General Study of the Distribution of N-Tuples of Letters or Words Based on the Distributions of the Single Letters or Words

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Abstract—This paper establishes the general relation between the distribution of N-tuples of letters (e.g., N-truncations, N-grams) or words (e.g., N-word phrases) and the distributions of the single letters or words. Here the very general case is treated: the case where there is dependence on the place i in the N-tuple ($i = 1, \ldots, N$) in the sense that, for each $i = 1, \ldots, N$, a different distribution of the letters or words is supposed.

Concrete calculations are performed in the important case of Zipfian distributions (i.e., power laws) for the single letters or words. In this case, we prove that the distribution of the N-tuples (N-fixed) is the sum of power laws. © 2000 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

The problem of studying the distribution of multiword phrases (i.e., chains of words, say of fixed length $N \in \mathbb{N}$) has been introduced in [1] and solved in [2] in the case that, in an N-word phrase, the words appear independently of each other. Of course, as also noted in [2], this is not correct and the model only served as a first approximation. It must be noted that the model developed in [2] is already much more general than, e.g., Mandelbrot's (see [3] or the more readable proof of it in [4] where one supposes (in the framework of N-tuples of letters) that the letters all have an equal chance to occur). This was not supposed in [2] nor in [5], where one gives a proof of the distribution of N-grams (N fixed) based on the distributions of the letters. In this framework, independence of letters in N-grams is supposed but also proven to be exact in the case of redundant N-grams. For more details on N-grams (especially on their importance in virtually every aspect of information science), we refer the reader to [6–9] or to [5] in which these works are briefly described in the introduction.

The independence assumption may then well be true for N-grams. It is still an approximation of reality in the cases of N-word phrases and of texts consisting of words truncated to the length N

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(right or left truncated). Indeed, in the former case, words do not occur independently of each other, and in the latter case, in truncated words (as is the case in normal words in texts), the letters do not occur independently of each other. At the time of the writing of the papers [2,5], it was not clear, however, how to deal with the dependent case, and more in particular, it was not clear to generalize the intricate proofs (as given in the appendices of [2] and [5]) to the dependent case. This paper presents the solution to this problem, not in the least stimulated by a positive review of [2] by R. Marcus, see [10], in which the problem of knowing distributions of N-tuples of objects based on the objects' distributions has been recognized as very important. We present the solution in the next section (and the Appendix).

Section 3 presents applications of this general result to the cases of N-tuples of letters, and words: N-word phrases are discussed, texts of N-truncated words (i.e., left or right truncated words up to length N) are discussed as well as—for the sake of completeness—redundant N-grams, where we repeat briefly the results of [5]. Section 4 presents some conclusions and open problems.

2. THE DISTRIBUTION OF N-TUPLES OF OBJECTS BASED ON THE DISTRIBUTIONS OF THE OBJECTS THEMSELVES—DEPENDENT CASE

Since we want to capture the topics: N-word phrases and words consisting of N letters, we will use henceforth the terminology: N-tuples of objects. In the case of N-word phrases, the N objects are consecutive words; in the case of words consisting of N letters, the objects clearly are consecutive letters. Note also that, in this paper, N will be fixed in $\mathbb{N} = \{1, 2, 3, \dots\}$.

For each $i \in \{1, \dots, N\}$, we denote by S_i the object set, i.e., the set of objects that can be used on the ith place. Usually the S_is are equal (e.g., the alphabet, the set of possible words, ...) but this is not supposed here. It is even so that the cardinals $\#S_i$ (i.e., the number of elements in S_i) do not have to be equal.

Denote, for each $i \in \{1, \dots, N\}$, $P(r_i \mid i)$ the rank-frequency distribution on "coordinate" i of the objects in S_i. So we clearly work here in a framework of i-dependence. Of course, one more generalization could be possible, namely working with $P(r_i \mid r_1, \dots, r_{i-1})$, the probability to have the symbol in S_i on rank r_i, given that on the jth place in the N-tuple ($j = 1, \dots, i-1$) we have the symbol on rank r_j in S_j. We leave open this very general case of dependence and will work only with dependence on the place $i \in \{1, \dots, N\}$. In other words, we put

