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The Theorem of Fellman and Jakobsson: a new Proof and dual Theory

by

L. Egghe

Universiteit Hasselt (UHasselt), Agoralaan, B-3590 Diepenbeek, Belgium[1]
and
Universiteit Antwerpen (UA), Campus Drie Eiken, Universiteitsplein 1, B-2610 Wilrijk, Belgium

leo.egghe@uhasselt.be

ABSTRACT

The theorem of Fellman and Jakobsson of 1976 deals with transformations φ of the rank-frequency function g and with their Lorenz curves $L(\varphi \circ g)$ and $L(g)$. It states (briefly) that $L(\varphi \circ g)$ is monotonous (in terms of the Lorenz dominance order) with $\frac{\varphi(x)}{x}$. In this paper we present a new, elementary proof of this important result.

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[1] Permanent address

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Acknowledgement. The author is indebted to prof. Dr. J. Fellman for pointing out the exact "history" of his theorem – which is also correctly described in Lambert (2001).

The main part of the paper is devoted to the dual transformation $g\circ\psi^{-1}$, where ψ is a transformation acting on source densities (instead of item densities as is the case with the transformation φ). We prove that, if the average number of items per source is changed after application of the transformation ψ, we always have that $L(g\circ\psi)$ and $L(g)$ intersect in an interior point of $[0,1]$, i.e. the theorem of Fellman and Jakobsson is not true for the dual transformation. We also show that this includes all convex and concave transformations. We also show that all linear transformations ψ yield the same Lorenz curve.

We also indicate the importance of both transformations φ and ψ in informetrics.

I. Introduction

Econometric as well as informetric production processes are, classically, described by sources producing items ands by two, equivalent functions: the size-frequency function f and the rank-frequency function g:

$$f:[a,\rho_m]\to\mathbb{R}^+ \tag{1}$$

such that, for all $j\in[a,\rho_m]$

$$\int_j^{\rho_m} f(j)dj \tag{2}$$

is the number of sources with production density larger than or equal to j and

$$g:[0,T]\to\mathbb{R}^+ \tag{3}$$

is such that for all $r\in[0,T]$ (T = total number of sources), $j=g(r)$ denotes the item density in the source density r. By their very definition we have that, for all $j\in[a,\rho_m]$, $r\in[0,T]$:

$$r = g^{-1}(j) = \int_j^{\rho_m} f(j)\,dj \tag{4}$$

We only work with decreasing functions f and g: for f this means, in practise, that the higher the source productivity, the less sources we have, while for g this means that we arranged the sources in decreasing order of productivity.

Such a production system can be described in terms of its "inequality" properties, e.g. using Lorenz curves (Lorenz (1905)). To answer the question "How unequal is the production of the sources?" we can construct the Lorenz curve of g, denoted $L(g)$, defined on $[0,1]$ as follows: using the bijective relation between $r \in [0,T]$ and $y \in [0,1]$ via $r = yT$ we define

$$L(g)(y) = \frac{\int_0^{yT} g(r)\,dr}{\int_0^{T} g(r)\,dr} \tag{5}$$

, i.e. the fraction of the items contained in a fraction y of the sources, cf. Egghe (2005a,b) (there we increased the ranks with 1 for reasons that are of no importance here, but, of course, leading to the same Lorenz curve $L(g)$). In terms of the size-frequency function f we have that

$$L(g)(y) = \frac{\int_j^{\rho_m} j' f(j')\,dj'}{\int_a^{\rho_m} j' f(j')\,dj'} \tag{6}$$

where

$$y = \frac{\int_j^{\rho_m} f(j')\,dj'}{\int_a^{\rho_m} f(j')\,dj'} \tag{7}$$

Indeed, by (4), (7) gives the fraction $y \in [0,1]$ of the sources with item density j or higher and, by definition of f, (6) gives the corresponding fraction of the items, hence $L(g)(y)$. It is hence clear that $L(g)$ is the same as the Lorenz curve introduced e.g. in Lambert (2001) in econometrics except for one point: in the above approach we use the sources in decreasing rank of production while in Lambert (2001) one uses (see e.g. formula (2.11)) the increasing order of sources with respect to their production: this leads to completely equivalent Lorenz curves: in our case concavely increasing curves $L(g)$, connecting $(0,0)$ with $(1,1)$ while in the latter case we have convexly increasing curves connecting $(0,0)$ with $(1,1)$. The former curves are all situated above the line of equality (the straight line connecting $(0,0)$ with $(1,1)$) while the latter curves are all situated below this line of equality. Our concave curves have a bijective relation with the convex ones by applying two mirroring transformations: one over the line of equality and one over the straight line which is perpendicular to this line, as is well-known and trivial to see. Because of this, the two theories are completely equivalent.

