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# BRS-Compactness in Networks: Theoretical Considerations Related to Cohesion in Citation Graphs, Collaboration Networks and the Internet

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**Abstract**—Compactness as introduced by Botafogo, Rivlin and Shneiderman, in short: BRS-compactness, is studied in general, as it can be used to describe the cohesion of parts of the internet or collaboration networks, and in the particular case of a unidirectional network, such as a citation graph. It is shown that the connection coefficient is an upper bound for the BRS-compactness value of a network. During our investigations, we derive an upper bound for the generalized Wiener index of a directed graph. Several networks are constructed and their BRS-compactness values are calculated.


**Keywords**—BRS-compactness, Networks, Hyperlinks, Internet, Citation networks, Collaboration graphs, Generalized Wiener index, Sum of distances in a graph.

## 1. INTRODUCTION

The pages and hyperlinks of the world wide web may be viewed as nodes and edges in a directed graph [1,2]. The degree of the interconnectedness of a hypertext or similar graph-like entities can be expressed using cohesion measures. One of these is compactness as introduced by Botafogo, Rivlin and Shneiderman [3]. As the word 'compactness' has several meanings in mathematics and graph theory, we will refer to the compactness notion as introduced by the fore-mentioned authors as BRS-compactness. An exact definition follows later.

BRS-compactness is a measure which tries to capture how well connected a hyperdocument or a network is. As a measure of cohesion, its value can be used as a guideline for hypertext authoring

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systems [4]. It has been studied and discussed in many other works, see e.g., [5–9]. Indeed, the density and cohesion of links in a hypermedia environment influences the retrieval efficiency of users [10]. Leazer *et al.* [11] study compactness in the context of textual identity networks, i.e., a set of documents that share a common semantic or linguistic form. They, moreover, compare BRS-compactness with other so-called topological indices such as the Wiener index, stratum, and Randić's index [12].

In informetric studies, publications, citations, cocitations [13,14] as well as collaborations give rise to networks [15–17]. A citation network is clearly not symmetric (if article $A$ cites article $B$, then $B$ normally does not cite $A$), while a collaboration network definitely is: if author $X$ collaborates with author $Y$, then automatically author $Y$ has collaborated with $X$. Note, that recently also other collaborations, such as actor collaborations have inspired fellow scientists [18]. Citation links have been inspirational to web search techniques such as those used by the Clever algorithm and by Google [19–21]. Moreover, the 'hubs' and 'authorities' approach is related to the Pinski-Narin influence weight citation measure [22] and mimics the idea of 'highly' cited documents (authorities) and reviews (hubs). The exact relation between the older citation-based measures, such as the Pinski-Narin weights, including Geller's modification [23], and the newer hypertext and www-based approach is clearly described by Kleinberg [24].

In this article, we study the compactness of a general network and show how this web metric may be used in citation analysis and the study of collaboration networks. Indeed, de Bra [25] observed that when studying the literature of a field, large differences in the density of citations may be found. Sometimes we see densely connected citation clusters with little or no links to other clusters. De Bra suggests that the BRS-compactness measure can be used to identity research fields with a similar citation behavior. This, in turn, could be a factor in research evaluation exercises. For all these reasons, we think it is necessary to have a closer look at the notion of BRS-compactness, to study its properties and to construct some more examples, besides those given by [3,5].

## 2. SOME NOTIONS FROM GRAPH THEORY

A directed graph $G$, in short: digraph, consists of a set of nodes, denoted as $N(G)$, and a set of links (also called arcs or edges), denoted as $L(G)$. In this text, the words 'network' and 'graph' are synonymous. A link $e$ is an *ordered pair* $(a, b)$ representing a connection from node $a$ to node $b$. Node $a$ is called the initial node of link $e$, $a = \text{init}(e)$, and node $b$ is called the final node of the link: $b = \text{fin}(e)$. The out-degree of a node $b$ is the number of arcs leading out from it, i.e., the number of arcs $e$ such that $\text{init}(e) = b$. Similarly, the in-degree of a node, $b$, is the number of arcs $e$ such that $\text{fin}(e) = b$ [26, p. 371]. A path from node $a$ to node $b$ is a sequence of *distinct* links $(a, u_1), (u_1, u_2), \ldots, (u_k, b)$. The length of this path is the number of links (here, $k + 1$). Note that, in general, a path from $a$ to $b$ does not necessarily imply a path from $b$ to $a$. A cycle is a path of length $> 1$, beginning and ending in the same node. A graph that does not contain any cycle is called an acyclic graph. In this paper, we will always assume that edges are unweighted, or, equivalently, have a weight equal to one. We assume in this paper that there exists at most one direct link between two nodes. Further, nodes will often receive an index number and will be identified through this number.

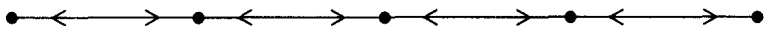
Two graphs $G$ and $H$ are isomorphic if there exists a bijection $f$ from $G$ to $H$ such that if $h_1 = f(g_1)$ and $h_2 = f(g_2)$, with $g_i \in N(G)$ and $h_i \in N(H)$, $i = 1, 2$, and if there exists a link in $G$ between $g_1$ and $g_2$, then there exists a corresponding link in $H$ between $h_1 = f(g_1)$ and $h_2 = f(g_2)$ (in that order), and vice versa [27,28].

A unidirectional graph is a graph in which a link between nodes $a$ and $b$, implies that there is not a (direct) link from $b$ to $a$. In a unidirectional graph, cycles may exist, but the smallest possible length is 3. If there are nodes $a$ and $b$ such that the whole graph consists of exactly one path of length $N - 1$ from $a$ to $b$, we will refer to such a linear graph as a unidirectional $N$-chain.

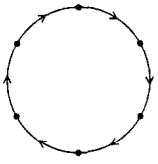
We will say, that a graph $T$ is a tree if it is unidirectional, acyclic and there exists exactly one point, called the root, from which each other point can be reached. The distance from the root to a node $t$ in a tree is called the depth of $t$. If each node in the tree has the same number of children (at least those which have children), this number is called the branching factor of the tree. Nodes without children are called terminal nodes or leaves. The length of a longest path from the root to a leaf is called the tree-depth. A tree is balanced if, at the same depth, all nodes have the same number of children. Hence, in a balanced tree, no leaf is further away from the root than any other leaf.

If the existence of a link between nodes $a$ and $b$ necessarily implies the existence of a link from $b$ to $a$, we say that this network is a bidirectional graph. If a bidirectional graph consists of exactly one path of length $N-1$, then we will refer to such a graph as a bidirectional $N$-chain. Figure 1 shows a unidirectional $N$-chain, a bidirectional $N$-chain and a unidirectional $N$-loop.

(a) Unidirectional chain.



(b) Bidirectional chain.



(c) Loop.

Figure 1.

The distance from node $a$ to node $b$ is the smallest length of all the paths that join $a$ to $b$. If such a path does not exist the length is infinite. A strongly connected component of a digraph is a set of nodes such that any two of them are joined by a path. Different strongly connected components in a network consists of disjoint sets of nodes. If a digraph consists of one strongly connected component, it is said to be strongly connected.

An undirected graph consists of a set of nodes and a set of edges, each of which is an *unordered pair* of nodes. Any bidirectional digraph can be considered as an undirected graph. A collaboration network is an example of such a graph: if author $A$ coauthored an article with author $B$, then author $B$ coauthored an article with $A$. Hence, most of the results obtained in this paper can be applied to collaboration networks as studied, e.g., in [29].

When applying our ideas to citation networks (a network where nodes are articles and a link from article $a$ to article $b$ means that article $a$ refers $b$), we always assume that these are unidirectional, although this is in reality not always the case (due, e.g., to the existence of invisible colleges). All citation networks considered in this paper are moreover assumed to be acyclic.