$$P(r_i \mid i) = P(r_i \mid r_1, \dots, r_{i-1}) \tag{1}$$

for all $r_1, \dots, r_i$, being at least an important generalization of the independent case (there we would put $P(r_i \mid i) = P(r_i)$ where all S_i are equal and $r_i \in \{1; \dots, \#S_i\}$). Let $P(r_1, \dots, r_N)$ denote the probability to have an N-tuple where, for each $i = 1, \dots, N$, we use the object on rank r_i. By the very definition of conditional probability, applied repeatedly, we have

$$\begin{aligned}
P(r_1, \dots, r_N) &= P(r_N \mid r_1, \dots, r_{N-1})P(r_1, \dots, r_{N-1}) \\
&= P(r_N \mid r_1, \dots, r_{N-1})P(r_{N-1} \mid r_1, \dots, r_{N-2}).P(r_1, \dots, r_{N-2}) \\
&= \vdots \\
&= P(r_N \mid r_1, \dots, r_{N-1})P(r_{N-1} \mid r_1, \dots, r_{N-2}) \dots P(r_2 \mid r_1)P(r_1) \\
&= P(r_N \mid N)P(r_{N-1} \mid N-1) \dots P(r_2 \mid 2)P(r_1),
\end{aligned} \tag{2}$$

using (1).

Denote by $P_N(r)$ the rank-frequency distribution of the N-tuples. It takes only a moment's reflection to see that P_N is obtained as

$$x = P_N(r), \tag{3}$$

where

$$r = \#\{(r_1, \ldots, r_N) \parallel P(r_1)P(r_2 \mid 2) \ldots P(r_N \mid N) \geq x\}. \tag{4}$$

Formulae (3) and (4) are the basic parts in the solution of the problem dealt with in this paper. They provide the device to derive the N-tuple distribution from the objects distributions. Of course, in order to be able to produce "concrete" functional relationships for (3), we must put in "concrete" functional relationships for the conditional probabilities $P(r_i \mid i)$, $i = 2, \ldots, N$ and for $P(r_1)$. For the case of N-word phrases, we assume the word distributions to follow Zipfian distributions. It is well known that this is the most evident assumption. But also in the case of N-tuples of letters, Egghe in [5] shows that Zipfian distributions can be used: especially for Asiatic languages the fit is almost perfect, but for the other languages the model is certainly acceptable. We therefore put

$$P(r_i \mid i) = \frac{C_i}{r_i^{\beta_i}}, \tag{5}$$

$i = 1, \ldots, N$ as the distributions of the objects on "coordinate" i. Here $r_i \in \mathbb{R}^+$. Indeed we will adopt the continuous setting for the ease of calculation. On the same lines, (4) will be calculated by replacing $\#$ by vol, the volume of the N-dimensional set appearing in formula (4). We can prove the following theorem.

THEOREM 2.1. *Let $N \in \mathbb{N}$ be fixed and let (5) denote the conditional distributions of the objects on coordinate $i \in \{1, \ldots, N\}$. Then $x = P_N(r)$ is the distribution of the N-tuples, where*

$$r = \sum_{j=1}^{N} \frac{D_j}{x^{1/\beta_j}}, \tag{6}$$

where D_j are constants. Hence, the inverse of (3) is a sum of Zipfian (i.e., power law) distributions.

Since the proof is elaborate and unrelated with the paper's topic, we present it in the Appendix. Since Theorem 2.1 deals with the dependent case, we supposed $\beta_i \neq \beta_j$ for all $i \neq j$. This will always be the case in practice. If some β_i are equal (but not all), we leave the result as an open problem. For the sake of completeness, if all β_i are equal, i.e., the Zipfian distribution of the objects is independent from the coordinate $i \in \{1, \ldots, N\}$, we repeat here the result of [5].