Inequality is now described in terms of increasing or decreasing Lorenz curves. Let us have two situations as above, described by two rank-frequency functions g and g^*. We say that the $*$-situation is more unequal (more concentrated) than the other one if

$$L(g^*) > L(g) \tag{8}$$

as functions, i.e.

$$L(g^*)(y) \geq L(g)(y) \tag{9}$$

for all $y \in]0,1[$ where we also have > in at least 1 point (hence in an infinite number of points). Note that higher concentration is described via higher Lorenz curves (which we prefer); in the other (equivalent) definition (as e.g. in Lambert (2001)), higher concentration is described via lower Lorenz curves.

It is well-known (see e.g. Egghe and Rousseau (1990a), Lambert (2001)) that Lorenz dominance (8) describes the overall inequality of a system (e.g. income inequality in a company or country, publication or citation, including well-known “good” principles such as the transfer principle or the principle of nominal increase (see also Egghe and Rousseau (1990b, 1991)).

Lorenz concentration theory has become a basic econometric theory and has also lead to important applications in e.g. informetrics (measuring inequality in information production – see the above references) and biomathematics (measuring diversity, the opposite of concentration – see e.g. Rousseau and Van Hecke (1999)). Lorenz theory, which development started in the beginning of the 20th Century, experienced a continuous interest since then. A (rather late) development came in 1976 when, independently, two econometricians: Fellman and Jakobsson studied the following problem: what is the influence of applying transformations to (in our terminology and notation) the rank-frequency function g to the Lorenz curve? In other words: if we transform a function g into $g^* = \varphi \circ g$ what is the relation between $L(g)$ and $L(g^*) = L(\varphi \circ g)$? This has applications in econometrics (see e.g. Lambert (2001)) where φ is the function that calculates the taxes based on the income $g(r)$. The answer to the above problem is the so-called theorem of Fellman and Jakobsson which reads as follows:

Theorem (Fellman (1976), Jakobsson (1976)):
In the above notation we have

$$L(\varphi \circ g) \geq L(g) \text{ iff } \frac{\varphi(x)}{x} \text{ increases}$$

$$L(\varphi \circ g) = L(g) \text{ iff } \frac{\varphi(x)}{x} \text{ is constant}$$

$$L(\varphi \circ g) \leq L(g) \text{ iff } \frac{\varphi(x)}{x} \text{ decreases}$$

and strict inequalities apply if the monotonicity of $\frac{\varphi(x)}{x}$ is strict.

As said above, Fellman (only proving the – most important – “if” part) and Jakobsson proved this theorem independently in 1976. One year later, Kakwani also presented this result with a new, elegant, proof (see Kakwani (1977) but also Lambert (2001) – note again that here the

Lorenz order is reversed as explained above). In the next section we will present a new, short, and direct proof of this important theorem.

The above theorem has also been applied in informetrics, where one talks about positive reinforcement (of the source production) by applying transformation φ (e.g. such that $\varphi(x) \geq x$ for all x) – see Rousseau (1992), or where one makes the link with linear, three-dimensional informetrics, see Egghe (2004). Still in informetrics, the transformation φ can explain dynamical aspects of information production processes or demographical evolutions such as the evolution to which sources can only have a large number of items as e.g. is the case with database sizes or village/city or country sizes – see Egghe and Rousseau (2005).

While studying the above informetric applications, the present author came across the following substantial extension of the transformation φ: the transformation φ can be considered as a transformation of the item densities. It is, however, equally important to allow for a transformation of the source densities. In other words, one should not only allow for a (differentiable) transformation φ, working on $j = g(r)$:

$$\begin{aligned} \varphi : [a, \rho_m] &\rightarrow \left[a^*, \rho_m^*\right] \\ j &\rightarrow j^* = \varphi(j) \end{aligned} \tag{10}$$

but also one should consider a possible (differentiable) transformation

$$\begin{aligned} \psi : [0, T] &\rightarrow \left[0, T^*\right] \\ r &\rightarrow r^* = \psi(r) \end{aligned} \tag{11}$$

acting on source densities $r \in [0, T]$. Both transformations together yield the general equation

$$g^*\left(r^*\right) = g^*(\psi(r)) = \varphi(g(r)) \tag{12}$$

for all $r \in [0, T]$, yielding

$$g^*: [0, T^*] \to \mathbb{R}^+ \qquad (13)$$
$$r^* \to g^*(r^*)$$

Transformations as in (12) have been used in Egghe (2006) to model the dynamics of information production processes such as networks (with nodes as sources and with in- or out-links as items) in which not only links can be added or disappear (transformation φ) but where even sources can be added or destroyed (transformation ψ). As proved in Egghe (2006), the size-frequency function f^*, equivalent with g^* above is given by the following theorem:

