

NON-COMMUTATIVE CREPANT RESOLUTIONS (WITH SOME CORRECTIONS)

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ABSTRACT. We introduce the notion of a “non-commutative crepant” resolution of a singularity and show that it exists in certain cases. We also give some evidence for an extension of a conjecture by Bondal and Orlov, stating that different crepant resolutions of a Gorenstein singularity have the same derived category.

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1. INTRODUCTION

Let k be an algebraically closed field and let R be an integral Gorenstein k -algebra of dimension n . Put $X = \text{Spec } R$ and let $f : Y \rightarrow X$ be a resolution of singularities. f is a crepant resolution of X if $f^*\omega_X = \omega_Y$. Crepant resolutions do not always exist and they are usually not unique. A conjecture of Bondal and Orlov states that if $f_1 : Y_1 \rightarrow X$, $f_2 : Y_2 \rightarrow X$ are different crepant resolutions of X then $D^b(\text{coh}(Y_1)) \cong D^b(\text{coh}(Y_2))$.

In this paper we study certain non-commutative analogues of crepant resolutions and we call these “non-commutative crepant resolutions”. A non-commutative

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crepant resolution of R is an algebra $A = \text{End}_R(M)$ where M is a reflexive R -module and where A has finite global dimension and is a (maximal) Cohen-Macaulay R -module. For some general discussion of non-commutative resolutions we refer to [9, §5]

A standard example of a non-commutative crepant resolution is the following:

Example 1.1. Let G be a finite group and let V be a finite dimensional G -representation such that $G \subset \text{Sl}(V)$. Put $S = \text{Sym}(V)$ and $R = S^G$. Then $A = \text{End}_R(S) \cong S * G$ is a non-commutative crepant resolution of R .

In [43] we made the following conjecture.

Conjecture 1.2. If R is three dimensional and has terminal singularities then it has a non-commutative crepant resolution if and only if it has a crepant commutative resolution.

We prove this conjecture below (Theorem 6.6.3). We also show that a version of the Bondal-Orlov conjecture holds in the sense that the two resolutions in Conjecture 1.2 are derived equivalent. One direction in the proof of Conjecture 1.2 is obtained from the beautiful version of the McKay correspondence given in [15] (with virtually the same proof).

In addition to proving Conjecture 1.2 we will give two other instances where a non-commutative resolution exists:

- (1) Cones over Del Pezzo surfaces (see §7).
- (2) Invariants of a one dimensional torus acting linearly on a polynomial ring (see §8).

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The main reason for this new version is that the argument for the existence of non-commutative crepant resolutions for cones of Del Pezzo surfaces was incorrect in the published version of this paper [42]. Luckily the statement follows easily from the work of Kuleshov and Orlov. This approach was suggested to the author by Tom Bridgeland.

Besides this we have updated a few references but we have made no attempt to cover recent developments (this would have required extensive changes to the paper).

Finally we have corrected a few other minor errors.

- The proof of Proposition 3.3 was slightly faulty.
- There was a disastrous typo in the statement of Lemma 7.1(5) (pointed out by Orlov).

2. CONVENTIONS

Throughout k is an algebraically closed field of characteristic zero. All rings are k -algebras. Modules are right modules and Cohen-Macaulay means maximal Cohen-Macaulay. When we say that R is complete local then we mean that R is a (commutative) complete local k -algebra with residue field k .

Below when we say “reduction to the complete case” we usually mean completing the strict Henselization of a commutative noetherian ring at a prime ideal and then lifting the residue field. Replacing k with this lifted field we are in a “complete local” situation, in the sense of the previous paragraph.

We also use it in the following sense: let R be a commutative noetherian ring and let $R \rightarrow K$ be a map of R to a field with kernel P . Then a complete local situation may be obtained by replacing R with a suitable flat complete local extension of the completion of R at P which has residue field K and which also contains K .

Below we will sometimes use the theory of dualizing complexes for schemes and for non-commutative coherent sheaves of rings over these. When the underlying scheme is of finite type or when it is the spectrum of a complete local ring then there is a canonical dualizing complex which we will call the ‘‘Grothendieck dualizing complex’’ and which we denote by D . One possible way of characterizing this dualizing complex is through the theory of ‘‘rigid dualizing complexes’’ [41, 45, 46] (and its topological variant). The duality functor associated to D will be denoted by \mathbb{D} .

We frequently use the following result by Keller [24, Thm 4.3]:

Theorem 2.1. *Let \mathcal{E} be a Grothendieck category and assume that $\mathcal{A} = D(\mathcal{E})$ is generated by a compact object E (i.e. $E^\perp = 0$ and $\mathrm{Hom}_{\mathcal{A}}(E, -)$ commutes with direct sums). Then $\mathcal{A} = D(\Lambda)$ where Λ is a DG-algebra whose cohomology is given by $\mathrm{Ext}^*(E, E)$.*

In our case we will always have $\mathrm{Ext}^i(E, E) = 0$ for $i \neq 0$ and hence Λ is a true algebra. In that case we refer to E as a *tilting object*.

3. BACKGROUND ON HOMOLOGICALLY HOMOGENEOUS RINGS

It is well-known that for a non-commutative ring the condition of being of finite global dimension is very weak. In order to obtain a good homological theory generalizing the commutative case, a stronger condition is needed. Such a condition was introduced in [18].

Let R be a commutative noetherian k -algebra and let A be a module-finite R -algebra. We will say that A is *homologically homogeneous* if for all $P \in \mathrm{Spec} R$ we have

$$(H1) \quad \mathrm{gl\,dim} A_P = \dim R_P.$$

$$(H2) \quad A_P \text{ is (maximal) Cohen-Macaulay.}$$

If R is equidimensional of dimension n this is equivalent to the following condition given in [18].

$$(H1') \quad \text{All simple } A\text{-modules have the same projective dimension } n.$$

Example 3.1. Consider the ring

$$A = \begin{pmatrix} R & m \\ R & R \end{pmatrix}$$

with $R = k[x, y]$, $m = (x, y) \subset R$. One has $\mathrm{gl\,dim} A = 2$ but A is not Cohen-Macaulay. So A is *not* homologically homogeneous.

Zero-dimensional h.h. rings are semi-simple. One-dimensional h.h. rings are classical hereditary orders [30]. Two dimensional h.h. rings over a complete local ring were classified in [2, 31]. The results in [3, 4, 5, 38, 39] may be viewed as a classification of three-dimensional graded local h.h. rings.

Not everything about homologically homogeneous rings is understood. For example it is an interesting question to understand precisely which rings can occur as centers of homologically homogeneous rings. In particular we would like to know the following:

Question 3.2. If A is homologically homogeneous and R is the center of A , is it true that R has rational singularities?

This question is answered affirmatively in complete generality in [34]. Since we don't want to change too much we have kept the discussion below although it is somewhat obsolete.

In the published version of this paper [42] we were able to give a positive answer to Question 3.2 in the graded case. Unfortunately as written the proof was not quite correct. So below we give a corrected proof.

Proposition 3.3. *Assume that A is a finitely generated positively graded homologically homogeneous graded k -algebra with semi-simple degree zero part. Let R be the center of A and assume that R has an isolated singularity. Then R has rational singularities.*

Proof. The part of degree zero of R is a direct sum of fields. Using the resulting central idempotents we may reduce to the case where R is connected (i.e. $R_0 = k$). Put $n = \dim R$. If $n = 0$ then there is nothing to prove so we assume $n > 0$. Put $m = R_{>0}$. $M = A_{>0}$. Since R has an isolated singularity, it will have a rational singularity if and only if the following condition holds [44]:

$$(3.1) \quad \begin{aligned} H_m^i(R) &= 0 & \text{for } i < n \\ H_m^n(R)_u &= 0 & \text{for } u \geq 0 \end{aligned}$$

Since we are in characteristic zero, R is a direct summand of A and hence it is certainly Cohen-Macaulay. Thus $H_m^i(R) = 0$ for $i < n$.

Let S be a graded simple right A -module. From the theory of homologically homogeneous rings [18, 34] we obtain $\text{Ext}_A^n(S, A)$ is a graded simple left A -module. Replacing S by a minimal (finite!) projective A -resolution we easily see that $\text{Ext}_A^n(S, A)$ is concentrated in strictly negative degree.

From this we easily obtain that $H_M^n(A) = \text{inj lim Ext}_A^n(A/M^n, A)$ lives in strictly negative degree. Since $H_M^n(A) = H_m^n(A)$ (see e.g. [1]) and $H_m^n(R)$ is a direct summand of $H_m^n(A)$ we obtain that $H_m^n(R)$ also lives in strictly negative degree. \square

In [31] it is shown that every two-dimensional homologically homogeneous k -algebra over a complete local ring is obtained as the completion of a graded one. This is unfortunately false in higher dimension (otherwise Proposition 3.3 would yield an affirmative answer to Question 3.2).