THEOREM 2.2. *Let $N \in \mathbb{N}$ be fixed and assume that*

$$P(r_i) = \frac{C}{r_i^{\beta}} \tag{7}$$

denotes the object distribution (the same for all $i = 1, \ldots, N$). Then $x = P_N(r)$ is the distribution of the N-tuples, where

$$r = \frac{D}{x^{1/\beta}} \frac{\ln^{N-1}\left(D/x^{1/\beta}\right)}{(N-1)!}, \tag{8}$$

and where D is a constant. Hence,

$$r = \frac{f_N(y)}{(N-1)!},$$

where $f_N(y) = y \ln^{N-1}(y)$ and $y = D/(x^{1/\beta})$. It is shown in [2,5] that (8) above. for large r. reduces to

$$P_N(r) \approx \frac{E}{(\Psi_N(r))^{\beta}}, \tag{9}$$

where E is a constant and where $\Psi_N = f_N^{-1}$, the inverse of f_N.

Theorem 2.1 above represents the case of dependence on the coordinate $i \in \{1, \ldots, N\}$ of the object distributions; Theorem 2.2 represents the independent case. Both theorems have direct applications, as will be seen in the next section.

3. APPLICATIONS

We start by repeating the independence result of [2,5], where Theorem 2.2 can be used. We include it here for the sake of completeness.

CASE 3.1. THE CASE OF REDUNDANT N-GRAMS. We refer to [5] and the already-mentioned references on N-grams. Here we will suffice by explaining what redundant N-grams are. N-grams are, simply, words consisting of N letters. N-grams can be generated from normal texts by replacing each word (e.g., the word SYMBOL) by the string of N-grams, e.g., for $N = 2$,

$$\star S \text{ SY YM MB BO OL } L\star$$

and, e.g., for $N = 3$,

$$\star\star S \text{ } \star SY \text{ SYM YMB MBO BOL } OL\star \text{ } L\star\star.$$

The use of the stars makes sure that all letters in all words appear an equal number of times in the N-grams. This is the most important case for applications and are called redundant N-grams.

Note that in this case, Theorem 2.2 applies since the distribution of the letters is the same (hence, independent) on the coordinate $i \in \{1, \ldots, N\}$. We can conclude that redundant N-grams (as, e.g., derived from texts) have a rank-frequency distribution as in (8), hence, not a power law.

NOTE. A general note is in order here. Both Theorems 2.1 and 2.2 show that the rank-frequency distribution P_N of N-tuples is not a power law, even when we supposed the object distributions to be power laws. In Theorem 2.1, we obtained for P_N essentially a power law of the inverse of the function $f_N(y) = y \ln^{N-1}(y)$. But since this last function very much resembles a power law (in the sense of statistical fitting), it will be perfectly possible to fit $P_N(r)$ by a power law. The same remark goes for the result in Theorem 2.2: there one essentially has that P_N is the inverse of a sum of power laws, which resembles a power law in the statistical sense. In this paper, we showed that—as a mathematical model—the power law is not correct for N-tuples, when the object distributions are power laws, in all cases (dependence on the place i or not).

CASE 3.2. THE CASE OF N-WORD PHRASES. Here Theorem 2.1 applies since we can assume that words in N-word phrases occur according to Zipfian laws but where these laws are all different since words do not occur independently from each other. Hence, (6) represents the rank frequency distribution (where $x = P_N(r)$) for N-word phrases.

This extends the result of [2].

CASE 3.3. THE CASE OF N-TRUNCATED WORDS. For, e.g., information retrieval and indexing purposes, it is interesting to know what is the distribution of truncated words (say up to length N, N fixed). Indeed the knowledge of such a distribution gives information on the inequality between occurrences of different N-truncated words. This, in turn, can be used in measures of concentration or diversity (as, e.g., the entropy measure—see [11] or [12]) which are used (by applying some threshold on the concentration or diversity values) to determine the "selectivity" power of a key word.

Of course, contrary to the case of redundant N-grams, the occurrence of letters in an N-truncated word is dependent on the place (the coordinate) in the N-tuple. Hence, here Theorem 2.1 applies and formula (6) can be used. This is also the case for nonredundant N-grams.

4. CONCLUSIONS, OPEN PROBLEMS

In this paper, we have established the rank-frequency distribution of N-tuples of objects, given the conditional object distributions in coordinate $i \in \{1, \ldots, N\}$. We proved that this distribution is of the form $x = P_N(r)$, where

$$r = \sum_{i=1}^{N} \frac{D_i}{x^{1/\beta_i}} \tag{10}$$

in case the object distributions on coordinate i are Zipfian with exponent β_i, and where $i \neq j \Rightarrow$ $\beta_i \neq \beta_j$. The case where all $\beta_i s$ are equal, dealt with in [2,5], was repeated here for the sake of completeness.