Theorem (Egghe (2006)):

If φ and ψ are differentiable and $\varphi' \neq 0$, then for all $j \in [a, \rho_m]$ and $j^* \in [a^*, \rho_m^*]$ such that $j^* = \varphi(j)$, we have

$$f^*(j^*) = f(j)\frac{\psi'(g^{-1}(j))}{\varphi'(j)} \qquad (14)$$

Proof:

We present the proof since it is short and for the sake of completeness. The defining relation (4) (and the similar one for the transformed system, indicated with $*$) yield:

$$r = g^{-1}(j) = \int_j^{\rho_m} f(k)\,dk \qquad (15)$$

$$r^* = g^{*-1}(j^*) = \int_{j^*}^{\rho_m^*} f^*(k^*)\,dk^* \qquad (16)$$

hence, by (10) and (11),

$$\psi\left(\int_j^{\rho_m} f(k)\,dk\right) = \int_{\varphi(j)}^{\rho_m^*} f^*(k^*)\,dk^* \qquad (17)$$

Differentiating both sides of (17) with respect to j yields:

$$-\psi'\left(\int_j^{\rho_m} f(k)dk\right) f(j) = -f^*(\varphi(j))\varphi'(j)$$

whence (14), using (10) and (15). W

Clearly, equation (12) is the most general formalism for the dynamics of information production processes but we are convinced that it must have applications in other fields (such as econometrics and biometrics) as well: all these fields are also faced with a variable number of sources (transformation ψ) as well as of items (transformation φ). The theorem of Fellman and Jakobsson clearly only deals with transformation φ: in the next section we will present a new, very simple, proof of this theorem. This result could be considered as a theory of (12) where ψ is taken as the identical transformation: $\psi = \mathrm{Id}$.

Therefore, in Section III, we will study the general relation (12) but now taking $\varphi = \mathrm{Id}$, hence focusing completely on the transformation ψ. In Section III we will show that the theorem of Fellman and Jakobsson does not hold for transformation ψ: we will show that, if $\mu \neq \mu^*$ ($\mu =$ the average number of items per source before the transformation, μ^* = the average number of items per source after the transformation), we always have that the Lorenz curves $L(g)$ and $L(g^*) = L(g \circ \psi^{-1})$ intersect in a point in $]0,1[$. We also show that all strictly convex and all strictly concave transformations ψ satisfy $\mu \neq \mu^*$ and hence yield intersecting (on $]0,1[$) Lorenz curves $L(g)$ and $L(g^*)$. While leaving the limiting case $\mu = \mu^*$ open for a complete study we also show that, for linear transformations ψ:

$$\psi : [0,T] \to [0,T^*] \qquad (18)$$
$$r = yT \to r^* = yT^*$$

$(T^* \geq T \text{ or } T^* \leq T)$, we always have that $L(g^*) = L(g)$. This result can then be combined with the theorem of Fellman and Jakobsson to yield a full treatment of the case (12), where ψ is as in (18).

II. A new, simple, proof of the Theorem of Fellman and Jakobsson

Although the theorem of Fellman and Jakobsson is a very important result, it can be proved in a very simple way as the next proof shows. The new proof is given for the "if" part (the "only if" part is a simple consequence of it as indicated in Lambert (2001) and is not modified here).

