol{x}), g)$ is a g-closed set, and by the continuity of h, the set $G = h^{-1}(\mathrm{Orb}(h(\boldsymbol{x}), g))$ is f-closed. Since $\boldsymbol{x} = h^{-1}(h(\boldsymbol{x}))$ is an element of G, we have that $\mathrm{Orb}(\boldsymbol{x}, f) \subseteq G$. Similarly, by the continuity of h^{-1}, $h(\mathrm{Orb}(\boldsymbol{x}, f))$ is g-closed and contains $\mathrm{Orb}(h(\boldsymbol{x}), g)$. Hence, $h(\mathrm{Orb}(\boldsymbol{x}, f)) \subseteq h(G) = \mathrm{Orb}(h(\boldsymbol{x}), g)$ and also $\mathrm{Orb}(h(\boldsymbol{x}), g) \subseteq h(\mathrm{Orb}(\boldsymbol{x}, f))$. This implies that $h(\mathrm{Orb}(\boldsymbol{x}, f)) = \mathrm{Orb}(h(\boldsymbol{x}), g)$ and also the only-if direction is proven. $\square$

Property 3. *Let f and g be two functions from $\mathbb{R}^n$ to $\mathbb{R}^n$. If f and g are congruent by a mapping h, i.e., $h \circ f = g \circ h$, then the topological spaces $(\mathbb{R}^n, C_f)$ and $(\mathbb{R}^n, C_g)$ are homeomorphic by the mapping h.*

Proof. By Lemma 2 it suffices to verify that for all $\boldsymbol{x} \in \mathbb{R}^n$, $h(\mathrm{Orb}(\boldsymbol{x}, f)) = \mathrm{Orb}(h(\boldsymbol{x}), g)$. But this follows directly from the fact that $h(f^k(\boldsymbol{x})) = g^k(h(\boldsymbol{x}))$ for any $\boldsymbol{x} \in \mathbb{R}^n$. □

This property shows that termination properties such as mortality, nilpotency, termination and point-to-fixed-point are shared by congruent functions.

Monks states, without proof, that the converse of Property 3 only holds for acyclic functions, i.e., functions where the only cyclic points are the fixed points [8] (more precisely, a function $f : \mathbb{R}^n \to \mathbb{R}^n$ is called *acyclic* if for any $\boldsymbol{x} \in \mathbb{R}^n$, $f^d(\boldsymbol{x}) = \boldsymbol{x}$ implies that d is 1). For reasons of completeness, we here give the proof of this result and give for any cycle length greater than one examples of functions for which that converse of Property 3 does not hold.

Property 4. *Let f and g be acyclic functions. If the topological spaces $(\mathbb{R}^n, C_f)$ and $(\mathbb{R}^n, C_g)$ are homeomorphic by a mapping h, then f and g are congruent by the mapping h, i.e., $h \circ f = g \circ h$.*

Proof. Let f and g be acyclic functions and assume that $h : (\mathbb{R}^n, C_f) \to (\mathbb{R}^n, C_g)$ is a homeomorphism. We have to show that for any $\boldsymbol{x} \in \mathbb{R}^n$, $h(f(\boldsymbol{x})) = g(h(\boldsymbol{x}))$.

By Lemma 2, for all $\boldsymbol{x} \in \mathbb{R}^n$, $h(\mathrm{Orb}(\boldsymbol{x}, f)) = \mathrm{Orb}(h(\boldsymbol{x}), g)$. Therefore, for any $\boldsymbol{x} \in \mathbb{R}^n$, there exists a natural number k such that $h(f(\boldsymbol{x})) = g^k(h(\boldsymbol{x}))$. Denote by $k_{\boldsymbol{x}}$ the minimal such natural number. We distinguish between three cases: $k_{\boldsymbol{x}} = 0$, $k_{\boldsymbol{x}} = 1$ and $k_{\boldsymbol{x}} > 1$.