We formulate here the problem of extending the conditional distributions $P(r_i \mid i)$ (as used here) to distributions of the form $P(r_i \mid i, N)$ where the probabilities also depend on the length N of the N-tuple and to prove results similar to the ones here but using $P(r_i \mid i, N)$ instead of the simpler $P(r_i \mid i)$. In this connection also, a theory of truncation (right, left) would be interesting. A further extension could involve $P(r_i \mid r_1, \ldots, r_{i-1})$ for all $i = 1, \ldots, N$.

It would also be nice to develop an axiomatic theory of texts which consist of words and where words consist of letters, i.e., a kind of "composition" of the separate results as described in Section 3.

APPENDIX

THEOREM. *Let $N \in \mathbb{N}$ be fixed and denote by*

$$P(r_i \mid i) = \frac{C_i}{r_i^{\beta_i}} \tag{A1}$$

the conditional distributions of the objects on coordinate $i \in \{1, \ldots, N\}$. Then $x = P_N(r)$ is the distribution of the N-tuples where

$$r = \sum_{j=1}^{N} \frac{D_j}{x^{1/\beta_j}}, \tag{A2}$$

where the $D_j s$ are constants.

PROOF. In the continuous setting, we have that $x = P_N(r)$ (the distribution of the N-tuples) iff (cf. (4))

$$r = \text{vol} \{(r_1, \ldots, r_N) \parallel P(r_1)P(r_2 \mid 2) \ldots P(r_N \mid N) \geq x\}. \tag{A3}$$

The inequality in the set in (A3) yields

$$\frac{C_1 \ldots C_N}{r_1^{\beta_1} \ldots r_N^{\beta_N}} \geq x. \tag{A4}$$

Hence, denoting $a = (C_1 \ldots C_N)/x$

$$1 \leq r_1 \leq \frac{a^{1/\beta_1}}{\left(r_2^{\beta_2} \ldots r_N^{\beta_N}\right)^{1/\beta_1}}. \tag{A5}$$

Hence, the maximal value of r_1 is

$$\frac{a^{1/\beta_1}}{\left(r_2^{\beta_2} \ldots r_N^{\beta_N}\right)^{1/\beta_1}},$$

where $r_2, \ldots, r_N$ can vary as follows. (A5) implies

$$r_2^{\beta_2/\beta_1} \leq \frac{a^{1/\beta_1}}{\left(r_3^{\beta_3} \ldots r_N^{\beta_N}\right)^{1/\beta_1}}.$$

Hence,

$$1 \leq r_2 \leq \frac{a^{1/\beta_2}}{\left(r_3^{\beta_3} \ldots r_N^{\beta_N}\right)^{1/\beta_2}}. \tag{A6}$$

In the same way, we have

$$1 \leq r_3 \leq \frac{a^{1/\beta_3}}{\left(r_4^{\beta_4} \ldots r_N^{\beta_N}\right)^{1/\beta_3}}, \tag{A7}$$

$$\vdots$$

$$1 \leq r_{N-1} \leq \frac{a^{1/\beta_{N-1}}}{\left(r_N^{\beta_N}\right)^{1/\beta_{N-1}}}, \tag{A8}$$

and

$$1 \leq r_N \leq a^{1/\beta_N}. \tag{A9}$$

The above arguments yield

$$r = a^{1/\beta_1} \int_{r_N=1}^{r_N=a^{1/\beta_N}} \frac{dr_N}{r_N^{\beta_N/\beta_1}} \int_{r_{N-1}=1}^{r_{N-1}=\frac{a^{1/\beta_{N-1}}}{r_N^{\beta_N/\beta_{N-1}}}} \frac{dr_{N-1}}{r_{N-1}^{\beta_{N-1}/\beta_N}} \cdots \int_{r_2=1}^{r_2=\frac{a^{1/\beta_2}}{\left(r_3^{\beta_3}\ldots r_N^{\beta_N}\right)^{1/\beta_2}}} \frac{dr_2}{r_2^{\beta_2/\beta_1}}. \tag{A10}$$

