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Peer-reviewed author version

EGGHE, Leo (2008) The measures precision, recall, fallout and miss in function of the number of retrieved documents and their mutual interrelations. In: INFORMATION PROCESSING & MANAGEMENT, 44(2). p. 856-876.

DOI: 10.1016/j.ipm.2007.03.014

Handle: <http://hdl.handle.net/1942/817>

The measures precision, recall, fallout and miss as a function of the number of retrieved documents and their mutual interrelations

by

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ABSTRACT

In this paper, for the first time, we present global curves for the measures precision, recall, fallout and miss in function of the number of retrieved documents. Different curves apply for different retrieved systems, for which we give exact definitions in terms of a retrieval density function: perverse retrieval, perfect retrieval, random retrieval, normal retrieval, hereby extending results of Buckland and Gey and of Egghe in the following sense: mathematically

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Acknowledgement: The author is grateful to Prof. Dr. R. Rousseau for interesting discussions on the topic of this paper.

Key words and phrases: precision, recall, fallout, miss, retrieved documents, perverse retrieval, perfect retrieval, random retrieval, normal retrieval.

more advanced methods yield a better insight into these curves, more types of retrieval are considered and, very importantly, the theory is developed for the “complete” set of measures: precision, recall, fallout and miss.

Next we study the interrelationships between precision, recall, fallout and miss in these different types of retrieval, hereby again extending results of Buckland and Gey (incl. a correction) and of Egghe. In the case of normal retrieval we prove that precision in function of recall and recall in function of miss is a concavely decreasing relationship while recall in function of fallout is a concavely increasing relationship. We also show, by producing examples, that the relationships between fallout and precision, miss and precision and miss and fallout are not always convex or concave.

I. Introduction

This paper deals with the simplest information retrieval (IR) systems, i.e. systems for which we have a clear dichotomous division between retrieved documents (ret) and non retrieved documents (\emptyset ret) and the same for relevant documents (rel) and non relevant documents (\emptyset rel). This includes Boolean IR systems in which one has a clear set of retrieved documents and in which we assume that we can make a clear decision on which documents are relevant and which ones are not. Hence it excludes so-called fuzzy IR systems where the sets of retrieved documents are fuzzy and the same goes for the set of relevant documents. It also excludes IR systems in which retrieved documents are presented in a ranked way (e.g. in search engines).

It is, however, evident that, as long as the simplest IR systems are not fully understood or described, those studies remain very important (in itself and for the purpose of further extension to more advanced IR systems).

So we simply suppose we have a “universe” Ω (i.e. the documentary system or IR system), a topic (determining relevant and non relevant documents) and a query in the form of a formal command (determining in Ω , using its retrieval software, the retrieved and non retrieved documents). The following measures are well-known – see e.g. Boyce, Meadow and Kraft

(1995), Frants, Shapiro and Voiskunskii (1997), Grossman and Frieder (1998), Heaps (1978), Losee (1998), Salton and McGill (1987), Tague-Sutcliffe (1995) and Van Rijsbergen (1979): Let us denote by ret the set of retrieved documents, by $\emptyset ret$ the set of non retrieved documents, by rel the set of relevant documents and by $\emptyset rel$ the set of non relevant documents. Then the precision (P) is given by

$$P = \frac{|ret \cap rel|}{|ret|} \quad (1)$$

(here $|\cdot|$ denotes the cardinality of the set, i.e. the number of elements in the set). Hence P is the fraction of the retrieved documents that are relevant. The fraction of relevant documents that are retrieved is called recall (R):

$$R = \frac{|ret \cap rel|}{|rel|} \quad (2)$$

Most of the above references also define fallout (F) being the fraction of non relevant documents that are retrieved:

$$F = \frac{|ret \cap \emptyset rel|}{|\emptyset rel|} \quad (3)$$

Much less known, introduced to the English speaking community in Heine (1984) (see also Egghe (2004, 2005)) and (re-)introduced in Egghe (2004) is the measure miss (M) being the fraction of non retrieved documents that are relevant:

$$M = \frac{|\emptyset ret \cap rel|}{|\emptyset ret|} \quad (4)$$

Note that all four of the above expressions can be interpreted as conditional probabilities, e.g. for (1): $P = p(rel|ret)$, the probability for a document to be relevant, given that it is retrieved (and similar for the other 3-expressions).

Four other “fraction” measure can be defined (see e.g. Egghe (2004)) but they lead to the measures $1 - P, 1 - R, 1 - F, 1 - M$, hence they are known, once P, R, F and M are fully described.

However, the description of the measures P, R, F and M is far from complete. What do we mean by “describing” these measures? First of all, we can study these measures by indicating their relationship with the number of retrieved documents $t = |\text{ret}|$, giving certain IR types, i.e. orders in which relevant documents appear as t increases. We will study perverse retrieval (first all non relevant documents are retrieved and then all relevant documents), perfect retrieval (first all relevant documents are retrieved and then all non relevant documents), random retrieval, where we retrieve relevant and non relevant in a non preferential role (exact definitions follow) and finally “normal” retrieval, where one retrieves a higher density of relevant documents for small $t = |\text{ret}|$ than for high t . Perverse, perfect and random retrieval were (heuristically) studied in Buckland and Gey (1994) for the measures P and R only; normal retrieval was studied in Egghe (1992), also only for the measures P and R . In fact – besides Yates-Mercer (1976), where some scatter diagrams are given between P and R and between R and F – we only know of studies (theoretical and or practical) that relate P and R : besides the above references we can mention: Cleverdon (1972), Heine (1973), Bollmann (1977), Bellardo and Saracevic (1987), Gordon (1989), McCarn and Lewis (1990), Deerwester et al. (1990), Egghe and Rousseau (1998), Gebhardt (1975), Salton and McGill (1987) and the in LISA (Library and Information Science Abstracts) described papers in Chinese: Deng, Wang and Wang (2000) and Deng, Wang and Mang (2000).

In the present paper we will, first of all, study all these different types of retrieval in terms of the curves P, R, F and M in function of $t = |\text{ret}|$, hereby considerably extending the results in Buckland and Gey (1994) and Egghe (1992). Also more advanced techniques (than in Egghe (1992)) will be used to determine the fine shapes of P, R, F and M in function of $t = |\text{ret}|$. We prove that, in the case of normal retrieval, $P(t)$ is concavely decreasing, $R(t)$ is concavely increasing, $F(t)$ is convexly increasing and $M(t)$ is concavely decreasing.

Next we can also study the mutual interrelations between the measures P, R, F and M for the mentioned IR types. For normal retrieval we prove that P is concavely decreasing in R (a

result already proved in Egghe (1992) but here we give a shorter proof), that R is concavely increasing in F and that R is concavely decreasing in M . From this also the relations $R(P)$, $F(R)$ and $M(R)$ follow. For the relations between P and F , P and M and between F and M we produce examples that, in general, no convex or concave relationship exists. These examples constitute real examples of IR systems, based on the theory developed recently in Egghe (2005). We also use these examples to illustrate and to confirm the theoretically established relations $P(R)$, $R(F)$ and $R(M)$.

The paper is organised as follows. In the next section we present the general IR model in terms of the retrieval density function $h = h(t)$, i.e. the function indicating in what order the relevant documents are retrieved, in function of the retrieved quantity $t = |\text{ret}|$. In the third section we present the case of perverse retrieval: exact definitions are given and we refine the heuristic models of Buckland and Gey (1994) for $P(t)$, $R(t)$ and correct their result for $P(R)$ and, of course, complete the theory by proving the graphs of $F(t)$ and $M(t)$ and the interrelations between P , R , F and M . In section IV we present exact definitions and results for perfect retrieval, again refining the models for $P(t)$, $R(t)$ and $P(R)$, but now also including the results on $F(t)$, $M(t)$ and the interrelations between P , R , F and M . In section V the same is done for random retrieval.

In section VI we, firstly, define, in an exact way, what “normal” retrieval means: this is done by proposing properties in terms of derivatives of the retrieval density function h . The second part of Section VI is devoted to the study of $P(t)$, $R(t)$, $F(t)$ and $M(t)$. Their complete functional behaviour will be described. In the third part of Section VI we study the relations between the measures P , R , F and M in the case of normal retrieval, as described above and in the fourth part we present concrete IR examples of all interrelations between P , R , F and M (as described above). Section VII is a conclusion section, including the description of some open problems.