Example 3.4. We construct a three-dimensional homologically homogeneous k -algebra which is not the completion of a graded one. Assume that R is a complete local ring with a three-dimensional terminal Gorenstein singularity which is *not* quasi-homogeneous but which is such that $X = \text{Spec } R$ has a crepant resolution. Then it follows from Theorem 5.1 below that there is a reflexive R -module such that $A = \text{End}_R(M)$ is homologically homogeneous. Clearly this A is not the completion of a graded ring since otherwise the same would be true for its center R also.

To make the example explicit consider $R = k[[x, y, z, t]]/(f)$ where

$$f = xy - (z - t^2)(z - t^3)(z - t^4)$$

This is a compound Du Val singularity which has a crepant resolution. Since R has a hypersurface singularity it will be quasi-homogeneous if and only if the Milnor number $\mu(f)$ and the Tyurina number $\tau(f)$ of f are equal [32]. Using the computer

algebra program SINGULAR [20] we compute: $\mu(f) = 12$, $\tau(f) = 11$. So R is *not* quasi-homogeneous.

A more sensible approach to Question 3.2 is probably through the following:

Question 3.5. If A is a homologically homogeneous ring over a complete local ring, is there a separated filtration $A = A_0 \supset A_1 \supset \cdots$ on A with $A_1 = \text{rad}(A)$ (maybe the $\text{rad}(A)$ -adic filtration(?)) such that $\text{gr } A$ is still homologically homogeneous?

Since a deformation of a rational singularity is rational [19], it not hard to see that an affirmative answer to this question combined with Proposition 3.3 would yield an affirmative answer to Question 3.2.

Remark 3.6. It is tempting to try to develop directly a theory of rational singularities for non-commutative rings. For example (3.1) seems to make perfect sense for non-commutative graded rings. Unfortunately it is easy to see that (3.1) is not invariant under Morita-equivalence so there are certainly some problems.

A related objection is that there seems to be no obvious non-commutative analogue of the Grauert-Riemenschneider vanishing theorem. It seems unlikely that there is a good theory of rational singularities without the Grauert-Riemenschneider theorem.

4. NON-COMMUTATIVE CREPANT RESOLUTIONS

Unless otherwise specified, in the rest of this paper R is a normal Gorenstein domain and $X = \text{Spec } R$. Let $f : Y \rightarrow X$ be a projective morphism with Y regular. We say that f is *crepant* if $f^*\omega_X = \omega_Y$.

Bondal and Orlov conjecture in [10] that different crepant resolutions of X have equivalent bounded derived categories of coherent sheaves. If X is projective then it is known that they have the same Hodge numbers [7, 26].

We propose a definition of a non-commutative crepant resolution. The motivation is mainly philosophical and the main example is given by the McKay correspondence [15] (see also [22] in dimension two).

Definition 4.1. A *non-commutative crepant resolution* of R is an homologically homogeneous R -algebra of the form $A = \text{End}_R(M)$ where M is a reflexive R -module.

This definition will serve as a vehicle for introducing the various examples we discuss in this paper. Some obvious variations are possible (see Remark 4.3 below).

The following remark will be convenient.

Lemma 4.2. *If $A = \text{End}_R(M)$ for a reflexive R -module M and if $\text{gl dim } A < \infty$ and A is Cohen-Macaulay then A is homologically homogeneous (and hence A is a non-commutative crepant resolution).*

Proof. We may assume that R is local. Put $n = \dim R$. Using the fact that $\text{Ext}_R^i(A, R) = 0$ for $i > 0$ (since A is Cohen-Macaulay) we easily deduce for $M \in \text{mod}(A)$: $\text{Ext}_A^i(M, A) = \text{Ext}_R^i(M, R)$. Thus the injective dimension of A_A is $\leq n$. Applying the formula with M a simple A -module obtain that the injective dimension of A_A is precisely n .

Since the global dimension of A equal to the injective dimension of A_A (given that it is finite) we find $\text{gl dim } A \leq n$. \square

The point of Definition 4.1 is that it provides reasonable non-commutative substitutes for “regularity”, “birationality” and “crepancy”. Obviously regularity correspond to the condition $\text{gl dim } A < \infty$.

Let K be the function field of R . To have a substitute for birationality we note that in non-commutative geometry it is customary to replace a ring with its module category. So birationality should be expressed by the fact that $A \otimes_R K$ is Morita equivalent to K . I.e. we should have $A \otimes_R K = M_n(K)$.

Assume that R is complete local of dimension n and let ω_R be the dualizing module of R . By the Gorenstein hypotheses ω_R is an invertible R -module. The Grothendieck dualizing complex of R is given by $D_R = \omega_R[n]$. By the adjunction formula A has a dualizing complex given by $D_A = \text{RHom}(A, D_R)$. It follows that A has a dualizing complex concentrated in one degree if and only if $\text{Ext}_R^i(A, \omega_R) = 0$ for $i > 0$, i.e. if and only if A is Cohen-Macaulay. If A is Cohen-Macaulay then in particular it is reflexive.

If A is Cohen-Macaulay and has finite global dimension then it is homologically homogeneous. Hence it follows from [18] that if π is a height one prime in R then A_π is a hereditary order in $M_n(K)$ over the discrete valuation ring by R_π . We denote the ramification index [30] of A_π by $e(\pi)$.

If A is Cohen-Macaulay then the dualizing module of A is given by $\omega_A = \text{Hom}_R(A, \omega_R)$ and it is easy to prove that the latter is equal to the tensor product of ω_R with $(\otimes_\pi \pi^{-1+e(\pi)})^{**}$ where the product runs over all height one primes in R . It follows that ω_A will be generated by ω_R (a substitute for crepancy) if and only if $e(\pi) = 1$ for all π , i.e. if and only if A is a maximal order. According to [6] all maximal orders in $M_n(K)$ are of the form $\text{End}_R(M)$ for a reflexive R -module M . This finishes our motivating discussion.

Remark 4.3. It is tempting to weaken Definition 4.1 in such a way as to require only that A is an unramified maximal order over R . The proof Theorem 6.3.1 below does not work in this added generality.

Remark 4.4. In all our examples the R -module M may be taken to be itself Cohen-Macaulay but we have no proof that this is always possible. Note that the implication $\text{End}_R(M)$ Cohen-Macaulay $\Rightarrow M$ Cohen-Macaulay is false.

Remark 4.5. If our basescheme X is not affine then we may define a non-commutative crepant resolution of X as a stack of abelian categories which is, locally on an affine open $\text{Spec } R$, of the form $\text{Mod}(A)$ where A is a non-commutative crepant resolution of R .

The following conjecture is inspired by the conjecture of Bondal and Orlov.

Conjecture 4.6. All crepant resolutions of X (commutative as well as non-commutative ones) are derived equivalent.

In subsequent sections we will give some examples of non-commutative crepant resolutions and in addition we will give some evidence for Conjecture 4.6. In particular we will prove Conjecture 4.6 for three dimensional terminal Gorenstein singularities.

5. RESOLUTIONS WITH FIBERS OF DIMENSION ≤ 1

The following result was proved in [43] (our hypotheses here are slightly more general but this does not affect the proof).

Theorem 5.1. *Assume that there exists a crepant resolution of singularities $f : Y \rightarrow X$ such that the dimensions of the fibers of f are ≤ 1 and such that the exceptional locus of f has codimension ≥ 2 . Then R has a non-commutative crepant resolution $A = \text{End}_R(M)$ where M is in addition Cohen-Macaulay. Furthermore Y and A are derived equivalent.*

Let us briefly recall how M is constructed. The Grauert-Riemenschneider theorem implies $H^i(Y, \mathcal{O}_Y) = 0$ for $i > 0$ (thus X has rational singularities).

Let \mathcal{L} be an ample line bundle on Y generated by global sections. The hypotheses on the fibers of f imply that $(\mathcal{O}_Y \oplus \mathcal{L})^\perp = 0$ in $D(\text{Qch}(Y))$. Take an extension

$$0 \rightarrow \mathcal{O}_Y^r \rightarrow \mathcal{M}' \rightarrow \mathcal{L} \rightarrow 0$$

associated to a set of r generators of $\text{Ext}_Y^1(\mathcal{L}, \mathcal{O}_Y)$ as R -module and put $\mathcal{M} = \mathcal{M}' \oplus \mathcal{O}_Y$. Then $\mathcal{M}^\perp = (\mathcal{O}_Y \oplus \mathcal{L})^\perp = 0$ and furthermore $\text{Ext}_A^i(\mathcal{M}, \mathcal{M}) = 0$ for $i > 0$.