We suppose here that $\beta_i \neq \beta_j$, $\forall i, j = 1, \ldots, N$, $i \neq j$. We will evaluate the last integral, get an idea of what the general induction step will be and then complete the proof by complete induction.

$$\int_{r_2=1}^{r_2=\frac{a^{1/\beta_2}}{\left(r_3^{\beta_3}\ldots r_N^{\beta_N}\right)^{1/\beta_2}}} \frac{dr_2}{r_2^{\beta_2/\beta_1}} = \frac{1}{1-\beta_2/\beta_1} \left[\frac{a^{1/\beta_2-1/\beta_1}}{r_3^{\beta_3(1/\beta_2-1/\beta_1)} \ldots r_N^{\beta_N(1/\beta_2-1/\beta_1)}} - 1 \right].$$

Substituting this into (A10) yields the form

$$r = \gamma_2 a^{1/\beta_2} \int_{r_N} \frac{dr_N}{r_N^{\beta_N/\beta_2}} \int_{r_{N-1}} \frac{dr_{N-1}}{r_{N-1}^{\beta_{N-1}/\beta_2}} \cdots \int_{r_3} \frac{dr_3}{r_3^{\beta_3/\beta_2}}$$

$$+ \gamma_1 a^{1/\beta_1} \int_{r_N} \frac{dr_N}{r_N^{\beta_N/\beta_1}} \int_{r_{N-1}} \frac{dr_{N-1}}{r_{N-1}^{\beta_{N-1}/\beta_1}} \cdots \int_{r_3} \frac{dr_3}{r_3^{\beta_3/\beta_1}}. \tag{A11}$$

Here the bounds in all integrals are the same as in (A10) and

$$\gamma_2 = \frac{1}{1-\beta_2/\beta_1} = -\gamma_1.$$

Suppose now, by complete induction that

$$r = \sum_{j=1}^{i} \gamma_j^{(i)} a^{1/\beta_j} \int_{r_N} \frac{dr_N}{r_N^{\beta_N/\beta_j}} \int_{r_{N-1}} \frac{dr_{N-1}}{r_{N-1}^{\beta_{N-1}/\beta_j}} \cdots \int_{r_{i+1}} \frac{dr_{i+1}}{r_{i+1}^{\beta_{i+1}/\beta_j}}, \tag{A12}$$

where $\gamma_j^{(i)}$ are constants. The last integral yields

$$\int_{r_{i+1}=1}^{r_{i+1}=\frac{a^{1/\beta_{i+1}}}{\left(r_{i+2}^{\beta_{i+2}}\ldots r_N^{\beta_N}\right)^{1/\beta_{i+1}}}} \frac{dr_{i+1}}{r_{i+1}^{\beta_{i+1}/\beta_j}}$$

$$= \frac{1}{1-\beta_{i+1}/\beta_j} \left[\frac{a^{1/\beta_{i+1}-1/\beta_j}}{r_{i+2}^{\beta_{i+2}/\beta_{i+1}-\beta_{i+2}/\beta_j} \ldots r_N^{\beta_N/\beta_{i+1}-\beta_N/\beta_j}} - 1 \right]. \tag{A13}$$

Substituting this in (A12) and regrouping the terms yields (A12) but now for i replaced by $i+1$. Of course, this also proves that

$$r = \sum_{j=1}^{N-1} \gamma_j^{(N-1)} a^{1/\beta_j} \int_{r_N=1}^{r_N=a^{1/\beta_N}} \frac{dr_N}{r_N^{\beta_N/\beta_j}}, \tag{A14}$$

which can be rewritten as (and denoting γ_j for $\gamma_j^{(N)}$)

$$r = \sum_{j=1}^{N} \gamma_j a^{1/\beta_j}. \tag{A15}$$

Substituting the value of a and denoting the new constant D_j, this gives

$$r = \sum_{j=1}^{N} \frac{D_j}{x^{1/\beta_j}} \tag{A16}$$

completing the proof. Note that the $\beta_j s$ are the Zipf-law powers with which we started in (A1). Law (A16), interpreted inversely as $x = P_N(r)$ is the rank frequency distribution for N-tuples. Note also that these calculations are exact in the sense that no approximations are used. ∎

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