For more information on graphs, we refer the reader to [26,27,30–33].

## 3. COMPACTNESS

DEFINITION 3.1. THE BRS NETWORK MATRIX. (See [3].) Any (finite) network can be described by a matrix $D$ such that its element on the $i^{\text{th}}$ row and $j^{\text{th}}$ column, denoted as $d(i,j)$, is equal to the shortest distance between the $i^{\text{th}}$ and the $j^{\text{th}}$ node of the network. If node $j$ cannot be reached from node $i$, then $d(i,j)=\infty$. In their analysis of hypertexts and hyperlinks, Botafogo *et al.* [3] introduced the following convention: if node $j$ cannot be reached from node $i$, then $d(i,j)$ is not put equal to $\infty$, but takes as its value the number of nodes in the analyzed network,

see also [11]. This representation will be called the BRS representation and the associated matrix is denoted as $D_B$. Reference [3] refers to this matrix as the converted distance matrix. We define the generalized Wiener index of a general digraph, denoted by $W$, as the sum of all elements of the converted distance matrix. In the case of an undirected, strongly connected graph, this sum divided by two is known as the Wiener index, after the chemist Wiener [34].

DEFINITION 3.2. BRS-COMPACTNESS. The BRS-compactness value, $C$, of a network consisting of $N \geq 2$ nodes, is calculated using a formula having the following general structure:

$$C = \frac{\max - \sum_{i,j=1}^{N} d(i,j)}{\max - \min}, \tag{1}$$

where $d(i,j)$ denotes an element of the network matrix under study while max and min denote the maximum and the minimum sum for the corresponding $N$-node network [3]. We see that compactness is the normalized, generalized Wiener index. If $N = 1$ (a network consisting of just one node), $C$ is not defined.

### 3.1. The Compactness Formula for a General Digraph and the Connection Coefficient

In the BRS-representation, two unconnected nodes are attributed a distance value equal to $N$. There seems, however, to be no *a priori* reason why the value $N$ must be used. Hence, we will just assume that this value is a function of the number of nodes in the citation network. This value is denoted as $\varphi(N)$ ([3] denotes this value by $K$). We will certainly put $\varphi(N) \geq N$, otherwise unconnected pairs could have a smaller distance than connected ones. This agreement leads to the following compactness formula for a general network.

**The general BRS-compactness formula [3]**

$$C = \frac{(N^2 - N)\phi(N) - \sum_{i,j=1}^{N} d(i,j)}{(N^2 - N)(\phi(N) - 1)}, \tag{2}$$

here max is obtained in the case that no two pairs are connected. This gives $N^2 - N$ times the largest value, namely $\varphi(N)$. min is obtained when every two pairs of different nodes are connected. This gives a value of $N^2 - N$ multiplied by 1.

DEFINITION 3.3. CONNECTION COEFFICIENT. Now let $\beta$, $\beta \in [0,1]$, be the fraction of all pairs $(i,j)$ (with $i \neq j$) that are connected and let $A_\beta$ denote the set of those pairs $(i,j)$ for which this happens, i.e., for which $d(i,j) < \varphi(N)$. The fraction $\beta$ will be called the *connection coefficient* of the network. The connection coefficient is either zero (and then $C = 0$) or it satisfies the following inequality:

$$\frac{1}{N(N-1)} \leq \beta \leq 1. \tag{3}$$

If the compactness value $C$ is one (every two nodes have distance 1), then $\beta$ is one also. The converse is not true: $\beta = 1$ simply means that every two nodes have a finite distance in the matrix $D_B$ (this means that the graph is strongly connected). For a unidirectional chain $\beta = 1/2$, while for a bidirectional chain, and for a unidirectional loop the $\beta$-value is 1.

If a network has $N$ nodes, then *a priori*, the largest possible distance between two connected nodes is $N-1$. If, however, we know that its connection coefficient is $\beta$, then the largest (possible) distance between two connected nodes is $L_\beta = \min(N-1, \beta N(N-1))$. Following [15], we may say that in a communication network, a high value of the connection coefficient improves the level of accessibility between nodes, and hence, the transfer of information.

### 3.2. A Decomposition of the Compactness Measure

Using the connection coefficient, the BRS-compactness formula can be rewritten as

$$C = \frac{(N^2 - N)\phi(N) - (1 - \beta)(N^2 - N)\phi(N) - \sum_{(i,j)\in A_\beta} d(i,j)}{(N^2 - N)(\phi(N) - 1)}. \tag{4}$$

This leads to the following decomposition of (2) in two parts. The first is determined by the upper limit for a network with a connection coefficient $\beta$, the second part reduces this value further depending on the degree of connectedness,

$$C = \frac{\beta\phi(N)}{\phi(N) - 1} - \frac{\sum_{(i,j)\in A_\beta} d(i,j)}{(N^2 - N)(\phi(N) - 1)}. \tag{5}$$

Because there are $\beta(N^2 - N)$ pairs $(i,j) \in A_\beta$ (pairs for which $d(i,j) < \varphi(N)$), we immediately see that

$$\beta\,(N^2 - N) \leq \sum_{(i,j)\in A_\beta} d(i,j). \tag{6}$$

Consequently, for fixed $\beta$

$$C \in [0, \beta]. \tag{7}$$

Note, that the upper bound, $\beta$, can actually be reached, namely when all pairs $(i,j) \in A_\beta$ are at distance 1 (they are directly connected). We next derive a (much) better lower bound. Yet, relation (7) is all we need to study the limiting behavior of the following examples.

### 3.3. A Limiting Procedure for Trees and Disjoint Unions of Networks

We remind the reader that trees are important concepts in the information sciences. Distances between nodes in a tree, representing a hierarchical thesaurus, have been studied in the context of knowledge-based information retrieval [35]. Let $T$ be a balanced tree with branching factor $b > 1$ (Figure 2). The number of nodes in such a tree with depth $d$ is

$$N_d = 1 + b + b^2 + \cdots + b^d = \frac{b^{d+1} - 1}{b - 1}. \tag{8}$$

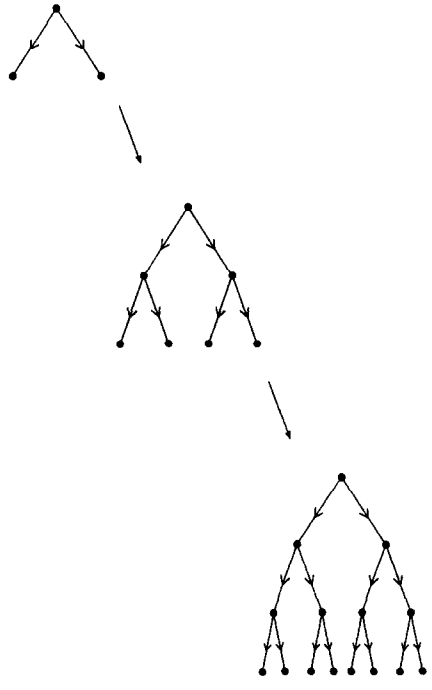


Figure 2. Construction of a tree with branching factor 2.

In order to find the connection $\beta_d$ of a tree at depth $d$, we proceed step by step. At depth 1, there are $b$ links of length 1. Expanding the tree and reaching depth 2 leads to $b^2$ links of length 2, plus $b^2$ new links of length 1. At the next expansion, $b^3$ links of length 1, of length 2, and of length 3 are created. This yields at depth $d$ a total number of links equal to

$$\sum_{i=1}^{d} ib^i = \frac{db^{d+2} - (d+1)b^{d+1} + b}{(b-1)^2}. \tag{9}$$

Consequently,

$$\beta_d = \frac{(db^{d+2} - (d+1)b^{d+1} + b)/(b-1)^2}{((b^{d+1}-1)/(b-1))\,((b^{d+1}-1)/(b-1)-1)} = \frac{db^{d+1} - (d+1)b^d + 1}{(b^{d+1}-1)(b^d-1)}. \tag{10}$$

The connection coefficient $\beta_d$ is clearly smaller than 1. Moreover, $\lim_{d\to\infty} \beta_d = 0$ which proves, by (7), that the limiting compactness value of this balanced tree with fixed branching factor ($b > 1$) is zero.