II. General IR model using the retrieval density function

As discussed in the Introduction, we have a “universe” Ω representing the set of all the documents in the IR system. Let us have retrieved $t = |\text{ret}|$ documents. In the real discrete case t can range in the set $\{1, 2, \dots, |\Omega| = N\}$ but we will use the continuous model in which $t \in [0, N]$. For each t , we can, in the discrete case, count the total number of relevant documents in the first t retrieved documents. Let us denote this by $H(t)$, which we also use in the continuous case: then we suppose that H is differentiable and we denote $H' = h$. Hence, since $H(0) = 0$, we have

$$H(t) = \int_0^t h(i) di \quad (5)$$

The function h denotes the density of the relevant documents at each t , i.e. the number of relevant documents in the range $[t, t + \Delta t]$, comprising Δt retrieved documents, divided by Δt retrieved documents (and then $\Delta t \rightarrow 0$): indeed

$$h(t) = H'(t) = \lim_{\Delta t \rightarrow 0} \frac{H(t + \Delta t) - H(t)}{\Delta t} \quad (6)$$

Hence $h(t) \in [0, 1]$, for all $t \in [0, N]$, by definition.

This is the framework in which we will discuss the graphs of $P(t)$, $R(t)$, $F(t)$, $M(t)$, i.e. precision, recall, fallout and miss when there are $t = |\text{ret}| \in [0, N]$ retrieved documents. The function h will determine the retrieval type: we will go into this in the next sections but here we will give an indication of how we will define exactly (in the next sections) the different retrieval types:

1. Perverse retrieval will be characterised by a density function h that is 0 for low t and 1 for high t , i.e. we first retrieve all non relevant documents and then the relevant ones.
2. Perfect retrieval will be characterised by a density function h that is 1 for low t and 0 for high t , i.e. we first retrieve all relevant documents and then the non relevant ones (note that in all retrieval types we will always consider the full range $t \hat{\in} [0, N]$, where $N = |\Omega|$).
3. Random retrieval does not give a preference of relevant above non relevant documents, hence the density function is constant here.
4. Normal retrieval allows for a wide range of retrieval types but we always have that, for t increasing, the density function h decreases concavely, meaning that, relatively speaking, more relevant documents are retrieved in the beginning (low t) and that this number of relevant documents decreases more rapidly, the higher t . We prefer to work in this general setting and not adopting special examples such as “parabolic” retrieval in Buckland and Gey (1994) where they adopt non-explained quadratic functions and apply them directly e.g. on $R(t)$. We prefer the general setting for h and then prove general properties of $P(t)$, $R(t)$, $F(t)$ and $M(t)$ (and, later on, interrelations between P , R , F and M).

Note that, in this model, we fix $l = |\text{rel}|$ (otherwise stated: we fix any query) but allow l to be any number in $[0, N]$, just like $t = |\text{ret}|$ but t is treated as a variable (and l as a parameter). Hence, at each moment, we fix the topic and consider retrieval of $t \hat{\in} [0, N]$ documents. Note, however, that we do not consider the retrieved set as a set of ordered documents: for each $t = |\text{ret}|$, the set ret is an ordinary set in which each element has a membership value of 1. The same is true for the set rel , the set of relevant documents.

Using the definitions (1)-(4) of P , R , F and M , we have the following general formulae in the present setting (keeping $l \hat{\in} [0, N]$ fixed):

$$P = P(t) = \frac{\int_0^t h(i) di}{t} \quad (7)$$

$$R = R(t) = \frac{\int_0^t h(i) di}{1} \quad (8)$$

$$F = F(t) = \frac{t - \int_0^t h(i) di}{N - 1} \quad (9)$$

$$M = M(t) = \frac{1 - \int_0^t h(i) di}{N - t} \quad (10)$$

III. Perverse retrieval

Although, in practise, this is an unimportant example, it is interesting from a mathematical point of view and since it represents a limiting case, opposite to perfect retrieval, to be discussed in the next section. Perverse retrieval was also considered in Buckland and Gey (1994): we will rekind (and considerably extend) their heuristic findings and correct one of their results.

Perverse retrieval is defined via the retrieval density function h as follows

$$\begin{cases} h(i) = 0, & 0 \leq i \leq N-1 \\ h(i) = 1, & N-1 < i \leq N \end{cases} \quad (11)$$

, i.e. the $1 = |\text{rel}|$ relevant documents are retrieved last. Otherwise stated: the documents in Ω are arranged so that nonrelevant documents are grouped first.

It is easy to see that

$$\int_0^t h(i) di = \int_{\min(t, N-1)}^t 1 di$$

$$\int_0^t h(i) di = t - \min(t, N-1) \quad (12)$$

Hence we have the following formulae, using (7)-(10):

$$P(t) = \frac{t - \min(t, N-1)}{t} \quad (13)$$

$$R(t) = \frac{t - \min(t, N-1)}{1} \quad (14)$$

$$F(t) = \frac{\min(t, N-1)}{N-1} \quad (15)$$

$$M(t) = \frac{1 - t + \min(t, N-1)}{N-t} \quad (16)$$

For $0 \leq t \leq N-1$ this yields

$$P(t) = 0 = R(t) \quad (17)$$

$$F(t) = \frac{t}{N-1} \quad (18)$$

$$M(t) = \frac{1}{N-t} \quad (19)$$

For $t > N-1$ this yields

$$P(t) = \frac{t - N + 1}{t} \quad (20)$$

$$R(t) = \frac{t - N + 1}{1} \quad (21)$$

$$F(t) = 1 = M(t) \quad (22)$$

We hence have obtained the graph in Fig. 1.

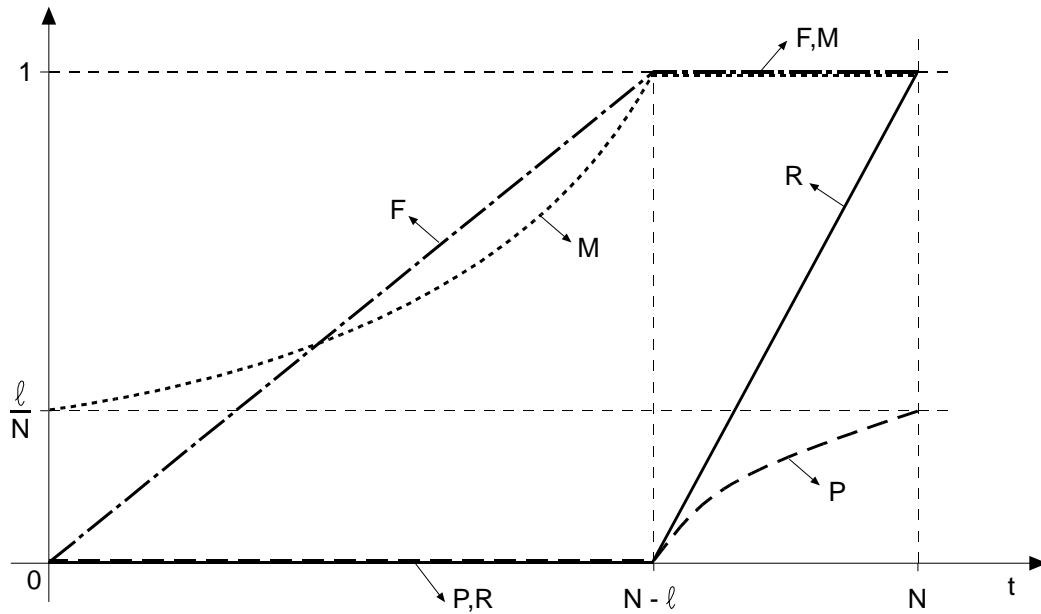


Fig. 1. Graphs of $P(t)$, $R(t)$, $F(t)$ and $M(t)$ in case of perverse retrieval.

The graphs of R and P can be found in Buckland and Gey (1994) but no calculations are given and there $\frac{l}{N} = 0.1$.

Now we will calculate the interrelations between P , R , F and M . P and R are both zero or (20) and (21) applies. Formula (21) yields

$$t = 1R + N - 1$$

which in (20) yields

$$P = \frac{1R}{1R + N - 1} \quad (23)$$

which is a concavely increasing function of R as graphed in Fig. 2. This corrects the $P(R)$ graph in Buckland and Gey (1994) (they present a linear relationship between P and R which is clearly false, by (23)).

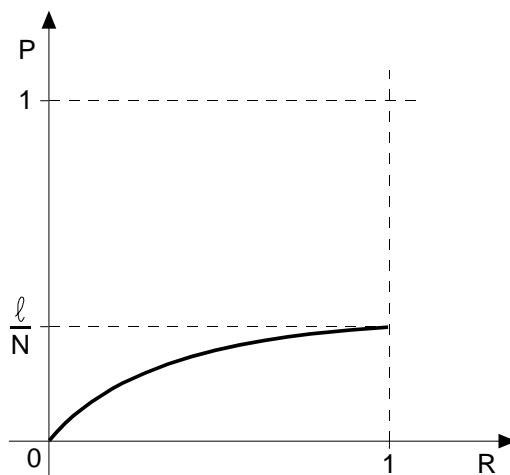


Fig. 2. Graph of $P(R)$ in case of perverse retrieval.

From Fig. 1, the $R(F)$, $P(M)$, $M(R)$ and $P(F)$ curves are straightforward – see Figs. 3-6.