Proof:

Since the transformation φ only works on the densities $j \in [1,\rho_m]$ we have that the rank densities r are unchanged (in our notation: $\psi = \mathrm{Id}$). Hence, for every $y \in [0,1]$, by (5):

$$L(g)(y) = \frac{\int_0^{yT} g(r)dr}{\int_0^T g(r)dr} \tag{19}$$

$$L(\varphi^\circ g)(y) = \frac{\int_0^{yT} (\varphi^\circ g)(r)dr}{\int_0^T (\varphi^\circ g)(r)dr} \tag{20}$$

Hence

$$L(\varphi^\circ g)(y) - L(g)(y) = \frac{1}{N}\left[\int_{r=0}^{r=yT}\int_{r'=0}^{r'=T} \varphi(g)(r)g(r')drdr' - \int_{r=0}^{r=yT}\int_{r'=0}^{r'=T} g(r)\varphi(g)(r')drdr'\right]$$

, where

$$N = \int_0^T g(r)dr \int_0^T \varphi(g)(r)dr > 0$$

So

$$L(\varphi\circ g)(y) - L(g)(y)$$

$$= \frac{1}{N}\left[\int_{r=0}^{yT}\int_{r'=0}^{yT} \varphi(g)(r)g(r')\,dr\,dr' + \int_{r=0}^{yT}\int_{r'=yT}^{T} \varphi(g)(r)g(r')\,dr\,dr'\right.$$

$$\left. - \int_{r=0}^{yT}\int_{r'=0}^{yT} g(r)\varphi(g)(r')\,dr\,dr' - \int_{r=0}^{yT}\int_{r'=yT}^{T} g(r)\varphi(g)(r')\,dr\,dr'\right]$$

$$= \frac{1}{N}\left[\int_{r=0}^{yT}\int_{r'=yT}^{T} g(r)g(r')\left(\frac{\varphi(g)(r)}{g(r)} - \frac{\varphi(g)(r')}{g(r')}\right)dr\,dr'\right]$$

If $\frac{\varphi(x)}{x}$ (strictly) increases, we have, since g strictly decreases, that, since $r \in [0, yT]$ and $r' \in [yT, T]$

$$\frac{\varphi(g)(r)}{g(r)} > \frac{\varphi(g)(r')}{g(r')}$$

and hence

$$L(\varphi\circ g) > L(g).$$

The other assertions of the theorem of Fellman and Jakobsson are proved in the same way. W

III. Concentration aspects of the transformation $g \to g^* = g\circ\psi$

We repeat that the Fellman-Jakobsson theorem deals with the concentration properties of the transformation

$$g \to \varphi\circ g = g^* \tag{21}$$

where $\psi = \mathrm{Id}$ in the relation (12)

$$g^*\left(r^*\right) = g^*\left(\psi(r)\right) = \varphi\left(g(r)\right) \tag{22}$$

for $r \in [0, T]$.

Now we take $\varphi = \mathrm{Id}$ in the above relation (22), yielding

$$g^*\left(r^*\right) = g^*\left(\psi(r)\right) = g(r) \tag{23}$$

or the transformation

$$g \to g\circ\psi^{-1} = g^* \tag{24}$$

Note that, since $\varphi = \mathrm{Id}$, $a = a^*$, $\rho_m = \rho_m^*$ (see (10)) and we, naturally, assume (see (11)): $\psi(0) = 0$, $\psi(T) = T^*$.

We can prove the following theorem.

Theorem III.1:

Let μ (respectively μ^*) denote the average number of items per source in the original system (respectively in the transformed system) and let $\mu \neq \mu^*$. Then $L(g)$ and $L(g^*) = L(g \circ \psi^{-1})$ always intersect in a point $y \in]0,1[$, i.e. the theorem of Fellman-Jakobsson is not true for the transformation (24).

Proof:

By (5) we have, for every $y \in [0,1]$:

$$L(g)(y) = \frac{\int_0^{yT} g(r)\,dr}{\int_0^{T} g(r)\,dr} \tag{25}$$

and similarly for the transformed system:

$$L(g^*)(y) = \frac{\int_0^{yT^*} g^*(r^*)\,dr^*}{\int_0^{T^*} g^*(r^*)\,dr^*} \tag{26}$$

We have:

$$L'(g)(y) = \frac{Tg(yT)}{\int_0^{T} g(r)\,dr} = \frac{g(yT)}{\mu} \tag{27}$$

since

$$A = \int_0^{T} g(r)\,dr \tag{28}$$

is the total number of items and T is the total number of sources. Likewise, for every $y \in [0,1]$:

$$L'\left(g^*\right)(y) = \frac{T^* g^*\left(yT^*\right)}{A^*} = \frac{g^*\left(yT^*\right)}{\mu^*} \tag{29}$$

Now note that, by (23) and (14):

$$L'(g)(0) = \frac{g(0)}{\mu} = \frac{\rho_m}{\mu} \tag{30}$$

$$L'(g)(1) = \frac{g(T)}{\mu} = \frac{a}{\mu} \tag{31}$$

Likewise, for the transformed system:

$$L'\left(g^*\right)(0) = \frac{g^*(0)}{\mu^*} = \frac{\rho_m}{\mu^*} \tag{32}$$

and

$$L'\left(g^*\right)(1) = \frac{g^*\left(T^*\right)}{\mu^*} = \frac{a}{\mu^*} \tag{33}$$

since $a = a^*$, $\rho_m = \rho_m^*$ (since $\varphi = \mathrm{Id}$).