Consider now a network consisting of $N$ nodes. Starting from this network, we consider the following construction of an infinite network. In a first step, we construct a $2N$-node network by adding, in a disconnected way, a copy of the first one, i.e., there are no connections between the first copy and the second. The resulting graph is called the disjoint union of this network with itself. Then, we iterate this procedure, leading to a network with $4N, 8N$, and in general $2^m N$ nodes. We will show that the limiting BRS-compactness value ($m$ tending to infinity) of this network is zero.

Let $\beta$ be the connection coefficient of the original network. Then, $\beta_1$, the corresponding coefficient after the first iteration is equal to

$$\beta_1 = \beta\frac{N-1}{2N-1} < \beta\frac{1}{2}.$$

In general, when $\beta_{m-1}$ is the connection coefficient for the network after $m-1$ iterations, then $\beta_m$, the connection coefficient after $m$ iterations is

$$\beta_m = \beta_{m-1}\frac{2^{m-1}N-1}{2^m N-1} < \beta\left(\frac{1}{2}\right)^m. \tag{11}$$

As the connection coefficient is the upper bound for the compactness value of any network (7), this proves that the limiting compactness value of this infinite network is zero, and hence, such a construction yields increasingly sparse networks.

## 4. BOUNDS FOR THE SUM OF DISTANCES BETWEEN CONNECTED NODES

THEOREM 4.1. *Given a network with $N$ nodes and with connection coefficient $\beta$, if the length of the longest used path in the distance matrix is $k_L$, then*

$$\sum =: \sum_{(i,j)\in A_\beta} d(i,j) \le \frac{k_L\left(3\beta\left(N^2-N\right) - \left(k_L^2-1\right)\right)}{3}. \tag{12}$$

PROOF. If there exists a path of length $k_L$, then there also exist two paths of length $(k_L-1)$, three paths of length $(k_L-2)$, and so on, ending with $k_L$ paths of length 1. Note, that all these paths are used in the distance matrix, otherwise $k_L$ would not be the longest one. This yields

$k_L(k_L+1)/2$ pairs $(i,j)$ for which we know the exact distance $d(i,j)$. We obtain an upper bound for $\Sigma$ by taking all $d(i,j)$ equal to $k_L$, except the $k_L(k_L+1)/2$ ones mentioned above. This yields

$$\begin{aligned}\sum_{(i,j)\in A_\beta} d(i,j) &\le \left(\beta\left(N^2-N\right)-\frac{k_L(k_L+1)}{2}\right)k_L+\sum_{j=1}^{k_L} j(k_L+1-j)\\ &= \left(\beta\left(N^2-N\right)-\frac{k_L(k_L+1)}{2}\right)k_L+\frac{k_L\left(k_L^2+3k_L+2\right)}{6}\\ &= \frac{k_L\left(6\beta\left(N^2-N\right)-\left(3k_L^2+3k_L-k_L^2-3k_L-2\right)\right)}{6}\\ &= \frac{k_L\left(3\beta\left(N^2-N\right)-\left(k_L^2-1\right)\right)}{3}.\end{aligned}$$

This proves the theorem.

COROLLARY 1. *If an $N$-node network with connection coefficient $\beta$ has a maximum path length equal to $k_L$, then its BRS-compactness value $C$ satisfies the following inequality:*

$$\frac{\beta\varphi(N)}{\varphi(N)-1}-\frac{k_L\left(3\beta\left(N^2-N\right)-k_L^2+1\right)}{3\left(N^2-N\right)\left(\varphi(N)-1\right)}\le C\le\beta. \tag{13}$$

*Note, that if $k_L=1$, the lower bound for $C$ becomes equal to the upper bound $\beta$.*

COROLLARY 2. *A unidirectional $N$-chain is characterized by the following parameters:*

$$\beta=\frac{1}{2},\qquad \Sigma=\frac{N\left(N^2-1\right)}{6},$$
$$C=\frac{3\left(N^2-N\right)\varphi(N)-N\left(N^2-1\right)}{6\left(N^2-N\right)\left(\varphi(N)-1\right)}.$$

PROOF. Clearly,

$$\beta=\frac{1+2+\cdots+(N-1)}{N^2-N}=\frac{1}{2}.$$

The fact that $\Sigma$ is equal to $(N(N^2-1))/6$ follows from the proof of Theorem 4.1, noting that $k_L=N-1$, and hence, the inequality in Theorem 4.1 becomes an equality for a unidirectional chain. Finally,

$$\begin{aligned}C &= \beta\frac{\varphi(N)}{\varphi(N)-1}-\frac{\Sigma}{\left(N^2-N\right)\left(\varphi(N)-1\right)}\\ &= \frac{\varphi(N)}{2\left(\varphi(N)-1\right)}-\frac{N\left(N^2-1\right)}{6\left(N^2-N\right)\left(\varphi(N)-1\right)}\\ &= \frac{3\left(N^2-N\right)\varphi(N)-N\left(N^2-1\right)}{6\left(N^2-N\right)\left(\varphi(N)-1\right)}.\end{aligned}$$

PROPOSITION 4.1. *Given a bidirectional network with $N$ nodes and with connection coefficient $\beta$. If the length of the longest used path in the distance matrix is $k_L$, then*

$$\Sigma=:\sum_{(i,j)\in A_\beta} d(i,j)\le\frac{k_L\left(3\beta\left(N^2-N\right)-2\left(k_L^2-1\right)\right)}{3}. \tag{14}$$

PROOF. If there exists a path of length $k_L$, then there exists a second one, by the fact that the graph is bidirectional. There, similarly exist four paths of lengths $(k_L-1)$, six paths of length

$(k_L - 2)$, and so on, ending with $2k_L$ paths of length 1. This yields $k_L(k_L + 1)$ pairs $(i, j)$ for which we know the exact distance $d(i, j)$. Again, we obtain an upper bound for $\Sigma$ by taking all $d(i, j)$ equal to $k_L$, except the $k_L(k_L + 1)$ ones mentioned above. This yields

$$\begin{aligned}\sum_{(i,j)\in A_\beta} d(i,j) &\leq \left(\beta\left(N^2 - N\right) - k_L\left(k_L + 1\right)\right) k_L + 2\sum_{j=1}^{k_L} j(k_L + 1 - j) \\ &= \left(\beta\left(N^2 - N\right) - k_L(k_L + 1)\right) k_L + \frac{k_L\left(k_L^2 + 3k_L + 2\right)}{3} \\ &= \frac{k_L\left(3\beta\left(N^2 - N\right) - \left(3k_L^2 + 3k_L - k_L^2 - 3k_L - 2\right)\right)}{3} \\ &= \frac{k_L\left(3\beta\left(N^2 - N\right) - 2\left(k_L^2 - 1\right)\right)}{3}.\end{aligned}$$

This proves the proposition.

COROLLARY 3. *(See [36].) A bidirectional N-chain is characterized by the following parameters:*

$$\beta = 1, \qquad \Sigma = \frac{N\left(N^2 - 1\right)}{3},$$
$$C = \frac{3\left(N^2 - N\right)\varphi(N) - N\left(N^2 - 1\right)}{3\left(N^2 - N\right)\left(\varphi(N) - 1\right)}.$$

PROOF. This follows immediately from Proposition 4.2.

Best lower and upper bounds for $\Sigma$ are known in graph theory [36–39]. The $\Sigma$-value for a bidirectional chain is a best upper bound for a graph with $N$ vertices.