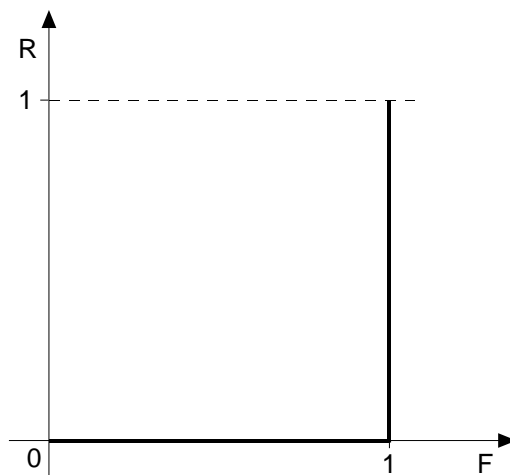


Fig. 3. Graph of $R(F)$ in case of perverse retrieval.

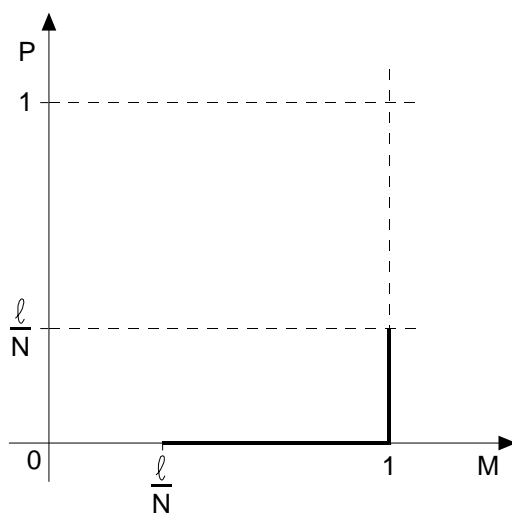


Fig. 4. Graph of $P(M)$ in case of perverse retrieval.

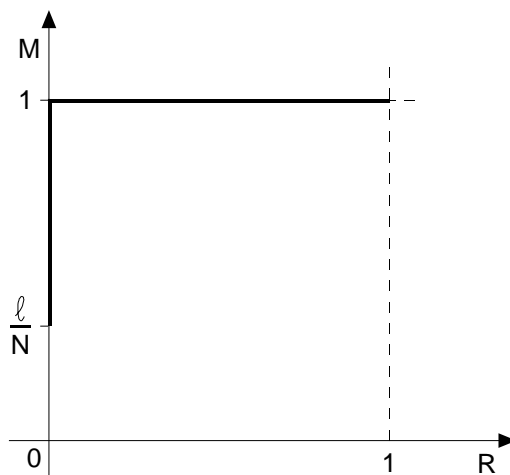


Fig. 5. Graph of $M(R)$ in case of perverse retrieval.

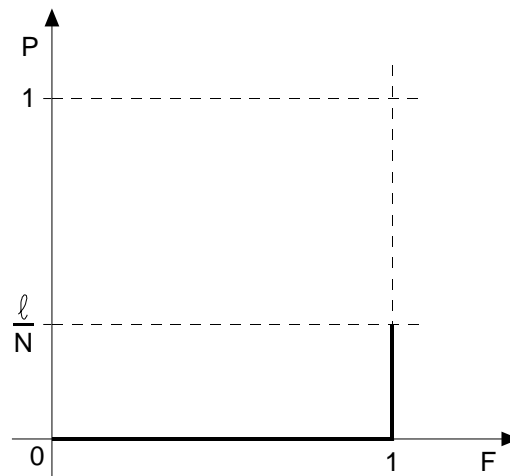


Fig. 6. Graph of $P(F)$ in case of perverse retrieval.

Finally, for the $F(M)$ curve we note that, if F and M are not both 1, we have (18) and (19).

Equation (19) yields

$$t = \frac{MN - 1}{M}$$

which in (18) yields

$$F = \frac{MN - 1}{M(N - 1)} \quad (24)$$

which is a concavely increasing function of M as graphed in Fig. 7. This ends our study of the perverse retrieval case.

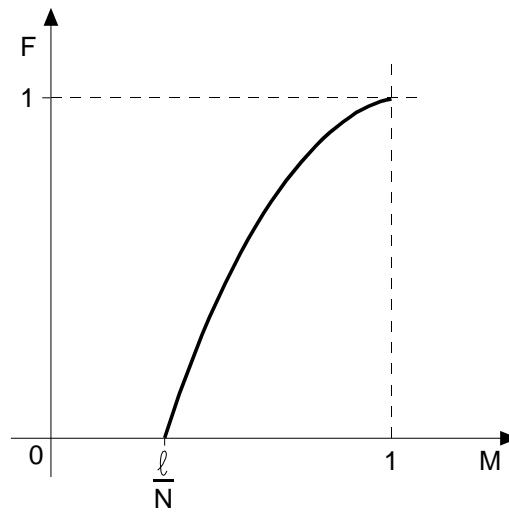


Fig. 7. Graph of $F(M)$ in case of perverse retrieval.

We now study the “dual” case: perfect retrieval.

IV. Perfect retrieval

Perfect retrieval is defined via the retrieval density function as follows.

$$\begin{cases} h(i) = 1, & 0 \leq i \leq 1 = |\text{rel}| \\ h(i) = 0, & 1 < i \leq N \end{cases} \quad (25)$$

It is easy to see that

$$\int_0^t h(i) di = \int_0^{\min(t,1)} 1 di$$

$$\int_0^t h(i) di = \min(t,1) \quad (26)$$

Hence we have the following formulae, using (7)-(10):

$$P(t) = \frac{\min(t,1)}{t} \quad (27)$$

$$R(t) = \frac{\min(t,1)}{1} \quad (28)$$

$$F(t) = \frac{t - \min(t,1)}{N - 1} \quad (29)$$

$$M(t) = \frac{1 - \min(t,1)}{N - t} \quad (30)$$

For $0 \leq t \leq 1$ this yields

$$P(t) = 1 \quad (31)$$

$$R(t) = \frac{t}{1} \quad (32)$$

$$F(t) = 0 \quad (33)$$

$$M(t) = \frac{1 - t}{N - t} \quad (34)$$

For $t > 1$ this yields

$$P(t) = \frac{1}{t} \quad (35)$$

$$R(t) = 1 \quad (36)$$

$$F(t) = \frac{t-1}{N-1} \quad (37)$$

$$M(t) = 0 \quad (38)$$

These formulae yield the graph in Fig. 8.

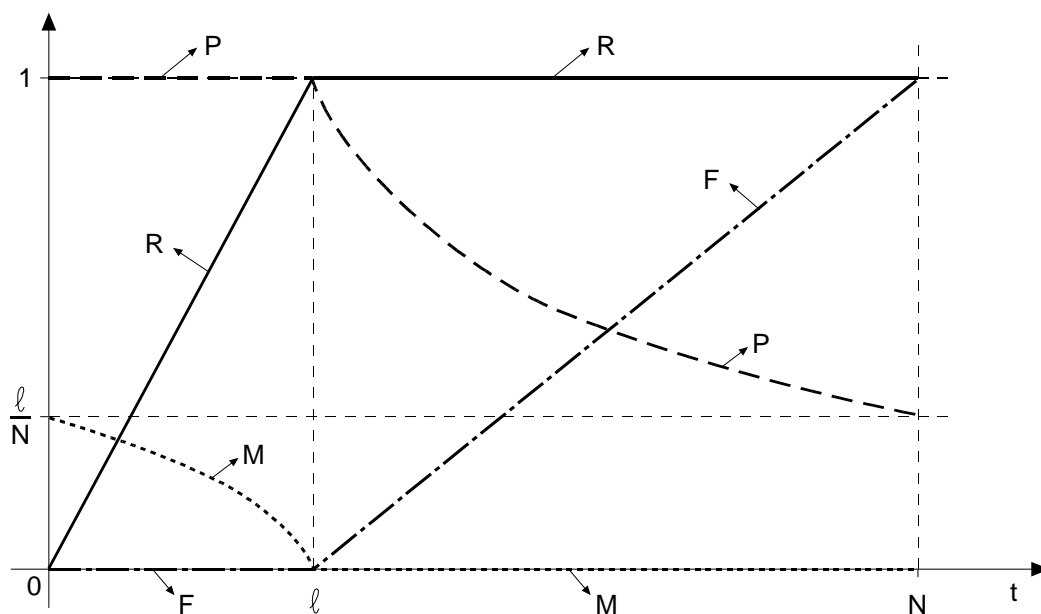


Fig. 8. Graphs of $P(t)$, $R(t)$, $F(t)$ and $M(t)$ in case of perfect retrieval.

The graphs of P and R can be found in Buckland and Gey (1994) for $\frac{1}{N} = 0.1$.

Now we will calculate the interrelations between P , R , F and M . Formulae (31), (32), (35) and (36) yield the $P(R)$ graph which is also appearing in Buckland and Gey (1994) – see Fig. 9.