Put $M = \Gamma(Y, \mathcal{M})$. The hypotheses imply that $A = \text{End}_R(M) = \text{End}_Y(\mathcal{M})$. Hence Y is derived equivalent to A . In particular A has finite global dimension. For the fact that A and M are Cohen-Macaulay we refer to [43].

6. CONSTRUCTION A CREPANT RESOLUTION STARTING FROM A NON-COMMUTATIVE ONE

6.1. Introduction. In this section we assume that $A = \text{End}_R(M)$ is an arbitrary non-commutative crepant resolution of R . We will show that the beautiful approach in [15] to the McKay correspondence generalizes to this situation. In fact an almost literal copy of the proof works, but we will nevertheless give a summary of it, in order to convince the reader (and ourselves!) that nothing specific to the situation of G -equivariant sheaves is used in [15] (see §6.3-6.6). The only part that is specific to our more general situation is in §6.3.

An obvious first step in our more general situation is to take for Y some type of A -Hilbert-scheme of M . This would work provided that M is Cohen-Macaulay. Unfortunately we don't know if this is always the case (but as already said, it is true in all examples we know).

Therefore we take a slightly different approach. We construct Y as a moduli-space of certain stable A -representations. It is standard how to do this but since our base ring R is somewhat more general than usual we recall the necessary steps in the next section.

6.2. Moduli spaces of representations. In this section R is a commutative noetherian k -algebra where as usual k is algebraically closed of characteristic zero. Put $X = \text{Spec } R$. Let A be an R -algebra which is finitely generated as R -module. Let $(e_i)_{i=1, \dots, p}$ be pairwise orthogonal idempotents in A such that $1 = \sum_i e_i$ and let $D = \oplus_i R e_i \subset A$ be the corresponding diagonal subalgebra.

For a map $R \rightarrow K$ with K a field and V a finite dimensional $A \otimes_R K$ representation we write $\underline{\dim} V = (\dim_K V e_i)_i \in \mathbb{Z}^p$. We put on \mathbb{Z}^p the ordinary Cartesian scalar product.

Pick $\lambda \in \mathbb{Z}^p$ and let $\alpha = \underline{\dim} V$. Following [25] We say that V is *(semi-)stable* (with respect to λ) if

$$(6.1) \quad (\lambda, \alpha) = 0$$

and if for any proper subrepresentation W of V with $\beta = \underline{\dim} W$ we have

$$(6.2) \quad (\lambda, \beta) (\geq) > 0$$

Note that if

$$(6.3) \quad (\lambda, \beta) \neq 0 \quad \text{for } 0 < \beta < \alpha$$

then stability and semi-stability are equivalent. For a fixed α there will exist λ satisfying (6.1) and (6.2) if and only if the greatest common divisor of all α_i is one.

An *affine family* of A representations with dimension vector α is a commutative R -algebra T together with a finitely generated $A \otimes_R T$ module P which is projective as T -module such that Pe_i has constant rank α_i for all i . For such a P we write $\underline{\dim} P = \alpha$. This is equivalent to saying that for any map of T to a field K we have $\underline{\dim}(P \otimes_T K) = \alpha$. We say that P is (semi-)stable if for any K we have that $P \otimes_R K$ is (semi-)stable. Non-affine families are defined in the obvious way by gluing affine families. We call families *equivalent* if they are locally isomorphic.

Let $\alpha = (\alpha_i)_{i=1, \dots, p}$ be natural numbers which are relatively prime and pick a λ satisfying (6.1) and (6.3). Consider the functor \mathbf{R}^s which assigns to a commutative R -scheme Z the following set:

$$\left\{ \begin{array}{l} \text{equivalence classes of families of } \lambda\text{-stable } A\text{-representations} \\ \text{over } Z \text{ with dimension vector } \alpha \end{array} \right\}$$

Proposition 6.2.1. *The functor \mathbf{R}^s is representable by a projective scheme over X .*

We need the following lemma.

Lemma 6.2.2. *Assume that R' is a commutative R -algebra and that T is a commutative R' -algebra. Put $A' = A \otimes_R R'$. Assume that P is a T -family of stable A' -representations. Then P is also stable as a family of A -representations.*

Proof. This is trivial from the definition. \square

If we temporarily write \mathbf{R}_R^s for \mathbf{R}^s then this lemma implies that $\mathbf{R}_{R'}^s = \mathbf{R}_R^s \times_{\text{Spec } R} \text{Spec } R'$. Hence if \mathbf{R}_R^s is representable by a projective scheme over R then so is $\mathbf{R}_{R'}^s$.

We use this as follows: we may find a finitely generated subring R_0 of R and a module finite R_0 algebra A_0 such that $A = A_0 \otimes_{R_0} R$. It is then sufficient to prove that $\mathbf{R}_{R_0}^s$ is representable by a projective scheme over R_0 . Hence without loss of generality we may (and we will) assume that R is finitely generated over k .

Put $\bar{\alpha} = \sum_i \alpha_i$. Define A' as the centralizer of $M_{\bar{\alpha}}(k)$ in $M_{\bar{\alpha}}(R) *_D A$. So we have

$$(6.4) \quad M_{\bar{\alpha}}(R) *_D A = M_{\bar{\alpha}}(k) \otimes A'$$

where the obvious copy of $M_{\bar{\alpha}}(k)$ on the left and right is the same.

Note that A' is an R -algebra. Since $M_{\bar{\alpha}}(R) *_D A = M_{\bar{\alpha}}(A')$ is finitely generated the same is true for A' .

Put $S' = A' / [A', A']$ and $W' = \text{Spec } S'$. W' is an affine R -scheme of finite type representing the functor [33] which assigns to a commutative R -algebra T :

$$(6.5) \quad \left\{ \begin{array}{l} A\text{-module structures on } T^{\bar{\alpha}} \text{ which commute with the} \\ T\text{-structure such that } e_i \text{ acts as } \text{diag}(0^{\alpha_1 + \dots + \alpha_{i-1}}, 1^{\alpha_i}, 0^{\alpha_{i+1} + \dots + \alpha_p}) \end{array} \right\}$$

The corresponding universal bundle is given by $U'_0 = k^{\bar{\alpha}} \otimes_k S$ with the right action of A obtained via the composition $A \rightarrow M_{\bar{\alpha}}(R) *_D A = M_{\bar{\alpha}}(A') \rightarrow M_{\bar{\alpha}}(S')$.

Let $G = \prod_i \mathrm{GL}_{\alpha_i}(k)$ and let PG be equal to G modulo its center. Conjugation on the first factor induces a G action on $M_{\bar{\alpha}}(R) *_D A$, leaving the elements of A invariant.

The G action on $M_{\bar{\alpha}}(R) *_D A$ leaves $M_{\bar{\alpha}}(k)$ stable and hence it induces rational G -actions on A' , S' and W' . The G -action on the righthand side of (6.4) is the diagonal one.

Putting $g(v \otimes \gamma) = vg^{-1} \otimes g\gamma$ defines a rational G -action on U'_0 . The center of G acts trivially on S' and via scalar multiplication on U'_0 . Let $a \in \mathbb{Z}^p$ be such that $(\alpha, a) = 1$ and define $U' = U'_0 \otimes_S \bigotimes_i \wedge^{\alpha_i} (U'_0 e_i)^{\otimes -a_i}$. U' is still a $A \otimes_R S'$ module which is projective as S' -module and which has dimension vector α . The center of G now acts trivially on U' .

As usual the stability condition (6.2) corresponds to a character $\chi : G \rightarrow k^*$ [25]. This character defines a \mathbb{Z} -grading on S' .

Let $G_0 = \ker \chi$ and put $S = (S')^{G_0}$, $U = (U')^{G_0}$. Then S is still \mathbb{Z} -graded and U is a graded S -module. Put $W = \mathrm{Proj} S_{\geq 0}$ and let $f : W \rightarrow X$ be the structure map. If W'^s is the open subset of W corresponding to the complement of the closed subscheme defined by $S_{>0}$ then it follows from the Luna slice theorem [28] that $W'^s \rightarrow W$ is a principal PG -fiber bundle.

Let \mathcal{U} be the coherent sheaf on W which corresponds to U . \mathcal{U} is a sheaf of right A -modules. It follows from standard descent theory that the pullback of \mathcal{U} to W'^s is the restriction of U' (considered as a coherent sheaf on W'). In particular \mathcal{U} is a vector bundle on W .

It is now standard that W represents the functor \mathbf{R}^s and that \mathcal{U} is the corresponding universal bundle. See [29] for the case of vector bundles.