### 4.1. Acceptable Functions for $\varphi(N)$ in the Compactness Formula

We explained already that we will always choose $\varphi(N) \geq N$. If now $\varphi(N) = N^\alpha$ $(\alpha \geq 1)$, then a bidirectional $N$-chain has, by the previous corollary, a $C$-value

$$C = \frac{3N(N-1)N^\alpha - N\left(N^2 - 1\right)}{3N(N-1)\left(N^\alpha - 1\right)}. \tag{15}$$

If $N$ tends to infinity, this value tends to $2/3$, if $\alpha = 1$, and to 1 if $\alpha > 1$. This would mean that the compactness value of a network where some nodes may have arbitrarily large distances can be as close to one as one likes. This is counterintuitive and provides a good argument for taking $\alpha = 1$. This example does not rule out the possibility of taking $\varphi(N) = cN$ $(c > 1)$, yielding a compactness value for the bidirectional chain of $(3c-1)/3c$. Such a value is not *a priori* excluded or counterintuitive. Yet, following [3], from now on we will take $\varphi(N) = N$. This leads to the following formula for $C$:

$$C = \frac{\beta N}{N-1} - \frac{\sum_{(i,j)\in A_\beta} d(i,j)}{N(N-1)^2}. \tag{16}$$

We note that even if $\varphi(N) = N$, the limiting value (for $N \to \infty$) of a bidirectional star (a single root, connected by bidirectional links to all other nodes) is one [3]. In this graph, the distance between two nodes (except if one of the nodes is the root) is equal to two. Such a bidirectional star is a model for a totally centralized network, see Figure 3.

NOTE. In a unidirectional, acyclic network, the connection coefficient $\beta$ is at most $1/2$. Hence, the BRS-compactness value of such a network is at most 0.5.

PROPOSITION 4.2. *Adding a new link between existing nodes in any network always increases the compactness value C.*

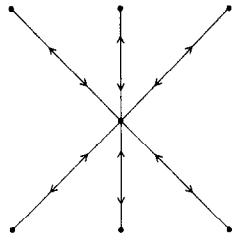


Figure 3. A bidirectional star.

This is trivial. Adding an extra link decreases the value of $\Sigma$, and as $N$ stays constant, this means that $C$ increases. The increase in $C$ is at least equal to

$$\frac{1}{N(N-1)^2}. \tag{17}$$

### 4.2. The Meaning of Cohesion and the Importance of 'Central' Links

From Corollary 2 of Theorem 4.1, we know that $\Sigma$, the sum of the distances of all connected nodes of a unidirectional $N$-chain is $(N(N-1)(N+1))/6$, and that its BRS-compactness value is $(2N-1)/(6(N-1))$ (take $\varphi(N) = N$ in Corollary 2). Note, in particular, that the first (or the last) link in a chain has a contribution in $\Sigma$ equal to $N-1$. Indeed, this link participates once in the set of links of length 1, once in the set of links of length 2, and so on, ending with links of length $N-1$. Assuming $N$ to be even we see, however, that the link in the middle, connecting node $N/2$ with node $(N/2)+1$, has a participation of 1 in the set of links of length 1, a participation of 2 in the set of links with length 2, increasing to a participation of $N/2$ in the set of links with length $N/2$, and then again a decreasing participation. Hence, the middle link contributes $N^2/4$ to the Wiener index. This calculation illustrates the fact that a central link plays a more important role in the determination of the cohesion, as measured by BRS-compactness, than a more peripheral one. This is a desired property of a cohesion measure. If one were interested in replacing the BRS-compactness measure by another measure of cohesion, then this measure must have at least a similar property.

### 4.3. Weaker Inequalities for the Sum of All Distances That Are Strictly Smaller than $N$

In this section, we will derive weaker inequalities than inequality (12). Although weaker, they have the advantage that they depend on less parameters and, hence, can be used when certain data (such as $k_L$) are not known. Recall, that with $\sum_{(i,j)\in A_\beta} d(i,j)$ denoted as $\Sigma$, we have (12). As $\beta \leq 1$, we always have

$$\Sigma \leq \beta \frac{k_L\left(3\left(N^2-N\right)-\left(k_L^2-1\right)\right)}{3}. \tag{18}$$

Considering the second factor on the left-hand side as a function of $k_L$ (with $N$ fixed), we see that it is increasing for $k_L < k_0 = \sqrt{N(N-1)+1/3}$. Because $k_0 > N-1$, and $1 \leq k_L \leq N-1$, we may replace $k_L$ by $N-1$. This leads to the following (weaker) inequality:

$$\Sigma \leq \beta \frac{(N-1)N(2N-1)}{3}. \tag{19}$$

Inequality (19) leads to the following inequality between $\beta$ and $C$.

THEOREM 4.2. *Given a network with $N$ nodes, connection coefficient $\beta$ and compactness $C$, then*

$$C \leq \beta \leq \min\left(\frac{3(N-1)}{N+1}C, 1\right) \leq \min(3C, 1). \tag{20}$$

PROOF. The first inequality in formula (20) follows from (13). The last one is trivial, and so is the fact that $\beta$ and $C$ are always smaller than or equal to 1. So, we only have to show that $\beta \leq (3(N-1)/(N+1))C$. This inequality follows from the chain of relations

$$\begin{aligned} C &= \frac{\beta N}{N-1} - \frac{\Sigma}{N(N-1)^2} && \text{(by (16))} \\ &\geq \frac{\beta N}{N-1} - \frac{\beta(N-1)N(2N-1)}{3N(N-1)^2} && \text{(by (19))} \\ &= \beta\left(\frac{3N-2N+1}{3(N-1)}\right) = \beta\frac{N+1}{3(N-1)}. \end{aligned}$$

Consequently,

$$\beta \leq \frac{3(N-1)}{N+1}C.$$

## 5. ADDING ONE NODE: ITS INFLUENCE ON THE COMPACTNESS VALUE

In this section, we show that adding one node, disconnected from all others, lowers the compactness value of the network. Adding, however, a node that is connected to all others increases the compactness value. This shows that the compactness measure as proposed by [3] has nice (and expected) properties.

### 5.1. Adding One Node Disconnected from All Other Ones

The reader will notice, that the proof of this result is surprisingly difficult (or at least more complicated than the authors expected). If the compactness value was zero before the expansion, it stays zero, and if the compactness value was 1, it certainly decreases. We next consider compactness values that lie strictly between 0 and 1, hence with $\beta$-values strictly between 0 and 1.

If the compactness value before the expansion was as in (16), with $\Sigma$ the sum of all $d(i,j)$ not equal to $N$, then its compactness value after the expansion is

$$C' = \frac{\beta'(N+1)}{N} - \frac{\Sigma}{(N+1)N^2}, \tag{21}$$

where $\beta'$ denotes the connection coefficient of the new, expanded network. Now, $\beta' = \beta((N-1)/(N+1))$, so that we have to show

$$\begin{aligned} &\frac{\beta N}{N-1} - \frac{\Sigma}{N(N-1)^2} > \frac{\beta(N-1)}{N} - \frac{\Sigma}{(N+1)N^2} \quad \text{or} \\ &\frac{\beta N}{N-1} - \frac{\beta(N-1)}{N} > \frac{\Sigma}{N(N-1)^2} - \frac{\Sigma}{(N+1)N^2}. \end{aligned}$$

This inequality reduces, after some simple algebra to

$$\Sigma(3N-1) < \beta\left(2N^4 - N^3 - 2N^2 + N\right) = \beta N\left(N^2-1\right)(2N-1). \tag{22}$$

Applying inequality (19) gives, that it is sufficient to prove

$$\beta\frac{(N-1)N(2N-1)}{3}(3N-1) \leq \beta N(N-1)(N+1)(2N-1),$$

or, equivalently $3N-1 \leq 3(N+1)$. This inequality is clearly true, proving that adding one node, disconnected from all others decreases the compactness value of the network.