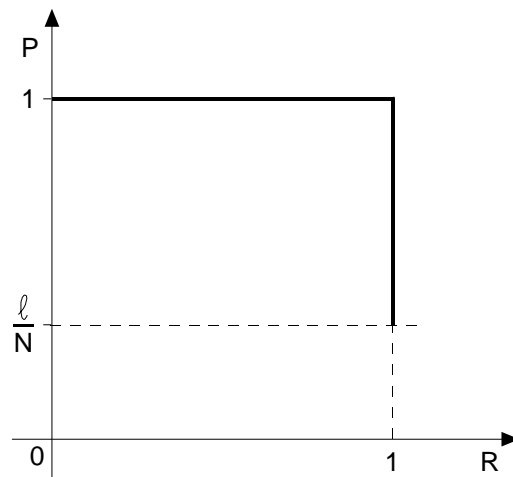


Fig. 9. Graph of $P(R)$ in case of perfect retrieval.

Also the $P(M)$, $F(M)$ and $R(F)$ graphs are straightforward from Fig. 8 – see Figs. 10-12.

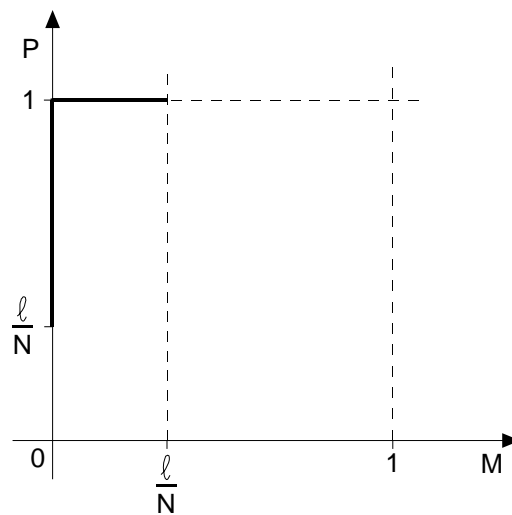


Fig. 10. Graph of $P(M)$ in case of perfect retrieval.

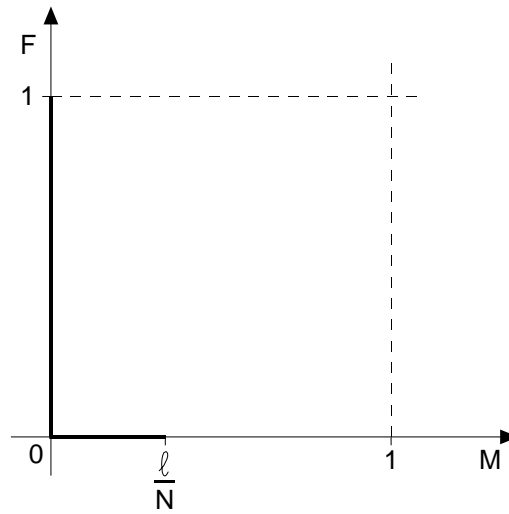


Fig. 11. Graph of $F(M)$ in case of perfect retrieval.

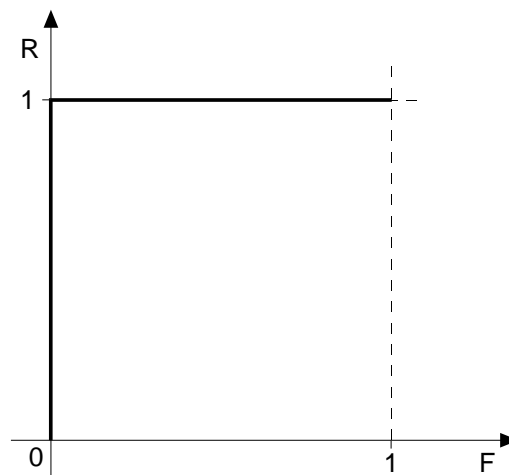


Fig. 12. Graph of $R(F)$ in case of perfect retrieval.

For the $M(R)$ graph, note that if $(R, M)^1 (1, 0)$ then (32) and (34) apply. From (32) we have

$$t = 1R$$

which in (34) yields

$$M = \frac{1 - 1R}{N - 1R} \quad (39)$$

which is a concavely decreasing function – see Fig. 13.

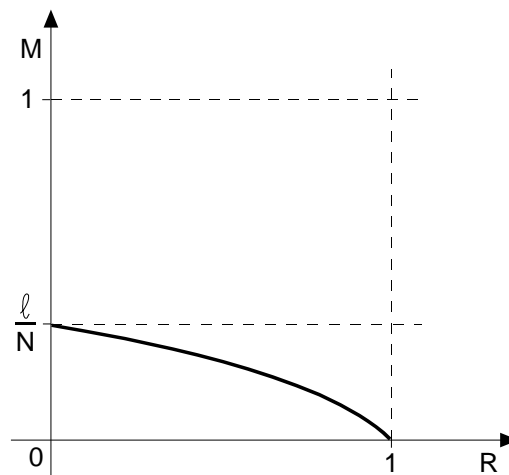


Fig. 13. Graph of $M(R)$ in case of perfect retrieval.

For the $P(F)$ graph, note that if $(F, P) \in (0, 1)$, then (35) and (37) apply. From (37) we have

$$t = 1 + (N - 1)F$$

which in (35) yields

$$P = \frac{1}{1 + (N - 1)F} \quad (40)$$

which is a convexly decreasing function – see Fig. 14.

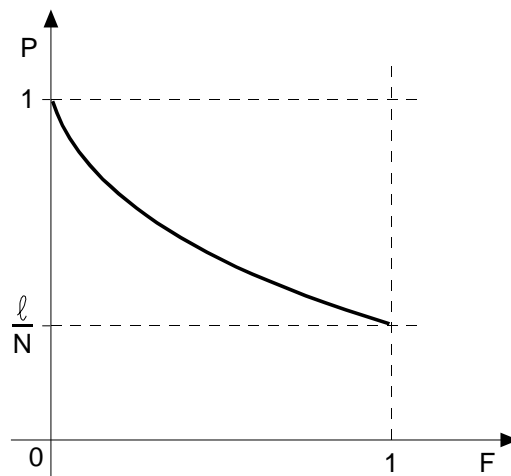


Fig. 14. Graph of $P(F)$ in case of perfect retrieval.

This ends the case of perfect retrieval.

V. Random retrieval

Random retrieval is defined via the retrieval density function as follows:

$$h(i) = \frac{1}{N} \quad (41)$$

for all $i \in [0, N]$. Note that $\frac{1}{N}$ expresses the fraction of relevant documents. In random retrieval, the retrieval density function indicates that we retrieve relevant documents in a “speed” equal to the probability of randomly picking a relevant document from Ω . It is clear that

$$\int_0^t h(i) di = \frac{1t}{N} \quad (42)$$

and hence we have the following formulae, using (7)-(10):

$$P(t) = \frac{1}{N} = M(t) \quad (43)$$

$$R(t) = \frac{t}{N} = F(t) \quad (44)$$

Hence we have equality of P and M and of R and F. Formulae (43) and (44) yield the graph in Fig. 15.

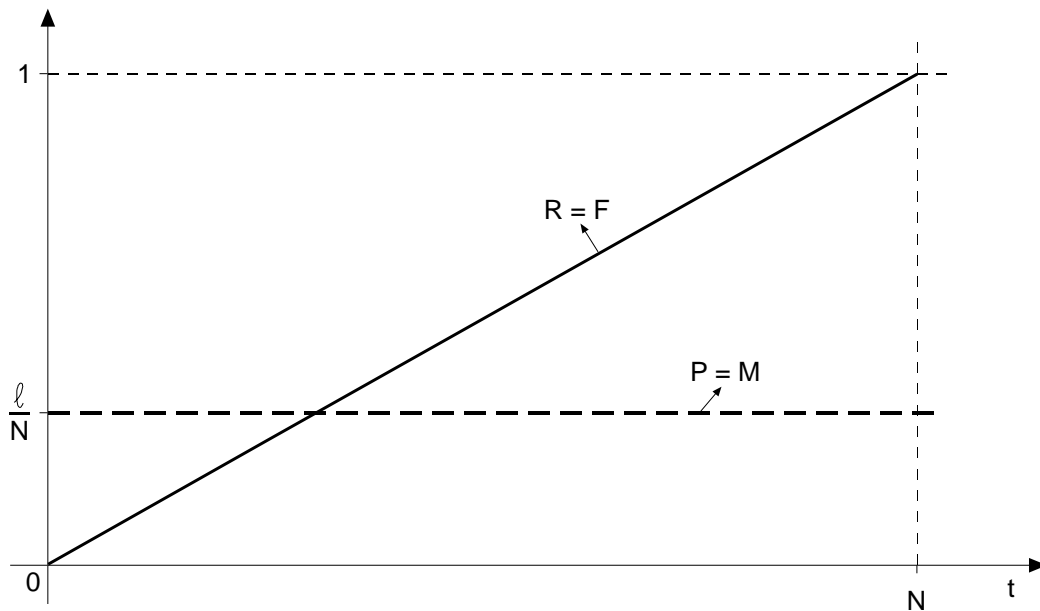


Fig. 15. Graphs of $P(t) = M(t)$ and of $R(t) = F(t)$
in case of random retrieval.