Lemma 6.2.3. *If $A = M_{\bar{\alpha}}(R)$ then the map $W \rightarrow X$ is an isomorphism.*

Proof. It follows from Morita theory that in this case the functor \mathbf{R}^s is represented by X . So this proves what we want. \square

6.3. Application to our situation. Now we return to our standard assumption that R is a normal Gorenstein domain and we let $A = \mathrm{End}_R(M)$ be an arbitrary non-commutative crepant resolution of R .

Since A has finite global dimension there is a A -resolution

$$(6.6) \quad 0 \rightarrow P \rightarrow A^a \rightarrow \dots \rightarrow A^b \rightarrow M \rightarrow 0$$

with P projective. By the reflexive Morita correspondence we have $P = \mathrm{Hom}_R(M, M_1)$ where

$$(6.7) \quad M_1 \oplus M_2 \cong M^{\oplus c}$$

for some $c \in \mathbb{N}$ and some other reflexive R -module M_2 .

From (6.6) we obtain $\mathrm{rk} M = d \mathrm{rk} A \pm \mathrm{rk} P$ for some $d \in \mathbb{Z}$, where “ $\mathrm{rk} U$ ” denotes the rank of U as R -module. Simplifying we find $1 = d \mathrm{rk} M \pm \mathrm{rk} M_1$ and hence the ranks of M and M_1 are coprime. Increasing c if necessary we may also assume that the ranks of M_1 and M_2 are coprime.

We now replace M by $M^{\oplus c}$. This changes A into something Morita equivalent and the decomposition (6.7) becomes $M = M_1 \oplus M_2$.

Let us more generally consider a decomposition $M = \bigoplus_{i=1}^p M_i$. Denote the rank of M_i by α_i and put $\alpha = (\alpha_i)_i$. Take λ satisfying (6.1) and (6.3) and let $f : W \rightarrow X$ and \mathcal{U} be as in the previous section.

If we let $X_1 \subset X = \text{Spec } R$ be the locus where M is locally free then it follows from lemma 6.2.3 that $f^{-1}(X_1) \rightarrow X_1$ is an isomorphism. We define Y as the unique irreducible component of W mapping onto X . We will denote the restriction of f to Y also by f . We let \mathcal{M} be the restriction of \mathcal{U} to Y . \mathcal{M} is a sheaf of A -modules on Y .

Following [15] we now define a pair of adjoint functors between $D^b(\text{coh}(Y))$ and $D^b(\text{mod}(A))$.

$$\begin{aligned}\Phi : D^b(\text{coh}(Y)) &\rightarrow D^b(\text{mod}(A)) : C \mapsto R\Gamma(C \overset{L}{\otimes}_{\mathcal{O}_Y} \mathcal{M}) \\ \Psi : D^b(\text{mod}(A)) &\rightarrow D^b(\text{coh}(Y)) : D \mapsto D \overset{L}{\otimes}_A \mathcal{M}^*\end{aligned}$$

The following is a straightforward generalization of [15].

Theorem 6.3.1. *Assume that for any point $x \in X$ of codimension n the fiber product $(Y \times_X Y) \times_X \text{Spec } \mathcal{O}_{X,x}$ has dimension $\leq n + 1$. Then $f : Y \rightarrow X$ is a crepant resolution of X and Φ and Ψ are inverse equivalences.*

To give the proof we need to say something about spanning classes and Serre functors. This is done in the next section.

6.4. Relative Serre duality. Assume that $\mathcal{D} \subset \mathcal{C}$ is a full faithful inclusion of k -linear triangulated categories such that for $C \in \mathcal{C}$, $D \in \mathcal{D}$ we have $\sum_i \dim \text{Ext}^i(C, D) < \infty$, $\sum_i \dim \text{Ext}^i(D, C) < \infty$. We say that an auto-equivalence of triangulated categories $S : \mathcal{D} \rightarrow \mathcal{D}$ is a relative Serre functor for the pair $(\mathcal{D}, \mathcal{C})$ if the following properties hold:

- (1) S leaves \mathcal{D} stable.
- (2) For $C \in \mathcal{C}$, $D \in \mathcal{D}$ there are natural isomorphisms

$$(6.8) \quad \eta_{D,C} : \text{Hom}(D, C) \rightarrow \text{Hom}(C, SD)^*$$

Thus in particular S is Serre functor for \mathcal{D} .

We will use this in the following situation.

Lemma 6.4.1. *Assume that $X = \text{Spec } R$ where R is a complete local noetherian ring and let $f : Y \rightarrow X$ be a projective map. Let \mathcal{A} be a coherent sheaf of \mathcal{O}_Y algebras such that for every $y \in Y$ the stalk of \mathcal{A} at y has finite global dimension. Let \mathcal{C} be the bounded derived category of $\text{coh}(\mathcal{A})$ and let \mathcal{D} be the full subcategory of complexes whose homology has support in $Y_0 = f^{-1}(x)$ where $x \in X$ is the closed point. Then the pair $(\mathcal{D}, \mathcal{C})$ has a relative Serre functor given by tensoring with the Grothendieck dualizing complex $D_{\mathcal{A}} = R\mathcal{H}om_Y(\mathcal{A}, D_Y)$ of \mathcal{A} .*

Proof. We claim first that if $D \in \mathcal{D}$, $C \in \mathcal{C}$ then we have the following identity:

$$(6.9) \quad \text{Hom}(D, D_Y) \cong \Gamma(Y, D)^*$$

Let Y_n be the n 'th formal neighborhood of Y_0 and let $j_n : Y_n \rightarrow X$ be the inclusion map.

It is easy to see that there is some n and some $D_n \in D^b(\text{coh}(Y_n))$ such that $D = Rj_{n*}D_n$. Then we have $\text{Hom}(D, D_Y) = \text{Hom}(Rj_{n*}D_n, D_Y) = \text{Hom}(D_n, j_n^!(D_Y)) = \text{Hom}(D_n, D_{Y_n})$. Now since Y_n is proper it satisfies classical Serre duality. I.e. there are isomorphisms $\text{Hom}(D_n, D_{Y_n}) \cong \Gamma(Y_n, D_n)^* = \Gamma(Y, D)^*$. It is easy to see that the resulting isomorphism $\text{Hom}(D, D_Y) \cong \Gamma(Y, D)^*$ is independent of n . This finishes the proof of (6.9).

We compute for $D \in \mathcal{D}$, $C \in \mathcal{C}$

$$(6.10) \quad \begin{aligned} \mathrm{Hom}_{\mathcal{A}}(D, C) &= \Gamma(Y, \mathrm{RHom}_{\mathcal{A}}(D, C)) \\ &\cong \mathrm{Hom}(\mathrm{RHom}_{\mathcal{A}}(D, C), D_Y)^* \quad (\text{by (6.9)}) \end{aligned}$$

We claim that there are natural isomorphism

$$(6.11) \quad \mathrm{RHom}(\mathrm{RHom}_{\mathcal{A}}(D, C), D_Y) \cong \mathrm{RHom}_{\mathcal{A}}(C, D \overset{L}{\otimes}_{\mathcal{A}} D_{\mathcal{A}})$$

This is easily proved by replacing D_Y with a bounded injective complex and D with a locally projective complex (using the projectivity of f).

So combining (6.10) and (6.11) we obtain an isomorphism $\mathrm{Hom}_{\mathcal{A}}(D, C) \cong \mathrm{Hom}_{\mathcal{A}}(C, D \overset{L}{\otimes}_{\mathcal{A}} D_{\mathcal{A}})^*$ which finishes the proof. \square

6.5. Spanning classes. If \mathcal{C} is a triangulated category then a *spanning class* [13] $\Omega \subset \mathcal{C}$ is a set of objects such that $\Omega^{\perp} = 0$ and ${}^{\perp}\Omega = 0$.

Example 6.5.1. Let $f : Y \rightarrow X$ be a *proper* map where X is the spectrum of a noetherian local ring with algebraically closed residue field k . Then $\Omega = \{\mathcal{O}_y \mid y \in Y(k)\}$ is a spanning class for $D^b(\mathrm{coh}(Y))$.

Assume that $F : \mathcal{C} \rightarrow \mathcal{E}$ is a functor between pairs of triangulated categories. Assume that F has a left adjoint G and a right adjoint H and that $\Omega \subset \mathcal{C}$ is a spanning class.

Lemma 6.5.2. [12][13] $F : \mathcal{C} \rightarrow \mathcal{E}$ is fully faithful if and only if the map

$$(6.12) \quad \mathrm{Ext}_{\mathcal{C}}^i(\omega, \omega') \rightarrow \mathrm{Ext}_{\mathcal{E}}^i(F\omega, F\omega')$$

is an isomorphism for all $\omega, \omega' \in \Omega$.