We observe that eliminating $\beta'$ and $\Sigma$ from (16) and (21) leads to the following formula expressing $C'$ as a function of $C$, $N$, and $\beta$:

$$C' = C\frac{(N-1)^2}{N(N+1)} + \beta\frac{N-1}{N(N+1)}. \tag{23}$$

### 5.2. Bidirectionally Adding One Node Connected to All Other Ones

If the compactness value before the expansion was

$$\frac{N^3 - N^2 - S}{N^3 - 2N^2 + N}, \tag{24}$$

where $S$ denotes the sum of all $d(i,j)$, then its compactness value after the expansion is at least

$$\frac{(N+1)^3 - (N+1)^2 - (S+2N)}{(N+1)^3 - 2(N+1)^2 + (N+1)}.$$

Hence, we have to show that

$$\frac{N^3 - N^2 - S}{N^3 - 2N^2 + N} \leq \frac{N^3 + 2N^2 - N - S}{N^3 + N^2} \quad \text{or}$$
$$\left(N^3 - N^2 - S\right)\left(N^3 + N^2\right) \leq \left(N^3 + 2N^2 - N - S\right)\left(N^3 - 2N^2 + N\right).$$

After some calculations, this leads to

$$N^2(N-1)(3N-1) \leq SN(3N-1) \quad \text{or} \quad N(N-1) \leq S.$$

Because $N(N-1)$ is the smallest possible value for $S$, this proves the increase in the compactness value.

## 6. A NEW FORMULA FOR BRS-COMPACTNESS

Consider a network with $N$ nodes and with connection coefficient $\beta$. Then, we introduce the following definition.

DEFINITION 6.1. Let $\delta_k$, $k = 1, 2, \ldots, L_\beta$ (recall, that $L_\beta$ denotes the largest possible distance between two nodes, given the connection coefficient $\beta$) be the fraction of the nodes in $A_\beta$ for which $d(i,j) = k$. As the $\delta_k$ are fractions, we have

$$\sum_{k=1}^{L_\beta} \delta_k = 1. \tag{25}$$

Consequently, this leads to the following new formula for $C$:

$$C = \frac{\beta N}{N-1} - \frac{\sum_{k=1}^{L_\beta} k\delta_k \beta\left(N^2 - N\right)}{N(N-1)^2} = \frac{\beta}{N-1}\left(N - \sum_{k=1}^{L_\beta} k\delta_k\right). \tag{26}$$

We know, that $\beta = 0.5$ for a unidirectional $N$-chain. In this case, we further have

$$\delta_1 = \frac{2}{N} > \delta_2 = \frac{2(N-2)}{N(N-1)} > \delta_3 > \cdots > \delta_{N-1} = \frac{2}{N(N-1)}. \tag{27}$$

For a bidirectional $N$-chain $\beta = 1$, but the $\delta_k$ are the same as for a unidirectional one. For a unidirectional $N$-loop, $\beta = 1$, and all $\delta_k$ are equal to $1/(N-1)$.

This leads to the following research problem. For which networks is

$$\delta_{L_\beta} \leq \delta_{L_\beta - 1} \leq \cdots \leq \delta_1. \tag{28}$$

Note, that it is easy to find networks, unidirectional as well as bidirectional ones where (28) is not satisfied. Indeed, for the following network with seven nodes (Figure 4), we have

$$\beta = \frac{5}{14}, \qquad \delta_1 = \frac{2}{5}, \qquad \delta_2 = \frac{3}{5}, \qquad \text{and} \qquad C = \frac{9}{28}.$$

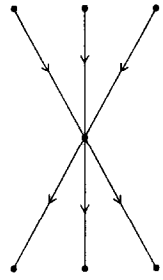


Figure 4. A unidirectional network where inequality (28) is not satisfied.

## 7. DISJOINT UNIONS OF ARBITRARY NETWORKS

Let $G_1$ and $G_2$ be two disjoint networks, the first having $N_1$ ($> 1$) nodes, the second one having $N_2$ ($> 1$) nodes. We next consider their (disjoint) union. The aim of this section is to obtain the compactness value $C$ and connection coefficient $\beta$ of this union, as a function of the compactness values of $G_1$ and $G_2$ (denoted, respectively, as $C_1$ and $C_2$), their connection coefficients $\beta_1$ and $\beta_2$ and the number of nodes $N_1$ and $N_2$.

LEMMA 7.1. *With the notations introduced above, we have*

$$\beta = \frac{\beta_1 N_1(N_1-1) + \beta_2 N_2(N_2-1)}{(N_1+N_2)(N_1+N_2-1)}. \tag{29}$$

PROOF. Graph $G_1$ contains, by the definition of the connection coefficient, $\beta_1(N_1^2-N_1)$ connected pairs of nodes. Similarly, $G_2$ contains $\beta_2(N_2^2-N_2)$ connected pairs of nodes. Then $\beta$, the connection coefficient of $G$, the disjoint union of $G_1$ and $G_2$, is

$$\beta = \frac{\beta_1 N_1(N_1-1) + \beta_2 N_2(N_2-1)}{(N_1+N_2)(N_1+N_2-1)}.$$

COROLLARY. *The connection coefficient* $\beta = \lambda_1\beta_1 + \lambda_2\beta_2$ *with* $\lambda_1, \lambda_2 \in [0,1]$, *and* $\lambda_1 + \lambda_2 < 1$.

PROOF. Clearly,

$$\lambda_j = \frac{N_j^2 - N_j}{N_1^2 + 2N_1N_2 + N_2^2 - N_1 - N_2},$$

with $j = 1, 2$. Now,

$$\lambda_1 + \lambda_2 = \frac{N_1^2 + N_2^2 - N_1 - N_2}{N_1^2 + N_2^2 - N_1 - N_2 + 2N_1N_2} < 1. \tag{30}$$

This result is not unexpected. $\beta$ is not a convex combination of $\beta_1$ and $\beta_2$ ($\lambda_1 + \lambda_2 \neq 1$) as there is a disjoint union involved. Intuitively, there is a loss in cohesion. This corresponds to a decrease in compactness (at least if $G_1$ and $G_2$ are 'similar') as will be shown shortly.

THEOREM 7.1. *Using the notation introduced above, we find for the value* $C$, *of the compactness of a disjoint union*

$$C = \frac{C_1N_1(N_1-1)^2 + C_2N_2(N_2-1)^2 + N_1N_2(\beta_1(N_1-1) + \beta_2(N_2-1))}{(N_1+N_2)(N_1+N_2-1)^2}. \tag{31}$$

PROOF. Denoting in the first graph $\sum_{\substack{d(i,j)<N_1 \\ i\neq j}} d(i,j)$ by $\Sigma_1$, $\sum_{\substack{d(i,j)<N_2 \\ i\neq j}} d(i,j)$ by $\Sigma_2$ in the second one, and $\sum_{\substack{d(i,j)<N_1+N_2 \\ i\neq j}} d(i,j)$ by $\Sigma$ in the union, we have

$$\begin{aligned} C_1 &= \frac{\beta_1 N_1}{N_1-1} - \frac{\Sigma_1}{N_1(N_1-1)^2}, \\ C_2 &= \frac{\beta_2 N_2}{N_2-1} - \frac{\Sigma_2}{N_2(N_2-1)^2}, \\ C &= \frac{\beta(N_1+N_2)}{N_1+N_2-1} - \frac{\Sigma}{(N_1+N_2)(N_1+N_2-1)^2}. \end{aligned} \tag{32}$$