Now there is only one P, R, F, M interrelations graph $P(R) = P(F) = M(R) = M(F)$, namely the constant function $\frac{1}{N}$ - see Fig. 16.

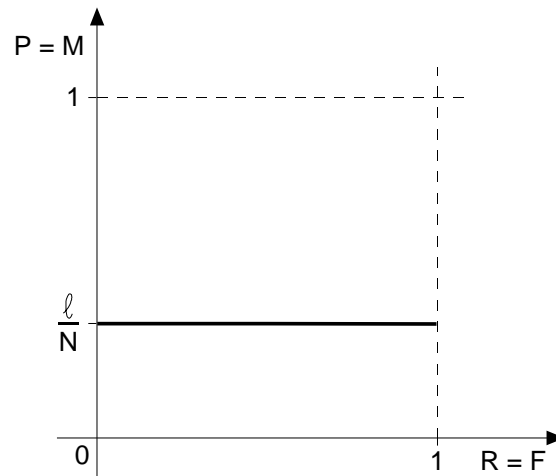


Fig. 16. Graph of $P(R) = P(F) = M(R) = M(F)$
in case of random retrieval.

We now turn our attention to the most important retrieval type: normal retrieval.

VI. Normal retrieval

VI.1 The retrieval density function

Normal retrieval is defined via the retrieval density function h as follows: we require, very generally $h(0) = 1$, $h(N) = 0$ and

$$h' < 0 \quad (45)$$

$$h'' < 0 \quad (46)$$

i.e., we require a concavely decreasing density function. This is the most “normal” way of retrieving: the density of relevant documents is decreasing with $t = |\text{ret}|$ and this decrease is faster, the higher t . Indeed: this concave decreasing behavior expresses that the “price” we

have to pay for retrieving more relevant documents is that we will retrieve relatively more and more non-relevant documents. The conditions (45) and (46) allow for many functions h , hence we comprise a wide class of retrieval actions. So do we accept both types of functions h as depicted in Figs. 17,18: $h''' < 0$ respectively $h''' > 0$. Although we think that the case $h''' < 0$ is the most natural one (h' decreasing faster the higher t) we will allow, in this article both cases, hence we do not put requirements on h''' since we do not need them in the sequel.

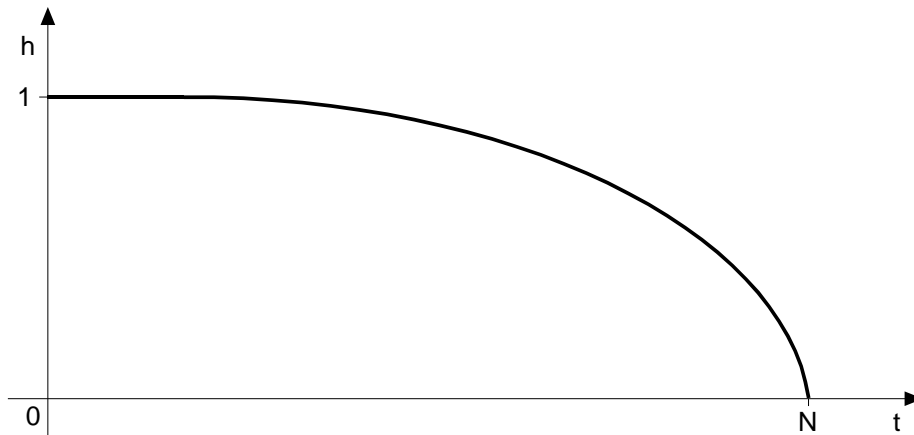


Fig. 17. Graph of h such that $h(0)=1$, $h(N)=0$, $h' < 0$, $h'' < 0$,

$$h''' < 0. \text{ Example: } h(t) = \frac{\ln(N+1-t)}{\ln(N+1)}.$$

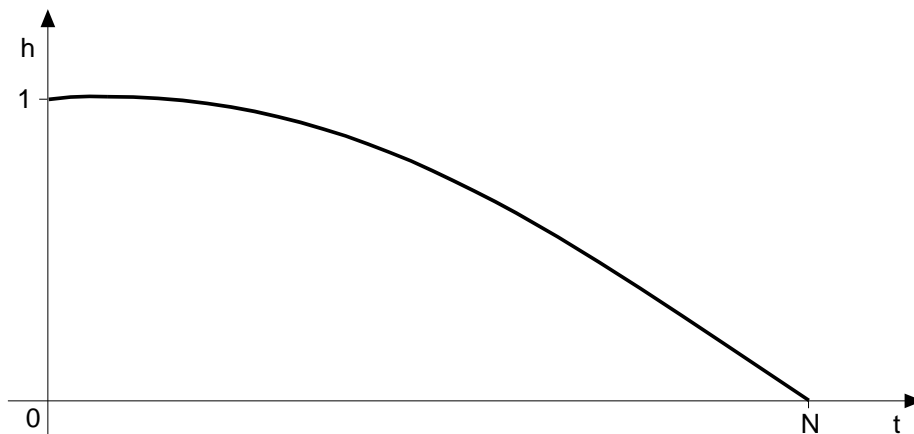


Fig. 18. Graph of h such that $h(0)=1$, $h(N)=0$, $h' < 0$, $h'' < 0$,

$$h''' > 0. \text{ Example: } h(t) = \cos\left(\frac{\pi t}{2N}\right).$$

The general formulae (7)-(10) apply and based on the above properties we will study the properties of the functions $P(t)$, $R(t)$, $F(t)$ and $M(t)$. This will be done in the next subsection. Later on we will study the interrelations between P , R , F and M .

VI.2 The properties of the functions $P(t)$, $R(t)$, $F(t)$ and $M(t)$

We start with a Lemma which will be applied twice further on.

Lemma VI.2.1: Let

$$\psi(x) = \frac{\varphi(x)}{x} \quad (47)$$

for $x \in]0, N]$ such that $\varphi(0) = 0$.

- (i) If $\varphi'' < 0$ and $\varphi''' < 0$ then ψ is concavely decreasing (strictly)
- (ii) If $\varphi'' < 0$ and $\varphi''' > 0$ then ψ is convexly decreasing (strictly)
- (iii) If $\varphi'' > 0$ and $\varphi''' < 0$ then ψ is concavely increasing (strictly)
- (iv) If $\varphi'' > 0$ and $\varphi''' > 0$ then ψ is convexly increasing (strictly).

Proof: From (47) it follows that

$$\psi'(x) = \frac{x\varphi'(x) - \varphi(x)}{x^2} < 0 \quad (48)$$

iff

$$\varphi'(x) < \frac{\varphi(x)}{x} = \frac{\int_0^x \varphi'(i) di}{x} \quad (49)$$

If $\varphi'' < 0$ then φ' is strictly decreasing and hence (49) is proved since $\varphi'(x)$ is the smallest value for $\varphi'(i)$ for $i \in [0, x]$ and since the right hand side of (49) is the average of these $\varphi'(i)$ values. Hence ψ decreases (strictly). For ψ to increase strictly we need the opposite inequality sign in (48) which is so if $\varphi'' > 0$ since then φ' increases strictly, hence (49) with opposite inequality sign, hence $\psi' > 0$.

Since

$$\psi'(x) = \frac{x\varphi'(x) - \varphi(x)}{x^2}$$

we have

$$\psi''(x) = \frac{x^2\varphi''(x) - 2x\varphi'(x) + 2\varphi(x)}{x^3} \quad (50)$$

We apply the Taylor expansion of φ (up to φ''' in the rest term) on the interval $[i, x]$ for any $i \geq 0, x > 0, i < x$:

$$\varphi(i) = \varphi(x) + \varphi'(x)(i-x) + \frac{\varphi''(x)}{2!}(i-x)^2 + \frac{\varphi'''(x_0)}{3!}(i-x)^3 \quad (51)$$

for $x_0 \in [i, x]$. Taking $i = 0$ we have, since $\varphi(0) = 0$,

$$0 = \varphi(0) = \varphi(x) - x\varphi'(x) + \frac{\varphi''(x)}{2}x^2 - \frac{\varphi'''(x_0)}{3!}x^3$$

So

$$2\varphi(x) - 2x\varphi'(x) + \varphi''(x)x^2 = \frac{\varphi'''(x_0)}{3}x^3 \quad (52)$$

If $\varphi''' < 0$ then (52) and (50) imply that $\psi'' < 0$, hence ψ is concave. If $\varphi''' > 0$ then we have that $\psi'' > 0$, hence ψ is convex. \square

Based on (5) and (7) we have that

$$P(t) = \frac{H(t)}{t} \quad (53)$$

where $t \in]0, N]$, $H(0) = 0$, $H'(t) = h(t) > 0$, $H''(t) = h'(t) < 0$ (by (45)) and

$H'''(t) = h''(t) < 0$ (by (46)) so that, according to the above Lemma, we have that $P(t)$ is

concavely decreasing. Of course, $P(0) = \lim_{t \rightarrow 0^+} \frac{H(t)}{t} = \lim_{t \rightarrow 0^+} h(t)$ by de l'Hôpital's rule. So

$P(0) = h(0) = 1$. Further $P(N) = \frac{H(N)}{N} = \frac{\int_0^N h(i) di}{N} = \frac{1}{N}$, since $\int_0^N h(i) di$ is the total

number of relevant documents, hence $1 = |\text{rel}|$. Note also that it follows from (48) that

$$P'(0) = \frac{h'(0)}{2} < 0 \text{ and } P'(N) = -\frac{1}{N^2} < 0.$$

Hence the graph of $P(t)$ can be depicted as in Fig. 19.