Suppose now that F is actually a compatible pair of fully faithful functors $(\mathcal{D}, \mathcal{C}) \rightarrow (\mathcal{F}, \mathcal{E})$ between pairs of triangulated categories satisfying the hypotheses given in the beginning of §6.4 which in addition have relative Serre functors $S_{\mathcal{C}}$ and $S_{\mathcal{E}}$. Assume $\Omega \subset \mathcal{D}$.

Lemma 6.5.3. *Assume that \mathcal{C} is not trivial and that \mathcal{E} is connected. Assume in addition that $S_{\mathcal{E}}F\omega = FS_{\mathcal{C}}\omega$ for $\omega \in \Omega$. Then F is an equivalence of categories.*

Proof. The proof is the same as in [13] with Serre functors being replaced by relative ones. \square

6.6. Proof of the main theorem. We now complete the proof of Theorem 6.3.1. We need to show that the canonical natural transformations $\Psi\Phi \rightarrow \mathrm{id}$ and $\mathrm{id} \rightarrow \Phi\Psi$ are isomorphisms. Since everything is compatible with base change we may reduce to the case where R is a complete local ring containing a copy of its algebraically closed residue field. We will denote this new residue field also by k . We let x be the unique closed point of $X = \mathrm{Spec} R$.

The functors Φ and Ψ have versions for left modules, denoted by Φ° and Ψ° respectively, which are given by the formulas $R\Gamma(Y, \mathcal{M}^* \overset{L}{\otimes}_{\mathcal{O}_Y} -)$ and $\mathcal{M} \overset{L}{\otimes}_{\mathcal{A}} -$. It is easy to see that $\Phi^{\circ} = \mathbb{D}_{\mathcal{A}} \circ \Phi \circ \mathbb{D}_Y$. Hence Φ also has a right adjoint given by $\mathbb{D}_Y \circ \Psi^{\circ} \circ \mathbb{D}_{\mathcal{A}}$. Furthermore as in Example 6.5.1 the objects \mathcal{O}_y form a spanning class for $D^b(\mathrm{coh}(Y))$.

For $y \in Y(k)$ denote by $\mathcal{M}_y = \Phi(\mathcal{O}_y)$ the fiber of \mathcal{M} at y . To prove that Φ is fully faithful we need to prove that the canonical maps

$$(6.13) \quad \text{Ext}_Y^i(\mathcal{O}_y, \mathcal{O}_{y'}) \rightarrow \text{Ext}_A^i(\mathcal{M}_y, \mathcal{M}_{y'})$$

are isomorphisms for $y, y' \in Y$.

What do we know already?

- (1) (6.13) is certainly an isomorphism for $i = 0$.
- (2) By Serre duality for A and the fact that $D_A \cong A[n]$ (lemma (6.4.1)) (6.13) is an isomorphism if $i = n$ and $y \neq y'$.

There is one more subtle piece of information that may be obtained. Recall that Y is a closed subscheme of the scheme W representing the functor of stable A -representations. This means that there is an injection (the Kodaira-Spencer map):

$$\phi : \text{Ext}_Y^1(\mathcal{O}_y, \mathcal{O}_y) = T_{Y,y} \hookrightarrow T_{W,y} = \text{Ext}_W^1(\mathcal{M}_y, \mathcal{M}_y)$$

where $T_{*,y}$ denotes the tangent space at y . The map ϕ is constructed as follows. An element of $T_{Y,y}$ corresponds to a map $u : \text{Spec } k[\epsilon]/(\epsilon^2) \rightarrow Y$ and hence to an extension E of \mathcal{O}_y with itself. Then $\phi(u) = u^*(E)$. Thus the Kodaira-Spencer map coincides with (6.13) for $i = 1$ and $y = y'$. Hence we have

- (3) (6.13) is an injection for $i = 1$ and $y = y'$.

Using an amazing trick based on the intersection theorem in commutative algebra it is shown in [15] that (1),(2) and (3) are sufficient to prove that Φ is fully faithful (under the standing hypothesis $\dim Y \times_X Y \leq n + 1$).

We need a succinct description of $\Psi\Phi$. Let $Y \hat{\times} Y = (Y \times Y)_{X \times X} \text{Spec } \hat{\mathcal{O}}_{X \times X, (x,x)}$. This a noetherian scheme proper over $\text{Spec } \hat{\mathcal{O}}_{X \times X, (x,x)}$. We denote the projections $Y \hat{\times} Y \rightarrow Y$ by $\text{pr}_{1,2}$.

$Y \times_X Y$ may be considered as a closed subscheme of $Y \hat{\times} Y$ (no need to complete). We define $\mathcal{M} \boxtimes_A \mathcal{M}^*$ as the coherent sheaf on $Y \hat{\times} Y$ such that for affine opens $U, V \subset Y$ we have $(\mathcal{M} \boxtimes_A \mathcal{M}^*)(U \hat{\times} V) = \mathcal{M}(U) \otimes_A \mathcal{M}(V)^*$. $\mathcal{M} \boxtimes_A \mathcal{M}^*$ is clearly supported on $Y \times_X Y$. Using suitable flat resolutions we may define the analogous derived object $\mathcal{Q} = \mathcal{M} \overset{L}{\boxtimes}_A \mathcal{M}^*$ which is also supported on $Y \times_X Y$. Then it is easy to see that

$$\Psi\Phi = R \text{pr}_{2*}(L \text{pr}_1^*(-) \otimes_{Y \hat{\times} Y} \mathcal{Q})$$

We have $\mathcal{O}_{y,y'} \otimes_{Y \hat{\times} Y} \mathcal{Q} = \mathcal{M}_y \overset{L}{\otimes}_A \mathcal{M}_{y'}^*$. Using lemma 6.4.1 for A we have $\mathcal{M}_{y'}^* = \text{Hom}_A(A, \mathcal{M}_{y'})^* = \text{Hom}_A(\mathcal{M}_{y'}, A[n])$ and since the righthand side of the last equality has homology only in degree zero, it is equal to $\text{RHom}(\mathcal{M}_{y'}, A[n])$. Thus $\mathcal{M}_y \overset{L}{\otimes}_A \mathcal{M}_{y'}^* = \text{RHom}_A(\mathcal{M}_{y'}, \mathcal{M}_y)[n]$.

Thus if $y \neq y'$ then $H^{-n}(\mathcal{O}_{y,y'} \overset{L}{\otimes}_{Y \hat{\times} Y} \mathcal{Q}) = 0$ and using relative Serre duality for A again we also have $H^0(\mathcal{O}_{y,y'} \overset{L}{\otimes}_{Y \hat{\times} Y} \mathcal{Q}) = \text{Ext}_A^n(\mathcal{M}_y, \mathcal{M}_{y'}) = \text{Hom}_A(\mathcal{M}_{y'}, \mathcal{M}_y) = 0$.

So the range of possible non-zero values for $H^i(\mathcal{O}_{y,y'} \overset{L}{\otimes}_{Y \hat{\times} Y} \mathcal{Q}) = 0$ has size $n - 1$. But this implies by the intersection theorem (see [15, 16]) that on the complement of the diagonal \mathcal{Q} has support of dimension $\geq \dim(Y \hat{\times} Y) + 1 - (n - 1) = n + 2$ (if non-empty). But by hypotheses the support of \mathcal{Q} has dimension less than or equal the dimension of $Y \times_X Y$ which is $n + 1$. We conclude that \mathcal{Q} is supported on Δ .

Consider $\Psi\Phi\mathcal{O}_y$. By the previous paragraph this is a complex supported on y living in non-positive degree and we have $\text{Ext}^i(\Psi\Phi\mathcal{O}_y, \mathcal{O}_y) = \text{Ext}^i(\mathcal{M}_y, \mathcal{M}_y)$ which is non-zero only in degrees $[0, \dots, n]$. We claim $H^0(\Psi\Phi\mathcal{O}_y) = \mathcal{O}_y$.

Let $c_y[1]$ be the cone over $\Psi\Phi\mathcal{O}_y \rightarrow \mathcal{O}_y$. Thus we have a triangle

$$(6.14) \quad c_y \rightarrow \Psi\Phi\mathcal{O}_y \rightarrow \mathcal{O}_y$$

By the previous paragraph c_y is supported in y . It is also easy to see that the homology of c_y is concentrated in non-positive degree. There is an exact sequence

$$0 \rightarrow \text{Hom}(\mathcal{O}_y, \mathcal{O}_y) \rightarrow \text{Hom}(\mathcal{M}_y, \mathcal{M}_y) \rightarrow \text{Hom}(c_y, \mathcal{O}_y) \rightarrow \text{Ext}^1(\mathcal{O}_y, \mathcal{O}_y) \rightarrow \text{Ext}^1(\mathcal{M}_y, \mathcal{M}_y)$$

and using (1)(3) we conclude $\text{Hom}(c_y, \mathcal{O}_y) = 0$ and hence $H^0(c_y) = 0$. The fact that $H^0(\Psi\Phi\mathcal{O}_y) = \mathcal{O}_y$ now follows from (6.14). By the intersection theorem (see [15, 16]) we conclude that Y is regular at y and that $\Psi\Phi\mathcal{O}_y = \mathcal{O}_y$.