It is clear that $\Sigma = \Sigma_1 + \Sigma_2$ because we have a disjoint union. Substituting the value of $\beta$ (29) and this sum in expression (32) yields

$$C = \frac{(N_1+N_2)}{N_1+N_2-1}\,\frac{\beta_1 N_1(N_1-1)+\beta_2 N_2(N_2-1)}{(N_1+N_2)(N_1+N_2-1)} - \frac{\Sigma_1+\Sigma_2}{(N_1+N_2)(N_1+N_2-1)^2}.$$

Rearranging terms gives

$$C = \frac{\beta_1 N_1}{N_1-1}\left(\frac{N_1-1}{N_1+N_2-1}\right)^2 + \frac{\beta_2 N_2}{N_2-1}\left(\frac{N_2-1}{N_1+N_2-1}\right)^2 - \frac{\Sigma_1}{N_1(N_1-1)^2}\,\frac{N_1(N_1-1)^2}{(N_1+N_2)(N_1+N_2-1)} - \frac{\Sigma_2}{N_2(N_2-1)^2}\,\frac{N_2(N_2-1)^2}{(N_1+N_2)(N_1+N_2-1)}.$$

As

$$\frac{\Sigma_j}{N_j(N_j-1)^2} = \frac{\beta_j N_j}{N_j-1} - C_j, \qquad \text{for } j = 1,2,$$

we obtain

$$\begin{aligned} C = {} & C_1\frac{N_1}{N_1+N_2}\left(\frac{N_1-1}{N_1+N_2-1}\right)^2 + C_2\frac{N_2}{N_1+N_2}\left(\frac{N_2-1}{N_1+N_2-1}\right)^2 \\ & + \frac{\beta_1 N_1}{N_1-1}\left[\left(\frac{N_1-1}{N_1+N_2-1}\right)^2 - \frac{N_1(N_1-1)^2}{(N_1+N_2)(N_1+N_2-1)^2}\right] \\ & + \frac{\beta_2 N_2}{N_2-1}\left[\left(\frac{N_2-1}{N_1+N_2-1}\right)^2 - \frac{N_2(N_2-1)^2}{(N_1+N_2)(N_1+N_2-1)^2}\right]. \end{aligned}$$

Simplifying this expression leads to

$$\begin{aligned} C = {} & C_1\frac{N_1}{N_1+N_2}\left(\frac{N_1-1}{N_1+N_2-1}\right)^2 + C_2\frac{N_2}{N_1+N_2}\left(\frac{N_2-1}{N_1+N_2-1}\right)^2 \\ & + \frac{\beta_1 N_1}{N_1-1}\,\frac{(N_1-1)^2 N_2}{(N_1+N_2)(N_1+N_2-1)^2} + \frac{\beta_2 N_2}{N_2-1}\,\frac{(N_2-1)^2 N_1}{(N_1+N_2)(N_1+N_2-1)^2} \end{aligned}$$

or

$$C = \frac{C_1 N_1(N_1-1)^2 + C_2 N_2(N_2-1)^2 + N_1 N_2(\beta_1(N_1-1)+\beta_2(N_2-1))}{(N_1+N_2)(N_1+N_2-1)^2}. \tag{31}$$

This proves the theorem.

### 7.1. Some Special Cases

(1) Taking $N_1 = N_2 = n$, $C_1 = C_2 = c$, and $\beta_1 = \beta_2 = b$ gives

$$C = \frac{c(n-1)^2 + bn(n-1)}{(2n-1)^2}. \tag{33}$$

(2) Taking $N_2 = 1$ (and leaving $\beta_2$ and $C_2$ unspecified, but finite) gives

$$C = \frac{C_1(N_1-1)^2 + \beta_1(N_1-1)}{(N_1+1)N_1},$$

which is exactly formula (23). This shows that, although formula (23) does not follow from the proof of Theorem 7.1, it does follow from formula (31), showing that formula (31) is also correct if one of the two (or even both!) networks consists of one point.

(3) Taking $\beta_1 = \beta_2 = 1$ and $C_1 = C_2 = 1$ gives the disjoint union of two complete networks. Its compactness is

$$C = \frac{N_1(N_1-1)^2 + N_2(N_2-1)^2 + N_1N_2(N_1+N_2-2)}{(N_1+N_2)(N_1+N_2-1)^2} = \frac{N_1(N_1-1)+N_2(N_2-1)}{(N_1+N_2)(N_1+N_2-1)}, \tag{34}$$

as obtained by [3]. If, moreover, $N_1 = N_2 = n$, then the BRS-compactness is equal to $(n-1)/(2n-1)$, which tends to $1/2$ if $n$ tends to infinity.

THEOREM 7.2. *If $N_1 = N_2 = N$, $C_1 = C_2 = c$, and $\beta_1 = \beta_2 = b$, then*

$$C = \frac{c(N-1)^2 + bN(N-1)}{(2N-1)^2} < c. \tag{35}$$

PROOF. We know by (20) that $b \leq (3(n-1)/(n+1))c$, hence,

$$C = \frac{n-1}{(2n-1)^2}(c(n-1)+bn) \leq \frac{n-1}{(2n-1)^2}\left(c(n-1) + n\frac{3(n-1)}{n+1}c\right) = \left(\frac{n-1}{2n-1}\right)^2 c\left(1+\frac{3n}{n+1}\right) < \left(\frac{1}{2}\right)^2 c4 = c.$$

We observe that this result is derived using formula (20), which in itself is a result of the general formula (19) giving an upper bound for the sum of all distances between connected nodes in a network.

We end this section by showing that the conditions $N_1 = N_2 = N$, $C_1 = C_2 = c$, and $\beta_1 = \beta_2 = b$ do not imply that the two graphs are isomorphic. This implies also that the result of Theorem 7.2 is not only valid for identical graphs, but also for some nonisomorphic ones.

Consider graphs $G_1$ and $G_2$ (see Figure 5). All links are assumed to be bidirectional. They have both seven nodes ($n = 7$) and clearly have a connection coefficient of 1 ($b = 1$). Finally, their $\Sigma$-values are 64, so that they have the same compactness value $c = 7/6 - (64/7 \cdot 6^2) = 115/126$. Moreover, the two graphs $G_1$ and $G_2$ are nonisomorphic as $G_1$ has a point with out-degree 1 (namely point 2), while $G_2$ does not have such a point. Using the same construction, both now with unidirectional links leads to another example.

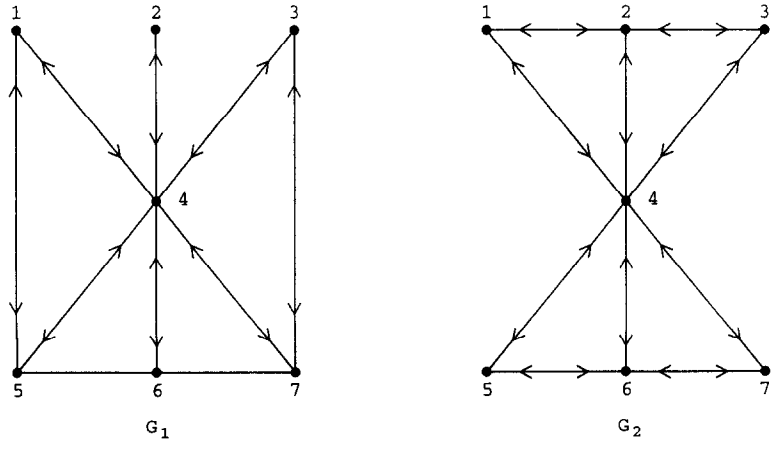



Figure 5. Two nonisomorphic graphs with the same compactness value.

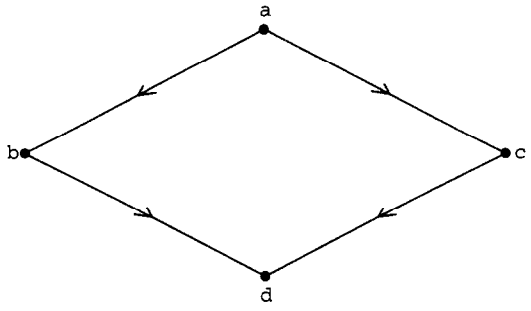



Figure 6. A network used to illustrate the difference between the general and the de Bra approach.