If $k_{\boldsymbol{x}} = 0$, then $h(f(\boldsymbol{x})) = h(\boldsymbol{x})$ and thus $f(\boldsymbol{x}) = \boldsymbol{x}$. Therefore, $\mathrm{Orb}(\boldsymbol{x}, f) = \{\boldsymbol{x}\}$ and, by Lemma 2, $\mathrm{Orb}(h(\boldsymbol{x}), g) = \{h(\boldsymbol{x})\}$. Thus, $g(h(\boldsymbol{x})) = h(\boldsymbol{x}) = h(f(\boldsymbol{x}))$. If $k_{\boldsymbol{x}} = 1$, then we immediately have $h(f(\boldsymbol{x})) = g(h(\boldsymbol{x}))$. Finally, assume that $k_{\boldsymbol{x}} > 1$. From the minimality of $k_{\boldsymbol{x}}$ it follows that $g^\ell(h(\boldsymbol{x})) \neq h(f(\boldsymbol{x}))$ for all $0 \leqslant \ell < k_{\boldsymbol{x}}$. Also, because g is acyclic, for all $0 \leqslant \ell < k_{\boldsymbol{x}}$ and all integers $p \geqslant 0$, $g^\ell(h(\boldsymbol{x})) \neq g^{k_{\boldsymbol{x}}+p}(h(\boldsymbol{x}))$. Therefore,

$$g^\ell\big(h(\boldsymbol{x})\big) \notin \mathrm{Orb}\big(g^{k_{\boldsymbol{x}}}\big(h(\boldsymbol{x})\big), g\big) = h\big(\mathrm{Orb}\big(f(\boldsymbol{x}), f\big)\big).$$

But $g^\ell(h(\boldsymbol{x})) \in \mathrm{Orb}(h(\boldsymbol{x}), g) = h(\mathrm{Orb}(\boldsymbol{x}, f))$. We can therefore conclude that $g^\ell(h(\boldsymbol{x})) = h(\boldsymbol{x})$ for all $0 \leqslant \ell < k_{\boldsymbol{x}}$, in particular for $\ell = k_{\boldsymbol{x}} - 1$. This implies that $g(h(\boldsymbol{x})) = g(g^{k_{\boldsymbol{x}}-1}(h(\boldsymbol{x}))) = g^{k_{\boldsymbol{x}}}(h(\boldsymbol{x})) = h(f(\boldsymbol{x}))$. This contradicts the minimality of $k_{\boldsymbol{x}}$ and makes the third case impossible. We have shown that for any $\boldsymbol{x} \in \mathbb{R}^n$, $h(f(\boldsymbol{x})) = g(h(\boldsymbol{x}))$. □

Finally, we show that for any cycle length $d > 1$, there are non-congruent functions $f_d, g_d : \mathbb{R} \to \mathbb{R}$, such that $(\mathbb{R}, C_{f_d})$ and $(\mathbb{R}, C_{g_d})$ are homeomorphic. Consider the functions f_d and g_d defined by $f_d(i) = g_d(i) = i + 1$ for $i = 1, \ldots, d-1$; $f_d(d) = g_d(d) = 1$; $f_d(\frac{1}{2}) = g_d(\frac{1}{2}) = 1$; $f_d(\frac{1}{3}) = g_d(\frac{1}{3}) = 2$; $f_d(\frac{1}{4}) = g_d(\frac{1}{4}) = \frac{1}{3}$; $f_d(\frac{1}{5}) = 2$; $g_d(\frac{1}{5}) = 1$; and both f_d and g_d constant 0 elsewhere. It is clear that both f_d and g_d have a cycle of length d. Using Lemma 2, it is readily verified that f_d and g_d give rise to homeomorphic topologies $(\mathbb{R}, C_{f_d})$ and $(\mathbb{R}, C_{g_d})$. It is also easy to see that f_d and g_d are non-congruent functions.

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