Since this is true for all y we now know that Y is regular and that (6.13) holds. Thus Φ is faithful.

To prove that Φ is an equivalence we use lemma 6.5.3 since $D^b(\text{mod}(A))$ is trivially connected. We need that $\Phi S_Y \mathcal{O}_y \cong S_A \mathcal{M}_y$. Now S_A is just shifting n places to the left, and since Y is regular ω_Y is invertible and thus $S_Y \mathcal{O}_y$ is just $\mathcal{O}_y[n]$.

Finally we prove that f is crepant. It is sufficient to prove that $\omega_Y \cong \mathcal{O}_Y$. Indeed if this is the case then $f_*\omega_Y \cong \mathcal{O}_X$ is reflexive and hence it is equal to ω_X . Furthermore it is then also clear that $f^*\omega_X = \omega_Y$.

Let $D_x^b(\text{coh}(Y))$ the full subcategory of $D^b(\text{coh}(Y))$ consisting of complexes supported in $f^{-1}(x)$. Similarly let $D_x^b(\text{mod}(A)) \subset D^b(\text{mod}(A))$ be the complexes supported on x . The functor Φ and Ψ define inverse equivalences between $D_x^b(\text{coh}(Y))$ and $D_x^b(\text{mod}(A))$.

On $D_x^b(\text{coh}(Y))$ we have that $S_Y[-n]$ is isomorphic to the identity functor since the same holds for $D_x^b(\text{mod}(A))$. Thus if $Y_0 = f^{-1}(x)$ then $\omega_Y/\omega_Y(-nY_0) \cong \mathcal{O}_Y/\mathcal{O}_Y(-nY_0)$. Hence if \hat{Y} is the formal scheme associated to Y_0 then $\hat{\omega}_Y \cong \hat{\mathcal{O}}_Y$. But by the Grothendieck existence theorem this implies $\omega_Y \cong \mathcal{O}_Y$.

Remark 6.6.1. As noted in [15] the fact that Y is smooth and the fact that for $y \in Y$ we have $T_{Y,y} = \text{Ext}_Y^1(\mathcal{O}_y, \mathcal{O}_y) \cong \text{Ext}_A^1(\mathcal{M}_y, \mathcal{M}_y) = T_{W,y}$ implies that Y is a connected component of W . If $\dim X = 3$ then it is shown that actually $W = Y$. This result generalizes probably not to our current situation. However if R is complete local and we take a decomposition of M into indecomposables (unique up to non-unique isomorphism by the Krull-Schmidt-theorem) then the proof goes through virtually unmodified.

Remark 6.6.2. Varying λ we get many different crepant resolutions of X . They are all derived equivalent since they are all derived equivalent to A . This gives another instance where the Bondal-Orlov conjecture is true.

Now we may prove.

Theorem 6.6.3. *Assume that R is three-dimensional and has terminal singularities.*

- (1) *R has a non-commutative crepant resolution if and only if it has a commutative one.*
- (2) *Conjecture 4.6 is true in this case.*

- Proof.* (1) This follows from Theorems 5.1 and 6.3.1.
 (2) Let $Y \rightarrow X$ be a crepant resolution of singularities and let A be a non-commutative one. By Theorem 6.3.1 there is another crepant resolution $Y' \rightarrow X$ of X associated to A . The resulting birational map $Y \rightarrow Y'$ is a composition of flops. Hence by [14] Y and Y' are derived equivalent. Since Y' and A are also derived equivalent, we are done. \square

7. CONES OVER DEL PEZZO SURFACES

A standard example of a canonical singularity which is not terminal is the cubic cone $w^3 + x^3 + y^3 + z^3 = 0$. This singularity has a crepant resolution obtained by blowing up the origin. Our aim in this section is to show that it also has a non-commutative crepant resolution. The method used applies more generally to cones over Del-Pezzo surfaces (see Proposition 7.3) below.

Below Z is regular connected projective scheme of dimension $n - 1 > 0$ with an ample line bundle \mathcal{L} . We denote the homogeneous coordinate ring $\oplus_i \Gamma(Z, \mathcal{L}^i)$ of Z corresponding to \mathcal{L} by R . R is a finitely generated normal graded ring. Put $X = \text{Spec } R$. We will call R a cone over Z . X has unique singularity at the origin o . This singularity has a standard resolution by $Y = \text{Spec } S(\mathcal{L})$. Denote the structure map $Y \rightarrow Z$ by π and let f be the canonical map $Y \rightarrow X$. Thus we have the following diagram

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & Z \\ f \downarrow & & \\ & & X \end{array}$$

Let $X' \subset X$, $Y' \subset Y$ be the open subsets respectively defined by the ideals $R_{>0}$ and $(S\mathcal{L})_{>0}$. X' is regular and $X - X' = \{o\}$. We also have $Y - Y' = E$ where E is the image of the zero section on π . The map f restricts to an isomorphism $Y' \rightarrow X'$ and so the exceptional locus of f is given by E .

We now state a list of properties of X which are well-known and which are easy to prove.

Lemma 7.1. (1) X is Cohen-Macaulay if and only if

$$H^i(Z, \mathcal{L}^j) = 0 \quad \text{for all } 0 < i < n - 1, j \geq 0$$

(2) X has rational singularities if and only if

$$H^i(Z, \mathcal{L}^j) = 0 \quad \text{for all } i > 0, j \geq 0$$

(3) X has an invertible canonical bundle if and only if $\omega_Z = \mathcal{L}^{-m}$ for $m \in \mathbb{Z}$. In that case $\omega_Y = f^*\omega_X((m - 1)E)$.

(4) If X has Gorenstein rational singularities then $\omega_Z = \mathcal{L}^{-m}$ for some $m > 0$.

(5) If X has Gorenstein rational singularities then f is not crepant (or equivalently X is terminal) if and only if $\omega_Z \not\cong \mathcal{L}^{-1}$.

Now assume that there is a vector bundle \mathcal{E}_0 on Z which is a tilting object. Put $\mathcal{E} = \pi^*\mathcal{E}_0$. Then \mathcal{E} is a tilting object on Y if and only if

$$(7.1) \quad H^i(Z, \text{End}(\mathcal{E}_0, \mathcal{E}_0) \otimes \mathcal{L}^j) = 0 \quad \text{for } i > 0 \text{ and } j \geq 0$$

Thus if (7.1) holds and we put $A = \text{End}(\mathcal{E})$ then A and Y are derived equivalent.

We claim the following result.

Proposition 7.2. *Assume that \mathcal{E}_0 is a vector bundle on Z which is a generator of $D(\text{Qch}(Z))$ such that $\mathcal{E} = \pi^*\mathcal{E}_0$ is a tilting object on Y . Then $A = \text{End}(\mathcal{E})$ is a non-commutative crepant resolution of X if and only if the following condition holds:*

$$(7.2) \quad H^i(Z, \mathcal{A}_0 \otimes \mathcal{L}^j) = 0 \quad \text{for } i < n - 1 \text{ and } j < 0$$

for $\mathcal{A}_0 = \mathcal{E}nd(\mathcal{E}_0)$.

If $\omega_Z = \mathcal{L}^{-1}$ then (7.2) is always satisfied.

Proof. For $\mathcal{M} \in \text{Qch}(Z)$ put $\underline{\Gamma}(Z, \mathcal{M}) = \bigoplus_{j \in \mathbb{Z}} \Gamma(Z, \mathcal{M} \otimes \mathcal{L}^j)$ and denote the derived functors of $\underline{\Gamma}$ by \underline{H}^* .

If \mathcal{M} is associated to $M \in \text{Gr}(R)$ then it is well-known that we have

$$(7.3) \quad \underline{H}^i(Z, \mathcal{M}) = H_{R>0}^{i+1}(M) \quad \text{for } i > 0$$

and there is a long exact sequence

$$(7.4) \quad 0 \rightarrow H_{R>0}^0(M) \rightarrow M \rightarrow \underline{\Gamma}(Z, \mathcal{M}) \rightarrow H_{R>0}^1(M) \rightarrow 0$$

Now we prove the first part of the proposition. Since A is derived equivalent to Y we already know $\text{gldim } A < \infty$, so we only have to be concerned with the Cohen-Macaulayness of A . The latter is equivalent to $H_{R>0}^i(A) = 0$ for $i \leq n - 1$ (since the dimension of R is n).