## 8. UNIDIRECTIONAL NETWORKS

For unidirectional, acyclic networks, such as citation networks, [25] introduced another convention, leading to the following definition.

DEFINITION 8.1. DE BRA'S SYMMETRIC CITATION DISTANCE MATRIX. Given a set of $N$ documents, then [25] defines the citation distance matrix $D_{\mathrm{DB}}$ as follows. $d(i,j)$ is equal to the length of the shortest path (in number of links) in the citation network from document $i$ to document $j$, if such a path exists. Further, $d(i,j) = d(j,i)$ and $d(i,i) = 0$. Finally, all other entries of the $D_{\mathrm{DB}}$-matrix are equal to $N$. Note, that the matrix $D_{\mathrm{DB}}$ is not the $D_B$-matrix of the corresponding undirected network. The relation between the $D_{\mathrm{DB}}$-matrix, the $D_B$-matrix and the $D_B$-matrix of the corresponding undirected network is illustrated in the following example (Figure 6).

The $D_B$, $D_{\mathrm{DB}}$, and $D_B$-matrix of the corresponding undirected network are as follows.

| | a b c d | a b c d | a b c d |
|---|---|---|---|
| a | 0 1 1 2 | 0 1 1 2 | 0 1 1 2 |
| b | 4 0 4 1 | 1 0 4 1 | 1 0 2 1 |
| c | 4 4 0 1 | 1 4 0 1 | 1 2 0 1 |
| d | 4 4 4 0 | 2 1 1 0 | 2 1 1 0 |

De Bra made the citation matrix artificially symmetric. We show now that, for acyclic unidirectional networks, this is not really necessary.

### 8.1. Different Representations of an Acyclic Unidirectional Network

An acyclic unidirectional network such as a citation network, can be described in the following three ways:

(1) as a general network using the BRS distance matrix (where the network just happens to be unidirectional and acyclic),
(2) using the BRS distance matrix, but it is given that the network is unidirectional and acyclic (this condition influences the min-value in the compactness formula),
(3) using de Bra's conventions.

In all three cases, $\max = N(N^2 - N)$. In the first case, $\min = N^2 - N$, in the second one, it is $N(N^2 - 1)/2$, and in the last one, it is again $N^2 - N$. This leads to the following compactness formulae (denoted, respectively, as $C$, $C_{B|U}$, and $C_{\mathrm{DB}}$):

$$C = \frac{N^3 - N^2 - \left( \sum_{(i,j) \in A_\beta} d(i,j) + \left( \left(N^2 - N\right)/2 \right) N \right)}{\left(N^3 - N^2\right) - \left(N^2 - N\right)}, \tag{36}$$

$$C_{B|U} = \frac{N^3 - N^2 - \left( \sum_{(i,j)\in A_\beta} d(i,j) + \left( \left(N^2 - N\right)/2 \right) N \right)}{\left(N^3 - N^2\right) - \left(N\left(N^2 - 1\right)\right)/2}, \tag{37}$$

and, finally,

$$C_{\mathrm{DB}} = \frac{N^3 - N^2 - \left( 2 \sum_{(i,j)\in A_\beta} d(i,j) \right)}{\left(N^3 - N^2\right) - \left(N^2 - N\right)}. \tag{38}$$

These three formulae are all special cases of the general form introduced in formula (2).

The next theorem shows the relation between the three compactness formulae.

THEOREM 8.1. *For unidirectional, acyclic networks such as citation networks, we have*

(1) $C_{B|U} = C_{\mathrm{DB}}$.
(2) $2C = C_{B|U} = C_{\mathrm{DB}}$.

PROOF. Denoting $\sum_{(i,j)\in A_\beta} d(i,j)$ simply by $\Sigma'$ and $N^2 - N$ by $m$, we have

$$C_{B|U} = \frac{Nm - \Sigma' - Nm/2}{Nm - m(N+1)/2} = \frac{Nm - 2\Sigma'}{Nm - m} = C_{\mathrm{DB}}.$$

Further,

$$C = \frac{Nm - \Sigma' - Nm/2}{Nm - m} = \frac{1}{2}\frac{Nm - 2\Sigma'}{Nm - m} = \frac{1}{2}C_{\mathrm{DB}} = \frac{1}{2}C_{B|U}.$$

This proves the theorem.

### 8.2. Corollaries and Comments

It is easy to check that the previous result is also true in general, i.e., with $\varphi(N)$ instead of $N$.

As the BRS-compactness value $C$ of an acyclic, unidirectional network is at most 0.5, this also implies that de Bra's measure can be considered as a renormalization (resulting again in values between 0 and 1) of the BRS-value for the case of acyclic, unidirectional networks.

In [40], the compactness of some small lattice citation networks has been calculated using de Bra's formula [25].

### 8.3. The Nonuniqueness of the de Bra Matrix Description

If one reverses all arrows in a digraph, then the new network will be called the *reversed network*. The operation of reversing all arrows in a network is called *reversion*. It is clear that the de Bra matrix of a citation network and that of its reversion are the same. Observe that, generally, citation networks that are each other's reversion, are nonisomorphic.

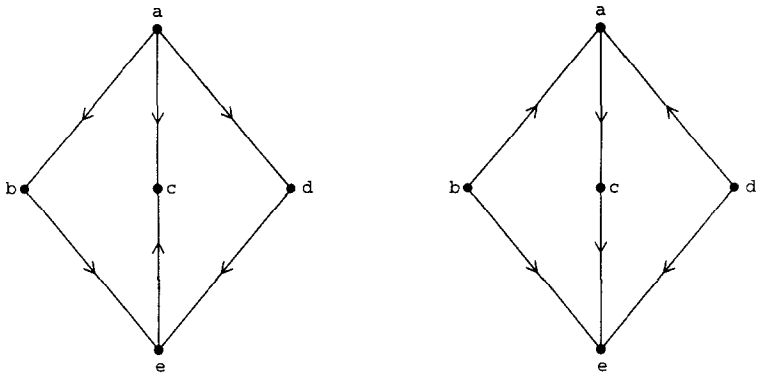
PROPERTY. De Bra's description is nonunique. By this, we mean that there exist citation networks that are nonisomorphic and are not each other's reversion, and yet yield the same de Bra matrix representation.

It suffices to give an example. The citation networks represented by Figures 7a and 7b are clearly nonisomorphic and not each other's reversion. Yet, they both have the following de Bra matrix representation.

Matrix representation

$$\begin{pmatrix} & a & b & c & d & e \\ a & 0 & 1 & 1 & 1 & 2 \\ b & 1 & 0 & 2 & 5 & 1 \\ c & 1 & 2 & 0 & 2 & 1 \\ d & 1 & 5 & 2 & 0 & 1 \\ e & 2 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

It is clear, that this nonuniqueness shows the nonoptimality of de Bra's representation. Consequently, it seems better to stick to the original BRS matrix representation.




(a) (b)

Figure 7. Two nonisomorphic graphs with the same deBra matrix representation.