<u>(i) Let $\mu > \mu^*$</u>

Then

$$L'(g)(0) < L'\left(g^*\right)(0) \tag{34}$$

and

$$L'(g)(1) < L'\left(g^*\right)(1) \tag{35}$$

This implies that $L(g)$ and $L(g^*)$ intersect in a point $y \in]0,1[$, by the Lemma below.

<u>(ii) Let $\mu < \mu^*$</u>

Then

$$L'(g)(0) > L'(g^*)(0) \tag{36}$$

and

$$L'(g)(1) > L'(g^*)(1) \tag{37}$$

This also implies that $L(g)$ and $L(g^*)$ intersect in a point $y \in]0,1[$, by the Lemma below.

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<u>Lemma III.2</u>:

Let L_1 and L_2 be two Lorenz curves. Suppose that

$$L_1'(0) < L_2'(0) \tag{38}$$

and

$$L_1'(1) < L_2'(1) \tag{39}$$

Then we have that L_1 and L_2 intersect in a point $y \in]0,1[$ and the number of intersections is odd.

<u>Proof</u>:

Inequality (38) implies that there exists a right neighbourhood of 0 such that on this interval: $L_1 < L_2$. Inequality (39) implies that there exists a left neighbourhood of 1 such that on this interval: $L_1 > L_2$. This implies that L_1 and L_2 intersect at least once and that the number of intersections (on $]0,1[$) cannot be even. See Fig. 1 for an illustration of this proof. This concludes the proof of this Lemma. W

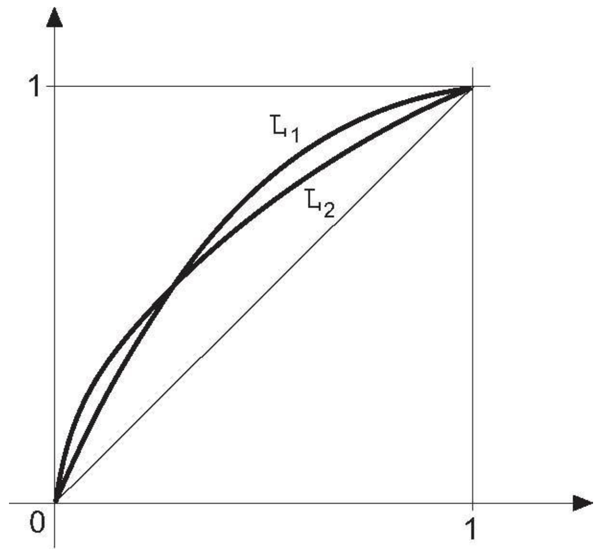

Fig. 1. Illustration of Lemma III.2.

Theorem III.1 shows that, for “almost all” transformations ψ the Theorem of Fellman and Jakobsson is not valid. In the next theorem we show that all convex and all concave functions ψ satisfy the condition $\mu\,^1\,\mu^*$ of Theorem III.1.

<u>**Theorem III.3**</u>:

(i) Let ψ be a strictly convex function. Then

$$\mu > \mu^* . \tag{40}$$

(ii) Let ψ be a strictly concave function. Then

$$\mu < \mu^* . \tag{41}$$

<u>Proof</u>:

It is intuitively clear (but see Proposition II.1.2.1 in Egghe (2005) for an exact proof – the extension to a general lower bound a being trivial) that

$$A = \int_a^{\rho_m} j f(j) dj \tag{42}$$

is the total number of items. It also follows directly from (4) that

$$T = \int_a^{\rho_m} f(j) dj \tag{43}$$

is the total number of sources. Hence the average number of items per source (μ) is given by

$$\mu = \frac{A}{T} = \frac{\int_a^{\rho_m} j f(j) dj}{\int_a^{\rho_m} f(j) dj} \tag{44}$$

Likewise, for the transformed system (24) we have (since $\varphi = \mathrm{Id}$)

$$\mu^* = \frac{A^*}{T^*} = \frac{\int_a^{\rho_m} j f^*(j) dj}{\int_a^{\rho_m} f^*(j) dj} \tag{45}$$