Since we assume (7.1) the condition (7.2) is equivalent to

$$(7.5) \quad \begin{aligned} \underline{H}^i(Z, \mathcal{A}_0) &= 0 & \text{for } 0 < i < n - 1 \\ \underline{\Gamma}(Z, \mathcal{A}_0)_{<0} &= 0 \end{aligned}$$

Using (7.3) and (7.4) with $M = A$ we see that A is Cohen-Macaulay if and only if $\underline{H}^i(Z, \mathcal{A}_0) = 0$ for $0 < i < n - 1$ and $A = \underline{\Gamma}(Z, \mathcal{A}_0)$. These conditions correspond precisely to the conditions given in (7.5).

Now assume that (7.5) holds and put $E = \underline{\Gamma}(\mathcal{E}_0)^{**}$ where $(-)^{**}$ denotes the R -bidual. Then A and $\text{End}_R(E)$ are reflexive R -modules which have the same restriction to X' (when considered as sheaves). Hence they are equal.

If $\omega_Z = \mathcal{L}^{-1}$ then by Serre duality we have $H^i(Z, \mathcal{A}_0 \otimes \mathcal{L}^j) = H^{n-1-i}(Z, \mathcal{A}_0^* \otimes \mathcal{L}^{-j} \otimes \omega_Z)^* = H^{n-1-i}(Z, \mathcal{A}_0 \otimes \mathcal{L}^{-j-1})^*$. If $j < 0$ and $i < n - 1$ then $-j - 1 \geq 0$ and $i > 0$. I.e. (7.2) follows from (7.1). \square

Our aim is now to apply this is the case that Z is a surface with ample anti-canonical bundle (i.e. a Del-Pezzo surface). Recall that by [8] Z is either $\mathbb{P}^1 \times \mathbb{P}^1$ or else is obtained by blowing up \mathbb{P}^2 in ≤ 8 points in general position.

Proposition 7.3. *Let Z be a Del-Pezzo surface and let R be a cone over Z with trivial canonical bundle. Then R has a non-commutative crepant resolution.*

Proof. Let us first discuss the cases where ω_Z is a proper multiple of a line bundle. If F is an exceptional curve on a surface Z then $\deg(\omega_Z | F) = -1$ and so ω_Z cannot be a proper multiple of a line bundle. The Del-Pezzo surfaces without exceptional curves are \mathbb{P}^2 and $\mathbb{P}^1 \times \mathbb{P}^1$. In the first case $\omega_Z = \mathcal{O}_Z(-3)$ and hence $\mathcal{L} = \mathcal{O}_Z(1)$ and in the second case $\omega_Z = \mathcal{O}_Z(-2, -2)$ and hence $\mathcal{L} = \mathcal{O}_Z(1, 1)$.

In the first case the cone over Z is a polynomial ring so this is trivial. In the second case the cone is given by $R = k[u, v, x, y]/(uv - xy)$ which is standard. The

non-commutative crepant resolution is given by

$$\begin{pmatrix} R & I \\ I^{-1} & R \end{pmatrix}$$

where $I = (u, x)$.

So from now on we assume $\mathcal{L} = \omega_Z^{-1}$. We will construct a generator for $D(\text{Qch}(Z))$ satisfying condition (7.1). In fact this follows easily from the results in [27] (I thank Tom Bridgeland for pointing this out to me). Let K be the canonical divisor.

It is easy to construct an exceptional collection of vector bundles on Z generating the derived category $D(\text{Qch}(Z))$. By [27, Claim 6.5] one may, using a process called “mutation”, construct from the initial exceptional collection a new exceptional collection E_1, \dots, E_n of vector bundles on Z generating the derived category $D(\text{Qch}(Z))$ such that the slope function

$$\mu(E) = -\frac{c_1(E) \cdot K}{r(E)}$$

is non-decreasing on the sequence

$$\dots, E_{n-1}(K), E_n(K), E_n, \dots, E_n, E_1(-K), E_2(-K), \dots$$

and such that any interval of length n is an exceptional collection of “type Hom” (i.e. it is an exceptional collection without higher Ext’s; this is also called a strong exceptional collection).

We claim that the whole sequence has no forward Ext^2 ’s and no backward Hom’s and furthermore that all Ext^1 ’s vanish. To simplify the notation we write $E_i(-jK) = E_{i+jn}$.

Let us first consider Ext^1 . By Serre duality we have

$$\dim \text{Ext}^1(E_u, E_v) = \dim \text{Ext}^1(E_v, E_{u-n})$$

According to [27, Lemma 3.7] we have for $v \geq u$ (as $\mu(v) \geq \mu(u)$):

$$\text{Ext}^1(E_v, E_u) = 0 \Rightarrow \text{Ext}^1(E_u, E_v) = 0$$

and thus

$$\text{Ext}^1(E_u, E_{v-n}) = 0 \Rightarrow \text{Ext}^1(E_u, E_v) = 0$$

$$\text{Ext}^1(E_v, E_u) = 0 \Rightarrow \text{Ext}^1(E_v, E_{u-n}) = 0$$

It is now easy to see that the fact $\text{Ext}^1(E_u, E_v) = 0$ for $|u - v| < n$ implies that all Ext^1 ’s are zero.

Now we consider Hom’s. We need $\text{Hom}(E_v, E_u) = 0$ for $v > u$. Since $-K$ is effective on a Del Pezzo surface we have $E_{u-n} \hookrightarrow E_u$. Thus

$$(7.6) \quad \text{Hom}(E_v, E_u) = 0 \Rightarrow \text{Hom}(E_v, E_{u-n}) = 0$$

We have $\text{Hom}(E_v, E_u) = 0$ for $v - n < u < v$ as E_{v-n+1}, \dots, E_v is exceptional and also $\text{Hom}(E_v, E_u) = 0$ for $u = v - k$ by Serre duality. Hence (7.6) implies $\text{Hom}(E_v, E_u) = 0$ for all $u < v$.

Now we consider Ext^2 . Again by Serre duality we have

$$\dim \text{Ext}^2(E_u, E_v) = \dim \text{Hom}(E_v, E_{u-n})$$

Hence we need $\text{Hom}(E_v, E_{u-n}) = 0$ for $v \geq u$. Since $v \geq u$ implies $v > u - n$ this follows from the vanishing of Hom as discussed in the previous paragraph.

Put $\mathcal{E}_0 = E_1 \oplus \dots \oplus E_n$. The clearly \mathcal{E}_0 satisfies the conditions of (7.1). \square

Remark 7.4. In the published version of this paper [42] we considered a more naive tilting object. Assume that Z is obtained by blowing up \mathbb{P}^2 in x_1, \dots, x_p with $p \leq 8$ and denote the corresponding exceptional curves by F_1, \dots, F_p . Let $\alpha : Z \rightarrow \mathbb{P}^2$ be the structure map. In [42] we took $\mathcal{E}_0 = \alpha^*(\mathcal{O}_{\mathbb{P}^2}) \oplus \alpha^*(\mathcal{O}_{\mathbb{P}^2}(1)) \oplus \alpha^*(\mathcal{O}_{\mathbb{P}^2}(2)) \oplus \mathcal{O}_Z(F_1) \oplus \dots \oplus \mathcal{O}_Z(F_p)$ as tilting object on Z . In contrast to what we stated this tilting object does not satisfy the conditions of (7.1). Indeed for $p > 4$ we have $H^1(Z, \mathcal{E}nd(\mathcal{E}_0, \mathcal{E}_0) \otimes_Z \omega_Z^{-1}) \neq 0$ as one easily checks that $H^1(Z, \mathcal{H}om(\alpha^*(\mathcal{O}_{\mathbb{P}^2}(2)), \mathcal{O}_Z(F_1) \otimes_Z \omega_Z^{-1})) \neq 0$.

8. ONE-DIMENSIONAL TORUS INVARIANTS

Let $T = k^*$ be a one-dimensional torus acting on a finite dimensional vector space V . We may choose a basis x_1, \dots, x_n for V such that T acts diagonally: $z \cdot x_i = z^{a_i} x_i$ for some $a_i \in \mathbb{Z}$. Put $S = \text{Sym}(V)$ and $R = S^T$. In order to avoid trivialities we assume that there are at least two strictly positive and two strictly negative a_i 's and that the greatest common divisor of the a_i 's is one. Put $N^+ = \sum_{a_i > 0} a_i$, $N^- = -\sum_{a_i < 0} a_i$ and $N = \min(N^+, N^-)$.