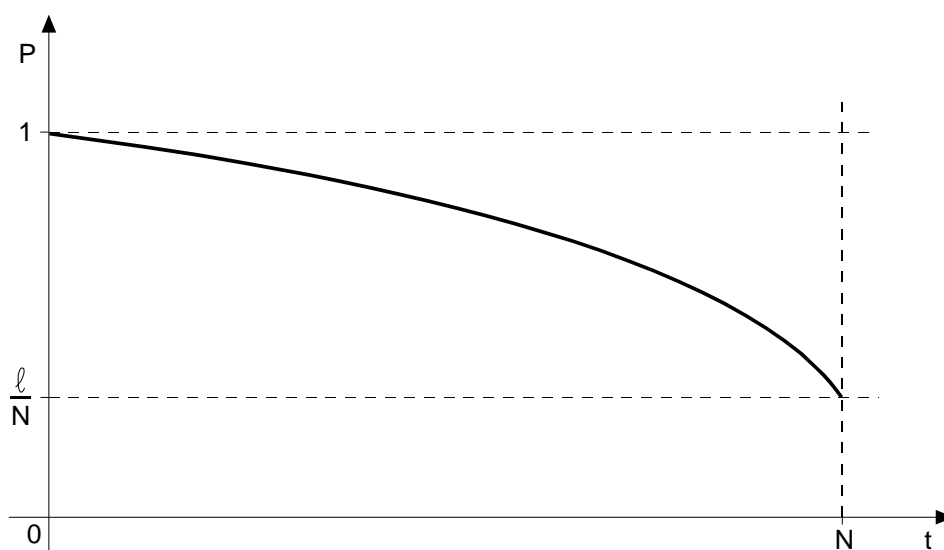


Fig. 19. Graph of $P(t)$ in case of normal retrieval.

For $R(t)$ we have, based on (5) and (8) that

$$R(t) = \frac{H(t)}{1} \quad (54)$$

so that

$$R'(t) = \frac{h(t)}{1} > 0$$

and

$$R''(t) = \frac{h'(t)}{1} < 0$$

so that R is concavely increasing. We even have that

$$R'''(t) = \frac{h''(t)}{1} < 0$$

which implies the shape in Fig. 20. Fig. 20 already appeared in Egghe (1992) without the

proof that $R''' < 0$. Note also that $R(0) = 0$ and $R(N) = \frac{\int_0^N h(i) di}{1} = 1$. Furthermore

$R'(0) = \frac{1}{1}$ (since $h(0) = 1$) and $R'(N) = 0$ (since $h(N) = 0$).

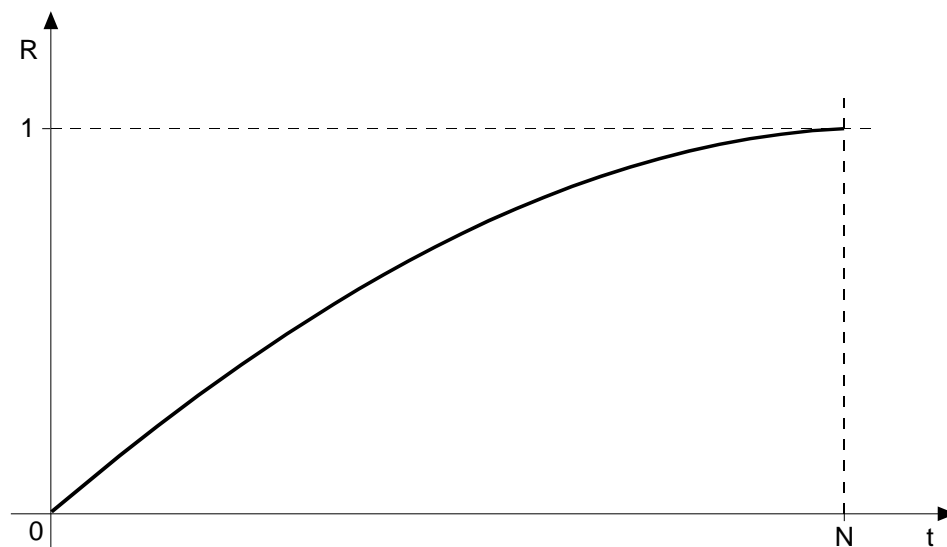


Fig. 20. Graph of $R(t)$ in case of normal retrieval.

For $F(t)$ we have, by (9),

$$F(t) = \frac{1}{N-1} \int_0^t (1-h(i)) di \quad (55)$$

so that

$$F'(t) = \frac{1}{N-1} (1-h(t)) > 0$$

$$F''(t) = -\frac{h'(t)}{N-1} > 0$$

by the assumptions (45) and (46) on h and the fact that $h(0) = 1$, $h(N) = 0$. Hence F is convexly increasing. Note that $F(0) = 0$, $F(N) = 1$ (since $\int_0^N h(i) di = 1$). Furthermore

$F'(0) = 0$, $F'(N) = \frac{1}{N-1}$. We have a graph as in Fig. 21.

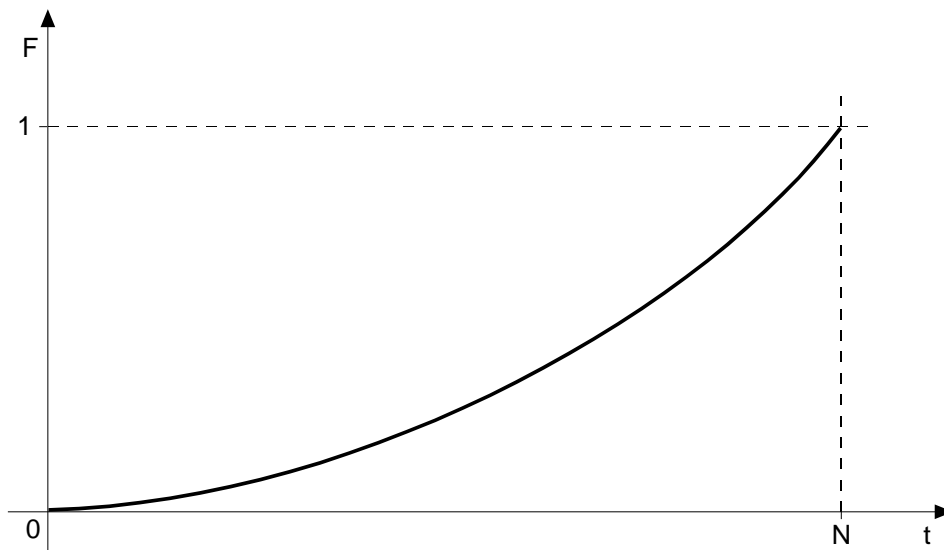


Fig. 21. Graph of $F(t)$ in case of normal retrieval.

Finally we study the graph of $M(t)$. From (10) we have

$$M(t) = \frac{\int_0^N h(i) di}{N - t} \quad (56)$$

since $1 = \int_0^N h(i) di$. Hence

$$M(t) = \frac{\int_{N-t}^N h(i) di}{N - t}$$

$$M(t) =: \frac{\varphi(N - t)}{N - t} \quad (57)$$

We have

$$M'(t) = - \frac{d}{d(N-t)} \frac{\varphi(N-t)}{N-t} \quad (58)$$

$$M''(t) = \frac{d^2}{dt^2} \frac{\phi(N-t)}{e^{N-t}} \quad (59)$$

Now

$$\begin{aligned} & \frac{d}{d(N-t)} \frac{\phi(N-t)}{e^{N-t}} \\ &= \frac{d}{dx} \frac{\phi(x)}{e^x} \end{aligned} \quad (60)$$

and

$$\begin{aligned} & \frac{d^2}{d(N-t)^2} \frac{\phi(N-t)}{e^{N-t}} \\ &= \frac{d^2}{dx^2} \frac{\phi(x)}{e^x} \end{aligned} \quad (61)$$

For (60) and (61) we apply Lemma VI.2.1. We have $\phi(0) = 0$ since $t = N$ then.