By (14) and using that $\varphi'(j) = 1$ for all $j \in [1, \rho_m]$, we have

$$\mu^* = \frac{\int_a^{\rho_m} j f(j) \psi'\left(g^{-1}(j)\right) dj}{\int_a^{\rho_m} f(j) \psi'\left(g^{-1}(j)\right) dj} \tag{46}$$

We will have proved that $\mu > \mu^*$ if we have

$$\frac{\int_1^{\rho_m} j f(j)dj}{\int_1^{\rho_m} f(j)dj} > \frac{\int_1^{\rho_m} j f(j)\psi'\left(g^{-1}(j)\right)dj}{\int_1^{\rho_m} f(j)\psi'\left(g^{-1}(j)\right)dj}$$

or

$$\int_{j=1}^{j=\rho_m}\int_{j'=1}^{j'=\rho_m} j f(j) f(j')\psi'\left(g^{-1}(j')\right)dj dj' > \int_{j=1}^{j=\rho_m}\int_{j'=1}^{j'=\rho_m} j f(j)\psi'\left(g^{-1}(j)\right) f(j')dj dj'$$

Which is equivalent with

$$\int_{j=1}^{j=\rho_m}\int_{j'=1}^{j'=\rho_m} j f(j) f(j')\left[\psi'\left(g^{-1}(j')\right)-\psi'\left(g^{-1}(j)\right)\right]dj dj' > 0 \qquad (47)$$

Since the integrand is 0 for $j = j'$ we have that (47) is equivalent with (integration is still over $(j, j') \in [1, \rho_m] x [1, \rho_m]$ and the same integrand as in (47) is used)

$$\iint_{\substack{(j,j')\\ j>j'}} + \iint_{\substack{(j,j')\\ j<j'}} > 0$$

Applying the change of notation $j \leftrightarrow j'$ in the second double integral we obtain the equivalent condition

$$\iint_{\substack{(j,j')\\ j>j'}} j f(j) f(j')\left[\psi'\left(g^{-1}(j')\right)-\psi'\left(g^{-1}(j)\right)\right]dj dj' + \iint_{\substack{(j,j')\\ j>j'}} j' f(j') f(j)\left[\psi'\left(g^{-1}(j)\right)-\psi'\left(g^{-1}(j')\right)\right]dj' dj > 0$$

which is, in turn, equivalent with

$$\iint_{\substack{(j,j')\\ j>j'}} (j-j') f(j) f(j')\left[\psi'\left(g^{-1}(j')\right)-\psi'\left(g^{-1}(j)\right)\right]dj dj' > 0 \qquad (48)$$

If we suppose ψ to be strictly convex we have that ψ' strictly increases, hence, since g (hence g^{-1}) strictly decreases, we have that $\psi' \circ g^{-1}$ strictly decreases. Consequently

$$\psi'\left(g^{-1}(j')\right) > \psi\left(g^{-1}(j)\right) \tag{49}$$

on the domain of integration in (48).

Inequality (49) now implies that (48) is true, hence $\mu > \mu^*$.

The proof that $\mu < \mu^*$ for a strictly concave function ψ follows the same lines. W

<u>Corollary III.4</u>:

(i) Let ψ be a strictly convex function. Then

$$L'(g)(0) < L'\left(g^*\right)(0)$$

and

$$L'(g)(1) < L'\left(g^*\right)(1)$$

and hence $L(g)$ and $L\left(g^*\right)$ intersect on a value $y \in]0,1[$.

(ii) Let ψ be a strictly concave function. Then

$$L'(g)(0) > L'\left(g^*\right)(0)$$

and

$$L'(g)(1) > L'\left(g^*\right)(1)$$

and hence $L(g)$ and $L(g^*)$ intersect on a value $y \in]0,1[$.

Proof:
The proof follows from equations (30), (31), (32) and (33), Theorem III.3 and Lemma III.2.
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The following two examples illustrate Corollary III.4.

Example III.5:
Let $\psi(x) = x^2$. Hence $\psi(0) = 0$. Let T^* be such that $T^* = \psi(T) = T^2$ and take $T = 3$, hence $T^* = 9$. Note that $\psi' > 0$ and that ψ is convex. Furthermore, $\psi^{-1}(x) = \sqrt{x}$. For the function g we will use the simple Zipf function with exponent (in the denominator) equal to one:

$$g(r) = \frac{A}{r+1} \tag{50}$$

$(r \in [0,T] = [0,3])$.