It will be convenient to use the Artin-Zhang Proj of a graded ring [1]. If T is a noetherian \mathbb{Z} -graded ring then $X = \text{Proj}_{AZ} T$ is the Grothendieck category $\text{Gr}(T)/\text{Tors}(T)$ where $\text{Tors}(T)$ is the localizing subcategory of $\text{Gr}(T)$ given by the graded modules which are limits of right bounded ones. Below we denote the quotient functor $\text{Gr}(T) \rightarrow \text{Gr}(T)/\text{Tors}(T)$ by π . We write $\mathcal{O}_X = \pi S$ and $\underline{\Gamma}(X, \mathcal{M}) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}(\mathcal{O}_X, \mathcal{M}(n))$ where the functor $\mathcal{M} \mapsto \mathcal{M}(n)$ is obtained from the corresponding functor on $\text{Gr}(T)$. The derived functors of $\underline{\Gamma}$ are denoted by \underline{H}^* . We write $\text{coh}(X)$ for the category of noetherian objects in X .

It is shown in [1] that $\text{Proj}_{AZ} T = \text{Proj}_{AZ} T_{\geq 0}$ so in principle one may restrict to \mathbb{N} -graded rings but below it will be more convenient to work with \mathbb{Z} -graded rings.

Putting $\deg x_i = a_i$ defines a \mathbb{Z} -grading on S such that $R = S_0$. By [21] R is Cohen-Macaulay. The other homogeneous components S_a of S are finitely generated R -modules. It is known *precisely* when they are Cohen-Macaulay [35, 40]. In particular one has the following result.

Lemma 8.1. *S_a is Cohen-Macaulay for $-N^+ < a < N^-$.*

Let $S^+ = S$ as graded rings and let S^- be the ring S with modified grading given by $S_n^- = S_{-n}$. We define $X^\pm = \text{Proj}_{AZ} S^\pm$. Thus if I^+ , I^- are respectively the ideals generated by $S_{>0}$ and $S_{<0}$ then X^\pm are simply the Grothendieck categories $\text{Gr}(S^\pm)/\text{Tors}(S^\pm)$ where $\text{Tors}(S^\pm)$ consists of the objects in $\text{Gr}(S^\pm)$ which are (elementwise) annihilated by powers of I^\pm . Note that I^+ , I^- are the graded ideals in S respectively generated by $(x_i)_{a_i > 0}$ and $(x_i)_{a_i < 0}$. From the description of $\text{Tors}(S^\pm)$ as torsion theories associated to ideals, it follows that $\text{Tors}(S^\pm)$ is stable. I.e. $\text{Tors}(S^\pm)$ is closed under injective hulls [37]. In particular the following lemma follows.

Lemma 8.2. *X^+ , X^- have finite global dimension.*

Lemma 8.3. *We have $\underline{H}^i(X^\pm, \mathcal{O}_{X^\pm}) = H_{I^\pm}^{i+1}(S^\pm)$ for $i > 0$ and $S^\pm = \underline{\Gamma}(X^\pm, \mathcal{O}_{X^\pm})$.*

Proof. This is a variant on (7.3)(7.4). Note that $H_{I^\pm}^{0,1}(S^\pm) = 0$ because of the hypotheses on the weights $(a_i)_i$. \square

Lemma 8.4. *Let $0 \leq i, j < N^\pm$. Then $\text{Ext}_{X^\pm}^p(\mathcal{O}_{X^\pm}(m), \mathcal{O}_{X^\pm}(n)) = 0$ for $p > 0$ and $\text{Hom}_{X^\pm}(\mathcal{O}_{X^\pm}(m), \mathcal{O}_{X^\pm}(n)) = S_{n-m}^\pm$.*

Proof. By the previous lemma we have $\text{Ext}_{X^\pm}^p(\mathcal{O}_{X^\pm}(m), \mathcal{O}_{X^\pm}(n)) = H_{I^\pm}^{p+1}(S^\pm)_{n-m}$ for $p > 0$. A standard computation reveals that $H_{I^\pm}^u(S^\pm)$ is zero in degrees $> -N^\pm$ which is what we want.

Also by the previous lemma we find $\text{Hom}_{X^\pm}(\mathcal{O}_{X^\pm}(m), \mathcal{O}_{X^\pm}(n)) = S_{n-m}^\pm$. \square

Lemma 8.5. *The objects $\mathcal{O}_{X^\pm}(m)$, $m = 0, \dots, N^\pm - 1$ form a generating family for $D(X^\pm)$.*

Proof. Let us work with X^+ . With a variant of [11, Lemma 4.2.2] we see that $(\mathcal{O}_{X^+}(n))_n$ generates $D(X^+)$. Now we look at the Koszul exact sequence associated to $(x_i)_{a_i > 0}$.

$$0 \rightarrow S(-N^+) \rightarrow \cdots \rightarrow \bigoplus_{a_i > 0} S(-a_i) \rightarrow S \rightarrow S/I^+ \rightarrow 0$$

Note that S/I^+ is right bounded. Shifting and applying π we obtain exact sequences

$$0 \rightarrow \mathcal{O}_{X^+}(n - N^+) \rightarrow \cdots \rightarrow \bigoplus_{a_i > 0} \mathcal{O}_{X^+}(n - a_i) \rightarrow \mathcal{O}_{X^+}(n) \rightarrow 0$$

Hence it follows that all $\mathcal{O}_{X^+}(n)$ may be obtained using triangles from $\mathcal{O}_{X^+}(m)$, $m = 0, \dots, N^+ - 1$ which finishes the proof. \square

To simplify the notations below we will now invert, if necessary, the signs of the a_i 's, to insure that $N^+ \leq N^-$. Thus $N = N^+$.

Define $\mathcal{E}^\pm = \bigoplus_{m=0, \dots, N-1} \mathcal{O}_{X^\pm}(m)$, According to lemma 8.4 we have $A \stackrel{\text{def}}{=} \text{End}_{X^+}(\mathcal{E}^+) = \text{End}_{X^-}(\mathcal{E}^-)$.

Lemmas 8.4,8.5 now yield the following

Theorem 8.6. *The functor $\text{RHom}(\mathcal{E}^+, -)$ defines an equivalence $D(X^+) \rightarrow D(A)$ and the functor $-\otimes_A \mathcal{E}^-$ defines a full faithful embedding $D(A) \rightarrow D(X^-)$. In particular there is a full faithful embedding $D(X^+) \rightarrow D(X^-)$. All embeddings restrict to embeddings between the corresponding bounded derived categories of coherent objects. If $N^- = N^+$ (or equivalently $\sum_i a_i = 0$) then all embeddings are equivalences.*

The existence of an embedding/equivalence $D(X^+) \rightarrow D(X^-)$ was proved by Kawamata in [23] (in a slightly more general situation).

We also have

Proposition 8.7. *A is Cohen-Macaulay.*

Proof. According to lemma 8.4 we have $A = (S_{n-m})_{0 \leq m, n < N}$. Cohen-Macaulauness now follows from lemma 8.1. \square

Now we restrict to the case $N^+ = N^-$, or equivalently $\sum_i a_i = 0$. Under these hypotheses we have

Lemma 8.8. *S_0 is Gorenstein and the S_a are reflexive S_0 -modules, satisfying $(S_a S_b)^{**} = S_{a+b}$.*

Proof. S_0 is Gorenstein because of [36, 13.3]. The hypotheses that the greatest common divisor of the $(a_i)_i$ is one implies that the generic stabilizer of the T -action on $W = \text{Spec Sym}(V) = V^*$ is trivial.

In general if $(\zeta_i)_i$ is a point in W then the order of its stabilizer is equal to the greatest common divisor of the a_i 's such that $\zeta_i \neq 0$. From the relation $\sum_i a_i = 0$ we then deduce that the complement of the locus $W' \subset W$ where T has trivial stabilizer, has codimension at least two. The equality $(S_a S_b)^{**} = S_{a+b}$ is now true

on W' and hence it is true on $W'//T$ since $W' \rightarrow W'//T$ is a principal T -bundle. Then it is also true on $W//T$ since $W \rightarrow W//T$ contracts no divisor and hence all modules of covariants are reflexive [17, Section 1.3]. \square

We may now state our final result.

Theorem 8.9. *Let the notations be as above and assume $\sum_i a_i = 0$. Then $R = S^T$ is a Gorenstein ring with a non-commutative crepant resolution given by $A = \text{End}_R(\bigoplus_{a=0}^{N-1} S_a)$ where $N = \sum_{a_i > 0} a_i = -\sum_{a_i < 0} a_i$.*

Proof. We have $A = \bigoplus_{m,n=0,\dots,N-1} (S_{n-m})$ which by lemma 8.8 is equal to $\text{End}_R(\bigoplus_{a=0}^{N-1} S_a)$. \square

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