Furthermore, since

$$\phi(x) = \int_{N-x}^N h(i) di = - \int_N^{N-x} h(i) di \quad (62)$$

we have that $\phi'(x) = h(N-x) = h(t) > 0$, $\phi''(x) = -h'(N-x) = -h'(t) > 0$,

$\phi'''(x) = h''(N-x) = h''(t) < 0$. According to the Lemma we have that the function

$x \otimes \frac{\phi(x)}{x}$ is concavely increasing. Using (58) and (59) we have that $M(t)$ is concavely

decreasing. Note that $M(0) = \frac{1}{N}$ and that $M(N) = h(N) = 0$, using (56) and de l'Hôpital's rule. Finally

$$M'(t) = \frac{-h(t)(N-t) + \int_0^N h(i) di}{(N-t)^2}$$

so that $M'(0) = \frac{1-N}{N^2} < 0$ (since $h(0) = 1$ and $1 = \int_0^N h(i) di$) and, using de l'Hôpital's rule

$$\begin{aligned} M'(N) &= \lim_{t \rightarrow N} \frac{-h'(t)(N-t) + h(t) - h(t)}{-2(N-t)} \\ &= \frac{h'(N)}{2} < 0. \end{aligned}$$

We have the graph of $M(t)$ in Fig. 22

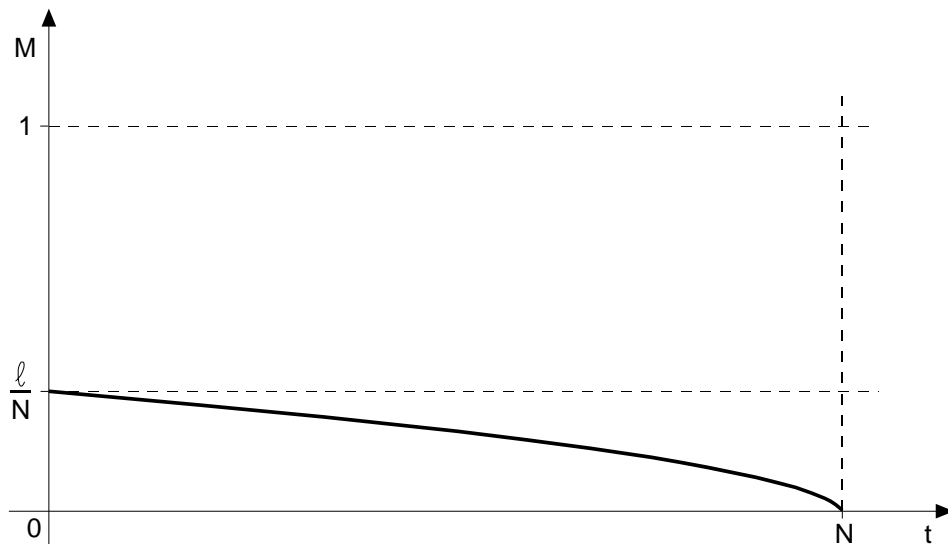


Fig. 22. Graph of $M(t)$ in case of normal retrieval.

This concludes the study of $t = |\text{ret}|$ -dependencies of the measures P , R , F and M . In the next subsection we will study the interrelationships between these measures.

VI.3 Interdependency results between the measures P , R , F and M

In the previous subsection we were able to determine the functions $t \rightarrow P(t)$, $t \rightarrow R(t)$, $t \rightarrow F(t)$ and $t \rightarrow M(t)$. We have seen that, using the assumptions on h , that all these functions have second derivatives and that they all are injective. This implies that the functional relations $R \rightarrow P(R)$, $F \rightarrow P(F)$, $M \rightarrow P(M)$, $F \rightarrow R(F)$, $M \rightarrow R(M)$ and $M \rightarrow F(M)$ (and of course also their inverses) are determined via these parameter equations in t . Hereby we use the following Theorem on functions determined by parameter equations, which can e.g. be found in Courant (1968) (p. 262-265) and Adams (1999) (p. 488-498). We give the proof since it is straightforward and short and for the sake of completeness.

Theorem VI.3.1: Let $t \rightarrow x(t)$ and $t \rightarrow y(t)$ be two functions (on a domain comprising an interval) such that $x''(t)$ and $y''(t)$ exist and such that x is injective in t . Then we have that the function $x \rightarrow f(x) = (y \circ x^{-1})(x)$ is twice differentiable and we have

$$f'(x) = \frac{y'(t)}{x'(t)} \quad (63)$$

$$f''(x) = \frac{x'(t)y''(t) - y'(t)x''(t)}{(x'(t))^3} \quad (64)$$

Proof:

$$\begin{aligned} f'(x) &= \frac{df(x)}{dx} \\ &= \frac{dy(t)}{dt} \frac{dx^{-1}(x)}{dx} \end{aligned}$$

where $t = x^{-1}(x)$ (i.e. $x = x(t)$). Hence (63) follows. Further, by (63)

$$f''(x) = \frac{d}{dx} \left(\frac{y'(t)}{x'(t)} \frac{dx^{-1}(x)}{dx} \right)$$

$$= \frac{x'(t)y''(t) - y'(t)x''(t)}{(x'(t))^2} \cdot \frac{1}{x'(t)}$$

whence (64). □

We will now use the above Theorem to determine the inter measures relationships. Let us start with $R \otimes P(R)$. We have $P'(t) < 0$, $P''(t) < 0$, $R'(t) > 0$, $R''(t) < 0$. Hence (63) implies that $P'(R) < 0$ and (64) implies that $P''(R) < 0$. Hence $R \otimes P(R)$ is a concavely decreasing function. Figs. 19 and 20 also show that $P(R = 0) = 1$ and $P(R = 1) = \frac{1}{N}$. Also,

$$\text{by (63), } P'(R = 0) = \frac{P'(t = 0)}{R'(t = 0)} = \frac{\frac{h'(0)}{2}}{\frac{1}{1}} = \frac{1h'(0)}{2} < 0 \text{ and}$$

$$P'(R = 1) = \frac{P'(t = N)}{R'(t = N)} = \frac{-\frac{1}{N^2}}{0} = -\infty \text{ (from the previous subsection). Hence the graph}$$

$R \otimes P(R)$ can be depicted as in Fig. 23 (it appeared already in Egghe (1992) – see also Gebhardt (1975)).

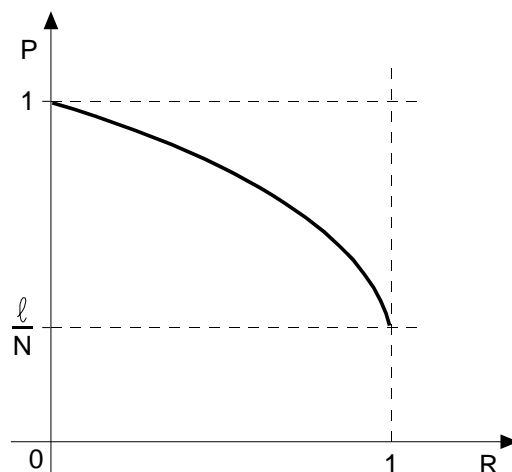


Fig. 23. Graph of $R \otimes P(R)$ in case of normal retrieval.

The next graph that we study is $F \otimes P(F)$. We have $P'(t) < 0$, $P''(t) < 0$, $F'(t) > 0$, $F''(t) > 0$. Hence $P'(F) < 0$ by (63). However (64) does not give a clear sign for $P''(F)$ since we have the subtraction of two negative numbers: in the next subsection we will present an example showing that $F \otimes P(F)$ is neither convex nor concave (in general). So, we can only conclude here that $P(F)$ is a decreasing function of F .

Now we study the function $M \otimes P(M)$. We have $P'(t) < 0$, $P''(t) < 0$, $M'(t) < 0$, $M''(t) < 0$. Hence (63) implies that $P'(M) > 0$ but (64) does not give a clear sign for $P''(M)$ since we have the subtraction (in the nominator) of two positive numbers: in the next subsection we will present an example showing that $M \otimes P(M)$ is neither convex nor concave (in general). So, we can only conclude that $P(M)$ is an increasing function of M .

Now we consider the function $F \otimes R(F)$. We have $R'(t) > 0$, $R''(t) < 0$, $F'(t) > 0$, $F''(t) > 0$. Hence (63) implies that $R'(F) > 0$. Now (64) gives that $R''(F) < 0$ so that the function $F \otimes R(F)$ is a concavely increasing function of F . Figs. 20 and 21 also show that

$R(F=0)=0$ and $R(F=1)=1$. Also, by (63), $R'(F=0)=\frac{R'(t=0)}{F'(t=0)}=\frac{1}{0}=+\infty$ and

$R'(F=1)=\frac{R'(t=N)}{F'(t=N)}=0$ (from the previous section). Hence the graph $F \otimes R(F)$ can be

depicted as in Fig. 24.