## 9. CALCULATION OF THE COMPACTNESS OF A BALANCED TREE

In Section 3, we calculated the connection coefficient of a balanced tree of depth $d$. We will now continue the calculations in order to obtain this tree's compacted value. First, we need its $\Sigma$-value, denoted as $\Sigma_d$. This is obtained as follows:

$$\begin{aligned}\Sigma_d &= \sum_{i=1}^{d} b^i + 2\sum_{i=2}^{d} b^i + \cdots + d\sum_{i=d}^{d} b^i = \frac{\left(b^{d+1}-b\right)+2\left(b^{d+1}-b^2\right)+\cdots+\left(b^{d+1}-b^d\right)}{b-1} \\ &= \frac{(d(d+1)/2)b^{d+1} - \left(db^{d+2} - (d+1)b^{d+1} + b\right)/(b-1)^2}{b-1} \\ &= \frac{db^{d+3}(d+1) - 2db^{d+2}(d+2) + b^{d+1}(d+1)(d+2) - 2b}{2(b-1)^3}.\end{aligned} \tag{39}$$

This leads to the following compactness value:

$$\begin{aligned}C_d &= \frac{\left(db^{d+1} - (d+1)b^d + 1\right)\left(b^{d+1}-1\right)}{\left(b^{d+1}-1\right)\left(b^d-1\right)b\left(b^d-1\right)} \\ &\quad - \frac{\left(db^{d+3}(d+1) - 2db^{d+2}(d+2) + b^{d+1}(d+1)(d+2) - 2b\right)(b-1)^3}{2(b-1)^3\left(b^{d+1}-1\right)b^2\left(b^d-1\right)^2} \\ &= \frac{b^{d-1}\left(2b^{d+1}(bd-d-1) - b^2d(d+1) + 2b\left(d^2+d+1\right) - d(d+1)\right)}{2\left(b^d-1\right)^2\left(b^{d+1}-1\right)}.\end{aligned} \tag{40}$$

Taking $d = 1$ in equation (40) gives $C_1 = 1/(b+1)$. If, moreover, we take the limit for $b$ tending to 1 in (40), we find the compactness value of a unidirectional chain of length $d$, namely $(2d+1)/6d$ (checked by computer).

## 10. CALCULATION OF THE COMPACTNESS OF AN ENSEMBLE

In this section, we present another construction of a unidirectional network based on simple building blocks. This construction generalizes the unidirectional chain. We will compute its BRS-compactness and study its limiting behavior. We are convinced that examples such as this one are important in order to gain experience with this measure of cohesion. Moreover, the more complicated an example is, the more it resembles real-world networks, and, hence, can be used for modelling purposes.

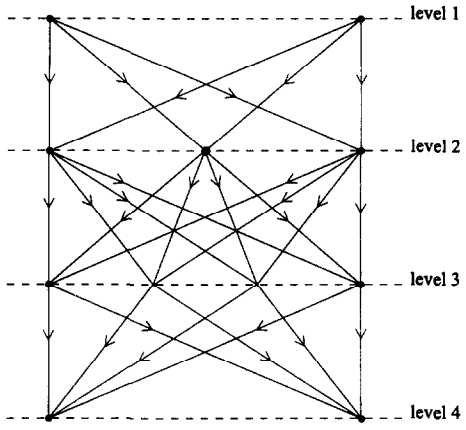



Figure 8. Ensemble with four levels.

## Construction of an ensemble

Consider $L$ 'levels'. Each level $j$ consists of $n_j$ nodes. Nodes at a fixed level are disconnected between each other, but are connected to each node at level $j+1$ (except, of course, nodes at level $L$). Connections are unidirectional and no other connections exist. This graph will be called an ensemble. An example, with four levels, is given in Figure 8.

The total number of nodes in the ensemble is $N = \sum_{j=1}^{L} n_j$. We now determine the $\Sigma$-value of the BRS-representation. The nodes at level 1 contribute

$$n_1 \times ([n_2 + 2n_3 + \cdots + (L-1)n_L] + (n_1 - 1) \times N),$$

nodes at level 2 contribute

$$n_2 \times ([n_3 + 2n_4 + \cdots + (L-2)n_L] + (n_2 - 1 + n_1) \times N),$$

in general, nodes at level $j$ contribute $(j = 1, \ldots, L-1)$,

$$n_j \times ([n_{j+1} + 2n_{j+2} + \cdots + (L-j)n_L] + (n_j - 1 + n_1 + \cdots + n_{j-1}) \times N),$$

finally, at level $L$, we have $n_L((n_L - 1) + n_1 + \cdots + n_{L-1}) \times N$.

This leads to the following total:

$$N \sum_{j=1}^{L} n_j \left( \sum_{k=1}^{j} n_k - 1 \right) + \sum_{j=1}^{L-1} n_j \left( \sum_{k=1}^{L-j} k n_{j+k} \right).$$

We next consider the special case that all $n_j$ are equal, hence $N = nL$. Then, the total is

$$nL \sum_{j=1}^{L} n(nj-1) + \sum_{j=1}^{L-1} n \left( \sum_{k=1}^{L-j} kn \right) = n^3 L \frac{L(L+1)}{2} - n^2 L^2 + n^2 \sum_{j=1}^{L-j} \frac{(L-j)(L-j+1)}{2}.$$

Putting $L - j = k$ leads to

$$n^2 L^2 \left( n \frac{L+1}{2} - 1 \right) + n^2 \sum_{k=1}^{L-1} \frac{k(k+1)}{2} = n^2 L^2 \left( n \frac{L+1}{2} - 1 \right) + n^2 \frac{(L-1)L(L+1)}{6}$$

$$= n^2 L \left( \frac{nL(L+1)}{2} - L + \frac{L^2 - 1}{6} \right).$$

Consequently, the BRS-compactness value is

$$C = \frac{n^3L^3 - n^2L^2 - n^2L\left(nL(L+1)/2 - L + \left(L^2-1\right)/6\right)}{n^3L^3 - 2n^2L^2 + nL}$$
$$= \frac{n(3nL^2 - 3nL - L^2 + 1)}{6(n^2L^2 - 2nL + 1)}.$$

In particular, if $n = 1$ (a unidirectional chain consisting of $L$ nodes), $C$ is equal to

$$\frac{2L-1}{6(L-1)}.$$

This result is in agreement with Corollary 2 of Theorem 4.1, with $\varphi(N) = N = L$. Hence, we see that a chain consisting of two nodes has a BRS-compactness value of 0.5 (the maximum value for a unidirectional network), a chain consisting of three nodes of 5/12, for four nodes it is 7/18, and so on, with a limiting value of 1/3.

We fix $n$ and consider the limit for $L \to \infty$. Then, the limiting value is

$$\frac{3n^2 - n}{6n^2} = \frac{1}{2} - \frac{1}{6n}.$$

For $n = 1$, this is 2/6, for $n = 2$, it is 5/12, for $n = 3$, it is 8/18 and so on. If also, $n$ tends to infinity, we find the value 0.5, as expected for a unidirectional network.

We next fix $L$, and consider the limit for $n \to \infty$. This limit value is equal to

$$\frac{3L^2 - 3L}{6L^2} = \frac{1}{2} - \frac{1}{2L}.$$

If now, $L$ tends to infinity, we find (again) 0.5. For $L = 1$, we find 0, also as it is expected to be.

## 11. CONCLUSION

The internet, citation networks, as well as scientific collaboration networks, are nowadays in the center of attention. We hope that the structural measure of cohesion, namely BRS-compactness, studied in this article will prove to be a useful element for their description. The fact that this measure has the well-known Wiener index as the main component leads to the suggestion to find and apply more topological indices. These indices play an important role in the description of molecular graphs in computational and mathematical chemistry [33,41,42]. It has, moreover, been shown that the Wiener index is correlated to a large number of physiochemical properties such as boiling point, melting point, refractive index, surface tension and viscosity of chemical molecules. There seems to be no reason why they could not play an equally important role to characterize networks in the context of the information sciences.

One clear restriction of the measure studied here and by [3] is the fact that it relates to unweighted networks. Yet, there are usually many links between the nodes in a graph, be it authors that receive many citations from the same colleagues, or sites on the internet that are connected through many links. This leads to a weighted graph structure that will be studied in a following paper [43].

We conclude with an open problem. Given a rational number between 0 and 1, does there exist a graph with that particular BRS-compactness value?

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