By (25) we have, for all $y \in [0,1]$:

$$L(g)(y) = \frac{\int_0^{3y} \frac{A}{r+1} dr}{\int_0^{3} \frac{A}{r+1} dr}$$

$$L(g)(y) = \frac{\ln(3y+1)}{\ln 4} \tag{51}$$

and by (26):

$$L(g^*)(y) = \frac{\int_0^{9y} g^*(r^*) dr^*}{\int_0^{9} g^*(r^*) dr^*}$$

$$L\left(g^*\right)(y)=\frac{\int_{\psi^{-1}(0)}^{\psi^{-1}(9y)} g^*\left(\psi(r)\right)\psi'(r)dr}{\int_0^3 g^*\left(\psi(r)\right)\psi'(r)dr}$$

$$=\frac{\int_0^{\sqrt{9y}} g(r)rdr}{\int_0^3 g(r)rdr}$$

by (23) and since $\psi'(x)=2x$. Hence

$$L\left(g^*\right)(y)=\frac{\int_0^{\sqrt{9y}}\left(1-\frac{1}{r+1}\right)dr}{\int_0^3\left(1-\frac{1}{r+1}\right)dr}$$

$$L\left(g^*\right)(y)=\frac{\sqrt{9y}-\ln\left(1+\sqrt{9y}\right)}{3-\ln 4} \qquad (52)$$

Note that $L(g)(0)=L\left(g^*\right)(0)=0$ and $L(g)(1)=L\left(g^*\right)(1)=1$. Furthermore

$$L(g)(0.01)=0.0213222<L\left(g^*\right)(0.01)=0.0233226$$

$$L(g)(0.9)=0.9437626>L\left(g^*\right)(0.9)=0.9289199$$

Hence $L(g)$ and $L\left(g^*\right)$ intersect on a value $y\in\,]0,1[$, which agrees with the above results (note that, first, $L(g)$ is below $L\left(g^*\right)$ and that, later, $L(g)$ is above $L\left(g^*\right)$ as predicted in Corollary III.4).

Example III.6:

Let $\psi(x) = \sqrt{x}$. Hence $\psi(0) = 0$. Let T^* be such that $T^* = \psi(T) = \sqrt{T}$ and take $T = 9$, hence $T^* = 3$. Note that $\psi' > 0$ and that ψ is concave. Furthermore, $\psi^{-1}(x) = x^2$. Using (50) again we have, for all $y \in [0,1]$:

$$L(g)(y) = \frac{\int_0^{9y} \frac{A}{r+1} dr}{\int_0^{9} \frac{A}{r+1} dr}$$

$$L(g)(y) = \frac{\ln(9y+1)}{\ln 10} \tag{53}$$

and

$$L(g^*)(y) = \frac{\int_0^{(3y)^2} \frac{1}{(r+1)2\sqrt{r}} dr}{\int_0^{9} \frac{1}{(r+1)2\sqrt{r}} dr}$$

since $\psi'(x) = \frac{1}{2\sqrt{x}}$.

But

$$\int \frac{dr}{(r+1)2\sqrt{r}} = \int \frac{d\sqrt{r}}{r+1}$$

$$= \int \frac{dx}{1+x^2} = \operatorname{Arc}\tan(x)$$

Hence

$$L\left(g^*\right)(y) = \frac{\operatorname{Arctan}\left[(3y)^2\right]}{\operatorname{Arctan} 9} \tag{54}$$

since $\operatorname{Arctan}(0) = 0$. Note again that $L(g)(0) = L\left(g^*\right)(0) = 0$ and $L(g)(1) = L\left(g^*\right)(1) = 1$. We now have

$$L(g)(0.1) = 0.2787536 > L\left(g^*\right)(0.1) = 0.0614723$$

$$L(g)(0.9) = 0.9590413 < L\left(g^*\right)(0.9) = 0.9824221$$

Again $L(g)$ and $L\left(g^*\right)$ intersect on a value $y \in \left]0,1\right[$. Note that, first, $L(g)$ is above $L\left(g^*\right)$ and that, later, $L(g)$ is below $L\left(g^*\right)$ as predicted in Corollary III.4.