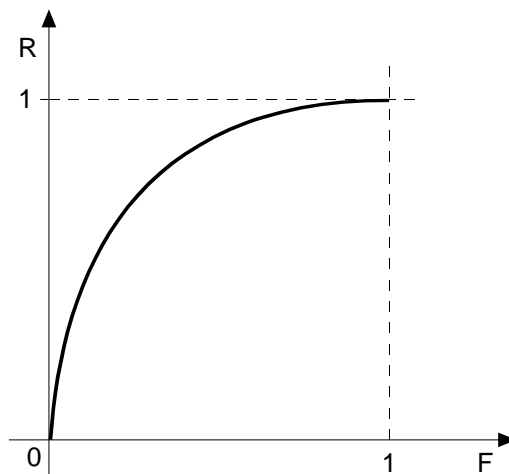


Fig. 24. Graph of $F \otimes R(F)$ in case of normal retrieval.

Next we consider the function $M \otimes R(M)$. We have $R'(t) > 0$, $R''(t) < 0$, $M'(t) < 0$,

$M''(t) < 0$. Hence (63) implies that $R'(M) < 0$. Also (64) gives a decision: $R''(M) < 0$ so

that the function $M \otimes R(M)$ is a concavely decreasing function of M . Figs. 20 and 22 also

show that $R(M=0)=1$, $R''_{CM} = \frac{1 \cdot \ddot{\theta}}{N \cdot \ddot{\theta}} = 0$. Also, by (63), $R'(M=0) = \frac{1}{\frac{h'(N)}{2}} = \frac{2}{h'(N)} < 0$

and $R''_{CM} = \frac{1 \cdot \ddot{\theta}}{N \cdot \ddot{\theta}} = \frac{1}{1-N} = \frac{N^2}{1(1-N)} < 0$, using results of the previous section. Hence the

graph of $M \otimes R(M)$ can be depicted as in Fig. 25.

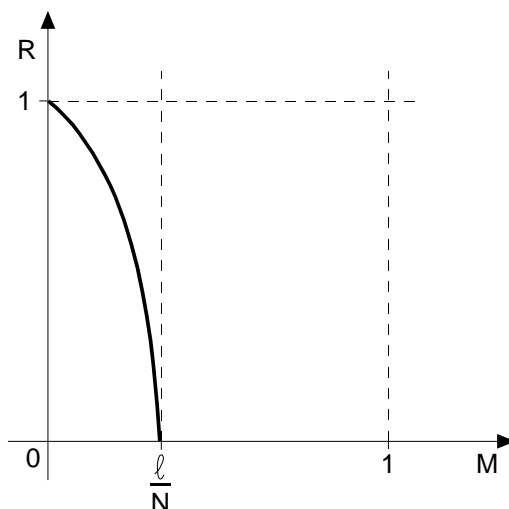


Fig. 25. Graph of $M \otimes R(M)$ in case of normal retrieval.

We, finally, study the case of the function $M \otimes F(M)$. We have $F'(t) > 0$, $F''(t) > 0$, $M'(t) < 0$, $M''(t) < 0$. Hence $F'(M) < 0$ by (63). The equation (64) yields (in the nominator) a difference of two negative numbers, hence not a clear sign for $F''(M)$: in the next subsection we will present an example showing that $F = F(M)$ is neither convex nor concave (in general). So we can only conclude that $F(M)$ is a decreasing function of M .

Note: These models are somewhat related to the one studied in Robertson (1975). There one defines 3 types of variables that can change the outcome of an IR process (comparable with our variable t). In this connection one defines a type C variable as one leading to $\frac{d^2R}{dF^2} < 0$. Our model hence shows that our variable t is of type C, in Robertson's terminology.

This completes our study of the interrelations of the measures P , R , F and M in the case of normal retrieval. In the next subsection we present the promised counterexamples showing that $F \otimes P(F)$, $M \otimes P(M)$ and $M \otimes F(M)$ are (in general) neither convex nor concave. We will use the same examples to illustrate that the function $R \otimes R(P)$ is concavely decreasing, that the function $F \otimes F(R)$ is concavely increasing and that the function $M \otimes R(M)$ is concavely decreasing, as generally shown above.

VI.4 Counterexamples to concavity and convexity and illustration of the obtained results

We will take $N = 8$ and we will present four tables: of $P(t)$ (concavely decreasing by Subsection VI.2), of $R(t)$ (concavely increasing by Subsection VI.2), of $F(t)$ (convexly increasing by Subsection VI.2) and of $M(t)$ (concavely decreasing by Subsection VI.2), for $t = 0, 1, 2, 3, 4, 5, 6, 7, 8$. We start from $t = 0$ just to compare with the theoretically obtained curves for $t \in [0, N]$. For visual purposes we will, in fact, also connect the obtained discrete points. Alternatively one could, of course, denote the points as $t = 1, 2, \dots, 9$. These functions will be combined, two by two, to yield graphs $R \otimes P(R)$, $F \otimes P(F)$, $M \otimes P(M)$, $F \otimes R(F)$, $M \otimes R(M)$ and $M \otimes F(M)$ confirming the findings of the previous subsection. We invoke a result of Egghe (2005) that expresses that any two numbers in $]0, 1[\cap \mathbb{Q}$ (i.e. any two rational numbers in $]0, 1[$) are values of any selection of two numbers from the set $\{P, R, F, M\}$ of an existing IR system. So, the examples do not only represent mathematical properties of the above mentioned functions but they also represent real IR systems (two by two).

Table 1. Example of functions $P(t), R(t), F(t), M(t)$ for
 $t = 0, 1, 2, 3, 4, 5, 6, 7, 8$.

t	P(t)	R(t)	F(t)	M(t)
0	1	0	0	0.1
1	1	0.2	0	0.095
2	0.95	0.4	0.05	0.09
3	0.9	0.6	0.1	0.085
4	0.8	0.7	0.2	0.08
5	0.7	0.8	0.4	0.07
6	0.5	0.9	0.6	0.05
7	0.3	0.95	0.8	0.03
8	0.1	1	1	0

Note that here $\frac{1}{N} = P(8) = 0.1$. One can check that $P(t)$ is concavely decreasing, $R(t)$ is concavely increasing, $F(t)$ is convexly increasing and $M(t)$ is concavely decreasing.

This yields the curves for $R \otimes P(R)$, $F \otimes P(F)$, $M \otimes P(M)$, $F \otimes R(F)$, $M \otimes R(M)$ and $M \otimes F(M)$: one can check that $R \otimes P(R)$ and $M \otimes R(M)$ are concavely decreasing and that $F \otimes R(F)$ is concavely increasing (illustrating the above theory) while the curves of $F \otimes P(F)$, $M \otimes P(M)$ and $M \otimes F(M)$ are non convex and non concave, showing that the above theory cannot be improved or extended.

VII. Conclusions and open problems

This paper “catalogues” the relations between the measures precision, recall, fallout and miss in function of $t = |\text{ret}| =$ the number of retrieved documents. In terms of the retrieval density function h we give exact definitions of perverse, perfect, random and normal retrieval. We re-find some results in Egghe (1992) and Buckland and Gey (1994) on recall and precision, we give exact proofs and extend the study to the relations of fallout and miss in function of t . We also present the interrelations graphs (for all types of retrieval) between precision, recall, fallout and miss, based on these relations with t . In case of normal retrieval we prove that precision is a concavely decreasing function of recall (this is a reprove), that recall is a concavely increasing function of fallout and that recall is a concavely decreasing function of miss. We also show by example that the relations between precision and fallout, precision and miss and fallout and miss are, in general, neither convex nor concave.

We leave it as an open problem to characterise the different retrieval types (perverse, perfect, random and normal) in terms of “domains” that are taken by the quadruple (P, R, F, M) on the universal \mathbb{R} surface

$$\frac{P}{1-P} + \frac{1-R}{R} + \frac{F}{1-F} + \frac{1-M}{M} = 1 \quad (65)$$

which was proved in Egghe (2004, 2005) to be the necessary and sufficient condition for a quadruple in (P, R, F, M) to represent the precision, recall, fallout and miss of an IR system.

Equation (65) can be proved, based on the definitions of P, R, F and M (see Egghe (2004)). Using also Egghe (2005) we see that every point of this surface is a “real” (P,R,F,M) point meaning that it represents a case of an existing precision, recall, fallout and miss value of an IR system. Adopting certain limitations on the retrieval system (e.g. perverse, perfect, random or normal retrieval) might limit the range for the possible points (P,R,F,M) on this surface, which we leave here as an open problem.

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