We end this paper with the case $\mu = \mu^*$. We leave open the general study of this case but remark that the general linear transformation.

$$\psi : [0, T] \to \left[0, T^*\right] \tag{55}$$

$$r = yT \to \psi(r) = yT^*$$

($T^* < T$, $T^* = T$ or $T^* > T$ are allowed) satisfies $\mu = \mu^*$. Indeed, using (28) (for g as well as for g^*) we have

$$A = \int_0^T g(r)dr \tag{56}$$

$$A^* = \int_0^{T^*} g^*\left(r^*\right)dr^* \tag{57}$$

But

$$A^* = \int_0^{T^*} g^*(r^*)dr^*$$

$$= \int_0^{T} g(r)\psi'(r)dr \tag{58}$$

by (23) and the fact that $\psi(0) = 0$ and $\psi(T) = T^*$. But, by (55)

$$\psi(yT) = \psi(r) = yT^* = r\frac{T^*}{T},$$

hence

$$\psi'(r) = \frac{T^*}{T} \tag{59}$$

Formula (59) in (58) yields

$$A^* = \left(\int_0^{T} g(r)dr\right)\frac{T^*}{T}$$

whence, by (56),

$$A^* = A\frac{T^*}{T}$$

hence

$$\mu^* = \frac{A^*}{T^*} = \frac{A}{T} = \mu.$$

We have the following proposition:

<u>Proposition III.7</u>:

For ψ as above we have that (with $\varphi = \mathrm{Id}$):

$$L(g) = L(g^*) = L(g \circ \psi^{-1}) \tag{60}$$

<u>Proof</u>:

By (23) and (26) we generally have, for every $y \in [0,1]$:

$$L(g \circ \psi^{-1})(y) = L(g^*)(y) = \frac{\int_0^{yT^*} g^*(r^*)dr^*}{\int_0^{T^*} g^*(r^*)dr^*}$$

$$= \frac{\int_{\psi^{-1}(0)}^{\psi^{-1}(yT^*)} g(r)\psi'(r)dr}{\int_{\psi^{-1}(0)}^{\psi^{-1}(T^*)} g(r)\psi'(r)dr}$$

, by (23). But by (55) and (59) we have

$$L(g^*)(y) = \frac{\int_0^{yT} g(r)dr}{\int_0^{T} g(r)dr} = L(g)(y)$$

by (25). Hence we have proved (60). $\square$

<u>Corollary III.8</u>:

For the general transformation (12):

$$g^*(r^*) = g^*(\psi(r)) = \varphi(g(r))$$

and ψ as in (55) we have that the theorem of Fellman and Jakobsson is valid.

Proof:

This follows trivially from the above proposition, applied to $\varphi \circ g$ instead of g:

$$L(g^*) = L(\varphi \circ g)$$

and on $L(\varphi \circ g)$ we can apply the theorem of Fellman and Jakobsson. W

Note also that, if $\mu = \mu^*$, $\varphi = \mathrm{Id}$ and $L(g) = L(g^*)$, then, for all $r \in [0,T]$:

$$\psi(r) = r\frac{T^*}{T}.$$

Indeed: $L(g) = L(g^*)$ implies $L'(g) = L'(g^*)$, hence, by (27) and (28) (and $\mu = \mu^*$):
$g(yT) = g^*(yT^*)$ for all $y \in [0,1]$. Hence, by (24)

$$g(yT) = g\left(\psi^{-1}\left(yT^*\right)\right)$$

Since g is injective we hence also have

$$yT = \psi^{-1}\left(yT^*\right)$$

so

$$\psi(yT) = yT^*$$

for all $y \in [0,1]$, whence (putting $r = yT$: this can be done for all $r \in [0,T]$)

$$\psi(r) = r\frac{T^*}{T}.$$

IV. Conclusions

In this paper we gave a new, short proof of the well-known theorem of Fellman and Jakobsson on the relation between $L(g)$ and $L(\varphi \circ g)$, where g is the rank-frequency function of a system and φ is a transformation. We prove that this theorem is false for the dual transformation $g \circ \psi^{-1}$ at least in all cases yielding a different value (after transformation) of the average number of items per source. This case comprises all convex and concave transformations ψ.

In the case $\mu = \mu^*$ we prove that the linear transformation

$$\psi(r) = r\frac{T^*}{T}$$

satisfies

$$L(g \circ \psi^{-1}) = L(g)$$

in which case the general transformation (12)

$$g^*(r^*) = g^*(\psi(r)) = \varphi(g(r))$$

yields

$$L(g^*) = L(\varphi \circ g)$$

and hence, in this case, the theorem of Fellman and Jakobsson is applicable.

We leave open the general study of the special case $\mu = \mu^*$.

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